THE JOINT HEDGING AND LEVERAGE DECISION

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Abstract

The validating roles of hedging and leverage as value-adding corporate strategies arise from their beneficial manipulation of deadweight market impositions such as taxes and financial distress costs. These roles may even be symbiotic in their value-adding effects, but they are antithetic in their effects on company risk. This study's modelling analysis indicates that hedging and leverage do interact for net benefit to company value; for sensible base-case exogenous parameters, the optimal (value-maximising) joint hedging and leverage strategy increases company value by about 4.0% compared to the unhedged optimal leverage strategy, by about 1.3% compared to the unlevered optimal hedge strategy, and by about 4.0% compared to the company being unlevered and unhedged. Furthermore an optimal joint hedging and leverage strategy is less financially risky than an unhedged optimal leverage strategy or an unhedged and unlevered strategy, and is often less financially risky than an unlevered optimal hedge strategy. Interestingly, the optimal joint hedging and leverage strategy entails some risk-seeking hedge reversal in response to weak price outcomes for production output.

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1. INTRODUCTION

The seemingly patent roles of corporate leverage and hedging are respectively to provide finance and reduce financial risk. However, in getting beyond a Modigliani and Miller (1958) irrelevance argument, their validating roles as value-adding corporate strategies arise from their beneficial manipulation of deadweight market impositions such as taxes and financial distress costs. Expositions in this regard include trade-off theory with respect to leverage, and the work of Smith and Stulz (1985) with respect to hedging.

Ross (1996) took the further step of considering the interrelation of the hedging and capital structure decisions and proposed that hedging facilitates higher optimal leverage and thereby allows firms to access greater tax shield benefits. While hedging and leverage are potentially symbiotic in their value-adding effects, they are antithetic in their effects on company risk: hedging reduces a company's financial risk ceteris paribus, but it is indeterminate whether the optimal joint hedging and leverage decision should be associated with increased, decreased, or unchanged financial risk compared to that when the company is optimally unhedged and levered, or hedged and unlevered, or unhedged and unlevered. This motivates the following research aims:

1. To verify or otherwise the value-adding benefit of the joint hedging and leverage decision (compared variously to the unhedged leverage decision, the unlevered hedging decision, and the unhedged and unlevered decision).

2. To indicate how the value-maximising joint hedging and leverage decision affects leverage (compared to the unhedged leverage decision).

3. To indicate how the value-maximising joint hedging and leverage decision affects hedging (compared to the unlevered hedging decision).

4. To determine whether the value-maximising joint hedging and leverage decision (compared variously to the unhedged leverage decision, the unlevered hedging decision, and the unhedged and unlevered decision) is associated with higher, lower or generally unchanged financial risk. Five measures are used to assess financial risk: the value of equity's comprehensive limited liability option, equity's value-at-risk, equity's beta with respect to the underlying production output price, and conditional and unconditional probability of bankruptcy.

The research method utilises a multi-period model for a company subject to respectively hedgeable and unhedgeable production output price and quantity risk variables, endogenously derived deadweight costs, and the tandem availability of risky leverage and flexible hedging control variables. With due concern for the realism of exogenous parameter values, the model is applied as a theoretical tool to investigate both the value and risk impacts of a joint hedging and leverage decision in the presence of deadweight impositions in the forms of taxation, agency costs of free cash-flow, and costs of financial distress and bankruptcy. The model's representation of hedging and leverage control and motivation offers favourable innovation compared with previous modelling approaches concerned with the hedging decision in the presence of leverage (e.g. Leland (1998), Mello and Parsons (2000) and Fehle and Tsyplakov (2005)). The modelling method crucially accommodates intertemporal interdependence of optimal hedging and optimal leverage. That is, optimal hedging and optimal leverage are not only enabled to interact contemporaneously given underlying price and risk conditions for production output; they are also enabled to interact intertemporally with historic hedging and leverage decisions and the expectation of future hedging and leverage decisions.

For sensible base-case exogenous parameters, it is found that the optimal (value-maximising) joint hedging and leverage strategy increases company value by about 4.0% compared to the unhedged optimal leverage strategy, by about 1.3% compared to the unlevered optimal hedge strategy, and by about 4.0% compared to the company being unlevered and unhedged. Also found is that optimal leverage is usually much higher in conjunction with optimal hedging than with no hedging, but the relationship is not purely a matter of higher hedging facilitating higher leverage.

At outset a jointly optimal hedging and leverage strategy markedly boosts the levels of both hedging and leverage, compared to optimal hedging without leverage, and optimal leverage without hedging respectively. Then as long as the price for production output is at least rising modestly, the hedging level is largely maintained, while leverage tends to fall due to maturing-hedge losses substituting for debt as a tax shield. For conversely weak output price outcomes close to the unit cost of production, the ongoing operational value of the company is low, but the hedge portfolio is deep in-the-money and facilitative of increased leverage, driven by pecking order roll-over of hedging-boosted debt. Low ongoing operational value imposes a net cost/benefit asymmetry with respect to subsequent output price outcome, because an output price fall (below unit production cost) will lead to abandonment/bankruptcy regardless of hedge portfolio value, but an output price rise will warrant continued operation and production. However the operational benefit of a subsequent output price rise is vulnerable to the financial risk posed by high leverage (in the presence of unhedgeable production quantity risk) and the fact that the hedge portfolio would lose value; that is, an output price rise actually entails adversely high risk of bankruptcy and consequential foregone profitable production. To counter this, some reversal of the overall hedge position increases the likelihood that the company will avoid bankruptcy in event of an output price rise and thereby benefit from ongoing profitable production.[1] The value-maximising optimality of this risk-seeking behaviour arises in conjunction with fair compensation to debt-holders and other non-equity stakeholders for limited liability risk.

For base-case exogenous parameters and a majority of sensitivity analysis scenarios, optimal hedging with or without optimal leverage always entails less financial risk (or equal financial risk at a minimum value of zero for some measures) than the unhedged strategies with or without leverage. Furthermore, while output price is generally rising, jointly optimal hedging and leverage is generally less risky or no more risky than optimal hedging without leverage; but is more risky for weak output price outcomes, for which jointly optimal hedging and leverage is associated with high leverage and hedge reducing (i.e. risk-seeking) behaviour.

A generally qualitative overview of the model company's design features and application set-up is presented in the following Chapter 2 for consideration with reference to the exacting design details provided in Appendix A. Chapter 3 presents and analyses the results of application of the model, and Chapter 4 concludes the thesis.

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[1] This somewhat counterintuitive behaviour corresponds with empirical evidence from Graham and Rogers (2002) who found that firms reduce hedging in response to accumulated operating losses, and Brown, Crabb and Haushalter (2006) who found a tendency for gold miners to reduce hedging with falling gold price.

2. MODELLING APPROACH

I construct a multi-period model of a company, ostensibly a resource producer, subject to hedgeable price risk and unhedgeable quantity risk for its production output. The company has to contend with income tax, agency costs of free cash-flow, and costs of financial distress and bankruptcy, and can manipulate its exposure to these deadweight costs via hedging and leverage control decisions. The model is detailed in Appendix A. The intention in the model design is to define the company's equity and non-equity stakes as complex, interdependent, controlled contingent claims (effectively American-style options) on the underlying production output price and quantity random variables (p_t and q_t respectively). Thereby an option-pricing approach can be used to value equity and non-equity and to assess the effects of hedging and leverage control decisions.

The model company's non-equity stakeholders are debt finance providers, hedge contract providers, providers of production labour and equipment, and providers of direct bankruptcy services. All non-equity stakeholders provide valuable service without certainty of full compensation in event of bankruptcy, thereby providing equity with its limited liability option. A prominent and important feature of the model design is that at the beginning of every production period, equity purchases from non-equity a fairly priced comprehensive limited liability option exercisable for the ensuing period. The periodic up-front expense for the limited liability option will vary depending on the financial risk being faced by non-equity each period; this expense effectively grosses together the risk premiums that would be charged by individual non-equity stakeholders. The overall non-equity stake is then defined to be a combination of the risk-free value of the company's debt plus the short value of the comprehensive limited liability option (which acts to deduct debt's risk premium as well as the risk premiums of all other non-equity stakeholders from the risk-free debt valuation); the non-equity value is therefore a broader measure than risky debt value.

Risk variables

The model company faces two sources of uncertainty: at time t, being the end of a discrete production period of duration Δt years, the production output for the period is an uncertain quantity of q_t units, and the price obtainable for the production output is an uncertain amount of p_t dollars per unit. The bivariate price and quantity process is assumed to be lognormal Markovian. Hedging can only be contracted with respect to price uncertainty, hence the price risk is hedgeable and the quantity risk is not.

The company is assumed to have uniform production periods, and periodic expected production quantity is assumed to be independent of previous unexpected production quantity deviations. This allows the company to be specified as blind to the history of the stochastic component of production quantities, which advantageously allows equity and non-equity, conditional on price and control variables, to have generalisable analytic valuation solutions with respect to quantity uncertainty.

While the market price for production output (p_t) is defined as a lognormal process, for the sake of model implementation it is approximated by a binomial process. Within production periods, p_t evolves by an n-step recombining binomial tree. However, at the end of each production period the price-tree is specified as non-recombining so that path-dependent hedging and leverage control behaviour is achievable (i.e. there are $(n+1)$ price-nodes at the end of the first production period, $(n+1)^2$ after period two, and so on up to $(n+1)^N$ after period N). Correlation between production output price and quantity (ρ) is used to represent the preference and ability of the company to adjust expected production quantity concurrently with the trend of the market price for its output.

Control variables

The model company is specified to have N production periods for which control decisions can be made, each of duration Δt years. Time (t) is denominated accordingly: $t \in \{0, \Delta t, 2\Delta t, ..., N\Delta t\}$. The company has available to it a flexible set of hedging and leverage control variables. The periodic leverage decision allows the issue of risky zero-coupon bonds with different maturities and individually specifiable face-values. Similarly the periodic hedging decision allows individually specifiable hedge quantities for different maturities. The available hedge contracts are short forwards, long put options or a ratio combination of the two (however a negative hedge quantity implies an 'anti'-hedge consisting of long forwards and/or short put options).

The computational complexity of optimising for all the possible hedging and leverage choices available to the model company compels some limits on control behaviour. The number of controlled production periods is limited to three ($N = 3$). The number of possible output price outcomes for each period is limited to five (i.e. the price evolves by a four-step binomial tree within each period). Hence there are $5^3 = 125$ non-recombining price-paths for the three controlled production periods. Also the periodic leverage decision is simplified by restricting the allowable debt maturities to a single production period; effectively the total face-value of outstanding debt is renegotiated each period with a commensurately adjusted (i.e. floating) limited liability risk premium.

Reiterating from Appendix A, define $y_{t,t+\Delta t} \geq 0$ to be the total face-value of zero-coupon debt issued at time t and maturing after a single production period; $x_{t,t+\kappa\Delta t}$ to be the hedge quantity contracted at time t and maturing after κ production periods; $0 \leq w_{t,t+\kappa\Delta t} \leq 1$ to be the ratio choice anywhere between an all-put hedge ($w_{t,t+\kappa\Delta t} = 0$) and an all-short forward hedge ($w_{t,t+\kappa\Delta t} = 1$); and $z_{t,t+\kappa\Delta t} > 0$ to be the strike price of the put options. The complete set of control variables is: at $t = 0$, one specification of $\{ (w_{0,\Delta t}, w_{0,2\Delta t}, w_{0,3\Delta t})$; $(x_{0,\Delta t}, x_{0,2\Delta t}, x_{0,3\Delta t})$; $y_{0,\Delta t}$; $(z_{0,\Delta t}, z_{0,2\Delta t}, z_{0,3\Delta t}) \}$; at $t = \Delta t$, five path-dependent specifications of $\{ (w_{\Delta t,2\Delta t}, w_{\Delta t,3\Delta t})$; $(x_{\Delta t,2\Delta t}, x_{\Delta t,3\Delta t})$; $y_{\Delta t,2\Delta t}$; $(z_{\Delta t,2\Delta t}, z_{\Delta t,3\Delta t}) \}$; and, at $t = 2\Delta t$, 25 path-dependent specifications of $\{ w_{2\Delta t,3\Delta t}$; $x_{2\Delta t,3\Delta t}$; $y_{2\Delta t,3\Delta t}$; $z_{2\Delta t,3\Delta t} \}$. Noteworthy is that, each period, the maturities of new hedge positions (with positive or negative hedge quantities) may overlap with previously established hedge positions so as to increase or decrease the overall hedge position for any particular maturity.

Each production period the company can abandon operations by making an ‘extreme’ hedging or leverage control decision which radically increases financial risk and makes the required up-front expense for the comprehensive limited liability option unviable, thereby triggering immediate bankruptcy (in which case the extreme hedging or leverage decision does not actually get instigated). Such voluntary bankruptcy (i.e. abandonment) will not necessarily result in loss for non-equity stakeholders and may be desirable when the output price drops so low as to make ongoing production economically unviable.

2.1. Leverage and hedging imperatives

The modelling approach developed for this study is attractive for its inclusion of a range of features that allow the model company to be controlled in respect of several theoretical imperatives. Modigliani and Miller's (1958) demonstration of the irrelevance of capital structure under the condition of a 'perfect market' dictates that acceptance of capital structure relevance must presume deviation(s) from the perfect market condition. Likewise must be the presumption for acceptance of corporate hedging relevance. To this effect, the model company is established as subject to: deadweight cost (taxation) of income attributable to equity; deadweight cost of financial distress; deadweight cost of bankruptcy; and deadweight cost of free cash-flow.

