ON THE ORDER OF BOREL SUBGROUPS OF GROUP AMALGAMS AND AN APPLICATION TO LOCALLY-TRANSITIVE GRAPHS

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Abstract. A permutation group is called *semiprimitive* if each of its normal subgroups is either transitive or semiregular. Given nontrivial finite transitive permutation groups L_1 and L_2 with L_1 not semiprimitive, we construct an infinite family of rank two amalgams of permutation type $[L_1, L_2]$ and Borel subgroups of strictly increasing order. As an application, we show that there is no bound on the order of edge-stabilisers in locally $[L_1, L_2]$ graphs.

We also consider the corresponding question for amalgams of rank $k \geq 3$. We completely resolve this by showing that the order of the Borel subgroup is bounded by the permutation type $[L_1, \ldots, L_k]$ only in the trivial case where each of $L_1, \ldots, L_k$ is regular.

1. Introduction

All graphs in this paper are connected, simple and locally finite. Let Γ be a graph, let v be a vertex of Γ and let G be a group of automorphisms of Γ. We denote by $\Gamma(v)$ the neighbourhood of v, by G_v the stabiliser of v in G, and by $G_v^{\Gamma(v)}$ the permutation group induced by G_v on $\Gamma(v)$. We say that Γ is G-locally-transitive if $G_v^{\Gamma(v)}$ is transitive for every vertex v of Γ. (This is easily seen to imply that G is transitive on the edges of Γ.)

The starting point for our investigations is a classical result of Goldschmidt [9], a consequence of which states that in a finite G-locally-transitive graph of valency three, the edge-stabilisers have order dividing 128. Inspired by this result, we introduce the following terminology.

Let L_1 and L_2 be finite transitive permutation groups, let $[L_1, L_2]$ denote the multiset containing L_1 and L_2 and let Γ be a G-locally-transitive graph. We say that (Γ, G) is *locally* $[L_1, L_2]$ if, for some edge $\{u, v\}$ of Γ, we have permutation isomorphisms $G_u^{\Gamma(u)} \cong L_1$ and $G_v^{\Gamma(v)} \cong L_2$.

Definition 1.1. The multiset $[L_1, L_2]$ is *locally-restrictive* if there exists a constant $c \in \mathbb{N}$ such that, if Γ is a finite G-locally-transitive graph with (Γ, G) locally $[L_1, L_2]$ and $\{u, v\}$ is an edge of Γ, then $|G_{uv}| \leq c$.

With this terminology, Goldschmidt's result implies that, if L_1 and L_2 are transitive permutation groups of degree three then $[L_1, L_2]$ is locally-restrictive. A related conjecture of Goldschmidt-Sims states that if L_1 and L_2 are both primitive permutation groups then

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$[L_1, L_2]$ is locally-restrictive. Whilst there has been some progress on the Goldschmidt-Sims Conjecture (see [4, 5, 8, 12, 16, 21]), it remains open. Although the focus of the Goldschmidt-Sims Conjecture is on primitive permutation groups, it is still possible for $[L_1, L_2]$ to be locally-restrictive even when neither L_1 nor L_2 is primitive. For example, it is easy to see that if L_1 and L_2 are both regular permutation groups then $[L_1, L_2]$ is locally-restrictive. We therefore pose the following problem.

Problem 1.2. Determine which pairs of finite transitive permutation groups are locally-restrictive.

Our main result, Theorem 1.3, is a significant step towards solving Problem 1.2.

Theorem 1.3. *Let L_1 and L_2 be nontrivial finite transitive permutation groups. If one of L_1 or L_2 is not semiprimitive then $[L_1, L_2]$ is not locally-restrictive.*

(A permutation group is called *semiregular* if the identity is the only element of the group that fixes a point and *semiprimitive* if each of its normal subgroups is either transitive or semiregular.) In view of Theorem 1.3, we are naturally led to pose the following question, the answer to which we believe to be positive.

Question 1.4. *Does the converse of Theorem* 1.3 *hold? In other words, if L_1 and L_2 are finite transitive semiprimitive permutation groups, is $[L_1, L_2]$ locally-restrictive?*

Our notion of locally-restrictive is to locally-transitive graphs what the notion of graph-restrictive (see [13, Definition 2]) is to arc-transitive graphs. Many of the concepts and results we have discussed so far have well-known analogues in the arc-transitive case. For example, Goldschmidt's Theorem can be seen as the locally-transitive version of Tutte's famous result on arc-transitive graphs of valency three [23, 24]. Similarly, the Goldschmidt-Sims Conjecture corresponds to the long-standing Weiss Conjecture [26] which asserts that primitive groups are graph-restrictive. The recent Potočnik-Spiga-Verret Conjecture [13, Conjecture 3] asserts that a permutation group is graph-restrictive if and only if it is semiprimitive. Remarkable evidence towards this conjecture can be found in [20], where the intransitive case is dealt with. For recent progress on the transitive case, see [6, 7, 14, 15, 18].

We remark that an affirmative answer to Question 1.4 would show the validity of both the Weiss and Potočnik-Spiga-Verret Conjectures. In fact, these conjectures can easily be rephrased using our terminology. Indeed, if (Γ, G) is locally $[L, \mathrm{C}_2]$ (where C_2 denotes the cyclic group of order 2) then Γ is simply the barycentric subdivision of a graph $\tilde{\Gamma}$ on which G acts faithfully and arc-transitively and such that $(\tilde{\Gamma}, G)$ is locally $[L, L]$. There is an obvious converse to this procedure, thus $[L, \mathrm{C}_2]$ is locally-restrictive if and only if L is graph-restrictive.

The theory of groups acting on trees due to Bass-Serre allows us to interpret Question 1.4 in terms of locally-transitive discrete subgroups of the automorphism group of a bi-regular tree. This equivalence will be proved in Section 2. For now, let us simply point out that, under this equivalence, Theorem 1.3 is equivalent to Theorem 1.5 below.

Theorem 1.5. *Let L_1 and L_2 be nontrivial finite transitive permutation groups and let $\mathfrak{T}$ be the bi-regular tree with valencies the degrees of L_1 and L_2. If one of L_1 and L_2 is not semiprimitive then, for every integer c, there exists a group G of automorphisms of $\mathfrak{T}$ such that $(\mathfrak{T}, G)$ is locally $[L_1, L_2]$ and $c \leq |G_{uv}| < \infty$ for some edge $\{u, v\}$ of $\mathfrak{T}$.*

Theorem 1.5 is a significant improvement of [1, (7.14)] which shows that the automorphism group of a bi-regular tree with composite valencies contains strictly ascending chains of locally-transitive discrete subgroups, but with no control over the local permutation groups.

