Discrete self-similarity of multiscale materials and systems. Universality of scaling exponents

Abstract

Materials often possess complex microstructure showing structural hierarchy covering a wide range of scales. An example of material with multiscale structure is the Earth's crust. The simplest model of such material structure is self-similarity making the scale dependent quantities to be expressed by the power law. Somewhat more realistic model is the discrete self-similarity that is the material structure is self-similar only over a discrete (and self-similar) set of scales. We show that in discrete self-similarity the power law dependence is preserved but only over the discrete sets of scales. The main result of this work is that as long as discrete-scaling quantities are in a linear relationship, they must either scale with the same exponent or vanish; we call this property the Universality of scaling exponents. In particular, non-zero components of the same tensor must scale with the same exponent. This leads to the same scaling of the elastic moduli and, consequently, to the same scaling of the wave velocities. Average stress and strain and their higher statistical moments also scale by power law; simple relationships are identified between the scaling exponents of average strain, stress, their higher moments and the elastic moduli. As the scaling of average strain could be determined from observations and scaling moduli can be inferred from the measured wave velocities, the derived relationships can be used to infer the scaling of the statistical parameters of stress distributions. The presented concept will be useful in characterising properties of both the meta- and hybrid materials and geomaterials.

Keywords: Power law, Universality of scaling, Multiscale modelling, Effective characteristics, Wave velocities, Average stress, Average strain.

1 Introduction

Engineering and natural materials often possess structural hierarchy (see e.g., reviews in Lakes, 1993; Pan, 2014, Lim et al., 2015). The main example is the geomaterials that can incorporate a wide range of scales. The properties of such materials are strongly scale-dependent exhibiting the scale effect, that is the reduction in values of the characteristics (e.g., the strengths) with scale increase (e.g. Jaeger, 1979; van Mier, 1992; Carpinteri et al., 1995). Even the results of stress measurements can be scale dependent, Cuisiat & Haimson (1992). As another example one can consider hybrid materials, where the scale can serve as a design parameter thus expanding the space of design parameters (in the spirit of Ashby and Bréchet, 2003).

Quantifying the scale dependence both experimentally, observationally or theoretically meets considerable technical difficulties. This circumstance prompted search for simplified methods of incorporating the scale factor into the description of the material structure. Staring with the seminal

works of Mandelbrot there has been a considerable body of research on fractals (and multi fractals) and their applications in engineering to model different features of the materials with multiscale microstructure (e.g. Katz & Thompson, 1985; Jones, et al., 1989; Barton and Zoback, 1992; Bruno et al., 1992; Ouchi & Matsushita, 1992; Turcotte, 1993; Roach & Fowler, 1993; Carpintery, 1994; Saouma & Barton, 1994; Yokoi, et al., 1995; Dubois, 1998; Cheng, 1999; Parsons, 2008; Ghanbarian & Hunt, 2017) as well as fractures in the materials (e.g., Herrmann, 1989; Mecholsky et al., 1989; Gol'dstein & Mosolov, 1991; Mosolov, 1991a,b; Mosolov & Borodich, 1992; Borodich, 1992, 1994, 1997; Bazant, 1993, 1997; Issa & Hammad, 1993; Xie & Sanderson, 1995; Balankin, 1997; Dyskin, 1999a,b, 2000, 2001, 2002a,b, 2003, 2006, 2007, 2008; Sahouryeh & Dyskin, 2001; Weiss, 2001; Carpenteri & Cornetti, 2002; Yavari et al., 2002a,b; Drazer, et. al., 2004; He et al., 2019). Central to this approach are the observations and measurements of patterns in the materials and especially in the Earth's crust suggesting the presence of self-similarity (e.g., Shearer, 2012) often associated with some types of self-organised criticality (e.g., Turcotte, 2001).

Sometimes term "fractal" is used as a synonym of for global self-similarity (e.g., Hartmann, 1969; Babaie, et al., 1985; Scholz and Aviles, 1986; Scholz, 1990; Redner, 1990; Turcotte, 1993; Gillespie et al., 1993; Sornette, 2000; Shen, 2011) although fractals per se possess self-similarity in the limiting case of the scale tending to zero. We however concentrate on the materials, e.g., the Earth's crust whose structure ranges across the scales and hence can provide a proper object for the self-similar approximation. The self-similarity then forms a basis for quantification of scaling of various properties. Of interest here are the mechanical properties; the concept of self-similarity will be used for determining the material properties at the scales restricted for direct measurements, such as microscopic scales in engineering materials or large scales in geomaterials.

One of the types of multiscale structure is the presence of fractures of different scales. These obviously affect both deformation moduli, wave velocities and strength parameters. However the presence of fractures hardly affect the material fractal dimension, which remains to be approximately 3. Therefore the attempts to relate the deformation and fracture behaviour of material by directly relating their properties to fractal dimensions (e.g., Balankin, 1992; Bazant, 1993; Mosolov, 1993; Carpinteri, 1994; Cherepanov, et al., 1995) is insufficient; the relationship between the mechanical and geometric scaling laws is more subtle. This will be discussed later in the paper.

If a system can be regarded as self-similar, then any continuously differentiable function of scale, H, can only be expressed by a power law, $f(H) \propto H^{\alpha}$, (e.g., Barenblatt & Botvina, 1980; Zosimov & Lyamishev, 1995). Gelikman & Pisarenko (1989) showed that it is sufficient for $f(H)$ be bounded on an interval $[H_1, H_2]$.) for the power law property to be upheld, The power law property can be expressed as

$$\frac{f(H)}{f(H_0)} = \left(\frac{H}{H_0}\right)^{\kappa} \tag{1}$$

where H_0 is a reference scale, κ is the scaling exponent.