Taxation

The assumption of asymmetric corporate taxation, entailing tax deductibility of interest payments to lenders but (some degree of) non-deductibility of income attributable to equity, potentially allows firms to add value by using debt finance to reduce their corporate tax burdens. The actual value benefit to any firm of debt finance as a tax shield depends on the corporate tax rate and system (generally classical versus imputation systems), and the personal tax rates for debt and equity income for a 'marginal' investor indifferent between debt and equity income.[2] The 'effective' rate of tax shielded by an additional unit of corporate debt derives from the additional post-tax value of attributing a dollar of pre-tax corporate income to the marginal investor as an interest payment for debt finance as opposed to a dividend plus capital gain return for equity finance. Under a classical tax system this marginal tax advantage of debt finance over equity finance is:

$$A = (1-\tau_{PD}) - (1-\tau_C)(1-\tau_{PE})$$

where τ_C is the marginal expected corporate tax rate, and τ_{PD} and τ_{PE} are respectively the personal tax rates on debt and equity income for the marginal investor. How equity

[2] Although firms and investment intermediaries can take debt and equity stakes in each other, the marginal investor is an individual amongst all of the individuals that are the ultimate suppliers of all debt and equity finance (n.b. government can arguably be considered to also be amongst these individuals).

income is split between dividends and capital gains and the investor's preferences for realising capital gains add complexity to the determination of τ_{PE}.

For a dividend imputation tax system, corporate taxation can be claimed by (some) investors as a credit for personal taxation on grossed-up dividend income comprising the cash dividend plus imputed tax credit. The combined corporate and personal tax burden for equity finance depends on the firm's dividend payout ratio (θ), the imputation credit ratio (ω), and the ability of investors to access the face-value of dividend imputation credits; defining ν as the marginal investor's valuation ratio for imputation credits, the marginal tax advantage of debt finance is:

$$A_{imputation} = (1-\tau_{PD}) - \left\{(1-\theta)(1-\tau_C)\left(1-\tau_{PE(capital\,gains)}\right) + \theta\left[1-(1-\omega\nu)\tau_C\right]\left(1-\tau_{PE(dividends)}\right)\right\}.$$

Miller (1977) argued that firms, desirous of debt finance over equity finance, will act competitively and increase the interest rates they offer to attract more lenders until the marginal investor offers no tax benefit for debt over equity (i.e. for whom τ_{PD} and τ_{PE} are such that, for *any* individual firm, regardless of its 'before debt finance' marginal corporate tax rate ($\tau_{C,before\,debt\,finance}$), there will be no 'before debt finance' marginal tax advantage of debt finance over equity finance (symbolised as $A(\tau_{C,before\,debt\,finance}) \leq 0$ for all firms individually)). Consequently lenders capture all of the corporate tax shield benefit of debt and, although an equilibrium aggregate level of debt across all firms obtains, individual firms receive no benefit from their capital structure choices.

DeAngelo and Masulis (1980) counter-argued Miller (1977) by considering the availability of non-debt tax shields combined with the potential for excessive tax shields to go unutilised. Non-debt tax shields like depreciation expenses and carry-forward losses will arise in conjunction with the normal operations of firms and will therefore tend to be available to reduce taxable income ahead of the discretionary issue of debt for tax shield purposes. Furthermore, as a generalisation, tax law restricts the realisation of the nominal value of tax shields when taxable income is negative (possibly in association with bankruptcy). Thus, for individual firms, after firstly allowing for non-debt tax shields to be claimed against income, the marginal expected benefit of debt as a tax shield may decline rapidly as debt level increases; since higher debt makes an earnings loss (negative taxable income) more probable and tax shields less valuable at the margin. Such considerations will cause individual firms to limit their demand for debt finance such that the aggregate may potentially be satisfied by lenders up to a marginal investor who does offer a tax benefit for debt over equity (i.e. for whom $A(\tau_{C,before\,debt\,finance}) > 0$ for at least some firms). The critical issue is whether the supply of investors that satisfy $A(\tau_{C,before\,debt\,finance}) > 0$ for at least some firms is sufficient for all firms to reach their individual leverage equilibriums before Miller's (1977) aggregate equilibrium is reached.

The actual process by which any tax benefit of debt financing is capitalised into the value of a firm has two opposing elements: the cash expected to be saved from corporate tax (i.e. higher expected post-corporate tax, pre-financing cash-flows); and a market cost for debt finance that has been grossed-up to reflect the marginal investor's personal tax penalty for debt income relative to equity income (assuming $\tau_{PD} > \tau_{PE}$).[3] To establish whether there is any tax benefit to be had from debt financing, it is problematic that the marginal investor's characteristics are not specifically ascertainable. Additionally confounding is that the relevant corporate tax rate is the marginal expected rate after consideration of all in-place and expected tax shields. An individual firm's marginal expected corporate tax rate will be an idiosyncratic function of the convolutions of tax law, such as carry-forward and carry-back tax shield provisions for earnings losses, applied to past and expected future earnings.

Under the taxation regime established for the model company, so long as the company is not bankrupt, positive earnings before tax is subject to the full corporate tax rate (α), while negative earnings before tax is subject to a partial corporate tax rate ($\lambda\alpha$, where $0 \le \lambda \le 1$ represents the claimability of a tax refund in event of an earnings loss). That is, instead of allowing carry-forward or carry-back of earnings losses as tax shields, the model's taxation regime gives the company a partial but immediate tax benefit (refund) for an earnings loss. The tax shields that are available each production period are simply the period-specific expenses used to calculate earnings. The resulting model set-up is such that the company's marginal expected corporate tax rate arises and adjusts endogenously through time in respect of the risk variables and in response to control decisions.

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[3] As a generalisation of tax law, relatively favourable personal tax conditions for capital gains mean that the overall personal tax rate for equity income can be expected to be lower than for debt income.

The model company's primary non-debt tax shield is total production costs. The company can be considered to lease (for operation) rather than own the necessary physical assets for production, hence depreciation tax shields are implicitly included in total production costs. Hedging outcomes net of transaction costs are also included in taxable earnings. Otherwise the company's debt tax shield is incorporated as part of a broader expense calculated each period and symbolised by ($y_{t,t+\Delta t} - Y_t$) at time t, where: $y_{t,t+\Delta t}$ is the face-value of newly issued debt (with maturity of a single production period); and Y_t, termed the risky measure of new debt proceeds, equals the short value of equity's comprehensive limited liability option ($O_{t+} \leq 0$) plus the risk-free value of the newly issued debt with personal tax penalty adjustment. Effectively the interest expense for debt is calculated and tax deducted immediately upon issue in combination with the expense for the limited liability option each period.

The effective marginal debt tax shield rate (A) for a firm with high demand for tax shields (i.e. for which the marginal expected corporate tax rate equals the full corporate tax rate, $\tau_C = \alpha$) indicates the upper limit for that part of the full corporate tax rate that effectively gets shielded by debt finance relative to equity finance after lenders are compensated for their personal tax penalty. Symbolising this upper tax shield effectiveness limit for A (with $\tau_C = \alpha$) as A_α, the part of the full corporate tax rate that is converse to A_α (i.e. $(\alpha - A_\alpha)$) is that part of the full corporate tax rate shield that gets passed on to lenders as a higher rate of return in compensation for their personal tax penalty. Dictating the model company to be valued on a pre-personal tax on equity basis, the risky measure of new debt proceeds incorporates a deduction for debt's personal tax penalty:[4]

$$Y_t = y_{t,t+\Delta t} e^{-r\Delta t} + O_{t+} - (y_{t,t+\Delta t} - Y_t)(\alpha - A_\alpha) = \frac{O_{t+} + y_{t,t+\Delta t}\left[e^{-r\Delta t} - (\alpha - A_\alpha)\right]}{1 - (\alpha - A_\alpha)}$$

where: r is termed the risk-free interest rate but is more correctly described as the pre-personal tax risk-free rate of return for equity; and ($y_{t,t+\Delta t} - Y_t$) represents taxable income to non-equity comprising risk-free return on the debt plus receipt for the compre-

[4] Equation (A.15) in Appendix A provides the more general formulation for debt made up of multiple face-values with differing maturities.

hensive limited liability option purchased by equity. Because the comprehensive limited liability option grosses together the premiums charged by all non-equity stakeholders (not just lenders) for limited liability risk, application of debt's personal tax penalty rate ($\alpha - A_\alpha$) within the risky measure of new debt proceeds assumes that all non-equity income is subject to a personal tax regime equivalent to that for debt income.

As a result of the model's overall taxation set-up, no direct assumption is made about whether there will be any net tax benefit from any level of debt finance at any stage of the company's operations. The net tax benefit to be had from each incremental dollar of debt finance, whether positive, zero or negative, arises endogenously, but is, however, affected by the exogenous choices for the λ, α and A_α parameters. A higher value for the claimability of a tax refund for a loss (λ) increases the marginal expected corporate tax rate and makes debt finance more attractive (as a result of the tax rate for a loss being higher, meaning a higher tax refund in event of a loss and lower potential financial penalty from being highly levered); similarly a higher value for the full corporate tax rate (α) makes debt finance more attractive (for its tax shield effect); whereas a higher value for the personal tax penalty against debt's corporate tax shield rate ($\alpha - A_\alpha$) increases the cost of debt finance and makes it less attractive.

Due to the tangle of details routinely associated with tax law, an empirically appropriate value for α may be less than perfectly manifest. For example, under the US classical tax system the federal corporate tax rate varies across rising earnings bands to become a flat rate of 35% (as of 1993) for earnings above a relatively modest threshold; additionally more irksome is the raft of different state corporate tax rates that can apply (but which are deductible at the federal level). Under an imputation tax system, additional empirical ambiguity arises because it is appropriate to adjust α according to dividend payout ratio (θ), the imputation credit ratio (ω), and the marginal investor's imputation credit valuation ratio (v):

$$\alpha_{imputation} = (1-\theta)\alpha + \theta(1-\omega v)\alpha .$$

There is also considerable empirical ambiguity about what are appropriate values for λ and A_α. The intention for λ is to reflect, on average, corporate tax law provisions for the carry-forward and carry-back of earnings losses for tax shield purposes. Under the US tax system an earnings loss can be carried backward for up to three years and forward for up to 15 years. Hence a US company that is generally profitable suffers relatively little opportunity cost for excess tax shields in event of an earnings loss, which tends to support a relatively high value for λ; but in turn this may encourage more aggressive use of tax shields (i.e. leverage), making earnings losses more likely and increasing the expected opportunity cost of excess tax shields, thereby lowering the appropriate value for λ.

Evidence on historic values for A for the US tax system was obtained by Graham (1999) using a simulation procedure and meticulous application of historic tax laws to estimate marginal expected corporate tax rates for individual firms on the COMPUSTAT database for each year from 1980 to 1994. This was done with careful consideration of non-debt tax shields for both a before and after debt financing basis. Each year's median value for the marginal expected corporate tax rate after non-debt tax shields but before debt financing was consistently at or very close to the top corporate tax rate (i.e. $\tau_{C,before\,debt\,finance} \approx \alpha$). With further careful assumptions about the marginal investor's personal tax rates, Graham also estimated the yearly marginal tax advantage of debt finance for individual firms before any debt finance ($A(\tau_{C,before\,debt\,finance})$) and after actual historic debt finance ($A(\tau_{C,after\,debt\,finance})$). The median value of $A(\tau_{C,before\,debt\,finance})$ ranged between 0.075 and 0.102 for the years 1982 to 1994, but was considerably lower, though still positive, for 1980 and 1981; this indicates a generally consistent and substantial tax advantage for debt finance. Furthermore, the median value of $A(\tau_{C,after\,debt\,finance})$ ranged between -0.002 and 0.018 for the years 1986 to 1994, and was 0.012 for 1980 and 1981, but was notably higher for 1982 to 1985; this is generally consistent with a trade-off theory explanation being that firms take on debt finance up to the point where the marginal tax benefit is zero or, due to offsetting marginal disbenefits, slightly positive.[5]

While leverage can offer benefit as a tax shield, hedging can offer benefit by reducing exposure to adverse tax asymmetry as demonstrated by Smith and Stulz (1985). In the

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[5] That the trade-off equilibrium for leverage should occur when the marginal tax benefit of debt is zero or *slightly* positive is consistent with Miller's (1977) suspicion that the expected bankruptcy cost disbenefit of debt is a minor concern. Specifically Miller described the capital structure trade-off between tax gains and bankruptcy costs as looking like "the recipe for the fabled horse-and-rabbit stew - one horse and one rabbit"!

case of the model company, when the claimability of a tax refund for a loss (λ) is less than one, the company faces an adverse (convex) tax refund versus tax expense asymmetry. That is, the tax refund for bad future financial states (negative earnings) is outweighed by the tax expense for equivalently probable good future financial states (positive earnings). Hedging, in essence, reduces the variability of future aggregate cashflows and thereby reduces the likelihood of *both* good and bad future financial states (i.e. hedging reduces financial risk). By this effect, hedging can be used to reduce the company's expected tax burden.

Financial distress and bankruptcy

Trade-off theory suggests that firms will individually take on debt finance up until the marginal expected benefit equals marginal expected disbenefits. The most prominently espoused disbenefit to firms of higher debt levels is higher likelihood of suffering costs of financial distress and bankruptcy.[6] Andrade and Kaplan (1998) investigated a sample of firms that became financially distressed after undertaking highly leveraged transactions (HLTs, i.e. capital restructuring that greatly increased leverage) during the 1980s. To isolate financial distress from economic distress, Andrade and Kaplan limited their sample to firms that maintained positive operating margins while financially distressed. They estimated the costs of financial distress to be about 10% to 20% of pre-distress firm value. For comparison's sake, in contrast to the broad costs of financial distress, the direct costs of bankruptcy were estimated by Weiss (1990) to be 3.1% of firm value on average. While these cost estimates clearly represent a considerable ex-post encumbrance, the probability of a typical firm experiencing financial distress or bankruptcy is small, and thus so are the expected costs of financial distress and bankruptcy.

