

# **ECONOMICS**

## **UNCERTAINTY IN CURRENCY MISPRICING**

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**DISCUSSION PAPER 16.03**

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#### Abstract

Declaring a currency to be mispriced is fraught with uncertainties. In this paper, these uncertainties are explicitly recognised in a model of pricing a homogeneous commodity around the world. This allows for a common driver of prices, due to a base-currency effect, and country-specific factors that lead to departures from absolute purchasing power parity on account of income differences, local taxes and charges, etc. This approach leads to estimates of currency mispricing whose significance can be tested in the usual way. Using Big Mac prices, we show that the approach has advantages over the popular Big Mac Index to currency valuation.

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## 1. INTRODUCTION

In 1972, just prior to the collapse of the Bretton-Woods system of fixed exchange rates, the US dollar cost about 40 British pence. By 1985, the dollar had appreciated to 90 pence, but in September 2015 it had fallen back to 65 pence. Should these substantial currency movements be a concern for policy makers? How do we assess whether these movements are “excessive”, or “rational” and reflect economic fundamentals? Is the Chinese RMB still undervalued? Despite the notorious volatility of currency values in the short run and their long-run swings, it is now widely accepted that exchange rates revert back to an equilibrium over the long run, so that departures from equilibrium signal temporary misalignment.

Exchange-rate mispricing is unobservable and has to be estimated on the basis of an equilibrium rate. One of the most prominent approaches to the equilibrium rate is purchasing power parity (PPP), whereby the exchange rate is the ratio between prices at home and abroad.<sup>1</sup> But even here there can be considerable scope for disagreement and uncertainties. For example, what price should be used in PPP calculations? Price indexes are frequently used, but this entails the additional complication that the underlying baskets should be identical in the two countries, a requirement rarely met in practice.<sup>2</sup> The Big Mac Index (BMI), published by The Economist magazine, uses hamburger prices as the basis for equilibrium exchange rates. Its advantage lies in the ingredients of a Big Mac hamburger being approximately the same globally and the ready availability of its price in many countries. Therefore, the BMI largely avoids the identical-baskets problem. Cumby (1996) and Ong (1997, 2003) are among the first to investigate the performance of the BMI; see Clements et al. (2012) for a review.

Under the BMI, the equilibrium exchange rate equates the cost of a Big Mac hamburger in the two countries, so that deviations from absolute parity are identified as currency mispricing. In this paper, we improve on the BMI in two ways. First, we allow for

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<sup>1</sup> Other approaches include the Fundamental Equilibrium Exchange Rate (FEER) model of Wren-Lewis (1992), Williamson (1994), Cline (2008) and Cline and Williamson (2008, 2010); the Behavioural Equilibrium Exchange Rate (BEER), introduced by Edwards (1989); and the Natural Real Exchange Rate (NATREX), proposed by Stein (1995a, 1995b). The International Monetary Fund employs related concepts (see, e. g., Lee, et. al, 2008). For reviews, see MacDonald and Stein (1999) and MacDonald (2000). Recent research on PPP includes Sarno and Valente (2006), Chortareas and Kapetanios (2009), Taylor (2010) and Zhou and Kutan (2011); see also Taylor and Taylor (2004) and Manzur (2008).

<sup>2</sup> The data derived from surveys conducted by the International Comparison Program are widely used in PPP-based studies. For example, Frankel (2006), Cheung et al. (2007, 2010) and Coudert and Couharde (2007), among others, use these data to measure the undervaluation of the Chinese Renminbi. For details of the most recent rendition of the ICP data, see World Bank (2015).

the substantial bias in the BMI. According to the BMI, the currencies of some countries are continuously over- or under-valued for lengthy periods of time.<sup>3</sup> It is widely known that Big Mac prices are heavily influenced by a range of country-specific factors, including taxes, labour costs and the cost of non-tradable inputs such as electricity and rents. This means that Big Mac prices are cheaper in poorer countries than in richer ones, so that currencies of low-income countries appear undervalued and vice versa, along the lines of the Balassa-Samuelson productivity-bias hypothesis.<sup>4</sup> Rather than allowing these long-term factors to represent sustained departures from parity, we incorporate them in a more realistic pricing model. When the long-term deviations from parity are removed, in essence the exchange rate becomes proportional to the price ratio, which amounts to relative parity. There is also an analogous time effect, whereby in a given year, there are common shocks to markets that affect all currencies. As all BMI currencies have the US dollar as their base, this most likely reflects common shocks to the US economy and/or shifts in sentiment away from/towards the US dollar.

Our second improvement is to recast mispricing as an econometric estimation problem that explicitly recognises the inherent uncertainties involved in valuing a currency. In view of the substantial scope for disagreement regarding the valuation of currencies, focusing on confidence intervals and other uncertainty measures of mispricing, rather than simple point estimates, would seem beneficial. This innovation draws on the stochastic approach to index numbers, which treats the measurement of inflation as a signal-extraction problem: Inflation is the common component that has to be extracted from the noisy prices of individual goods.<sup>5</sup>

The structure of the paper is as follows. Sections 2 and 3 set out a simple model that leads to stochastic measures of currency mispricing. As this involves the extension of the BMI to eliminate bias (over countries and years) and the application of an econometric framework to allow for uncertainty, we call the new index the “enhanced” BMI (EBMI). Sections 4 and 5 apply the framework to 24 currencies over the last two decades, compare the EBMI to the BMI and analyse the degree of predictability of future currency movements. Section 6 contains concluding comments.

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<sup>3</sup> For example, as shown below, on the basis of the BMI, the Australian dollar was undervalued for at least 15 consecutive years up to 2010.

<sup>4</sup> Since 2012, *The Economist* has published an “adjusted” BMI to take into consideration income differences between countries. This is an attempt to deal with the issue of systematic departures from parity. As far as we are aware, no studies have evaluated the performance of the adjusted index, possibly because of its short life.

<sup>5</sup> For details of the stochastic approach, see, for example, Clements et al. (2006), Diewert (2005) and Selvanathan and Rao (1994).

## 2. A GLOBAL PRICING MODEL

Suppose MacDonald's is opening its first store in a country. How might it price hamburgers? A reasonable starting point might be to look at the price of hamburgers in the US and the other country, and then make appropriate adjustments for local conditions. These adjustments could include using an appropriate exchange rate to convert from US dollars to the domestic currency, as well as allowing for differences in local costs and the impact of any impediments to international trade that give rise to a wedge between local and US prices. The exchange rate could reflect the longer term and recognise systematic under- or over-valuation of currencies. These considerations can be summarised in the following simple pricing model:

$$(1) \quad p_{it} = p_t^* + \tilde{s}_{it} + \beta_i + u_{it}.$$

Here,  $p_{it}$  is the logarithm of the domestic-currency price of a Big Mac in country  $i$  for year  $t$ ,  $p_t^*$  is the price in the US in terms of US dollars (also in logs),  $\tilde{s}_{it}$  is the appropriate exchange rate (the domestic-currency cost of one US dollar, in logs);  $\beta_i$  is a country-specific effect; and  $u_{it}$  is a random term measuring the effects of omitted factors.

According to this model, the local price is the sum of four factors:

$$p_{it} = \text{head-office price in \$US} + \text{conversion to domestic currency} \\ + \text{country effect} + \text{everything else.}$$

These factors can be explained as follows:

- Head-office price (HO),  $p_t^*$ : The HO price is the price of a Big Mac in the US.
- Currency conversion factor,  $\tilde{s}_{it}$ : The HO price needs to be converted from dollars to the currency of country  $i$  by adding the log of the appropriate exchange rate  $\tilde{s}_{it}$  (an increase implies a depreciation of the domestic currency). Due to the possibility of currency misalignment, the exchange rate  $\tilde{s}_{it}$  is not necessarily the observed rate and is defined as  $\tilde{s}_{it} = s_{it} + \alpha_t$ , where  $s_{it}$  is the actual rate and  $\alpha_t$  is an adjustment factor, common to all countries, that measures the under or overvaluation of the dollar. If, for example, the dollar were under-valued, then using the actual rate to convert the HO price from dollars to domestic currency terms,  $s_{it} + p_t^*$ , would give too low a value as the starting point for local pricing – perhaps so low to be unprofitable. In such a case, the required adjustment factor  $\alpha_t > 0$ . This adjustment factor can vary from year to

year, according to developments in currency markets. But as  $\alpha_t$  refers to the value of the dollar, the base currency, the adjustment factor affects prices in all non-US countries equiproportionally. Thus, we could call  $\tilde{s}_{it}$  the “true” exchange rate, with  $\alpha_t = \tilde{s}_{it} - s_{it}$  the deviation of the true value from the actual rate. If purchasing power parity held in absolute form, then  $\alpha_t = 0$ . As  $\alpha_t$  is the deviation from the perspective of the non-US countries, it is convenient to switch signs and refer to  $-\alpha_t$  as the departure from parity for the dollar: The dollar is thus undervalued if  $-\alpha_t < 0$ , and overvalued if  $-\alpha_t > 0$ .

- Country effect,  $\beta_i$ : This allows for the impacts of local costs, transport costs and all other barriers to trade that prevent domestic prices fully reflecting US prices.
- Everything else,  $u_{it}$ : There are many forces other than head office effects, currency values and country effects that determine prices. These might include discounting as a marketing strategy in a certain period, changes in the competitive environment faced by the local producer, one-off weather conditions that may influence costs and/or demand for hamburgers in the country, and so on. These effects are included in the random term  $u_{it}$ , which is taken to have an expectation of zero.

Using  $\tilde{s}_{it} = s_{it} + \alpha_t$ , equation (1) becomes  $p_{it} = p_t^* + s_{it} + \alpha_t + \beta_i + u_{it}$ , or  $p_{it} - p_t^* - s_{it} = \alpha_t + \beta_i + u_{it}$ . Define country  $i$ 's deviation from absolute parity as

$$q_{it} = \log \left( \frac{P_{it}}{S_{it} P_t^*} \right) = p_{it} - p_t^* - s_{it},$$

where uppercase letters are antilogs of their lowercase counterparts, so that equation (1) can be expressed as

$$(2) \quad q_{it} = \alpha_t + \beta_i + u_{it}, \quad i = 1, \dots, n \text{ countries}; t = 1, \dots, T \text{ years.}$$

If  $q_{it} > 0$ , prices in country  $i$  are too high relative to those of the US and according to the original BMI, the currency is overvalued, and vice versa. In what follows, we use model (2) to enhance this “raw” measure of mispricing.

### 3. STOCHASTIC MISPRICING

In equation (2),  $-\alpha_t$  is the under- or over-valuation of the US dollar, so that detrended mispricing,  $q_{it} - \alpha_t = \beta_i + u_{it}$ , has the expectation of  $\beta_i$ . As countries vary greatly in

economic size, an equally-weighted approach would not be a satisfactory representation of this heterogeneity. To make things representative, let  $w_i$  be the weight to be accorded to country  $i$  that reflects its relative economic size, with  $\sum_{i=1}^n w_i = 1$ . The weighted least-squares estimators of  $\alpha_t$  and  $\beta_i$ , subject to the identification constraint  $\sum_{i=1}^n w_i \beta_i = 0$ , are (Clements and Izan, 1987)

$$(3) \quad \hat{\alpha}_t = \sum_{i=1}^n w_i q_{it}, \quad \hat{\beta}_i = \sum_{t=1}^T \phi_t (q_{it} - \hat{\alpha}_t),$$

where  $\phi_t = (1/\theta_t^2) / \sum_{t=1}^T (1/\theta_t^2)$  and  $\theta_t^2 = \sum_{i=1}^n w_i \left[ q_{it} - \hat{\alpha}_t - (1/T) \sum_{i=1}^n (q_{it} - \hat{\alpha}_t) \right]^2$  is a type of weighted residual sum of squares. Thus, the estimated common component of the  $n$  currencies,  $\hat{\alpha}_t$ , is a weighted average of the raw mispricings,  $q_{1t}, \dots, q_{nt}$ . The weights are the shares of each country in the world economy,  $w_1, \dots, w_n$ . The country effect,  $\hat{\beta}_i$ , is a weighted average over all  $T$  years of the de-trended mispricing for currency  $i$ ,  $q_{i1} - \alpha_1, \dots, q_{iT} - \alpha_T$ . To interpret the weights  $\phi_t$  in  $\hat{\beta}_i$ , note that  $1/\theta_t^2$  is a measure of goodness-of-fit of the model in year  $t$ , while  $\sum_{t=1}^T (1/\theta_t^2)$  is the same measure for all years combined. Accordingly, as the weight  $\phi_t$  is the proportion of the overall goodness-of-fit attributable to year  $t$ ,  $\hat{\beta}_i$  accords more weight to years in which the model more accurately reflects observed mispricing. The sampling variances are  $\text{var}(\hat{\alpha}_t) = \theta_t^2 / (n-1)$ ,  $\text{var}(\hat{\beta}_i) = (1/w_i - 1) \sum_{t=1}^T (1/\theta_t^2) / (n-1)$ , which also have simple interpretations.

As the error term in equation (2) has a zero-mean,  $E(q_{it}) = \alpha_t + \beta_i$ , which accords with relative PPP in an expected value sense. That is, the nominal exchange rate is proportional to the ratio of prices,  $E(S_{it}) = A_{it} (P_{it}/P_t^*)$ , where  $A_{it} = \exp(-\alpha_t - \beta_i)$ . The error term in equation (2) thus represents the departure from relative parity and the proportional deviation is

$$\frac{S_{it} - E(S_{it})}{E(S_{it})} = \frac{e^{-\alpha_t - \beta_i - u_{it}} (P_{it}/P_t^*)}{e^{-\alpha_t - \beta_i} (P_{it}/P_t^*)} - 1 = e^{-u_{it}} - 1 \approx u_{it}.$$

This constitutes a new measure of mispricing, which is adjusted for both the time and the country effects. We dub this index the “enhanced” BMI (EBMI).<sup>6</sup>

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<sup>6</sup> Our EBMI is different from the “adjusted” index published by [The Economist](#) since 2012, as mentioned above. The “adjusted” index takes account of criticisms that Big Macs are cheaper in poorer countries where the

To estimate the EBMI for currency  $j$  in year  $\tau$ , we add the term  $\gamma$  to model (2)

$$(4) \quad q_{j\tau} = a_\tau + b_j + \gamma + \varepsilon_{j\tau},$$

where  $a_t$  and  $b_i$  are the corresponding currency conversion and country effects and  $\varepsilon_{it}$  is a disturbance term. Equation (4) applies for currency  $j$  and year  $\tau$  only, while (2) refers to all other pairs of currencies and years. Defining  $\delta_{pq}$  as the Kronecker delta ( $\delta_{pq} = 1$  if  $p = q$ , 0 otherwise), these two equations can be combined as

$$(5) \quad q_{it} = a_t + b_i + \delta_{ij}\delta_{t\tau}\gamma + \varepsilon_{it}, \quad i = 1, \dots, n; t = 1, \dots, T,$$

where  $a_t = \alpha_t, t \neq \tau, b_i = \beta_i, i \neq j$ . The weighted least-squares estimate of the EBMI,  $\gamma$  is approximately equal to the estimated error term in model (2) for  $i = j$  and  $t = \tau$ , viz.,  $q_{j\tau} - \hat{\alpha}_\tau - \hat{\beta}_j$ ; see Appendix A1 for details. It is also shown in Appendix A1 that the exact expressions for the estimators are

$$(6) \quad \hat{a}_t = \begin{cases} \hat{\alpha}_t & \text{when } t \neq \tau \\ \hat{\alpha}_\tau + \frac{1}{(1-\phi_\tau)(1-1/w_j)}(q_{j\tau} - \hat{\alpha}_\tau - \hat{\beta}_j) & \text{when } t = \tau \end{cases}$$

$$\hat{b}_i = \begin{cases} \hat{\beta}_i + \frac{1}{(1-1/\phi_\tau)(1-1/w_j)}(q_{j\tau} - \hat{\alpha}_\tau - \hat{\beta}_j) & \text{when } i \neq j \\ \hat{\beta}_i + \frac{1}{(1-1/\phi_\tau)}(q_{j\tau} - \hat{\alpha}_\tau - \hat{\beta}_j) & \text{when } i = j \end{cases}$$

$$\hat{\gamma} = \frac{1}{(1-\phi_\tau)(1-w_j)}(q_{j\tau} - \hat{\alpha}_\tau - \hat{\beta}_j).$$

Here,  $\hat{\alpha}_t$ ,  $\hat{\beta}_i$ ,  $\phi_\tau$  and  $\theta_t^2$  are as defined in and below equation (3). The sampling variance of the estimated EBMI is  $\text{var}(\hat{\gamma}) = \theta_\tau^2 / [w_j(1-w_j)(1-\phi_\tau)(n-1)]$ .

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labour costs are lower and vice versa. The underlying model is  $P_i/S_i = \delta + \lambda \text{GDP}_i + e_i$ , where  $P_i$ ,  $S_i$  and  $\text{GDP}_i$  are the Big Mac price, the exchange rate and GDP per capita in country  $i$ . The Economist calculates the adjusted Big Mac index as  $(\hat{P}^*/\hat{P}_i)(P_i/P^*) - 1$ , where  $\hat{P}^* = \hat{\delta} + \hat{\lambda} \text{GDP}^*$  and  $\hat{P}_i = S_i(\hat{\delta} + \hat{\lambda} \text{GDP}_i)$  are the fitted prices in the US and country  $i$ , with  $\hat{\delta}$  and  $\hat{\lambda}$  being the estimated intercept and slope.



#### 4. ESTIMATED MISPRICING

Big Mac prices are available for each year between 1994 and 2014 in 24 currencies.<sup>7</sup> Figure 1 gives the information on the distribution of the original BMI, or raw mispricing, in logarithmic terms, that is,  $q_{ct}$  for each year (100 times  $q_{ct}$  is the approximate percentage deviation). The means and medians are negative in all years, suggesting that the US dollar is over-valued in the whole period. The interquartile range varies from 38 to 65 percent, which is substantial. The dollar is least over-valued in 1995, when the mean  $q_{i,1995}$  is -7 percent.

##### Country and Time Effects

We initially estimate model (5) with the mispricing parameter  $\gamma$  omitted, so that it reverts to model (2). The weights are each country's share in the total trade of the 24-country group and since they vary over time, we divide the 21-year period into four sub-periods, with six years in the first sub-period and five years in each of the subsequent sub-periods, and use the mean shares.<sup>8</sup> Figure 2 presents the estimates of the country components of mispricing,  $\hat{b}_i$ . As can be seen, the largest are for those countries with historically strong currencies, Switzerland, Denmark and Sweden, while China, Hong Kong and Malaysia are at the other end. In general, currencies of developed countries in Europe have the largest  $\hat{b}_i$ , while those of developing countries/areas in the far-east have the smallest. There is only a moderate amount of variation between sub-periods, but this is mostly confined to currencies of countries in the middle of the figure, such New Zealand, Australia and Argentina.<sup>9</sup>

Figure 3 presents the estimates of  $\hat{a}_t$  in the form of a fan chart. The width of the error band varies over time, reflecting how changes in the underlying uncertainty of currency markets: The two-standard-error band ranges from something of the order of 10 percentage points in some years to less than half that in others. Since  $-\hat{a}_t$  represents mispricing of the US dollar, that currency is not significantly different from parity if the two-standard-error band includes zero. Figure 3 shows this is the case for 1994 and 1996. In all other years except 1995, the estimates of  $-\hat{a}_t$  are positive, suggesting that the US dollar has been systematically overvalued in most years. This pattern can be compared with that of the raw

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<sup>7</sup> The data are from The Economist (2015).

<sup>8</sup> Total trade is the sum of current-price exports and imports of goods in US dollars, from the IMF (2015). For details of the data, see Appendix A2.

<sup>9</sup> Interestingly, these three are commodity exporters. There is a similar pattern for Brazil, also a commodity exporter.

BMI in Figure 1, where the dots refer to the unweighted means. In broad terms, the patterns in the two figures are similar.<sup>10</sup>

Next, we estimate  $\gamma$ , mispricing of currency  $j$  in year  $\tau$ , which is the EBMI. In the previous section this mispricing referred to the specific pair  $(j, \tau)$  only. To investigate possible mispricing of each currency and in each years, we estimate model (5)  $n \cdot T$  times, by successively setting the currency index  $j$  to  $1, \dots, n$  and the year index  $\tau = 1, \dots, T$ . Figure 4 contains a fan chart of the estimated EBMI for the 24 currencies in 2014. The point estimates are in the range of -20 to 20 percent, except for the Argentina peso that is undervalued by 57 percent. About one-half of the currencies are significantly mispriced in this year. A comparison of the ranking of countries in Figures 2 and 4 reveals there is little or no relationship between mispricing, as measured by the EBMI, and the country component of raw mispricing (the  $\hat{b}_i$ ).<sup>11</sup>

#### Comparison with the Raw Index

The raw Big Mac Index is  $q_{it} = \log \left[ P_{it} / (S_{it} P_t^*) \right]$ . The estimated EBMI of equation (6) is  $\hat{\gamma} = (q_{j\tau} - \hat{\alpha}_\tau - \hat{\beta}_j) / [(1 - \phi_\tau)(1 - w_j)]$ , which is the raw BMI stripped of the year and country effects,  $q_{j\tau} - \hat{\alpha}_\tau - \hat{\beta}_j$  scaled by  $1 / [(1 - \phi_\tau)(1 - w_j)]$ . As  $\phi_t$  and  $w_i$  are both positive fractions,  $1 / [(1 - \phi_\tau)(1 - w_j)] \approx 1 / (1 - \phi_\tau - w_j) \approx 1 + \phi_\tau + w_j > 1$ . Other things equal, larger year and county effects lead to a lower value of the adjusted index, and vice versa. Figure 5 presents histograms of the BMI and EBMI for all  $n \cdot T$  observations. The raw index clearly exhibits substantially more dispersion: Its standard deviation of 36 percent is more than two-and-one-half times larger than that of the EBMI of 14 percent.

Figure 6 compares the evolution of two indices for Australia. According to the raw BMI, the Australian dollar was undervalued for 15 years up to 2009, whereas the EBMI more reasonably fluctuates around zero. For some years, the contrast is stark. Take as an example 2004, arguably the beginning of the Millennium Commodity Boom, from which Australia, as a commodity exporter, was a huge beneficiary. In this year, the Australian dollar was

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<sup>10</sup> We have carried out some sensitivity analysis by using two other sets of weights, those based on GDP with (i) market exchange rates used to convert to US dollars, and (ii) PPP exchange rates from the World Economic Outlook published by the International Monetary Fund. Although there are changes, in the main, the results are not overly sensitive to the choice of weights. When trade weights are used (the case reported above), the overvaluation of the dollar lies between that from the other two other cases. For details, see Appendix A2.