Power law (1) corresponds to a linear relationship between log $f(H)$ and log H ranging from zero scale to infinity (from negative infinity to infinity in the log-log plot). Then the power law can be treated as a linear approximation for scale-dependent functions (in the log-log plots, Fig. 1, Dyskin

2004). Hence it could be considered as representing the real scale dependence only within a certain range of scales.

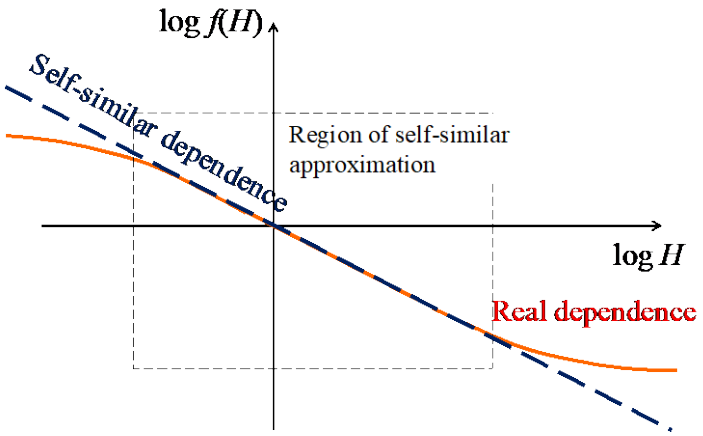

Figure 1. Relationship between self-similar approximation (power law) and the real scale-dependent function in log-log scale.

It is necessary to emphasise that this approach cannot be reduced to the fractal theory where the power law and the fractal dimension refer to the limit $H \to 0$ ($\log H \to -\infty$ in Fig. 1) only, and for that reason can find itself outside the region of approximation.

The assumption that the self-similarity is continuous is restrictive. More common situations would correspond to the self-similarity valid only at a discrete set of scales (see Sadovskiy, 1983 on a mechanism of formation of discrete self-similarity). The main property of such systems is (Sornette, 2000)

$$f(\lambda_0 H_0) = \mu_0 f(H_0) \tag{2}$$

where H_0 is a proper reference scale (that is the scale belonging to the set the discrete self-similarity applies to) and λ_0, μ_0 are the scaling factors.

Discrete self-similarity corresponds to a discrete set of scales, $\{H_n\}$:

$$H_n = \lambda_0^n H_0 \tag{3}$$

which implies that

$$\frac{f(H_n)}{f(H_0)} = \left(\frac{H_n}{H_0}\right)^{\kappa}, \qquad \kappa = \frac{\ln\mu_0}{\ln\lambda_0} \tag{4}$$

This is a power law similar to eq. (1) but valid only for discrete values of H given by (3). (In order to reduce the power law (4) to continuous scaling, that is the scaling valid for all scales H, Sornette, 2000 attempted to introduce a periodic approximation of (4): $f(H)/f(H_0) = (H_n/H_0)^{\kappa} P(H/H_0)$,

where P was an interpolating function. Then extending it by periodicity and developing into Fourier series Sornette, 2000 arrived at a power law (1) with complex exponents. However, in line with (4), it is ln$P$, which allows relevant periodic extension; after this correction the complex exponents are no longer derivable.) This paper will analyse discrete power law (4) and derive important scaling properties.

2 Universality of scaling in tensors

The power law (both continuous and discrete) imposes strong restrictions on the scale dependence of the tensorial properties as these satisfy what we call the *Universality of scaling exponents*.

Universality of scaling exponents. Consider a vector (tensor) whose components depend upon scale only, whose scaling is governed by the discrete power law (4). Then all non-zero components of the vector (tensor) scale with the *same scaling exponent* κ. In particular, all elastic moduli either scale with the same exponent or zero. In other words, the scaling of the moduli (rather than Poisson's ratios which are coefficients) is isotropic, only the prefactors are the bearers of anisotropy. (For the case of continuous self-similarity this property was shown by Dyskin, 2005.)

Proof of Universality of scaling exponents

We start with the following Lemma: K functions (continuous or discrete) forming set $\{h^{\kappa_1}, h^{\kappa_2}, \dots h^{\kappa_K}\}, \kappa_1 < \kappa_2 < \cdots < \kappa_K$ are linearly independent, that is equation $\sum_{k=1}^{K} C_k h^{\kappa_k} = 0$ allows only one solution, $C_k = 0,\ k = 1, \dots, K$. For continuous functions this lemma was proven by Dyskin (2005). For discrete functions pertinent to this paper, the proof is presented in Appendix A.

The direct consequence of this Lemma is that if discrete power law functions are connected by a linear equation $\sum_{k=1}^{K} C_k h^{\kappa_k} = 0$, all non-zero terms must scale with the same exponent. The pre-factors C_k are not independent but satisfy a linear equation $\sum_{k=1}^{K} C_k = 0$.

Now consider tensor **T** and introduce a co-ordinate frame (x_1, x_2, x_3). Let its components in this co-ordinate frame be $T_{i_1,i_2,\dots,i_n},\ \ i_1, i_2, \dots, i_n = 1,2,3,\ \ n \geq 1$. Now rotate the co-ordinate frame to become (y_1, y_2, y_3). Then the components of the tensor in the new coordinate frame read $Y_{i_1,i_2,\dots,i_n} = R_{i_1,j_1} R_{i_2,j_2} \cdots R_{i_n,j_n}\, T_{j_1,j_2,\dots,j_n}$, where $R_{i_1,j_1}, R_{i_2,j_2}, \dots, R_{i_n,j_n}$ are the components of a rotation matrix (summation over repeated indexes is presumed). Applying scaling law (4) and noting that the rotation matrix is chosen to be scale-independent one obtains the following scaling equations

$$h^{\kappa(i_1,i_2,\dots,i_n)} Y_{i_1,i_2,\dots,i_n}(H_0) = R_{i_1,j_1} R_{i_2,j_2} \cdots R_{i_n,j_n} h^{\kappa(j_1,j_2,\dots,j_n)}\, T_{j_1,j_2,\dots,j_n}(H_0) \tag{5}$$

Here $h = H/H_0$; brackets around the indexes denote the absence of summation.