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[6] Financial distress entails difficulty or inability to satisfy financial obligations in a timely manner. There can be several mechanisms by which financial distress reduces the value of a firm: trade creditors, customers and employees who have a stake in the firm as a going concern and fear bankruptcy may behave more restrictively towards the firm; management time and resources will be diverted to the financial predicament; some debt covenants may be broken leading to financial or operational penalties; profitable investment opportunities may have to be foregone since new finance will be more difficult to obtain; and valuable assets may have to be sold at depressed prices. These financial distress costs will also generally apply once an official state of external administration or bankruptcy is declared. There will also be additional explicit (direct) legal and administrative costs associated with bankruptcy.

For the model company, occurrence of financial distress or bankruptcy, as distinct from a state of solvency, is signalled by the value of periodic free cash-flow (F_t), being the overall net cash-flow from operations, hedging, debt and taxation resulting ultimately from control decisions and the outcomes for production output price and quantity. Positive or zero free cash-flow is a state of solvency, in which case any positive free cash-flow is paid out as a dividend to equity (i.e. it is assumed that the company does not retain any earnings or any capital from the issue of debt). Negative free cash-flow signals financial distress and necessitates new equity finance (i.e. a negative dividend, which can be conceptualised as a rights issue). Bankruptcy (and consequential liquidation) occurs when the free cash-flow shortfall is so large that the required amount of new equity finance cannot be justified by the ongoing equity value (E_{t+}). By this method the states of financial distress and bankruptcy are determined endogenously.

In event of negative free cash-flow ($F_t < 0$), for the model company to be able to access future earnings potential it must finance the free cash-flow shortfall ($-F_t$); in such case the trade-off optimum for new debt finance takes into account the costliness of resorting to new equity finance. Note that free cash-flow includes cash-flow from new debt finance but excludes cash-flow from new equity finance. The model is set up to appeal to Myers' (1984) pecking order theory to the extent that endogenous resorting to new equity finance is defined to signify financial distress, and accordingly a financial distress cost is applied with such occurrence. This pushes the leverage trade-off optimum more in favour of debt finance to cover a free cash-flow shortfall and makes new equity finance more of a last resort.[7] The financial distress cost is applied as a factor ($\gamma \geq 0$) of the free cash-flow shortfall so that equity-holders must invest $-F_t(1+\gamma)$ to finance a

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[7] The leverage trade-off optimum will also be forward looking and give balance to the risk and cost of having to seek new equity finance in the future. Thus there can be trade-off value in maintaining pecking order borrowing capacity. Titman and Tsyplakov (2004) intuited benefit in having "an option to issue debt in the future". It may even sometimes be optimal to break pecking order and defer new debt finance for new equity finance, despite the deadweight cost, so as to avoid a more severe dependence on new equity finance in the future. Myers (1984) noted that "financial slack (liquid assets or reserve borrowing power) is valuable, and the firm may rationally issue stock to acquire it". Fama and French (2002) attributed this as forward-looking, "complex" pecking order behaviour entailing "soft", one-sided leverage targeting (i.e. the financial slack argument will never advocate an increase in leverage).

free cash-flow shortfall of $-F_t$ so as to access an ongoing equity value worth E_{t+}. The financial distress cost ($-\gamma F_t$) is a penalty to equity that represents, for instance, the direct transaction costs of the new equity issue and the operational difficulties that may arise when stakeholders fear imminent bankruptcy; or, more characteristic of pecking order theory, it can represent the costs of assuaging principal-agent information asymmetry.

The endogenous occurrence of bankruptcy for the model company represents equity exercising its limited liability option to refuse new equity finance and liquidate the company. This will occur when the required amount of new equity finance inclusive of financial distress costs ($-F_t(1+\gamma)$ given $F_t < 0$) is greater than the company's ongoing equity value (i.e. when $F_t < -E_{t+}/(1+\gamma)$). In this case the company is liquidated for the current period's operating profit and net payoff of maturing hedge contracts, plus the net market value of any non-maturing hedge contracts, minus the face-value of outstanding debt factored up by a bankruptcy cost rate (b). A positive liquidation cash-flow is a remainder after the claims of all non-equity stakeholders have been satisfied in full; positive liquidation cash-flow is taxed and then paid as a liquidating dividend to equity. A negative liquidation cash-flow indicates a combined loss suffered by non-equity stakeholders due to their claims being partly or wholly unpaid.

It is assumed that only debt-holders are willing to pay anything for bankruptcy proceedings, hence the model's bankruptcy cost is applied as a factor (b) of the face-value of outstanding debt. Consequently if there is no outstanding debt, there will be no bankruptcy cost. And because the conditions of financial distress and bankruptcy cannot occur concurrently for a production period, there is no financial distress cost applied with the occurrence of bankruptcy. Thus it may be desirable to set the bankruptcy cost rate to a level that incorporates some degree of financial distress cost associated with bankruptcy. Nevertheless, because the financial distress cost rate lowers the level of free cash-flow shortfall at which bankruptcy occurs, there is an effective financial distress opportunity cost associated with bankruptcy (i.e. the company gets liquidated 'too early' at a free cash-flow shortfall of $E_{t+}/(1+\gamma)$ instead of E_{t+}).

Deadweight financial distress and bankruptcy costs cause an asymmetry in the possible financial outcomes for a firm such that bad future financial states will be more value-destroying than good future states will be value-enhancing. As previously discussed, hedging reduces the likelihood of both good and bad future financial states. This reduces exposure to financial outcome asymmetry and adds value equal to the reduction in expected financial distress and bankruptcy costs as demonstrated by Smith and Stulz (1985). Furthermore, by reducing financial risk, hedging also ramps-up leverage's trade-off equilibrium. Ross (1996) proposed that hedging facilitates higher optimal leverage and thereby allows firms to access greater tax shield benefits. Ross explained that hedging "enables the firm to substitute tax-benefitted risk, in the form of leverage, for non-tax-benefitted risk". Graham and Rogers (2002) were the first to empirically investigate the hedging and leverage decisions jointly and, in favour of Ross's proposition, they found that firms do hedge to increase debt capacity and that this behaviour is tax motivated. Additionally their evidence indicated that firms also hedge in response to large expected financial distress costs. Graham and Rogers concluded that "a complete modelling of corporate debt policy should control for the influence of hedging decisions".

Free cash-flow misappropriation

In addition to its role as a tax shield, another avenue by which debt may add value is as a shield against agency costs. This potential benefit arises out of Jensen's (1986) free cash-flow theory which posits that the cash-flow discipline required to sustain high leverage means that there is less free cash-flow available for self-interested managers to squander. Myers (2001) suggested that, with hindsight, it seems clear that the 1980s spate of leveraged buyouts was a manifestation of Jensen's free cash-flow problem. Jensen defined free cash-flow to be "cash flow in excess of that required to fund all projects that have positive net present values", which includes excess financing cash-flows. Herein is a problem for the discipline-of-debt argument: while the interest payments on debt (net of the tax shield benefit) and eventual repayment of principal reduce free cash-flow, the initial proceeds of debt add to free cash-flow. When conditions are bullish for a firm, cash-flow discipline is highly warranted and, seemingly opportunely, increased debt financing is easily obtained; but increased debt financing will up-front *boost* free cash-flow. Hence Jensen's caveat is that the debt creation must be "without retention of the proceeds of the issue". But even if the proceeds are not retained, they may still be squandered. Indicative of such occurrence in the lead up to the leveraged buyout bust of the early 1990s, Kaplan and Stein (1993) found that management and other interested parties such as investment bankers and deal promoters took out more money up front in the leveraged buyout deals of the late 1980s compared to those done earlier.

Hedging also has implications for free cash-flow. When external finance is excessively costly compared to internally generated funds, Froot, Scharfstein and Stein (1993) showed that hedging for income management can offer benefit by increasing the certainty that internal funds will be available to finance future investment opportunities (this motivation for hedging accords with pecking-order theory). However Tufano (1998) cautioned that such a strategy protects managers from the discipline of external finance and may consequently entail agency costs for equity.

In consideration of Jensen's free cash-flow theory, the model company is subject to a deadweight cost applied as a factor ($0 \le a \le 1$) of positive free cash-flow. Hence under condition of solvency (i.e. positive or zero free cash-flow, $F_t \ge 0$), management misappropriates an amount equal to aF_t, and the dividend paid out to equity equals $(1-a)F_t$. Recall that the model company's free cash-flow is defined to include the net proceeds from debt finance. Consequently 'excess' new debt finance (i.e. that part, if any, of new debt finance that contributes to positive free cash-flow) is subject to misappropriation. Since positive free cash-flow after misappropriation becomes a dividend to equity, excess new debt finance signifies leverage-increasing capital restructure. The implication for the company is that the optimal leverage decision must balance the up-front misappropriation associated with excess new debt finance against reduced expected future misappropriation due to the commitment to repay the debt with interest which thusly reduces future free cash-flow.

Agency theories

Notably not represented in this study's modelling approach are prominent agency based theories for corporate hedging. For example, Smith and Stulz (1985) provided theoretical overview and analysis of the influence of managerial risk aversion and performance-linked compensation on the discretionary hedging choices of managers. Taking an information asymmetry perspective, DeMarzo and Duffie (1995) suggested that, because hedging reduces earnings variability attributable to systematic risk factors beyond management control, managers may hedge to improve the informativeness of earnings as a signal of management and project quality. And Tufano (1998) demonstrated that managers can use hedging to increase their certainty of access to future discretionary cash-flows from which private benefits may be obtained.

If management is predominantly aligned with the interests of equity ahead of debt, a levered firm will tend to avoid investment in low risk, positive net present value (NPV) projects as per Myers' (1977) underinvestment hypothesis; but may invest in high risk, negative NPV projects as per Jensen and Meckling's (1976) asset substitution hypothesis. Such behaviour reduces the firm's overall value, but increases the value of equity's limited liability option at the expense of debt's short position in the option. A policy of hedging tends to equalise the risk of existing and new projects and thereby decreases the sensitivity of equity's limited liability option to investment decisions. By this effect hedging benefits the firm by reducing the incentive to underinvest (see Bessembinder (1991)) or asset substitute. Nevertheless there will be incentive to renege on a policy of hedging since doing so is equivalent in effect to asset substitution and benefits equity. Naturally lenders will demand a premium rate of return or debt covenants to compensate for or restrict any form of asset substitution. Thus equity has to bear (financially or strategically) the expected agency costs of debt whether or not any agency costs of debt materialise. However, if the firm can credibly commit to maintaining hedging, expected agency costs of debt will be lower and accordingly the cost or covenants of debt finance will be less onerous (see Campbell and Kracaw (1990)).

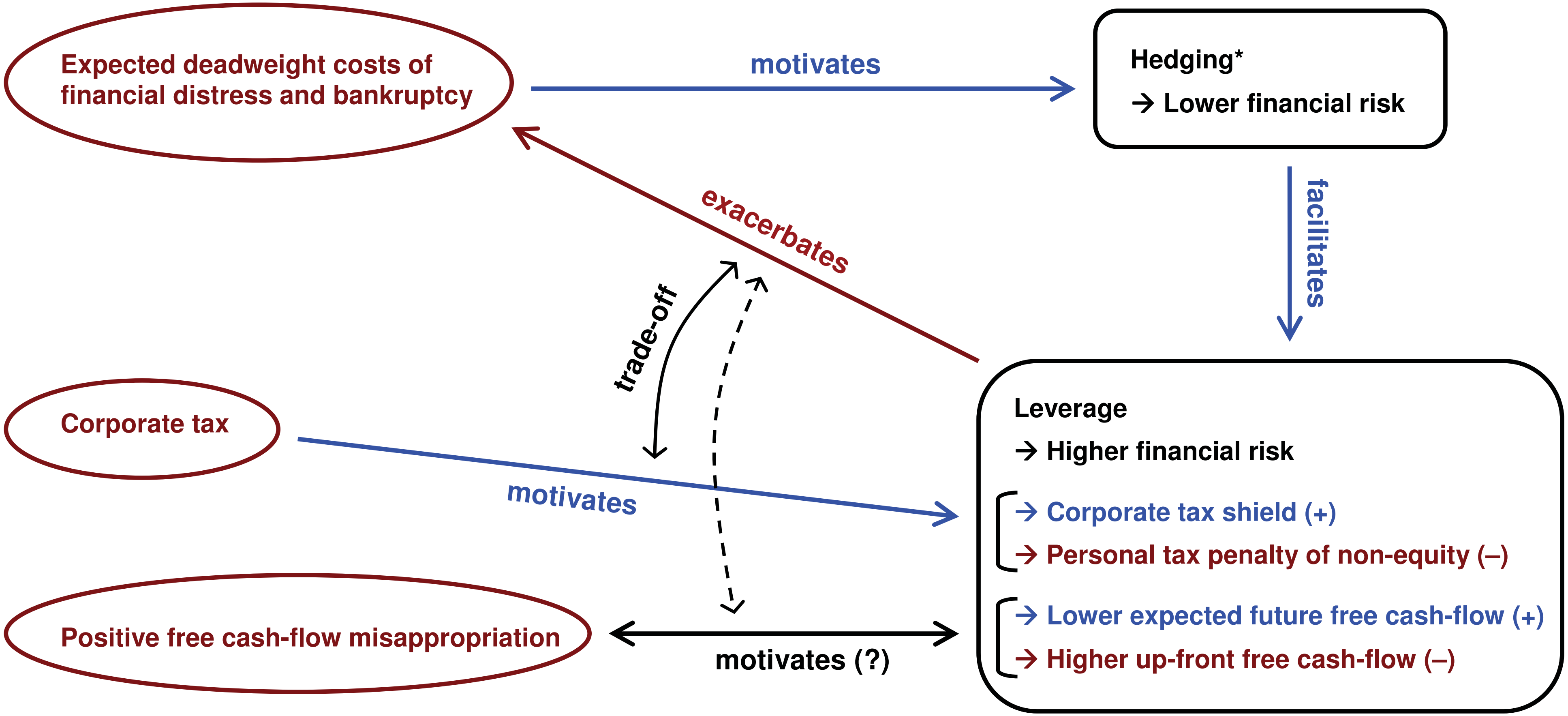
The model company is not subject to agency motivations for either hedging or leverage control. Although the company is specified to be subject to free cash-flow misappropriation, this is treated as a deadweight cost to be reduced by hedging and leverage control rather than a management benefit to be maximised by hedging and leverage control. To incorporate separation of the benefits of ownership and control, a management utility function would need to be specified and maximised. To incorporate debt versus equity conflict, some degree of lender 'naivety' would need to be included in debt contracting to create divergence between equity-maximising and total value-maximising control. Furthermore, inclusion of a meaningful investment decision in the control set-up to reflect underinvestment and asset substitution motivations would require considerable extension of the model. Deliberation on these matters is left for future study. This study assumes that hedging and leverage control is motivationally aligned with maximisation of ownership value; and that debt and other non-equity stakeholders are fairly compensated for their limited liability risk so that equity-maximising control is aligned with total value-maximising control.

