QUANTITATIVE PESIN THEORY FOR ANOSOV DIFFEOMORPHISMS AND FLOWS

SÉBASTIEN GOUËZEL AND LUCHEZAR STOYANOV

Abstract. Pesin sets are measurable sets along which the behavior of a matrix cocycle above a measure preserving dynamical system is explicitly controlled. In uniformly hyperbolic dynamics, we study how often points return to Pesin sets under suitable conditions on the cocycle: if it is locally constant, or if it admits invariant holonomies and is pinching and twisting, we show that the measure of points that do not return a linear number of times to Pesin sets is exponentially small. We discuss applications to the exponential mixing of contact Anosov flows, and counterexamples illustrating the necessity of suitable conditions on the cocycle.

1. Introduction and main results

Uniformly hyperbolic dynamical systems are very well understood. An approach to study more general systems is to see to what extent they resemble uniformly hyperbolic ones. A very fruitful approach in this respect is the development of Pesin theory, that requires hyperbolic features (no zero Lyapunov exponent) almost everywhere with respect to an invariant measure, and constructs from these local stable and unstable manifolds, then leading to results such as the ergodicity of the system under study.

A basic tool in Pesin theory is the notion of Pesin sets, made of points for which, along their orbits, the Oseledets decomposition is well controlled in a quantitative way. Their existence follows from general measure theory argument, but they are not really explicit. Even in uniformly hyperbolic situations, Pesin sets are relevant objects as the control of the Oseledets decomposition gives directions in which the dynamics is close to conformal. In particular, the second author has shown in [Sto13b] that Pesin sets could be used, in contact Anosov flows, to study the decay of correlations: he proved that, if points return exponentially fast to Pesin sets, then the correlations decay exponentially fast.

Our goal in this article is to investigate this question, for Anosov diffeomorphisms and flows. We do not have a complete answer, but our results indicate a dichotomy: if the dynamics is not too far away from conformality (for instance in the case of the geodesic flow on a 1/4-pinched compact manifold of negative curvature), points return exponentially fast to Pesin sets for generic metrics (in a very strong sense), and possibly for all metrics. On the other hand, far away from conformality, this should not be the case (we have a counter-example in a related setting, but with weaker regularity).

Such statements are related to large deviations estimates for matrix cocycles, i.e., products of matrices governed by the dynamics (for Pesin theory, the cocycle is simply the differential of the map). Indeed, we will show that such large deviations estimates make it

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possible to control the returns to Pesin sets, by quantifying carefully some arguments in the proof of Oseledets theorem.

Let $T : X \to X$ be a measurable map on a space X, preserving an ergodic probability measure μ. Consider a measurable bundle E over X, where each fiber is isomorphic to $\mathbb{R}^d$ and endowed with a norm. A linear cocycle is a measurable map M on E, mapping the fiber above x to the fiber above Tx in a linear way, through a matrix $M(x)$. We say that the cocycle is log-integrable if $\int \log \max(\|M(x)\|, \|M(x)^{-1}\|) \,\mathrm{d}\mu(x) < \infty$. In this case, it follows from Kingman's theorem that one can define the Lyapunov exponents of the cocycle, denoted by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d$. They are the growth rate of vectors under iteration of the cocycle, above μ-almost every point. The sum $\lambda_1 + \cdots + \lambda_i$ is also the asymptotic exponential growth rate of the norm of the i-th exterior power $\Lambda^i M^n(x)$, for μ-almost every x.

The main condition to get exponential returns to Pesin sets is an exponential large deviations condition.

Definition 1.1. *Consider a transformation T preserving a probability measure μ, and a family of functions $u_n : X \to \mathbb{R}$. Assume that, almost everywhere, $u_n(x)/n$ converges to a limit λ. We say that the family has exponential large deviations if, for any $\varepsilon > 0$, there exists $C > 0$ such that, for all $n \geqslant 0$,*

$$\mu\{x \,:\, |u_n(x) - n\lambda| \geqslant n\varepsilon\} \leqslant Ce^{-C^{-1}n}.$$

This general definition specializes to several situations that will be relevant in this paper:

Definition 1.2. *Consider an integrable function u above an ergodic transformation (T, μ). We say that u has exponential large deviations if its Birkhoff sums $S_n u$ have exponential large deviations in the sense of Definition 1.1, i.e., for any $\varepsilon > 0$, there exists $C > 0$ such that, for all $n \geqslant 0$,*

$$\mu\{x \,:\, |S_n u(x) - n \int u| \geqslant n\varepsilon\} \leqslant Ce^{-C^{-1}n}.$$

Definition 1.3. *Consider a log-integrable linear cocycle M above a transformation (T, μ), with Lyapunov exponents $\lambda_1 \geqslant \cdots \geqslant \lambda_d$. We say that M has exponential large deviations for its top exponent if the family of functions $u_n(x) = \log\|M^n(x)\|$ (which satisfies $u_n(x)/n \to \lambda_1$ almost everywhere) has exponential large deviations in the sense of Definition 1.1, i.e., for any $\varepsilon > 0$, there exists $C > 0$ such that, for all $n \geqslant 0$,*

$$\mu\{x \,:\, |\log\|M^n(x)\| - n\lambda_1| \geqslant n\varepsilon\} \leqslant Ce^{-C^{-1}n}.$$

We say that M has exponential large deviations for all exponents if, for any $i \leqslant d$, the functions $\log\|\Lambda^i M^n(x)\|$ satisfy exponential large deviations in the sense of Definition 1.1, i.e., for any $\varepsilon > 0$, there exists $C > 0$ such that, for all $n \geqslant 0$,

$$\mu\{x \,:\, |\log\|\Lambda^i M^n(x)\| - n(\lambda_1 + \cdots + \lambda_i)| \geqslant n\varepsilon\} \leqslant Ce^{-C^{-1}n}. \tag{1.1}$$

We will explain in the next paragraph that many linear cocycles above subshifts of finite type have exponential large deviations for all exponents, see Theorem 1.5 below. This builds on techniques developed by Bonatti, Viana and Avila (see [BV04, AV07]). The main novelty of our work is the proof that such large deviations imply exponential returns to Pesin sets, as we explain in Paragraph 1.2. The last paragraph of this introduction discusses consequences of these results.

1.1. **Sufficient conditions for large deviations for linear cocycles.** In this paragraph, we consider a (bilateral) transitive subshift of finite type $T : \Sigma \to \Sigma$, together with a Gibbs measure μ for a Hölder potential. Let E be a continuous $\mathbb{R}^d$-bundle over Σ, endowed with a continuous linear cocycle M on E over T. For instance, one may take $E = \Sigma \times \mathbb{R}^d$, then $M(x)$ is simply an invertible $d \times d$ matrix depending continuously on x. We describe in Theorem 1.5 various conditions under which such a cocycle has exponential large deviations for all exponents, in the sense of Definition 1.3. Through the usual coding process, similar results follow for hyperbolic basic sets of diffeomorphisms, and in particular for Anosov or Axiom A diffeomorphisms.

We show in Appendix A the existence of a continuous linear cocycle above a subshift of finite type which does not have exponential large deviations for its top exponent. Hence, additional assumptions are needed for this class of results (contrary to the case of Birkhoff sums, where all Birkhoff sums of continuous functions over a transitive subshift of finite type have exponential large deviations). These assumptions, as is usual in the study of linear cocycles, are defined in terms of holonomies. In a geometric context, holonomies are usually generated by connections. In the totally disconnected context of subshifts of finite type, connections do not make sense, but the global notion of holonomy does.

The local stable set of x is the set $W^s_{\mathrm{loc}}(x) = \{y : y_n = x_n \text{ for all } n \geqslant 0\}$. In the same way, its local unstable set is $W^u_{\mathrm{loc}}(x) = \{y : y_n = x_n \text{ for all } n \leqslant 0\}$. By definition, $W^s_{\mathrm{loc}}(x) \cap W^u_{\mathrm{loc}}(x) = \{x\}$.

An unstable holonomy is a family of isomorphisms $H^u_{x \to y}$ from $E(x)$ to $E(y)$, defined for all x and y with $y \in W^u_{\mathrm{loc}}(x)$. We require the compatibility conditions $H^u_{x\to x} = \mathrm{Id}$ and $H^u_{y \to z} \circ H^u_{x\to y} = H^u_{x \to z}$ for any x, y and z on the same local unstable set. Moreover, we require the continuity of $(x, y) \mapsto H^u_{x\to y}$ (globally, i.e., not only along each leaf).

In the same way, one defines a stable holonomy as a family of maps $H^s_{x\to y}$ from $E(x)$ to $E(y)$ when x and y belong to the same local stable set, with the same equivariance and continuity requirements as above.

Definition 1.4. *A linear cocycle admits invariant continuous holonomies if there exist two stable and unstable continuous holonomies, denoted respectively by H^s and H^u, that are equivariant with respect to the cocycle action. More precisely, for any x, for any $y \in W^s_{\mathrm{loc}}(x)$, and any $v \in E(x)$, one should have*

$$M(y)H^s_{x\to y}v = H^s_{Tx\to Ty}M(x)v.$$

Similarly, for any x, for any $y \in W^u_{\mathrm{loc}}(x)$, and any $v \in E(x)$, one should have

$$M(y)^{-1}H^u_{x\to y}v = H^u_{T^{-1}x\to T^{-1}y}M(x)^{-1}v.$$

Stable holonomies give a canonical way to trivialize the bundle over local stable sets. Thus, to trivialize the whole bundle, one may choose an arbitrary trivialization over an arbitrary local unstable set, and then extend it to the whole space using the holonomies along the local stable sets. In this trivialization, the cocycle is constant along local stable sets, i.e., it only depends on future coordinates. Symmetrically, one can trivialize the bundle first along a stable set, and then using unstable holonomies along the local unstable sets. In this trivialization, the cocycle is constant along unstable sets, and depends only on past coordinates. Note that these two trivializations do not coincide in general, unless the stable and unstable holonomies commute: In this case, the cocycle only depends on the coordinate x_0 in the resulting trivialization, i.e., it is locally constant. Conversely, a locally constant cocycle admits the identity as stable and unstable invariant commuting holonomies.

We say that a linear cocycle is *pinching and twisting in the sense of Avila-Viana* [AV07] if it has invariant continuous holonomies, and if there exist a periodic point p (of some period k) and a point q which is asymptotic to p both in the past and in the future (i.e., $q \in W^u_{\mathrm{loc}}(p)$ and $T^i q \in W^s_{\mathrm{loc}}(p)$ for some i which is a multiple of k), such that

- All the eigenvalues of $M^k(p)$ are real and different.
- Define a map $\Psi : H^s_{T^i q \to p} \circ M^i(q) \circ H^u_{p \to q}$ from $E(p)$ to itself. Then, for any subspaces U and V of $E(p)$ which are invariant under $M^k(p)$ (i.e., which are union of eigenspaces) with $\dim U + \dim V = \dim E$, then $\Psi(U) \cap V = \{0\}$. In other words, the map Ψ puts the eigenspaces of $M^k(p)$ in general position.

This condition ensures that the Lyapunov spectrum of any Gibbs measure is simple, by the main result of [AV07]. In the space of fiber-bunched cocycles (which automatically admit invariant continuous holonomies), this condition is open (this is clear) and dense (this is harder as there might be pairs of complex conjugate eigenvalues at some periodic points, which need more work to be destroyed, see [BV04, Proposition 9.1]).

Theorem 1.5. *Let T be a transitive subshift of finite type on a space Σ, and μ a Gibbs measure for a Hölder-continuous potential. Consider a continuous linear cocycle M on a vector bundle E above T. Then M has exponential large deviations for all exponents in the following situations:*

1. *If all its Lyapunov exponents coincide.*
2. *If there us a continuous decomposition of E as a direct sum of subbundles $E = E_1 \oplus \dots \oplus E_k$ which is invariant under M, such that the restriction of M to each E_i has exponential large deviations for all exponents.*
3. *More generally, if there is an invariant continuous flag decomposition $\{0\} = F_0 \subseteq F_1 \subseteq \dots \subseteq F_k = E$, such that the cocycle induced by M on each F_i/F_{i-1} has exponential large deviations for all exponents.*
4. *If the cocycle M is locally constant in some trivialization of the bundle E (this is equivalent to the existence of invariant continuous holonomies which are commuting).*
5. *If the cocycle M admits invariant continuous holonomies, and if it is pinching and twisting in the sense of Avila-Viana.*
6. *If the cocycle M admits invariant continuous holonomies, and the bundle is 2-dimensional.*

The first three points are easy, the interesting ones are the last ones. The various statements can be combined to obtain other results. For instance, if each (a priori only measurable) Oseledets subspace is in fact continuous (for instance if the Oseledets decomposition is dominated), then the cocycle has exponential large deviations for all exponents: this follows from points (1) and (2) in the theorem. We expected that our techniques would show a result containing (4–6): if a cocycle admits invariant continuous holonomies, then it should have exponential large deviations for all exponents. However, there is a difficulty here, see Remark 3.9. Points (1-3) are proved on Page 15, (4) on Page 22, (5) on Page 25 and (6) on

Page 25. The proofs of (4), (5) and (6) follow the same strategy, we will insist mainly on (4) and indicate more quickly the modifications for (5) and (6). These proofs are essentially applications of the techniques in [BV04, AV07].

Remark 1.6. In Theorem 1.5, exponential large deviations are expressed in terms of matrix norms: one should choose on each $E(x)$ a norm, depending continuously on x, and then $\|M^n(x)\|$ is the operator norm of $M^n(x)$ between the two normed vector spaces $E(x)$ and $E(T^n x)$. The above statement does not depend on the choice of the norm (just as the value of the Lyapunov exponents) as the ratio between two such norms is bounded from above and from below by compactness. Hence, we may choose whatever norm we like most on E. For definiteness, we use a Euclidean norm.

The above theorem shows that, in most usual topologies, generic linear cocycles have exponential large deviations for all exponents. Indeed, for generic cocycles in the C^0 topology, the Oseledets decomposition is dominated (see [Via14, Theorem 9.18]), hence (1) and (2) in the theorem yield exponential large deviations. For generic cocycles in the Hölder topology among fiber bunched cocycles (the most tractable Hölder cocycles), pinching and twisting are generic, hence (5) also gives exponential large deviations.

1.2. **Quantitative Pesin theory from large deviations for linear cocycles.** Let T be an invertible continuous map on a compact metric space X, preserving an ergodic probability measure μ. Let M be a continuous cocycle above T, on the trivial bundle $X \times \mathbb{R}^d$. Denote by $\lambda_1 \geqslant \cdots \geqslant \lambda_d$ its Lyapunov exponents, and $I = \{i \,:\, \lambda_i < \lambda_{i-1}\}$. Then $(\lambda_i)_{i \in I}$ are the distinct Lyapunov exponents. Denote by E_i the corresponding Oseledets subspace, its dimension d_i is $\mathrm{Card}\{j \in [1, d] \,:\, \lambda_j = \lambda_i\}$. The subspaces $E_i(x)$ are well-defined on an invariant subset X' of X with $\mu(X') = 1$ and $E_i(T(x)) = E_i(x)$ for all $x \in X'$. Moreover $\frac{1}{n} \log\|M^n(x)v\| \to \lambda_i$ as $n \to \pm\infty$ for all $v \in E_i(x) \setminus \{0\}$. With this notation, the space $E_i(x)$ is repeated d_i times. The distinct Oseledets subspaces are $(E_i(x))_{i \in I}$.

Let $\varepsilon > 0$. The basic ingredient in Pesin theory is the function

$$\begin{aligned} A_\varepsilon(x) &= \sup_{i \in I} A_\varepsilon^{(i)}(x) \\ &= \sup_{i \in I} \sup_{v \in E_i(x) \setminus \{0\}} \sup_{m,n \in \mathbb{Z}} \frac{\|M^n(x)v\|}{\|M^m(x)v\|} e^{-(n-m)\lambda_i} e^{-(|n|+|m|)\varepsilon/2} \in [0, \infty]. \end{aligned} \tag{1.2}$$

This function is slowly varying, i.e.,

$$e^{-\varepsilon} A_\varepsilon(x) \leqslant A_\varepsilon(Tx) \leqslant e^{\varepsilon} A_\varepsilon(x),$$

as the formulas for x and Tx are the same except for a shift of 1 in n and m. Moreover, for all $k \in \mathbb{Z}$ and all $v \in E_i(x)$,

$$\|v\| A_\varepsilon(x) e^{-|k|\varepsilon} \leqslant \frac{\|M^k(x)v\|}{e^{k\lambda_i}} \leqslant \|v\| A_\varepsilon(x) e^{|k|\varepsilon},$$

where one inequality follows by taking $m = 0$ and $n = k$ in the definition of A_ε, and the other inequality by taking $m = k$ and $n = 0$. The almost sure finiteness of A_ε follows from Oseledets theorem.

Pesin sets are sets of the form $\{x : A_\varepsilon(x) \leqslant C\}$, for some constant $C > 0$. Our main goal is to show that most points return exponentially often to some Pesin set. This is the content of the following theorem.

Theorem 1.7. *Let T be a transitive subshift of finite type on a space Σ, and μ a Gibbs measure for a Hölder-continuous potential. Consider a continuous linear cocycle M on the trivial vector bundle $\Sigma \times \mathbb{R}^d$ above T. Assume that M has exponential large deviations for all exponents, both in positive and negative times.*

Let $\varepsilon > 0$ and $\delta > 0$. Then there exists $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\mu\{x : \operatorname{Card}\{j \in [0, n-1] : A_\varepsilon(T^j x) > C\} \geqslant \delta n\} \leqslant C e^{-C^{-1} n}.$$

One difficulty in the proof of this theorem is that the function A_ε is defined in terms of the Lyapunov subspaces, which are only defined almost everywhere, in a non constructive way. To get such controls, we will need to revisit the proof of Oseledets to get more quantitative bounds, in Section 5.1, showing that an explicit control on the differences $\left(\left|\log\|\Lambda^i M^n(x)\| - n(\lambda_1 + \dots + \lambda_i)\right|\right)_{n\in\mathbb{Z}}$ at some point x implies an explicit control on $A_\varepsilon(x)$ in Theorem 5.1. Then, the number of returns to the Pesin sets is estimated using an abstract result in subadditive ergodic theory, interesting in its own right, Theorem 4.1. These two statements are finally combined in Section 6 to prove Theorem 1.7.

1.3. **Applications.** In this paragraph, we describe several systems to which our results on large deviations and exponential returns to Pesin sets apply.

First, coding any Anosov or Axiom A diffeomorphism thanks to a Markov partition, then the above theorems apply to such maps, provided the matrix cocycle has exponential large deviations. Hence, one needs to check the conditions in Theorem 1.5.

The main class of cocycles admitting stable and unstable holonomies is the class of *fiber bunched cocycles*, see [AV07, Definition A.5].