<sup>11</sup> The rank correlation is -0.25.

undervalued by about 20 percent on the basis of the BMI, but the EBMI declares it to be overvalued by almost the same amount. This overvaluation is more plausible for a commodity currency at the start of a boom, such as the \$A in that year. The BMI is a lagging indicator as it signals the \$A first became overvalued as late as 2012, when the Millennium Boom was ending.<sup>12</sup>

Figure 7 is a scatter plot of the two indices and as can be seen, there is no noticeable relationship between them. The majority of points lie below the 45-degree line, so the raw BMI is smaller than the EBMI, on average. It is also clear that there is greater dispersion along the vertical axis relative to the horizontal, again indicating the lower variance of the EBMI.

## 5. PREDICTING FUTURE CURRENCY MOVEMENTS

An over- (under-) valued currency should subsequently depreciate (appreciate). Does this in fact occur? And if it does, how long does it take for the mispricing to disappear? These issues are investigated in this section.<sup>13</sup>

In Figure 8, we plot the future change in the BMI over a  $h$ -year horizon  $\Delta_h q_{i,t+h} = q_{i,t+h} - q_{i,t}$  against the current EBMI  $\hat{\gamma}_{it}$ . We use a rolling window of five years so that the EBMI is based only on past information of the most up-to-date kind. The figure also contains the regression line corresponding to

$$(7) \quad \Delta_h q_{i,t+h} = k_0^{(h)} + k_1^{(h)} \hat{\gamma}_{it} + e_{it}^{(h)},$$

where  $k_0^{(h)}$  and  $k_1^{(h)}$  are coefficients for horizon  $h$  and  $e_{it}^{(h)}$  is a disturbance. In this context, ordinary least-squares estimates are unbiased but the standard errors are incorrect due to the two special characteristics of our data: Overlapping observations and clustering by year and country. We thus first transform the data to put them on a non-overlapping basis, following Britton-Jones et al. (2011). We then apply Thompson's (2011) clustering approach to the transformed data to obtain robust standard errors that are adjusted for correlation across years and countries.

Under the purchasing power parity hypothesis, deviations are temporary and over time, the rate returns to equilibrium. If, for example, the currency is currently undervalued, a future

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<sup>12</sup> The difference between the two indices is approximately the sum of the estimated year and country effects,  $\hat{a}_t + \hat{b}_i$ . Hence, in Figure 6 the two almost coincide in 2010, when  $\hat{a}_{2010}$  almost offsets  $\hat{b}_{AU}$ . For comparisons of the two indices for each of the 24 currencies, see Appendix A3.

<sup>13</sup> For analyses along similar lines, see Cumby (1996) and Clements et al. (2012).

appreciation is to be expected, so that  $q_{i,t+h}$  would increase relative to  $q_{it}$ . The h-year subsequent change is  $\Delta_h q_{i,t+h} = (p_{i,t+h} - p_{it}) - (p_{t+h}^* - p_t^*) - (s_{i,t+h} - s_{it})$ , so that  $\Delta_h q_{i,t+h} > 0$  would mean either inflation at home, an appreciation of the nominal rate (a fall in  $s$ ), or any combination of the two. Measuring mispricing by  $\hat{\gamma}_{it}$  means that the points in Figure 8 should mostly lie in the second and fourth quadrants and the regression line have a negative slope. Although there is considerable randomness, these predictions seem to be mostly borne out in the figure, especially as the forecast horizon lengthens.

There is a further implication of pricing model (4). That model implies that the h-year change in  $q_{it}$  is

$$(8) \quad \Delta_h q_{i,t+h} = a_{t+h} - a_t + \gamma_{i,t+h} - \gamma_{it} + \varepsilon_{i,t+h} - \varepsilon_{it},$$

so the currency effect drops out. Suppose that PPP holds h years into the future, that is  $\gamma_{i,t+h} = 0$ . Equation (8) then takes the form of (7) with the slope coefficient  $k_1^{(h)} = -1$ , the intercept  $k_0^{(h)}$  now varying with time and equal to  $a_{t+h} - a_t$ , and the disturbance  $e_{i,t+h} = \varepsilon_{i,t+h} - \varepsilon_{it}$ . In Figure 8, for simplicity, the h-year change in the year effect is taken to be a constant. Figure 9 is a fan-chart of the estimated slopes and as can be seen, the slopes are all negative and mostly decrease with the horizon. The 95-percent confidence interval (based on robust standard errors) always includes the value of -1 for horizons greater than 5 years, which could be taken as an estimate of the long run for PPP.<sup>14</sup> As PPP is usually thought to refer to the long run, a five-year adjustment period is not implausible.<sup>15</sup>

Next, we compare the predictive properties of the EBMI with those of the BMI. Panel A of Table 1 reproduced the estimates of model (7) from Figure 8, while panel B contains similar regressions with the BMI replacing the EBMI. Compared to the BMI, for each horizon, the slope coefficient for the EBMI is closer to -1. As the -1 value means full adjustment to current mispricing, this result points to the advantage of the EBMI. What if we use a linear combination of the two methods? Panel C reveals this approach has some benefit as the estimated BMI weight is of the order of one-half, which means that the optimal

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<sup>14</sup> We also experimented with separate intercepts for each year. Qualitatively, the results for the slope are almost the same.

<sup>15</sup> Note that if the disturbances in model (4),  $\varepsilon_{it}$ , are serially uncorrelated, then the h-year differences in (8),  $\varepsilon_{i,t+h} - \varepsilon_{it}$ , will not be. While this is ignored, any serial correlation declines as the horizon increases, that is, as PPP becomes more likely to hold.

predictor is an unweighted average of the EBMI and the BMI. It is also noteworthy that the slope coefficients are now even closer to the PPP value of -1.

Finally, panel A of Figure 10 contains the root-mean-squared errors (RMSEs) of the three prediction methods. For all horizons, the EBMI and the average prediction are quite similar with RMSEs usually in the range of about 20-25 percent, while that of the BMI is about 15 points higher at around 40 percent. When the overlapping observations are dropped (panel B), there is more variability, but the results are qualitatively similar. Figure 11 shows the sensitivity of the results for the weighted average by varying the weights: As the weight accorded to the BMI increases, the quality of the weighted average prediction almost always deteriorates.

In summary, the EMBI is an imperfect measure, but it seems to do a reasonable job predicting future currency movements, especially over the longer term. While the EBMI dominates the BMI, the BMI still has some predictive value when averaged with the EBMI.

## 6. CONCLUDING COMMENTS

Understanding the valuation of currencies is a significant intellectual challenge and of great importance for policy, the efficient functioning of financial markets and the financial management of international companies. In view of the volatility of exchange rates, measures of the over-/under-valuation of currencies should come with considerable scepticism. There will always be debate (disputation?) regarding the underlying theory – Purchasing Power Parity (PPP), the Fundamental Equilibrium Exchange Rate, the Behavioural Equilibrium Exchange Rate and the Natural Real Exchange Rate are popular models that all have their own strengths and weaknesses, supporters and detractors. Even once a model is agreed upon, there is the further issue of uncertainty of the numerical values of parameters, which usually have to be estimated econometrically. Notwithstanding the difficulties, there is still a strong demand for these measures from financial markets, importers and exporters, and policy makers.

In this paper, we explicitly acknowledged at least some of the inherent uncertainties in currency valuation by using an approach in which mispricing is stochastic. Mispricing has a probability distribution, the dispersion of which reflects the extent to which the underlying model adequately describes the data. We use a weak version of PPP theory (the “relative” version) in the form of a model of the price of a homogeneous product (Big Mac hamburgers) in different countries. Departures from the model signal currency misalignment and our

approach means that it is possible to test if this is significantly different from zero. We dub this new measure the “extended Big Mac Index”. The results indicate substantial differences from the conventional Big Mac Index of The Economist magazine.

We conclude with a recapitulation of our approach to currency mispricing that can be summarised in seven elements:

1. Absolute parity: We start with the Big Mac Index (BMI) that uses absolute PPP with the exchange rate equalising the dollar price of a Big Mac hamburger in the home country with that in the US. That is, if  $P_{it}$  and  $P_t^*$  are the prices for year  $t$  in country  $i$  and the US respectively, then  $P_{it}/S_{it} = P_t^*$ , where  $S_{it}$  is the exchange rate (the domestic-currency cost of the dollar). The BMI uses the observed rate to measure currency mispricing as the proportionate departure from parity,  $(P_{it}/S_{it} - P_t^*)/P_t^*$ . For reasons of convenience, we use a logarithmic formulation of the deviation from parity,  $\log(P_{it}/S_{it}P_t^*) \approx (P_{it}/S_{it} - P_t^*)/P_t^*$ . If  $q_{it} = \log(P_{it}/S_{it}P_t^*)$ , then parity holds when  $q_{it} = 0$ .
2. Relative parity: As the BMI is known to systematically under or over value certain currencies for sustained periods, it is a biased measure of mispricing. Thus, we de-bias the BMI for each currency by subtracting its average deviation from parity.
3. Dollar shocks: As the US dollar is the base currency, fluctuations in the dollar that represent departures from parity impart shocks of the opposite sign to all other currencies. We measure mispricing of the dollar in a given year by averaging over currencies the departures from parity in that year.
4. The extended Big Mac Index: The new measure of mispricing is the extended BMI (EBMI). This accounts for currency and dollar effects -- to be denoted by  $bias_i$  and  $(\$US\ mispricing)_t$ , respectively -- that represent systematic departures from absolute parity. Accordingly,

Mispricing of currency  $i$  in year  $t =$

$$k_{it} \cdot [q_{it} - bias_i + (\$US\ mispricing)_t], \quad \text{with } k_{it} = \frac{1}{(1 - \phi_t)(1 - w_i)}.$$

As this is in logarithmic terms, multiplication by 100 is the percentage mispricing.

5. The scaling factor: The terms  $\phi_t$  and  $w_i$  in the scaling factor  $k_{it}$  are positive fractions, so  $k_{it} \approx 1 + \phi_t + w_i > 1$ . The fraction  $\phi_t$  is the relative importance of year  $t$

in the overall period under consideration; if, for example, the residual variance is lower in  $t$ , then  $\phi_t$  takes a larger value. The fraction  $w_i$  measures the relative size of country  $i$  in the world economy. Accordingly,  $k_{it}$  increases with the importance of the year in question and the share of world economy accounted for by the country under consideration. Currency mispricing is thus proportional to the BMI after allowing for currency biases and base-currency shocks.

6. Hypothesis testing: Precise expressions are given in the paper for each of the terms above, as well as their standard errors. This means that the significance of any mispricing, and its components, can be investigated.
7. Predicting exchange rates: If, for example,  $EBMI_{it} > 0$ , then domestic prices are too high, the currency is over-valued and can be expected to adjust in the future either by a nominal depreciation or higher inflation, or by some combination of the two. The EBMI was found to predict future exchange rates better than the BMI. There is also some evidence in favour of averaging the two indexes.

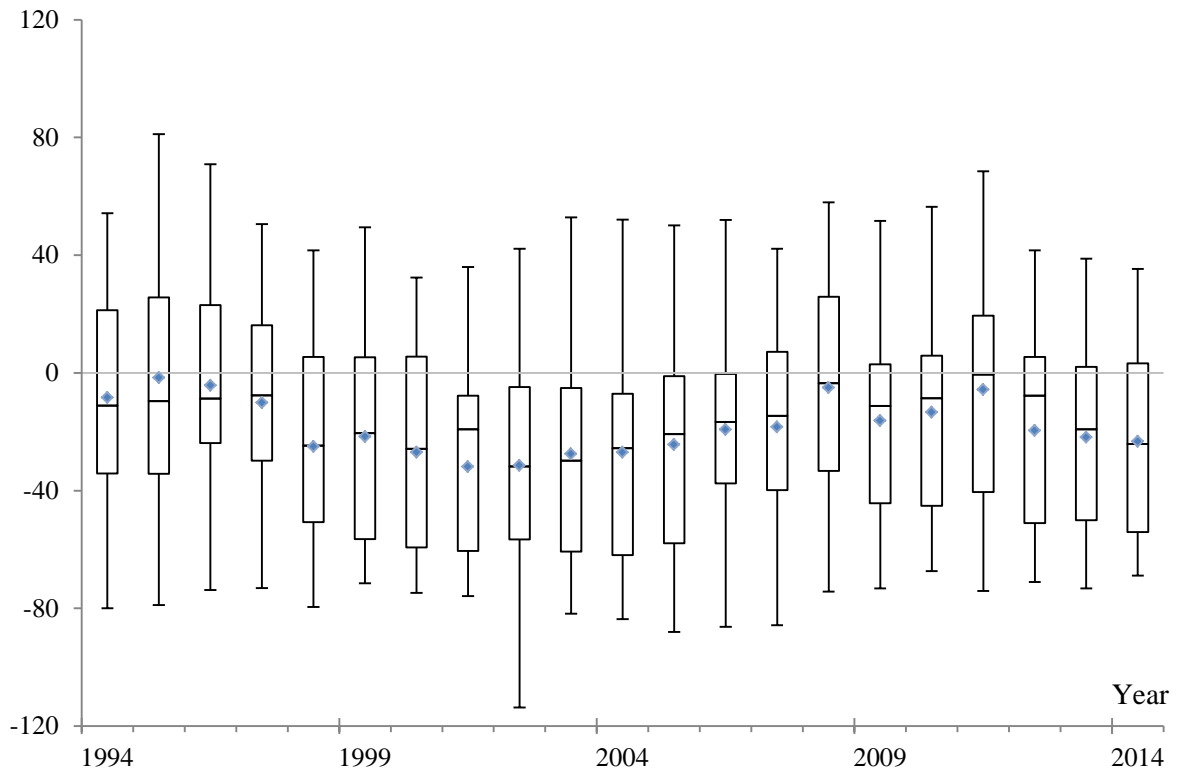
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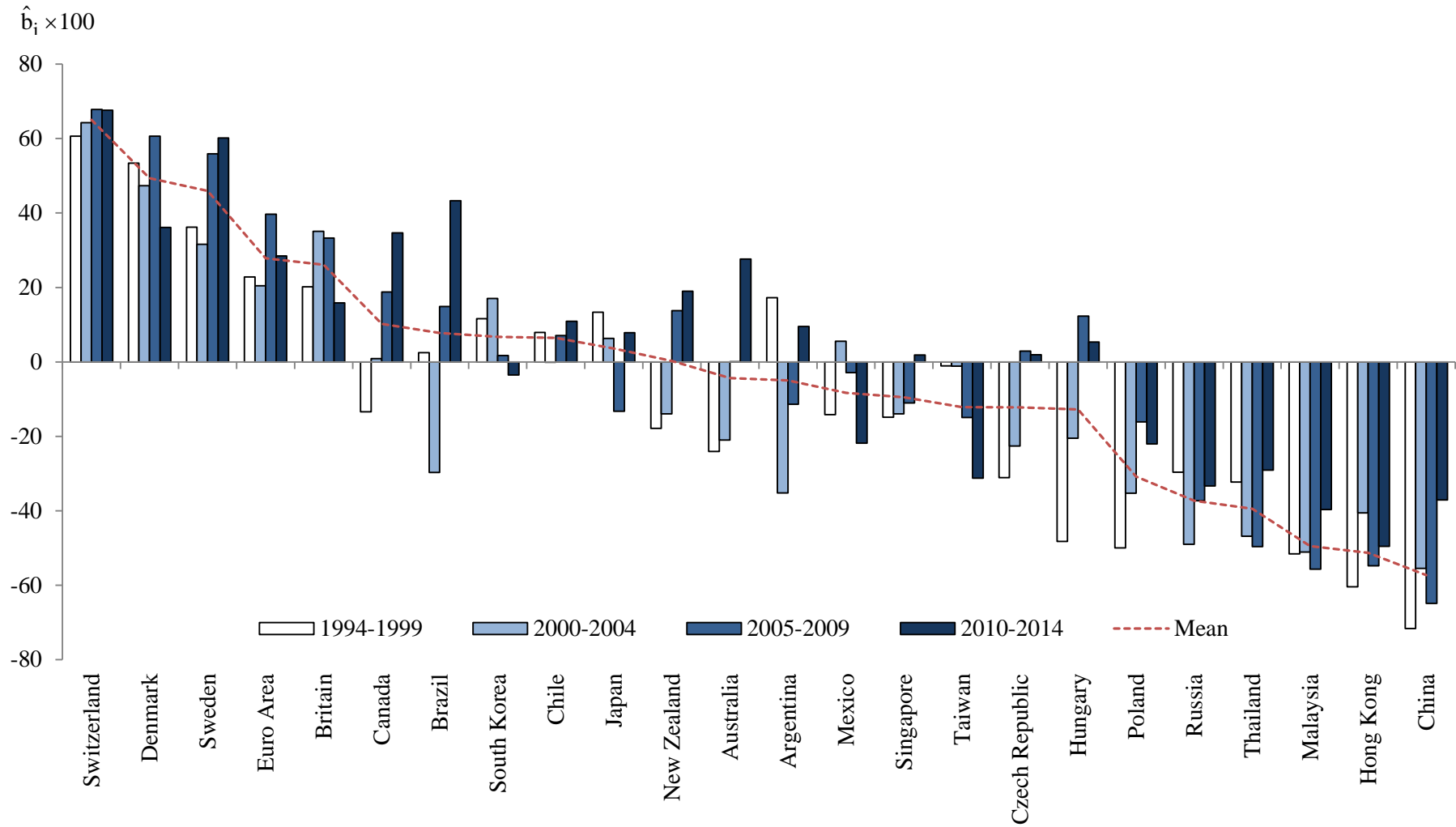
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**Figure 1** Raw Currency Mispricing of 24 Currencies, 1994-2014



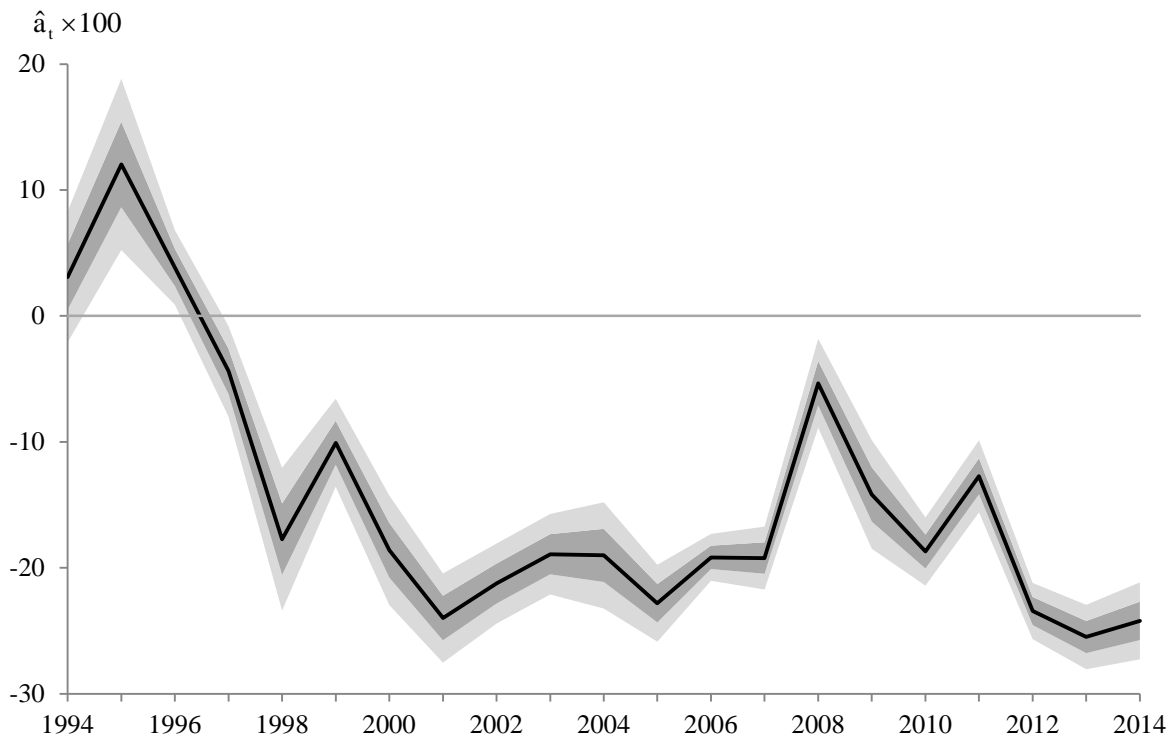
Note: This figure contains boxplots for 24 currencies in each year of the BMI, or raw mispricing, in logarithmic form, that is,  $q_{it} = \log(P_{it}/S_{it}P_t^*)$ , where, for year  $t$ ,  $P_{it}$  and  $P_t^*$  are the prices of a Big Mac in country  $i$  and the US, respectively; and  $S_{it}$  is the exchange rate (the cost in terms of currency  $i$  of \$US1). If  $q_{it} > 0$ , prices in country  $i$  are too high relative to those of the US and according to the BMI, the currency of  $i$  is overvalued, and vice versa. All entries are  $\times 100$ .

**Figure 2** Country Components of Raw Currency Mispricing in 24 Countries, 1994-2014



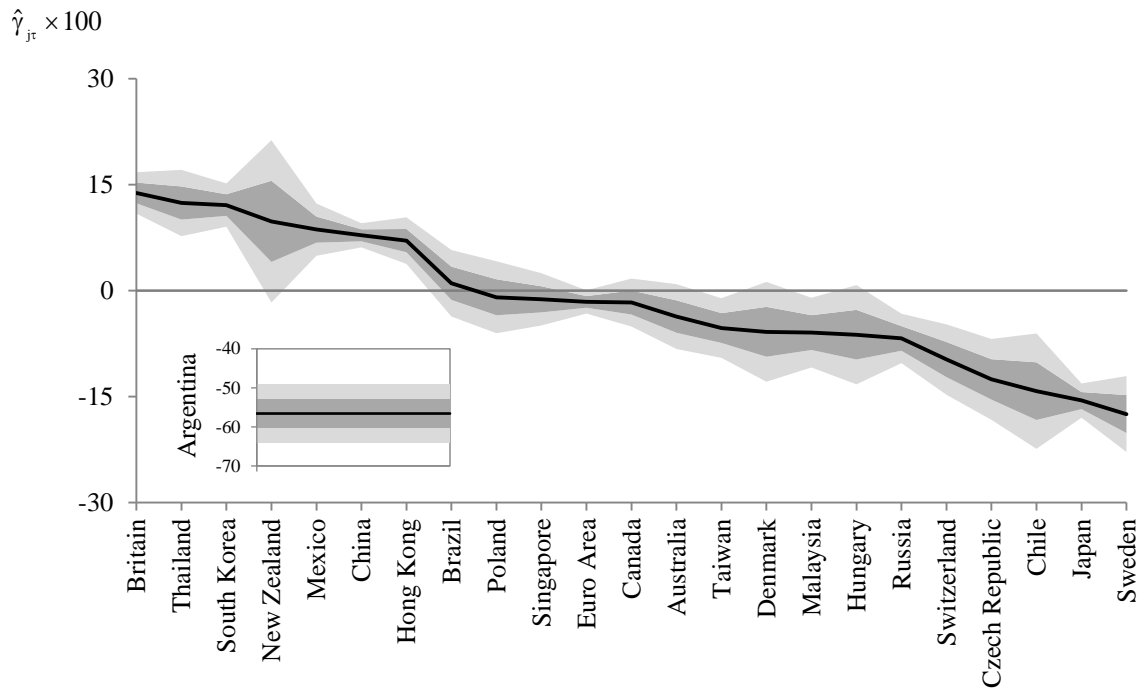
Note: This figure plots estimates of the currency component,  $\beta_i$ , in  $q_{it} = \alpha_t + \beta_i + u_{it}$ ,  $i = 1, \dots, 24$  currencies,  $t = 1, \dots, T$  years. Here,  $q_{it} = \log(P_{it}/S_{it}P_t^*)$  is the BMI or raw mispricing. If  $q_{it} > 0$ , the currency is overvalued vis-a-vis the dollar.