Summary

The model company serves as a framework by which several prominent corporate finance theories can interact and counteract. Fundamentally it allows playing-out of the trade-off theory of leverage with deference to pecking order theory in regard to dependence on new equity finance. Very importantly the trade-off set-up is augmented to allow playing-out of the effects of hedging: directly on expected financial distress and bankruptcy costs as per Smith and Stulz (1985); and in feedback to trade-off leverage capacity as per Ross (1996). Additionally the trade-off set-up is subject to a deadweight cost of positive free cash-flow as per Jensen's (1986) free cash-flow theory. Figure 1 depicts these hedging and leverage motivations and interactions. Under sway of these theoretical influences, the model company's available control behaviour appeals to realism via multi-period incorporation of risky leverage and flexible hedging control variables subject to hedgeable and unhedgeable risk factors.

Figure 1 – Hedging and leverage motivations and interactions

Theoretical hedging and leverage motivations and interactions incorporated into the model company set-up.



* Hedging will also directly impact expected corporate taxation and free cash-flow misappropriation.

2.2. Valuation method

Assuming a complete market for the output price and perfectly diversifiable production quantity risk, risk-neutral valuation is used to obtain market values for equity and for non-equity's combined debt plus short limited liability option position. The valuation process works backwards through the binomial price-tree with the production quantity risk incorporated analytically into the valuations at each price-node.

Given the specification of three controlled production periods ($N = 3$) and a four-step binomial price process within each period ($n = 4$), Figure 2 depicts the price-tree branching structure for the model company's production output incorporating non-recombination between periods. With no preconceptions about optimal hedging and leverage control behaviour along any price-path, the choice of $N = 3$ and $n = 4$ represents a compromise between computational expediency and expositional comprehensiveness. A four-step binomial price process within each production period gives a spread of five price outcomes each period, which, with foresight of the results, is a reasonable minimum for ascertaining price outcome differentiation of optimal hedging and leverage decisions and financial risk. Meanwhile the merit of at least three controlled production periods is that the model company can be assessed and controlled at an intermediate stage of its productive life (i.e. at commencement of the second controlled production period) in the light of historic control decisions and in the shadow of future control uncertainty.

The risk-neutral binomial process that grows the tree for production output price (p_t) is:

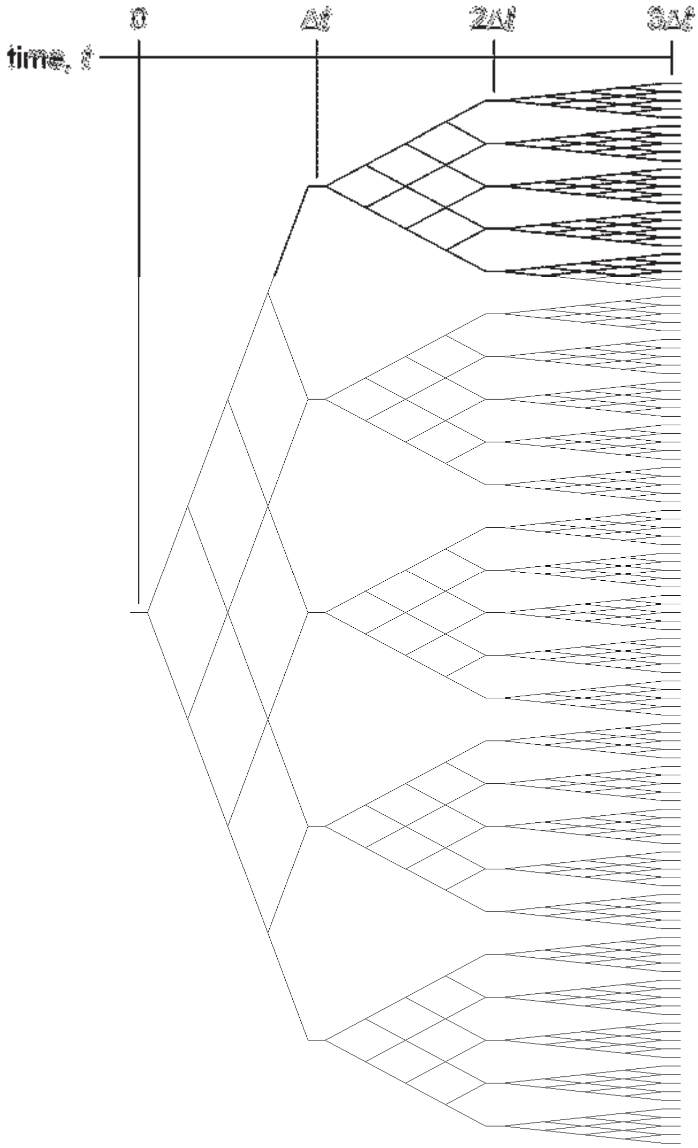
$$p_t = p_{t-\Delta t}\left[e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/4)+\sigma_p\sqrt{\Delta t/4}}\right]^{\mathrm{P}_t}\left[e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/4)-\sigma_p\sqrt{\Delta t/4}}\right]^{(4-\mathrm{P}_t)},$$

$$\mathrm{P}_t \sim \mathrm{BIN}\left(4, \frac{e^{(r-\delta)(\Delta t/4)}-e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/4)-\sigma_p\sqrt{\Delta t/4}}}{e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/4)+\sigma_p\sqrt{\Delta t/4}}-e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/4)-\sigma_p\sqrt{\Delta t/4}}}\right)$$

where: $t \in \{\Delta t, 2\Delta t, 3\Delta t\}$ denominates time; Δt is the duration of each production period; r is the annual, continuously compounding risk-free interest rate; δ is the annual, continuously compounding convenience yield of the production output; and σ_p^2 is the annual variance rate for the output price.

Figure 2 – Price-tree

Binomial price-tree for the model company's production output. The price-tree is four-step recombining within each controlled production period ($n=4$) and non-recombining between the three controlled production periods ($N=3$). Hence there are five price-nodes at end-of-period one (time $t=\Delta t$), 25 price-nodes with unique price-paths at end-of-period two (time $t=2\Delta t$) and 125 price-nodes with unique price-paths at end-of-period three (time $t=3\Delta t$).



Cash-flow formulations

At each output price outcome the cash-flows to equity and to non-equity's debt plus short comprehensive limited liability option are conditional on the values of earnings before tax (EBT_t), free cash-flow (F_t) and liquidation cash-flow (L_t). The sign of EBT_t determines whether corporate tax is payable or refundable, which impacts on the value of F_t. If F_t is positive (the condition of solvency), an amount is deducted for management misappropriation and the remainder is paid out as a dividend to equity. If F_t is negative (the condition of financial distress), a financial distress cost is incurred and new equity finance is required (via a negative dividend, i.e. a rights issue) to fund the free cash-flow shortfall plus financial distress cost. If the required amount of new equity finance exceeds the ongoing value of equity, the condition of bankruptcy is instead instigated. In event of bankruptcy, L_t determines whether all liabilities are covered and whether there will be a liquidating dividend for equity. In total there are eight critical conditions: solvency and financial distress each with either positive or negative EBT_t; and bankruptcy with combinations of either positive or negative EBT_t and either L_t able to or not able to cover all liabilities. Formula definitions for these eight cash-flow conditions are derived from the following formula components specified at the end of each production period:

- $p_t q_t$ is cash revenue from production output price and quantity;
- $f(p_t)$ is total due pre-tax liabilities for production costs and maturing debt face-value, less net payoff of maturing hedge contracts, less new debt proceeds (Y_t, which incorporates deductions for the cost of the comprehensive limited liability option and the personal tax penalty of debt) (see equation (A.4) in Appendix A for the formulation);
- $d_t(y_\eta)$ is debt refinancing change in outstanding debt face-value (i.e. new debt face-value minus maturing debt face-value); and
- $g(p_t)$ is net liquidation liabilities comprising production costs, plus face-value of outstanding debt, plus bankruptcy costs, less net payoff of maturing hedge contracts, less net fair value of live hedge contracts (see equation (A.8) in Appendix A for the formulation).

Free cash-flow before tax (FBT_t) equals $p_t q_t$ minus $f(p_t)$. EBT_t can be obtained from FBT_t by adding back the face-value of maturing debt and deducting from the value of new debt finance its face-value so as to incorporate up-front deduction of interest expense:

$$EBT_t = p_t q_t - f(p_t) - d_t(y_\eta).$$

Positive EBT_t is taxed at the corporate tax rate ($0 \le \alpha < 1$) and negative EBT_t is subject to the tax refund claimability rate ($0 \le \lambda \le 1$). Hence F_t equals FBT_t adjusted for corporate tax payable or refundable on EBT_t:

$$F_t = \begin{cases} (1-\alpha)\left[p_t q_t - f(p_t)\right] + \alpha d_t(y_\eta) & , \quad EBT_t \ge 0 \\ (1-\alpha\lambda)\left[p_t q_t - f(p_t)\right] + \alpha\lambda d_t(y_\eta) & , \quad EBT_t < 0 . \end{cases}$$

A negative value for F_t indicates a free cash-flow shortfall, which is subject to the financial distress cost rate ($\gamma \ge 0$) and necessitates new equity finance if bankruptcy is to be avoided. If free cash-flow shortfall plus financial distress cost ($-F_t(1+\gamma)$) is greater than ongoing equity value (E_{t+}), new equity finance will not be forthcoming and bankruptcy occurs. In event of bankruptcy: if L_t (equal to $p_t q_t$ minus $g(p_t)$) is positive, all non-equity claims are satisfied in full and L_t is taxed and paid to equity; whereas if L_t is negative, there is a shortfall in payment of non-equity's claims, which represents a claim by equity against its limited liability option.

Conditional on the output price at each node of the price-tree, analytical valuations for equity and non-equity with respect to production quantity uncertainty are to be obtained. Thus the eight cash-flow conditions dependent on EBT_t, F_t and L_t must be specified as functions of production quantity (q_t). To simplify notation for that purpose, define χ_1, χ_2, χ_3, χ_4 and χ_5:

$$\chi_1 = \frac{f(p_t) + d_t(\mathbf{y}_\eta)}{p_t}, \quad \chi_2 = \frac{(1-\alpha)f(p_t) - \alpha d_t(\mathbf{y}_\eta)}{(1-\alpha)p_t},$$

$$\chi_3 = \frac{(1+\gamma)(1-\alpha)f(p_t) - E_{t+} - \alpha(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha)p_t}, \quad \chi_4 = \frac{(1-\alpha\lambda)f(p_t) - \alpha\lambda d_t(\mathbf{y}_\eta)}{(1-\alpha\lambda)p_t},$$

$$\chi_5 = \frac{(1+\gamma)(1-\alpha\lambda)f(p_t) - E_{t+} - \alpha\lambda(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha\lambda)p_t}.$$

Thus: $q_t < \chi_1$ if and only if EBT_t is negative; $q_t < \chi_2$ if and only if F_t after tax payable is negative; $q_t < \chi_3$ if and only if F_t after tax payable is too negative for a successful rights issue; $q_t < \chi_4$ if and only if F_t after tax refundable is negative; $q_t < \chi_5$ if and only if F_t after tax refundable is too negative for a successful rights issue; and $q_t < g(p_t)/p_t$ if and only if L_t is too low to cover total outstanding liabilities.

Table 1 defines the eight critical cash-flow conditions in terms of compound indicator functions (the indicator function, $\mathrm{I}_{logical\ statement}$, equals one if the *logical statement* is true and zero otherwise). Also shown in Table 1 are the formulas for the cash-flows to equity (G_t and GL_t) and to non-equity's debt plus short comprehensive limited liability option (H_t) for each of the eight conditions. For convenience of derivation in Appendix A, cash-flow to equity under condition of non-bankruptcy (G_t) and cash-flow to equity under condition of bankruptcy (GL_t) are specified separately; but for convenience of display in Table 1, they are combined ($G_t + GL_t$).

Under condition of solvency ($F_t \geq 0$), the positive free cash-flow is reduced by the misappropriation rate (a) and the remainder ($(1-a)F_t$) becomes cash-flow to equity. Under condition of financial distress ($-E_{t+}/(1+\gamma) \leq F_t < 0$), the free cash-flow shortfall ($-F_t$) is increased by the financial distress cost rate (γ) and the aggregate ($-(1+\gamma)F_t$) becomes cash-flow *from* equity. Under condition of bankruptcy ($F_t < -E_{t+}/(1+\gamma)$): if liquidation cash-flow is positive ($L_t \geq 0$), it is reduced by the corporate tax rate (α) and the remainder ($(1-\alpha)L_t$) becomes cash-flow to equity; but if liquidation cash-flow is negative ($L_t < 0$), there is no cash-flow to equity.