A ν-Hölder continuous cocycle M over a hyperbolic map T on a compact space is s-fiber bunched if there exists $\theta \in (0,1)$ such that $d(Tx, Ty) \leqslant \theta d(x,y)$ and $\|M(x)\|\|M(y)^{-1}\|\theta^\nu < 1$, for all x, y on a common local stable set (or more generally if this property holds for some iterate of the map and the cocycle). This means that the expansion properties of the cocycle are dominated by the contraction properties of the map T. This results in the fact that $M^n(y)^{-1}M^n(x)$ converges exponentially fast when $n \to \infty$, to a map which is a continuous invariant stable holonomy, see [AV07, Proposition A.6]

In the same way, one defines u-fiber bunched cocycles. Finally, a cocycle is fiber-bunched if it is both s and u-fiber bunched. For instance, if T and ν are fixed, then a cocycle which is close enough to the identity in the C^ν topology is fiber bunched. Our results apply to such cocycles if they are pinching and twisting, which is an open and dense condition among fiber bunched cocycles.

Our results also apply to generic cocycles in the C^0 topology. Indeed, the Oseledets decomposition is then dominated (see [Via14, Theorem 9.18]), hence (1) and (2) in the theorem yield exponential large deviations, and from there one deduces exponential returns to Pesin sets by Theorem 1.7.

The main application we have in mind is to flows. The second author proves in [Sto13b] the following theorem:

Theorem 1.8 (Stoyanov [Sto13b]). *Let g_t be a contact Anosov flow on a compact manifold X, with a Gibbs measure μ_X.*

Consider the first return map to a Markov section T, the corresponding invariant measure μ, and the corresponding derivative cocycle M, from the tangent space of X at x to the tangent space of X at Tx. Assume that (T, M, μ) has exponential returns to Pesin sets as in the conclusion of Theorem 1.7.

Then the flow g_t is exponentially mixing: there exists $C > 0$ such that, for any C^1 functions u and v, for any $t \geqslant 0$,

$$\left| \int u \cdot v \circ g_t \, \mathrm{d}\mu_X - \int u \, \mathrm{d}\mu_X \cdot \int v \, \mathrm{d}\mu_X \right| \leqslant C \|u\|_{C^1} \|v\|_{C^1} e^{-C^{-1} t}.$$

By a standard approximation argument, exponential mixing for Hölder continuous functions follows.

This statement is the main motivation to study exponential returns to Pesin sets. We deduce from Theorem 1.7 the following:

Theorem 1.9. *Consider a contact Anosov flow with a Gibbs measure, for which the derivative cocycle has exponential large deviations for all exponents. Then the flow is exponentially mixing.*

To apply this theorem in concrete situations, we have to check whether the sufficient conditions of Theorem 1.5 for exponential large deviations hold. The main requirement is the existence of stable and unstable holonomies. Unfortunately, we only know their existence when the foliation is smooth:

Lemma 1.10. *Consider a contact Anosov flow for which the stable and unstable foliations are C^1. Then the derivative cocycle admits continuous invariant holonomies with respect to the induced map on any Markov section.*

Proof. It suffices to show that the flow admits continuous invariant holonomies along weak unstable and weak stable manifolds, as they descend to the Markov section.

We construct the holonomy along weak unstable leaves, the holonomy along weak stable leaves being similar. Consider two points x and y on a weak unstable leaf. Then the holonomy of the weak unstable foliation gives a local C^1 diffeomorphism between $W^s(x)$ to $W^s(y)$, sending x to y. The derivative of this map is a canonical isomorphism between $E^s(x)$ and $E^s(y)$, which is clearly equivariant under the dynamics. There is also a canonical isomorphism between the flow directions at x and y. What remains to be done is to construct an equivariant isomorphism between $E^u(x)$ and $E^u(y)$.

For this, we use the fact that the flow is a contact flow, i.e., there exists a smooth one-form α, invariant under the flow, with kernel $E^s \oplus E^u$, whose derivative $\mathrm{d}\alpha$ restricts to a symplectic form on $E^s \oplus E^u$. We get a map φ from E^s to $(E^u)^*$, mapping v to $\mathrm{d}\alpha(v, \cdot)$. This map is one-to-one: a vector v in its kernel satisfies $\mathrm{d}\alpha(v, w) = 0$ for all $w \in E^u$, and also for all $w \in E^s$ as E^s is Lagrangian. Hence, v is in the kernel of $\mathrm{d}\alpha$, which is reduced to 0 as $\mathrm{d}\alpha$ is a symplectic form. As E^s and E^u have the same dimension, it follows that φ is an isomorphism.

Consider now x and y on a weak unstable leaf. We have already constructed a canonical isomorphism between $E^s(x)$ and $E^s(y)$. With the above identification, this gives a canonical isomorphism between $(E^u(x))^*$ and $(E^u(y))^*$, and therefore between $E^u(x)$ and $E^u(y)$. This identification is equivariant under the flow, as α is invariant. □

For instance, for the geodesic flow on a compact Riemannian manifold with negative curvature, the stable and unstable foliations are smooth if the manifold is 3-dimensional or the curvature is strictly 1/4-pinched, i.e., the sectional curvature belongs everywhere to an interval $[-b^2, -a^2]$ with $a^2/b^2 > 1/4$, by [HP75]. Hence, we deduce the following corollary from Theorem 1.5 (1), (6) and (5) respectively:

Corollary 1.11. *Consider the geodesic flow g_t on a compact riemannian manifold X with negative curvature. Assume one of the following conditions:*

1. *X is of dimension 3.*
2. *X is of dimension 5 and the curvature is strictly 1/4 inched.*
3. *X has any dimension, the curvature is strictly 1/4 pinched, and moreover the flow is pinching and twisting.*

Then the flow is exponentially mixing for any Gibbs measure.

However, these results were already proved by the second author, under weaker assumptions: exponential mixing holds if the curvature is (not necessarily strictly) 1/4-pinched, in any dimension (without twisting and pinching). This follows from the articles [Sto11], in which it is proved that a contact Anosov flow with Lipschitz holonomies and satisfying a geometric condition is exponentially mixing for all Gibbs measure, and from [Sto13a] where the aforementioned geometric condition is proved to be satisfied in a class of flows including geodesic flows when the curvature is 1/4-pinched.

On the opposite side, the techniques of [Liv04] or [FT13] prove exponential mixing for any contact Anosov flow, without any pinching condition, but for Lebesgue measure (or for Gibbs measure whose potential is not too far away from the potential giving rise to Lebesgue measure): they are never able to handle all Gibbs measure.

The hope was that Theorem 1.8 would be able to bridge the gap between these results and the results of Dolgopyat, proving exponential mixing for all contact Anosov flows and all Gibbs measures. However we still need geometric conditions on the manifold to be able to proceed. The counterexample in the Appendix A shows that in general exponential large deviations do not hold. Whether one can design similar counterexamples for nice systems, e.g. contact Anosov flows, remains unknown at this stage. It is also unknown whether one can prove a result similar to Theorem 1.9 without assuming exponential large deviations for all exponents.

2. Preliminaries

2.1. **Oseledets theorem.**

Let A be a linear transformation between two Euclidean spaces of the same dimensions. We recall that, in suitable orthonormal bases at the beginning and at the end, A can be put in diagonal form with entries $s_1 \geqslant \cdots \geqslant s_d \geqslant 0$. The s_i are the *singular values* of A. They are also the eigenvalues of the symmetric matrix $\sqrt{A^t \cdot A}$. The largest one s_1 is the norm of A, the smallest one s_d is its smallest expansion. The singular values of A^{-1} are $1/s_d \geqslant \cdots \geqslant 1/s_1$. For any $i \leqslant d$, denote by $\Lambda^i A$ the i-th exterior product of A, given by

$$(\Lambda^i A)(v_1 \wedge v_2 \wedge \cdots \wedge v_i) = Av_1 \wedge Av_2 \wedge \cdots \wedge Av_i.$$

Then

$$\|\Lambda^i A\| = s_1 \cdots s_i,$$

as $\Lambda^i A$ is diagonal in the corresponding orthonormal bases.

Consider a transformation T of a space X, together with a finite dimensional real vector bundle E above X: all the fibers are isomorphic to $\mathbb{R}^d$ for some d and the bundle is locally trivial by definition. For instance, E may be the product bundle $X \times \mathbb{R}^d$, but general bundles are also allowed. In our main case of interest, T will be a subshift of finite type. In this case, any such continuous vector bundle is isomorphic to $X \times \mathbb{R}^d$: by compactness, there is some $N > 0$ such that the bundle is trivial on all cylinders $[x_{-N}, \dots, x_N]$. As these (finitely many) sets are open and closed, trivializations on these cylinders can be glued together to form a global trivialization of the bundle. In the course of the proof, even if we start with the trivial bundle, we will have to consider general bundles, but they will be reducible to trivial bundles thanks to this procedure.

A cocycle is a map M associating to $x \in X$ an invertible linear operator $M(x) : E(x) \to E(Tx)$ (where $E(x)$ denotes the fiber of the fiber bundle above x). When $E = X \times \mathbb{R}^d$, then $M(x)$ is simply a $d \times d$ matrix. The iterated cocycle is given by $M^n(x) = M(T^{n-1}x) \cdots M(x)$ for $n \geqslant 0$, and by $M^{-n}(x) = M(T^{-n}x)^{-1} \cdots M(T^{-1}x)$. It maps $E(x)$ to $E(T^n x)$ in all cases. Be careful that, with this notation, $M^{-1}(x) \neq M(x)^{-1}$: the first notation indicates the inverse of the cocycle, with the intrinsic time shift, going from $E(x)$ to $E(T^{-1}x)$, while the second one is the inverse of a linear operator, so it goes from $E(Tx)$ to $E(x)$. In general, $M^{-n}(x) = M^n(T^{-n}x)^{-1}$.

Assume now that T is invertible, that it preserves an ergodic probability measure, and that the cocycle M is log-integrable. For any $i \leqslant d$, the quantity $x \mapsto \log\|\Lambda^i(M^n(x))\|$ is a subadditive cocycle. Hence, by Kingman's theorem, $\log\|\Lambda^i(M^n(x))\|/n$ converges almost surely to a constant quantity that we may write as $\lambda_1 + \dots + \lambda_i$, for some scalars λ_i. These are called the *Lyapunov exponents* of the cocycle M with respect to the dynamics T and the measure μ. Let $I = \{i : \lambda_i < \lambda_{i-1}\}$ parameterize the distinct Lyapunov exponents, and let $d_i = \mathrm{Card}\{j : \lambda_j = \lambda_i\}$ be the multiplicity of λ_i.

In this setting, the Oseledets theorem asserts that the λ_i are exactly the asymptotic growth rates of vectors, at almost every point. Here is a precise version of this statement (see for instance [Arn98, Theorem 3.4.11]).

Theorem 2.1 (Oseledets Theorem). *Assume that the cocycle M is log-integrable. Then:*

(1) *For $i \in I$, define $E_i(x)$ to be the set of nonzero vectors $v \in E(x)$ such that, when $n \to \pm\infty$, then $\log\|M^n(x)v\|/n \to \lambda_i$, to which one adds the zero vector. For μ-almost every x, this is a vector subspace of $E(x)$, of dimension d_i. These subspaces satisfy $E(x) = \bigoplus_{i \in I} E_i(x)$. Moreover, $M(x)E_i(x) = E_i(Tx)$.*

(2) *Almost surely, for any $i \in I$, one has $\log\|M^n(x)_{|E_i(x)}\|/n \to \lambda_i$ when $n \to \pm\infty$, and $\log\|M^n(x)^{-1}_{|E_i(x)}\|/n \to -\lambda_i$.*

In other words, the decomposition of the space $E(x) = \bigoplus_{i \in I} E_i(x)$ gives a block-diagonal decomposition of the cocycle M, such that in each block the cocycle has an asymptotic behavior given by $e^{n\lambda_i}$ up to subexponential fluctuations.

The spaces $E_i(x)$ can be constructed almost surely as follows. Let $t_1^{(n)}(x) \geqslant \cdots \geqslant t_d^{(n)}(x)$ be the singular values of $M^n(x)$. They are the eigenvalues of the symmetric matrix $\sqrt{M^n(x)^t \cdot M^n(x)}$, the corresponding eigenspaces being orthogonal. Write $t_i^{(n)}(x) = e^{n\lambda_i^{(n)}(x)}$. Then $\lambda_i^{(n)}(x)$ converges to λ_i for almost every x. In particular, for $i \in I$, one has $t_{i-1}^{(n)}(x) > t_i^{(n)}(x)$ for large enough $n > 0$. It follows that the direct sum of the eigenspaces of $\sqrt{M^n(x)^t \cdot M^n(x)}$ for the eigenvalues $t_i^{(n)}(x), \dots, t_{i+d_i-1}^{(n)}(x)$ is well defined. Denote it by $F_i^{(n)}(x)$. We will write $F_{\geqslant i}^{(n)}$ for $\bigoplus_{j \geqslant i, j \in I} F_j^{(n)}(x)$, and similarly for $F_{\leqslant i}^{(n)}$. In the same way, we define similar quantities for $n < 0$.

Theorem 2.2. *Fix $i \in I$. With these notations, $F_i^{(n)}(x)$ converges almost surely when $n \to \infty$, to a vector subspace $F_i^{(\infty)}(x) \subseteq E(x)$. In the same way, $F_i^{(-n)}$ converges almost surely to a space $F_i^{(-\infty)}(x)$. Moreover, the direct sums $F_{\geqslant i}^{(\infty)}(x)$ and $F_{\leqslant i}^{(-\infty)}(x)$ are almost surely transverse, and their intersection is $E_i(x)$.*

See [Arn98, Theorem 3.4.1 and Page 154]. One can reformulate the theorem as follows. The subspaces $F_{\geqslant i}^{(n)}(x)$ (which are decreasing with i, i.e., they form a flag) converge when $n \to \infty$ to the flag $E_{\geqslant i}(x)$. Note that $F_{\geqslant i}^{(n)}(x)$ is only defined in terms of the positive times of the dynamics, hence this is also the case of $E_{\geqslant i}(x)$: this is the set of vectors for which the expansion in positive time is at most $e^{n\lambda_i}$, up to subexponential fluctuations (note that this condition is clearly stable under addition, and therefore defines a vector subspace, contrary to the condition that the expansion would be bounded *below* by $e^{n\lambda_i}$). In the same way, $F_{\leqslant i}^{(-n)}(x)$ converges when $n \to \infty$ to $E_{\leqslant i}(x)$, which therefore only depends on the past of the dynamics. On the other hand, $E_i(x)$, being defined as the intersection of two spaces depending on positive and negative times, depends on the whole dynamics and is therefore more difficult to analyze. We emphasize that $E_i(x)$ is in general different from $F_i^{(\infty)}(x)$ or $F_i^{(-\infty)}(x)$.

In the above theorem, when we mention the convergence of subspaces, we are using the natural topology on the *Grassmann manifold* of linear subspaces of some given dimension p. It comes for instance from the following distance, that we will use later on:

$$\mathbf{d}(U, V) = \|\pi_{U \to V^\perp}\| = \max_{u \in U, \|u\|=1} \|\pi_{V^\perp} u\|, \tag{2.1}$$

where $\pi_{U \to V^\perp}$ is the orthogonal projection from U to the orthogonal $V^\perp$ of V. It is not completely obvious that this formula indeed defines a distance. As $\mathbf{d}(U, V) = \|\pi_{V^\perp} \pi_U\|$, the triangular inequality follows from the following computation (in which we use that orthogonal projections have norm at most 1):

$$\begin{aligned} \mathbf{d}(U, W) &= \|\pi_{W^\perp} \pi_U\| = \|\pi_{W^\perp}(\pi_V + \pi_{V^\perp})\pi_U\| \leqslant \|\pi_{W^\perp}\pi_V\pi_U\| + \|\pi_{W^\perp}\pi_{V^\perp}\pi_U\| \\ &\leqslant \|\pi_{W^\perp}\pi_V\| + \|\pi_{V^\perp}\pi_U\| = \mathbf{d}(V, W) + \mathbf{d}(U, V). \end{aligned}$$

For the symmetry, we note that $\mathbf{d}(U, V) = \sqrt{1 - \|\pi_{U \to V}\|_{\min}^2}$, where $\|M\|_{\min}$ denotes the minimal expansion of a vector by a linear map M. This is also its smallest singular value.

As $\pi_{V\to U} = \pi_{U\to V}^t$, and a (square) matrix and its transpose have the same singular values, it follows that $\|\pi_{U\to V}\|_{\min} = \|\pi_{V\to U}\|_{\min}$, and therefore that $\mathbf{d}(U,V) = \mathbf{d}(V,U)$.

2.2. **Oseledets decomposition and subbundles.**

The following lemma follows directly from Oseledets theorem, by considering the Oseledets decomposition in each subbundle.

Lemma 2.3. *Consider a log-integrable cocycle M on a normed vector bundle E, over an ergodic probability preserving dynamical system T. Assume that E splits as a direct sum of invariant subbundles E_i. Then the Lyapunov spectrum of M on E is the union of the Lyapunov spectra of M on each E_i, with multiplicities.*

The same holds if M, instead of leaving each E_i invariant, is upper triangular. While this is well known, we give a full proof as this is not as trivial as one might think.

Lemma 2.4. *Consider a log-integrable cocycle M on a normed vector bundle E, over an ergodic probability preserving dynamical system T. Assume that there is a measurable invariant flag decomposition $\{0\} = F_0(x) \subseteq F_1(x) \subseteq \cdots \subseteq F_k(x) = E(x)$. Then the Lyapunov spectrum of M on E is the union of the Lyapunov spectra of M on each F_i/F_{i-1}, with multiplicities.*

Equivalently, considering E_i a complementary subspace to F_{i-1} in F_i, then the matrix representation of M in the decomposition $E = E_1 \oplus \cdots \oplus E_k$ is upper triangular, and the lemma asserts that the Lyapunov spectrum of M is the union of the Lyapunov spectra of the diagonal blocks.

Proof. Passing to the natural extension if necessary, we can assume that T is invertible.

Let us first assume that $k = 2$, and that there is only one Lyapunov exponent λ in E_1 and one Lyapunov exponent μ in E_2, both with some multiplicity. In matrix form, M can be written as $\left(\begin{smallmatrix} A_1 & B \\ 0 & A_2 \end{smallmatrix}\right)$, where the growth rate of A_1^n and A_2^n are respectively given by $e^{\lambda n}$ and $e^{\mu n}$. Then

$$M^n(x) = \begin{pmatrix} A_1^n(x) & \sum_{k=1}^n A_1^{n-k}(T^k x) B(T^{k-1}x) A_2^{k-1}(x) \\ 0 & A_2^n(x) \end{pmatrix}. \tag{2.2}$$

As M is a log-integrable cocycle, $\log\|M(T^n x)\|/n$ tends almost surely to 0 by Birkhoff theorem. Hence, the growth of $\|B(T^n x)\|$ is subexponential almost surely.

Assume first $\lambda > \mu$. Define a function $\Phi(x) : E_2(x) \to E_1(x)$ by

$$\Phi(x) = -\sum_{k=0}^{\infty} A_1^{k+1}(x)^{-1} B(T^k x) A_2^k(x).$$

The series converges almost surely as $\|A_1^{k+1}(x)^{-1} B(T^k x) A_2^k(x)\| \leqslant C e^{(\mu-\lambda)k + \varepsilon k}$ and $\mu - \lambda < 0$. This series is designed so that $A_1(x)\Phi(x) + B(x) = \Phi(Tx)A_2(x)$, i.e., so that the subspace $\tilde{E}_2(x) = \{(\Phi(x)v, v) \,:\, v \in E_2(x)\}$ is invariant under M. We have obtained a decomposition $E = E_1 \oplus \tilde{E}_2$, on which the cocycle acts respectively like A_1 and A_2. Hence, the result follows from Lemma 2.3.