**Figure 3** Time Component of Raw Currency Mispricing in 24 Countries, 1994-2014



Note: The solid dark line represents the estimated time components,  $\alpha_t$ , in  $q_{it} = \alpha_t + \beta_i + u_{it}$ ,  $i = 1, \dots, 24$  currencies,  $t = 1, \dots, T$  years. The dark grey band is  $\pm$  one standard error, and the light grey band is  $\pm$  two standard errors. When, for some  $t$ , the light grey band includes 0, we are unable to reject  $H_0: \alpha_t = 0$ .

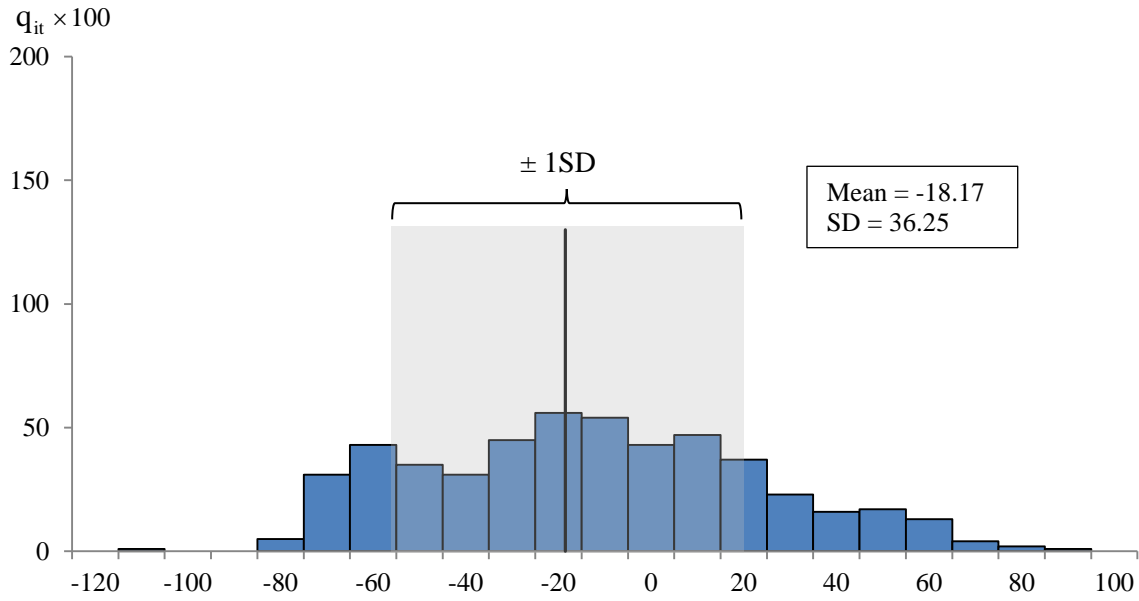
**Figure 4** Currency Mispricing, the Enhanced Big Mac Index, 2014



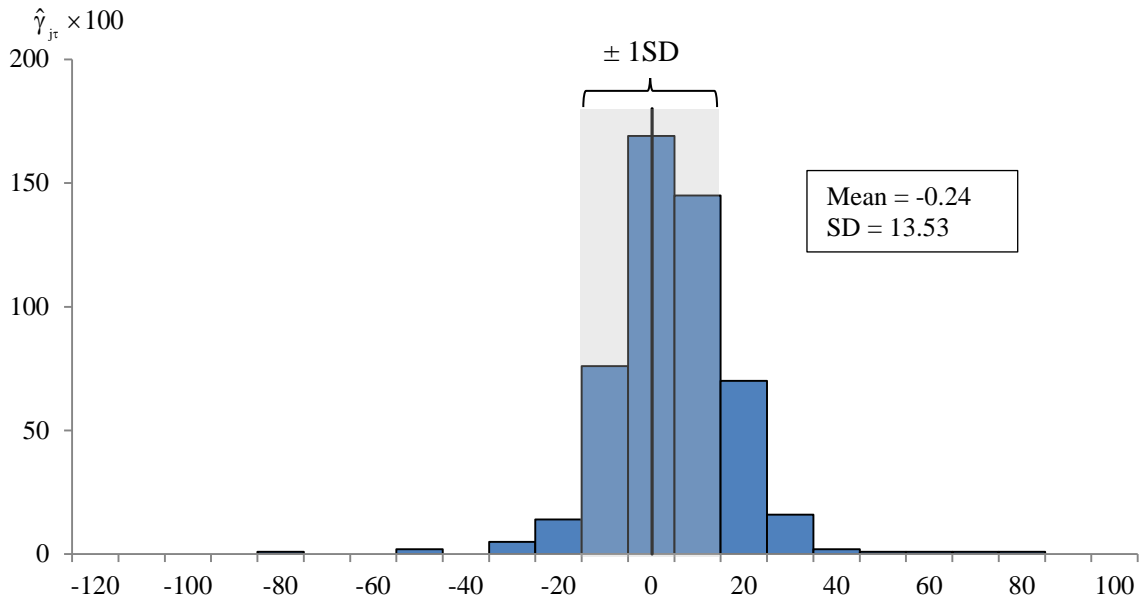
Note: The solid dark line represents the enhanced BMI (EBMI) for each currency in 2014, that is, the estimate of  $\gamma$  in  $q_{it} = a_i + b_i + \delta_{ij} \delta_{pq} \gamma + \varepsilon_{it}$ , where  $\delta_{pq}$  is the Kronecker delta ( $\delta_{pq} = 1$  if  $p = q$ , 0 otherwise). The dark grey band is the EBMI  $\pm$  one standard error, and the light grey band is  $\pm$  two standard errors. When, for some currency, the light grey band includes 0, we are unable to reject  $H_0: \gamma = 0$ .

**Figure 5** Two Measures of Mispricing, 24 Currencies, 1994-2014

**A. The BMI**



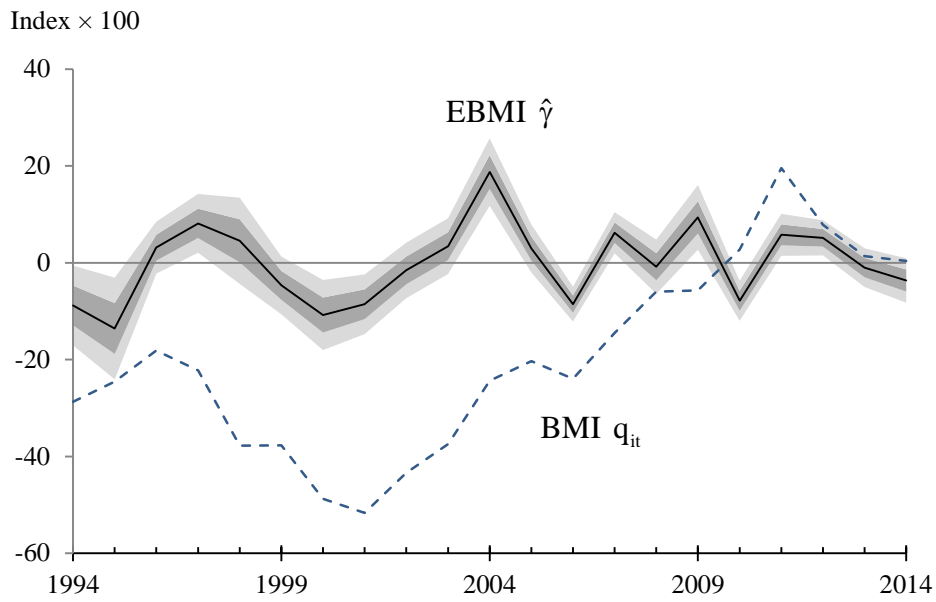
**B. The Enhanced BMI**



Notes:

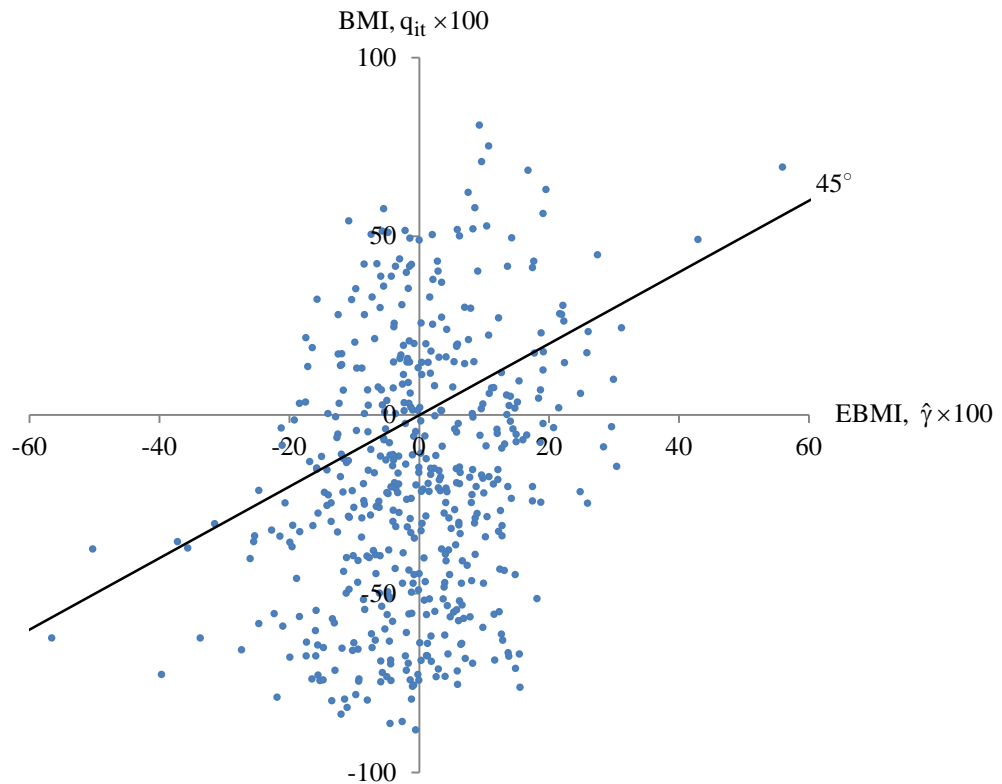
1. Panel A is a histogram of the BMI for all currencies and years. That is,  $q_{it} = \log(P_{it}/S_{it}P_t^*)$  for  $i = 1, \dots, 24$  countries and  $t = 1994, \dots, 2014$ . For year  $t$ ,  $P_{it}$  and  $P_t^*$  are the prices of a Big Mac in country  $i$  and the US; and  $S_{it}$  is the exchange rate. According to the BMI, the currency is overvalued if  $q_{it} > 0$  and vice versa.
2. Panel B is a similar histogram for the EBMI, which is the estimate of  $\gamma$  in  $q_{it} = a_t + b_i + \delta_{ij}\delta_{t\tau}\gamma + \varepsilon_{it}$ . This equation is estimated  $24 \times 21$  times with the subscripts  $(j, \tau)$  successively taking each currency-year pair,  $j = 1, \dots, 24$ ,  $\tau = 1994, \dots, 2014$ .

**Figure 6** Two Measures of Mispricing, Australian Dollar



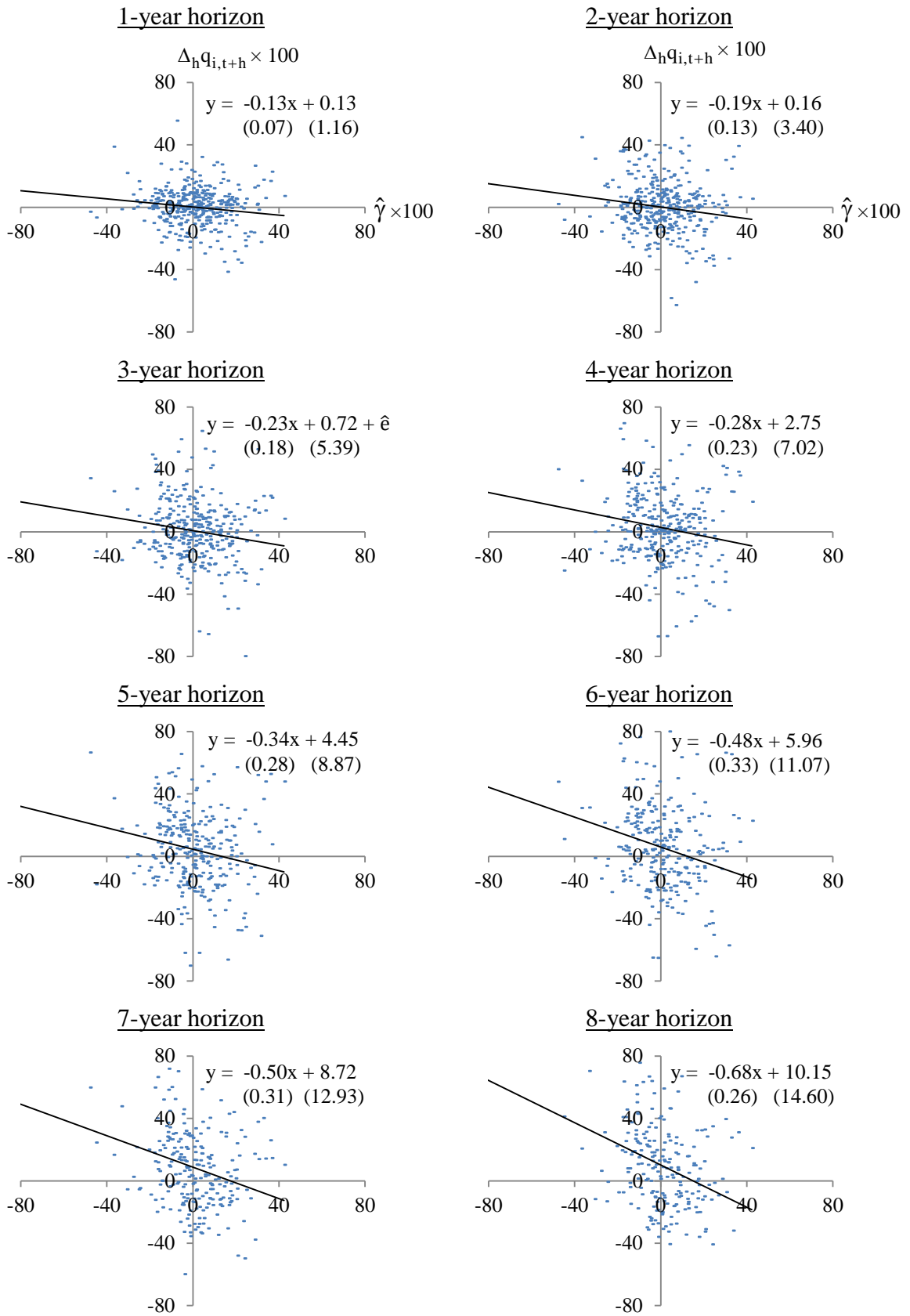
Note: The solid dark line represents the enhanced BMI (EBMI) for Australia. This is the estimate of  $\gamma$ ,  $\hat{\gamma}$ , in  $q_{it} = a_t + b_i + \delta_{ij}\delta_{it}\gamma + \varepsilon_{it}$ , when the model is estimated 21 times with the subscript pair  $(j, \tau)$  referring to Australia in each year successively; that is, for  $j = \text{Australia}$  and  $\tau = 1, \dots, 21$  years. The dark grey band is the EBMI  $\pm$  one standard error, and the light grey band is  $\pm$  two standard errors. When the light grey band includes 0, we are unable to reject  $H_0: \gamma = 0$ ; and when the band includes the BMI, we are unable to reject  $H_0: \gamma = \text{BMI}$ .

**Figure 7** Scatter Plot of Two Measures of Mispricing, 24 Currencies, 1994-2014



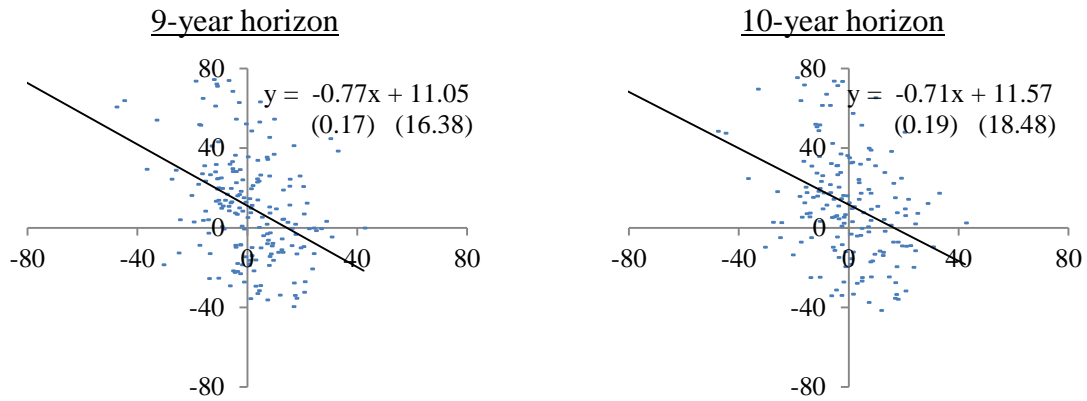
Note: See notes to Figure 5.

**Figure 8** Predictions of Future Currency Movements, 24 Currencies, 1994-2014



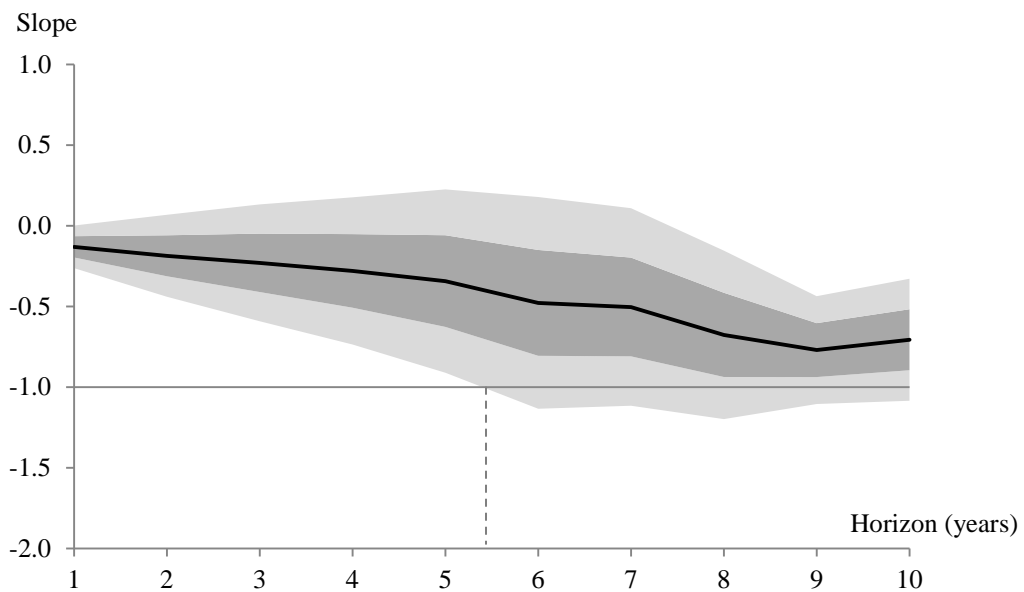
(continued on next page)

**Figure 8** Predictions of Future Currency Movements, 24 Currencies, 1994-2014 (continued)



Notes: Each panel of this figure is a scatter plot of changes in the rate from year  $t$  to  $t+h$ ,  $\Delta_h q_{i,t+h}$ , with  $h$  fixed, against mispricing in  $t$ , as measured by the EBMI. Robust standard errors are in parentheses. For presentation purposes, observations have been truncated at  $[-80, 80]$ .

**Figure 9** Slope Coefficients for Future Currency Movements

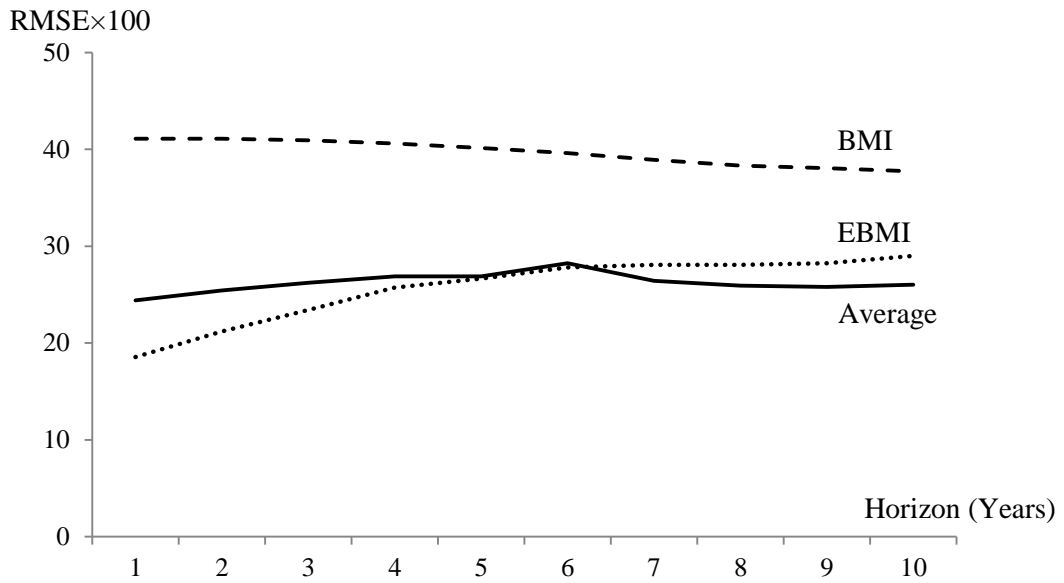


Notes: The solid dark line represents the estimated slope coefficients from Figure 8. The dark grey band is the coefficients  $\pm$  one standard error, and the light grey band is  $\pm$  two standard errors. Standard errors are robust.