Under condition of either solvency or financial distress (i.e. non-bankruptcy), the cash-flow to debt equals total due face-value ($(\mathbf{e}_t^T \mathbf{y}_\eta)\mathbf{1}$). Under condition of bankruptcy: if liquidation cash-flow is positive, the cash-flow to debt equals total outstanding face-value ($\sum_{i=t/\Delta t}^{N} (\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta)\mathbf{k}_t$); but if liquidation cash-flow is negative, the cash-flow to debt and against the comprehensive limited liability option equals debt's total outstanding face-value reduced by the liquidation cash-flow shortfall ($L_t + \sum_{i=t/\Delta t}^{N} (\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta)\mathbf{k}_t$). Because, for this analysis, the periodic leverage control decision has been limited to the issue of debt with single-period maturity only, $(\mathbf{e}_t^T \mathbf{y}_\eta)\mathbf{1}$ and $\sum_{i=t/\Delta t}^{N} (\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta)\mathbf{k}_t$ are equivalently represented as $y_{t-\Delta t,t}$.

To obtain the unconditional formulas for $G_t + GL_t$ and H_t, the products of the compound indicator function and cash-flow for each condition are summed. After some algebra, the formula for $G_t + GL_t$ will reduce to the sum of equations (A.6) and (A.9) from Appendix A. Similarly the formula for H_t reduces to equation (A.13) from Appendix A. As shown in Appendix A, the formulations can be expressed as functions of put option payoffs dependent on production quantity (q_t).

From the joint distribution of production output price and quantity (p_t and q_t) specified in Appendix A, at the beginning of each production period, end-of-period q_t has the following lognormal risk-neutral distribution conditional on end-of-period p_t:

$$\left(\ln q_t \mid p_t\right)_{t-\Delta t} \sim \mathrm{N}\left(\ln \overline{q}_0 + \rho\left(\frac{\sigma_q}{\sigma_p}\right)\left[\ln\left(\frac{p_t}{p_0}\right) - \left(r - \delta - \frac{\sigma_p^2}{2}\right)t\right], \sigma_q^2(1-\rho^2)\Delta t\right)$$

where: r is the annual, continuously compounding risk-free interest rate; δ is the annual, continuously compounding convenience yield of the production output; p_0 and $\overline{q}_0$ are respectively the initial (start-of-period one) output price and the initial median periodic production quantity; σ_p^2 and σ_q^2 are the annual variance rates for the output price and production quantity respectively; and ρ is the correlation between output price and production quantity, which can be set to represent the preference and ability of the model company to adjust expected production quantity in step with changes in output price. That is, the expected production quantity equals a time-zero base level plus an adjustment (dependent on ρ) for the output price level deviation from time-zero price net of the expected price drift, representing a resource producer that adjusts its production plans to follow the absolute price level for its production output.[8] The subscript

[8] A non-zero value (in particular, a positive value) for the 'in-period' price/quantity correlation (ρ) effectively assumes that the model company's expected production quantity can respond instantaneously to output price fluctuations, to a limit specified by the scale of ρ. This is plausible if there is short-term (in-period) predictability/momentum for the output price. Despite allowing for such concurrent production quantity response to output price variability, the model does not allow a concurrent hedging response to output price variability. By this I am supposing a company that makes long-term (period-to-period) hedging decisions ignorant of long-term future output price changes, but that makes short-term (in-period) expected production quantity changes with confidence about short-term future output price changes.

$t-\Delta t$ in the representation $(\ln q_t \mid p_t)_{t-\Delta t}$ is used to indicate that there is an entire period's uncertainty (from time $t-\Delta t$ to t) in the probability distribution for $(\ln q_t \mid p_t)$ (i.e. the variance of $(\ln q_t \mid p_t)$ is scaled by the production period duration, Δt).

It is then a simple matter of applying a Black (1976) based valuation to obtain periodic expected values for $G_t + GL_t$ and H_t conditional on p_t: the formula for $\hat{\mathrm{E}}_{t-\Delta t}[G_t + GL_t \mid p_t]$ is given by the sum of equations (A.7) and (A.10) in Appendix A; and the formula for $\hat{\mathrm{E}}_{t-\Delta t}[H_t \mid p_t]$ is given by equation (A.14) in Appendix A; where $\hat{\mathrm{E}}_t[\cdot]$ is the risk-neutral expectation operator. Thus quantity risk is incorporated analytically into the equity and non-equity cash-flow formulations at each output price-node of the price-tree.

Table 1 – Summary of conditional cash-flows to equity and to debt plus short limited liability option

Cash-flows to equity (G_t and GL_t) and to debt plus short limited liability option (H_t), and the conditions for which the cash-flows occur in terms of compound indicator functions for eight critical combinations of earnings before tax (EBT_t), free cash-flow (F_t) and liquidation cash-flow (L_t).

	Condition			Compound indicator function for each condition: if condition is true (false), compound indicator function equals '1' ('0')	Cash-flows to equity and to debt plus short limited liability option	
	EBT_t	F_t	L_t		$G_t + GL_t$	H_t
Solvency conditions	≥ 0	≥ 0	n.a.	$\mathrm{I}_{q_t \geq \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}} \mathrm{I}_{q_t \geq \frac{(1-\alpha) f(p_t)-\alpha d_t(\mathbf{y}_\eta)}{(1-\alpha) p_t}} = 1 - \mathrm{I}_{q_t<\chi_1} - \mathrm{I}_{q_t<\chi_2} + \mathrm{I}_{q_t<\min[\chi_2,\chi_1]}$	$(1-a)\left\{(1-\alpha)\left[p_t q_t - f(p_t)\right] + \alpha d_t(\mathbf{y}_\eta)\right\} \geq 0$	$\left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}$
	< 0	≥ 0	n.a.	$\mathrm{I}_{q_t < \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}} \mathrm{I}_{q_t \geq \frac{(1-\alpha\lambda) f(p_t)-\alpha\lambda d_t(\mathbf{y}_\eta)}{(1-\alpha\lambda) p_t}} = \mathrm{I}_{q_t<\chi_1} - \mathrm{I}_{q_t<\min[\chi_4,\chi_1]}$	$(1-a)\left\{(1-\alpha\lambda)\left[p_t q_t - f(p_t)\right] + \alpha\lambda d_t(\mathbf{y}_\eta)\right\} \geq 0$	$\left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}$
Financial distress conditions	≥ 0	< 0, $\geq \frac{-E_{t+}}{(1+\gamma)}$	n.a.	$\mathrm{I}_{q_t \geq \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}} \mathrm{I}_{q_t \geq \frac{(1+\gamma)(1-\alpha) f(p_t)-E_{t+}-\alpha(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha) p_t}} \mathrm{I}_{q_t < \frac{(1-\alpha) f(p_t)-\alpha d_t(\mathbf{y}_\eta)}{(1-\alpha) p_t}} = \mathrm{I}_{q_t<\chi_2} - \mathrm{I}_{q_t<\min[\chi_2,\chi_1]} - \mathrm{I}_{q_t<\chi_3} + \mathrm{I}_{q_t<\min[\chi_3,\chi_1]}$	$(1+\gamma)\left\{(1-\alpha)\left[p_t q_t - f(p_t)\right] + \alpha d_t(\mathbf{y}_\eta)\right\} < 0$	$\left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}$
	< 0	< 0, $\geq \frac{-E_{t+}}{(1+\gamma)}$	n.a.	$\mathrm{I}_{q_t < \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}} \mathrm{I}_{q_t \geq \frac{(1+\gamma)(1-\alpha\lambda) f(p_t)-E_{t+}-\alpha\lambda(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha\lambda) p_t}} \mathrm{I}_{q_t < \frac{(1-\alpha\lambda) f(p_t)-\alpha\lambda d_t(\mathbf{y}_\eta)}{(1-\alpha\lambda) p_t}} = \mathrm{I}_{q_t<\min[\chi_4,\chi_1]} - \mathrm{I}_{q_t<\min[\chi_5,\chi_1]}$	$(1+\gamma)\left\{(1-\alpha\lambda)\left[p_t q_t - f(p_t)\right] + \alpha\lambda d_t(\mathbf{y}_\eta)\right\} < 0$	$\left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}$

Table 1 (continued) – Summary of conditional cash-flows to equity and to debt plus short limited liability option

Cash-flows to equity (G_t and GL_t) and to debt plus short limited liability option (H_t), and the conditions for which the cash-flows occur in terms of compound indicator functions for eight critical combinations of earnings before tax (EBT_t), free cash-flow (F_t) and liquidation cash-flow (L_t).

	Condition			Compound indicator function for each condition: if condition is true (false), compound indicator function equals '1' ('0')	Cash-flows to equity and to debt plus short limited liability option	
	EBT_t	F_t	L_t		$G_t + GL_t$	H_t
Bankruptcy/liquidation conditions	≥ 0	$< \dfrac{-E_{t+}}{(1+\gamma)}$	≥ 0	$\mathrm{I}_{q_t \geq \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}} \mathrm{I}_{q_t < \frac{(1+\gamma)(1-\alpha)f(p_t)-E_{t+}-\alpha(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha)p_t}} \mathrm{I}_{q_t \geq \frac{g(p_t)}{p_t}}$ $= \mathrm{I}_{q_t<\chi_3} - \mathrm{I}_{q_t<\min[\chi_3,\chi_1]} - \mathrm{I}_{q_t<\min[g(p_t)/p_t,\chi_3]} + \mathrm{I}_{q_t<\min[g(p_t)/p_t,\chi_3,\chi_1]}$	$(1-\alpha)\left(p_t q_t - g(p_t)\right) \geq 0$	$\sum_{i=t/\Delta t}^{N} \left(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta\right)\mathbf{k}_t$
	≥ 0	$< \dfrac{-E_{t+}}{(1+\gamma)}$	< 0	$\mathrm{I}_{q_t \geq \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}} \mathrm{I}_{q_t < \frac{(1+\gamma)(1-\alpha)f(p_t)-E_{t+}-\alpha(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha)p_t}} \mathrm{I}_{q_t < \frac{g(p_t)}{p_t}}$ $= \mathrm{I}_{q_t<\min[g(p_t)/p_t,\chi_3]} - \mathrm{I}_{q_t<\min[g(p_t)/p_t,\chi_3,\chi_1]}$	0	$p_t q_t - g(p_t) + \sum_{i=t/\Delta t}^{N} \left(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta\right)\mathbf{k}_t$
	< 0	$< \dfrac{-E_{t+}}{(1+\gamma)}$	≥ 0	$\mathrm{I}_{q_t < \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}} \mathrm{I}_{q_t < \frac{(1+\gamma)(1-\alpha\lambda)f(p_t)-E_{t+}-\alpha\lambda(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha\lambda)p_t}} \mathrm{I}_{q_t \geq \frac{g(p_t)}{p_t}}$ $= \mathrm{I}_{q_t<\min[\chi_5,\chi_1]} - \mathrm{I}_{q_t<\min[g(p_t)/p_t,\chi_5,\chi_1]}$	$(1-\alpha)\left(p_t q_t - g(p_t)\right) \geq 0$	$\sum_{i=t/\Delta t}^{N} \left(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta\right)\mathbf{k}_t$
	< 0	$< \dfrac{-E_{t+}}{(1+\gamma)}$	< 0	$\mathrm{I}_{q_t < \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}} \mathrm{I}_{q_t < \frac{(1+\gamma)(1-\alpha\lambda)f(p_t)-E_{t+}-\alpha\lambda(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha\lambda)p_t}} \mathrm{I}_{q_t < \frac{g(p_t)}{p_t}}$ $= \mathrm{I}_{q_t<\min[g(p_t)/p_t,\chi_5,\chi_1]}$	0	$p_t q_t - g(p_t) + \sum_{i=t/\Delta t}^{N} \left(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta\right)\mathbf{k}_t$

Table 1 (continued) – Summary of conditional cash-flows to equity and to debt plus short limited liability option

Cash-flows to equity (G_t and GL_t) and to debt plus short limited liability option (H_t), and the conditions for which the cash-flows occur in terms of compound indicator functions for eight critical combinations of earnings before tax (EBT_t), free cash-flow (F_t) and liquidation cash-flow (L_t).

Notes:

G_t is the cash-flow to equity under condition of non-bankruptcy; GL_t is the liquidating dividend to equity under condition of bankruptcy; H_t is the cash-flow to debt and against the short limited liability option of non-equity stakeholders.

$p_t q_t$ is cash revenue from output price and quantity; $f(p_t)$ is pre-tax net due liabilities; $d_t(\mathbf{y}_\eta)$ is debt refinancing change in total outstanding debt face-value; $g(p_t)$ is net liquidation liabilities.

$EBT_t = p_t q_t - f(p_t) - d_t(\mathbf{y}_\eta)$; $0 \le \alpha < 1$ is the corporate tax rate; $0 \le \lambda \le 1$ is the tax refund claimability rate; $(1-\alpha)\left[p_t q_t - f(p_t)\right] + \alpha d_t(\mathbf{y}_\eta)$ is F_t after tax; $(1-\alpha\lambda)\left[p_t q_t - f(p_t)\right] + \alpha\lambda d_t(\mathbf{y}_\eta)$ is F_t after tax adjusted for tax refund claimability; $L_t = p_t q_t - g(p_t)$.

$\gamma \ge 0$ is the financial distress cost rate; E_{t+} is ongoing equity value; $0 \le a \le 1$ is the free cash-flow misappropriation rate; $\left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}$ is total due debt face-value; $\sum_{i=t/\Delta t}^{N}\left(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta\right)\mathbf{k}_t$ is total outstanding debt face-value.