Question 1.4 also has a natural formulation in terms of group amalgams of rank two. Before presenting our result using this language, we first define amalgams, following [10].

Definition 1.6. Let $k \geq 2$. A *rank k amalgam* is a finite set $\mathcal{A}$ together with a set of k subsets $P_1, \dots, P_k$, where each P_i forms a group, $\bigcup_{i=1}^k P_i = \mathcal{A}$, $\bigcap_{i=1}^k P_i \neq \emptyset$ and, for every $i, j \in \{1, \dots, k\}$ the group operations defined on P_i and P_j coincide when restricted to $P_i \cap P_j$.

The *Borel subgroup* of $\mathcal{A}$ is $\bigcap_{i=1}^k P_i$ and is denoted $\mathcal{B}(\mathcal{A})$. If there is no nontrivial subgroup of $\mathcal{B}(\mathcal{A})$ that is normalised by each of $P_1, \dots, P_k$ then we say that $\mathcal{A}$ is *faithful*. The *permutation type* of $\mathcal{A}$ is the multiset $[L_1, \dots, L_k]$ where L_i is the permutation group induced by P_i in its action on the right cosets of $\mathcal{B}(\mathcal{A})$ in P_i.

In Section 2 we will show the equivalence between Theorem 1.3 and Theorem 1.7 below.

Theorem 1.7. *Let L_1 and L_2 be nontrivial finite transitive permutation groups. If one of L_1 and L_2 is not semiprimitive then, for every integer c, there exists a rank two faithful amalgam of permutation type $[L_1, L_2]$ with Borel subgroup of order at least c.*

Our proof of Theorem 1.7 is constructive and can be found in Section 3. The construction used in the proof is a generalisation of the construction that appeared in [13, Section 4] (in fact, Theorem 1.3 generalises [13, Theorem 4]) which in turn was inspired by the so-called wreath extension construction [11, Chapter IV, 8.1]. A precursory idea to this construction can also be traced to [19, Section 4].

Theorem 1.7 naturally leads one to wonder about the corresponding statement for amalgams of rank greater than two. Here is the complete answer in this case.

Theorem 1.8. *Let $k \geq 3$ and let $L_1, \dots, L_k$ be nontrivial finite transitive permutation groups. The following are equivalent:*

(1) *One of $L_1, \dots, L_k$ is not regular.*
(2) *For every integer c, there exists a rank k faithful amalgam of permutation type $[L_1, \dots, L_k]$ with Borel subgroup of order at least c.*

In fact, it is easy to see that if $L_1, \dots, L_k$ are regular then a faithful amalgam $\mathcal{A}$ of permutation type $[L_1, \dots, L_k]$ must have $\mathcal{B}(\mathcal{A}) = 1$ (see Section 4). The real meat of Theorem 1.8 is therefore the statement that for rank at least three, these trivial examples are the only ones which admit upper bounds on $|\mathcal{B}(\mathcal{A})|$ depending upon the permutation type alone. This is in sharp contrast with the situation in the rank two case; Goldschmidt's result has the highly nontrivial consequence that a rank two faithful amalgam of permutation type $[\mathrm{Sym}(3), \mathrm{Sym}(3)]$ has Borel subgroup of order at most 128. In particular, the naïve "$k = 2$" version of Theorem 1.8 is false. We find the relative simplicity of the higher rank case rather surprising. Note that under additional assumptions, the order of the Borel subgroup of a rank three amalgam can be bounded, see for example [22].

2. Equivalence of Theorems 1.3, 1.5 and 1.7

Lemma 2.1. *Let L_1 and L_2 be finite transitive permutation groups and let B be a finite group. The following are equivalent:*

1. *There exists a rank two faithful amalgam of permutation type $[L_1, L_2]$ with Borel subgroup B.*
2. *There exists a locally $[L_1, L_2]$ pair $(\mathfrak{T}, G)$ such that $\mathfrak{T}$ is an infinite tree and $G_{uv} = B$ for some edge $\{u, v\}$ of $\mathfrak{T}$.*
3. *There exists a locally $[L_1, L_2]$ pair (Γ, H) such that Γ is finite and $H_{uv} = B$ for some edge $\{u, v\}$ of Γ.*

Proof. (1) $\Longrightarrow$ (2) : Let $\mathcal{A}$ be a rank two faithful amalgam of permutation type $[L_1, L_2]$ with Borel subgroup B. Let P_1 and P_2 be the two groups involved in $\mathcal{A}$ and let $G = P_1 *_B P_2$ (that is, G is the free product of P_1 and P_2 amalgamated over B). By [17, I.4.1, Theorem 7], there exists an infinite tree $\mathfrak{T}$ on which G acts faithfully, edge- but not vertex-transitively, and an edge $\{u, v\}$ of $\mathfrak{T}$ such that $G_{uv} = B$, $G_u = P_1$ and $G_v = P_2$. As $\mathfrak{T}$ is G-edge- but not G-vertex-transitive, it must be G-locally-transitive. Since $\mathcal{A}$ is of permutation type $[L_1, L_2]$, it follows that $(\mathfrak{T}, G)$ is locally $[L_1, L_2]$.

(2) $\Longrightarrow$ (3) : Let G' be the largest subgroup of G that preserves the bipartition of $\mathfrak{T}$. Note that $(\mathfrak{T}, G')$ is locally $[L_1, L_2]$ and $G'_{uv} = B$. By replacing G with G', we may thus assume that G is not vertex-transitive. By [17, I.4.1, Theorem 6], it follows that G is isomorphic to $G_u *_B G_v$.

By [17, I.4.1, Proof of Theorem 7], we may assume that the vertex set of $\mathfrak{T}$ is the disjoint union of the right coset spaces G/G_u and G/G_v, with two vertices being adjacent if they have nonempty intersection, and that the action of G on $\mathfrak{T}$ is given by right multiplication. In particular, G_u and G_v are adjacent when viewed as vertices of $\mathfrak{T}$. Since $\mathfrak{T}$ is G-locally-transitive it follows that the neighbourhood of G_u is $\{G_v g \mid g \in G_u\}$.