By moving all parts of eq. (5) to the left hand side and recalling that according to the Lemma the discrete power functions are linearly independent we see that the only non-zero coefficients at $h^{\kappa(i_1,i_2,\dots,i_n)}$ are those multiplied by the power functions with the same exponent. Given that

$R_{i_1,j_1}R_{i_2,j_2}\cdots R_{i_n,j_n} \neq 0$ one arrives at the conclusion that the corresponding tensorial components are zero, which completes the proof.

3 Scaling of moduli and wave velocities. Isotropic case

Mechanical properties such as for instance elastic moduli are defined for continua and thus can be called the continuum characteristics. Obviously, there is no direct way of applying continuum characteristics to self-similar materials having discontinuities ranging across the scales. In other words, at any scale *H* there will be discontinuities both smaller and larger than *H*. This means that the structure does not satisfy *separation of scales property* that requires the representative volume element to be large compared with the microscopic sizes and small compared to the characteristic length of the variation of external fields (e.g., Scipio, 1967; Batchelor, 1974; Hunter, 1976; Krajcinovic, 1996).

Continuum modelling of these deeply discontinuous objects is nevertheless possible by assigning a *series of continua* each representing its own scale, which can be regarded as a kind of multiscale modelling (e.g., Dyskin et al., 1992). This can be accomplished by introducing a series of representative volume elements of different sizes *H*, either continuous (for continuous self-similarity) or discrete, $H_n = \lambda_0^n H_0$ (for discrete self-similarity). Each representative volume element size *H* defines its own equivalent continuum that corresponds to the state variables averaged over the volume elements of size *H* in a usual way (e.g., Scipio, 1967; Batchelor, 1974; Hunter, 1976; Krajcinovic, 1996; Jeronimidis, 2000).

The procedure for self-similar continuum modelling can be described as follows. Consider a material with self-similar structure. In the case of discrete self-similarity each scale is characterised by a discrete set of sizes, $\{L_n : L_n/L_{n-1} = \lambda = const\}$, where L_n is the maximum size of discontinuities (or heterogeneities) of scale *n* (see a 2D illustration in Fig. 2a). Introduce a set of sizes of representative volume elements, $\{H_n = \alpha L_n,\ 1 < \alpha < \lambda,\ \alpha = const\}$. Now, following the methodology of multiscale modelling (Dyskin at al., 1992), we introduce a set of continua $\{C_n\}$, called *equivalent continua*. Continuum C_n is constructed by removing all discontinuities of size greater than L_n and assigning the characteristics (*effective characteristics*) that relate the fields averaged over volume elements of size H_n (Fig. 2b). Set $\{C_n\}$ constructed in such a way approximates the behaviour of the multiscale material asymptotically as $\alpha \to \infty, \lambda/\alpha \to \infty$.

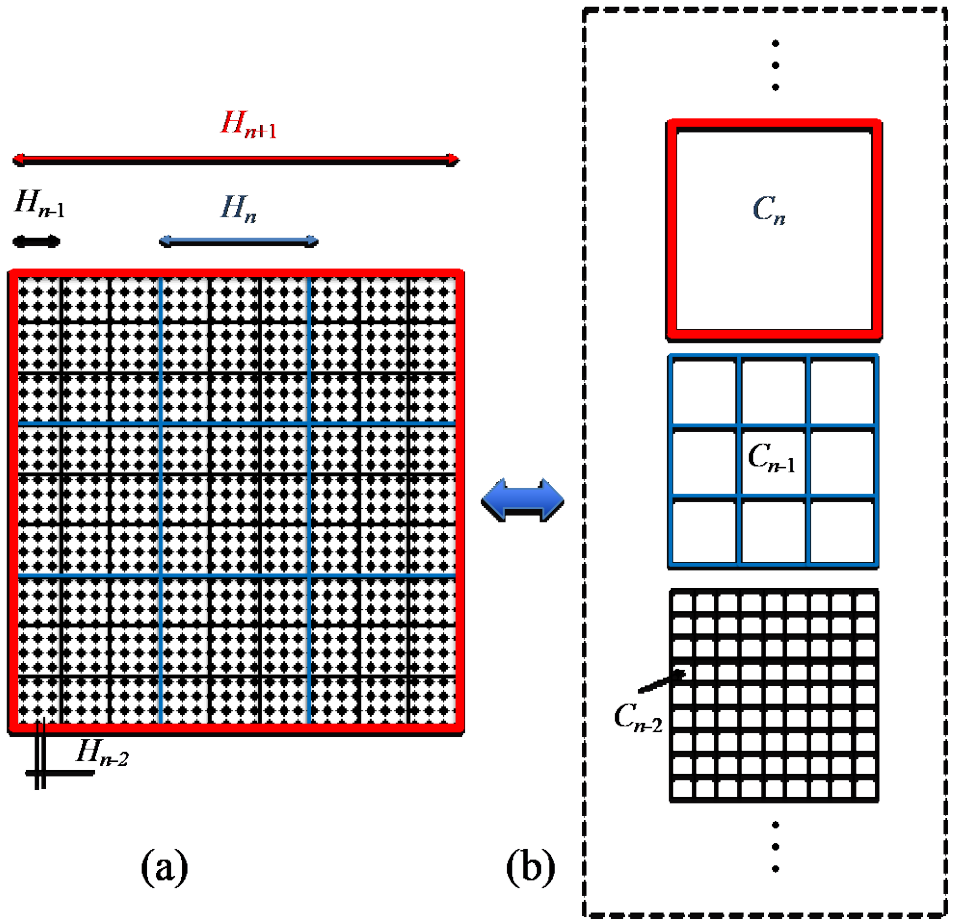

Figure 2. Multiscale continuum modelling; 2D illustration: (a) a material with discrete self-similar set of discontinuities; (b) the self-similar set of equivalent continua modelling the material.