Assume now $\lambda < \mu$. Then one can solve again the equation $A_1(x)\Phi(x) + B(x) = \Phi(Tx)A_2(x)$, this time going towards the past, by the converging series

$$\Phi(x) = \sum_{k=0}^{\infty} A_1^{-k}(x)^{-1} B(T^{-k}x) A_2^{-k-1}(x).$$

Then, one concludes as above.

Finally, assume $\lambda = \mu$. For any typical x, any n and any $k \leqslant n$, we have

$$\begin{aligned}\|A_1^{n-k}(T^k x)\| = \|A_1^n(x)A_1^k(x)^{-1}\| &\leqslant \|A_1^n(x)\|\|A_1^k(x)^{-1}\| \\ &\leqslant Ce^{\lambda n + \varepsilon n/4} \cdot e^{-\lambda k + \varepsilon k/4} \leqslant Ce^{\lambda(n-k)+\varepsilon n/2}.\end{aligned}$$

Hence, one deduces from the expression (2.2) of $M^n(x)$ that its norm grows at most like $ne^{n\lambda + n\varepsilon}$ almost surely. Hence, all its Lyapunov exponents are $\leqslant \lambda$. The same argument applied to the inverse cocycle, for T^{-1}, shows that all the Lyapunov exponents are also $\geqslant \lambda$, concluding the proof in this case.

We turn to the general case. Subdividing further each F_i/F_{i-1} into the sum of its Oseledets subspaces, we may assume that there is one single Lyapunov exponent in each F_i/F_{i-1}. Then, we argue by induction over k. At step k, the induction assumption ensures that the Lyapunov spectrum L_2 of M in E/F_1 is the union of the Lyapunov spectra in the F_i/F_{i-1} for $i > 1$. Denoting by L_1 the Lyapunov spectrum in F_1 (made of a single eigenvalue λ with some multiplicity), we want to show that the whole Lyapunov spectrum is $L_1 \cup L_2$, with multiplicities. Using the Oseledets theorem in E/F_1 and lifting the corresponding bundles to E, we obtain subbundles $G_2, \dots, G_I$ such that, in the decomposition $E = F_1 \oplus G_2 \oplus \cdots \oplus G_I$, the matrix M is block diagonal, except possibly for additional blocks along the first line. Each block G_i in which the Lyapunov exponent is not λ can be replaced by a block $\tilde{G}_i$ which is really invariant under the dynamics, as in the $k = 2$ case above. We are left with F_1 and possibly one single additional block, say G_i, with the same exponent λ. The $k = 2$ case again shows that all the Lyapunov exponents in $F_1 \oplus G_i$ are equal to λ, concluding the proof. □

3. Exponential large deviations for norms of linear cocycles

3.1. **Gibbs measures.** In this section, we recall basic properties of Gibbs measures, as explained for instance in [Bow75] and [PP90]. By *Gibbs measure*, we always mean in this article Gibbs measure with respect to some Hölder continuous potential.

Let φ be a Hölder-continuous function, over a transitive subshift of finite type $T : \Sigma \to \Sigma$. The *Gibbs measure* associated to φ, denoted by μ_φ, is the unique T-invariant probability measure for which there exist two constants P (the *pressure* of φ) and $C > 0$ such that, for any cylinder $[a_0, \dots, a_{n-1}]$, and for any point x in this cylinder,

$$C^{-1} \leqslant \frac{\mu_\varphi[a_0, \dots, a_{n-1}]}{e^{S_n\varphi(x) - nP}} \leqslant C. \tag{3.1}$$

The Gibbs measure only depends on φ up to the addition of a coboundary and a constant, i.e., $\mu_\varphi = \mu_{\varphi + g - g\circ T + c}$.

Here is an efficient way to construct the Gibbs measure. Any Hölder continuous function is cohomologous to a Hölder continuous function which only depends on positive coordinates of points in Σ. Without loss of generality, we can assume that this is the case of φ, and also that $P(\varphi)=0$. Denote by $T_+ : \Sigma_+ \to \Sigma_+$ the unilateral subshift corresponding to T. Define the transfer operator $\mathcal{L}_\varphi$, acting on the space C^α of Hölder continuous functions on Σ_+ by

$$\mathcal{L}_\varphi u(x_+) = \sum_{T_+ y_+ = x_+} e^{\varphi(y_+)} v(y_+).$$

Then one shows that this operator has a simple eigenvalue 1 at 1, finitely many eigenvalues of modulus 1 different from 1 (they only exist if T is transitive but not mixing) and the rest of its spectrum is contained in a disk of radius <1. One deduces that, for any $v \in C^\alpha$, then in C^α one has $\frac{1}{N}\sum_{n=0}^{N-1} \mathcal{L}_\varphi^n u \to \mu^+(v) v_0$, where v_0 is a (positive) eigenfunction corresponding to the eigenvalue 1, and μ^+ is a linear form on C^α. One can normalize them by $\mu^+(1)=1$. By approximation, it follows that this convergence also holds in C^0 for $v \in C^0$. Moreover, μ^+ extends to a continuous linear form on C^0, i.e., it is a probability measure.

Replacing φ with $\varphi + \log v_0 - \log v_0 \circ T_+$, one replaces the operator $\mathcal{L}_\varphi$ (with eigenfunction v_0) with the operator $\mathcal{L}_{\varphi+\log v_0 - \log v_0 \circ T_+}$, with eigenfunction 1. Hence, without loss of generality, we can assume that $v_0 = 1$. With this normalization, one checks that the measure μ^+ is T_+-invariant. It is the Gibbs measure for T_+, satisfying the property (3.1). Its natural T-invariant extension μ to Σ is the Gibbs measure for T. We have for any $v \in C^0(\Sigma_+)$

$$\frac{1}{N}\sum_{n=0}^{N-1} \mathcal{L}^n v(x_+) \to \int v \,\mathrm{d}\mu^+, \quad \text{uniformly in } x_+ \in \Sigma_+. \tag{3.2}$$

It follows from the construction above that the jacobian of μ^+ with respect to T_+ is given by $J(x_+) = \frac{\mathrm{d}T^*\mu_+}{\mu_+}(x) = e^{-\varphi(x_+)}$.

Consider the disintegration of μ with respect to the factor μ^+: there exists a family of measures $\mu^-_{x_+}$ on $W^s_{\mathrm{loc}}(x_+)$ for $x_+ \in \Sigma_+$, such that $\mu = \int \mu^-_{x_+} \,\mathrm{d}\mu_+(x_+)$. Formally, we write $\mu = \mu_+ \otimes \mu^-_{x_+}$, even though this is not a direct product. These measures can in fact be defined for all x_+ (instead of almost all x_+) in a canonical way, they depend continuously on x_+, they belong to the same measure class when the first coordinate $(x_+)_0$ is fixed, and moreover their respective Radon-Nikodym derivatives are continuous in all variables. See for instance [AV07, Section A.2].

Geometrically, the picture is the following. Consider some point $x_+ \in \Sigma_+$. It has finitely many preimages $y^1_+, \dots, y^i_+$ under T_+. Then $W^s_{\mathrm{loc}}(x_+) = \bigcup_i T(W^s_{\mathrm{loc}}(y^i_+))$, and

$$\mu^-_{x_+} = \sum_i \frac{1}{J(y^i_+)} T_* \mu^-_{y^i_+} = \sum_i e^{\varphi(y^i_+)} T_* \mu^-_{y^i_+}. \tag{3.3}$$

3.2. **First easy bounds.** In this paragraph, we prove (1-3) in Theorem 1.5.

Lemma 3.1. *Let $a_n(x) = a(n,x)$ be a subadditive cocycle which is bounded in absolute value for any n. Then, for any N, there exists $C>0$ with*

$$a(n,x) \leqslant S_n(a_N/N)(x) + C,$$

for any $n \in \mathbb{N}$ and any $x \in \Sigma$.

Proof. This is clear for $n \leqslant 2N$ as all those quantities are bounded. Consider now $n \geqslant 2N$, consider p such that $n = Np + r$ with $r \in [N, 2N]$. For any $j \in [0, N-1]$, one may write $n = j + Np + r$ with $r \in [0, 2N]$. Thus,

$$a(n,x) \leqslant a(j,x) + \sum_{i=0}^{p-1} a(N, T^{iN+j}x) + a(r, T^{pN+j}x) \leqslant C + \sum_{i=0}^{p-1} N(a_N/N)(T^{iN+j}x).$$

Summing over $j \in [0, N-1]$, we get

$$\begin{aligned} Na(n,x) &\leqslant NC + \sum_{j=0}^{N-1}\sum_{i=0}^{p-1} N(a_N/N)(T^{iN+j}x) = NC + NS_{Np}(a_N/N)(x) \\ &\leqslant C' + NS_n(a_N/N)(x). \end{aligned}$$

This proves the claim. □

Lemma 3.2. *Let (T,μ) be a transitive subshift of finite type with a Gibbs measure, and $a(n,x)$ a subadditive cocycle above T such that $a(n,\cdot)$ is continuous for all n. Let λ be the almost sure limit of $a(n,x)/n$, assume $\lambda > -\infty$. Then, for any $\varepsilon > 0$, there exists $C > 0$ such that, for all $n \geqslant 0$,*

$$\mu\{x \,:\, a(n,x) \geqslant n\lambda + n\varepsilon\} \leqslant Ce^{-C^{-1}n}.$$

Proof. By Kingman's theorem, $a(n,x)/n$ converges to λ almost everywhere and in L^1. Thus, one can take N such that $\int a_N/N \,\mathrm{d}\mu(x) \leqslant (\lambda+\varepsilon/2)N$. From the previous lemma, we obtain a constant C such that, for all n and x,

$$a(n,x) \leqslant S_n(a_N/N)x + C.$$

Thus,

$$\{x \,:\, a(n,x) \geqslant n\lambda + n\varepsilon\} \subseteq \{x \,:\, S_n(a_N/N)x \geqslant n\int (a_N/N) + n\varepsilon/2 - C\}.$$

By the large deviations inequality for continuous functions[1], this set has exponentially small measure. This proves the lemma. □

Proposition 3.3. *Let (T,μ) be a transitive subshift of finite type with a Gibbs measure, and M a continuous cocycle above T with Lyapunov exponents $\lambda_1 \geqslant \cdots \geqslant \lambda_d$. For any $\varepsilon > 0$, there exists $C > 0$ such that, for all $n \geqslant 0$ and all $i \leqslant d$,*

$$\mu\{x \,:\, \log\|\Lambda^i M^n(x)\| \geqslant n(\lambda_1 + \cdots + \lambda_i) + n\varepsilon\} \leqslant Ce^{-C^{-1}n}.$$

Proof. Fix $i \leqslant d$. Then the result follows from the previous lemma applied to $a(n,x) = \log\|\Lambda^i M^n(x)\|$. □

This proposition shows one of the two directions in Theorem 1.5, without any assumption on the cocycle. Hence, to prove this theorem, it will suffice to prove the corresponding lower bound

$$\mu\{x \,:\, \log\|\Lambda^i M^n(x)\| \leqslant n(\lambda_1 + \cdots + \lambda_i) - n\varepsilon\} \leqslant Ce^{-C^{-1}n}, \tag{3.4}$$

[1] This holds for continuous functions in transitive subshifts of finite type, by reduction to the mixing setting after taking a finite iterate of the map, and by reduction to Hölder continuous functions by uniform approximation.

under the various possible assumptions of this theorem. As is usual with subadditive ergodic theory, this lower bound is significantly harder than the upper bound. Indeed, the analogue of Lemma 3.2 for the lower bound is false, see Proposition A.1 in Appendix A

We already have enough tools to prove the easy cases of Theorem 1.5.

Proof of Theorem 1.5 (1-3). First, we prove (1): assuming that $\lambda_1 = \ldots = \lambda_d = \lambda$, we have to prove that

$$\mu\{x \,:\, \log\|\Lambda^i M^n(x)\| \leqslant ni\lambda - n\varepsilon\} \leqslant Ce^{-C^{-1}n},$$

Let $s_i(x,n)$ be the i-th singular value of $M^n(x)$. Then

$$\|\Lambda^i M^n(x)\| = s_1(x,n)\cdots s_i(x,n) \geqslant s_d(x,n)^i = \|M^n(x)^{-1}\|^{-i}.$$

Hence, to conclude, it suffices to show that

$$\mu\{x \,:\, \log\|M^n(x)^{-1}\| \geqslant -n\lambda + n\varepsilon\} \leqslant Ce^{-C^{-1}n}.$$

This follows from Proposition 3.3 applied to the cocycle $\tilde{M}(x) = (M(x)^{-1})^t$, whose Lyapunov exponents are all equal to $-\lambda$.

Let us now prove (3), for $k = 2$ as the general case then follows by induction over k. Assume that E_1 is an invariant continuous subbundle such that, on E_1 and on E/E_1, the induced cocycle has exponential large deviations for all exponents. Denote by L_1 and L_2 the Lyapunov exponents of the cocycle on these two bundles, then the Lyapunov spectrum on E is $L_1 \cup L_2$ with multiplicity, by Lemma 2.4. Let E_2 be the orthogonal complement to E_1. We want to show (3.4), for some i. In $\lambda_1, \ldots, \lambda_i$, some of these exponents, say a number i_1 of them, are the top exponents in L_1. Denote their sum by Σ_1. The remaining $i_2 = i - i_1$ exponents are the top exponents in L_2, and add up to a number Σ_2.

In the decomposition $E = E_1 \oplus E_2$, the matrix M is block diagonal, of the form $\begin{pmatrix} M_1 & B \\ 0 & M_2 \end{pmatrix}$. One has $\|\Lambda^i M(x)\| \geqslant \|\Lambda^{i_1} M_1(x)\|\|\Lambda^{i_2} M_2(x)\|$: considering v_1 and v_2 that are maximally expanded by $\Lambda^{i_1} M_1(x)$ and $\Lambda^{i_2} M_2(x)$, the expansion factor of $\Lambda^i M(x)$ along $v_1 \wedge v_2$ is at least $\|\Lambda^{i_1} M_1(x)\|\|\Lambda^{i_2} M_2(x)\|$ thanks to the orthogonality of E_1 and E_2, and the block-diagonal form of $M(x)$. Therefore,

$$\begin{aligned}
&\{x \,:\, \log\|\Lambda^i M^n(x)\| \leqslant n(\lambda_1 + \cdots + \lambda_i) - n\varepsilon\} \\
&\qquad \subseteq \{x \,:\, \log\|\Lambda^{i_1} M_1^n(x)\| + \log\|\Lambda^{i_2} M_2^n(x)\| \leqslant n\Sigma_1 + n\Sigma_2 - n\varepsilon\} \\
&\qquad \subseteq \{x \,:\, \log\|\Lambda^{i_1} M_1^n(x)\| \leqslant n\Sigma_1 - n\varepsilon/2\} \cup \{x \,:\, \log\|\Lambda^{i_2} M_2^n(x)\| \leqslant n\Sigma_2 - n\varepsilon/2\}.
\end{aligned}$$

The last sets both have an exponentially small measure, as we are assuming that the induced cocycles on E_1 and E/E_1 have exponential large deviations for all exponents. Hence, $\mu\{x \,:\, \log\|\Lambda^i M^n(x)\| \leqslant n(\lambda_1 + \cdots + \lambda_i) - n\varepsilon\}$ is also exponentially small. This concludes the proof of (3).

Finally, (2) follows from (3) by taking $F_i = E_1 \oplus \cdots \oplus E_i$. □

3.3. u**-states.** Consider a cocycle M admitting invariant continuous holonomies. We define a fibered dynamics over the projective bundle $\mathbb{P}(E)$ by

$$T_{\mathbb{P}}(x,[v]) = (Tx, [M(x)v]).$$

Let $\pi_{\mathbb{P}(E)\to\Sigma} : \mathbb{P}(E) \to \Sigma$ be the first projection.

In general, $T_{\mathbb{P}}$ admits many invariant measures which project under $\pi_{\mathbb{P}(E)\to\Sigma}$ to a given Gibbs measure μ. For instance, if the Lyapunov spectrum of M is simple, denote by $v_i(x)$ the vector in $E(x)$ corresponding to the i-th Lyapunov exponent, then $\mu\otimes\delta_{[v_i(x)]}$ is invariant under $T_{\mathbb{P}}$. By this notation, we mean the measure such that, for any continuous function f,

$$\int f(x,v)\,\mathrm{d}(\mu\otimes\delta_{[v_i(x)]})(x,v) = \int f(x,[v_i(x)])\,\mathrm{d}\mu(x).$$

More generally, if m_x is a family of measures on $\mathbb{P}(E(x))$ depending measurably on x such that $M(x)_* m_x = m_{Tx}$, then the measure $\mu\otimes m_x$ (defined as above) is invariant under $T_{\mathbb{P}}$. Conversely, any $T_{\mathbb{P}}$-invariant measure that projects down to μ can be written in this form, by Rokhlin's disintegration theorem.

To understand the growth of the norm of the cocycle, we need to distinguish among those measures the one that corresponds to the maximal expansion, i.e., $\mu\otimes\delta_{[v_1]}$. This measure can be obtained as follows, assuming that λ_1 is simple. Start from a measure on $\mathbb{P}(E)$ that is of the form $\mu\otimes\nu_x$ where the measures ν_x depend continuously on x and give zero mass to all hyperplanes. Then

$$(T_{\mathbb{P}}^n)_*(\mu\otimes\nu_x) = \mu\otimes(M^n(T^{-n}x)_*\nu_{T^{-n}x}).$$

By Oseledets theorem, the matrix $M^n(T^{-n}x)$ acts as a contraction on $\mathbb{P}(E(T^{-n}x))$, sending the complement of a neighborhood of some hyperplane to a small neighborhood of $[v_1(x)]$. As ν_y gives a small mass to the neighborhood of the hyperplane (uniformly in y), it follows that $(M^n(T^{-n}x)_*\nu_{T^{-n}x})$ converges to $\delta_{[v_1(x)]}$. Thus,

$$\mu\otimes\delta_{[v_1]} = \lim (T_{\mathbb{P}}^n)_*(\mu\otimes\nu_x).$$

There is a remarkable consequence of this construction. We can start from a family of measure ν_x which is invariant under the unstable holonomy $H^u_{x\to y}$, i.e., such that $(H^u_{x\to y})_*\nu_x = \nu_y$. Then the same is true of all the iterates $(M^n(T^{-n}x)_*\nu_{T^{-n}x})$. In the limit $n\to\infty$, it follows that $\delta_{[v_1]}$ is also invariant under unstable holonomies. (There is something to justify here, as it is not completely straightforward that the holonomy invariance is invariant under weak convergence: The simplest way is to work with a one-sided subshift, and then lift things trivially to the two-sided subshift, see [AV07, Section 4.1] for details). This remark leads us to the following definition.