Let $X = G_u G_v G_u \cup G_v G_u G_v$. Note that, as L_1, L_2 and B are finite, so are G_u, G_v and X. By [3, Theorem 2], G is residually finite and hence there exists a normal subgroup R of finite index in G with $R \cap X = \{1\}$. Let $H = G/R$ and let Γ be the normal quotient graph $\mathfrak{T}/R$. (The vertices of Γ are the R-orbits on the vertex set of $\mathfrak{T}$, with two such R-orbits adjacent in Γ if there is an edge between them in $\mathfrak{T}$.) Note that Γ is H-locally-transitive and finite.

Since $\mathfrak{T}$ is locally-transitive, it is bi-regular. We now show that Γ is bi-regular with the same valencies as $\mathfrak{T}$. We argue by contradiction and suppose, without loss of generality, that the R-orbit of G_u, viewed as a vertex of Γ, has valency strictly less than G_u, viewed as a vertex of $\mathfrak{T}$. It follows from the definition of Γ that the vertex G_u of $\mathfrak{T}$ must have two distinct neighbours in the same R-orbit. Recall that the neighbourhood of G_u in $\mathfrak{T}$ is $\{G_v g \mid g \in G_u\}$. In particular, there exist $g, h \in G_u$ and $r \in R$ such that $G_v g \neq G_v h$ and $G_v gr = G_v h$. This implies that $r \in G_u G_v G_u \subseteq X$ and hence $r \in R \cap X = \{1\}$, which is a contradiction.

Let K/R be the kernel of the action of $H = G/R$ on Γ. By the previous paragraph, Γ is bi-regular with the same valencies as $\mathfrak{T}$ and a standard argument yields that K is semiregular on $\mathfrak{T}$. In particular, $K = R$ (and $\mathfrak{T}$ is a regular cover of Γ) and H acts faithfully on Γ. It follows that the stabiliser in H of the vertex $G_u R$ of Γ is $G_u R/R \cong G_u$, the stabiliser of the vertex $G_v R$ is $G_v R/R \cong G_v$ and the stabiliser of the edge $\{G_u R, G_v R\}$ is $BR/R \cong B$. Since $(\mathfrak{T}, G)$ is locally $[L_1, L_2]$, this implies that (Γ, H) is locally $[L_1, L_2]$.

(3) $\implies$ (1) : Let $\mathcal{A}$ be the rank two amalgam of the groups H_u and H_v with Borel subgroup H_{uv}. Since Γ is H-locally-transitive, the group generated by H_u and H_v is transitive on edges of Γ. In particular, any subgroup of H_{uv} that is normalised by both H_u and H_v must be trivial. This shows that $\mathcal{A}$ is faithful. Clearly, $\mathcal{A}$ has permutation type $[L_1, L_2]$. □

The following is an immediate corollary to Lemma 2.1.

Corollary 2.2. *Let L_1 and L_2 be finite transitive permutation groups. The following are equivalent:*

1. *For every integer c, there exists a rank two faithful amalgam of permutation type $[L_1, L_2]$ with Borel subgroup of order at least c.*
2. *For every integer c, there exists a locally $[L_1, L_2]$ pair $(\mathfrak{T}, G)$ such that $\mathfrak{T}$ is an infinite tree and $c \leq |G_{uv}| < \infty$ for some edge $\{u, v\}$ of $\mathfrak{T}$.*
3. *$[L_1, L_2]$ is not locally-restrictive.*

The equivalence of Theorems 1.3, 1.5 and 1.7 follows immediately from Corollary 2.2.

3. Proof of Theorem 1.7

All groups mentioned in the next two sections are finite. We adopt the notation and hypothesis of Theorem 1.7 and, without loss of generality, we assume that L_1 is not semiprimitive. To simplify notation, we write $L = L_1$ and $R = L_2$. Let m_2 be the degree of R, let ℓ be a positive integer, let $m = \ell m_2$ and let $\Omega = \{(y, z) \mid 1 \leq y \leq m_2, 1 \leq z \leq \ell\}$. Observe that $|\Omega| = \ell m_2 = m$ and that the action of R on $\{1, \ldots, m_2\}$ induces an action of R on Ω: for $r \in R$ and $(y, z) \in \Omega$, we set

$$(y, z)^r = (y^r, z).$$

We endow the set Ω with its natural lexicographic order, that is $(y, z) < (y', z')$ if either $y < y'$, or $y = y'$ and $z < z'$. This total ordering allows us to identify Ω with $\{0, \ldots, m-1\}$ in a natural way : $(1, 1)$ is identified with 0, (m_2, ℓ) with $m - 1$, etc. We extend the action of R on $\Omega = \{0, \ldots, m-1\}$ to an action of R on $\{0, \ldots, m\}$ by letting the point m be fixed by every element of R.

Since L is not semiprimitive, there exists a normal subgroup K of L that is neither transitive nor semiregular. Denote by Δ the set of orbits of K and let K' be the kernel of the action of L on Δ. Note that K' is a normal subgroup of L having the same orbits as K that is neither transitive nor semiregular. We may thus assume that $K = K'$ without loss of generality. Let S denote the permutation group induced by the action of L on Δ and let $\pi : L \to S$ be the canonical projection with kernel K. Fix $\delta \in \Delta$ and $\lambda \in \delta$. Since K is transitive on δ, we have $L_\delta = KL_\lambda$ and $S_\delta \cong L_\delta/K = KL_\lambda/K \cong L_\lambda/(K \cap L_\lambda) = L_\lambda/K_\lambda$. We sometimes denote by π the restriction $\pi|_{L_\lambda} : L_\lambda \to S_\delta$, slightly abusing notation.

Fix $\mathcal{T}$ a transversal for the set of right cosets of S_δ in S with $1 \in \mathcal{T}$. For every $s \in S$, there exists a unique element of $\mathcal{T}$, which we denote by s^τ, such that $S_\delta s = S_\delta s^\tau$. The correspondence $s \mapsto s^\tau$ defines a map $\tau : S \to \mathcal{T}$ with $1^\tau = 1$.