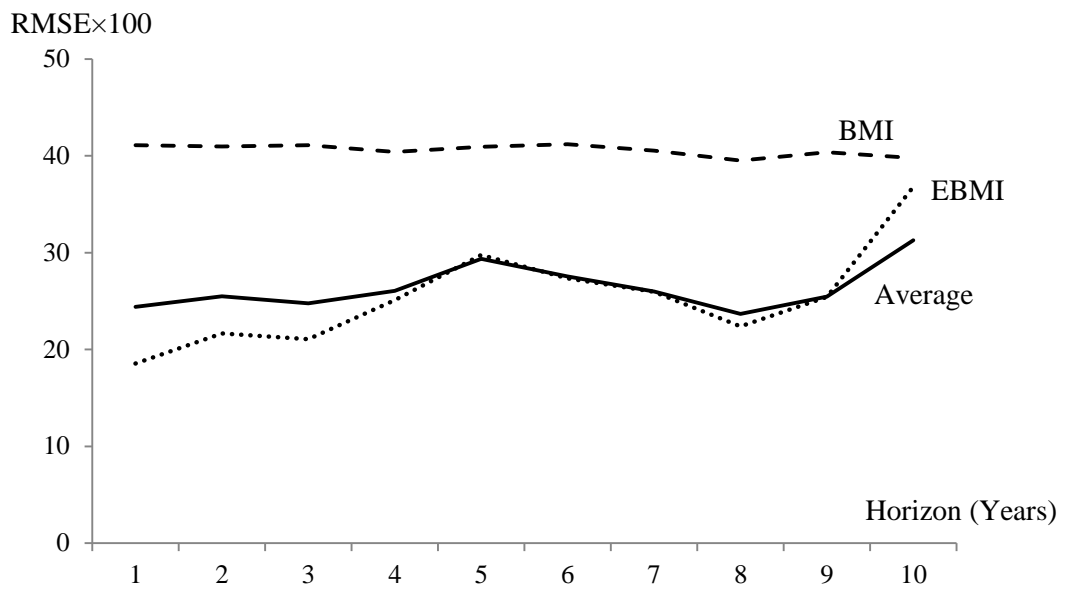


**Figure 10** Quality of Exchange-Rate Predictions

**A. With overlapping observations**

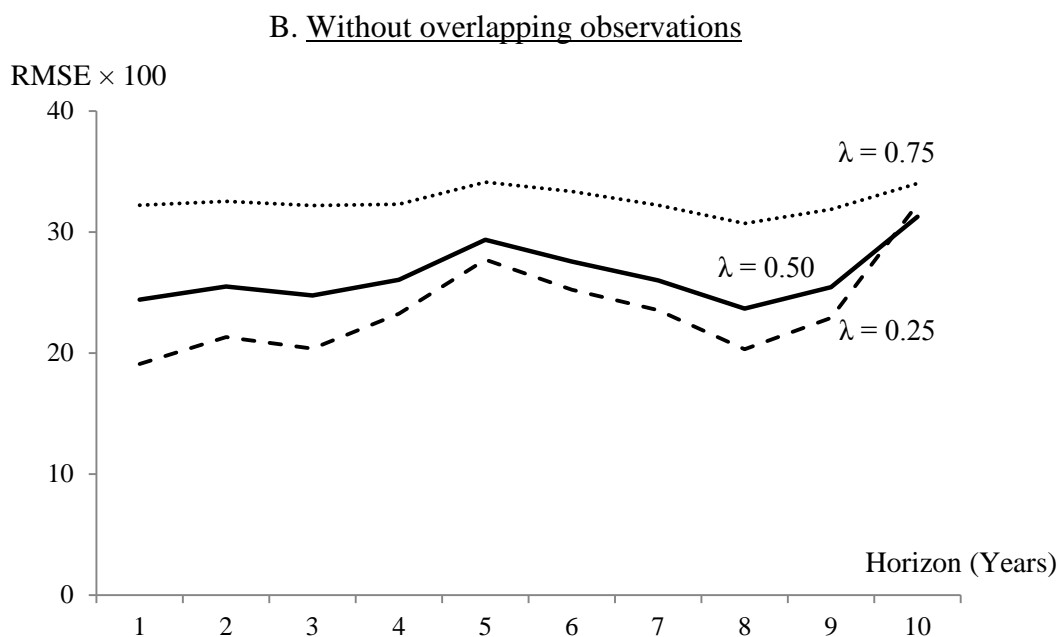
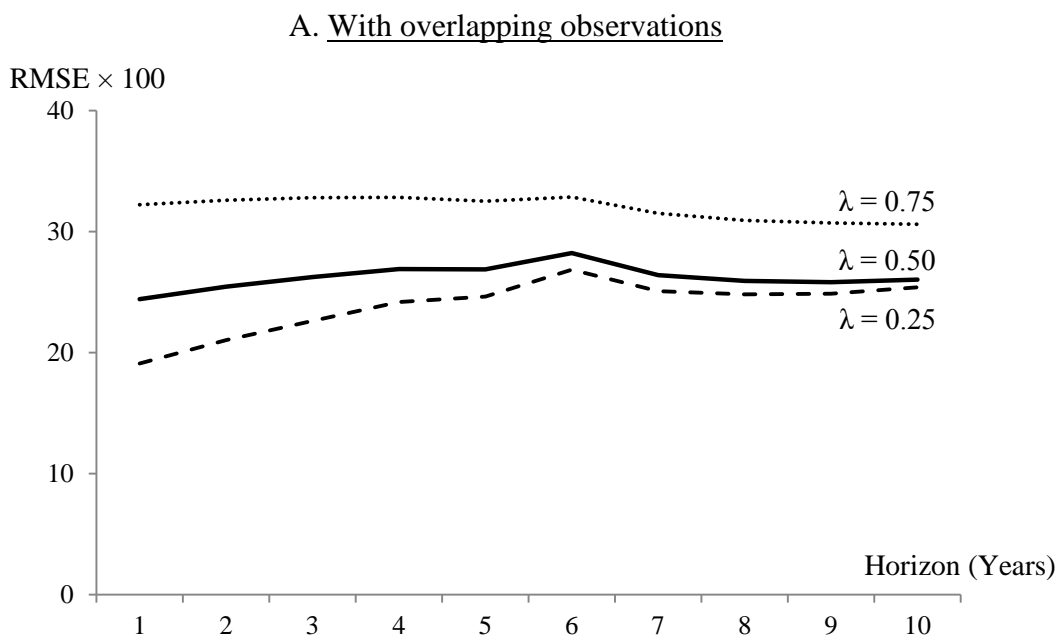


**B. Without overlapping observations**



Note: The “average” prediction is the equally weighted average of the BMI and EBMI.

**Figure 11** Sensitivity of Exchange-Rate Predictions



Note: The predictions here are the weighted average of the BMI (which has weight  $\lambda$ ) and the EBMI (weight  $1-\lambda$ ).

**Table 1** Exchange-Rate Predictions

$$\Delta q_{i,t+h} = k_1^{(h)} + k_1^{(h)} x_{it} + e_{it}^{(h)}$$

Prediction horizon h	Intercept $k_0^{(h)}$	Slope $k_1^{(h)}$	Intercept $k_0^{(h)}$	Slope $k_1^{(h)}$	BMI weight $\lambda^{(h)}$
(1)	(2)	(3)	(4)	(5)	(6)
	<u>A. <math>x_{it} = \text{Enhanced Big Mac Index}</math></u>		<u>C. <math>x_{it} = \text{Weighted Average, } \lambda^{(h)} \text{ BMI} + (1 - \lambda^{(h)}) \cdot \text{EBMI}</math></u>		
1	0.131 (1.158)	-0.131 (0.066)	-1.190 (2.083)	-0.142 (0.066)	0.431 (0.160)
2	0.157 (3.402)	-0.186 (0.127)	-1.974 (4.269)	-0.234 (0.117)	0.519 (0.160)
3	0.724 (5.386)	-0.230 (0.181)	-2.094 (6.646)	-0.311 (0.172)	0.579 (0.165)
4	2.746 (7.018)	-0.280 (0.228)	-0.876 (8.906)	-0.377 (0.239)	0.567 (0.151)
5	4.446 (8.869)	-0.343 (0.284)	0.130 (11.022)	-0.429 (0.274)	0.533 (0.135)
6	5.965 (11.067)	-0.478 (0.328)	0.298 (13.345)	-0.538 (0.310)	0.459 (0.104)
7	8.718 (12.929)	-0.503 (0.306)	2.449 (15.062)	-0.549 (0.287)	0.414 (0.097)
8	10.147 (14.598)	-0.676 (0.261)	2.463 (16.095)	-0.646 (0.280)	0.329 (0.073)
9	11.054 (16.382)	-0.771 (0.167)	2.431 (17.191)	-0.691 (0.237)	0.297 (0.067)
10	11.569 (18.484)	-0.706 (0.189)	2.913 (18.760)	-0.665 (0.169)	0.332 (0.079)
	<u>B. <math>x_{it} = \text{Big Mac Index}</math></u>				
1	-1.516 (2.168)	-0.078 (0.034)			
2	-2.655 (4.388)	-0.133 (0.060)			
3	-3.120 (6.841)	-0.180 (0.089)			
4	-2.150 (9.052)	-0.220 (0.123)			
5	-1.208 (10.933)	-0.247 (0.133)			
6	-1.049 (13.127)	-0.295 (0.145)			
7	1.354 (14.652)	-0.288 (0.124)			
8	1.928 (15.607)	-0.320 (0.120)			
9	2.402 (16.901)	-0.340 (0.111)			
10	2.893 (18.804)	-0.346 (0.101)			

Notes: Standard errors are in parentheses and have been adjusted for time and country clustering, as well as overlapping observations. Using the weighted average, the weight accorded to the BMI for horizon h,  $\lambda^{(h)}$ , is estimated from the equation  $\Delta q_{i,t+h} = k_0^{(h)} + k_1^{(h)} \cdot [\lambda^{(h)} \cdot \text{BMI}_{it} + (1 - \lambda^{(h)}) \cdot \text{EBMI}_{it}] + e_{it}^{(h)}$  in the form  $\Delta q_{i,t+h} = k_0^{(h)} + k_1^{(h)} \cdot \text{EBMI}_{it} + k_1^{(h)} \lambda^{(h)} \cdot (\text{BMI}_{it} - \text{EBMI}_{it}) + e_{it}^{(h)}$ , so that  $\lambda^{(h)}$  is the ratio of the coefficient of BMI-EBMI to that of EBMI.

## APPENDICES

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## A1. DERIVATIONS FOR THE ENHANCED BMI

The enhanced Big Mac Model is equation (5) of the text:

$$(A1.1) \quad q_{it} = a_t + b_i + \delta_{ij}\delta_{t\tau}\gamma + \varepsilon_{it}, \quad i = 1, \dots, n \text{ currencies; } t = 1, \dots, T \text{ years.}$$

For identification, we take the share weighted-average of the country effects to be zero,

$$(A1.2) \quad \sum_{i=1}^n w_i b_i = 0,$$

where  $w_i$  is the share of country  $i$  in the world economy, with  $\sum_{i=1}^n w_i = 1$ . This appendix extends Clements and Izan (1987) to derive the expressions for the estimators of this model given as equation (6). We also derive simple scalar expressions for the corresponding standard errors. We proceed in several steps.

### A1.1 First-Round Estimators

We consider estimating (A1.1) by least squares with each country weighted to reflect its relative size in the world economy, as measured by  $w_i$ . Initially, any heteroscedasticity is ignored. Thus, we multiple both sides of equation (A1.1) by  $\sqrt{w_i}$ :

$$(A1.3) \quad y_{it} = a_t x_i + b_i x_i + \delta_{ij}\delta_{t\tau}\gamma x_i + \varepsilon'_{it},$$

where  $y_{it} = \sqrt{w_i}q_{it}$ ,  $x_i = \sqrt{w_i}$  and  $\varepsilon'_{it} = \sqrt{w_i}\varepsilon_{it}$ . To minimise the sum of squared disturbances, subject to (A1.2), we form the Lagrangean function:

$$\underbrace{\frac{1}{2} \sum_{i=1}^n \sum_{t=1}^T (y_{it} - a_t x_i - b_i x_i)^2 - \frac{1}{2} (y_{j\tau} - a_\tau x_j - b_j x_j)^2}_{nT-1 \text{ terms}} + \underbrace{\frac{1}{2} (y_{j\tau} - a_\tau x_j - b_j x_j - \delta_{ij}\delta_{t\tau}\gamma x_j)^2}_{(j, \tau)^{\text{th}} \text{ term}} - \eta \sum_{i=1}^n w_i b_i,$$

where  $\eta$  is a Lagrange multiplier.

#### *Estimator of $a_t$*

We differentiate the Lagrangean with respect to  $a_t$  for  $t \neq \tau$  and equate the derivative to 0, to give  $-\sum_{i=1}^n x_i (y_{it} - a_t^* x_i - b_i x_i) = 0$ , or  $-\sum_{i=1}^n w_i q_{it} + a_t^* \sum_{i=1}^n w_i + \sum_{i=1}^n w_i b_i = 0$ . In view of (A1.2) and  $\sum_{i=1}^n w_i = 1$ , this simplifies to  $a_t^* = \sum_{i=1}^n w_i q_{it}$ ,  $t \neq \tau$ . Next, we differentiate with respect to  $a_t$  for  $t = \tau$  to obtain  $-\sum_{i=1}^n x_i (y_{i\tau} - a_\tau^* x_i - b_i x_i) + \gamma x_j^2 = 0$ , or  $a_\tau^* = \sum_{i=1}^n w_i q_{i\tau} - \gamma w_j$ . Therefore,

$$(A1.4) \quad a_t^* = \sum_{i=1}^n w_i q_{it} - \delta_{t\tau} w_j \gamma,$$

where  $\delta_{t\tau}$  is the Kronecker delta ( $\delta_{t\tau} = 1$  if  $t = \tau$ , 0 otherwise). For future reference, we sum both sides of equation (A1.4) over  $t = 1, \dots, T$ , omit the  $*$  from  $a_t^*$  and rearrange, to obtain

$$(A1.5) \quad -\sum_{t=1}^T \sum_{i=1}^n w_i q_{it} + \sum_{t=1}^T a_t + \gamma w_j = 0.$$

*Estimator of  $b_i$*

Differentiating the Lagrangean with respect to  $b_i$  and then setting the result to zero, gives

$$(A1.6) \quad -w_i \sum_{t=1}^T q_{it} + w_i \sum_{t=1}^T a_t + T w_i b_i + \delta_{ij} \gamma w_j = \eta w_i.$$

In view of (A1.2) and  $\sum_{i=1}^n w_i = 1$ , summing both sides of the above over  $t = 1, \dots, T$ , yields

$$-\sum_{t=1}^T \sum_{i=1}^n w_i q_{it} + \sum_{t=1}^T a_t + \gamma w_j = \eta.$$

As the left-hand side of this equation is equivalent to that of (A1.5), the Lagrange multiplier  $\eta = 0$ . Substituting  $\eta = 0$  back into (A1.6) and rearranging gives the estimator of  $b_i$  :

$$b_i^* = \frac{1}{T} \sum_{t=1}^T (q_{it} - a_t^*) - \delta_{ij} \frac{1}{T} \gamma,$$

or, in view of (A1.4) and using  $\sum_{t=1}^T \delta_{t\tau} \gamma w_j = \gamma w_j$ ,

$$(A1.7) \quad b_i^* = \frac{1}{T} \sum_{t=1}^T (q_{it} - \sum_{i=1}^n w_i q_{it}) - \frac{1}{T} (\delta_{ij} - w_j) \gamma.$$

*Estimator of  $\gamma$*

Finally, we differentiate the Lagrangean with respect to  $\gamma$  and equate to zero to obtain  $\gamma^* = q_{j\tau} - a_\tau^* - b_j^*$ . Using equations (A1.4) for  $t = \tau$  and (A1.7) for  $i = j$ , the estimator is

$$(A1.8) \quad \gamma^* = \frac{1}{(1-w_j)(1-1/T)} \left[ q_{j\tau} - \sum_{i=1}^n w_i q_{i\tau} - \frac{1}{T} \sum_{t=1}^T (q_{jt} - \sum_{i=1}^n w_i q_{it}) \right].$$

*Summary*

Substituting (A1.8) into (A1.4) and (A1.7) for  $\gamma$  gives the first-round estimators

$$\begin{aligned}
a_t^* &= \sum_{i=1}^n w_i q_{it} + \delta_{tr} \frac{1}{(1-1/w_j)(1-1/T)} \left[ q_{jt} - \sum_{i=1}^n w_i q_{it} - \frac{1}{T} \sum_{t=1}^T (q_{jt} - \sum_{i=1}^n w_i q_{it}) \right] \\
b_i^* &= \frac{1}{T} \sum_{t=1}^T (q_{it} - \sum_{i=1}^n w_i q_{it}) - \frac{1}{T} \frac{\delta_{ij} - w_j}{(1-w_j)(1-1/T)} \left[ q_{jt} - \sum_{i=1}^n w_i q_{it} - \frac{1}{T} \sum_{t=1}^T (q_{jt} - \sum_{i=1}^n w_i q_{it}) \right] \\
\gamma^* &= \frac{1}{(1-w_j)(1-1/T)} \left[ q_{jt} - \sum_{i=1}^n w_i q_{it} - \frac{1}{T} \sum_{t=1}^T (q_{jt} - \sum_{i=1}^n w_i q_{it}) \right],
\end{aligned}$$

where the first-round estimate of  $\gamma^*$ , equation (A1.8), has been repeated for convenience.

## A1.2 Second-Round Estimators

We now allow for heteroscedasticity on the basis that currency markets are more turbulent in some years than in others. Let  $\theta_t^2 = \sum_{i=1}^n (y_{it} - a_t^* x_i - b_i^* x_i - \delta_{ij} \delta_{tr} \gamma^* x_i)^2$  be the sum of squared residuals from the first-round estimators of model (A1.3). We divide both sides of (A1.3) by  $\theta_t$  to give

$$(A1.9) \quad \tilde{y}_{it} = a_t \tilde{x}_{it} + b_i \tilde{x}_{it} + \delta_{ij} \delta_{tr} \gamma \tilde{x}_{it} + \tilde{\varepsilon}_{it},$$

where  $\tilde{y}_{it} = y_{it}/\theta_t$ ,  $\tilde{x}_{it} = x_{it}/\theta_t$  and  $\tilde{\varepsilon}_{it} = \varepsilon'_{it}/\theta_t$ . Replacing  $y_{it}$  and  $x_{it}$  in the Lagrangean function of the previous sub-section with  $\tilde{y}_{it}$  and  $\tilde{x}_{it}$ , repeating the same steps for  $a_t$ , we obtain

$$-\sum_{i=1}^n w_i q_{it} / \theta_t^2 + a_t \sum_{i=1}^n w_i / \theta_t^2 + \sum_{i=1}^n w_i b_i / \theta_t^2 + \delta_{tr} \gamma w_j / \theta_t^2 = 0.$$

The third term disappears due to constraint (A1.2). This simplifies to yield for the second-round estimator,  $a_t^{**}$ , similar to the first-round version (A1.4),  $a_t^*$ .

$$(A1.10) \quad a_t^{**} = \sum_{i=1}^n w_i q_{it} - \delta_{tr} w_j \gamma.$$

Differentiating the new Lagrangean function with respect to  $b_i$  gives

$$-w_i \sum_{t=1}^T q_{it} / \theta_t^2 + w_i \sum_{t=1}^T a_t / \theta_t^2 + w_i b_i \sum_{t=1}^T (1/\theta_t^2) + \delta_{ij} \gamma w_j / \theta_t^2 = \eta w_i.$$

As before, the multiplier  $\eta = 0$ . Using (A1.10),

$$(A1.11) \quad b_i^{**} = \sum_{t=1}^T \phi_t (q_{it} - \sum_{i=1}^n w_i q_{it}) - (\delta_{ij} - w_j) \phi_t \gamma,$$

where  $\phi_t = (1/\theta_t^2) / \sum_{t=1}^T (1/\theta_t^2)$  is a weight applied to the  $t^{\text{th}}$  observation.

Finally, differentiating the Lagrangean with respect to  $\gamma$  gives  $\gamma^{**} = q_{j\tau} - a_{\tau}^{**} - b_j^{**}$ .

Substituting the above expressions for  $a_{\tau}^{**}$  and  $b_j^{**}$  yields

$$(A1.12) \quad \gamma^{**} = \frac{1}{(1-w_j)(1-\phi_{\tau})} \left[ q_{j\tau} - \sum_{i=1}^n w_i q_{i\tau} - \sum_{t=1}^T \phi_t \left( q_{jt} - \sum_{i=1}^n w_i q_{it} \right) \right].$$

This expression has a simple interpretation. The right-hand side is proportional to

$$\underbrace{q_{j\tau} - \sum_{i=1}^n w_i q_{i\tau}}_A - \underbrace{\sum_{t=1}^T \phi_t \left( q_{jt} - \sum_{i=1}^n w_i q_{it} \right)}_B.$$

The first term is the raw deviation from parity of currency  $j$  in year  $\tau$ ,  $q_{j\tau}$ . This raw deviation is just the BMI. The second term is  $\sum_{i=1}^n w_i q_{i\tau}$ , which is a weighted average of the deviations of the  $n$  currencies. The weight given to currency  $i$  is  $w_i$ , which is the share of currency  $i$  in the 24-currency basket. Thus, term A is the deviation from raw parity of currency  $j$  in year  $\tau$  relative to the weighted mean of all such deviations. Term B is a weighted mean of the relative deviations in all  $T$  years. Year  $t$  observations receive the weight  $\phi_t$ , which is proportional to the inverse of the residual variance in that year, so that more variable years receive a lower weight. Accordingly, the difference A-B is a double-weighted mean, a mean involving both currencies and time, which reflects the two dimensional structure of the problem. In words, A-B is mispricing of  $j$  in  $\tau$  relative to that of all currencies in that year, as compared to the weighted mean of that concept in all years. In view of equation (3) of the text, A-B is also interpreted as the residual from model (2) for  $i = j$  and  $t = \tau$ ,  $q_{j\tau} - \hat{\alpha}_{\tau} - \hat{\beta}_j$ .