Definition 3.4. *Consider a probability measure ν on $\mathbb{P}(E)$ which projects to μ under π. It is called a u-state if, in the fiberwise decomposition $\nu = \mu\otimes\nu_x$, the measures ν_x are μ-almost surely invariant under unstable holonomies. It is called an invariant u-state if, additionally, it is invariant under the dynamics.*

The invariant u-states can be described under an additional irreducibility assumption of the cocycle, strong irreducibility.

Definition 3.5. *We say that a cocycle M with invariant continuous holonomies over a subshift of finite type is* not strongly irreducible *if there exist a dimension $0 < k < d = \dim E$, an integer $N > 0$, and for each point $x \in \Sigma$ a family of distinct k-dimensional vector subspaces $V_1(x), \dots, V_N(x)$ of $E(x)$, depending continuously on x, with the following properties:*

- *the family as a whole is invariant under M, i.e., for all x,*
$$M(x)\{V_1(x), \dots, V_N(x)\} = \{V_1(Tx), \dots, V_N(Tx)\}.$$
- *the family as a whole is invariant under the holonomies, i.e., for all x and all $y \in W^u_{\mathrm{loc}}(x)$ one has $H^u_{x\to y}\{V_1(x), \dots, V_N(x)\} = \{V_1(y), \dots, V_N(y)\}$, and the same holds for the stable holonomies.*

Otherwise, we say that M is strongly irreducible.

In a locally constant cocycle, where holonomies commute (and can therefore be taken to be the identity), then the holonomy invariance condition reduces to the condition that each V_i is locally constant, i.e., it only depends on x_0.

The following theorem is the main result of this paragraph. It essentially follows from the arguments in [BV04, AV07].

Theorem 3.6. *Consider a transitive subshift of finite type T with a Gibbs measure μ. Let M be a locally constant cocycle on a bundle E over T, which is strongly irreducible and has simple top Lyapunov exponent. Then the corresponding fibered map $T_{\mathbb{P}}$ has a unique invariant u-state, given by $\mu \otimes \delta_{[v_1]}$ where $v_1(x)$ is a nonzero vector spanning the 1-dimensional Oseledets subspace for the top Lyapunov exponent at x.*

Note that we are assuming that the cocycle is locally constant: This theorem is wrong if the cocycle only has invariant continuous holonomies, see Remark 3.9 below.

The rest of this subsection is devoted to the proof of this theorem. We have already seen that $\mu \otimes \delta_{[v_1]}$ is an invariant u-state, what needs to be shown is the uniqueness. Starting from an arbitrary u-state ν, we have to prove that it is equal to $\mu \otimes \delta_{[v_1]}$.

As the cocycle is locally constant, one can quotient by the stable direction, obtaining a unilateral subshift $T_+ : \Sigma_+ \to \Sigma_+$ with a Gibbs measure μ_+, a vector bundle E_+ and a cocycle M_+. The measure $\nu^+ = (\pi_{E\to E_+})_*\nu$ is then invariant under the fibered dynamics $T_{+,\mathbb{P}}$. It can be written as $\mu_+ \otimes \nu^+_{x_+}$ for some measurable family $\nu^+_{x_+}$ of probability measures on $\mathbb{P}(E_+(x_+))$.

The following lemma is [AV07, Proposition 4.4].

Lemma 3.7. *Assume that ν is an invariant u-state. Then the family of measures $\nu^+_{x_+}$, initially defined for μ_+-almost every x_+, extends to a (unique) family that depends continuously in the weak topology on all $x_+ \in \Sigma_+$.*

For completeness, we sketch the proof, leaving aside the technical details.

Proof. The measure $\nu^+_{x_+}$ is obtained by averaging all the conditional measures ν_x over all points x which have the future x_+, i.e., over the points (x_-, x_+), with respect to a conditional measure $d\mu^-_{x_+}(x_-)$. If y_+ is close to x_+, one has $y_0 = x_0$, so the possible pasts of y_+ are the same as the possible pasts of x_+. For any continuous function f on projective space, we obtain

$$\int f \,\mathrm{d}\nu^+_{x_+} = \int \left(\int f \,\mathrm{d}\nu_{x_-,x_+} \right) \mathrm{d}\mu^-_{x_+}(x_-), \qquad \int f \,\mathrm{d}\nu^+_{y_+} = \int \left(\int f \,\mathrm{d}\nu_{x_-,y_+} \right) \mathrm{d}\mu^-_{y_+}(x_-).$$

When y_+ is close to x_+, the measures $\mathrm{d}\mu^-_{x_+}$ and $\mathrm{d}\mu^-_{y_+}$ are equivalent, with respective density close to 1, as we explained in Paragraph 3.1. Moreover, by holonomy invariance of the conditional measures of ν,

$$\int f \,\mathrm{d}\nu_{x_-,y_+} = \int f \circ H^u_{(x_-,x_+)\to(x_-,y_+)} \,\mathrm{d}\nu_{x_-,x_+}.$$

By continuity of the holonomies, the function $f \circ H^u_{(x_-,x_+)\to(x_-,y_+)}$ is close to f if y_+ is close to x_+. It follows that $\int f \,\mathrm{d}\nu^+_{y_+}$ is close to $\int f \,\mathrm{d}\nu^+_{x_+}$, as desired. Details can be found in [AV07, Section 4.2]. □

Henceforth, we write $\nu^+_{x_+}$ for the family of conditional measures, depending continuously on x_+. The next lemma is a version of [AV07, Proposition 5.1] in our setting.

Lemma 3.8. *Assume that M is strongly irreducible in the sense of Definition 3.5. Let ν be an invariant u-state, write $\nu^+_{x_+}$ for the continuous fiberwise decomposition of Lemma 3.7. Then, for any x_+, for any hyperplane $L \subset \mathbb{P}(E_+(x_+))$, one has $\nu^+_{x_+}(L) = 0$.*

Proof. Assume by contradiction that $\nu^+_{x_+}$ gives positive mass to some hyperplane, for some x_+. We will then construct a family of subspaces as in Definition 3.5, contradicting the strong irreducibility of the cocycle.

Let k be the minimal dimension of a subspace with positive mass at some point. Let γ_0 be the maximal mass of such a k-dimensional subspace. By continuity of $x_+ \mapsto \nu^+_{x_+}$ and compactness, there exist a point a_+ and a k-dimensional subspace V with $\nu^+_{a_+}(V) = \gamma_0$ ([AV07, Lemma 5.2])

Let $\mathcal{V}(x_+)$ be the set of all k-dimensional subspaces V of $E_+(x_+)$ with $\nu^+_{x_+}(V) = \gamma_0$. Two elements of $\mathcal{V}(x_+)$ intersect in a subspace of dimension $< k$, which has measure 0 by minimality of k. Hence, $\gamma_0 \operatorname{Card} \mathcal{V}(x_+) = \nu^+_{x_+}(\bigcup_{V\in\mathcal{V}(x_+)} V)$. As this is at most 1, the cardinality of $\mathcal{V}(x_+)$ is bounded from above, by $1/\gamma_0$.

Consider a point b_+ where the cardinality N of $\mathcal{V}(b_+)$ is maximal. For each $V \in \mathcal{V}(b_+)$, $\nu^+_{b_+}(V)$ is an average of $\nu^+_{x_+}(M(x_+)^{-1}V)$ over all preimages x_+ of b_+ under T_+ (see [AV07, Corollary 4.7]). By maximality, all the $M(x_+)^{-1}V$ also have mass γ_0 for $\nu^+_{x_+}$. Iterating this process, one obtains for all points in $T_+^{-n}\{b_+\}$ at least N subspaces with measure γ_0 (and in fact exactly N by maximality). The set $\bigcup_n T_+^{-n}\{b_+\}$ is dense. Hence, any x_+ is a limit of a sequence x_n for which $\mathcal{V}(x_n)$ is made of N subspaces $V_1(x_n), \dots, V_N(x_n)$. Taking subsequences, we can assume that each sequence $V_i(x_n)$ converges to a subspace V_i, which belongs to $\mathcal{V}(y)$ by continuity of $y_+ \mapsto \nu^+_{y_+}$. Moreover, one has $V_i \neq V_j$ for $i \neq j$: otherwise, the corresponding space would have measure at least $2\gamma_0$, contradicting the definition of γ_0. This shows that the cardinality of $\mathcal{V}(x_+)$ is at least N, and therefore exactly N.

We have shown that the family $\mathcal{V}(x_+)$ is made of exactly N subspaces everywhere, that it depends continuously on x_+ and that it is invariant under $T_{+,\mathbb{P}}$. We lift everything to the bilateral subshift Σ, setting $\mathcal{V}(x) = \mathcal{V}(\pi_{\Sigma\to\Sigma_+}x)$. By construction, the family is invariant under the dynamics $T_{\mathbb{P}}$. As V_i does not depend on the past of the points, it is invariant under the stable holonomy (which is just the identity when one moves along stable sets, thanks to our choice of trivialization of the bundle).

The family $\mathcal{V}(x)$ only depends on x_+. We claim that, in fact, it only depends on x_0, i.e., it is also invariant under the unstable holonomy. Fix some x_+, and some y_+ with $y_0 = x_0$. Then $\gamma_0 = \nu^+_{x_+}(V_i(x_+))$ is an average of the quantities $\nu_{(x_-,x_+)}(V_i(x_+))$ over all possible pasts x_- of x_+. One deduces from this that $\nu_{(x_-,x_+)}(V_i(x_+)) = \gamma_0$ for almost every such x_-, see [AV07, Lemma 5.4]. As ν is invariant under unstable holonomy, we obtain $\nu_{(x_-,y_+)}(V_i(x_+)) = \gamma_0$ for almost every x_-. Integrating over x_-, we get $\nu^+_{y_+}(V_i(x_+)) = \gamma_0$. Hence, $V_i(x_+) \in \mathcal{V}(y_+)$. This shows that $\mathcal{V}(x_+) = \mathcal{V}(y_+)$ if $x_0 = y_0$ (almost everywhere and then everywhere by continuity). Hence, $\mathcal{V}$ is locally constant. This shows that M is not strongly irreducible. □

Let us explain how this proof fails if the cocycle are not locally constant, i.e., if the holonomies do not commute. Let us argue in a trivialization were the stable holonomies are the identity. The failure is at the end of the proof, when we show that the family $\mathcal{V}(x)$ is invariant under unstable holonomy. We can indeed prove that $\nu_{(x_-,x_+)}(V_i(x_+)) = \gamma_0$ for almost every x_-. Then, it follows that $\nu_{(x_-,y_+)}(H^u_{(x_-,x_+)\to(x_-,y_+)}V_i(x_+)) = \gamma_0$. The problem is that the subspaces $H^u_{(x_-,x_+)\to(x_-,y_+)}V_i(x_+)$ vary with x_-, so one can not integrate this equality with respect to x_-, to obtain a subspace V with $\nu^+_{y_+}(V) = \gamma_0$.

Proof of Theorem 3.6. Let ν be a u-state, let $\mu \otimes \nu_x$ be its fiberwise disintegration, and $\nu^+_{x_+}$ the conditional expectation of ν_x with respect to the future sigma-algebra. The martingale convergence theorem shows that, almost surely,

$$\nu_x = \lim M^n(T^{-n}x)_*\nu^+_{(T^{-n}x)_+}, \tag{3.5}$$

see [AV07, Proposition 3.1].

Let $\varepsilon > 0$. We may find δ such that, for any x_+ and any hyperplane $L \subseteq E_+(x_+)$, the δ-neighborhood of L in $\mathbb{P}(E_+(x))$ (for some fixed distance on projective space) satisfies $\nu^+_{x_+}(\mathcal{N}_\delta(L)) \leqslant \varepsilon$, thanks to Lemma 3.8 and continuity of the measures.

Let $E_1(x) = \mathbb{R}v_1(x)$ be the top Oseledets subspace of M, and $E_2(x)$ be the sum of the other subspaces. Let A be a compact subset of Σ with positive measure on which the decomposition $E(x) = E_1(x) \oplus E_2(x)$ is continuous and on which the convergence in Oseledets theorem is uniform. Fix $x \in A$. By Poincaré's recurrence theorem, there exists almost surely an arbitrarily large n such that $T^{-n}x \in A$. In the decomposition $E = E_1 \oplus E_2$, the cocycle $M^n(T^{-n}x)$ is block diagonal, with the first (one-dimensional) block dominating exponentially the other one. Hence, it sends $\mathbb{P}(E(T^{-n}x)) \setminus \mathcal{N}_\delta(E_2(T^{-n}x))$ (whose $\nu^+_{(T^{-n}x)_+}$-measure is at least $1-\varepsilon$ thanks to the choice of δ) to an ε-neighborhood of $E_1(x)$ if n is large enough. Therefore, $M^n(T^{-n}x)_*\nu^+_{(T^{-n}x)_+}(\mathcal{N}_\varepsilon([v_1(x)])) \geqslant 1-\varepsilon$. Letting ε tend to 0, we get $\nu_x([v_1(x)]) = 1$ thanks to (3.5). As the measure of A can be taken arbitrarily close to 1, we finally get that ν_x is almost everywhere equal to $\delta_{[v_1(x)]}$. □

Remark 3.9. Theorem 3.6 is wrong in general for cocycles which are not locally constant. The difficulty is in Lemma 3.8: If the cocycle M merely admits invariant continuous holonomies, there is no reason why the invariant family of subspaces $\mathcal{V}(x)$ we construct there should be invariant under the unstable holonomy, even though ν_x is. Here is an example of a strongly irreducible cocycle with simple Lyapunov exponents, over the full shift on two symbols endowed with any Gibbs measure, which admits two u-states.

Let Σ be the full shift, let $E = \Sigma \times \mathbb{R}^3$ and let M be the constant cocycle given by the matrix $\left(\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. We introduce the holonomies

$$H^u_{x\to y} = \begin{pmatrix} 1 & 0 & \sum_{n\geqslant 0} 3^{-n}(y_n - x_n) \\ 0 & 1 & \sum_{n\geqslant 0} 2^{-n}(y_n - x_n) \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$H^s_{x\to y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sum_{n\leqslant 0} 3^{n}(y_n - x_n) & \sum_{n\leqslant 0} 2^{n}(y_n - x_n) & 1 \end{pmatrix}.$$

One checks easily that they are indeed holonomies, and that they are invariant under T. Let e_i denote the i-th vector of the canonical basis. As e_1 and e_2 are invariant under the unstable holonomies, they give rise to two distinct u-states.

We claim that the cocycle is strongly irreducible. Indeed, consider a nonzero subbundle F of E which is invariant under T and the holonomies, we will show that $F = E$. Considering the Oseledets decomposition of F under the cocycle, it follows that F is spanned by some subfamily $(e_i)_{i\in I}$. If $1 \in I$ or $2 \in I$, then the invariance under stable holonomy implies that $3 \in I$, since $H^s e_1$ and $H^s e_2$ have a nonzero component along e_3. Hence, $e_3 \in F$ almost everywhere. Then, using the invariance under unstable holonomy, we deduce that $e_1 \in F$ and $e_2 \in F$ almost everywhere, as $H^u e_3$ has nonzero components along e_1 and e_2. Finally, $F = E$.

3.4. **The case of locally constant cocycles.** In this paragraph, we prove Theorem 1.5 (4): if a cocycle is locally constant above a transitive subshift of finite type, then it has exponential large deviations for all exponents. The main step is the following result:

Theorem 3.10. *Consider a continuous cocycle over a transitive subshift of finite type endowed with a Gibbs measure, admitting invariant continuous holonomies. Assume that it has a unique u-state. Then the cocycle has exponential large deviations for its top exponent.*

Before proving this theorem, let us show by successive reductions how it implies Theorem 1.5 (4).

Lemma 3.11. *Consider a locally constant cocycle which is strongly irreducible and has simple top Lyapunov exponent, above a subshift of finite type with a Gibbs measure. Then it has exponential large deviations for its top exponent.*

Proof. By Theorem 3.6, the cocycle admits a unique u-state. Hence, the result follows from Theorem 3.10. □

Lemma 3.12. *Consider a locally constant cocycle which has simple top Lyapunov exponent, above a subshift of finite type with a Gibbs measure. Then it has exponential large deviations for its top exponent.*

Proof. We argue by induction on dimension of the fibers of the cocycle. Consider a cocycle M on a bundle E over a subshift of finite type T, with simple top Lyapunov exponent. We will show that it has exponential large deviations for its top exponent, assuming the same results for all cocycles on fiber bundles with strictly smaller dimension. We will prove the lower bound (3.4) (with $i = 1$) for M.

If the cocycle M is strongly irreducible, then the result follows from Lemma 3.11, so assume that it is not. Consider a locally constant invariant family $V_1(x), \dots, V_N(x)$ as in Definition 3.5, such that N is minimal. Let $V(x)$ be the span of $V_1(x), \dots, V_N(x)$. It is also locally constant and invariant under the cocycle and the holonomies.

Assume first that the dimension of V is strictly smaller than that of E. Define a cocycle M_V as the restriction of M to V, and a cocycle $M_{E/V}$ as the cocycle induced by M on the quotient bundle E/V. These two cocycles are locally constant. By definition of the restriction norm and the quotient norm, one has

$$\|M^n(x)\| \geqslant \max(\|M_V^n(x)\|, \|M_{E/V}^n(x)\|). \tag{3.6}$$

Moreover, by Lemma 2.4, one of the two cocycles has λ_1 as a simple top Lyapunov exponent, and these two cocycles are locally constant and have strictly smaller fiber dimension. By our induction assumption, we deduce that

$$\mu\{x \,:\, \log\|M_W^n(x)\| \leqslant n\lambda_1 - n\varepsilon\} \leqslant Ce^{-C^{-1}n},$$

where W is either V or E/V. The same bound follows for M thanks to (3.6).

Assume now that the dimension of V is equal to that of E, i.e., $V = E$. Consider a new dynamics $\tilde{T}$, on $\tilde{\Sigma} = \Sigma \times \{1, \dots, N\}$, mapping (x, i) to (Tx, j) where $j = j(x, i)$ is the unique index such that $M(x)V_i(x) = V_j(Tx)$. As M and all the V_k only depend on x_0, the function j only depends on i, x_0 and x_1. Hence, $\tilde{T}$ is a subshift of finite type. As we chose N to be minimal, there is no invariant proper subfamily of $V_1, \dots, V_N$. Hence, $\tilde{T}$ is a transitive subshift. Let also $\tilde{\mu}$ be the product measure of μ and the uniform measure on $\{1, \dots, N\}$, it is again a Gibbs measure for $\tilde{T}$, therefore ergodic by transitivity.

Above $\tilde{\Sigma}$, we consider a new bundle $\tilde{E}(x, i) = V_i(x)$, and the resulting cocycle $\tilde{M}$ which is the restriction of M to V_i. On any $E(x)$, one can find a basis made of vectors in the subspaces $V_i(x)$, by assumption. It follows that $\|M^n(x)\| \leqslant C \max_i \|M^n(x)_{|V_i(x)}\|$, for some uniform constant C. Hence, the top Lyapunov exponent of $\tilde{M}$ is (at least, and therefore exactly) λ_1. Moreover, it is simple as the top Oseledets space for $\tilde{M}$ in $\tilde{E}(x, i)$ is included in the top Oseledets space for M in $E(x)$, which is one-dimensional by assumption.