Lemma 3.1. *If $x, s \in S$, then $(xs^{-1})^\tau s (x^\tau)^{-1} \in S_\delta$.*

Proof. We have $S_\delta xs^{-1} = S_\delta (xs^{-1})^\tau$ and hence $S_\delta x = S_\delta (xs^{-1})^\tau s$. Furthermore, as $S_\delta x = S_\delta x^\tau$, we obtain $S_\delta = S_\delta (xs^{-1})^\tau s (x^\tau)^{-1}$. □

Let V be the set of all functions from Δ to L_λ. Under point-wise multiplication, V is a group isomorphic to L_λ^Δ. Given $f \in V$ and $g \in L$, let f^g be the element of V defined by

$$f^g(\sigma) = f\left(\sigma^{(g^\pi)^{-1}}\right),\ \sigma \in \Delta. \tag{3.1}$$

This defines a group action of L on V and the semidirect product $V \rtimes L$ is isomorphic to the standard wreath product $L_\lambda \operatorname{wr}_\Delta L$. Moreover, by extending this action of L on V to the component-wise action of L on V^m, we obtain a semidirect product $L \ltimes V^m$ where the multiplication is given by

$$(g, f_1, \ldots, f_m)(g', f_1', \ldots, f_m') \quad = \quad (gg', f_1^{g'} f_1', \ldots, f_m^{g'} f_m'). \tag{3.2}$$

We now isolate some subgroups of $L \ltimes V^m$ that provide the backbone for our construction.

Definition 3.2. We define the following subsets of $L \ltimes V^m$:

$$\begin{aligned} A = &\quad \left\{(g, f_1, \ldots, f_m) \in L \ltimes V^m \;\middle|\; \begin{array}{c} (f_i(\delta^x))^\pi = (x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1} \\ \text{for every } i \in \{1, \ldots, m\} \text{ and for every } x \in S \end{array}\right\}, \\ C = &\quad \{(g, f_1, \ldots, f_m) \in A \mid g \in L_\lambda\}, \\ M = &\quad \{(1, f_1, \ldots, f_m) \in L \ltimes V^m \mid f_i(\Delta) \subseteq K_\lambda \text{ for every } i \in \{1, \ldots, m\}\}. \end{aligned}$$

Let $\varphi : A \to L$ be the map defined by $\varphi : (g, f_1, \ldots, f_m) \mapsto g$.

Note that, by Lemma 3.1, the element $(x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1}$ in the definition of A lies in S_δ.

Lemma 3.3. *The set A is a subgroup of $L \ltimes V^m$.*

Proof. Let $(g, f_1, \ldots, f_m), (g', f_1', \ldots, f_m') \in A$ and let $x \in S$. For every $i \in \{1, \ldots, m\}$, we have

$$\begin{aligned} ((f_i^{g'} f_i')(\delta^x))^\pi \quad &= \quad (f_i^{g'}(\delta^x))^\pi \cdot (f_i'(\delta^x))^\pi \\ &\overset{(3.1)}{=} \quad (f_i(\delta^{x(g'^\pi)^{-1}}))^\pi \cdot (f_i'(\delta^x))^\pi \\ &\overset{\text{Def. 3.2}}{=} \quad (((x(g'^\pi)^{-1})(g^\pi)^{-1})^\tau g^\pi ((x(g'^\pi)^{-1})^\tau)^{-1}) \cdot ((x(g'^\pi)^{-1})^\tau g'^\pi (x^\tau)^{-1}) \\ &= \quad (x((gg')^\pi)^{-1})^\tau (gg')^\pi (x^\tau)^{-1}. \end{aligned}$$

Using (3.2) and Definition 3.2, this shows that $(g, f_1, \ldots, f_m)(g', f_1', \ldots, f_m') \in A$. Clearly, the identity of $L \ltimes V^m$ is contained in A. Since $L \ltimes V^m$ is a finite group, this concludes the proof. □

Lemma 3.4. *The map φ is a surjective homomorphism.*

Proof. By (3.2), φ is a homomorphism. For each $s \in S_\delta$, choose an element s^ε of L_λ with $(s^\varepsilon)^\pi = s$. Since $\pi : L_\lambda \to S_\delta$ is surjective, $\varepsilon : S_\delta \to L_\lambda$ is well-defined. Let $g \in L$ and define $f_g : \Delta \to L_\lambda$ with

$$f_g(\delta^x) = ((x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1})^\varepsilon,$$

for $x \in S$. First, note that, by Lemma 3.1, $(x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1} \in S_\delta$ and thus $f_g(\delta^x) \in L_\lambda$. To see that f_g is well-defined, note that for every $y \in S_\delta$, we have $(yx)^\tau = x^\tau$ and $(yx(g^\pi)^{-1})^\tau = (x(g^\pi)^{-1})^\tau$, and hence $f_g(\delta^x) = f_g(\delta^{yx})$. Now

$$(f_g(\delta^x))^\pi = (((x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1})^\varepsilon)^\pi = (x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1},$$

and hence $(g, f_g, \ldots, f_g) \in A$ and $(g, f_g, \ldots, f_g)^\varphi = g$, which concludes the proof. □

Lemma 3.5. *The kernel of* φ *is* M *and* $M \cong K_\lambda^{|\Delta| m}$.

Proof. It is clear that $M \cong K_\lambda^{|\Delta| m}$. Suppose first that $(g, f_1, \ldots, f_m)$ is in the kernel of φ then $g = (g, f_1, \ldots, f_m)^\varphi = 1$. For every $i \in \{1, \ldots, m\}$ and every $x \in S$, it follows by Definition 3.2 that $(f_i(\delta^x))^\pi = (x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1} = 1$ and thus $f_i(\delta^x) \in K_\lambda$. It follows that $(g, f_1, \ldots, f_m) \in M$.

Conversely, if $(g, f_1, \ldots, f_m) \in M$ then $g = 1$ and $f_i(\delta^x) \in K_\lambda$ for every $i \in \{1, \ldots, m\}$ and every $x \in S$ and thus $(f_i(\delta^x))^\pi = 1 = (x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1}$. In particular, $(g, f_1, \ldots, f_m) \in A$ and hence $(g, f_1, \ldots, f_m)$ is in the kernel of φ. □

Lemma 3.6. *The set* C *is a subgroup of* A*,* M *is the core of* C *in* A *and the permutation group induced by the action of* A *on the right cosets of* C *is permutation isomorphic to* L.