Each equivalent continuum describes behaviour of the material (rock) at the corresponding scale, that is the behaviour of the material without discontinuities/heterogeneities of larger scales. Therefore, the scaling of the effective characteristics corresponds to the determination of their change from one equivalent continuum to another, which reflects the change of scale. This change is governed by pawer laws (4) (or power laws (1) in the case of continuous self-similarity). This allows bypassing technical difficulties associated with the actual averaging over representative volume elements. This concept will be illustrated using elastic moduli and compliances and then wave velocities, when all equivalent continua are isotropic, leaving orthotropic materials for Section 4.

3.1 *Scaling of isotropic moduli of classical continua*

The universality of scaling dictating that the scaling exponents of non-zero components of the tensor of moduli or compliances are the same, leads to the following scaling relations for the Young's, E and shear, G moduli:

$$E(H_n) = E(H_0)\left(\frac{H_n}{H_0}\right)^{\kappa},\ G(H_n) = G(H_0)\left(\frac{H_n}{H_0}\right)^{\kappa}\quad \nu(H_n) = \nu(H_0), \tag{6}$$

while κ is the exponent (in the case of discontinuities, $\kappa < 0$). The Poisson's ratio, being a bounded coefficient rather than a modulus, is scale-independent.

$$\nu(H_n) = \nu(H_0),\ \ 2[\nu(H_n) + 1] = E(H_n)/G(H_n), \tag{7}$$

In general, the theoretical determination of scaling exponent κ may not be an easy task, which makes it necessary to resort to observations and measurements. Yet there is a relatively simple case of the material structure represented by chaotically oriented disc-like cracks of discrete self-similar sets of radii when the scaling exponent can be derived.

Consider scale H_n. We assume that the cracks with sizes belonging to this scale are immersed in the continuum that corresponds to the previous scale, H_{n-1}. For isotropically oriented disc-like cracks the effective characteristics $E(H_n), \nu(H_n)$ are determined by the effective characteristics $E(H_{n-1}), \nu(H_{n-1})$ of the previous scale and the dimensionless crack concentration, $\mathrm{v}_0 = N\langle a^3\rangle$, the same at each scale (due to self-similarity). Here N is the number of cracks of each scale H_n per unit volume and $\langle a^3\rangle$ is the average cube of crack radius, where the averaging is over all cracks of the corresponding scale.

Using the differential self-consistent method (e.g., Salganik, 1973) the following system of differential equations can be obtained

$$\begin{cases} \frac{dE}{d\mathrm{v}} = -\frac{16}{45}E\cdot(10-3\nu)\frac{1-\nu^2}{2-\nu} \\ \frac{d\nu}{d\mathrm{v}} = -\frac{16}{15}\nu\cdot(3-\nu)\frac{1-\nu^2}{2-\nu} \\ E|_{\mathrm{v}=0} = E(H_{n-1}),\ \ \nu|_{\mathrm{v}=0} = \nu(H_{n-1}) \\ E(H_n) = E|_{\mathrm{v}=\mathrm{v}_0},\ \ \nu(H_n) = \nu|_{\mathrm{v}=\mathrm{v}_0} \end{cases} \tag{8}$$

According to eq. (7) the Poisson's ratio of a self-similar object must be constant. It is easy to see that the only constant satisfying the second equation of (8) is $\nu = 0$. Subsequently, the solution of the first equation of (8) is $E(\mathrm{v}) = E(H_{n-1})\exp\left(-\frac{16}{15}\mathrm{v}_0\right)$. From here

$$E(H_n) = E(H_{n-1})\exp\left(-\frac{16}{15}\mathrm{v}_0\right),\ \ \nu(H_n) = 0 \tag{9}$$

Therefore, $\lambda_0 = H_n/H_{n-1}$, $\mu_0 = \exp\left(-\frac{16}{15}\mathrm{v}_0\right)$ and

$$\frac{E(H_n)}{E(H_0)} = \left(\frac{H_n}{H_0}\right)^{\kappa},\quad \kappa = -\frac{16\mathrm{v}_0}{15\ln\lambda_0} \tag{10}$$

Another similar case is the case of randomly oriented disc-like frictionless shear cracks, that is the cracks with suppressed opening. Then, according to Appendix B,

$$\begin{cases} \frac{dE}{d\mathrm{v}}=-\frac{64}{45}E\cdot\frac{1-\nu^2}{2-\nu} \\ \frac{d\nu}{d\mathrm{v}}=-\frac{32}{45}(1-2\nu)\frac{1-\nu^2}{2-\nu} \\ E|_{\mathrm{v}=0}=E(H_{n-1}),\ \ \nu|_{\mathrm{v}=0}=\nu(H_{n-1}) \\ E(H_n)=E|_{\mathrm{v}=\mathrm{v}_0},\ \ \nu(H_n)=\nu|_{\mathrm{v}=\mathrm{v}_0} \end{cases} \tag{11}$$

Here the only value of Poisson's ratio satisfying the second equation of (11) is $\nu = 0.5$. That is the self-similar material with frictionless shear cracks is incompressible, which is a consequence of the fact that the shear cracks while reducing the Young's and shear moduli do not affect the bulk modulus. It should be noted that incompressible materials are intrinsically unstable (e.g. Dyskin et al., 2017), which indicates that pure shear cracks should not exist in nature. Indeed, the asperities and imperfections of the crack faces always introduce some normal compliance, which subsequently make the self-similar rock compressible. Yet, the value $\nu = 0.5$ can serve as a reasonable approximation for the moduli scaling.