The factor of proportionality for A-B in the expression for  $\gamma^{**}$  is

$$0 < \frac{1}{(1-w_j)(1-\phi_{\tau})} < 1,$$

as  $\phi_t$  and  $w_i$  are both positive fractions. As this is increasing in  $w_j$ , currencies that are relatively more important receive a larger weight. Similarly for years that are less variable. If volatility in year  $\tau$  is at the average value for the whole  $T$ -year period, then  $\phi_{\tau} = 1/T = 1/21$ ; and similarly when  $j$  is an average-sized country,  $w_j = 1/n = 1/24$ . In this case,

$$\frac{1}{(1-w_j)(1-\phi_{\tau})} = \frac{1}{\left(1 - \frac{1}{24}\right)\left(1 - \frac{1}{21}\right)} \approx 1.10,$$



so  $\gamma^{**}$  would be about 10 percent larger than the term in square brackets in (A1.12), which is the residual  $q_{jt} - \hat{\alpha}_t - \hat{\beta}_j$ .

Substituting the right-hand side of equation (A1.12) for  $\gamma$  in (A1.10), using  $-w_j / [(1-w_j)(1-\phi_\tau)] = 1 / [(1-1/w_j)(1-\phi_\tau)]$ , replacing  $\sum_{i=1}^n w_i q_{it}$  with  $\hat{\alpha}_t$  and  $\sum_{t=1}^T \phi_t (q_{jt} - \sum_{i=1}^n w_i q_{it})$  with  $\hat{\beta}_j$ , in accordance with equation (3) of the text, we obtain the expression for  $\hat{\alpha}_t$  given as the first member of equation (6) of the text. Similar substitutions confirm that  $b_i^{**}$  of equation (A1.11) and  $\gamma^{**}$  of (A1.12) coincide with  $\hat{b}_i$  and  $\hat{\gamma}$  of equation (6).

### A1.3 Matrix Formulation

The square of the standard error of  $\hat{\gamma}$ , or  $\gamma^{**}$ , is given below equation (6) of the text. Our next task is to derive this equation, as well as simple scalar expressions for the other standard errors. This involves first formulating the model as a matrix equation and then deriving the weighted least-squares estimators. While these estimators are, of course, the same as those of equations (A1.10)-(A1.12) above, this serves as a check on the calculations.

#### *Substituting Out the Constraint*

Given that  $w_i = x_{it}^2 = \tilde{x}_{it}^2 \theta_t^2$ , if we divide both sides of constraint (A1.2) by  $\theta_t^2$  we have  $\sum_{i=1}^n \tilde{x}_{it}^2 b_i = 0$ , or

$$b_n = \sum_{k=1}^{n-1} \left( -\frac{\tilde{x}_{kt}^2}{\tilde{x}_{nt}^2} \right) b_k .$$

By substituting for  $b_n$  in (A1.9) for  $i = n$ , we obtain

$$\tilde{y}_{nt} = a_t \tilde{x}_{nt} + \sum_{k=1}^{n-1} \left( -\frac{\tilde{x}_{kt}^2}{\tilde{x}_{nt}^2} \right) \cdot \tilde{x}_{nt} \cdot b_k + \gamma_{jt} \tilde{x}_{it} + \tilde{\epsilon}_{it} = a_t \tilde{x}_{nt} + \sum_{k=1}^{n-1} \left( -\frac{\tilde{x}_{kt}^2}{\tilde{x}_{nt}^2} \right) \cdot b_k + \delta_{ij} \delta_{t\tau} \gamma \tilde{x}_{it} + \tilde{\epsilon}_{it} .$$

#### *Notation*

Define the following vectors and matrices

$$\tilde{\mathbf{y}}_i = \begin{bmatrix} \tilde{y}_{i1} \\ \tilde{y}_{i2} \\ \vdots \\ \tilde{y}_{iT} \end{bmatrix}, \quad \tilde{\mathbf{y}} = \begin{bmatrix} \tilde{\mathbf{y}}_1 \\ \tilde{\mathbf{y}}_2 \\ \vdots \\ \tilde{\mathbf{y}}_{n-1} \\ \tilde{\mathbf{y}}_n \end{bmatrix}, \quad \tilde{\mathbf{X}}_i = \begin{bmatrix} \tilde{x}_{i1} & 0 & \cdots & 0 \\ 0 & \tilde{x}_{i2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \tilde{x}_{j\tau} & \vdots \\ 0 & 0 & \cdots & \tilde{x}_{iT} \end{bmatrix}, \quad \tilde{\mathbf{X}}_i \mathbf{1} = \begin{bmatrix} \tilde{x}_{i1} \\ \tilde{x}_{i2} \\ \vdots \\ \tilde{x}_{j\tau} \\ \vdots \\ \tilde{x}_{iT} \end{bmatrix},$$

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \tilde{\mathbf{d}}_j = \begin{bmatrix} 0 \\ \vdots \\ \tilde{x}_{j\tau} \\ \vdots \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{d}} = \begin{bmatrix} \mathbf{0}_{T \times 1} \\ \vdots \\ \tilde{\mathbf{d}}_j \\ \vdots \\ \mathbf{0}_{T \times 1} \end{bmatrix}, \quad \boldsymbol{\pi} = \begin{bmatrix} a_1 \\ \vdots \\ a_T \\ b_1 \\ \vdots \\ b_{n-1} \\ \gamma \end{bmatrix}$$

$$\mathbf{C} = -\tilde{\mathbf{X}}_n^{-1}(\tilde{\mathbf{X}}_1 \tilde{\mathbf{X}}_1 \mathbf{1} \quad \tilde{\mathbf{X}}_2 \tilde{\mathbf{X}}_2 \mathbf{1} \quad \cdots \quad \tilde{\mathbf{X}}_{n-1} \tilde{\mathbf{X}}_{n-1} \mathbf{1}) = [\mathbf{c}_{ti}],$$

where  $\mathbf{c}_{ti} = -\tilde{x}_{it}^2 / \tilde{x}_{nt} = -w_i / (\sqrt{w_n} \theta_t)$ ,

$$\mathbf{F} = \begin{bmatrix} \tilde{\mathbf{X}}_1 & \tilde{\mathbf{X}}_1 \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \tilde{\mathbf{X}}_2 & \mathbf{0} & \tilde{\mathbf{X}}_2 \mathbf{1} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{\mathbf{X}}_j & \vdots & \vdots & \tilde{\mathbf{X}}_j \mathbf{1} & \vdots & \tilde{\mathbf{d}}_j \\ \vdots & \vdots & \vdots & \vdots & \ddots & \mathbf{0} \\ \tilde{\mathbf{X}}_{n-1} & \mathbf{0} & \mathbf{0} & \cdots & \tilde{\mathbf{X}}_{n-1} \mathbf{1} & \vdots \\ \tilde{\mathbf{X}}_n & & & \mathbf{C} & & \mathbf{0} \end{bmatrix}.$$

With this notation, the model, with the constraint substituted out, can be expressed for all  $T$  years for the  $n$  currencies as  $\tilde{\mathbf{y}} = \mathbf{F}\boldsymbol{\pi} + \tilde{\boldsymbol{\varepsilon}}$ , where  $\tilde{\boldsymbol{\varepsilon}}$  is the disturbance vector. Thus, the least-squares estimator is  $\boldsymbol{\pi}^{**} = (\mathbf{F}'\mathbf{F})^{-1} \mathbf{F}'\tilde{\mathbf{y}}$  and under the assumption that weighting observations by  $\phi_t$  eliminates heteroscedasticity, the sampling variance is  $\text{var}(\boldsymbol{\pi}^{**}) = \frac{1}{n-1} (\mathbf{F}'\mathbf{F})^{-1}$ . In the next several subsections, we consider the components of these expressions.

### The Moment Matrix

Using the above expressions, the moment matrix is

$$\mathbf{F}'\mathbf{F} = \begin{bmatrix} \mathbf{B} & \mathbf{0} & \mathbf{D}_1 \\ \mathbf{0} & \Theta \left[ \mathbf{W} + \frac{1}{w_n} \mathbf{w}\mathbf{w}' \right] & \mathbf{D}_2 \\ \mathbf{D}_1' & \mathbf{D}_2' & w_j/\theta_\tau^2 \end{bmatrix}.$$

Here,

$$\mathbf{B} = \sum_{i=1}^n \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_i' = \begin{bmatrix} \sum_{i=1}^n \tilde{x}_{i1}^2 & 0 & \cdots & 0 \\ 0 & \sum_{i=1}^n \tilde{x}_{i2}^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sum_{i=1}^n \tilde{x}_{iT}^2 \end{bmatrix} = \begin{bmatrix} 1/\theta_1^2 & \cdots & 0 \\ \vdots & 1/\theta_2^2 & \\ & & \ddots \\ 0 & \cdots & & 1/\theta_T^2 \end{bmatrix},$$

$$\Theta = \sum_{t=1}^T (1/\theta_t^2), \quad \mathbf{W} = \text{diag}[w_1, w_2, \dots, w_{n-1}] \quad \text{and} \quad \mathbf{w} = [w_1 \quad \cdots \quad w_j \quad \cdots \quad w_{n-1}]',$$

$$\mathbf{D}_1 = \tilde{\mathbf{X}}_j \tilde{\mathbf{d}}_j = \begin{bmatrix} \tilde{x}_{j1} & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \tilde{x}_{jr} & \\ 0 & & & \ddots \\ & & & & \tilde{x}_{jT} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \tilde{x}_{jr} \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \tilde{x}_{jr}^2 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ w_j/\theta_\tau^2 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathbf{D}_2 = \begin{bmatrix} \mathbf{1}'\tilde{\mathbf{X}}_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{1}'\tilde{\mathbf{X}}_j & \\ 0 & \cdots & & \mathbf{1}'\tilde{\mathbf{X}}_{n-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \vdots \\ \tilde{\mathbf{d}}_j \\ \vdots \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{1}'\tilde{\mathbf{X}}_j \tilde{\mathbf{d}}_j \\ \vdots \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \tilde{x}_{jr}^2 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ w_j/\theta_\tau^2 \\ \vdots \\ 0 \end{bmatrix}.$$

### Partitioned Inversion of $\mathbf{F}'\mathbf{F}$

If the moment matrix is partitioned as

$$\mathbf{F}'\mathbf{F} = \begin{bmatrix} \mathbf{B} & \mathbf{0} & \mathbf{D}_1 \\ \mathbf{0} & \Theta \left[ \mathbf{W} + \frac{1}{w_n} \mathbf{w}\mathbf{w}' \right] & \mathbf{D}_2 \\ \mathbf{D}_1' & \mathbf{D}_2' & w_j/\theta_\tau^2 \end{bmatrix},$$

its inverse is then (Hua, 1996)

$$(\mathbf{F}'\mathbf{F})^{-1} = \left[ \begin{array}{cc|cc} & & & \begin{bmatrix} 0 \\ \vdots \\ -1/\phi_\tau \\ \vdots \\ 0 \end{bmatrix} \\ & \begin{bmatrix} (\mathbf{I}+\mathbf{J})\mathbf{B}^{-1} & \mathbf{B}^{-1}\mathbf{K}' \\ \mathbf{K}\mathbf{B}^{-1} & \Theta^{-1}(\mathbf{I}+\mathbf{K})(\mathbf{W}^{-1}-\mathbf{u}') \end{bmatrix} & & \begin{bmatrix} 1 \\ \vdots \\ 1-\frac{1}{w_j} \\ \vdots \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & \dots & -1/\phi_\tau & \dots & 0 \end{bmatrix} & \begin{bmatrix} 1 & \dots & 1-\frac{1}{w_j} & \dots & 1 \end{bmatrix} & & \frac{\theta_\tau^2}{w_j(1-w_j)(1-\phi_\tau)} \end{array} \right].$$

Here,

$$\mathbf{J} = \begin{bmatrix} 0 & & \dots & & 0 \\ & \ddots & & & \\ \vdots & & -\frac{1}{v_j(1-\phi_\tau)} & & \vdots \\ & & & \ddots & \\ 0 & & \dots & & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 0 & \dots & -\frac{1}{v_j h_\tau} & \dots & 0 \\ & \ddots & \vdots & & \\ \vdots & & -\frac{1}{h_\tau} & & \vdots \\ & & \vdots & \ddots & \\ 0 & \dots & -\frac{1}{v_j h_\tau} & \dots & 0 \end{bmatrix}, \quad v_j = 1 - \frac{1}{w_j}, \quad h_\tau = 1 - \frac{1}{\phi_\tau}.$$

The inverse is written out in full in Table A1.1 and verified in Section A1.6.

### *Partitioned Multiplication*

It follows from the definitions of  $\mathbf{F}$  and  $\tilde{\mathbf{y}}$  that

$$\mathbf{F}'\tilde{\mathbf{y}} = \begin{bmatrix} \sum_{i=1}^n \tilde{\mathbf{X}}_i \tilde{\mathbf{y}}_i \\ \hline \mathbf{v}'\tilde{\mathbf{X}}_1 \tilde{\mathbf{y}}_1 + \mathbf{c}'_1 \tilde{\mathbf{y}}_n \\ \vdots \\ \mathbf{v}'\tilde{\mathbf{X}}_j \tilde{\mathbf{y}}_j + \mathbf{c}'_j \tilde{\mathbf{y}}_n \\ \vdots \\ \mathbf{v}'\tilde{\mathbf{X}}_{n-1} \tilde{\mathbf{y}}_{n-1} + \mathbf{c}'_{n-1} \tilde{\mathbf{y}}_n \\ \hline \mathbf{d}'_j \tilde{\mathbf{y}}_j \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n (w_i q_{i1}) / \theta_1^2 \\ \vdots \\ \sum_{i=1}^n (w_i q_{i\tau}) / \theta_\tau^2 \\ \hline \sum_{t=1}^T (w_1 q_{1t} - w_1 q_{n\tau}) / \theta_t^2 \\ \vdots \\ \sum_{t=1}^T (w_j q_{jt} - w_j q_{n\tau}) / \theta_t^2 \\ \vdots \\ \sum_{t=1}^T (w_{n-1} q_{n-1,t} - w_{n-1} q_{n\tau}) / \theta_t^2 \\ \hline w_j q_{j\tau} / \theta_\tau^2 \end{bmatrix},$$

where  $\mathbf{c}_j$  is the  $j^{\text{th}}$  column of  $\mathbf{C}$ .

#### A1.4 The Estimator

The weighted least-squares estimator is  $\boldsymbol{\pi}^{**} = [\mathbf{a}_1^{**} \ \cdots \ \mathbf{a}_\tau^{**} \ \mathbf{b}_1^{**} \ \cdots \ \mathbf{b}_{n-1}^{**} \ \gamma^{**}]'$   
 $= (\mathbf{F}'\mathbf{F})^{-1} \mathbf{F}'\tilde{\mathbf{y}}$ . Using the above result, together with  $(\mathbf{F}'\mathbf{F})^{-1}$  from Table A1.1 and the identities  $v_j = 1 - 1/w_j$ ,  $h_\tau = 1 - 1/\phi_\tau$ ,  $\sum_{i=1}^{n-1} w_i = 1 - w_n$ , and  $\phi_t = 1/\Theta\theta_t^2$ , where  $\Theta = \sum_{t=1}^T (1/\theta_t^2)$ , we obtain the following scalar expressions.

#### *Year Components*

For  $t \neq \tau$ ,

$$\mathbf{a}_t^{**} = \theta_t^2 \sum_{i=1}^n \frac{w_i q_{it}}{\theta_t^2} = \sum_{i=1}^n w_i q_{it}.$$

For  $t = \tau$ ,

$$\mathbf{a}_\tau^{**} = \underbrace{\left(1 - \frac{1}{v_j(1-\phi_\tau)}\right) \sum_{i=1}^n w_i q_{i\tau}}_c - \underbrace{\frac{\theta_\tau^2}{v_j h_\tau} \sum_{i=1, i \neq j}^{n-1} \sum_{t=1}^T \frac{w_i q_{it} - w_i q_{n\tau}}{\theta_t^2} - \frac{\theta_\tau^2 w_j}{h_\tau} \sum_{t=1}^T \frac{(q_{jt} - q_{n\tau})}{\theta_t^2}}_d + \underbrace{\frac{\Theta \cdot \theta_\tau^2 w_j q_{j\tau}}{(1-w_j)h_\tau}}_e$$

We shall consider separately each of the three terms. The first is

$$c = \sum_{i=1}^n w_i q_{i\tau} - \frac{1}{(1-\phi_\tau)(1-1/w_j)} \sum_{i=1}^n w_i q_{i\tau}.$$

The second term is

$$\begin{aligned}
d &= -\frac{1}{\Theta\phi_\tau v_j h_\tau} \left[ \sum_{i=1, i \neq j}^{n-1} \sum_{t=1}^T \Theta\phi_t (w_i q_{it} - w_i q_{nt}) + w_j \left(1 - \frac{1}{w_j}\right) \sum_{t=1}^T \Theta\phi_t (q_{jt} - q_{nt}) \right] \\
&= \frac{1}{(1-\phi_\tau)(1-1/w_j)} \left[ \sum_{t=1}^T \phi_t \sum_{i=1}^{n-1} (w_i q_{it} - w_i q_{nt}) - \sum_{t=1}^T \phi_t (q_{jt} - q_{nt}) \right] \\
&= \frac{1}{(1-\phi_\tau)(1-1/w_j)} \left\{ \sum_{t=1}^T \phi_t \left[ \sum_{i=1}^{n-1} (w_i q_{it}) - q_{nt} (1-w_n) - (q_{jt} - q_{nt}) \right] \right\} \\
&= -\frac{1}{(1-\phi_\tau)(1-1/w_j)} \left[ \sum_{t=1}^T \phi_t \left( q_{jt} - \sum_{i=1}^n w_i q_{it} \right) \right].
\end{aligned}$$

The third term can be expressed as

$$e = \frac{\Theta \cdot \theta_\tau^2 w_j q_{j\tau}}{(1-w_j)h_\tau} = \frac{w_j q_{j\tau}}{\phi_\tau (1-w_j)h_\tau} = \frac{q_{j\tau}}{(1-\phi_\tau)(1-1/w_j)}.$$

Putting the three components together, we have

$$a_\tau^{**} = \sum_{i=1}^n w_i q_{i\tau} + \frac{1}{(1-\phi_\tau)(1-1/w_j)} \left[ q_{j\tau} - \sum_{i=1}^n w_i q_{i\tau} - \sum_{t=1}^T \phi_t \left( q_{jt} - \sum_{i=1}^n w_i q_{it} \right) \right].$$

This coincides with equation (A1.10), the estimator derived from the scalar formulation of the problem.

### Currency Components

Using similar steps to the above, for  $i \neq j$ , we have

$$\begin{aligned}
b_i^{**} &= \underbrace{-\frac{1}{v_j h_\tau} \sum_{i=1}^n w_i q_{i\tau}}_c - \underbrace{\frac{w_j q_{j\tau}}{(1-w_j)h_\tau}}_e \\
&\quad + \underbrace{\frac{1}{\Theta} \left[ \left( \frac{1}{v_j h_\tau} - 1 \right) \sum_{t=1}^T \sum_{i=1}^{n-1} \frac{w_i q_{it} - w_i q_{nt}}{\theta_t^2} + w_i (1-v_i) \sum_{t=1}^T \frac{q_{it} - q_{nt}}{\theta_t^2} + \frac{w_j}{h_\tau} \left(1 - \frac{1}{v_j}\right) \sum_{t=1}^T \frac{q_{jt} - q_{nt}}{\theta_t^2} \right]}_d.
\end{aligned}$$

The three terms here can be simplified as follows:

$$\begin{aligned}
c &= -\frac{1}{(1-1/\phi_\tau)(1-1/w_j)} \sum_{i=1}^n w_i q_{i\tau}, \\
e &= -\frac{w_j q_{j\tau}}{(1-w_j)h_\tau} = \frac{q_{j\tau}}{(1-1/\phi_\tau)(1-1/w_j)},
\end{aligned}$$

$$\begin{aligned}
d &= \left( \frac{1}{v_j h_\tau} - 1 \right) \sum_{t=1}^T \phi_t \sum_{i=1}^{n-1} (w_i q_{it} - w_i q_{nt}) + \sum_{t=1}^T \phi_t (q_{it} - q_{nt}) + \frac{w_j}{h_\tau} \left( \frac{1}{w_j - 1} \right) \sum_{t=1}^T \phi_t (q_{jt} - q_{nt}) \\
&= \left( \frac{1}{v_j h_\tau} - 1 \right) \sum_{t=1}^T \phi_t \left[ \sum_{i=1}^{n-1} (w_i q_{it}) - q_{nt} (1 - w_n) \right] + \sum_{t=1}^T \phi_t (q_{it} - q_{nt}) - \frac{1}{v_j h_\tau} \sum_{t=1}^T \phi_t (q_{jt} - q_{nt}) \\
&= \sum_{t=1}^T \phi_t \left[ q_{it} - q_{nt} - \sum_{i=1}^n (w_i q_{it}) + q_{nt} \right] + \frac{1}{v_j h_\tau} \sum_{t=1}^T \phi_t \left[ \sum_{i=1}^n (w_i q_{it}) - q_{nt} - (q_{jt} - q_{nt}) \right] \\
&= \sum_{t=1}^T \phi_t \left[ q_{it} - \sum_{i=1}^n w_i q_{it} \right] - \frac{1}{(1-1/\phi_\tau)(1-1/w_j)} \sum_{t=1}^T \phi_t \left[ q_{jt} - \sum_{i=1}^n (w_i q_{it}) \right].
\end{aligned}$$

Therefore,

$$b_i^{**} = \sum_{t=1}^T \phi_t \left( q_{it} - \sum_{i=1}^n w_i q_{it} \right) + \frac{1}{(1-1/\phi_\tau)(1-1/w_j)} \left[ q_{jt} - \sum_{i=1}^n w_i q_{it} - \sum_{t=1}^T \phi_t \left( q_{jt} - \sum_{i=1}^n w_i q_{it} \right) \right].$$

For  $i = j$ , we have

$$b_j^{**} = \underbrace{-\frac{1}{h_\tau} \sum_{i=1}^n w_i q_{it}}_c + \underbrace{\frac{1}{\Theta} \left[ \left( \frac{1}{h_\tau} - 1 \right) \sum_{t=1}^T \sum_{i=1}^{n-1} \frac{w_i q_{it} - w_i q_{nt}}{\theta_t^2} + w_j \left( \frac{1}{h_\tau} - 1 \right) (v_j - 1) \sum_{t=1}^T \frac{q_{jt} - q_{nt}}{\theta_t^2} \right]}_d + \underbrace{\frac{q_{jt}}{h_\tau}}_e,$$

with

$$\begin{aligned}
c &= -\frac{1}{(1-1/\phi_\tau)} \sum_{i=1}^n w_i q_{it}, \\
d &= \left( \frac{1}{h_\tau} - 1 \right) \sum_{t=1}^T \phi_t \sum_{i=1}^{n-1} (w_i q_{it} - w_i q_{nt}) + w_j \left( \frac{1}{h_\tau} - 1 \right) (v_j - 1) \sum_{t=1}^T \phi_t (q_{jt} - q_{nt}) \\
&= \left( \frac{1}{h_\tau} - 1 \right) \sum_{t=1}^T \phi_t \left[ \sum_{i=1}^{n-1} (w_i q_{it}) - q_{nt} (1 - w_n) - (q_{jt} - q_{nt}) \right] \\
&= \left( \frac{1}{h_\tau} - 1 \right) \sum_{t=1}^T \phi_t \left[ \sum_{i=1}^n (w_i q_{it}) - q_{jt} \right] \\
&= \sum_{t=1}^T \phi_t \left[ q_{jt} - \sum_{i=1}^n w_i q_{it} \right] - \frac{1}{(1-1/\phi_\tau)} \sum_{t=1}^T \phi_t \left[ q_{jt} - \sum_{i=1}^n w_i q_{it} \right], \text{ and} \\
e &= \frac{q_{jt}}{(1-1/\phi_\tau)}.
\end{aligned}$$

Therefore,

$$b_j^{**} = \sum_{t=1}^T \phi_t \left( q_{jt} - \sum_{i=1}^n w_i q_{it} \right) + \frac{1}{(1-1/\phi_\tau)} \left[ q_{jt} - \sum_{i=1}^n w_i q_{it} - \sum_{t=1}^T \phi_t \left( q_{jt} - \sum_{i=1}^n w_i q_{it} \right) \right].$$

These expressions coincide with equation (A1.11) above.