Proof. By Lemmas 3.4 and 3.5, φ is a surjective homomorphism with kernel M. In particular, $M \trianglelefteq A$ and $A/M \cong L$. Note that C is the pre-image of L_λ under φ and thus $M \leq C \leq A$. As $C^\varphi = L_\lambda$ and L_λ is core-free in L, it follows that M is the core of C in A. Finally, the action of A on the right cosets of C is permutation isomorphic to the action of $A^\varphi = L$ on the right cosets of $C^\varphi = L_\lambda$, that is, to L. □

We now introduce an alternative notation for the elements of A that will simplify some later computations. Let $a = (g, f_1, \ldots, f_m) \in A$. For $i \in \{1, \ldots, m\}$, we write $g_i = f_i(\delta)$ and $h_{i-1} = f_i|_{\Delta \setminus \{\delta\}}$. (Note that f_i is completely determined by (g_i, h_{i-1}).) We also write $g_0 = g$ and then denote a by

$$((g_0, \ldots, g_m), (h_0, \ldots, h_{m-1})).$$

Note that, with this notation, the multiplication is not component-wise (in contrast to (3.2)): indeed, if $a' = ((g_0', \ldots, g_m'), (h_0', \ldots, h_{m-1}'))$ is another element of A then

$$aa' = ((g_0 g_0', g_1^{g_0'} g_1' \ldots, g_m^{g_0'} g_m'), (h_0^{g_0'} h_0', \ldots, h_{m-1}^{g_0'} h_{m-1}')). \tag{3.3}$$

Using the above notation, for each $r \in R$ and $c = ((g_0, \ldots, g_m), (h_0, \ldots, h_{m-1})) \in C$, let

$$c^r = ((g_{0r^{-1}}, g_{1r^{-1}}, \ldots, g_{mr^{-1}}), (h_{0r^{-1}}, \ldots, h_{(m-1)r^{-1}})), \tag{3.4}$$

where, for $i \in \{0, \ldots, m\}$, we denote the image of i under r by ir.

Lemma 3.7. *Equation* (3.4) *defines a group action of* R *on the group* C.

Proof. In this proof, it is convenient to use both notations for elements of C. Let $c = (g, f_1, \ldots, f_m) = ((g_0, \ldots, g_m), (h_0, \ldots, h_{m-1})) \in C$. Since $c \in C$, we have $g_0 = g \in L_\lambda$ and hence $g^\pi \in S_\delta$. Since $c \in A$, for every $x \in S_\delta$ and $i \in \{1, \ldots, m\}$, we have $x^\tau = 1 = (x(g^\pi)^{-1})^\tau$ and thus

$$g_i^\pi = (f_i(\delta))^\pi = (f_i(\delta^x))^\pi \overset{\text{Def. 3.2}}{=} (x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1} = g^\pi. \tag{3.5}$$

Let $r \in R$ and write $v = r^{-1}$ and $c^r = (g_{0v}, f_1', \ldots, f_m')$. We first show that $c^r \in C$. For every $x \in S_\delta$ and $i \in \{1, \ldots, m\}$, we have

$$(f_i'(\delta^x))^\pi = (f_i'(\delta))^\pi \overset{(3.4)}{=} (g_{iv})^\pi \overset{(3.5)}{=} g_{0v}^\pi = (x(g_{0v}^\pi)^{-1})^\tau g_{0v}^\pi (x^\tau)^{-1}$$

where in the last equality $x, g_{0v}^\pi \in S_\delta$ is used. Similarly, for every $x \in S \setminus S_\delta$ and $i \in \{1, \ldots, m\}$, we have

$$(f_i'(\delta^x))^\pi \overset{(3.4)}{=} (h_{(i-1)v}(\delta^x))^\pi \overset{\text{Def. 3.2}}{=} (x(g^\pi)^{-1})^\tau g^\pi (x^\tau)^{-1} \overset{(3.5)}{=} (x(g_{0v}^\pi)^{-1})^\tau g_{0v}^\pi (x^\tau)^{-1}.$$

This shows that $c^r \in A$. Since $g_i \in L_\lambda$ for all $i \in \{0,\dots,m\}$, we have that $g_{0v} \in L_\lambda$ and thus $c^r \in C$. Let $d = ((y_0, y_1, \dots, y_m), (z_0, \dots, z_{m-1})) \in C$. Recall that $y_0 \in L_\lambda$. Hence, for $j \in \{1,\dots,m\}$, we have

$$g_j^{y_0} = f_j^{y_0}(\delta) \overset{(3.1)}{=} f_j(\delta^{(y_0^\pi)^{-1}}) = f_j(\delta) = g_j. \tag{3.6}$$

Now,

$$\begin{aligned} cd &\overset{(3.3)}{=} ((g_0y_0, g_1^{y_0}y_1, \dots, g_m^{y_0}y_m), (h_0^{y_0}z_0, \dots, h_{m-1}^{y_0}z_{m-1})) \\ &\overset{(3.6)}{=} ((g_0y_0, g_1y_1, \dots, g_my_m), (h_0^{y_0}z_0, \dots, h_{m-1}^{y_0}z_{m-1})) \end{aligned}$$

and thus

$$(cd)^r = ((g_{0v}y_{0v}, g_{1v}y_{1v}, \dots, g_{mv}y_{mv}), (h_{0v}^{y_0}z_{0v}, \dots, h_{(m-1)v}^{y_0}z_{(m-1)v})).$$

Recall that $c^r = ((g_{0v}, \dots, g_{mv}), (h_{0v}, \dots, h_{(m-1)v})) = (g_{0v}, f_1', \dots, f_m')$ and thus $f_i'(\delta) = g_{iv}$ and $f_i'(\sigma) = h_{(i-1)v}(\sigma)$ for $i \in \{1,\dots,m\}$ and $\sigma \in \Delta \setminus \{\delta\}$. Similarly, recall that $d^r = ((y_{0v}, \dots, y_{mv}), (z_{0v}, \dots, z_{(m-1)v}))$ and write $d^r = (y_{0v}, e_1', \dots, e_m')$. By (3.2), we have $c^rd^r = (g_{0v}y_{0v}, f_1'^{y_{0v}}e_1', \dots, f_m'^{y_{0v}}e_m')$. Since $y_{0v} \in L_\lambda$, we have $y_{0v}^\pi \in S_\delta$ and hence, for $i \in \{1,\dots,m\}$, we have

$$(f_i'^{y_{0v}}e_i')(\delta) = f_i'^{y_{0v}}(\delta)e_i'(\delta) \overset{(3.1)}{=} f_i'(\delta^{(y_{0v}^\pi)^{-1}})y_{iv} = f_i'(\delta)y_{iv} = g_{iv}y_{iv}.$$

Similarly, for $\sigma \in \Delta \setminus \{\delta\}$ and $i \in \{1,\dots,m\}$, we have

$$f_i'^{y_{0v}}(\sigma) \overset{(3.1)}{=} f_i'(\sigma^{(y_{0v}^\pi)^{-1}}) = h_{(i-1)v}(\sigma^{(y_{0v}^\pi)^{-1}}) \overset{(3.5)}{=} h_{(i-1)v}(\sigma^{(y_0^\pi)^{-1}}) \overset{(3.1)}{=} h_{(i-1)v}^{y_0}(\sigma)$$

and thus $(f_i'^{y_{0v}}e_i')(\sigma) = (h_{(i-1)v}^{y_0}z_{(i-1)v})(\sigma)$. This shows that $(cd)^r = c^rd^r$.