Substituting $\nu = 0.5$ into equation (11) one obtains

$$E(H_n) = E(H_{n-1})\exp\left(-\frac{32}{45}\mathrm{v}_0\right),\ \ \nu(H_n) = 0.5 \tag{12}$$

and

$$\frac{E(H_n)}{E(H_0)} = \left(\frac{H_n}{H_0}\right)^{\kappa},\quad \kappa = -\frac{32\mathrm{v}_0}{45\ln\lambda_0} \tag{13}$$

Further approximation can be achieved by assuming that the cracks at each scale are dense, such that they almost touch each other. In this case the average distance between the cracks is approximately $2a$, where a is the average crack radius. Therefore $\mathrm{v}_0 \approx 1/(2a)^3\langle a^3\rangle \approx 1/8$.

3.2 *Scaling of isotropic Cosserat moduli*

In particulate materials such as concrete, rocks, granular and segmented (fragmented) materials the heterogeneities, grains or segments (fragments) can rotate independently of the rotations associated with the displacement field. Rotations of the constituents of the material are observed at different scales from grains (e.g. Kishino and Thornton, 1999; Mueth, et al., 2000; Desrues and Viggiani, 2004; Bagi and Kuhn, 2004; Allonso-Marroquin, 2006; Teisseyre and Górski, 2007; Rechenmacher, et al., 2010) to masonry (e.g., Lourenço et al., 2005) to the Earth's crust (e.g. Burkart and Self, 1985; Fort, et al., 2004; Mukhamediev et al., 2006; McCaffrey et al., 2007; Kuzikova and Mukhamediev, 2010; Vikulin, et al., 2011). This gives rise to additional degrees of freedom, which are associated with the ability of the material constituents to rotate independently.

One of the purposes of developing the non-classical theories such as the Cosserat theory is to capture the effect of additional degrees of freedom related to the microstructure. This is done by introducing additional deformation and stress variables. As a result, the constitutive equations involve a series of characteristic lengths. As an example, the Cosserat theory includes three rotational degrees of freedom on top of the conventional translational ones and thus accounts for

independent rotations of the material constituents. In order to achieve this a vector of internal rotation, φ_i , i=1,2,3, is introduced on top of the displacement vector, u_i. Then the role of deformation measures is played by the strain γ_{ji} and curvature twist κ_{ji} tensors (e.g., Nowacki, 1970):

$$\gamma_{ji} = u_{i,j} - \varepsilon_{kji}\varphi_k, \;\; \kappa_{ji} = \varphi_{i,j} \tag{14}$$

where ε_{ijk} is the alternating tensor, index after comma ,j stands for differentiation over x_j and summation over repeated indexes is presumed. The elastic energy density expressed through these deformation measures reads

$$U = \frac{1}{2}\left(C_{ijkl}\gamma_{ij}\gamma_{kl} + B_{ijkl}\kappa_{ij}\kappa_{kl}\right) \tag{15}$$

where $\mathbf{C}=\left(C_{ijkl}\right)$ and $\mathbf{B}=\left(B_{ijkl}\right)$ are the tensors of Cosserat moduli. From these moduli the characteristic lengths can be formed as $l^2_{ijklmnpr} = C_{ijkl}/B_{mnpr}$. The characteristic lengths represent the structure at scale $H_{n\text{-}1}$. Therefore $l^2_{ijklmnpr} \sim H^2_{n-1}$ (see also Dyskin & Pasternak, 2008). Since, by definition $H_n/H_{n-1} = \lambda = const$, the characteristic lengths have the scaling exponent of 1. Subsequently is the non-zero components of moduli **C** scale with exponent κ, (as moduli of the classical continua), while moduli **B** scale with exponent $\kappa + 2$.

3.3 *Scaling of wave velocities and the set of maximum frequencies*

This section starts with discussing formal scaling laws for wave velocities leaving aside the question of whether or to what distances the waves can propagate in fractured material or rock. We concentrate on the case of self-similar fractured or blocked material (e.g., a blocky rock mass). As fractures little affect the density, in the self-similar approximation the density should be the same at all scales, $\rho \sim const$. The inertia moment of the elements of scale H_n needed when rotations are considered is $J \sim \rho H^2_{n-1}$ such that

$$\rho(H_n) = \rho(H_0), \; J(H_n) = J(H_0)\left(\frac{H_n}{H_0}\right)^2 \tag{16}$$

The wave velocities are controlled by the ratio of moduli, **C**, over density or rotational moduli, **B**, over the inertia moment. Therefore the scaling of all wave velocities is the same, which is another manifestation of the universality of scaling. The scaling of wave velocities reads

$$V(H_n) = V(H_0)\left(\frac{H_n}{H_0}\right)^{\kappa/2} \tag{17}$$

Wave propagation in a material with multiscale structure obviously depends upon the wave length. Indeed, on the one hand, the wave length should be greater than the maximum size of discontinuity. On the other hand, given the multiscale nature of the structure, the distance such a wave can propagate is controlled by the distance between the discontinuities of the scale greater than the wave length. In order to quantify these conditions consider scale H_n. The waves propagating at this scale should have wave lengths l such that at list $l > H_{n-1}$. The maximum distance the wave of length l can propagate is $L_n = min\{H_n > l\}$. The maximum (angular) frequency $\omega = V/l$ that corresponds to $l > H_{n-1}$ scales as

$$\omega_{max}(H_n) = \omega_{max}(H_0)\left(\frac{H_n}{H_0}\right)^{\kappa/2-1} \tag{18}$$

Therefore, the maximum frequencies form a discrete set

$$\omega_{max}^{(n)} = \omega_{max}^{(0)} q^n, \quad q = \lambda^{\kappa/2-1}, \tag{19}$$

where $H_n = H_0\lambda^n$ (for $\lambda = \sqrt{2}$, Kurlenya et al, 1996 called such a frequency set a canonical ensemble).