### Mispricing

Lastly,  $\gamma^{**}$  can be written as

$$\gamma^{**} = \underbrace{\frac{\Theta \cdot \theta_\tau^2}{(1-w_j)h_\tau} \sum_{i=1}^n w_i q_{i\tau}}_c - \underbrace{\frac{\theta_\tau^2}{(1-w_j)h_\tau} \sum_{t=1}^T \sum_{i=1, i \neq j}^{n-1} \frac{w_i q_{it} - w_i q_{nt}}{\theta_t^2}}_d + \underbrace{\frac{\theta_\tau^2}{h_\tau} \sum_{t=1}^T \frac{q_{jt} - q_{nt}}{\theta_t^2} + \frac{q_{j\tau}}{(1-w_j)(1-\phi_\tau)}}_e,$$

with

$$\begin{aligned} c &= \frac{1}{\phi_\tau(1-w_j)h_\tau} \sum_{i=1}^n w_i q_{i\tau} = -\frac{1}{(1-w_j)(1-\phi_\tau)} \sum_{i=1}^n w_i q_{i\tau}, \\ d &= -\frac{1}{\phi_\tau(1-w_j)h_\tau} \sum_{t=1}^T \phi_t \sum_{i=1, i \neq j}^{n-1} (w_i q_{it} - w_i q_{nt}) + \frac{1}{\phi_\tau h_\tau} \sum_{t=1}^T \phi_t (q_{jt} - q_{nt}) \\ &= \frac{1}{(1-w_j)(1-\phi_\tau)} \sum_{t=1}^T \phi_t \left[ \sum_{i=1, i \neq j}^{n-1} (w_i q_{it} - w_i q_{nt}) - (1-w_j)(q_{jt} - q_{nt}) \right] \\ &= \frac{1}{(1-w_j)(1-\phi_\tau)} \sum_{t=1}^T \phi_t \left[ \sum_{i=1}^{n-1} (w_i q_{it}) - q_{nt}(1-w_n) - (q_{jt} - q_{nt}) \right] \\ &= -\frac{1}{(1-w_j)(1-\phi_\tau)} \sum_{t=1}^T \phi_t \left( q_{jt} - \sum_{i=1}^n w_i q_{it} \right), \text{ and} \\ e &= \frac{q_{j\tau}}{(1-w_j)(1-\phi_\tau)}. \end{aligned}$$

Therefore,

$$\gamma^{**} = \frac{1}{(1-w_j)(1-\phi_\tau)} \left[ q_{j\tau} - \sum_{i=1}^n w_i q_{i\tau} - \sum_{t=1}^T \phi_t \left( q_{jt} - \sum_{i=1}^n w_i q_{it} \right) \right],$$

which is identical to equation (A1.12).

In summary, this sub-section has confirmed the previous expressions for the weighted least-squares estimators given in equations (A1.10)-(A1.12).

### A1.5 Sampling Variances

Using the above results, we can derive scalar expressions for the variance of the estimator,  $\text{var}(\boldsymbol{\pi}^{**}) = \frac{1}{n-1}(\mathbf{F}'\mathbf{F})^{-1}$ .



*Time and Currency Effects*

Using the block in the top right-hand corner of the inverse  $(\mathbf{F}\mathbf{F})^{-1}$  given in Table A1.1, we have

$$\text{var} \begin{bmatrix} a_1^{**} \\ a_2^{**} \\ \vdots \\ a_\tau^{**} \\ \vdots \\ a_T^{**} \end{bmatrix} = \frac{1}{n-1} \begin{bmatrix} \theta_1^2 & & \dots & & 0 \\ & \ddots & & & \\ & & \theta_\tau^2 \left[ 1 - \frac{1}{v_j(1-\phi_\tau)} \right] & & \\ & & & \ddots & \\ 0 & & \dots & & \theta_T^2 \end{bmatrix}.$$

So,

$$\text{var}(a_t^{**}) = \frac{\theta_t^2}{n-1} \left[ 1 - \delta_{t\tau} \frac{1}{(1-1/w_j)(1-\phi_\tau)} \right],$$

where  $\delta_{t\tau}$  is the Kronecker delta, and

$$\text{cov}(a_t^{**}, a_s^{**}) = 0, \quad \text{cov}(a_\tau^{**}, a_s^{**}) = 0, \quad \text{where } t \neq \tau \neq s.$$

Next, defining  $k = \Theta^{-1}/(n-1)$ , we have

$$\text{var} \begin{bmatrix} b_1^{**} \\ b_2^{**} \\ \vdots \\ b_j^{**} \\ \vdots \\ b_{n-1}^{**} \end{bmatrix} = k \begin{bmatrix} \frac{1}{v_j h_\tau} - v_1 & \frac{1}{v_j h_\tau} - 1 & \dots & \frac{1}{h_\tau} - 1 & \dots & \frac{1}{v_j h_\tau} - 1 \\ \frac{1}{v_j h_\tau} - 1 & \ddots & & \vdots & & \vdots \\ \vdots & & & & & \\ \frac{1}{h_\tau} - 1 & \dots & & \frac{v_j}{h_\tau} - v_j & \dots & \frac{1}{h_\tau} - 1 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \frac{1}{v_j h_\tau} - 1 \\ \frac{1}{v_j h_\tau} - 1 & \dots & & \frac{1}{h_\tau} - 1 & \dots & \frac{1}{v_j h_\tau} - v_{n-1} \end{bmatrix}.$$

Thus,

$$\text{var}(b_i^{**}) = k \left[ \frac{(1-1/w_j)^{2\delta_{ij}}}{(1-1/w_j)(1-1/\phi_\tau)} + \frac{1}{w_i} - 1 \right],$$

$$\text{cov}(b_i^{**}, b_{i'}^{**}) = k \left[ \frac{1}{(1-1/w_j)(1-1/\phi_\tau)} - 1 \right], \quad \text{cov}(b_i^{**}, b_j^{**}) = k \left[ \frac{1}{(1-1/\phi_\tau)} - 1 \right], \quad \text{for } i \neq i' \neq j.$$

### *The Last Currency*

The currency factor for the last currency,  $b_n^{**}$ , was omitted from the above analysis as the identification constraint equation (A1.2) was used to substitute it out. The variance of  $b_n^{**}$ , can be obtained from going back to the constraint (A1.2), which we expressed in the form  $b_n = -\sum_{k=1}^{n-1} (\tilde{x}_{kt}^2 / \tilde{x}_{nt}^2) b_k$ . If we write this as  $\mu = \zeta' \psi$ , where  $\zeta$  is a vector of  $n-1$  constants and  $\psi$  is a random vector with covariance matrix  $\text{var}(\psi)$ , then  $\text{var}(\mu) = \zeta' \text{var}(\psi) \zeta$ . After some algebra, application of this approach yields

$$\text{var}(b_n^{**}) = k \left[ \frac{1}{(1-1/w_j)(1-1/\phi_\tau)} + \frac{1}{w_n} - 1 \right],$$

which is the same form as  $\text{var}(b_i^{**})$  above with  $\delta_{ij} = 0$ .

### *Mispricing*

Lastly, the variance of  $\gamma^{**}$  is

$$\text{var}(\gamma^{**}) = \frac{\theta_\tau^2}{w_j(1-w_j)(1-\phi_\tau)(n-1)},$$

which is given below expression (6) of the text.

### A1.6 Verification of Inverse

The fourth subsection of Section A1.3 contains the partitioned inverse of the moment matrix  $\mathbf{F}'\mathbf{F}$ . The form of this inverse can be verified by establishing that  $(\mathbf{F}'\mathbf{F})(\mathbf{F}'\mathbf{F})^{-1} = \mathbf{I}$ , the identity matrix.

Using the expressions given above,  $\mathbf{F}'\mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}$  can be written as

$$\begin{array}{c}
\left[ \begin{array}{cc|c}
\mathbf{B} & \mathbf{0} & \mathbf{D}_1 \\
\mathbf{0} & \Theta \left[ \mathbf{W} + \frac{1}{w_n} \mathbf{w}\mathbf{w}' \right] & \mathbf{D}_2 \\
\hline
\mathbf{D}'_1 & \mathbf{D}'_2 & w_j/\theta_\tau^2
\end{array} \right]
\begin{array}{c}
\left[ \begin{array}{cc}
(\mathbf{I} + \mathbf{J})\mathbf{B}^{-1} & \mathbf{B}^{-1}\mathbf{K}' \\
\mathbf{K}\mathbf{B}^{-1} & \Theta^{-1}(\mathbf{I} + \mathbf{K})(\mathbf{W}^{-1} - \mathbf{u}')
\end{array} \right] \\
\left[ \begin{array}{ccc}
[0 \ \dots \ -1/\phi_\tau \ \dots \ 0] & [1 \ \dots \ 1 - \frac{1}{w_j} \ \dots \ 1] & \begin{bmatrix} 0 \\ \vdots \\ -1/\phi_\tau \\ 0 \\ 1 \\ \vdots \\ 1 - \frac{1}{w_j} \\ \vdots \\ 1 \end{bmatrix} \\
\hline
\frac{1}{\Theta(1-w_j)(1-\phi_\tau)} & \frac{1}{\Theta(1-w_j)(1-\phi_\tau)} & \frac{\theta_\tau^2}{w_j(1-w_j)(1-\phi_\tau)}
\end{array} \right]
\end{array}
\end{array}$$

$$= \mathbf{G} = \left[ \begin{array}{cc|c}
\mathbf{G}_{11} & \mathbf{G}_{12} & \mathbf{G}_{13} \\
\mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{G}_{23} \\
\hline
\mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{G}_{33}
\end{array} \right], \text{ say.}$$

The inverse is confirmed by establishing that  $\mathbf{G} = \mathbf{I}$ . To do that, we shall consider separately each of the three blocks of “rows” of  $\mathbf{G}$  to show

$$\left[ \begin{array}{cc|c}
\mathbf{G}_{11} & \mathbf{G}_{12} & \mathbf{G}_{13} \\
\mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{G}_{23} \\
\hline
\mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{G}_{33}
\end{array} \right] = \left[ \begin{array}{cc|c}
\mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{I}
\end{array} \right] = \mathbf{I}.$$

For what is to follow, it is useful to recall the definitions:

$$\mathbf{J} = \begin{bmatrix} 0 & & \dots & & 0 \\ & \ddots & & & \\ \vdots & & -\frac{1}{v_j(1-\phi_\tau)} & & \vdots \\ & & & \ddots & \\ 0 & & \dots & & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 0 & \dots & -\frac{1}{v_j h_\tau} & \dots & 0 \\ & \ddots & \vdots & & \\ \vdots & & -\frac{1}{h_\tau} & & \vdots \\ & & \vdots & \ddots & \\ 0 & \dots & -\frac{1}{v_j h_\tau} & \dots & 0 \end{bmatrix}, \quad v_j = 1 - \frac{1}{w_j}, \quad h_\tau = 1 - \frac{1}{\phi_\tau},$$

$$\mathbf{D}_1 = \begin{bmatrix} 0 \\ \vdots \\ w_j/\theta_\tau^2 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} 0 \\ \vdots \\ w_j/\theta_\tau^2 \\ \vdots \\ 0 \end{bmatrix}.$$

*Top Row of  $\mathbf{G}$*

We need to establish that the first row of  $\mathbf{G}$  takes the form  $[\mathbf{G}_{11} \ \mathbf{G}_{12} \ \mathbf{G}_{13}] = [\mathbf{I} \ \mathbf{0} \ \mathbf{0}]$ . Thus,

$$\begin{aligned} \mathbf{G}_{11} &= \mathbf{B}(\mathbf{I} + \mathbf{J})\mathbf{B}^{-1} + \mathbf{D}_1 \begin{bmatrix} 0 & \dots & \frac{\Theta \cdot \theta_\tau^4}{(1 - w_j)h_\tau} & \dots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ \vdots & & 1 - \frac{1}{v_j(1 - \phi_\tau)} & & \vdots \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} + \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ \vdots & & \frac{1}{v_j(1 - \phi_\tau)} & & \vdots \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix} \\ &= \mathbf{I}. \end{aligned}$$

$$\begin{aligned} \mathbf{G}_{11} &= \mathbf{B}(\mathbf{I} + \mathbf{J})\mathbf{B}^{-1} + \mathbf{D}_1 \begin{bmatrix} 0 & \dots & \frac{\Theta \cdot \theta_\tau^4}{(1 - w_j)h_\tau} & \dots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ \vdots & & 1 - \frac{1}{v_j(1 - \phi_\tau)} & & \vdots \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} + \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ \vdots & & \frac{1}{v_j(1 - \phi_\tau)} & & \vdots \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix} \\ &= \mathbf{I}. \end{aligned}$$

$$\begin{aligned} \mathbf{G}_{12} &= \mathbf{B}\mathbf{B}^{-1}\mathbf{K}' + \mathbf{D}_1 \begin{bmatrix} \frac{-\theta_\tau^2}{(1 - w_j)h_\tau} & \dots & \frac{\theta_\tau^2}{w_j h_\tau} & \dots & \frac{-\theta_\tau^2}{(1 - w_j)h_\tau} \end{bmatrix} \\ &= \mathbf{K}' + \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ & \ddots & \vdots & & \\ \frac{1}{v_j h_\tau} & & \frac{1}{h_\tau} & & \frac{1}{v_j h_\tau} \\ & & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix} \\ &= [\mathbf{0}]. \end{aligned}$$

$$\mathbf{G}_{13} = \begin{bmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = [\mathbf{0}].$$

*Middle Row of G*

The second row of  $\mathbf{G}$  needs to be of the form  $[\mathbf{G}_{21} \ \mathbf{G}_{22} \ \mathbf{G}_{23}] = [\mathbf{0} \ \mathbf{I} \ \mathbf{0}]$ . We proceed as before:

$$\mathbf{G}_{21} = \Theta \underbrace{\left[ \mathbf{W} + \frac{1}{w_n} \mathbf{w} \mathbf{w}' \right]}_{\mathbf{P}} \mathbf{K} \mathbf{B}^{-1} + \mathbf{D}_2 \begin{bmatrix} 0 & \cdots & \frac{\Theta \cdot \theta_\tau^4}{(1-w_j) \mathbf{h}_\tau} & \cdots & 0 \end{bmatrix},$$

Let the  $(i, k)^{\text{th}}$  element of the matrix  $\mathbf{P}$  be  $p_{ik}$ , which can be denoted by  $\mathbf{P} = [p_{ik}]$ . Two cases need to be considered for  $\mathbf{P}$ .

Case 1: For columns where  $k \neq j$ ,  $p_{ik} = 0$ .

Case 2: When column  $k = j$  and row  $i \neq j$ ,

$$\begin{aligned} p_{ik} &= \frac{1}{v_j(1-\phi_\tau)} \left( w_i + \frac{w_i}{w_n} \sum_{k=1, k \neq j}^{n-1} w_k \right) + \frac{1}{(1-\phi_\tau)} \left( \frac{w_i w_j}{w_n} \right) \\ &= \frac{1}{v_j(1-\phi_\tau)} \left[ w_i + \frac{w_i}{w_n} \sum_{k=1, k \neq j}^{n-1} (w_k) + (w_j - 1) \frac{w_i}{w_n} \right] \\ &= \frac{1}{v_j(1-\phi_\tau)} \left[ w_i + \frac{w_i}{w_n} (1 - w_n) - \frac{w_i}{w_n} \right] \\ &= 0. \end{aligned}$$

The next part of case 2 is when column  $k = j$  and row  $i = j$ ,

$$\begin{aligned} p_{jj} &= \frac{1}{v_j(1-\phi_\tau)} \left( \frac{w_j}{w_n} \sum_{k=1, k \neq j}^{n-1} w_k \right) + \frac{1}{(1-\phi_\tau)} \left( w_j + \frac{w_j w_j}{w_n} \right) \\ &= \frac{1}{v_j(1-\phi_\tau)} \left[ \frac{w_j}{w_n} \sum_{k=1, k \neq j}^{n-1} (w_k) + w_j \left( \frac{w_j - 1}{w_j} \right) \left( 1 + \frac{w_j}{w_n} \right) \right] \\ &= \frac{1}{v_j(1-\phi_\tau)} \left[ \frac{w_j}{w_n} (1 - w_n) + w_j - 1 - \frac{w_j}{w_n} \right] \\ &= -\frac{1}{v_j(1-\phi_\tau)}. \end{aligned}$$

Combining these results, we obtain

$$\begin{aligned}
\mathbf{G}_{21} &= \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & -\frac{1}{v_j(1-\phi_\tau)} & \vdots \\ 0 & \dots & 0 \end{bmatrix} + \mathbf{D}_2 \begin{bmatrix} 0 & \dots & \frac{\Theta \cdot \theta_\tau^4}{(1-w_j)h_\tau} & \dots & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & -\frac{1}{v_j(1-\phi_\tau)} & \vdots \\ 0 & \dots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \frac{1}{v_j(1-\phi_\tau)} & \vdots \\ 0 & \dots & 0 \end{bmatrix} \\
&= [\mathbf{0}].
\end{aligned}$$

Next,

$$\mathbf{G}_{22} = \underbrace{\left[ \mathbf{W} + \frac{1}{w_n} \mathbf{w} \mathbf{w}' \right]}_{\mathbf{A}} \left[ (\mathbf{I} + \mathbf{K})(\mathbf{W}^{-1} - \mathbf{u}') \right] + \mathbf{D}_2 \begin{bmatrix} \frac{-\theta_\tau^2}{(1-w_j)h_\tau} & \dots & \frac{\theta_\tau^2}{w_j h_\tau} & \dots & \frac{-\theta_\tau^2}{(1-w_j)h_\tau} \end{bmatrix},$$

with  $\mathbf{A} = [a_{ik}]$ . Three cases need to be considered.