It is clear from (3.4) that $(c^r)^{r'} = c^{rr'}$ for every $r' \in R$ and $c^r = 1$ if and only if $c = 1$, which concludes the proof. □

By Lemma 3.7, we can define the semidirect product $C \rtimes R$. Let

$$\begin{aligned} P_2 &= C \rtimes R, \\ B &= C \rtimes R_1, \end{aligned}$$

with B viewed as a subgroup of P_2. From our definitions we have:

Lemma 3.8. *The core of B in P_2 is C. Moreover, the permutation group induced by the action of P_2 on the right cosets of B is permutation isomorphic to R.*

From (3.4), R_1 inherits an action on C from R. We extend this to an action of R_1 on A in the following way: given $a = ((g_0, \dots, g_m), (h_0, \dots, h_{m-1})) \in A$ and $r \in R_1$, let

$$a^r = ((g_{0r^{-1}}, g_{1r^{-1}}, \dots, g_{mr^{-1}}), (h_{0r^{-1}}, \dots, h_{(m-1)r^{-1}})).$$

With minor changes, the proof of Lemma 3.7 can be adapted to show that this induces a group action of R_1 on A. (It is helpful to notice that for all $r \in R_1$ we have $0r = 0$.) Let

$$P_1 = A \rtimes R_1.$$

We view B as a subgroup of P_1 in the obvious way. (Note that the action of R on C cannot be extended to an action of R on A in any meaningful way.)

Lemma 3.9. *The core of B in P_1 is $M \rtimes R_1$ and the action of P_1 on the right cosets of B is permutation isomorphic to L.*

Proof. The proof follows with a computation and from Lemma 3.6. □

Let $\mathcal{A}$ be the rank two amalgam of the groups P_1 and P_2 with $\mathcal{B}(\mathcal{A}) = P_1 \cap P_2 = B$. Lemmas 3.8 and 3.9 show that the permutation type of $\mathcal{A}$ is $[L_1, L_2]$.

Proposition 3.10. *The amalgam $\mathcal{A}$ is faithful.*

Proof. Let N be a subgroup of B normal in P_1 and in P_2. We show that $N = 1$. By Lemma 3.8, the core of B in P_2 is C and hence $N \leq C$. By Lemma 3.9, the core of B in P_1 is $M \rtimes R_1$ and thus $N \leq C \cap (M \rtimes R_1) = M$.

For $i \in \{0, \ldots, m\}$, let $G(i)$ be the proposition: for every $((g_0, \ldots, g_m), (h_0, \ldots, h_{m-1})) \in N$, we have $g_i = 1$. Similarly, for $i \in \{0, \ldots, m-1\}$, let $H(i)$ be the proposition: for every $((g_0, \ldots, g_m), (h_0, \ldots, h_{m-1})) \in N$, we have $h_i = 1$. We prove the following preliminary claims.

Claim 1. Let $i \in \{1, \ldots, m\}$ and let $\sigma \in \Delta$. Suppose that, for every $(1, f_1, \ldots, f_m) \in N$, we have $f_i(\sigma) = 1$. Then $G(i)$ and $H(i-1)$ hold.

Let $(1, f_1, \ldots, f_m) \in N$ and let $\mu \in \Delta$. Since S is transitive on Δ and π is surjective, there exists $g \in L$ such that $\sigma^{(g^\pi)^{-1}} = \mu$. By Lemma 3.4, there exists $(f_1', \ldots, f_m') \in V^m$ with $(g, f_1', \ldots, f_m') \in A$. As $N \trianglelefteq A$,

$$(1, f_1, \ldots, f_m)^{(g, f_1', \ldots, f_m')} = (1, f_1'^{-1} f_1^g f_1', \ldots, f_m'^{-1} f_m^g f_m') \in N.$$

By hypothesis, we have $1 = (f_i'^{-1} f_i^g f_i')(\sigma) = f_i'(\sigma)^{-1} f_i(\sigma^{(g^\pi)^{-1}}) f_i'(\sigma) = f_i(\mu)$. Since μ is an arbitrary element of Δ we obtain $f_i = 1$. Since $(1, f_1, \ldots, f_m)$ was an arbitrary element of N, it follows that $G(i)$ and $H(i-1)$ hold. ■

Claim 2. Let $i \in \{1, \ldots, m\}$. Then $G(i) \iff H(i-1)$.

Suppose that $G(i)$ holds. Applying Claim 1 with $\sigma = \delta$, we immediately obtain $H(i-1)$. Conversely, if $H(i-1)$ holds then applying Claim 1 with some $\sigma \in \Delta \setminus \{\delta\}$, we obtain $G(i)$. ■

Claim 3. Let $i \in \{0, \ldots, m-1\}$ and let j be in the R-orbit of i. Then $G(i) \Longrightarrow G(j)$ and $H(i) \Longrightarrow H(j)$.

Assume that $G(i)$ holds and let $n = ((g_0, \ldots, g_m), (h_0, \ldots, h_{m-1})) \in N$. There exists $r \in R$ such that $ir^{-1} = j$. Since $R \leq P_2$, N is normalised by R and $n^r \in N$. By (3.4), this implies that $((g_{0r^{-1}}, g_{1r^{-1}}, \ldots, g_{mr^{-1}}), (h_{0r^{-1}}, \ldots, h_{(m-1)r^{-1}})) \in N$. Since $G(i)$ holds, we have that $g_j = g_{ir^{-1}} = 1$. As n was an arbitrary element of N, this shows that $G(j)$ holds. The proof that $H(i) \Longrightarrow H(j)$ is essentially the same and is omitted. ■

Claim 4. $G(i)$ holds for every $i \in \{0, \ldots, m\}$.