By our induction assumption, we obtain the bound (3.4) with $i = 1$ for the cocycle $\tilde{M}$ over the subshift $\tilde{T}$ and the measure $\tilde{\mu}$ (note that it is important there that we have formulated the induction assumption for all subshifts of finite type, not only the original one). The result follows for the original cocycle as $\|M^n(x)\| \geqslant \|\tilde{M}^n(x, 1)\|$ for all x. □

Lemma 3.13. *Consider a locally constant cocycle, above a subshift of finite type with a Gibbs measure. Then it has exponential large deviations for its top exponent.*

Proof. Consider a locally constant cocycle M for which the multiplicity d of the top Lyapunov exponent is > 1. Then the top Lyapunov exponent of $\Lambda^d M$ is simple, equal to $d\lambda_1$. Moreover, for any matrix A (with singular values $s_1 \geqslant s_2 \geqslant \dots$), we have $\|A\|^d = s_1^d \geqslant$

$\|\Lambda^d A\| = s_1 \cdots s_d$. Thus,

$$\{x \,:\, \log\|M^n(x)\| \leqslant n\lambda_1(M) - n\varepsilon\} \subseteq \{x \,:\, \log\|\Lambda^d M^n(x)\| \leqslant n\lambda_1(\Lambda^d M) - nd\varepsilon\}.$$

The last set has an exponentially small measure by Lemma 3.12, as $\Lambda^d M$ has a simple top Lyapunov exponent by construction, and is locally constant. The desired bound follows for M. □

Proof of Theorem 1.5 (4). Proving exponential large deviations for the cocycle M and some index i amounts to proving exponential large deviations for $\Lambda^i M$ and its top Lyapunov exponent. Hence, the theorem follows from Lemma 3.13. □

The rest of this paragraph is devoted to the proof of Theorem 3.10. The proof follows the classical strategy of Guivarc'h Le Page for products of independent matrices (with the uniqueness of the u-state replacing the uniqueness of the stationary measure), although the technical details of the implementation are closer for instance to [Dol04, Proof of Theorem 1].

Henceforth, we fix a transitive subshift of finite type $T : \Sigma \to \Sigma$ with a Gibbs measure μ, and a continuous cocycle $M : E \to E$ above T which admits a unique u-state denoted by ν_u. Changing coordinates in E using the unstable holonomy, we can assume without loss of generality that $M(x)$ only depends on the past x_- of x.

We denote by Σ_- the set of pasts of points in Σ. The left shift T does not induce a map on Σ_- (it would be multivalued, since there would be a choice for the zeroth coordinate), but the right shift T^{-1} does induce a map U on $\Sigma-$. This is a subshift of finite type, for which the induced measure $\mu_- = (\pi_{\Sigma \to \Sigma_-})_* \mu$ is invariant (and a Gibbs measure).

The measure μ has conditional expectations $\mu^+_{x_-}$ above its factor μ_-: it can be written as $\mu = \mu_- \otimes \mu^+_{x_-}$. The family of measures $\mu^+_{x_-}$ is canonically defined for all point $x_- \in \Sigma_-$, and varies continuously with x_-, as we explained in Paragraph 3.1 (for the opposite time direction).

To any point $(x, [v]) \in \mathbb{P}(E)$, we associate a measure $\nu_{(x,[v])}$ on $\mathbb{P}(E)$ as follows. There is a canonical lift to E of $W^u_{\mathrm{loc}}(x_-)$, going through v, given by $\{(x_-, y_+, H^u_{(x_-,x_+)\to(x_-,y_+)} v\}$. The measure $\mu^+_{x_-}$ on $W^u_{\mathrm{loc}}(x_-)$ can be lifted to this set, giving rise after projectivization to the measure $\nu_{(x,[v])}$. This measure is invariant under (projectivized) unstable holonomy, it projects to $\mu^+_{x_-}$ under the canonical projection $\mathbb{P}(E) \to \Sigma$, and it projects to δ_{x_-} under the canonical projection $\mathbb{P}(E) \to \Sigma_-$. By construction, for any x_-, x_+ and y_+,

$$\nu_{(x_-,x_+,[v])} = \nu_{(x_-,y_+,[H^u_{(x_-,x_+)\to(y_-,y_+)} v])}.$$

More generally, finite averages or even integrals of such measures are again H^u-invariant.

There is a natural Markov chain on Σ_-, defined as follows. A point x_- has several preimages y^i_- under U. By the invariance of the measure μ_-, the sum $1/J(y^i_-)$ is equal to 1, where J is the jacobian of U for μ_-. Hence, one defines a Markov chain, by deciding to jump from x_- to y^i_- with probability $1/J(y^i_i)$. The corresponding Markov operator is given by

$$\mathcal{L}v(x_-) = \sum_{U(y_-)=x_-} \frac{1}{J(y_-)} v(y_-).$$

This is simply the transfer operator of Paragraph 3.1 (for the map U instead of the map T). Replacing the potential φ which defines the Gibbs measure by a cohomologous potential, we may write $\frac{1}{J(y_-)} = e^{\varphi(y)}$.

Correspondingly, we define an operator $\mathcal{M}$ acting on measures on $\mathbb{P}(E)$, by

$$\mathcal{M}\nu = (T_{\mathbb{P}})_*\nu.$$

It maps $\nu_{(x,[v])}$ (supported on the lift V of $W^u_{\text{loc}}(x_-)$ through $[v]$) to a measure supported on $T_{\mathbb{P}}V$ (which is a lift of the union of the unstable manifolds $W^u_{\text{loc}}(y_-)$ for $U(y_-) = x_-$). Choose on each of these submanifolds a point $(y_-, y_+, [v_y])$ (where y_+ is arbitrary, and $[v_y]$ is the unique vector in TV above (y_-, y_+)). Then we have

$$\mathcal{M}\nu_{(x,[v])} = \sum e^{\varphi(y_-)}\nu_{(y_-,y_+,[v_y])}. \tag{3.7}$$

This follows from the equation (3.3) for the evolution of the conditional measures under the dynamics, and then the uniqueness of the H^u-invariant lift.

Proposition 3.14. *Let f be a continuous function on $\mathbb{P}(E)$. Then, uniformly in $x \in \Sigma$ and $v \in \mathbb{P}(E)$, when $N \to \infty$,*

$$\frac{1}{N}\sum_{n=0}^{N-1}\int f \,\mathrm{d}\mathcal{M}^n\nu_{(x,[v])} \to \int f \,\mathrm{d}\nu_u,$$

where ν_u is the unique invariant u-state of M.

Proof. It suffices to show that any weak limit ν_∞ of sequences of the form

$$\nu_N = \frac{1}{N}\sum_{n=0}^{N-1}\mathcal{M}^n\nu_{(x_N,[v_N])}$$

(where x_N and $[v_N]$ may vary with N) is an invariant u-state.

The invariance of the limiting measure is clear from the Cesaro-averaging and the definition $\mathcal{M}\nu = (T_{\mathbb{P}})_*\nu$. The H^u invariance also follows from the construction. It remains to show that ν_∞ projects to μ on Σ or, equivalently, that it projects to μ_- on Σ_-.

The projection of ν_N on Σ_- is the Cesaro average of $\sum_{U^n y_- = (x_N)_-} e^{S_{-n}\varphi(y_-)}\delta_{y_-}$, i.e., the position at time n of the Markov chain started from x_N at time 0. For any continuous function v on Σ_-, we get $\int v \,\mathrm{d}\pi_*\nu_N = \frac{1}{N}\sum_{n=0}^{N-1}\mathcal{L}^n v((x_N)_-)$. By a classical property of transfer operators (see (3.2)), this converges uniformly to $\int v \,\mathrm{d}\mu_-$. This proves that the only possible weak limit for $\pi_*(\nu_N)$ is μ_-. □

Fix once and for all $\varepsilon > 0$, for which we want to prove the inequality

$$\mu\{x \,:\, \log\|M^n(x)\| \leqslant n(\lambda_1 - \varepsilon)\} \leqslant Ce^{-C^{-1}n}. \tag{3.8}$$

Lemma 3.15. *Define a function g_0 on $\mathbb{P}(E)$ by*

$$g_0([x], v) = \log(\lambda_1 - \varepsilon) - \log(\|M(x)v\|/\|v\|),$$

where the last term in this formula does not depend on the choice of the lift v of $[v]$. Then there exist N and $\alpha, \beta > 0$ such that, for any x and v,

$$\int e^{\alpha S_N g_0}\,\mathrm{d}\nu_{(x,[v])} \leqslant e^{-\beta N}.$$

Proof. Define a function f_0 on $\mathbb{P}(E)$ by

$$f_0(x,[v]) = \log(\|M(x)v\|/\|v\|).$$

The integral of f_0 with respect to the unique invariant u-state ν_u measures the average expansion of a vector in the maximally expanded Oseledets subspace, which is by definition equal to the maximal Lyapunov exponent λ_1. Hence, it is not difficult to check the following formula, due to Furstenberg (see for instance [Via14, Proposition 6.5]):

$$\int f_0 \,\mathrm{d}\nu_u = \log \lambda_1.$$

It follows that

$$\int g_0 \,\mathrm{d}\nu_u = \log(\lambda_1 - \varepsilon) - \log \lambda_1 < 0.$$

Fix some $c_0 > 0$ such that $\int g_0 \,\mathrm{d}\nu_u < -c_0$. By Proposition 3.14, there exists an integer N such that, for any x and v,

$$\frac{1}{N}\sum_{n=0}^{N-1} \int g_0 \,\mathrm{d}\mathcal{M}^n \nu_{(x,[v])} \leqslant -c_0.$$

By definition of $\mathcal{M}$, we get

$$\int S_N g_0 \,\mathrm{d}\nu_{(x,[v])} = \sum_{n=0}^{N-1} \int g_0 \,\mathrm{d}\mathcal{M}^n \nu_{(x,[v])} \leqslant -c_0 N.$$

Using the inequality $e^t \leqslant 1 + t + t^2 e^{|t|}$, we obtain for any $\alpha \in (0,1)$

$$\int e^{\alpha S_N g_0} \,\mathrm{d}\nu_{(x,[v])} \leqslant 1 + \alpha \int S_N g_0 \,\mathrm{d}\nu_{(x,[v])} + \alpha^2 \int (S_N g_0)^2 e^{|S_N g_0|} \,\mathrm{d}\nu_{(x,[v])} \leqslant 1 - \alpha c_0 N + \alpha^2 C,$$

where C is a constant depending on N but not on α. (For the bound in the last term, note that the function $S_N g_0$ is uniformly bounded, as a continuous function on a compact space.) When α is small enough, the term $\alpha^2 C$ is negligible. Hence, we obtain for small enough α and for $\beta = \alpha c_0/2$ the inequality

$$\int e^{\alpha S_N g_0} \,\mathrm{d}\nu_{(x,[v])} \leqslant 1 - \beta N \leqslant e^{-\beta N}. \qquad \square$$

Lemma 3.16. *There exists a constant C such that, for any $n \in \mathbb{N}$ and any x and v, one has*

$$\int e^{\alpha S_n g_0} \,\mathrm{d}\nu_{(x,[v])} \leqslant C e^{-\beta n}.$$

Proof. It suffices to prove the lemma for times of the form nN, as the general case only results in an additional multiplicative constant.

Fix some n. Iterating (3.7), $(T_{\mathbb{P}}^{nN})_* \nu_{(x,[v])}$ is a finite linear combination of measures of the form $\nu_{(x_i,[v_i])}$, with some coefficients $c_i > 0$ adding up to 1. Then

$$\int e^{\alpha S_{(n+1)N} g_0} \,\mathrm{d}\nu_{(x,[v])} = \sum_i c_i \int e^{\alpha S_{nN} g_0 \circ T_{\mathbb{P}}^{-nN}} \cdot e^{\alpha S_N g_0} \,\mathrm{d}\nu_{(x_i,[v_i])}.$$

In each of the integrals, the term $e^{\alpha S_{nN} g_0 \circ T^{-nN}}$ is constant as g_0 and M only depend on the past of points in Σ. Hence, this integral is a constant multiple of $\int e^{\alpha S_N g_0} \,\mathrm{d}\nu_{(x_i,[v_i])}$, which is $\leqslant e^{-\beta N}$ by Lemma 3.15. We get

$$\int e^{\alpha S_{(n+1)N} g_0} \,\mathrm{d}\nu_{(x,[v])} \leqslant e^{-\beta N} \sum_i c_i \int e^{\alpha S_{nN} g_0 \circ T^{-nN}} \,\mathrm{d}\nu_{(x_i,[v_i])} = e^{-\beta N} \int e^{\alpha S_{nN} g_0} \,\mathrm{d}\nu_{(x,[v])}.$$

The conclusion then follows by induction on n. □

Proof of Theorem 3.10. Fix some vector v. Then the average $\int \nu_{(x,[v])} \,\mathrm{d}\mu(x)$ is a measure on $\mathbb{P}(E)$ that projects to μ. If $\log\|M^n(y)\| \leqslant n\lambda_1(M) - n\varepsilon$, then for any vector w one has $\log(\|M^n(y)w\|/\|w\|) \leqslant n(\lambda_1(M) - \varepsilon)$, i.e., $S_n g_0(y,w) \geqslant 0$. We obtain

$$\begin{aligned}\mu\{y \,:\, \log\|M^n(y)\| \leqslant n\lambda_1(M) - n\varepsilon\} &\leqslant \int 1(S_n g_0(y,w) \geqslant 0) \,\mathrm{d}\nu_{(x,[v])}(y,w) \,\mathrm{d}\mu(x) \\ &\leqslant \int \left(\int e^{\alpha S_n g_0} \,\mathrm{d}\nu_{(x,[v])} \right) \mathrm{d}\mu(x).\end{aligned}$$

By Lemma 3.16, the last integral is bounded by $Ce^{-\beta n}$. The upper bound (3.8) follows. □

3.5. **Proof of Theorem 1.5 (5).** Consider a cocycle M admitting invariant continuous holonomies, which is pinching and twisting in the sense of Avila-Viana. We want to show that it admits exponential large deviations for all exponents.

[AV07] shows that there is a unique invariant u-state on $\mathbb{P}(E)$, corresponding to the maximally expanded Oseledets subspace, see the first lines of Section 7 in [AV07]. Hence, Theorem 3.10 applies and shows that M has exponential large deviations for its top exponent.

To prove exponential large deviations for an exponent i, a natural strategy would be to consider the cocycle $\Lambda^i M$ and prove that it has exponential large deviations for its top Lyapunov exponent. However, there is no reason why $\Lambda^i M$ should be twisting and pinching. What Avila and Viana prove in [AV07], however, is that M has a unique u-state on the Grassmannian of i-dimensional subspaces. All the arguments in the proof of Theorem 3.10 go through if one replaces everywhere the space $\mathbb{P}(E)$ by the corresponding Grassmannian. Then the Grassmannian version of Theorem 3.10 shows that M has exponential large deviations for the exponent i. □

3.6. **Proof of Theorem 1.5 (6).** Consider a two dimensional cocycle M admitting continuous holonomies, we want to show that it satisfies exponential large deviations for all exponents i. For $i = 2$, the norm $\|\Lambda^i M(x)\|$ is the absolute value of the determinant of $M(x)$. The desired estimate (1.1) involves an additive cocycle, the Birkhoff sums of the continuous function $\log|\det M(x)|$. Hence, (1.1) follows from the large deviations estimate for Birkhoff sums.

The only non-trivial case is $i = 1$, i.e., exponential large deviations for $\|M^n(x)\|$. If M admits a unique invariant u-state on $\mathbb{P}(E)$, then the result follows from Theorem 3.10, and we are done. If the two Lyapunov exponents of M are equal, then the result follows from Theorem 1.5 (1). The last case is when the Lyapunov exponents are distinct, but there are two different invariant u-states. The non-uniqueness implies that something fails if we try to follow the proof of Theorem 3.6. The only place in the proof of this theorem where we used the fact that the cocycle is locally constant is in the proof of Lemma 3.8. Without this assumption, the proof of this lemma constructs a family $\mathcal{V}(x)$ of subspaces (which are necessarily one-dimensional), invariant under the dynamics and the stable holonomy, but not necessarily under the unstable holonomy. In general, this prevents us from implementing the induction argument of Lemma 3.12 as the induced cocycle on $\mathcal{V}$ and the quotient cocycle do not admit invariant continuous holonomies any more. However, in the specific case of 2-dimensional cocycles, the induced cocycles and the quotient cocycles are both 1-dimensional. Therefore, they satisfy exponential large deviations thanks to Theorem 1.5 (1). Hence, the argument in Lemma 3.12 goes through to prove that the original cocycle also satisfies exponential large deviations. □

4. Exponential returns to nice sets for subadditive cocycles

The main statement of this section is the following theorem. Note that the assumptions of the theorem ensure that the function F below is finite almost everywhere, although it can be infinite on points which are not typical for μ. We are trying to control how large it will be along typical orbits, in a quantitative sense.

Theorem 4.1. *Let $T : X \to X$ be a continuous map preserving an ergodic probability measure μ on a compact space. Consider a subadditive cocycle $u : \mathbb{N} \times X \to \mathbb{R}$, such that $u(n,x)/n$ converges almost everywhere to 0, and $u(n,\cdot)$ is continuous for all n. Let also $\varepsilon > 0$. Define a function*

$$F(x) = \sup_{n \geqslant 0} |u(n,x)| - \varepsilon n.$$

Assume that u has exponential large deviations, and that the Birkhoff sums of continuous functions also have exponential large deviations.

Let $\delta > 0$. Then there exists $C > 0$ such that, for any $n \geqslant 0$,

$$\mu\{x \,:\, \mathrm{Card}\{j \in [0,n-1] \,:\, F(T^j x) > C\} \geqslant \delta n\} \leqslant C e^{-C^{-1} n}.$$

In the applications we have in mind, u will be of the form $u(n,x) = \log\|\Lambda^i M^{(n)}(x)\| - n(\lambda_1 + \cdots + \lambda_i)$, for some cocycle M with Lyapunov exponents λ_k. The points where $F(x) \leqslant C$ are the points where all the iterates of the cocycle are well controlled. Essentially, they belong to some Pesin sets (see Proposition 5.1 below for a precise version of this statement). Hence, the lemma will imply that most iterates of a point return often to Pesin sets, if the matrix cocycle has exponential large deviations for all exponents.

The proof is most conveniently written in terms of superadditive cocycles. Note that, in the lemma below, the definition of G resembles that of F in the theorem above, except for the lack of absolute value. Hence, the following lemma applied to $v(n,x) = -u(n,x) - n\varepsilon$ proves one of two inequalities in Theorem 4.1.