Case 1: The main diagonal, where  $i = k, i \neq j$

$$\begin{aligned}
a_{ii} &= \left( \frac{1}{v_j h_\tau} - 1 \right) \left[ w_i + \frac{w_i}{w_n} \sum_{k=1, k \neq j}^{n-1} w_k \right] + (1 - v_i) \left( w_i + \frac{w_i w_i}{w_n} \right) + \left( \frac{1}{h_\tau} - 1 \right) \frac{w_i w_j}{w_n} \\
&= \frac{1}{v_j h_\tau} \left[ w_i + \frac{w_i}{w_n} \sum_{k=1, k \neq j}^{n-1} (w_k) \right] - w_i - \frac{w_i}{w_n} \sum_{k=1, k \neq j}^{n-1} (w_k) + 1 + \frac{w_i}{w_n} + \frac{1}{v_j h_\tau} \left( v_j \frac{w_i w_j}{w_n} \right) - \frac{w_i w_j}{w_n} \\
&= 1 + \frac{w_i}{w_n} - w_i - \frac{w_i}{w_n} \sum_{k=1}^{n-1} (w_k) + \frac{1}{v_j h_\tau} \left[ w_i + \frac{w_i}{w_n} \sum_{k=1}^{n-1} (w_k) - \frac{w_i}{w_n} \right] \\
&= 1 + \frac{w_i}{w_n} - w_i - \frac{w_i}{w_n} (1 - w_n) + \frac{1}{v_j h_\tau} \left[ w_i + \frac{w_i}{w_n} (1 - w_n) - \frac{w_i}{w_n} \right] \\
&= 1.
\end{aligned}$$

The next part of case 1 refers to the main diagonal with  $i = k = j$ :

$$\begin{aligned}
a_{jj} &= \left( \frac{1}{h_\tau} - 1 \right) \left[ \frac{w_j}{w_n} \sum_{k=1, k \neq j}^{n-1} w_k \right] + (v_j) \left( \frac{1}{h_\tau} - 1 \right) \left( w_j + \frac{w_j w_j}{w_n} \right) \\
&= \left( \frac{1}{h_\tau} - 1 \right) \left[ \frac{w_j}{w_n} \sum_{k=1, k \neq j}^{n-1} (w_k) + w_j - 1 + \frac{w_j w_j}{w_n} - \frac{w_j}{w_n} \right] \\
&= \left( \frac{1}{h_\tau} - 1 \right) \left[ \frac{w_j}{w_n} (1 - w_n) + w_j - 1 - \frac{w_j}{w_n} \right] \\
&= 1 - \frac{1}{h_\tau}.
\end{aligned}$$

Case 2: For off-diagonals, where  $i \neq j$ :

$$\begin{aligned}
a_{ik} &= \left( \frac{1}{v_j h_\tau} - 1 \right) \left[ w_i + \frac{w_i}{w_n} \sum_{k=1, k \neq j}^{n-1} w_k \right] + (1 - v_k) \left( \frac{w_i w_k}{w_n} \right) + \left( \frac{1}{h_\tau} - 1 \right) \frac{w_i w_j}{w_n} \\
&= \frac{1}{v_j h_\tau} \left[ w_i + \frac{w_i}{w_n} \sum_{k=1, k \neq j}^{n-1} (w_k) \right] - w_i - \frac{w_i}{w_n} \sum_{k=1, k \neq j}^{n-1} (w_k) + \frac{w_i}{w_n} + \frac{1}{v_j h_\tau} \left( v_j \frac{w_i w_j}{w_n} \right) - \frac{w_i w_j}{w_n} \\
&= -w_i - \frac{w_i}{w_n} \sum_{k=1}^{n-1} (w_k) + \frac{w_i}{w_n} + \frac{1}{v_j h_\tau} \left[ w_i + \frac{w_i}{w_n} \sum_{k=1, k \neq j}^{n-1} (w_k) - \frac{w_i}{w_n} \right] \\
&= \frac{w_i}{w_n} - w_i - \frac{w_i}{w_n} (1 - w_n) + \frac{1}{v_j h_\tau} \left[ w_i + \frac{w_i}{w_n} (1 - w_n) - \frac{w_i}{w_n} \right] \\
&= 0.
\end{aligned}$$

Case 3: For off-diagonal, when  $i = j$ :

$$\begin{aligned}
a_{jk} &= \left( \frac{1}{v_j h_\tau} - 1 \right) \left[ \frac{w_j}{w_n} \sum_{k=1, k \neq j}^{n-1} w_k \right] + (1 - v_k) \left( \frac{w_j w_k}{w_n} \right) + \left( \frac{1}{h_\tau} - 1 \right) \left( w_j + \frac{w_j w_j}{w_n} \right) \\
&= \frac{1}{v_j h_\tau} \left[ \frac{w_j}{w_n} \sum_{k=1, k \neq j}^{n-1} (w_k) \right] - \frac{w_j}{w_n} \sum_{k=1, k \neq j}^{n-1} (w_k) + \frac{w_j}{w_n} + \frac{1}{v_j h_\tau} \left[ v_j w_j \left( 1 + \frac{w_j}{w_n} \right) \right] - w_j - \frac{w_j w_j}{w_n} \\
&= -\frac{w_j}{w_n} \sum_{k=1}^{n-1} (w_k) + \frac{w_j}{w_n} - w_j + \frac{1}{v_j h_\tau} \left[ \frac{w_j}{w_n} \sum_{k=1}^{n-1} (w_k) + w_j - 1 - \frac{w_j}{w_n} \right] \\
&= -\frac{w_j}{w_n} (1 - w_n) + \frac{w_j}{w_n} - w_j + \frac{1}{v_j h_\tau} \left[ \frac{w_j}{w_n} (1 - w_n) + w_j - 1 - \frac{w_j}{w_n} \right] \\
&= -\frac{1}{v_j h_\tau}.
\end{aligned}$$

Combining, we obtain

$$\begin{aligned}
\mathbf{G}_{22} &= \mathbf{A} + \mathbf{D}_2 \begin{bmatrix} \frac{-\theta_\tau^2}{(1-w_j)h_\tau} & \cdots & \frac{\theta_\tau^2}{w_j h_\tau} & \cdots & \frac{-\theta_\tau^2}{(1-w_j)h_\tau} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ -\frac{1}{v_j h_\tau} & \cdots & 1 - \frac{1}{h_\tau} & \cdots & -\frac{1}{v_j h_\tau} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \frac{1}{v_j h_\tau} & \cdots & \frac{1}{h_\tau} & \cdots & \frac{1}{v_j h_\tau} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \end{bmatrix} \\
&= \mathbf{I}.
\end{aligned}$$

The last block of the middle row is

$$\mathbf{G}_{23} = \underbrace{\frac{1}{(1-w_j)(1-\phi_\tau)} \left[ \mathbf{W} + \frac{1}{w_n} \mathbf{w} \mathbf{w}' \right]}_{\mathbf{m}} \begin{bmatrix} 1 \\ \vdots \\ 1 - \frac{1}{w_j} \\ \vdots \\ 1 \end{bmatrix} + \frac{\theta_\tau^2}{w_j(1-w_j)(1-\phi_\tau)} \begin{bmatrix} 0 \\ \vdots \\ \frac{w_j}{\theta_\tau^2} \\ \vdots \\ 0 \end{bmatrix},$$

where  $\mathbf{m} = [m_i]$  is a column vector. For  $i \neq j$ ,

$$\begin{aligned}
m_i &= \frac{1}{(1-w_j)(1-\phi_\tau)} \left[ w_i + \frac{w_i}{w_n} \sum_{k=1, k \neq j}^{n-1} w_k + \left(1 - \frac{1}{w_j}\right) \left( \frac{w_i w_j}{w_n} \right) \right] \\
&= \frac{1}{(1-w_j)(1-\phi_\tau)} \left[ w_i + \frac{w_i}{w_n} \sum_{k=1}^{n-1} w_k - \frac{w_i}{w_n} \right] \\
&= \frac{1}{(1-w_j)(1-\phi_\tau)} \left[ w_i + \frac{w_i}{w_n} (1 - w_n) - \frac{w_i}{w_n} \right] \\
&= 0.
\end{aligned}$$

For  $i = j$ ,

$$\begin{aligned}
m_j &= \frac{1}{(1-w_j)(1-\phi_\tau)} \left[ \frac{w_j}{w_n} \sum_{k=1, k \neq j}^{n-1} w_k + \left(1 - \frac{1}{w_j}\right) \left( w_j + \frac{w_j w_j}{w_n} \right) \right] \\
&= \frac{1}{(1-w_j)(1-\phi_\tau)} \left[ \frac{w_j}{w_n} \sum_{k=1}^{n-1} w_k + w_j - 1 - \frac{w_j}{w_n} \right] \\
&= \frac{1}{(1-w_j)(1-\phi_\tau)} \left[ \frac{w_j}{w_n} (1 - w_n) + w_j - 1 - \frac{w_j}{w_n} \right] \\
&= -\frac{1}{(1-w_j)(1-\phi_\tau)}.
\end{aligned}$$

Therefore,



$$\mathbf{G}_{23} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \frac{1}{(1-w_j)(1-\phi_\tau)} + \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \frac{1}{(1-w_j)(1-\phi_\tau)} = [\mathbf{0}].$$

*Bottom Row of G*

The first two blocks of the bottom row,  $\mathbf{G}_{31}$  and  $\mathbf{G}_{32}$ , are row vectors whilst  $\mathbf{G}_{33}$  is a scalar. We need to confirm that  $[\mathbf{G}_{31} \ \mathbf{G}_{32} \ \mathbf{G}_{33}] = [\mathbf{0} \ \mathbf{0} \ 1]$ .

$$\mathbf{G}_{31} = \mathbf{D}'_1(\mathbf{I} + \mathbf{J})\mathbf{B}^{-1} + \mathbf{D}'_2\mathbf{B}^{-1}\mathbf{K} + \frac{w_j}{\theta_\tau^2} \cdot \begin{bmatrix} 0 & \dots & \frac{\theta_\tau^2}{(1-w_j)(1-\phi_\tau)} & \dots & 0 \end{bmatrix}.$$

Each of the three terms, the only non-zero element is  $j^{\text{th}}$ , which we add to give

$$\begin{aligned} w_j \left[ 1 - \frac{1}{v_j(1-\phi_\tau)} \right] - \frac{w_j}{h_\tau} - \frac{w}{(1-w_j)(1-\phi_\tau)} &= w_j + \frac{w_j^2}{(1-w_j)(1-\phi_\tau)} + \frac{w_j\phi_\tau}{(1-\phi_\tau)} - \frac{w_j}{(1-w_j)(1-\phi_\tau)} \\ &= \frac{w_j(1-w_j)(1-\phi_\tau) + w_j^2 + w_j\phi_\tau(1-w_j) - w_j}{(1-w_j)(1-\phi_\tau)} \\ &= 0. \end{aligned}$$

Therefore,  $\mathbf{G}_{31} = [\mathbf{0}]$ .

Next,

$$\mathbf{G}_{32} = \mathbf{D}'_1\mathbf{B}^{-1}\mathbf{K}' + \Theta^{-1}\mathbf{D}'_2(\mathbf{I} + \mathbf{K})(\mathbf{W}^{-1} - \mathbf{u}') + \frac{w_j}{\theta_\tau^2} \cdot \begin{bmatrix} \frac{-\theta_\tau^2}{(1-w_j)h_\tau} & \dots & \frac{\theta_\tau^2}{w_j h_\tau} & \dots & \frac{-\theta_\tau^2}{(1-w_j)h_\tau} \end{bmatrix}.$$

Considering each term separately,

$$\begin{aligned}
\mathbf{D}'\mathbf{B}^{-1}\mathbf{K}' &= \begin{bmatrix} 0 & \dots & \frac{w_j}{\theta_\tau^2} & \dots & 0 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \dots & 0 \\ \vdots & \theta_2^2 & \vdots \\ 0 & \dots & \ddots \\ 0 & \dots & \theta_\tau^2 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 \\ \ddots & \vdots & \\ -\frac{1}{v_j h_\tau} & \dots & -\frac{1}{h_\tau} & \dots & -\frac{1}{v_j h_\tau} \\ 0 & \dots & \vdots & \ddots & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \dots & w_j & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 \\ \ddots & \vdots & \\ -\frac{1}{v_j h_\tau} & \dots & -\frac{1}{h_\tau} & \dots & -\frac{1}{v_j h_\tau} \\ 0 & \dots & \vdots & \ddots & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{w_j}{v_j h_\tau} & \dots & -\frac{w_j}{h_\tau} & \dots & -\frac{w_j}{v_j h_\tau} \end{bmatrix}.
\end{aligned}$$

Using the central block matrix in Table A1.1 for  $\Theta^{-1}(\mathbf{I} + \mathbf{K})(\mathbf{W}^{-1} - \mathbf{u}')$ , we obtain

$$\begin{aligned}
\Theta^{-1}\mathbf{D}'_2(\mathbf{I} + \mathbf{K})(\mathbf{W}^{-1} - \mathbf{u}') &= \begin{bmatrix} 0 & \dots & \phi_\tau w_j & \dots & 0 \end{bmatrix} (\mathbf{I} + \mathbf{K})(\mathbf{W}^{-1} - \mathbf{u}') \\
&= \begin{bmatrix} \left(\frac{1}{h_\tau} - 1\right)\phi_\tau w_j & \dots & v_j \left(\frac{1}{h_\tau} - 1\right)\phi_\tau w_j & \dots & \left(\frac{1}{h_\tau} - 1\right)\phi_\tau w_j \end{bmatrix}.
\end{aligned}$$

The elements here are

$$\begin{aligned}
\left(\frac{1}{h_\tau} - 1\right)\phi_\tau w_j &= \frac{1}{h_\tau}(\phi_\tau w_j - \phi_\tau w_j h_\tau) = \frac{1}{h_\tau}[\phi_\tau w_j - w_j(\phi_\tau - 1)] = \frac{w_j}{h_\tau} = \frac{w_j - 1}{v_j h_\tau}, \text{ and} \\
v_j \left(\frac{1}{h_\tau} - 1\right)\phi_\tau w_j &= v_j \left(\frac{\phi_\tau}{\phi_\tau - 1}\right) w_j = v_j \left(\frac{w_j}{h_\tau}\right) = \frac{w_j - 1}{h_\tau}.
\end{aligned}$$

Therefore,

$$\Theta^{-1}\mathbf{D}'_2(\mathbf{I} + \mathbf{K})(\mathbf{W}^{-1} - \mathbf{u}') = \begin{bmatrix} \frac{w_j - 1}{v_j h_\tau} & \dots & \frac{w_j - 1}{h_\tau} & \dots & \frac{w_j - 1}{v_j h_\tau} \end{bmatrix}.$$

Lastly, the final term of  $\mathbf{G}_{32}$  is

$$\frac{w_j}{\theta_\tau^2} \cdot \begin{bmatrix} \frac{-\theta_\tau^2}{(1 - w_j)h_\tau} & \dots & \frac{\theta_\tau^2}{w_j h_\tau} & \dots & \frac{-\theta_\tau^2}{(1 - w_j)h_\tau} \end{bmatrix} = \begin{bmatrix} \frac{1}{v_j h_\tau} & \dots & \frac{1}{h_\tau} & \dots & \frac{1}{v_j h_\tau} \end{bmatrix}.$$

Combining these gives

$$\begin{aligned}\mathbf{G}_{32} &= \begin{bmatrix} -w_j + \frac{w_j - 1}{v_j h_\tau} + \frac{1}{v_j h_\tau} & \dots & -w_j + \frac{w_j - 1}{h_\tau} + \frac{1}{h_\tau} & \dots & -w_j + \frac{w_j - 1}{v_j h_\tau} + \frac{1}{v_j h_\tau} \\ \vdots & & \vdots & & \vdots \end{bmatrix} \\ &= [\mathbf{0}].\end{aligned}$$

The final block is  $\mathbf{G}_{33}$ , a scalar made up of the sum of three terms:

$$\mathbf{G}_{33} = \frac{1}{\Theta(1-w_j)(1-\phi_\tau)} \mathbf{D}'_1 \begin{bmatrix} 0 \\ \vdots \\ -1/\phi_\tau \\ \vdots \\ 0 \end{bmatrix} + \frac{\theta_\tau^2}{w_j(1-w_j)(1-\phi_\tau)} \mathbf{D}'_2 \begin{bmatrix} 1 \\ \vdots \\ 1 - \frac{1}{w_j} \\ \vdots \\ 1 \end{bmatrix} + \frac{w_j}{\theta_\tau^2} \cdot \frac{\theta_\tau^2}{w_j(1-w_j)(1-\phi_\tau)}$$

The first term is

$$\begin{aligned}\frac{1}{\Theta(1-w_j)(1-\phi_\tau)} \mathbf{D}'_1 \begin{bmatrix} 0 \\ \vdots \\ -1/\phi_\tau \\ \vdots \\ 0 \end{bmatrix} &= \frac{1}{\Theta(1-w_j)(1-\phi_\tau)} \begin{bmatrix} 0 & \dots & \frac{w_j}{\theta_\tau^2} & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ -1/\phi_\tau \\ \vdots \\ 0 \end{bmatrix} \\ &= \frac{1}{\Theta(1-w_j)(1-\phi_\tau)} \cdot \frac{-w_j}{\theta_\tau^2} \cdot \frac{1}{\phi_\tau} \\ &= \frac{-w_j \Theta}{\Theta(1-w_j)(1-\phi_\tau)} \\ &= \frac{-w_j}{(1-w_j)(1-\phi_\tau)}.\end{aligned}$$

Next,

$$\begin{aligned}\frac{1}{\Theta(1-w_j)(1-\phi_\tau)} \mathbf{D}'_2 \begin{bmatrix} 1 \\ \vdots \\ 1 - \frac{1}{w_j} \\ \vdots \\ 1 \end{bmatrix} &= \frac{1}{\Theta(1-w_j)(1-\phi_\tau)} \begin{bmatrix} 0 & \dots & \frac{w_j}{\theta_\tau^2} & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 - \frac{1}{w_j} \\ \vdots \\ 1 \end{bmatrix} \\ &= \frac{1}{\Theta(1-w_j)(1-\phi_\tau)} \cdot \frac{-(1-w_j)}{\theta_\tau^2} \\ &= \frac{-1}{\Theta \theta_\tau^2 (1-\phi_\tau)} \\ &= \frac{-\phi_\tau}{(1-\phi_\tau)}.\end{aligned}$$

Lastly,

$$\frac{w_j}{\theta_\tau^2} \cdot \frac{\theta_\tau^2}{w_j(1-w_j)(1-\phi_\tau)} = \frac{1}{(1-w_j)(1-\phi_\tau)}.$$

Combining gives

$$\mathbf{G}_{33} = \frac{-w_j}{(1-w_j)(1-\phi_\tau)} - \frac{\phi_\tau}{(1-\phi_\tau)} + \frac{1}{(1-w_j)(1-\phi_\tau)} = 1.$$

*Summary*

The analysis above establishes that

$$\mathbf{F}'\mathbf{F}(\mathbf{F}'\mathbf{F})^{-1} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} & \mathbf{G}_{13} \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{G}_{23} \\ \mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{G}_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} = \mathbf{I},$$

which serves to confirm the form of the inverse.

#### References

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## A2. THREE SETS OF WEIGHTS

For robustness, we have used two additional sets of weights -- PPP-based GDP and nominal GDP -- in addition to trade weights reported in the text. This Appendix gives the details for the calculation of these three sets of weights and the corresponding estimates.

### Trade Weights

The value of imports and exports in \$US for the 24 countries/areas from 1994 to 2014 are downloaded from the IMF International Financial Statistics website. The data for the Euro member countries are only available starting from 1999, the inception year. The country weights are computed as follows:

- For each year over the 16-year period from 1999 to 2014, we first calculate the weights for each of 24 countries, and then the 15-year average for the Euro area,  $\bar{w}_{\text{euro}}$ .
- For each year from 1994 to 1998, we replace the missing value for the euro area with its average  $\bar{w}_{\text{euro}}$ . The weights for the remaining 23 countries from 1994 to 1998 are normalised such that  $\sum_{i=1}^{23} w_i = 1 - \bar{w}_{\text{euro}}$ .
- Finally, the mean weight for each country is calculated as the average of the weights for the relevant sub-sample.

### GDP, PPP

GDP based on PPP exchange rates for the 24 countries are downloaded from IMF World Economic Outlook (WEO). Since data for the Czech Republic are unavailable for 1994 from the WEO, we use instead data from the World Bank (<http://data.worldbank.org/>). The Euro Area as a whole is not reported by the WEO so for each year, we calculate this as the sum of GDP of constituent countries adopting the Euro dollar as the official currency.<sup>16</sup> The corresponding weights are constructed following similar steps to those above for the missing pre-1999 Euro Area data.

### GDP, Market Exchange Rates

A third set of weights using GDP with market exchange rates (MER) are also constructed using data from the WEO. As before, data are only available up to 2014. We use the same approach to missing values and the calculation of weights as above.

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<sup>16</sup> The constituent countries making up the Euro Area are as follows: Starting from its inception in 1999, there countries are Austria, Belgium, Finland, France, Germany, Ireland, Italy, Luxembourg, Netherlands, Portugal and Spain. Later additions to the Euro Area are Greece (2001), Slovenia (2007), Cyprus (2008), Malta (2008), Slovakia (2009), Estonia (2011) and Latvia (2014).

## The Estimates

Table A2.1 gives the three sets of weights for the four sub-periods. Model (5) of the text is  $q_{it} = a_t + b_i + \delta_{ij}\delta_{t\tau}\gamma + \varepsilon_{it}$ , which we estimate without specifying a country and year for which mispricing is examined; that is, for  $i \neq j$  and  $t \neq \tau$ ,  $\delta_{ij} = \delta_{t\tau} = 0$ , so that the model to be estimated takes the form  $q_{it} = a_t + b_i + \varepsilon_{it}$ . The estimates are presented in Table A2.2. Panel A shows for the estimated year effects  $\hat{a}_t$ , the ranking is GDP-MER > Trade > GDP-PPP. Recall that  $-\hat{a}_t$  is interpreted as the mispricing of the dollar, that is, the dollar is over-valued if  $\hat{a}_t < 0$ . Thus, these results show that PPP-converted GDP leads to the dollar being over-valued by more than it is under the two other weighting schemes. In most cases, the standard errors of the  $\hat{a}_t$  's from three weights are of similar magnitude.

Panel B shows the country-specific estimates  $\hat{b}_i$  associated with the three sets of weights. In all years, when the GDP-PPP weights are used, these estimates are larger than those with when trade weights are used; and the trade-based estimates always exceed the ones with GDP-MER weights. However, for the majority of cases, the differences are not large. Generally speaking, the standard errors associated with the trade and MER weights are lower than those with the PPP weights.

## Summary

When the weights change, only very seldomly do the signs of the estimates change; and when they do, the change is usually not substantial. When trade weights are used (the case reported in the text), the overvaluation of the US dollar is somewhat larger relative to the case when GDP's at market exchange rates are used, but lower than when PPP GDP's are used. Regarding the estimates of the country effects, the choice of the weights has only modest effects.