We argue by contradiction and let z be minimal in $\{0, \ldots, m\}$ such that $G(z)$ does not hold. Since $N \leq M$, we have that $G(0)$ holds and thus $z \geq 1$. By Claim 2, we see that $H(z-1)$ does not hold.

Let $\mathcal{O}$ be the R-orbit on $\{0, \ldots, m\}$ containing z. By the minimality of z and Claim 3, we get that z is the minimum of $\mathcal{O}$. By examining the orbits of R on $\{0, \ldots, m\}$, we see that this implies that $z-1$ and $z-2$ are in the same R-orbit. Since $H(z-1)$ does not hold, Claim 3 implies that neither does $H(z-2)$. By Claim 2, neither does $G(z-1)$, contradicting the minimality of z. ■

Claim 2 together with Claim 4 implies that $H(i)$ holds for every $i \in \{0,\dots,m-1\}$ and thus $N=1$. This concludes the proof. □

Finally, we have $|B| \geq |C| \geq |M| = |K_\lambda|^{|\Delta|m}$, where the last equality follows by Lemma 3.5. Recall that $m = \ell m_2$. Since K is not semiregular, we have $|K_\lambda| \geq 2$ and thus $|B| \to \infty$ as $\ell \to \infty$. This concludes the proof of Theorem 1.7.

4. Proof of Theorem 1.8

As in the previous section, all groups considered are finite. Suppose first that $L_1,\dots,L_k$ are regular permutation groups and let $\mathcal{A} = \bigcup_{i=1}^k P_i$ be a rank k faithful amalgam of permutation type $[L_1,\dots,L_k]$. Since L_i is regular, we have $\mathcal{B}(\mathcal{A}) \trianglelefteq P_i$ for every $i \in \{1,\dots,k\}$. As $\mathcal{A}$ is faithful, this implies that $\mathcal{B}(\mathcal{A}) = 1$. This proves the implication (2) $\Longrightarrow$ (1) of Theorem 1.8.

We now turn to the proof of the implication (1) $\Longrightarrow$ (2). The following lemma will be needed.

Lemma 4.1. *Let H and K be transitive permutation groups on Δ and Λ, respectively. Let $\delta_0 \in \Delta$, $\lambda_0 \in \Lambda$ and let ℓ be a positive integer. If $|\Delta|,|\Lambda| \geq 2$ then there exist a set Ω of cardinality $\ell|\Delta||\Lambda|$, faithful group actions $\rho_H : H \to \mathrm{Sym}(\Omega)$ and $\rho_K : K \to \mathrm{Sym}(\Omega)$, and $\omega \in \Omega$ such that*

1. *$\rho_H(H)_\omega = \rho_H(H_{\delta_0})$ and $\rho_K(K)_\omega = \rho_K(K_{\lambda_0})$;*
2. *$\langle \rho_H(H), \rho_K(K)_\omega\rangle = \rho_H(H) \times \rho_K(K)_\omega$ and $\langle \rho_H(H)_\omega, \rho_K(K)\rangle = \rho_H(H)_\omega \times \rho_K(K)$;*
3. *$\langle \rho_H(H), \rho_K(K)\rangle$ is transitive on Ω.*

Proof. Let Ω be the set $\Delta \times \Lambda \times \mathbb{Z}_\ell$ and let $\omega = (\delta_0, \lambda_0, 0) \in \Omega$. Let $g \in \mathrm{Sym}(\Omega)$ be defined by

$$(\delta,\lambda,i)^g = \begin{cases} (\delta,\lambda,i) & \text{if } \delta \neq \delta_0 \text{ or } \lambda \neq \lambda_0, \\ (\delta_0,\lambda_0,i+1) & \text{if } \delta = \delta_0 \text{ and } \lambda = \lambda_0. \end{cases}$$

Define $\rho_H : H \to \mathrm{Sym}(\Omega)$ by setting $(\delta,\lambda,i)^{\rho_H(h)} = (\delta^h,\lambda,i)$ for every $h \in H$. Similarly, define $\rho_K, \rho_K' : K \to \mathrm{Sym}(\Omega)$ by setting $(\delta,\lambda,i)^{\rho_K'(k)} = (\delta,\lambda^k,i)$ and $\rho_K(k) = g^{-1}\rho_K'(k)g$ for every $k \in K$. It is easy to check that ρ_H and ρ_K define faithful group actions of H and K on Ω. A simple computation shows that $\rho_H(H)_\omega = \rho_H(H_{\delta_0})$ and $\rho_K(K)_\omega = \rho_K(K_{\lambda_0})$.

It is easy to check that if $k \in K_{\lambda_0}$ then $\rho_K(k) = \rho_K'(k)$. Since $\rho_H(H)$ centralises $\rho_K'(K)$, it centralises $\rho_K'(K_{\lambda_0}) = \rho_K(K_{\lambda_0}) = \rho_K(K)_\omega$. Similarly, since H_{δ_0} preserves $\{\delta_0\}$ and $\Delta \setminus \{\delta_0\}$ it follows that $\rho_K(K)$ centralises $\rho_H(H_{\delta_0}) = \rho_H(H)_\omega$. Clearly $\rho_H(H) \cap \rho_K(K) = 1$ and hence (2) is established.

As H and K are transitive, for every $(\delta,\lambda,i) \in \Omega$, we have

$$\begin{aligned} (\delta,\lambda,i)^{\rho_H(H)\rho_K(K)\rho_H(H)} &= (\Delta \times \{\lambda\} \times \{i\})^{\rho_K(K)\ rho_H(H)} \\ &\supseteq ((\Delta \setminus \{\delta_0\}) \times \{\lambda\} \times \{i\})^{\rho_K(K)\rho_H(H)} \\ &= ((\Delta \setminus \{\delta_0\}) \times \Lambda \times \{i\})^{\rho_H(H)} \\ &= \Delta \times \Lambda \times \{i\}. \end{aligned}$$

On the other hand, if $k \in K \setminus K_{\lambda_0}$ then $(\delta_0, \lambda_0, i)^{\rho_K(k)} = (\delta_0, \lambda_0^k, i-1)$. This shows that $\langle \rho_H(H), \rho_K(K) \rangle$ is transitive on Ω. □

Let k be a positive integer with $k \geq 3$ and let $L_1, \dots, L_k$ be nontrivial transitive permutation groups. For $i \in \{1, \dots, k\}$, let m_i denote the degree of L_i and denote by $\{0, \dots, m_i - 1\}$ the set acted upon by L_i. (Note that $m_i \geq 2$ since L_i is nontrivial.) Without loss of generality, we may assume that L_1 is not regular and thus $V := (L_1)_0 \neq 1$.