Lemma 4.2. *Let $T : X \to X$ preserve an ergodic probability measure μ on a compact space. Consider a superadditive cocycle $v : \mathbb{N} \times X \to \mathbb{R}$, such that $v(n,x)/n$ converges almost everywhere to $-\varepsilon < 0$, and $v(n,\cdot)$ is continuous for all n. Define a function*

$$G(x) = \sup_{n \geqslant 0} v(n,x).$$

Assume that v satisfies exponential large deviations, and that the Birkhoff sums of continuous functions also satisfy exponential large deviations.

Let $\delta > 0$. Then there exists $C > 0$ such that, for any $n \geqslant 0$,

$$\mu\{x \,:\, \mathrm{Card}\{j \in [0, n-1] \,:\, G(T^j x) > C\} \geqslant \delta n\} \leqslant C e^{-C^{-1} n}.$$

Proof. When N tends to $+\infty$, the sequence v_N/N tends almost surely to $-\varepsilon$. The convergence also holds in L^1 by Kingman's theorem. Then $v_N/N + \varepsilon$ tends almost surely and in L^1 to 0. Then $\min(v_N/N + \varepsilon, 0)$ tends almost surely and in L^1 to 0. Thus, we can take once and for all a large enough N so that

$$\int \min(v_N/N + \varepsilon, 0) \geqslant -\delta\varepsilon/10. \tag{4.1}$$

Let $w = v_N/N$.

By Lemma 3.1 applied to the subadditive cocycle $-v$, there exists a constant $C_0 > 0$ such that $v(n, x) \geqslant S_n w(x) - C_0$, for any $x \in X$ and any $n \in \mathbb{N}$. We will show that

$$\mu\{x \,:\, \mathrm{Card}\{j \in [0, n-1] \,:\, G(T^j x) > 2C_0\} \geqslant \delta n\} \leqslant C e^{-C^{-1} n}.$$

Assume first that x has an iterate where the cocycle is large along an extremely long interval, i.e., x belongs to

$$K_n = \bigcup_{t=0}^{n-1} (T^t)^{-1}\{y \,:\, \exists j \geqslant \delta n/2, v(j, y) > 0\}.$$

As v has exponential large deviations and converges to a negative constant, the last set has a measure which is exponentially small in n. As T is measure-preserving, it follows that $\mu(K_n)$ is also exponentially small.

Consider now $x \notin K_n$ such that $\mathrm{Card}\{j \in [0, n-1] \,:\, G(T^j x) > 2C_0\} \geqslant \delta n$. Then

$$\mathrm{Card}\{j \in [0, n-1-\delta n/2] \,:\, G(T^j x) > 2C_0\} \geqslant \delta n/2. \tag{4.2}$$

We define inductively a sequence of times t_k as follows. We start from $t_0 = 0$. If $G(T^{t_k} x) > 2C_0$ and $t_k \leqslant n - 1 - \delta n/2$, then we say that t_k belongs to the set U^+ of sum-increasing times. In this case, we can choose $n_k > 0$ such that $v(n_k, T^{t_k} x) > 2C_0$, by definition of H. Then we let $t_{k+1} = t_k + n_k$. Otherwise, we say that t_k belongs to the set U^- of sum-decreasing times, and we let $t_{k+1} = t_k + 1$. We stop at the first t_j where $t_j \geqslant n$.

Let $A^+ = \bigcup_{t_k \in U^+} [t_k, t_{k+1})$, and $A^- = [0, n-1] \backslash A^+$. As $x \notin K_n$, the lengths $n_k = t_{k+1} - t_k$ when $t_k \in U^+$ are all bounded by $\delta n/2$. Hence, A^+ is included in $[0, n-1]$. Moreover, the set of bad times, on the left of (4.2), is included in A^+. Therefore, $\mathrm{Card}\, A^+ \geqslant \delta n/2$, and $\mathrm{Card}\, A^- \leqslant (1 - \delta/2)n$.

We will also need to write the set A^- as a union of intervals $\bigcup [t'_j, t'_j + n'_j)$ over some index set J, i.e., we group together the times in U^- that are not separated by times in U^+.

Using the decomposition of $[0, n-1]$ as $A^+ \cup A^-$, the decomposition of these sets into intervals, and the superadditivity of the cocycle, we obtain the inequality

$$v(n, x) \geqslant \sum_{t_k \in U^+} v(n_k, T^{t_k} x) + \sum_{j \in J} v(n'_j, T^{t'_j} x) \geqslant \sum_{t_k \in U^+} 2C_0 + \sum_{j \in J} v(n'_j, T^{t'_j} x),$$

where the last inequality follows from the definition of U^+. Note that the right endpoint of an interval in A^- belongs to U^+, except for the last interval. It follows that $\mathrm{Card}\, J \leqslant$

$\operatorname{Card} U^+ + 1 \leqslant 2 \operatorname{Card} U^+$. Hence, the above inequality implies

$$v(n,x) \geqslant \sum_{j\in J} (C_0 + v(n'_j, T^{t'_j}x)).$$

Together with the definition of C_0, this gives

$$v(n,x) \geqslant \sum_{j\in J} S_{n'_j} w(T^{t'_j}x) = \sum_{k\in A^-} w(T^k x).$$

Now, let us introduce ε:

$$\begin{aligned} v(n,x) &\geqslant \sum_{k\in A^-} (w(T^k x) + \varepsilon) - \varepsilon \operatorname{Card}(A^-) \\ &\geqslant \sum_{k\in[0,n-1]} \min(w(T^k x) + \varepsilon, 0) - \varepsilon \operatorname{Card}(A^-) \\ &\geqslant \sum_{k\in[0,n-1]} \min(w(T^k x) + \varepsilon, 0) - \varepsilon(1-\delta/2)n, \end{aligned}$$

where the last inequality holds as $\operatorname{Card} A^- \leqslant (1-\delta/2)n$.

The continuous function $x \mapsto \min(w(x)+\varepsilon, 0)$ has exponential large deviations and integral $\geqslant -\delta\varepsilon/10$ by (4.1). Hence, we have $\sum_{k\in[0,n-1]} \min(w(T^k x) + \rho, 0) \geqslant -n\delta\varepsilon/5$ apart from an exponentially small set. Apart from this set, we obtain

$$v(n,x) \geqslant -\varepsilon n + (\delta/2 - \delta/5)\varepsilon n.$$

As v has exponential large deviations and asymptotic average $-\varepsilon$, it follows that this condition on x has exponentially small measure. □

Proof of Theorem 4.1. The function F is the maximum of the two functions

$$H(x) = \sup_{n\geqslant 0} -u(n,x) - n\varepsilon, \quad I(x) = \sup_{n\geqslant 0} u(n,x) - n\varepsilon.$$

We should show that each of these functions satisfies the conclusion of the theorem. For H, this follows from Lemma 4.2 applied to $v(n,x) = -u(n,x) - n\varepsilon$.

For I, let us consider $N > 0$ such that u_N/N has integral $< \varepsilon/2$. By Lemma 3.1, there exists a constant C_0 such that $u(n,x) \leqslant S_n(u_N/N) + C_0$ for all n. Let $w = u_N/N - \varepsilon$. Lemma 4.2 applied to the cocycle $S_n w$ shows that, for some constant $C_1 > 0$,

$$\mu\{x \,:\, \operatorname{Card}\{j \in [0,n-1] \,:\, \sup_n S_n w(T^j x) > C_1\} \geqslant \delta n\} \leqslant C e^{-C^{-1} n}.$$

If $u(n,x) - n\varepsilon \geqslant C_0 + C_1$, then $S_n w(x) \geqslant C_1$. Hence, the control on I follows from the previous equation. □

5. A deterministic control on the Pesin function

An important difficulty to prove Theorem 1.7 is that the Pesin function A_ε is defined in terms of the Oseledets subspaces $E_i(x)$, which vary only measurably with the point and for which we have no good control. On the other hand, Theorem 4.1 provides exponentially many returns for sets defined in terms of functions for which we have good controls, e.g., Birkhoff sums of continuous functions (by the large deviation principle) or norms of linear cocycles (if one can prove exponential large deviations for them, using for instance Theorem 1.5). Our goal in this section is to explain how controls on such quantities imply controls on the Pesin function A_ε. Then, Theorem 1.7 will essentially follow from Theorem 4.1. To prove such a result, we need to revisit the proof of Oseledets theorem and replace almost sure controls with more explicit bounds.

Consider an invertible map $T : X \to X$ preserving a probability measure μ, and a log-integrable linear cocycle M above T on $X \times \mathbb{R}^d$. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_d$ be its Lyapunov exponents, let $I = \{i : \lambda_i < \lambda_{i-1}\}$ be a set of indices for the distinct Lyapunov exponents, let E_i be the Lyapunov subspaces.

Given $\varepsilon > 0$, define functions

$$B_\varepsilon^+(x) = \sup_{i\in[1,d]} B_\varepsilon^{(i)+} = \sup_{i\in[1,d]} \sup_{n\geqslant 0} |\log\|\Lambda^i M^{(n)}(x)\| - n(\lambda_1 + \cdots + \lambda_i)| - n\varepsilon,$$

$$B_\varepsilon^-(x) = \sup_{i\in[1,d]} B_\varepsilon^{(i)-} = \sup_{i\in[1,d]} \sup_{n\leqslant 0} |\log\|\Lambda^i M^{(n)}(x)\| - n(\lambda_d + \cdots + \lambda_{d-i+1})| - |n|\varepsilon$$

and

$$B_\varepsilon(x) = \max(B_\varepsilon^+(x), B_\varepsilon^-(x)). \tag{5.1}$$

These are the functions we can control using the tools of the previous sections.

The following proposition asserts that a control on B_ε and a mild control on angles implies a control on $A_{\varepsilon'}$ for $\varepsilon' = 20d\varepsilon$. For $i \in I$, let us denote by $F_{\geqslant i}^{(m)}(x)$ the maximally contracted subspace of $M^{(m)}(x)$ of dimension $d - i + 1$, and by $F_{<i}^{(-m)}(x)$ the maximally contracted subspace of $M^{(-m)}(x)$ of dimension i, if these spaces are uniquely defined, as in the statement of Theorem 2.2.

Theorem 5.1. *Assume that $\|M(x)\|$ and $\|M(x)^{-1}\|$ are bounded uniformly in x. Consider $\varepsilon \in (0, \min_{i\neq j\in I}|\lambda_i - \lambda_j|/(20d))$ and $\rho > 0$ and $C > 0$. Then there exist $m_0 \in \mathbb{N}$ and $D > 0$ with the following properties.*

Consider a point x satisfying $B_\varepsilon(x) \leqslant C$. Then its subspaces $F_{\geqslant i}^{(n)}(x)$ and $F_{\leqslant i}^{(-n)}(x)$ are well defined for all $n \geqslant m_0$, and converge to subspaces $F_{\geqslant i}^{(\infty)}(x)$ and $F_{\leqslant i}^{(-\infty)}(x)$.

Assume additionally that, for all $i \in I$, there exists $m \geqslant m_0$ such that the angle between $F_{\geqslant i}^{(m)}(x)$ and $F_{<i}^{(-m)}(x)$ is at least ρ. Then the Oseledets subspace $E_i(x) = F_{\geqslant i}^{(\infty)}(x) \cap F_{\leqslant i}^{(-\infty)}(x)$ is a well-defined d_i-dimensional space for all $i \in I$. Moreover, the function $A_{20d\varepsilon}(x)$ (defined in (1.2) in terms of these subspaces) satisfies $A_{20d\varepsilon}(x) \leqslant D$.

Note that there is no randomness involved in this statement, it is completely deterministic.

The condition on B_ε controls separately what happens in the past and in the future. Oseledets subspaces are defined by intersecting flags coming from the past and from the future, as explained in Theorem 2.2. Therefore, it is not surprising that there should be an additional angle requirement to make sure that these flag families are not too singular one with respect to the other. Note that the angle requirement is expressed in terms of a fixed time m. Hence, it will be easy to enforce in applications.

In this section, we prove Theorem 5.1. Once and for all, we fix T, M and μ satisfying the assumptions of this theorem, and constants $C > 0$, $\varepsilon \in (0, \min_{i\neq j\in I}|\lambda_i - \lambda_j|/(20d))$ and

$\rho > 0$. Consider a point x satisfying $B_\varepsilon(x) \leqslant C$. We want to show that, if m is suitably large (depending only on C, ε and ρ), then the subspaces $F^{(m)}_{\geqslant i}(x)$ and $F^{(-m)}_{<i}(x)$ are well defined, and moreover if the angle between them is at least ρ, then $A_{20d\varepsilon}(x)$ is bounded by a constant D only depending on C, ε and ρ.

We will use the notations introduced before Theorem 2.2. In particular, $t^{(n)}_i(x) = e^{n\lambda^{(n)}_i(x)}$ is the i-th singular value of $M^n(x)$. We will essentially repeat the argument from the proof of a technical lemma in [Rue79]. A more detailed exposition is given in Section 2.6.2 in [Sar09].

Step 1: there exists $N_1 = N_1(C, \varepsilon)$ such that, if $n \geqslant N_1$, then $|\lambda^{(n)}_i(x) - \lambda_i| \leqslant 3\varepsilon$ for all i. In particular, thanks to the inequality $\varepsilon < \min_{i\neq j\in I}|\lambda_i - \lambda_j|/(20d)$, there is a gap between the eigenvalues $\lambda^{(n)}_j(x)$ in different blocks $\{i, \dots, i + d_i - 1\}$. (Note that the $20d$ is much larger than what we need here, 6 would be enough, but it will be important later on.) This implies that the different subspaces $(F^{(n)}_i(x))_{i\in I}$ are well defined.

Proof. We have $B_\varepsilon(x) \leqslant C$. Thanks to the equality $\log\|\Lambda^i M^n(x)\| = n(\lambda^{(n)}_1(x) + \dots + \lambda^{(n)}_i(x))$, and to the definition of B^+_ε, this gives for all i

$$n\left|\lambda^{(n)}_1(x) + \dots + \lambda^{(n)}_i(x) - (\lambda_1 + \dots + \lambda_i)\right| \leqslant \varepsilon n + C.$$

Subtracting these equations with indices i and $i - 1$, we get $|\lambda^{(n)}_i(x) - \lambda_i| \leqslant 2\varepsilon + 2C/n$. If n is large enough, this is bounded by 3ε as desired. □

From this point on, we will only consider values of n or m which are $\geqslant N_1$, so that the subspaces $F^{(n)}_i(x)$ are well defined. We will write $\Pi^{(n)}_i$ for the orthogonal projection on this subspace, and $\Pi^{(n)}_{\geqslant i}$ and $\Pi^{(n)}_{<i}$ for the projections on $\bigoplus_{j\in I, j\geqslant i} F^{(n)}_j(x)$ and $\bigoplus_{j\in I,j<i} F^{(n)}_j(x)$ respectively. They satisfy $\Pi^{(n)}_{\geqslant i} + \Pi^{(n)}_{<i} = \mathrm{Id}$.

Step 2: there exists a constant $K_1 = K_1(C, \varepsilon)$ such that, for all $m \geqslant n \geqslant N_1$, all $i > j$ in I and all $v \in F^{(n)}_{\geqslant i}(x)$, holds

$$\|\Pi^{(m)}_{\leqslant j} v\| \leqslant K\|v\|e^{-n(\lambda_j-\lambda_i-6(d-1)\varepsilon)}.$$

Proof. The proof is done in two steps.

First claim: there exists a constant K_0 such that, for $n \geqslant N_1$, $v \in F^{(n)}_{\geqslant i}(x)$ and $j < i$,

$$\|\Pi^{(n+1)}_j v\| \leqslant K_0\|v\|e^{-n(\lambda_j-\lambda_i-6\varepsilon)}.$$

Indeed, on the one hand, we have

$$\|M^{n+1}(x)v\| = \|M(T^n x)\cdot M^n(x)v\| \leqslant (\sup_y\|M(y)\|)\cdot\|M^n(x)v\| \leqslant (\sup_y\|M(y)\|)e^{n(\lambda_i+3\varepsilon)}\|v\|,$$

thanks to the first step and the fact that $v \in F^{(n)}_{\geqslant i}(x)$. On the other hand, as $M^{n+1}(x)$ respects the orthogonal decomposition into the spaces $F^{(n+1)}_k(x)$, we have

$$\|M^{n+1}(x)v\| \geqslant \|M^{n+1}(x)\Pi^{(n+1)}_j v\| \geqslant e^{(n+1)(\lambda_j-3\varepsilon)}\|\Pi^{(n+1)}_j v\|,$$

again thanks to the first step. Putting these two equations together gives the result.

Second claim: for all $j < i$ in I, there exists a constant $K_{i,j}$ such that, for all $m \geqslant n \geqslant N_1$ and all $v \in F^{(n)}_{\geqslant i}(x)$, we have

$$\|\Pi^{(m)}_{\leqslant j} v\| \leqslant K_{i,j} e^{-n(\lambda_j - \lambda_i - 6(i-j)\varepsilon)} \|v\|. \tag{5.2}$$

Once this equation is proved, then Step 2 follows by taking for K_1 the maximum of the $K_{i,j}$ over $j < i$ in I. To prove (5.2), we argue by decreasing induction over $j < i$, $j \in I$. Assume thus that the result is already proved for all $k \in I \cap (j, i)$, let us prove it for j.

Decomposing a vector v along its components on $F^{(m)}_{\leqslant j}(x)$, on $F^{(m)}_k(x)$ for $k \in I \cap (j,i)$ and on $F^{(m)}_{\geqslant i}(x)$, we get

$$\|\Pi^{(m+1)}_{\leqslant j} v\| \leqslant \|\Pi^{(m+1)}_{\leqslant j} \Pi^{(m)}_{\leqslant j} v\| + \sum_{k \in I \cap (j,i)} \|\Pi^{(m+1)}_{\leqslant j} \Pi^{(m)}_{k} v\| + \|\Pi^{(m+1)}_{\leqslant j} \Pi^{(m)}_{\geqslant i} v\|. \tag{5.3}$$

The first term is bounded by $\|\Pi^{(m)}_{\leqslant j} v\|$ as $\Pi^{(m+1)}_{\leqslant j}$ is a projection. The second term is bounded by $K_0 e^{-m(\lambda_j - \lambda_k - 6\varepsilon)} \|\Pi^{(m)}_k v\|$ thanks to the first claim applied to m and $\Pi^{(m)}_k v \in F^{(m)}_{\geqslant k}(x)$. The induction hypothesis asserts that $\|\Pi^{(m)}_k v\| \leqslant K_{k,i} e^{-m(\lambda_k - \lambda_i - 6(i-k)\varepsilon)} \|v\|$. Overall, we get for the second term a bound which is at most

$$\sum_{k \in I \cap (j,i)} K_0 K_{i,k} e^{-m(\lambda_j - \lambda_i - 6(i-k+1)\varepsilon)} \leqslant K' e^{-m(\lambda_j - \lambda_i - 6(i-j)\varepsilon)} \|v\|.$$

Finally, the third term in (5.3) is bounded by $K_0 e^{-m(\lambda_j - \lambda_i - 6\varepsilon)} \|\Pi^{(m)}_{\geqslant i} v\|$, by the first claim applied to m and $\Pi^{(m)}_{\geqslant i} v \in F^{(m)}_{\geqslant i}(x)$. This is bounded by $K_0 e^{-m(\lambda_j - \lambda_i - 6\varepsilon)} \|v\|$ as $\Pi^{(m)}_{\geqslant i}$ is a projection.