4 Material or rock with self-similar orthogonal sets of fractures

Consider now a material with three orthogonal discrete self-similar sets of fractures and approximate it with a discrete self-similar set of equivalent continua, as sketched in Fig. 3. Assume that the material between the discontinuities is isotropic. Owing to the assumed symmetry of the sets of discontinuities the equivalent continua will be orthotropic. In the elastic case the stress-strain relationship (Hooke's law) in each continuum reads (in the co-ordinate frame x_1,x_2,x_3 shown in Fig. 3) reads:

$$\begin{cases} \varepsilon_{11} = a_{11}(H_n)\sigma_{11} + a_{12}(H_n)\sigma_{22} + a_{13}(H_n)\sigma_{33} \\ \varepsilon_{22} = a_{12}(H_n)\sigma_{11} + a_{22}(H_n)\sigma_{22} + a_{23}(H_n)\sigma_{33} \\ \varepsilon_{33} = a_{13}(H_n)\sigma_{11} + a_{23}(H_n)\sigma_{22} + a_{33}(H_n)\sigma_{33} \\ \varepsilon_{23} = \frac{1}{2}a_{44}(H_n)\sigma_{23} \\ \varepsilon_{13} = \frac{1}{2}a_{55}(H_n)\sigma_{13} \\ \varepsilon_{12} = \frac{1}{2}a_{66}(H_n)\sigma_{12} \end{cases} \tag{20}$$

Here $a_{ij}(H_n)$ are compliances associated with equivalent continuum C_n and, according to the universality of scaling all non-zero components of compliances should scale with the same exponent, κ_a.

$$a_{ij}(H_n) = a_{ij}(H_0)\left(\frac{H_n}{H_0}\right)^{\kappa_a} \tag{21}$$

Since discontinuities can only soften the material (that is increase the compliances), $\kappa_a \geq 0$.

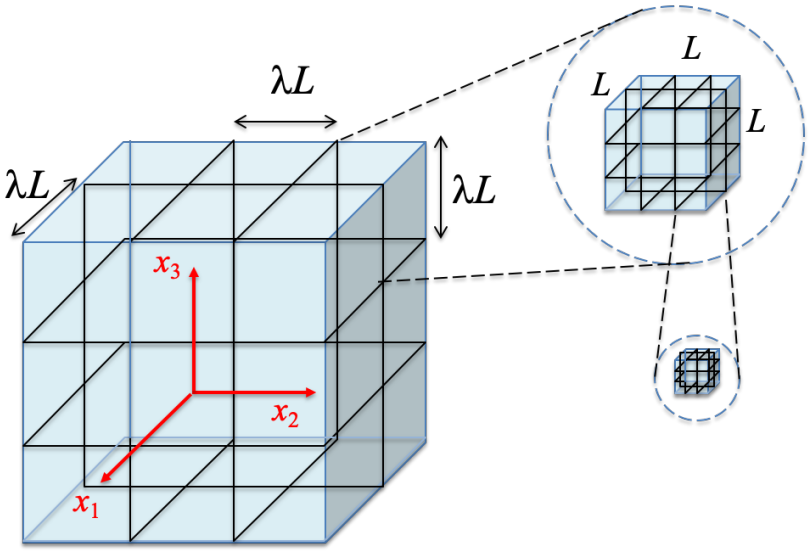

Figure 3. Sketch of a material with three orthogonal self-similar sets of fractures.

It is usually assumed that the discontinuities only affect the compliances that relate the strain-stress components corresponding to the planes of the discontinuities. Then two cases are possible.

Case 1, $\kappa_a = 0$. In this case the compliances are scale-invariant reflecting the situation where the cumulative effect of discontinuities of all scales is bounded. In other words the discontinuities can only provide limiting softening to the material.
Case 2, $\kappa_a > 0$. In this case the compliances increase with increase in scale. This leads to the following reduced Hooke's law in equivalent continua C_n:

$$\begin{cases} \varepsilon_{11} = a_{11}(H_n)\sigma_{11} \\ \varepsilon_{22} = a_{22}(H_n)\sigma_{22} \\ \varepsilon_{33} = a_{33}(H_n)\sigma_{33} \\ \varepsilon_{23} = \frac{1}{2}a_{44}(H_n)\sigma_{23} \\ \varepsilon_{31} = \frac{1}{2}a_{55}(H_n)\sigma_{31} \\ \varepsilon_{12} = \frac{1}{2}a_{66}(H_n)\sigma_{12} \end{cases} \tag{22}$$

An interesting situation occurs when the material has only one set of parallel discontinuities, say normal to x_3-axis. In this case constitutive relationship (22) reduces to

$$\begin{cases} \varepsilon_{33} = a_{33}(H_n)\sigma_{33} \\ \varepsilon_{23} = \frac{1}{2}a_{44}(H_n)\sigma_{23} \\ \varepsilon_{31} = \frac{1}{2}a_{55}(H_n)\sigma_{31} \end{cases} \tag{23}$$

which means that the strain-stress components other than listed in equation (23) are not determinable.

Therefore, in the case of one set of parallel fractures only scaling with exponent $\kappa_a = 0$ is possible. In other words one set of parallel fractures is insufficient to effect unlimiting increase of the components of compliances relating strain and stress components associated with the planes parallel to the discontinuities. A reason for this is the influence of the other components of compliances. For instance when the fractures are represented by finite cracks (normal to x_3-axis), vanishing of components a_{11}=a_{22}=0 makes the material infinitely stiff in x_1 and x_2 directions, which prevents the cracks from opening in x_3 direction, so the effect of cracks on a_{33} becomes negligible. As the assumption of dominance of compliances a_{33} and a_{44}=a_{55} is not valid, one has to resort to Case 1, so all the compliances are scale-invariant.