**Table A1.1** Inverse of Moment Matrix  $\mathbf{F}'\mathbf{F}$

T rows	$\theta_1^2$	0	...	0	0	$\vdots$	$\ddots$	...	0	$\vdots$	$\vdots$	
	0	$\ddots$			$\vdots$	$-\frac{\theta_\tau^2}{v_j h_\tau}$	...	$-\frac{\theta_\tau^2}{h_\tau}$	...	$-\frac{\theta_\tau^2}{v_j h_\tau}$	$\frac{\Theta \cdot \theta_\tau^4}{(1-w_j)h_\tau}$	
	$\vdots$		$\theta_\tau^2 \left[ 1 - \frac{1}{v_j(1-\phi_\tau)} \right]$	$\vdots$	$-\frac{\theta_\tau^2}{v_j h_\tau}$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	0	...	$\ddots$	0	$\theta_T^2$	0	...	$\ddots$	$\vdots$	0	0	
n-1 rows	0	...	$-\frac{\theta_\tau^2}{v_j h_\tau}$	...	0	$\frac{\Theta^{-1}}{v_j h_\tau} - \Theta^{-1} v_1$	$\frac{\Theta^{-1}}{v_j h_\tau} - \Theta^{-1}$	...	$\frac{\Theta^{-1}}{h_\tau} - \Theta^{-1}$	...	$\frac{\Theta^{-1}}{v_j h_\tau} - \Theta^{-1}$	$\frac{-\theta_\tau^2}{(1-w_j)h_\tau}$
	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\frac{\Theta^{-1}}{v_j h_\tau} - \Theta^{-1}$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$-\frac{\theta_\tau^2}{h_\tau}$	$\vdots$	$\vdots$	$\frac{\Theta^{-1}}{h_\tau} - \Theta^{-1}$	...	$\frac{\Theta^{-1} v_j}{h_\tau} - \Theta^{-1} v_j$	...	$\frac{\Theta^{-1}}{h_\tau} - \Theta^{-1}$	$\frac{\theta_\tau^2}{w_j h_\tau}$	
	0	...	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\frac{\Theta^{-1}}{v_j h_\tau} - \Theta^{-1}$	$\vdots$	$\vdots$	
1 row	0	...	$-\frac{\theta_\tau^2}{v_j h_\tau}$	...	0	$\frac{\Theta^{-1}}{v_j h_\tau} - \Theta^{-1}$	...	$\frac{\Theta^{-1}}{h_\tau} - \Theta^{-1}$	...	$\frac{\Theta^{-1}}{v_j h_\tau} - \Theta^{-1} v_{n-1}$	$\frac{-\theta_\tau^2}{(1-w_j)h_\tau}$	
	0	...	$\frac{\Theta \cdot \theta_\tau^4}{(1-w_j)h_\tau}$	...	0	$\frac{-\theta_\tau^2}{(1-w_j)h_\tau}$	...	$\frac{\theta_\tau^2}{w_j h_\tau}$	...	$\frac{-\theta_\tau^2}{(1-w_j)h_\tau}$	$\frac{\theta_\tau^2}{w_j(1-w_j)(1-\phi_\tau)}$	

**Table A2.1** Country Weights

Country/Weight	1994-1999			2000-2004			2005-2009			2010-2014		
	Trade	GDP-PPP	GDP-MER	Trade	GDP-PPP	GDP-MER	Trade	GDP-PPP	GDP-MER	Trade	GDP-PPP	GDP-MER
1. Argentina	0.0079	0.0176	0.0175	0.0057	0.0143	0.0108	0.0062	0.0152	0.0094	0.0068	0.0162	0.0125
2. Australia	0.0196	0.0202	0.0208	0.0183	0.0203	0.0221	0.0199	0.0194	0.0266	0.0231	0.0185	0.0325
3. Brazil	0.0168	0.0615	0.0399	0.0152	0.0573	0.0279	0.0181	0.0558	0.0397	0.0220	0.0563	0.0539
4. Britain	0.0883	0.0572	0.0722	0.0814	0.0557	0.0836	0.0661	0.0509	0.0785	0.0523	0.0438	0.0595
5. Canada	0.0642	0.0329	0.0331	0.0634	0.0328	0.0385	0.0502	0.0298	0.0409	0.0432	0.0271	0.0398
6. Chile	0.0054	0.0065	0.0039	0.0050	0.0065	0.0037	0.0067	0.0066	0.0048	0.0071	0.0068	0.0057
7. China	0.0495	0.1169	0.0459	0.0849	0.1520	0.0705	0.1333	0.2081	0.1060	0.1803	0.2772	0.1868
8. Czech Rep.	0.0081	0.0066	0.0032	0.0108	0.0062	0.0039	0.0143	0.0063	0.0054	0.0144	0.0055	0.0048
9. Denmark	0.0153	0.0066	0.0093	0.0135	0.0061	0.0091	0.0123	0.0054	0.0092	0.0096	0.0044	0.0075
10. Euro Area	0.2627	0.3099	0.3596	0.2589	0.3018	0.3580	0.2515	0.2686	0.3629	0.2245	0.2317	0.2941
11. Hong Kong	0.0594	0.0069	0.0084	0.0535	0.0065	0.0079	0.0450	0.0068	0.0061	0.0447	0.0067	0.0059
12. Hungary	0.0062	0.0054	0.0025	0.0096	0.0054	0.0033	0.0111	0.0051	0.0039	0.0097	0.0042	0.0030
13. Japan	0.1200	0.1325	0.2424	0.1032	0.1138	0.2062	0.0844	0.0976	0.1394	0.0754	0.0834	0.1215
14. Malaysia	0.0237	0.0107	0.0049	0.0225	0.0112	0.0052	0.0200	0.0117	0.0057	0.0199	0.0123	0.0069
15. Mexico	0.0338	0.0439	0.0250	0.0431	0.0423	0.0343	0.0349	0.0389	0.0292	0.0352	0.0363	0.0268
16. N. Zealand	0.0045	0.0032	0.0032	0.0039	0.0031	0.0032	0.0036	0.0030	0.0037	0.0036	0.0027	0.0039
17. Poland	0.0099	0.0156	0.0079	0.0130	0.0160	0.0096	0.0181	0.0160	0.0121	0.0187	0.0161	0.0116
18. Russia	0.0237	0.0585	0.0163	0.0238	0.0591	0.0177	0.0362	0.0649	0.0348	0.0398	0.0613	0.0421
19. Singapore	0.0382	0.0057	0.0047	0.0335	0.0060	0.0046	0.0355	0.0067	0.0050	0.0362	0.0075	0.0063
20. S. Korea	0.0411	0.0265	0.0265	0.0431	0.0295	0.0294	0.0458	0.0298	0.0297	0.0498	0.0298	0.0280
21. Sweden	0.0236	0.0095	0.0139	0.0209	0.0094	0.0137	0.0191	0.0087	0.0134	0.0159	0.0078	0.0123
22. Switzerland	0.0242	0.0107	0.0161	0.0217	0.0098	0.0149	0.0201	0.0089	0.0143	0.0192	0.0081	0.0150
23. Taiwan	0.0354	0.0165	0.0151	0.0334	0.0175	0.0152	0.0286	0.0178	0.0119	0.0273	0.0181	0.0111
24. Thailand	0.0187	0.0185	0.0078	0.0178	0.0176	0.0066	0.0189	0.0181	0.0074	0.0213	0.0180	0.0087

Note: GDP-PPP (MER) refers to the share of a country's GDP in the total when PPP (market) exchange rates are used to convert to US dollars.



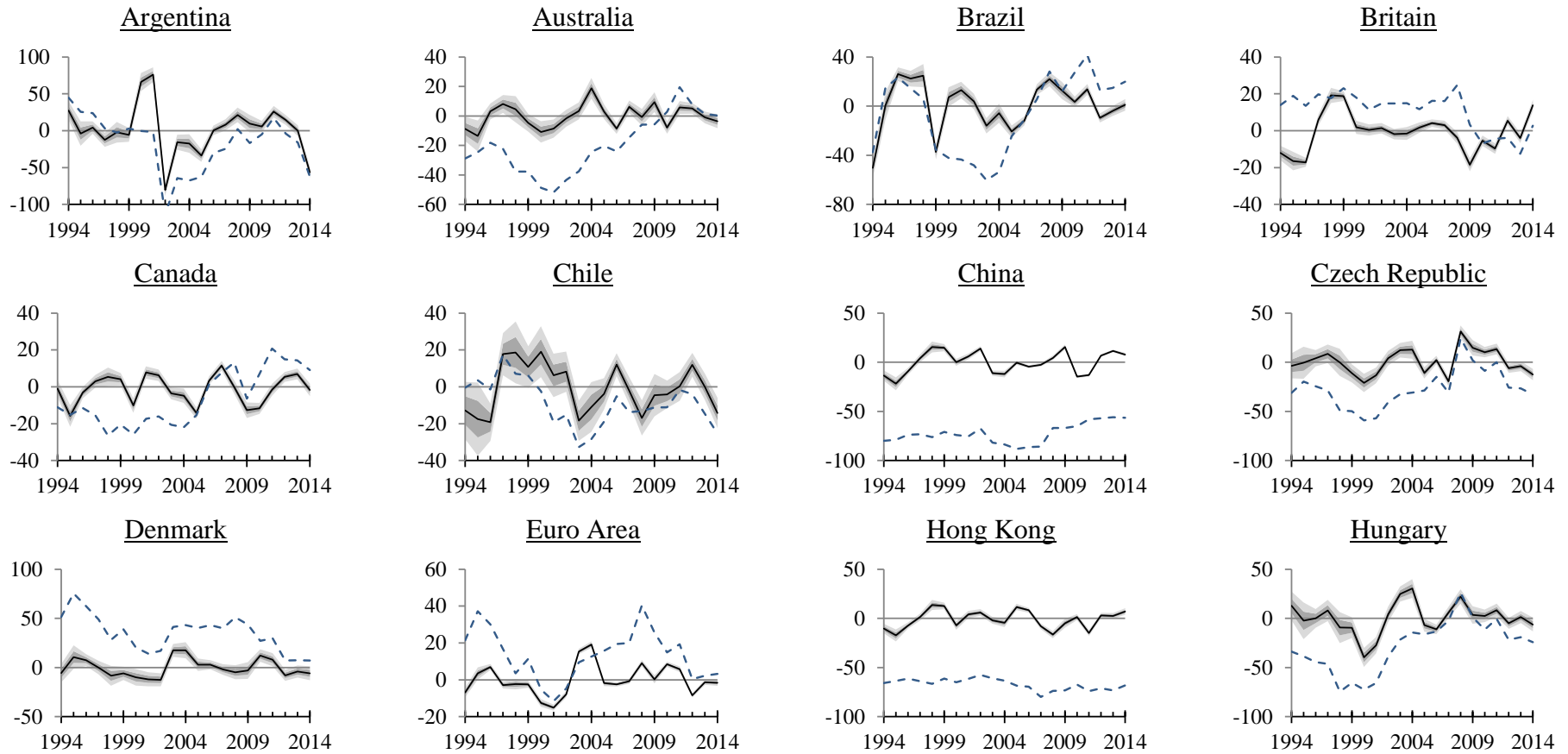
**Table A2.2** Estimates of Currency Pricing Model with Three Sets of Weights,  $q_{it} = a_t + b_i + \varepsilon_{it}$

	Sub-period 1						Sub-period 2					Sub-period 3					Sub-period 4							
	A. The time component $\hat{a}_t \times 100$																							
Weight/Year	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014			
Trade	0.03	0.12	0.04	-0.04	-0.18	-0.10	-0.19	-0.24	-0.21	-0.19	-0.19	-0.23	-0.19	-0.19	-0.05	-0.14	-0.19	-0.13	-0.23	-0.25	-0.24			
	(0.03)	(0.03)	(0.01)	(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)			
GDP-PPP	-0.01	0.10	0.02	-0.07	-0.18	-0.14	-0.23	-0.29	-0.27	-0.26	-0.25	-0.29	-0.25	-0.24	-0.08	-0.16	-0.21	-0.15	-0.25	-0.27	-0.27			
	(0.03)	(0.04)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)			
GDP-MER	0.15	0.29	0.13	0.02	-0.11	-0.03	-0.10	-0.17	-0.17	-0.13	-0.12	-0.14	-0.11	-0.10	0.05	-0.03	-0.11	-0.05	-0.17	-0.20	-0.18			
	(0.03)	(0.04)	(0.02)	(0.02)	(0.03)	(0.02)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)			
	B. The country component $\hat{b}_i \times 100$																							
Country/Weight	Trade		GDP-PPP		GDP-MER		Trade		GDP-PPP		GDP-MER		Trade		GDP-PPP		GDP-MER		Trade		GDP-PPP		GDP-MER	
1. Argentina	0.17	(0.09)	0.19	(0.07)	0.08	(0.07)	-0.35	(0.10)	-0.27	(0.07)	-0.42	(0.08)	-0.11	(0.07)	-0.06	(0.05)	-0.21	(0.06)	0.10	(0.07)	0.13	(0.05)	0.03	(0.06)
2. Australia	-0.24	(0.06)	-0.22	(0.07)	-0.33	(0.06)	-0.21	(0.06)	-0.15	(0.06)	-0.27	(0.06)	0.00	(0.04)	0.05	(0.04)	-0.09	(0.03)	0.28	(0.04)	0.30	(0.05)	0.21	(0.03)
3. Brazil	0.03	(0.06)	0.07	(0.04)	-0.07	(0.05)	-0.30	(0.06)	-0.25	(0.04)	-0.36	(0.05)	0.15	(0.04)	0.20	(0.02)	0.05	(0.03)	0.43	(0.04)	0.46	(0.03)	0.37	(0.03)
4. Britain	0.20	(0.03)	0.22	(0.04)	0.13	(0.03)	0.35	(0.03)	0.41	(0.04)	0.29	(0.03)	0.33	(0.02)	0.38	(0.03)	0.24	(0.02)	0.16	(0.02)	0.18	(0.03)	0.09	(0.02)
5. Canada	-0.13	(0.03)	-0.11	(0.05)	-0.22	(0.05)	0.01	(0.03)	0.06	(0.05)	-0.05	(0.04)	0.19	(0.03)	0.24	(0.03)	0.10	(0.03)	0.35	(0.03)	0.37	(0.04)	0.27	(0.03)
6. Chile	0.08	(0.11)	0.10	(0.12)	0.00	(0.15)	0.00	(0.11)	0.05	(0.11)	-0.07	(0.14)	0.07	(0.07)	0.12	(0.07)	-0.02	(0.08)	0.11	(0.07)	0.13	(0.07)	0.04	(0.08)
7. China	-0.72	(0.04)	-0.70	(0.03)	-0.79	(0.04)	-0.55	(0.03)	-0.51	(0.02)	-0.62	(0.03)	-0.65	(0.02)	-0.60	(0.01)	-0.74	(0.02)	-0.37	(0.01)	-0.35	(0.01)	-0.45	(0.01)
8. Czech Rep.	-0.31	(0.09)	-0.28	(0.11)	-0.40	(0.16)	-0.23	(0.08)	-0.17	(0.11)	-0.28	(0.14)	0.03	(0.05)	0.07	(0.07)	-0.06	(0.08)	0.02	(0.05)	0.04	(0.08)	-0.04	(0.09)
9. Denmark	0.53	(0.07)	0.56	(0.11)	0.45	(0.10)	0.47	(0.07)	0.55	(0.11)	0.42	(0.09)	0.61	(0.05)	0.65	(0.08)	0.52	(0.06)	0.36	(0.06)	0.39	(0.09)	0.30	(0.07)
10. Euro Area	0.23	(0.01)	0.25	(0.01)	0.14	(0.01)	0.20	(0.01)	0.27	(0.01)	0.15	(0.01)	0.40	(0.01)	0.44	(0.01)	0.30	(0.01)	0.28	(0.01)	0.31	(0.01)	0.22	(0.01)
11. Hong Kong	-0.60	(0.03)	-0.58	(0.11)	-0.68	(0.10)	-0.41	(0.03)	-0.35	(0.11)	-0.47	(0.10)	-0.55	(0.03)	-0.51	(0.07)	-0.64	(0.07)	-0.50	(0.03)	-0.48	(0.08)	-0.57	(0.08)
12. Hungary	-0.48	(0.10)	-0.46	(0.13)	-0.58	(0.19)	-0.21	(0.08)	-0.14	(0.12)	-0.25	(0.15)	0.12	(0.06)	0.17	(0.08)	0.03	(0.09)	0.05	(0.06)	0.08	(0.09)	-0.01	(0.11)
13. Japan	0.13	(0.02)	0.16	(0.02)	0.04	(0.02)	0.06	(0.02)	0.12	(0.02)	0.00	(0.02)	-0.13	(0.02)	-0.09	(0.02)	-0.23	(0.01)	0.08	(0.02)	0.10	(0.02)	0.01	(0.02)
14. Malaysia	-0.52	(0.05)	-0.49	(0.09)	-0.61	(0.13)	-0.51	(0.05)	-0.46	(0.08)	-0.57	(0.12)	-0.56	(0.04)	-0.51	(0.05)	-0.65	(0.07)	-0.40	(0.04)	-0.37	(0.06)	-0.46	(0.07)
15. Mexico	-0.14	(0.04)	-0.13	(0.04)	-0.22	(0.06)	0.06	(0.04)	0.10	(0.04)	-0.01	(0.05)	-0.03	(0.03)	0.02	(0.03)	-0.12	(0.03)	-0.22	(0.03)	-0.20	(0.03)	-0.29	(0.04)
16. N. Zealand	-0.18	(0.12)	-0.15	(0.17)	-0.26	(0.16)	-0.14	(0.13)	-0.06	(0.16)	-0.19	(0.15)	0.14	(0.10)	0.18	(0.11)	0.04	(0.09)	0.19	(0.10)	0.21	(0.12)	0.12	(0.10)
17. Poland	-0.50	(0.08)	-0.48	(0.07)	-0.58	(0.10)	-0.35	(0.07)	-0.30	(0.07)	-0.41	(0.09)	-0.16	(0.04)	-0.11	(0.05)	-0.25	(0.05)	-0.22	(0.04)	-0.20	(0.05)	-0.28	(0.06)
18. Russia	-0.30	(0.05)	-0.26	(0.04)	-0.39	(0.07)	-0.49	(0.05)	-0.43	(0.03)	-0.55	(0.06)	-0.37	(0.03)	-0.32	(0.02)	-0.46	(0.03)	-0.33	(0.03)	-0.31	(0.02)	-0.40	(0.03)
19. Singapore	-0.15	(0.04)	-0.12	(0.12)	-0.24	(0.14)	-0.14	(0.04)	-0.09	(0.11)	-0.20	(0.13)	-0.11	(0.03)	-0.06	(0.07)	-0.20	(0.08)	0.02	(0.03)	0.04	(0.07)	-0.05	(0.08)
20. S. Korea	0.12	(0.04)	0.14	(0.06)	0.03	(0.06)	0.17	(0.04)	0.23	(0.05)	0.11	(0.05)	0.02	(0.03)	0.07	(0.03)	-0.07	(0.03)	-0.04	(0.03)	-0.02	(0.04)	-0.10	(0.04)
21. Sweden	0.36	(0.05)	0.39	(0.10)	0.27	(0.08)	0.32	(0.05)	0.39	(0.09)	0.26	(0.07)	0.56	(0.04)	0.61	(0.06)	0.47	(0.05)	0.60	(0.05)	0.63	(0.07)	0.55	(0.06)
22. Switzerland	0.61	(0.05)	0.63	(0.09)	0.52	(0.07)	0.64	(0.05)	0.70	(0.09)	0.59	(0.07)	0.68	(0.04)	0.72	(0.06)	0.59	(0.05)	0.68	(0.04)	0.70	(0.07)	0.61	(0.05)
23. Taiwan	-0.01	(0.04)	0.01	(0.07)	-0.10	(0.07)	-0.01	(0.04)	0.04	(0.07)	-0.08	(0.07)	-0.15	(0.03)	-0.11	(0.04)	-0.24	(0.05)	-0.31	(0.03)	-0.29	(0.05)	-0.38	(0.06)
24. Thailand	-0.32	(0.06)	-0.30	(0.07)	-0.42	(0.10)	-0.47	(0.06)	-0.41	(0.07)	-0.53	(0.11)	-0.50	(0.04)	-0.45	(0.04)	-0.59	(0.07)	-0.29	(0.04)	-0.27	(0.05)	-0.37	(0.07)

Note: Standard errors in parentheses.

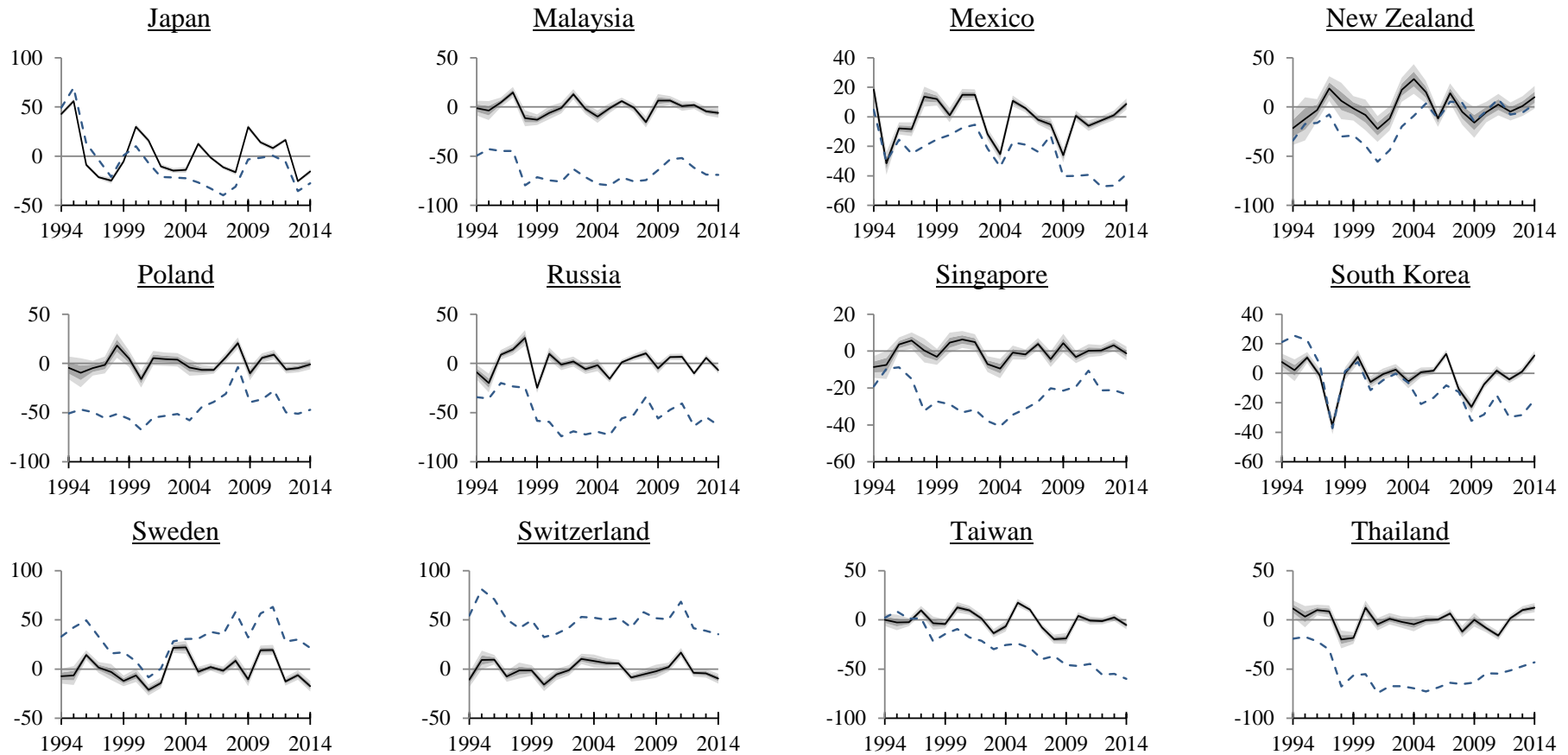
### A3. COMPARISON OF EBMI AND BMI

**Figure A3.1** EBMI vs. BMI, 24 Currencies, 1994-2014, All values  $\times 100$



(continued on next page)

**Figure A3.1** EBMI vs. BMI, 24 Currencies, 1994-2014, All values  $\times 100$  (continued)



Notes:

1. The broken line is the Big Mac Index (BMI). That is, for currency  $i$  and year  $t$ , the BMI is defined logarithmically as  $\log(P_{it}/S_{it}P_t^*)$ , where  $P_{it}$  and  $P_t^*$  are the prices of a Big Mac in country  $i$  and the US; and  $S_{it}$  is the exchange rate. According to the BMI, the currency is overvalued if  $\log(P_{it}/S_{it}P_t^*) > 0$  and vice versa.
2. The solid dark line is the enhanced BMI (EBMI), that is, the estimate of  $\gamma$  in  $q_{it} = a_i + b_i + \delta_{ij}\delta_{it}\gamma + \epsilon_{it}$ , where  $\delta_{pq}$  is the Kronecker delta ( $\delta_{pq} = 1$  if  $p = q$ , 0 otherwise). The dark grey band is the EBMI  $\pm$  one standard error, and the light grey band is  $\pm$  two standard errors. When the light grey band includes 0, we are unable to reject  $H_0: \gamma = 0$ ; and when the band includes the BMI, we are unable to reject  $H_0: \gamma = \text{BMI}$ . The model is estimated separately for each of the four sub-periods.

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