Let ℓ be a positive integer. By Lemma 4.1, there exist faithful actions of L_2 and L_3 on a set Ω of cardinality $\ell m_2 m_3$ with $\langle L_2, L_3 \rangle$ transitive on Ω. Moreover, there exists $\omega_0 \in \Omega$ such that $(L_2)_{\omega_0} = (L_2)_0$, $(L_3)_{\omega_0} = (L_3)_0$, $\langle (L_2)_{\omega_0}, L_3 \rangle = (L_2)_{\omega_0} \times L_3$ and $\langle L_2, (L_3)_{\omega_0} \rangle = L_2 \times (L_3)_{\omega_0}$.

Let $U = \prod_{\omega \in \Omega} V_\omega$ that is, U is the direct product of $|\Omega|$ copies of V, with the copies indexed by Ω. Observe that the action of $\langle L_2, L_3 \rangle$ on Ω gives rise to a natural group action of $\langle L_2, L_3 \rangle$ on U which enables us to construct the group $U \rtimes \langle L_2, L_3 \rangle$. Let $U' = \prod_{\omega \in \Omega \setminus \{\omega_0\}} V_\omega$, viewed as a subgroup of U in the natural way. Note that, by the previous paragraph, $(L_2)_0 \times (L_3)_0$ normalises U'. Now, consider the following abstract groups:

$$
\begin{aligned}
P_1 &:= L_1 \times \big(U' \rtimes ((L_2)_0 \times (L_3)_0)\big) \times (L_4)_0 \times \cdots \times (L_{k-1})_0 \times (L_k)_0, \\
P_2 &:= (U \rtimes (L_2 \times (L_3)_0)) \times (L_4)_0 \times \cdots \times (L_{k-1})_0 \times (L_k)_0, \\
P_3 &:= (U \rtimes ((L_2)_0 \times L_3)) \times (L_4)_0 \times \cdots \times (L_{k-1})_0 \times (L_k)_0, \\
P_4 &:= (U \rtimes ((L_2)_0 \times (L_3)_0)) \times L_4 \times \cdots \times (L_{k-1})_0 \times (L_k)_0, \\
&\;\;\vdots \\
P_{k-1} &:= (U \rtimes ((L_2)_0 \times (L_3)_0)) \times (L_4)_0 \times \cdots \times L_{k-1} \times (L_k)_0, \\
P_k &:= (U \rtimes ((L_2)_0 \times (L_3)_0)) \times (L_4)_0 \times \cdots \times (L_{k-1})_0 \times L_k, \\
B &:= (U \rtimes ((L_2)_0 \times (L_3)_0)) \times (L_4)_0 \times \cdots \times (L_{k-1})_0 \times (L_k)_0.
\end{aligned}
$$

Observe that, for every $i \in \{1, \dots, k\}$, there is an obvious embedding of B in P_i. (For $i = 1$, this is because $U = V \times U' \leq L_1 \times U'$.) Hence, in what follows, we regard B as a common subgroup of $P_1, \dots, P_k$. Let $\mathcal{A} = \bigcup_{i=1}^k P_i$. Thus $\mathcal{A}$ is a rank k amalgam of the groups $P_1, \dots, P_k$ with $\mathcal{B}(\mathcal{A}) = B$.

Lemma 4.2. *The permutation type of $\mathcal{A}$ is $[L_1, \dots, L_k]$.*

Proof. For every $i \in \{1, \dots, k\}$, it is immediate from the definitions that the permutation group induced by the action of P_i on the right cosets of B in P_i is permutation isomorphic to L_i. □

Lemma 4.3. *The amalgam $\mathcal{A}$ is faithful.*

Proof. Let N be a subgroup of B with $N \trianglelefteq P_i$ for every $i \in \{1, \ldots, k\}$. Let K_i denote the core of B in P_i. Clearly, we have

$$\begin{aligned}
K_1 &= \big(U' \rtimes ((L_2)_0 \times (L_3)_0)\big) \times (L_4)_0 \times \cdots \times (L_{k-1})_0 \times (L_k)_0,\\
K_2 &= (U \rtimes (1 \times (L_3)_0)) \times (L_4)_0 \times \cdots \times (L_{k-1})_0 \times (L_k)_0,\\
K_3 &= (U \rtimes ((L_2)_0 \times 1)) \times (L_4)_0 \times \cdots \times (L_{k-1})_0 \times (L_k)_0,\\
&\vdots\\
K_k &= (U \rtimes ((L_2)_0 \times (L_3)_0)) \times (L_4)_0 \times \cdots \times (L_{k-1})_0 \times 1,
\end{aligned}$$

and thus $N \leq \bigcap_{i=1}^{k} K_i = U'$. Let $n \in N$. As $N \leq U$, we may write $n = \prod_{\omega\in\Omega} n_\omega$ and, since $N \leq U'$, we have $n_{\omega_0} = 1$. Let $\omega \in \Omega$. Since $\langle L_2, L_3\rangle$ is transitive on Ω, there exists $x \in \langle L_2, L_3\rangle$ with $\omega^x = \omega_0$. Recall that $\langle L_2, L_3\rangle \leq \langle P_2, P_3\rangle$ hence $n^x \in N$ therefore $(n^x)_{\omega_0} = 1$. On the other hand $(n^x)_{\omega_0} = n_{\omega_0^{x^{-1}}} = n_\omega$. Since this holds for every $\omega \in \Omega$ and every $n \in N$, we have $N = 1$ and thus $\mathcal{A}$ is faithful. □

We have that $|\mathcal{B}(\mathcal{A})| = |B| \geq |U| = |V|^{|\Omega|} = |(L_1)_0|^{\ell m_2 m_3}$. Since L_1 is not regular, we have $|(L_1)_0| \geq 2$ and thus $|\mathcal{B}(\mathcal{A})| \to \infty$ as $\ell \to \infty$. This concludes the proof of Theorem 1.8.

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