The same conclusion is valid for two orthogonal sets of fractures, also leading to $\kappa_a = 0$.

Obviously, for scaling of moduli the exponent $\kappa = -\kappa_a$ and scaling of wave velocities and the frequencies will be given by equations (17) and (19).

5 Scaling of statistical parameters of stress-strain distributions

Understanding the scaling of stress and strain is of great importance for applications, however stress and strain are different in different points of the continuum, so it is impossible to determine their scaling. What is possible though is to investigate scaling of statistical or average properties of stress, σ_{ij}, and strain, ε_{ij} distributions. We will concentrate on scaling of averages. To this end we introduce global averages $\langle\sigma_{ij}\rangle, \langle\varepsilon_{ij}\rangle$. Since averaging is a linear operation the averages of stress and strain are tensors and therefore, due to the universality of scaling they scale with the same exponent. (We note that in our approach the average stress keeps the conventional units, as opposite to the results of stress renormalisation accounting for the fractal scaling of elementary area, Carpinteri, 1994, leading to non-traditional units of stress.) Subsequently, scaling of average stress and strain has the form

$$\langle\sigma_{ij}\rangle(H_n) = \langle\sigma_{ij}\rangle(H_0)\left(\frac{H_n}{H_0}\right)^{\kappa_\sigma}, \quad \langle\varepsilon_{ij}\rangle(H_n) = \langle\varepsilon_{ij}\rangle(H_0)\left(\frac{H_n}{H_0}\right)^{\kappa_\varepsilon} \tag{24}$$

Consider now the second central moment – the variance. Hereafter we investigate scaling of central moments of stresses noting that for strain the results shall be similar.

$$M_{ij}^{(2)} = \langle\sigma_{ij}^2\rangle - \langle\sigma_{ij}\rangle^2 \tag{25}$$

All three terms in (25) should scale according to the power law; the scaling exponent for the last term of (25) is known, it is $2\kappa_\sigma$. Let the exponents of the other two terms be different from $2\kappa_\sigma$. Then equation (25) is reduced to $M_{ij}^{(2)}(H_0)(H_n/H_0)^\alpha = \langle\sigma_{ij}^2\rangle(H_0)(H_n/H_0)^\beta - \langle\sigma_{ij}\rangle^2(H_0)(H_n/H_0)^{2\kappa_\sigma}$. Since the discrete power functions are linearly independent (Appendix A) the assumption that α and β are different from $2\kappa_\sigma$ leads to $M_{ij}^{(2)}(H_0) = \langle\sigma_{ij}^2\rangle(H_0) = \langle\sigma_{ij}\rangle^2(H_0) = 0$. Therefore for non-zero central moments

$$M_{ij}^{(2)}(H_n) = M_{ij}^{(2)}(H_0)(H_n/H_0)^{2\kappa_\sigma} \tag{26}$$

Similarly, the higher moments of the order of m scale as

$$M_{ij}^{(m)}(H_n) = M_{ij}^{(m)}(H_0)(H_n/H_0)^{m\kappa_\sigma} \tag{27}$$

Therefore, scaling of the averages determines the scaling of higher moments.

Furthermore, given that $\langle\sigma_{ij}\rangle(H_n) = C_{ijkl}(H_n)\langle\varepsilon_{kl}\rangle(H_n)$ and the components of the tensor of effective elastic moduli C_{ijkl} scale with exponent κ the following relationship between the scaling exponents can be written:

$$\kappa_\sigma = \kappa + \kappa_\varepsilon \tag{28}$$

where κ_σ and κ_ε are the scaling exponent for average stress and strain respectively.

Finally, scaling of the total elastic energy reads

$$U_T(H_n) = U_T(H_0)(H_n/H_0)^{\kappa_\sigma+\kappa_\varepsilon} = U_T(H_0)(H_n/H_0)^{\kappa+2\kappa_\varepsilon} \tag{29}$$

In the Cosserat continuum, the elastic energy density is given by equation (15). Since the total elastic energy contains terms from (29) it should scale the same way as (29). Then, given that the Cosserat moduli B_{ijkl} should scale with exponent $\kappa + 2$, the average curvature-twist tensor $\langle\kappa_{ij}\rangle$ scales with exponent $\kappa_\varepsilon - 1$.

6 Conclusions

Scaling of characteristics of materials or rocks with discrete self-similar structure can only be represented by power law with respect to discrete self-similar set of scales. The main (perhaps counterintuitive) result reported here is that as long as the discrete-scaling quantities are in a linear relationship, they must either scale with the same exponent or vanish (the *Universality of scaling exponent*). In particular, non-zero components of the same tensor must scale with the same exponent. This leads to the same scaling of elastic moduli (that is the scaling exponents are always isotropic) and, consequently to the same scaling of all wave velocities. Since each scale determines the minimum lengths of waves that can propagate, there exists a discrete set of maximum frequencies of propagating waves.

In isotropic material with isotropic distributions of discrete self-similar disc-like cracks scaling of Young's modulus is controlled by the concentration of cracks at each scale. The most interesting

is however the behaviour of the Poisson's ratio: self-similar sets of cracks capable of shearing and opening (or closing when the cracks have some initial aperture) lead to the Poisson's ratio equal to zero, while self-similar sets of frictionless shear cracks produce the Poisson's ratio equal to 0.5 making the material with such cracks incompressible.

Isotropic material with discrete self-similar parallel fractures forming one or two (mutually perpendicular) sets leaves the moduli scale invariant (scaling exponent is zero) suggesting that the combined effect of only one or two orthogonal sets of fractures is insufficient for the multiscale reduction of moduli. Only three mutually orthogonal sets of discontinuities are capable of producing self-similar reduction of moduli with the power law expressed by a negative exponent.