$\chi_1 = \left[f(p_t) + d_t(\mathbf{y}_\eta)\right]/p_t$; $\chi_2 = \left[(1-\alpha)f(p_t) - \alpha d_t(\mathbf{y}_\eta)\right]/(1-\alpha)p_t$; $\chi_3 = \left[(1+\gamma)(1-\alpha)f(p_t) - E_{t+} - \alpha(1+\gamma)d_t(\mathbf{y}_\eta)\right]/(1+\gamma)(1-\alpha)p_t$;
$\chi_4 = \left[(1-\alpha\lambda)f(p_t) - \alpha\lambda d_t(\mathbf{y}_\eta)\right]/(1-\alpha\lambda)p_t$; $\chi_5 = \left[(1+\gamma)(1-\alpha\lambda)f(p_t) - E_{t+} - \alpha\lambda(1+\gamma)d_t(\mathbf{y}_\eta)\right]/(1+\gamma)(1-\alpha\lambda)p_t$.

$q_t < \chi_1$ if and only if EBT_t is negative; $q_t < \chi_2$ if and only if F_t after tax payable is negative; $q_t < \chi_4$ if and only if F_t after tax refundable is negative; $q_t < \chi_3$ if and only if F_t after tax payable is too negative for a successful rights issue; $q_t < \chi_5$ if and only if F_t after tax refundable is too negative for a successful rights issue; $q_t < g(p_t)/p_t$ if and only if L_t is too low to cover total outstanding liabilities; by construction, $E_{t+} \ge 0$, therefore $\chi_3 \le \chi_2$ and $\chi_5 \le \chi_4$.

Price-tree application

Further to assuming that the model company will operate (while not bankrupt) for three consecutive controlled production periods, it is assumed that the company has presently completed an uncontrolled (i.e. without hedging or leverage) production period, and will operate (if not previously bankrupted) for a fourth and final production period uncontrolled. The presently (time $t = 0$) completed production period provides the company with current income which motivates an immediate tax-reducing capital restructure (because the interest expense for leverage is specified within the model as tax deductible in advance). The fourth production period provides an output 'bank' from which production can be brought forward or to which production can be delayed;[9] consequently it is necessary to specify an expected total output resource (R). The binomial price-tree process applied for the three controlled production periods is assumed to continue for the fourth period.

The valuation process requires predetermined valuations at the terminal price-nodes. To this end it is assumed that the remaining output resource expected to be produced in the final uncontrolled production period equals the initial total output resource minus the price-path-specific expected production quantities for the prior three controlled production periods.[10] The binomial price-tree is four-step recombining within production periods and non-recombining between production periods. Hence there are 125 path-specific price-nodes at the end of the third controlled production period. An ongoing equity value for each of these 125 price-nodes at end-of-period three (start-of-period four) is calculated according to an assumed all-equity payoff exposure for period four:

$$E_{3\Delta t+} = \max\left[0, e^{-r\Delta t}\left(R - \sum_{i=1}^{3} \hat{\mathrm{E}}_{(i-1)\Delta t}\left[q_{i\Delta t} \mid p_{i\Delta t}\right]\right)\hat{\mathrm{E}}_{3\Delta t}\left[(p_{4\Delta t} - c)\left(1 - \alpha\lambda - \alpha(1-\lambda)\mathrm{I}_{p_{4\Delta t} > c}\right)\right]\right] \tag{2.1}$$

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[9] Specification of the correlation between production quantity and output price (ρ) determines how the company shifts production through time.

[10] The final period 'wind-up' production quantity is uncorrelated with the final period output price outcome ($p_{4\Delta t}$).

where: $\hat{\mathrm{E}}_t[\cdot]$ is the risk-neutral expectation operator; the indicator function, $\mathrm{I}_{logical\ statement}$, equals one if the *logical statement* is true and zero otherwise; R is the expected total output resource for the three controlled and final uncontrolled production periods; r is the annual, continuously compounding risk-free interest rate; c is the total production cost per unit of expected production quantity; α is the corporate tax rate; and λ is the claimability of a tax refund in event of negative earnings.

For model set-up it has been assumed that deviation in production quantity from expectation (conditional on price outcome) for a production period does not impact on future production quantity expectations. For the context of a resource producer with a limited resource, the quantity risk does not represent variability of production capacity or efficiency, but rather represents variability of resource quality (e.g. ore concentration) during a production period, albeit with no implication for expectations of resource quality for future production periods. Hence the expected output resource remaining for fourth period production is calculated by subtracting from the initial expected total output resource (R) the (price-path conditional) expected rather than actual production quantities for the preceding three controlled production periods ($\sum_{i=1}^{3}\hat{\mathrm{E}}_{(i-1)\Delta t}\left[q_{i\Delta t} \mid p_{i\Delta t}\right]$). Because output price is modelled with a discrete binomial tree, the maximum value for $\sum_{i=1}^{3}\hat{\mathrm{E}}_{(i-1)\Delta t}\left[q_{i\Delta t} \mid p_{i\Delta t}\right]$ is ascertainable as a function of the correlation between output price and production quantity (ρ); appropriate choice for ρ ensures the expected fourth period production quantity cannot be negative (however, as the discrete price tree approaches the continuous time limit, such appropriate choice for ρ will approach zero).

Having calculated the 125 path-specific end-of-period three ongoing equity values ($E_{3\Delta t+}$), and given the 25 path-specific sets of control variables applicable to period three (i.e. 25 sets of $\{(w_{0,3\Delta t}, w_{\Delta t,3\Delta t}, w_{2\Delta t,3\Delta t})$; $(x_{0,3\Delta t}, x_{\Delta t,3\Delta t}, x_{2\Delta t,3\Delta t})$; $y_{2\Delta t,3\Delta t}$; $(z_{0,3\Delta t}$, $z_{\Delta t,3\Delta t}, z_{2\Delta t,3\Delta t})\}$), it is possible to calculate 125 corresponding price-conditional expected equity and non-equity valuations for the instant immediately prior to end-of-period three. These expected valuations, $\hat{\mathrm{E}}_{2\Delta t}[E_{3\Delta t-} \mid p_{3\Delta t}]$ and $\hat{\mathrm{E}}_{2\Delta t}[D_{3\Delta t-} + O_{3\Delta t-} \mid p_{3\Delta t}]$, are respectively obtained using equations (A.12) and (A.16) in Appendix A, where: $E_{3\Delta t-}$ is the total equity value immediately prior to any dividend payment or new equity finance; $D_{3\Delta t-} = y_{2\Delta t,3\Delta t}$ is the risk-free debt value immediately before settlement of the maturing debt; and $O_{3\Delta t-}$ is the short payoff value of the expiring comprehensive limited liability option. The price-conditional expectations are obtained analytically with respect to production quantity uncertainty and take into account the condition of the company as variously solvent, financially distressed or bankrupt across the spectrum of possible quantity production outcomes. The discounted unconditional expectations then give the 25 path-specific end-of-period two ongoing equity and non-equity values. These valuations, $E_{2\Delta t+}$ and $D_{2\Delta t+} + O_{2\Delta t+}$, are respectively obtained using equations (A.17) and (A.18) in Appendix A, where: $E_{2\Delta t+}$ is the ongoing equity value immediately after any dividend payment or new equity finance and implicitly conditional on bankruptcy having not occurred; $D_{2\Delta t+} = y_{2\Delta t,3\Delta t} e^{-r\Delta t}$ is the risk-free value of newly issued debt; and $O_{2\Delta t+}$ is the short initial value of the newly contracted comprehensive limited liability option.[11]

Now with the 25 path-specific values for $E_{2\Delta t+}$ and $D_{2\Delta t+} + O_{2\Delta t+}$, and given the five path-specific sets of control variables applicable to period two (i.e. five sets of $\{ (w_{0,2\Delta t}, w_{0,3\Delta t}, w_{\Delta t,2\Delta t}, w_{\Delta t,3\Delta t})$; $(x_{0,2\Delta t}, x_{0,3\Delta t}, x_{\Delta t,2\Delta t}, x_{\Delta t,3\Delta t})$; $y_{\Delta t,2\Delta t}$; $(z_{0,2\Delta t}, z_{0,3\Delta t}, z_{\Delta t,2\Delta t}, z_{\Delta t,3\Delta t}) \}$), the 25 path-specific values for $\hat{\mathrm{E}}_{\Delta t}[E_{2\Delta t-} \mid p_{2\Delta t}]$ and $\hat{\mathrm{E}}_{\Delta t}[D_{2\Delta t-} + O_{2\Delta t-} \mid p_{2\Delta t}]$ can be calculated (using again equations (A.12) and (A.16) in Appendix A), and thence the five path-specific values for $E_{\Delta t+}$ and $D_{\Delta t+} + O_{\Delta t+}$ obtain (using again equations (A.17) and (A.18) in Appendix A). Continuing in the same vein, E_{0+} and $D_{0+} + O_{0+}$ obtain.

With the further assumption of an uncontrolled production period having just been completed at time $t = 0$ with unresolved production quantity uncertainty, the values $\hat{\mathrm{E}}_{-\Delta t}[E_{0-} \mid p_0]$ and $\hat{\mathrm{E}}_{-\Delta t}[D_{0-} + O_{0-} \mid p_0]$ obtain with $D_{0-} = 0$ due to the production period being uncontrolled. The combined value of $\hat{\mathrm{E}}_{-\Delta t}[E_{0-} + O_{0-} \mid p_0]$ is then used to represent an all-equity valuation that encompasses the impacts of each price-path-dependent control decision going forward.

[11] Recall that each production period's leverage control decision has been restricted to the use of debt with a maturity of only a single period, and that, by model design, equity's comprehensive limited liability option must be newly contracted each period with a maturity of only a single period.

Break-down of valuation

Equation (A.20) in Appendix A gives the formula for total expected deadweight costs at the transition of production periods ($\hat{\mathrm{E}}_{t-\Delta t}[DWC_t \mid p_t]$). The total valuation of current and ongoing deadweight costs ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma DWC_t \mid p_t]$) cumulates backwards through the price-tree dependant on the probability of bankruptcy not occurring at each price-node:

$$\hat{\mathrm{E}}_{t-\Delta t}\left[\Sigma DWC_t \mid p_t\right] = \hat{\mathrm{E}}_{t-\Delta t}\left[DWC_t \mid p_t\right] + \left(1 - \hat{\mathrm{E}}_{t-\Delta t}\left[\mathrm{I}_{F_t < \frac{-E_{t+}}{(1+\gamma)}} \mid p_t\right]\right) e^{-r\Delta t} \hat{\mathrm{E}}_t\left[\Sigma DWC_{t+\Delta t}\right]$$

where $\hat{\mathrm{E}}_{t-\Delta t}[\mathrm{I}_{F_t < -E_{t+}/(1+\gamma)} \mid p_t]$ is the probability of bankruptcy ($F_t < -E_{t+}/(1+\gamma)$) occurring at a price-node due to production quantity risk (see equation (A.11) in Appendix A).

The constituents of $\hat{\mathrm{E}}_{t-\Delta t}[\Sigma DWC_t \mid p_t]$ can also be formulated. Equation (A.21) in Appendix A gives the formula for expected corporate taxation at the transition of production periods ($\hat{\mathrm{E}}_{t-\Delta t}[TAX_t \mid p_t]$). Thus the current and ongoing expected deadweight cost due to corporate taxation ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma TAX_t \mid p_t]$) is:

$$\hat{\mathrm{E}}_{t-\Delta t}\left[\Sigma TAX_t \mid p_t\right] = \hat{\mathrm{E}}_{t-\Delta t}\left[TAX_t \mid p_t\right] + \left(1 - \hat{\mathrm{E}}_{t-\Delta t}\left[\mathrm{I}_{F_t < \frac{-E_{t+}}{(1+\gamma)}} \mid p_t\right]\right) e^{-r\Delta t} \hat{\mathrm{E}}_t\left[\Sigma TAX_{t+\Delta t}\right].$$

Equivalently obtained are the formulas for current and ongoing expected deadweight costs due to the personal tax penalty of non-equity ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma PTP_t \mid p_t]$), free cash-flow misappropriation ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma MIS_t \mid p_t]$), financial distress ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma FND_t \mid p_t]$) or bankruptcy ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma BNK_t \mid p_t]$); where Equations (A.22), (A.23), (A.24) and (A.25) in Appendix A respectively give the formulas for $\hat{\mathrm{E}}_{t-\Delta t}[PTP_t \mid p_t]$, $\hat{\mathrm{E}}_{t-\Delta t}[MIS_t \mid p_t]$, $\hat{\mathrm{E}}_{t-\Delta t}[FND_t \mid p_t]$ and $\hat{\mathrm{E}}_{t-\Delta t}[BNK_t \mid p_t]$.