All in all, we have proved that

$$\|\Pi^{(m+1)}_{\leqslant j} v\| \leqslant (K' + K_0) e^{-m(\lambda_j - \lambda_i - 6(i-j)\varepsilon)} \|v\| + \|\Pi^{(m)}_{\leqslant j} v\|.$$

The estimate (5.2) then follows by induction over m, summing the geometric series starting from n as $\lambda_j - \lambda_i - 6(i-j)\varepsilon > 0$ thanks to the choice of ε. □

The second step controls projections from $F^{(n)}_i$ to $F^{(m)}_j$, for $m \geqslant n$, when $i > j$. The third step controls projections in the other direction, thus giving a full control of the respective projections of the spaces.

Step 3: for all $m \geqslant n \geqslant N_1$, all $i > j$ in I and all $v \in F^{(n)}_{\leqslant j}$, holds

$$\|\Pi^{(m)}_{\geqslant i} v\| \leqslant K_1 \|v\| e^{-n(\lambda_j - \lambda_i - 6(d-1)\varepsilon)}.$$

Proof. Define a new matrix cocycle by $\tilde{M}(x) = (M^{-1}(x))^t$, from $E^*(x)$ to $E^*(Tx)$. In coordinates (identifying $E(x)$ and $E^*(x)$ thanks to its Euclidean structure), it is given as follows. Write $M^n(x)$ as $k_1 A k_2$ where k_1 and k_2 are orthogonal matrices, and A is a diagonal matrix with entries $t^{(n)}_1(x) = e^{n\lambda_1(n)(x)}, \dots, t^{(n)}_d(x) = e^{n\lambda_d(n)(x)}$. Then $\tilde{M}^n(x) = k_1 A^{-1} k_2$. Hence, it has the same decomposition into singular spaces as $M^n(x)$, the difference being that the singular values of $M^n(x)$ are replaced by their inverses.

The proof in Step 2 only used the fact that the logarithms of the singular values were 3ε-close to λ_i, and the norm of the cocycle is uniformly bounded. All these properties are shared by $\tilde{M}$. Hence, the conclusion of Step 2 also applies to $\tilde{M}$, except that the inequality between i and j have to be reversed as the ordering of singular values of $\tilde{M}$ is the opposite of that of M. This is the desired conclusion. □

Overall, Steps 2 and 3 combined imply that the projection of a vector in $F_i^{(n)}(x)$ on $(F_i^{(m)}(x))^\perp = F_{<i}^{(m)}(x) \oplus F_{>i}^{(m)}(x)$ has a norm bounded by $2K_1 e^{-\delta n}$, for $\delta = \min_{k \neq \ell \in I} |\lambda_k - \lambda_\ell| - 6(d-1)\varepsilon > 0$. Hence, in terms of the distance $\mathbf{d}$ on the Grassmannian of d_i-dimensional subspaces defined in (2.1), we have $\mathbf{d}(F_i^{(n)}(x), F_i^{(m)}(x)) \leqslant 2K_1 e^{-\delta n}$. It follows that $F_i^{(n)}(x)$ is a Cauchy sequence, converging to a subspace $F_i^{(\infty)}(x)$ as claimed in the statement of the theorem.

Step 4: there exist $N_2 \geqslant N_1$ and a constant K_2 such that, for all $n \geqslant N_2$, all i in I and all $v \in F_i^{(\infty)}$ with norm 1, holds

$$K_2^{-1} e^{n(\lambda_i - 6d\varepsilon)} \leqslant \|M^n(x)v\| \leqslant K_2 e^{n(\lambda_i + 6d\varepsilon)}. \tag{5.4}$$

Proof. Take a vector $v \in F_i^{(\infty)}(x)$. For $j \in I$, the norm of the projection $\pi_{F_j^{(n)}(x) \to F_i^{(\infty)}(x)}$, as the limit of the projections $\pi_{F_j^{(n)}(x) \to F_i^{(m)}(x)}$, is bounded by $K_1 e^{-n(|\lambda_i - \lambda_j| - 6(d-1)\varepsilon)}$ thanks to Steps 2 and 3 (note that this bound is nontrivial only if $j \neq i$). Its transpose, the projection $\pi_{F_i^{(\infty)}(x) \to F_j^{(n)}(x)}$, has the same norm and therefore satisfies the same bound.

Writing $v_j = \pi_{F_i^{(\infty}(x) \to F_j^{(n)}(x)} v$, we have $M^n(x)v = \sum_{j \in I} M^n(x) v_j$. We have

$$\|v_j\| \leqslant K_1 e^{-n(|\lambda_i - \lambda_j| - 6(d-1)\varepsilon)}. \tag{5.5}$$

As $M^n(x)$ expands by at most $e^{n\lambda_j + 3\varepsilon}$ on $F_i^{(n)}(x)$, thanks to Step 1, we obtain

$$\|M^n(x)v_j\| \leqslant K_1 e^{-n(|\lambda_i - \lambda_j| - 6(d-1)\varepsilon)} e^{n\lambda_j + 3\varepsilon} \leqslant K_1 e^{n(\lambda_i + 6d\varepsilon)}.$$

Here, it is essential to have in Step 2 a control in terms of $\lambda_j - \lambda_i$, and not merely some exponentially decaying term without a control on the exponent. This proves the upper bound in (5.4).

For the lower bound, we write $\|M^n(x)v\| \geqslant \|M^n(x)v_i\|$ as all the vectors $M^{(n)}(x)v_j$ are orthogonal. This is bounded from below by $e^{n(\lambda_i - 3\varepsilon)}\|v_i\|$, by Step 1. To conclude, it suffices to show that $\|v_i\|$ is bounded from below by a constant if n is large enough. As $\|v_i\| \geqslant \|v\| - \sum_{j \neq i} \|v_j\|$, this follows from the fact that $\|v_j\|$ tends to 0 with n if $j \neq i$, thanks to (5.5). □

We recall that we are trying to control the behavior of $M^n(x)$ not on $F_i^{(\infty)}(x)$, but on the Oseledets subspace $E_i(x) = F_{\geqslant i}^{(\infty)}(x) \cap F_{\leqslant i}^{(\infty)}(x)$. To this effect, there is in the statement of Theorem 5.1 an additional angle assumption that we will use now. Let $\rho > 0$ be given as in the statement of the theorem. There exists $\delta > 0$ with the following property: if U and V are two subspaces of complementary dimension making an angle at least ρ, then any subspaces U' and V' with $\mathbf{d}(U, U') \leqslant \delta$ and $\mathbf{d}(V, V') \leqslant \delta$ make an angle at least $\rho/2$.

We fix once and for all $m_0 = m_0(C, \varepsilon, \delta) \geqslant N_2$ such that, for all $i \in I$ and all $m \geqslant m_0$, one has $\mathbf{d}(F_{\geqslant i}^{(m)}(x), F_{\geqslant i}^{(\infty)}(x)) \leqslant \delta$ and $\mathbf{d}(F_{\leqslant i}^{(-m)}(x), F_{\leqslant i}^{(-\infty)}(x)) \leqslant \delta$. Its existence follows from the convergence asserted at the end of Step 3 (and from the same result for T^{-1}).

Assume now (and until the end of the proof) that, for some $m \geqslant m_0$, the angle between $F^{(m)}_{\geqslant i}(x)$ and $F^{(-m)}_{<i}(x)$ is $\geqslant \rho$, as in the assumptions of the theorem. It follows then that the angle between $F^{(\infty)}_{\geqslant i}(x)$ and $F^{(-\infty)}_{<i}(x)$ is at least $\rho/2$. As a consequence, the spaces $F^{(\infty)}_{\geqslant i}(x)$ and $F^{(-\infty)}_{\leqslant i}(x)$ are transverse, and their intersection is a d_i-dimensional space $E_i(x)$.

Step 5: there exist constants $K_3 > 0$ and $N_3 \geqslant N_2$ such that, for all $n \geqslant N_3$, all $i \in I$ and all $v \in E_i(x)$ with norm 1, holds

$$K_3^{-1} e^{n(\lambda_i - 6d\varepsilon)} \leqslant \|M^n(x)v\| \leqslant K_3 e^{n(\lambda_i + 6d\varepsilon)}. \tag{5.6}$$

Proof. We have $v \in E_i(x) \subseteq F^{(\infty)}_{\geqslant i}(x)$. Decomposing the vector v along its components $v_j \in F^{(\infty)}_j(x)$ with $j \in I \cap [i, d]$ and using the upper bound of (5.4) for each v_j, the upper bound in (5.6) readily follows.

For the lower bound, we note that $E_i(x)$, being contained in $F^{(-\infty)}_{\leqslant i}(x)$, makes an angle at least $\rho/2$ with $F^{(\infty)}_{>i}(x)$. This implies that the norm of the projection v_i of v on $F^{(\infty)}_i(x)$ is bounded from below, by a constant $c_0 > 0$. Using both the upper and the lower bounds of Step 4, we obtain

$$\|M^n(x)v\| \geqslant \|M^n(x)v_i\| - \sum_{j \in I, j>i} \|M^n(x)v_j\| \geqslant c_0 K_2^{-1} e^{n(\lambda_i - 6d\varepsilon)} - \sum_{j \in I, j>i} K_2 e^{n(\lambda_j + 6d\varepsilon)}.$$

The choice of ε ensures that, for $j > i$ in I, one has $\lambda_i - 6d\varepsilon > \lambda_j + 6d\varepsilon$. Hence, the sum in this equation is asymptotically negligible, and we obtain a lower bound $c_0 K_2^{-1} e^{n(\lambda_i - 6d\varepsilon)}/2$ if n is large enough. □

Step 6: there exists a constant K_4 such that, for all $n \in \mathbb{Z}$, all $i \in I$ and all $v \in E_i(x)$ with norm 1, holds

$$K_4^{-1} e^{n\lambda_i - 6d\varepsilon|n|} \leqslant \|M^n(x)v\| \leqslant K_4 e^{n\lambda_i + 6d\varepsilon|n|}. \tag{5.7}$$

Proof. Step 5 shows that this control holds uniformly over $n \geqslant N_3$. The same argument applied to the cocycle M^{-1} and the map T^{-1} gives the same control for $n \leqslant -N_3$ (note that the function $B_\varepsilon(x)$, which is bounded by C by assumption, controls both positive and negative times). Finally, the control over $n \in (-N_3, N_3)$ follows from the finiteness of this interval, and the uniform boundedness of M and M^{-1}. □

We can finally conclude the proof of Theorem 5.1. We want to bound the quantity $A_{20d\varepsilon}(x)$ defined in (1.2). Fix $i \in I$, $v \in E_i(x) \setminus \{0\}$ and $m, n \in \mathbb{Z}$. Then, using the upper bound of (5.7) for $\|M^n(x)v\|$ and the lower bound for $\|M^m(x)v\|$, we get

$$\begin{aligned}
&\frac{\|M^n(x)v\|}{\|M^m(x)v\|} e^{-(n-m)\lambda_i} e^{-(|n|+|m|)(20d\varepsilon)/2} \\
&\qquad \leqslant K_4 e^{n\lambda_i + 6d\varepsilon|n|} \cdot K_4 e^{-m\lambda_i + 6d\varepsilon|m|} \cdot e^{-(n-m)\lambda_i} e^{-(|n|+|m|)(20d\varepsilon)/2} \\
&\qquad = K_4^2 e^{-(|n|+|m|)4d\varepsilon} \leqslant K_4^2.
\end{aligned}$$

Taking the supremum over $i \in I$, $v \in E_i(x) \setminus \{0\}$ and $m, n \in \mathbb{Z}$, this shows that $A_{20d\varepsilon}(x) \leqslant K_4^2$. This concludes the proof, for $D = K_4^2$. □

6. Exponential returns to Pesin sets

In this section, we prove Theorem 1.7. As in the assumptions of this theorem, let us consider a transitive subshift of finite type T, with a Gibbs measure μ and a Hölder cocycle M which has exponential large deviations for all exponents. Let $\delta > 0$. We wish to show that, for some $D > 0$, the set

$$\{x \,:\, \mathrm{Card}\{k \in [0, n-1] \,:\, A_\varepsilon(T^k x) > D\} \geqslant \delta n\}$$

has exponentially small measure. Reducing ε if necessary, we can assume $\varepsilon < |\lambda_i - \lambda_j|$ for all $i \neq j \in I$. Set $\varepsilon' = \varepsilon/(20d)$.

The angle between the Lyapunov subspaces is almost everywhere nonzero. In particular, given $i \in I$, the angle between $F^{(\infty)}_{\geqslant i}(x)$ and $F^{(-\infty)}_{<i}(x)$ is positive almost everywhere. On a set of measure $> 1 - \delta/2$, it is bounded from below by a constant $2\rho > 0$ for all i. These subspaces are the almost sure limit of $F^{(m)}_{\geqslant i}(x)$ and $F^{(-m)}_{<i}(x)$, according to Theorem 2.2. Hence, if m is large enough, say $m \geqslant m_1$, the set

$$U = U_m = \{x \in X \,:\, \forall i \in I, F^{(m)}_{\geqslant i}(x) \text{ and } F^{(-m)}_{<i}(x) \text{ are well defined} \\ \text{and } \angle(F^{(m)}_{\geqslant i}(x), F^{(-m)}_{<i}(x)) > \rho\}$$

has measure $> 1 - \delta/2$.

We will use the functions $B^{(i)\pm}_{\varepsilon'}$ defined before (5.1). For each $i \in [1, d]$ and $\sigma \in \{+, -\}$, there exists a constant $C_{i,\sigma}$ such that

$$\{x \,:\, \mathrm{Card}\{k \in [0, n-1] \,:\, B^{(i)\sigma}_{\varepsilon'}(T^k x) > C_{i,\sigma}\} \geqslant \delta n/(4d)\}$$

has exponentially small measure, by Theorem 4.1 and the assumption on exponential large deviations for all exponents. (For $\sigma = -$, this theorem should be applied to T^{-1}). Let $C' = \max C_{i,\sigma}$. As $B_{\varepsilon'}$ is the maximum of the functions $B^{(i)\sigma}_{\varepsilon'}$, it follows that

$$\{x \,:\, \mathrm{Card}\{k \in [0, n-1] \,:\, B_{\varepsilon'}(T^k x) > C'\} \geqslant \delta n/2\}$$

has exponentially small measure.

We apply Theorem 5.1 with $\varepsilon = \varepsilon'$ and $C = C'$ and ρ, obtaining some integer $m_0 \geqslant 1$ and some constant D with the properties described in Theorem 5.1. Let us fix until the end of the proof $m = \max(m_0, m_1)$.

The set $U = U_m$ is open by continuity of M^m and M^{-m}. In particular, it contains a set V which is a finite union of cylinders, with $\mu(V) > 1 - \delta/2$. To conclude, it suffices to show that

$$\{x \,:\, \mathrm{Card}\{k \in [0, n-1] \,:\, T^k x \notin V\} \geqslant \delta n/2\} \tag{6.1}$$

has exponentially small measure. Indeed, assume this holds. Then, apart from an exponentially small set, there are at most δn bad times k in $[0, n-1]$ for which $T^k x \notin V$ or $B_{\varepsilon'}(T^k x) > C'$. For the other good times, we have $T^k x \in V$ and $B_{\varepsilon'}(T^k x) \leqslant C'$. Then Theorem 5.1 shows that $A_\varepsilon(T^k x) = A_{20d\varepsilon'}(T^k x) \leqslant D$, as desired.

It remains to control (6.1). Let χ_V denote the characteristic function of V, it is a continuous function. The set in (6.1) is

$$\{x \,:\, S_n \chi_V(x) < (1 - \delta/2)n\}.$$

As $\int \chi_V = \mu(V) > 1 - \delta/2$ by construction, the large deviation principle for continuous functions shows that this set is indeed exponentially small. This concludes the proof of the theorem. □

Appendix A. Counterexamples to exponential large deviations

In this appendix, we give two counterexamples to exponential large deviations. The first easy one, in Proposition A.1, is for Hölder-continuous subadditive cocycles. The second harder one, in Theorem A.3, is in the more restrictive setting of norms of matrix cocycles (only continuous, although one expects that the same kind of result should hold for Hölder cocycles with small Hölder exponent).

Proposition A.1. *Let (T, μ) be an invertible subshift with an invariant ergodic measure μ which is not supported on a periodic orbit. Consider a positive sequence u_n tending to 0. There exists a subadditive cocycle $a(n, x)$ such that $a(n, \cdot)$ is Hölder continuous for any n, such that $a(n, x)/n \to 0$ almost everywhere, and such that, for infinitely many values of n,*

$$\mu\{x \,:\, a(n, x)/n \leqslant -1\} \geqslant u_n.$$

The proof uses the following easy variant of Rokhlin's lemma:

Lemma A.2. *Let $\delta > 0$ and $m > 0$. In a subshift in which the set of periodic points has measure 0, there exists a subset R made of finitely many cylinders such that the sets $(T^i R)_{0 \leqslant i < m}$ are pairwise disjoint and cover a measure at least $1 - \delta$.*

Proof. We may find a set S such that its m first iterates are disjoint and cover a measure $\geqslant 1 - \delta/2$, by Rokhlin's lemma. Let S' be a finite union of cylinders which approximates S so well that $\mu(S' \Delta S) \leqslant \rho$, for $\rho = \delta/(4m^2)$. Let $R = S' \setminus \bigcup_{0<i<m} T^i(S')$. It is a finite union of cylinder sets, and the sets $T^i R$ for $i < m$ are disjoint. We have

$$S' \cap T^i(S') \subseteq (S' \Delta S) \cup (S \cap T^i S) \cup (T^i S \Delta T^i S').$$

The middle set is empty, the other ones have measure at most ρ. Hence, the measure of this set is at most 2ρ. Finally, $\mu(R) \geqslant \mu(S') - 2(m-1)\rho \geqslant \mu(S) - 2m\rho$. Hence.

$$\mu\left(\bigcup_{0 \leqslant i < m} T^i R\right) = m\mu(R) \geqslant m\mu(S) - 2m^2\rho = \mu\left(\bigcup_{0 \leqslant i < m} T^i S\right) - 2m^2\rho \geqslant 1 - \delta/2 - 2m^2\rho.$$

The choice of ρ ensures that the last term is $1 - \delta$, as claimed. □

Proof of Proposition A.1. We will construct a sequence $n_i \to \infty$ and a sequence of functions f_i for $i \geqslant 1$ with the following properties:

1. Each f_i is Hölder continuous (in fact, it will only depend on finitely many coordinates).
2. We have $f_i(x) \leqslant 0$ for all x, and $\int f_i = -2^{-i}$.
3. We have $\mu\{x \,:\, S_{n_i} f_i(x) \leqslant -2n_i\} \geqslant u_{n_i}$.