While stress-strain distributions cannot be treated as self-similar, their averages and averages of higher moments are self-similar and scale by power law. This leads to simple relationships between scaling exponents of average strain, stress, their higher moments and the moduli. Since the scaling of average strain could be determined from observations or measurements and scaling moduli could in principle be inferred from the measured wave velocities, the derived relationships would produce the scaling of the statistical parameters of stress distributions. This can be used for estimating the risk of rock mass failures and assist in designing safe geotechnical structures.

Even the discrete self-similarity is a very strong property imposing severe limitations on scaling of the characteristics and statistical parameters of the mechanical (and physical) fields. Obviously, the self-similarity is only a first order approximation of scale dependence. Yet it can prove to be important for designing new multiscale materials as well as for rock characterisation in geosciences given that when the range of scales is considerable, the opportunities to conduct measurements at very small and very large scales are limited. Then the developed relationships between scaling exponents of the material characteristics and average stress and strain can provide the means for upscaling or downscaling (albeit approximately) of the laboratory data and thus offer insight into the material behaviour at micro- and macroscales.

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Appendix A

Here we prove the Lemma: K discrete functions of h forming set $\{h^{\kappa_1}, h^{\kappa_2}, \dots h^{\kappa_K}\}, \kappa_1 < \kappa_2 < \cdots < \kappa_K$ are linearly independent, that is equation

$$\sum_{k=1}^{K} C_k h^{\kappa_k} = 0 \tag{A1}$$

allows only one solution, $C_k = 0, k = 1, \dots, K$.

Indeed, consider a discrete self-similar set $h_p = q^p,\ p = 0, .. K-1$. Then eq. (A1) reads

$$\sum_{k=1}^{K} C_k q^{p\kappa_k} = 0 \tag{A2}$$

By introducing $x_k = q^{\kappa_k}$ equation (A2) can be turned into K linear homogeneous equations with respect to C_k

$$\sum_{k=1}^{K} C_k x_k^p = 0,\ p = 0, 1, .. K-1 \tag{A3}$$

The matrix of the system is the transposed Vandermonde matrix whose determinant $\det\left(x_k^p\right) = \prod_{i>k}(x_i - x_k) \neq 0$. Therefore, the only solution of (A3) is $C_k = 0, k = 1, \dots, K$, which finalized the proof.

Appendix B

Consider an isotropic material filled with isotropically distributed disc-like cracks. Suppose the concentration of the cracks is low such that their interaction can be neglected. (This is the simplest case comprehensively considered in the literature, e.g., Bristow, 1960; Walsh, 1965a,b,c.) It is convenient to use Salganik's (1973) solution for elliptical cracks as it is easy to modify to account for pure shear cracks. According to this solution the effective moduli can be expressed as follows

$$\begin{cases} E = E_m\left\{1 - \frac{8\pi}{45}(1-\nu_m^2)[C_1(\nu_m) + C_2(\nu_m) + 3C_3(\nu_m)]\mathrm{v}\right\} \\ \nu = \nu_m - \frac{4\pi}{45}(1-\nu_m^2)\{2(1+3\nu_m)C_3(\nu_m) - (1-2\nu_m)[C_1(\nu_m) + C_2(\nu_m)]\}\mathrm{v} \end{cases} \tag{B1}$$

Here E_m and ν_m are the Young's modulus and Poisson's ratio of the material between the cracks,

$$\mathrm{v} = N\langle a^3\rangle \tag{B2}$$

is the dimensionless crack concentration, N is the number of cracks per unit volume and $\langle a^3\rangle$ is the average cube of the crack radius.

Coefficients C_1, C_2, C_3 represent the contribution of shear (C_1, C_2) and normal (C_3) components of the displacement discontinuity through the cracks. For shear frictionless cracks we shall assume $C_3 = 0$, while the other coefficients, according to Salganik (1973) are

$$C_1 = C_2 = \frac{4}{\pi}\frac{1}{2-\nu} \tag{B3}$$

Then eqs. (B1), (B3) give

$$\begin{aligned} E &= E_m\left[1 - \frac{64}{45}\frac{1-\nu_m^2}{2-\nu_m}\mathrm{v}\right] \\ \nu &= \nu_m + \frac{32(1-\nu_m^2)(1-2\nu_m)}{45\quad 2-\nu_m}\mathrm{v} \end{aligned} \tag{B4}$$

Eq. (B4) is obtained for low crack concentrations, when the crack interaction can be neglected. For higher crack concentrations we use the differential self-consisted method according to which the cracks are introduced by portions of low concentrations each considering to be in the effective medium with moduli determined by the already introduced cracks. (This method was initially proposed by Bruggeman, 1936 and then used by Brinkman, 1952; Salganik, 1973, 1974, 1982; Neal and Nader, 1973; Vavakin and Salganik, 1975, 1978; Bruner, 1976; Henyey and Pomphrev, 1982; Nemat-Nasser and Horii, 1983; Zimmerman, 1984, 1991; Kemeny and Cook, 1986; Hashin, 1988; see also review by Kachanov, 1992. Dyskin, 1985; Germanovich & Dyskin, 1994 proved the validity of the differential self-consistent method for materials with cracks or heterogeneities of wide distribution of sizes; see Appendix A in Dyskin & Pasternak, 2019 for details.) The differential self-consistent method leads to the following system of differential equations

$$\begin{cases} \frac{dE}{d\mathrm{v}} = -E\,\frac{64}{45}\,\frac{1-\nu^2}{2-\nu} \\ \frac{d\nu}{d\mathrm{v}} = \frac{32}{45}\frac{(1-\nu^2)(1-\nu)}{2-\nu} \\ E|_{\mathrm{v}=0} = E_m \\ \nu|_{\mathrm{v}=0} = \nu_m \end{cases} \tag{B5}$$

After referring the material with properties E_m, ν_m to the n-1 scale, system of equations (11) is obtained.

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