The model company's value attributable to operations and hedging is given by current and ongoing expected earnings before interest and tax ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma EBIT_t \mid p_t]$):

$$\hat{\mathrm{E}}_{t-\Delta t}\left[\Sigma EBIT_t \mid p_t\right]=\hat{\mathrm{E}}_{t-\Delta t}\left[EBIT_t \mid p_t\right]+\left(1-\hat{\mathrm{E}}_{t-\Delta t}\left[\mathrm{I}_{F_t<\frac{-E_{t+}}{(1+\gamma)}} \mid p_t\right]\right)e^{-r\Delta t}\hat{\mathrm{E}}_t\left[\Sigma EBIT_{t+\Delta t}\right]$$

where $\hat{\mathrm{E}}_{t-\Delta t}[EBIT_t \mid p_t]$ is given by Equation (A.19) in Appendix A. The value attributable specifically to operations is determined from current and ongoing expected operating profit ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma OPP_t \mid p_t]$):

$$\hat{\mathrm{E}}_{t-\Delta t}\left[\Sigma OPP_t \mid p_t\right]=(p_t-c)\hat{\mathrm{E}}_{t-\Delta t}\left[q_t \mid p_t\right]+\left(1-\hat{\mathrm{E}}_{t-\Delta t}\left[\mathrm{I}_{F_t<\frac{-E_{t+}}{(1+\gamma)}} \mid p_t\right]\right)e^{-r\Delta t}\hat{\mathrm{E}}_t\left[\Sigma OPP_{t+\Delta t}\right].$$

For the uncontrolled fourth and final production period, corporate taxation is the only deadweight cost that applies (i.e. $\hat{\mathrm{E}}_{3\Delta t}[\Sigma PTP_{4\Delta t}]$, $\hat{\mathrm{E}}_{3\Delta t}[\Sigma MIS_{4\Delta t}]$, $\hat{\mathrm{E}}_{3\Delta t}[\Sigma FND_{4\Delta t}]$ and $\hat{\mathrm{E}}_{3\Delta t}[\Sigma BNK_{4\Delta t}]$ all equal zero). Thus, from equation (2.1), the expected end-of-period four deadweight cost is:

$$\begin{aligned}\hat{\mathrm{E}}_{3\Delta t}\left[\Sigma DWC_{4\Delta t}\right]&=\hat{\mathrm{E}}_{3\Delta t}\left[\Sigma TAX_{4\Delta t}\right]\\&=\left(R-\sum_{i=1}^{3}\hat{\mathrm{E}}_{(i-1)\Delta t}\left[q_{i\Delta t} \mid p_{i\Delta t}\right]\right)\hat{\mathrm{E}}_{3\Delta t}\left[(p_{4\Delta t}-c)\left(\alpha\lambda+\alpha(1-\lambda)\mathrm{I}_{p_{4\Delta t}>c}\right)\right]\mathrm{I}_{E_{3\Delta t+}>0}.\end{aligned}$$

Expected end-of-period four earnings before interest and tax and operating profit are both equal to the forward value of the uncontrolled, all-equity company value given by equation (2.1) with add-back of the expected deadweight cost due to taxation:

$$\hat{\mathrm{E}}_{3\Delta t}\left[\Sigma EBIT_{4\Delta t}\right]=\hat{\mathrm{E}}_{3\Delta t}\left[\Sigma OPP_{4\Delta t}\right]=e^{r\Delta t}E_{3\Delta t+}+\hat{\mathrm{E}}_{3\Delta t}\left[\Sigma DWC_{4\Delta t}\right].$$

2.3. Model application

To determine the valuation and financial risk differentiations for the model company between a joint hedging and leverage decision, an unhedged leverage decision, an unlevered hedging decision, and an unhedged and unlevered decision, four strategies for control behaviour are specified. Each strategy has the common aim of maximising the value of the company (signified by $\hat{\mathrm{E}}_{-\Delta t}[E_{0-}+O_{0-} \mid p_0]$), but with different limitations on the available control variables. The control decision to abandon (voluntarily liquidate) the company is made available to all control strategies by always allowing an extreme hedge choice of short forward 10 million units at the start of each production period (irrespective of whether an individual control strategy specifies availability of hedging control).[12] The four control strategies are:

1. *Unlevered & unhedged* strategy, which requires optimisation of the abandonment decision, with the absence of leverage or hedging.

2. *Levered & unhedged* strategy, which requires joint optimisation of the abandonment decision, and the 31 price-path-specific leverage control variables ($y_{t,t+\Delta t}$), with the absence of hedging.

3. *Unlevered & hedged* strategy, which requires joint optimisation of the abandonment decision, and the 114 price-path-specific hedging control variables: 38 each of hedge quantity ($x_{t,t+\kappa\Delta t}$), short forward versus put option ratio ($w_{t,t+\kappa\Delta t}$) and put option strike price ($z_{t,t+\kappa\Delta t}$) control variables, with the absence of leverage.

4. *Levered & hedged* strategy, which requires joint optimisation of the abandonment decision, and all 145 price-path-specific leverage and hedging control variables (31 of $y_{t,t+\Delta t}$, and 38 each of $x_{t,t+\kappa\Delta t}$, $w_{t,t+\kappa\Delta t}$ and $z_{t,t+\kappa\Delta t}$).

The value-maximising (optimised) control variable set for each control strategy is obtained via ongoing iterations of Microsoft Excel's 'Solver' algorithm applied to the control variables in overlapping subsets,[13] in sequence with linearly extrapolated value maximisation applied to the control variables individually, until the model company's value is maximised to a sensitivity of within a millionth of the value of each individual control variable (plus or minus a millionth if the control variable is not at a boundary

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[12] An extreme hedge choice radically increases financial risk and makes the required up-front expense for the comprehensive limited liability option unviable, which thereby triggers *immediate* bankruptcy, in which case the extreme hedge choice does not actually get instigated.

[13] Microsoft Excel's 'Solver' tool uses the Generalized Reduced Gradient nonlinear optimization code developed by Leon Lasdon, University of Texas at Austin, and Allan Warren, Cleveland State University. Over the course of model development, Matlab's 'fmincon' and 'ga' tools were also trialled during experimental efforts at optimising the control variables, but Microsoft Excel's 'Solver' was found to be more effective.

limit; plus (minus) a millionth if the control variable is at a lower (upper) boundary limit). To avoid localised maxima, the first-pass optimisation result is further refined and 'stress-tested' by deviating various solution components of the optimised control variable set and repeating the optimisation process; and so on with subsequent optimisation passes, until no further increments to value are deemed achievable.

Measures of leverage, hedging and financial risk

For different exogenous parameter scenarios, the four control strategies are assessed for differences in optimised leverage and hedging levels and financial risk. Appendix B details the formulations of the leverage, hedging and financial risk measures.

The leverage measure (ℓ_{t+}) is the risk-free valuation of newly issued debt ($D_{t+} = y_{t,t+\Delta t} e^{-r\Delta t}$) divided by ongoing equity value (E_{t+}). Recall that although the model accommodates risky debt, debt's risk premium is indistinctly incorporated into the comprehensive limited liability option purchased by equity from all non-equity stakeholders. For this reason ℓ_{t+} is calculated using risk-free debt valuation and will therefore tend to be a high measure in comparison to empirical leverage observations based on market or book values of debt.

Naive measures of the extent of hedging consider only the contracted price and quantity of individual hedge positions in a firm's overall hedge portfolio. An alternative approach is to calculate the delta of the hedge portfolio (i.e. the sensitivity of the hedge portfolio value with respect to the underlying asset price) which intrinsically takes into account any non-linearity of the individual hedge positions. The extent of hedging is then indicated by the negative of, the ratio of the hedge portfolio delta to the quantity of underlying asset to which the firm has financial exposure. This measure is here termed the hedge-delta ratio (h_{t+}) and is effectively equivalent to the delta-percentage measure defined by Tufano (1996). Because the model company has available to it the use of non-linear hedge contracts (specifically put options), the hedge-delta ratio is considered an appropriate hedging measure. The model company's hedge portfolio delta is calculated as a discrete measure in accordance with the assumed binomial price process; the formulation for the discrete binomial process delta of individual hedge positions ($\Delta x_{\tau,\tau+\kappa\Delta t} X_{t:\tau,\tau+\kappa\Delta t} / \Delta p_t$) is given by equation (B.1) in Appendix B. Summing the individual hedge position deltas to give the hedge portfolio delta, the hedge-delta ratios (h_{t+}) for each of the company's three controlled production periods are thus specified to be:

$$h_{0+} = \frac{-\left(\frac{\Delta}{\Delta p_0} x_{0,\Delta t} X_{0:0,\Delta t} + \frac{\Delta}{\Delta p_0} x_{0,2\Delta t} X_{0:0,2\Delta t} + \frac{\Delta}{\Delta p_0} x_{0,3\Delta t} X_{0:0,3\Delta t} \right)}{R}, \tag{2.2}$$

$$h_{\Delta t+} = \frac{-\left(\frac{\Delta}{\Delta p_{\Delta t}} x_{0,2\Delta t} X_{\Delta t:0,2\Delta t} + \frac{\Delta}{\Delta p_{\Delta t}} x_{0,3\Delta t} X_{\Delta t:0,3\Delta t} + \frac{\Delta}{\Delta p_{\Delta t}} x_{\Delta t,2\Delta t} X_{\Delta t:\Delta t,2\Delta t} + \frac{\Delta}{\Delta p_{\Delta t}} x_{\Delta t,3\Delta t} X_{\Delta t:\Delta t,3\Delta t} \right)}{R - \hat{\mathrm{E}}_0 \left[q_{\Delta t} \mid p_{\Delta t} \right]},$$

$$h_{2\Delta t+} = \frac{-\left(\frac{\Delta}{\Delta p_{2\Delta t}} x_{0,3\Delta t} X_{2\Delta t:0,3\Delta t} + \frac{\Delta}{\Delta p_{2\Delta t}} x_{\Delta t,3\Delta t} X_{2\Delta t:\Delta t,3\Delta t} + \frac{\Delta}{\Delta p_{2\Delta t}} x_{2\Delta t,3\Delta t} X_{2\Delta t:2\Delta t,3\Delta t} \right)}{R - \hat{\mathrm{E}}_0 \left[q_{\Delta t} \mid p_{\Delta t} \right] - \hat{\mathrm{E}}_{\Delta t} \left[q_{2\Delta t} \mid p_{2\Delta t} \right]}.$$

The denominator of the hedge-delta ratio is specified to be the remaining resource expected to be produced for all future production periods (i.e. the total remaining underlying asset). An alternative approach would be to specify that the denominator equal the expected production quantity for only the remaining controlled (hedgeable) production periods (i.e. excluding the uncontrolled fourth production period), which would match the production time-frame of the denominator with the maximum hedge maturity time-frame of the numerator. However, the delta measure indicates immediate hedge portfolio sensitivity to underlying price irrespective of specific hedge contract maturities or the time-frame for the underlying price exposure. Thus it is preferable that the total remaining production resource be incorporated in the hedge-delta ratio so as to indicate the extent of hedging for the entire spot position.[14]

[14] The discrete binomial process delta measure does not indicate instantaneous hedge portfolio sensitivity to underlying price, but instead indicates sensitivity for the next time-step of the binomial price process. Given equivalence of the price risk underlying the hedge and spot positions, a hedge-delta ratio of one would indicate a full hedge for the next time-step of the binomial price process. The model company is limited to being able to adjust its hedge portfolio only at the start of each production period, not each time-step of the binomial price process within each production period; thus a start-of-period full hedge (or any other hedge ratio) cannot be dynamically maintained throughout the ensuing production period. This is considered representative of the impossibility in practice of continuously maintaining any precise hedge position.

Five measures of financial risk are considered, the first being a limited liability risk measure equal to the value of equity's comprehensive limited liability option divided by ongoing equity value ($-O_{t+}/E_{t+}$, i.e. the proportion of equity's ongoing value attributable to the value of its comprehensive limited liability option). The value of the comprehensive limited liability option is a particularly meaningful indicator of financial risk as it combines consideration of both magnitude and probability of shortfall in satisfying due liabilities across all possible price and quantity outcomes for production output; dividing the option's value by ongoing equity value provides a relative measure.

The second financial risk measure is equity's relative value-at-risk from an extreme drop in output price for a production period (υ_{t+}):

$$\upsilon_{t+} = 1 - \frac{\hat{\mathrm{E}}_t \left[E_{(t+\Delta t)-} \mid p_{t+\Delta t} = p_t e^{\left(r-\delta-\sigma_p^2/2\right)\Delta t - 2\sigma_p \sqrt{\Delta t}} \right]}{E_{t+}}$$

where: $E_{(t+\Delta t)-}$ is the total equity value (dividend plus ongoing equity value) at the end of the ensuing production period; and, given the assumption of a four-step binomial price process within production periods, $p_t e^{\left(r-\delta-\sigma_p^2/2\right)\Delta t - 2\sigma_p \sqrt{\Delta t}}$ is the minimum possible output price at the end of the ensuing period.

As further measures of financial risk, two probability of bankruptcy ($\Pr B_{t-}$, $\hat{\Pr} B_{t+}$) calculations and an output price beta ($\hat{\beta}_{t+}$) calculation are considered (see equations (B.2), (B.3) and (B.4) respectively in Appendix B). $\Pr B_{t-}$ is the probability of bankruptcy for the preceding production period due to production quantity risk. $\hat{\Pr} B_{t+}$ and $\hat{\beta}_{t+}$ are risk-neutral measures of financial risk. $\hat{\Pr} B_{t+}$ is the probability of bankruptcy for the ensuing production period. $\hat{\beta}_{t+}$ is the sensitivity of equity's rate of return to the underlying output price rate of return for the ensuing production period (i.e. equity's output price beta). $\hat{\beta}_{t+} > 1$ (<1) implies that the return to equity is expected to be more (less) than one-for-one with the return to the underlying output price.

Exogenous parameters

To undertake a numerical analysis, the parameter values shown in Table 2 are chosen to represent the model company. Having already specified there to be three controlled production periods ($N = 3$), the time-span of each production period (Δt) is set to four years. The initial median periodic production quantity ($\overline{q}_0$) is standardised to 100 units, and commensurately the expected total output resource (R) for the three controlled production periods plus the uncontrolled fourth production period is assumed to be 400 units. Base-case ('mid') cost of production (c) is standardised to one dollar per unit of expected production quantity. For the purpose of sensitivity analysis, 'low' and 'top' values for c respectively equal to 0.5 and 1.5 dollars per unit of expected production quantity are also considered. The initial output price (p_0) is set to 1.5 dollars per unit; this represents an initial price margin of 50% over the base-case expected unit production cost.