Let also $f_0 = 1$ and $n_0 = 0$. Define then

$$a(n, x) = \sum_{i \,:\, n_i \leqslant n} S_n f_i(x).$$

As the $(f_i)_{i\geqslant 1}$ are nonpositive, this is a subadditive cocycle. Moreover, $\int a(n,x)/n = \sum_{n_i\leqslant n}\int f_i \to 0$. By Kingman's theorem, it follows that $a(n,x)/n$ tends to 0 almost surely. Moreover, if $S_{n_i}f_i(x) \leqslant -2n_i$, then by nonpositivity of all the f_j except for $j=0$,

$$a(n_i,x) \leqslant S_{n_i}f_0(x) + S_{n_i}f_i(x) \leqslant n_i - 2n_i \leqslant -n_i.$$

Hence, the third point in the definition of f_i ensures that $a(n_i,x)\leqslant -n_i$ with probability at least u_{n_i}, showing that a satisfies the conclusion of the proposition.

Let us now construct f_i and n_i as above. First, choose $n=n_i$ such that $u_{n_i}\leqslant 2^{-i-3}$. Then, let $K=2^{i+2}n_i$. We use a corresponding Rokhlin tower: by Lemma A.2, there exists a set R which is a finite union of cylinder sets such that $R,\dots,T^{K-1}R$ are disjoint, and their union covers a proportion $>1/2$ of the space. Then $\mu(R)\in(1/(2K),1/K]$. Define f_i to be equal to $-c_i$ on $\bigcup_{k<K/2^{i+1}}T^kR$ and 0 elsewhere, where c_i is chosen so that $\int f_i = -2^{-i}$. As $\mu(\bigcup_{k<K/2^{i+1}}T^kR) = (K/2^{i+1})\mu(R)\leqslant 2^{-i-1}$, it satisfies $c_i\geqslant 2$. For any $x\in\bigcup_{k<K/2^{i+2}}T^kR$, one has $f_i(T^kx)=-c_i$ for $k<K/2^{i+2}=n_i$, and therefore $S_{n_i}f_i(x) = -c_in_i\leqslant -2n_i$. The probability of this event is $\mu\left(\bigcup_{k<K/2^{i+2}}T^kR\right) = (K/2^{i+2})\mu(R)\geqslant 2^{-i-3}\geqslant u_{n_i}$, as desired. □

We will now construct a continuous cocycle taking values in $SL(2,\mathbb{R})$ without exponential large deviations for its top exponent. Note that a generic continuous cocycle away from uniform hyperbolicity has only zero Lyapunov exponents, by Bochi-Viana [BV05], so it has exponential large deviations by Theorem 1.5 (1). Hence, our construction can not be done using Baire arguments.

Theorem A.3. *Let u_n be any positive sequence tending to 0. Consider the full shift on two symbols with a fully supported invariant ergodic measure μ. Then there exists a continuous $SL(2,\mathbb{R})$-valued cocycle M with a positive top Lyapunov $\lambda_+(M)$ such that, for infinitely many values of n,*

$$\mu\{x\,:\,\log\|M^n(x)\|\leqslant n\lambda_+(M)/2\}\geqslant u_n.$$

If u_n tends to zero slower than exponentially, for instance $u_n=1/n$, then the cocycle M does not have exponential large deviations.

Let Σ be the full shift over two symbols 0 and 1, with a given invariant ergodic measure μ of full support (what we really need is that the support of μ contains a fixed point, or more generally a periodic orbit, but μ is not supported on this orbit). In this section, we will say that an object defined on Σ is locally constant if it only depends on $(x_n)_{|n|\leqslant N}$ for some N. Let $x_*\in\Sigma$ be the point with all coordinates equal to 1. We say that a cocycle M taking values in $SL(2,\mathbb{R})$ has property P_λ, for some $\lambda>0$, if it satisfies the following properties:

(1) The cocycle M is locally constant.
(2) Its largest Lyapunov exponent is $>\lambda$.
(3) Its Oseledets subspaces, initially defined μ-almost everywhere, are in fact locally constant (and therefore continuous).
(4) One has $M(x_*)=\mathrm{Id}$.

Define a cocycle M_0 by $M_0(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ if $x_0 = 0$, and $M_0(x) = \mathrm{Id}$ if $x_0 = 1$. Then its Oseledets subspaces are $\mathbb{R} \oplus \{0\}$ and $\{0\} \oplus \mathbb{R}$, and the corresponding Lyapunov exponents are nonzero. Hence, M_0 satisfies P_λ for some $\lambda > 0$.

The main lemma is the following:

Lemma A.4. *Let $\lambda > 0$ and $\varepsilon > 0$ and $n_0 > 0$. Let M be a cocycle with the property P_λ. Then there exist a time $n > n_0$ and another cocycle $\tilde{M}$, again having the property P_λ, with the following properties:*

(1) *For all x, one has $\|\tilde{M}(x) - M(x)\| \leqslant \varepsilon$.*
(2) *There exists a set A with measure $\geqslant u_n$ on which $\|\tilde{M}^n(x)\| < e^{\lambda n/2}$.*

Let us admit the lemma for the time being. We construct inductively a sequence of cocycles M_i, all with the property P_λ, starting with M_0 as above. Suppose that we have already constructed times $n_1, \dots, n_{i-1}$, sets $A_1, \dots, A_{i-1}$ with $\mu(A_j) \geqslant u_{n_j}$, and the cocycle M_{i-1} such that, for each $j < i$, $\|M_{i-1}^{n_j}(x)\| < e^{\lambda n_j/2}$ for all $x \in A_j$. We wish to construct a time $n_i > n_{i-1}$, a set A_i and a cocycle M_i that satisfies the same properties for all $j \leqslant i$. Note that, if $\varepsilon = \varepsilon_i$ is small enough, then any cocycle M_i with $\|M_i(x) - M_{i-1}(x)\| \leqslant \varepsilon$ for all x will satisfy the above properties for $j < i$, with the same sets A_j. Hence, it suffices to apply Lemma A.4 to $M = M_{i-1}$, with a sufficiently small ε, to get $M_i = \tilde{M}$.

We can require $\varepsilon_i \leqslant 1/2^i$. Then the sequence M_i converges uniformly, towards a limiting continuous cocycle $M(x)$. By semi-continuity of the Lyapunov exponents, $\lambda_+(M) \geqslant \limsup \lambda_+(M_i) \geqslant \lambda$. On the other hand, $\|M^{n_j}(x)\| \leqslant e^{\lambda n_j/2}$ for all $x \in A_j$, and this set has measure at least u_{n_j} as claimed. This concludes the proof of Theorem A.3. □

It remains to prove Lemma A.4. The main tool to modify the cocycle is the following lemma, due to Bochi.

Lemma A.5. *Assume that the cocycle M satisfies P_λ. Let $\varepsilon > 0$. Then, for almost every x, there exist $k(x) \in \mathbb{N}$ and matrices $Q_0, \dots, Q_{k-1}$ such that $\|Q_i - M(T^i x)\| \leqslant \varepsilon$ for all $i < k$, and the product $Q_{k-1} \cdots Q_0$ sends $E^u(x)$ to $E^s(T^k x)$, and $E^s(x)$ to $E^u(T^k x)$ (where E^s and E^u are the stable and unstable Oseledets directions of the cocycle M).*

Proof. The set A of points that satisfy the conclusion of the lemma is backwards invariant under the dynamics: if $Tx = y$ and the sequence of matrices $Q_0, \dots, Q_{k-1}$ works for y, then the sequence of matrices $\mathrm{Id}, Q_0, \dots, Q_{k-1}$ works for x, for $k(x) = k(y) + 1$. By ergodicity, it suffices to show that A has positive measure. This follows from [Via14, Proposition 9.10], as the cocycle M is not uniformly hyperbolic thanks to the condition $M(x_*) = \mathrm{Id}$ in $P(4)$. (In our case, there is a direct easy proof as the cocycle is the identity on a neighborhood of the fixed point x_*, so it can be replaced by a small rotation in suitable coordinates, on points whose orbit spends a long enough time close to x_*). □

Proof of Lemma A.4. The idea is to apply Lemma A.5 at some points, modifying the cocycle along a piece of orbit of length k, and then again the same lemma n steps later (for some n much larger than k), to put again E^s in line with E^s, and E^u in line with E^u. The norm of the new cocycle will essentially not increase along these n steps thanks to the cancellations between the stable and unstable directions, yielding the desired set A, while the Lyapunov exponent will essentially not be changed if these n steps are negligible compared to the whole dynamics. Making this precise requires the use of the Rokhlin tower provided by Lemma A.2, and some care when choosing the constants.

The cocycle M and its Oseledets subspaces are constant on cylinders of length $2N+1$, for some N, by assumption. Replacing the original subshift by a new subshift the symbols of which correspond to $2N+1$-cylinders of the original subshift, we may assume without loss of generality that $N=0$, i.e., the cocycle $M(x)$ and the Oseledets subspaces $E^s(x)$ and $E^u(x)$ only depend on the coordinate x_0 of x.

The minimal function $k(x)$ provided by Lemma A.5 is measurable. Hence, it is bounded on a set of arbitrarily large measure. We obtain an integer $k>0$, a set X with $\mu(X)>9/10$, and for each $x\in X$ a sequence of matrices $Q_0(x),\dots,Q_{k-1}(x)$ with

$$\|Q_i(x)-M(T^i x)\| \leqslant \varepsilon \tag{A.1}$$

whose product $Q_{k-1}(x)\cdots Q_0(x)$ maps $E^s(x)$ to $E^u(T^k x)$ and $E^u(x)$ to $E^s(T^k x)$.

Let $\lambda_+(M)>\lambda$ be the top Lyapunov exponent of M. Let $\delta>0$ be small enough so that $14\delta<\lambda$. For μ-almost every x, there exists a constant $C(x)<\infty$ such that, for all $\ell\in\mathbb{Z}$

$$C(x)^{-1}e^{-\delta|\ell|} \leqslant \frac{\|M^\ell(x)v^u(x)\|}{e^{\lambda_+(M)\ell}} \leqslant C(x)e^{\delta|\ell|}, \quad C(x)^{-1}e^{-\delta|\ell|} \leqslant \frac{\|M^\ell(x)v^s(x)\|}{e^{-\lambda_+(M)\ell}} \leqslant C(x)e^{\delta|\ell|},$$

where $v^u(x)$ and $v^s(x)$ are unit vectors in $E^u(x)$ and $E^s(x)$. Shrinking X just a little bit, we can assume that $C(x)$ is bounded by a constant C_0 on X, while retaining the estimate $\mu(X)>9/10$.

As the Oseledets subspaces depend continuously on the point, by $P_\lambda(3)$, the angle between $v^u(x)$ and $v^s(x)$ is bounded from below. Hence, increasing C_0 if necessary, we can ensure that, for any matrix A and any x,

$$\|A\| \leqslant C_0 \max(\|Av^u(x)\|, \|Av^s(x)\|). \tag{A.2}$$

Increasing C_0 and shrinking X if necessary, we can also assume that, for any $x\in X$, the global modification matrix at x given by $\tilde{Q}(x)=M^k(x)^{-1}Q_{k-1}(x)\cdots Q_0(x)$ (which exchanges $E^u(x)$ and $E^s(x)$) expands all vectors by at most C_0, and contracts them by at most C_0^{-1}.

Let $n\geqslant k$ be such that $C_0\leqslant e^{\delta n}$. Let $m=Kn$, where $K\geqslant 6$ will be chosen later, independently of n. Applying Lemma A.2, we obtain a set R which is a finite union of cylinders, whose first m iterates are disjoint and cover a measure $>9/10$. Subdividing R further if necessary, we may write it as a disjoint union of cylinders R_p of length $2r+1$, centered around 0, for some $r\geqslant m+k$. Let $O_p=\bigcup_{i<m}T^iR_p$, these sets are disjoint. We will make the modifications of the cocycle separately on each O_p.

The point x_* is in at most one O_p. If it belongs to O_1, say, then we remove R_1 from R. Increasing r if necessary, this removes an arbitrarily small measure from R, so the new R will still satisfy the condition $\mu(\bigcup_{j<m}T^jR)>9/10$. This means that modifying the cocycle on the sets O_p will not change its value on x_*, so that the condition $M(x_*)=\mathrm{Id}$ in $P_\lambda(4)$ will still be satisfied by the modified cocycle.

We say that a set O_p is modifiable if there exists an index $a_p\in[0,m-3n)$ such that $T^{a_p}R_p$ intersects $X\cap T^{-n}X$. If O_p is not modifiable, then the set $\tilde{O}_p=\bigcup_{a<m-3n}T^aR_p$ (whose measure is at least $\mu(O_p)/2$ as $m-3n\geqslant m/2$) does not intersect $X\cap T^{-n}X$. Hence,

the union of these $\tilde{O}_p$ has measure at most $1 - \mu(X \cap T^{-n}X) \leqslant 2/10$, the union of the corresponding O_p has measure at most $4/10$, and the measure of the union of the modifiable O_p is at least $9/10 - 4/10 = 1/2$.

Let O_p be modifiable. Choose a point $x_p \in T^{a_p}R_p \cap X \cap T^{-n}X$. On O_p, we define the cocycle $\tilde{M}$ to be equal to $Q_i(x_p)$ on $T^{a_p+i}R_p$ for $0 \leqslant i < k$, to $Q_i(T^n x_p)$ on $T^{a_p+n+i}R_p$ for $0 \leqslant i < k$, and to M elsewhere. The cocycle M is constant on each set $T^i R_p$ (as $M(x)$ only depends on x_0, and R_p is a cylinder of length $2r+1$ with $r > m$). Hence, it follows from (A.1) that $\|\tilde{M}(x) - M(x)\| \leqslant \varepsilon$ everywhere. Moreover, it is clear from the construction that $\tilde{M}$ is locally constant.

Let us show that the Lyapunov exponent of $\tilde{M}$ is $> \lambda$. Start from a point x which is not in the modified locus $\bigcup_p \bigcup_{a_p \leqslant i < a_p+n+k} T^i R_p$, we will estimate the expansion of $\tilde{M}^\ell(x) v^u(x)$ when ℓ tends to ∞. Except when $T^\ell x$ belongs to the modified locus, the vector $\tilde{M}^\ell(x) v^u(x)$ is a multiple of $v^u(T^\ell x)$, and undergoes the same expansion under M or $\tilde{M}$. The difference is the influence of the modified locus: when one enters this locus, then one should apply the modification operator $\tilde{Q}(x_p)$ which brings $v^u(x_p)$ to $v^s(x_p)$ (with an expansion at least C_0^{-1}), then the original cocycle $M^n(x_p)$ but on the vector $v^s(x_p)$, then the modification operator $\tilde{Q}(T^n x_p)$ that brings back $v^s(T^n x_p)$ to $v^u(T^n x_p)$ (again with an expansion at least C_0^{-1}). Then, one follows again the dynamics of the cocycle M. During such a visit to the modified locus, the expansion under $\tilde{M}$ is at least $C_0^{-1} \cdot C_0^{-1} e^{-\lambda_+(M)n - \delta n} \cdot C_0^{-1}$, while the expansion under M is at most $C_0 e^{\lambda_+(M)n + \delta n}$. Hence, the expansion loss for $\tilde{M}$ with respect to M is at most $C_0^{-4} e^{-2\lambda_+(M)n - 2\delta n} \geqslant e^{-2\lambda_+(M)n - 6\delta n}$. Moreover, such a loss happens at most once in every m steps, since a visit to O_p has length m by construction. We get

$$\lambda_+(\tilde{M}) \geqslant \lambda_+(M) - (2\lambda_+(M) + 6\delta)n/m.$$

By assumption, $\lambda_+(M) > \lambda$. If the ratio $K = m/n$ is large enough, it follows that one also has $\lambda_+(\tilde{M}) > \lambda$.

The same argument shows that, towards the past, $v^u(x)$ is exponentially contracted. Hence, $v^u(x)$ generates the Oseledets subspace $E^u(x)$ for $\tilde{M}$. This shows that, away from the modified locus, the Oseledets subspace is locally constant. Using its equivariance under $\tilde{M}$ and the fact that $\tilde{M}$ is locally constant, we deduce that the Oseledets subspace of $\tilde{M}$ is locally constant everywhere.

We have proved that $\tilde{M}$ satisfies P_λ. It remains to show the existence of a set A with measure $\geqslant u_n$ on which $\|\tilde{M}^n(x)\| < e^{n\lambda/2}$. We take for A the union of the sets $T^{a_p+i}R_p$ over $i \in [n/2 - \delta n/\lambda_+(M), n/2 + \delta n/\lambda_+(M)]$ and p such that O_p is modifiable. In each modifiable set, A takes a proportion $(2\delta n/\lambda_+(M))/m = 2\delta/(K\lambda_+(M))$. As the measure of modifiable sets O_p is at least $1/2$, we get $\mu(A) \geqslant \delta/(K\lambda_+(M))$, a number which is independent of n. In particular, if n is large enough, we get $\mu(A) \geqslant u_n$ as u_n tends to 0 with n.

Consider $x \in A$, let us show that $\|\tilde{M}^n(x)\| < e^{n\lambda/2}$ to conclude the proof. Consider p and $i = n/2 + j$ with $|j| \leqslant \delta n/\lambda_+(M)$ such that $x \in T^{a_p+i}R_p$. First, we estimate the norm of $\tilde{M}^n(x)v^s(x)$. This vector is obtained by iterating the original cocycle M during $n - i$ steps, then doing the modification $\tilde{Q}(T^n x_p)$ that brings it to $v^u(T^n x_p)$, and then iterating the original cocycle M during i steps. The first step results in an expansion by at most $C_0 e^{-\lambda_+(M)(n-i)+\delta(n-i)}$ (as $T^n x_p \in X$), the second one by an expansion at most C_0, and the third one by an expansion at most $C_0 e^{\lambda_+(M)i+\delta i}$. In the end, we obtain

$$\|\tilde{M}^n(x)v^s(x)\| \leqslant C_0^3 e^{\delta n} e^{-\lambda_+(M)(n-i)+\lambda_+(M)i} \leqslant C_0^3 e^{\delta n} e^{2\lambda_+(M)|j|} \leqslant C_0^3 e^{3\delta n}.$$

In the same way, $v^u(x)$ is expanded by at most $C_0 e^{\lambda_+(M)(n-i)+\delta(n-i)}$ during the first $n-i$ iterates, then by at most C_0 by the modification $\tilde{Q}(T^n x_p)$ that brings it to $v^s(T^n x_p)$, and then by at most $C_0 e^{-\lambda_+(M)i+\delta i}$ for the last i iterates. Hence,

$$\|\tilde{M}^n(x)v^u(x)\| \leqslant C_0^3 e^{\delta n} e^{\lambda_+(M)(n-i)-\lambda_+(M)i} \leqslant C_0^3 e^{3\delta n}.$$

With (A.2), this gives

$$\|\tilde{M}^n(x)\| \leqslant C_0^4 e^{3\delta n} \leqslant e^{7\delta n} < e^{n\lambda/2},$$

thanks to the choice of δ. □

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Laboratoire Jean Leray, CNRS UMR 6629, Université de Nantes, 2 rue de la Houssinière, 44322 Nantes, France

E-mail address: sebastien.gouezel@univ-nantes.fr

School of Mathematics and Statistics, University of Western Australia, Crawley 6009 WA, Australia
E-mail address: luchezar.stoyanov@uwa.edu.au