Output price volatility (σ_p) is assigned a base-case 'mid' value of 0.2 per year and a 'low' value of 0.1 per year. For empirical comparison, Slade and Thille's (2006) mean daily spot price volatility summary statistics translate into annualised price volatilities ranging between 16% and 27% for six metals traded on the London Metal Exchange during the 1990s (assuming 250 trading days a year). Production quantity uncertainty (σ_q) is assigned a base-case 'mid' value of 0.1 per year and a 'top' value of 0.2 per year. To provide further empirical perspective, data presented in Appendix C suggests that a typical gold mine is subject to annual production quantity uncertainty (i.e. standard deviation of the log difference between realised and forecast production quantity) of up to 17%; but diversification benefit applies to aggregate production quantity uncertainty for firms with more than one mine.[15]

With consideration of the recent history of US three-month Treasury bill rates, the risk-free interest rate (r) is set to 0.04 per year. The output convenience yield (δ) is set to 0.02 per year. For empirical comparison, Casassus and Collin-Dufresne (2005) esti-

[15] Note that the empirical production quantity uncertainty calculated in Appendix C includes any variability of production capacity/efficiency. As already discussed however, by model set-up, the production quantity uncertainty parameter notionally represents variability of resource quality/ concentration only.

mated long-term average annual convenience yields for silver, gold, copper and crude oil of about 0%, 1%, 6% and 11% respectively for the period 1990 to 2003.[16]

Table 2 – Exogenous parameter values

'Low', 'mid' (base-case) and 'top' parameter values chosen to represent the model company.

Exogenous parameter		'Low' value	Base-case 'mid' value	'Top' value
Number of controlled production periods	N	-	3	-
Production period time-span	Δt	-	4 years	-
Expected total output resource	R	-	400 units	-
Initial output price	p_0	-	$1.5 /unit	-
Initial median periodic production	$\bar{q}_0$	-	100 units	-
Output price volatility	σ_p	0.1 /year	0.2 /year	-
Production quantity uncertainty	σ_q	-	0.1 /year	0.2 /year
Output price and production quantity correlation	ρ	0	0.1	0.3
Output convenience yield	δ	-	0.02 /year	-
Production cost per unit of expected production quantity	c	$0.5 /unit	$1 /unit	$1.5 /unit
Risk-free interest rate	r	-	0.04 /year	-
Corporate tax rate	α	-	0.35	-
Personal tax penalty of non-equity	$\alpha - A_\alpha$	-	0.25	0.35
Claimability of a tax refund for a loss	λ	-	0.5	-
Financial distress cost rate	γ	-	0.3	1
Bankruptcy cost rate	b	-	0.4	-
Free cash-flow misappropriation rate	a	-	0	0.02
Hedge transaction cost rate	ε	0	0.005	-

The correlation between the output price and production quantity (ρ) is specified to have a base-case 'mid' value of 0.1, and also 'low' and 'top' values of zero and 0.3 respectively. It is intended that ρ be representative of an endogenous operational strategy

[16] An asset's convenience yield indicates the relative degree to which physical possession of the asset is not substitutable with a contract entitling future possession. Hence it is their nature that store-of-value resources have low or negligible convenience yields, and consumption resources have comparatively high convenience yields.

rather than an exogenous condition of the market for the company's production output. That is, the company is assumed to be a price-taker, and ρ parameterises a fixed strategy of shifting (expected) production quantity across time dependent on the output price level; presuming that production rate response to price change can occur concurrently with price change each production period. A strategy of high production when market price for output is high and vice versa (i.e. positive ρ) can be appropriate if there exists information asymmetry between managers and investors about the quality of the company. When investors are uncertain of the company's total resource quantity or production costs, the actual act of resource extraction will give proof to management claims about the company. The valuation benefit of such demonstration of credibility is traded-off against the time-value (opportunity) cost of early exercise of the real option to extract each unit of resource. If the credibility benefit increases with the rate of extraction, and given that the time-value cost of exercising the real option to extract a unit of resource decreases with the in-the-money price of the resource output, then the optimal extraction rate will be positively related to the output price (above the real option exercise price).[17] Nevertheless there will be physical constraints that limit the achievable extraction rate. The model set-up also entails a limit: so that the expected output resource remaining for fourth period production can never be less than zero, the maximum allowable value for ρ is 0.33 (when other relevant parameters are set to their base-case 'mid' values).

The corporate tax rate (α) is set to the top earnings level US federal rate of 0.35 (state taxes are ignored). The claimability of a tax refund in event of negative earnings (λ) is set to 0.5 (to represent an 'average' tax shield benefit from carry-forward and carry-back provisions for losses). The personal tax penalty acting against debt's role as a corporate tax shield is the corporate tax rate less the effective rate of combined personal and corporate tax being shielded by debt finance ($\alpha - A_\alpha$). Based on the results of Graham (1999), the base-case value of A_α is set to 0.1. The extreme of a full personal tax penalty is also considered. Hence the base-case 'mid' and 'top' values for ($\alpha - A_\alpha$) are set to 0.25 and 0.35 respectively.

[17] This is an extension of reasoning demonstrated by Grundy and Raaballe (2005).

The financial distress cost rate (γ) is assigned a base-case 'mid' value of 0.3. That is, base-case financial distress cost equals 30% of any shortfall in free cash-flow ($-0.3F_t$ given $F_t < 0$), up to the free cash-flow shortfall limit at which financial distress becomes bankruptcy, which entails a maximum for base-case financial distress cost equal to 23% of ongoing (post-distress) equity value ($-0.3F_t \leq 0.3E_{t+}/(1+0.3)$). This base-case maximum financial distress cost level is broadly congruent with the empirical evidence of Andrade and Kaplan (1998), who estimated the cost of financial distress to be of the order of 10% to 20% of *pre*-distress *total* company value. For sensitivity analysis, γ is also assigned a 'top' value of one.

The bankruptcy cost rate (b) applies as a factor of debt face-value. By model set-up, with occurrence of bankruptcy, all due liabilities are honoured (if possible) with pre-tax liquidation cash-flow. Thus to avoid instances of abandonment/bankruptcy being favourably used to return capital to debt-holders from pre-tax cash-flow, b is set to a value greater than the corporate tax rate (α). Furthermore, the financial distress cost rate does not get applied with the occurrence of bankruptcy (although it does have an opportunity cost effect via the free cash-flow shortfall limit at which bankruptcy is instigated). Hence b is set to a value that is considered to reflect financial distress costs, not just direct bankruptcy costs. Considering again the order of magnitude of Andrade and Kaplan's (1998) financial distress cost estimation relative to pre-distress total company value, b is set to a value of 0.4.

The rate of management misappropriation of positive free cash-flow (a) is arbitrarily assigned base-case 'mid' and 'top' values of zero and 0.02 respectively. The hedge transaction cost rate (ε) is assigned 'low' and base-case 'mid' values of zero and 0.005 respectively.

3. RESULTS

Figures 3a to 3i present the various total value, optimal leverage, optimal hedge and financial risk results for the model company at the start of each controlled production period, for each possible output price-path, and for each of the four control strategies, with the exogenous parameters set to the base-case 'mid' values provided in Table 2. At the end of the first controlled production period (at time $t = \Delta t$), there are five possible output price outcomes ($p_{\Delta t} \in \{0.67, 1.01, 1.50, 2.24, 3.34\}$). These five possible values for $p_{\Delta t}$ each branch out to five possible price outcomes at the end of the second controlled production period ($p_{2\Delta t}$). Thus there are five sets of five possible outcomes for $p_{2\Delta t}$ conditional on $p_{\Delta t}$. The sets of $p_{2\Delta t}$ conditional on $p_{\Delta t}$ are overlapping so that there are nine possible unconditional outcomes for $p_{2\Delta t}$ ($p_{2\Delta t} \in \{0.30, 0.45, 0.67, 1.01, 1.50, 2.24, 3.34, 4.98, 7.43\}$). For the base-case the model company is optimally abandoned/bankrupt for price outcome $p_{\Delta t} = 0.67$ at end-of-period one, consequently Figures 3a to 3i omit display of the redundant set of five possible $p_{2\Delta t}$ outcomes conditional on $p_{\Delta t} = 0.67$ (i.e. Figures 3a to 3i display only four of the five sets of five possible outcomes for $p_{2\Delta t}$ conditional on $p_{\Delta t}$). At each price-node (p_0, $p_{\Delta t}$ and $p_{2\Delta t}$ conditional on $p_{\Delta t}$) the results for the six control strategies are displayed as a bar graph and also given numerically.

To facilitate discussion of results, the five possible output price outcomes for the first controlled production period (at time $t = \Delta t$) are described as: 'large-up' (symbolised as p_{uu}), 'up' (p_u), 'middle' (p_m), 'down' (p_d) and 'large-down' (p_{dd}). The 25 possible path-specific price outcomes for the second controlled production period (at time $t = 2\Delta t$) are described in similar manner as: 'large-up, large-up' ($p_{uu,uu}$), 'large-up, up' ($p_{uu,u}$), 'large-up, middle' ($p_{uu,m}$), 'large-up, down' ($p_{uu,d}$), 'large-up, large-down' ($p_{uu,dd}$); 'up, large-up' ($p_{u,uu}$), 'up, up' ($p_{u,u}$), 'up, middle' ($p_{u,m}$), 'up, down' ($p_{u,d}$), 'up, large-down' ($p_{u,dd}$); et cetera.

In reviewing the results, be aware that at each price-node there exists the risk of bankruptcy due to production quantity uncertainty for the preceding production period. For each time t, being the instant when a production period finishes and another begins, define time $t+$ to occur instantaneously after time t. Given the control variables and knowing the output price (p_t) at a price-node, it is the production quantity outcome (q_t) that finally determines at time t whether a condition of solvency, financial distress or bankruptcy is in effect and the consequential cash-flows to be distributed to stakeholders. Thus at time $t+$ the company will be *ongoing* only if bankruptcy has not occurred at time t (or at an earlier instance). The probability of such an 'ex-post' bankruptcy (i.e. the probability of bankruptcy due to quantity risk for the period preceding the price-node) is given by the calculation of $\Pr B_{t-}$, for which the base-case results presented in Figure 3h show only a few instances where quantity risk is relatively so large as to cause notable uncertainty about whether the company will be ongoing at a price-node. Furthermore, result measures given the subscript $t+$ imply ongoing values, meaning they are conditional on bankruptcy having not occurred at time t (or earlier).

Base-case total value, leverage, hedging and financial risk

The valuation benefit of joint hedging and leverage is observable in the differences between the values of $\hat{\mathrm{E}}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t]$ at time $t = 0$ for each of the four control strategies as shown in Figure 3a. The $\hat{\mathrm{E}}_{-\Delta t}[E_{0-} + O_{0-} \mid p_0]$ measure (n.b. $D_{0-} = 0$ by model set-up) is the all-equity value of the company prior to any control decisions being instigated, which provides a uniform valuation basis for comparison of all four control strategies. That is, $\hat{\mathrm{E}}_{-\Delta t}[E_{0-} + O_{0-} \mid p_0]$ is the present value of expected cash-flows to the initial all-equity owners of the company, and differences in $\hat{\mathrm{E}}_{-\Delta t}[E_{0-} + O_{0-} \mid p_0]$ arise purely due to the cash-flow effects of differences in control strategy going forward in time. Figure 3a (at time $t = 0$) shows that the *levered & hedged* strategy is more valuable than the unlevered or unhedged strategy alternatives. Specifically, for the base-case, the *levered & hedged* strategy offers valuation premiums of 4.0%, 4.0% and 1.3% respectively relative to the *unlevered & unhedged* strategy, the *levered & unhedged* strategy and the *unlevered & hedged* strategy.

Figure 3b shows that the two hedging strategies with and without leverage benefit from a relative reduction in expected deadweight costs. The expected deadweight cost-to-value ratio ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma DWC_t \mid p_t] / \hat{\mathrm{E}}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t]$) indicates the ratio of: total current and ongoing expected deadweight costs due to corporate taxation, the personal tax penalty of non-equity, free cash-flow misappropriation, financial distress or bankruptcy; to expected end-of-period total company value immediately prior to any cash-flow distribution to equity or non-equity. Note that current and ongoing expected hedge transaction costs, and the expected production opportunity loss associated with the risk of untimely bankruptcy, are not included in the expected deadweight cost numerator, but do reduce the expected total company value denominator. For the base-case scenario at time $t = 0$, the expected deadweight cost-to-value ratio reduces from 0.57 for both the *unlevered & unhedged* strategy and the *levered & unhedged* strategy to 0.54 for the *unlevered & hedged* strategy and to 0.52 for the *levered & hedged* strategy. Figure 3b also shows the benefit of hedging, in terms of reduced relative expected deadweight costs, is most profound for 'very weak' output price outcomes ($p_t = 1.01$) close to the unit production cost and bordering on abandonment/bankruptcy.

The base-case optimal leverage results shown in Figure 3c indicate that optimal joint hedging and leverage almost always entails higher leverage than optimal leverage without hedging. Only in event of a very bullish price outcome ($p_{uu,uu} = 7.43$, $p_{uu,u} = 4.98$ or $p_{u,uu} = 4.98$) does the *levered & unhedged* strategy entail higher leverage than the *levered & hedged* strategy. Nevertheless, from comparison with the optimal hedge results shown in Figure 3d, it is evident that the relationship between hedging and leverage cannot always be straightforwardly described as facilitative (from hedging to leverage). That is, the *levered & hedged* strategy results in a distinct negative relationship between leverage and output price outcome; but when leverage is at its highest, hedging is actually reduced.

The base-case results given by Figures 3e to 3i show that the two hedging strategies with and without leverage always entail less financial risk (or equal financial risk at a minimum value of zero for some measures) than the two unhedged strategies with and without leverage. Meanwhile the *levered & hedged* strategy is more risky than the *unlevered & hedged* strategy for 'weak' output price outcomes ($p_{\Delta t} = 1.01$, $p_{2\Delta t} \leq 1.50$), but is generally less risky otherwise.

The following analysis considers the separate and joint hedging and leverage strategies and their valuation and financial risk implications with comparison to the positive and normative results of previous studies.

Figure 3a – Price-path-dependent total company value for base-case

Total value of the value-maximised model company at the start of each controlled production period instantaneously before new control decisions are instigated ($\hat{E}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t]$), for each possible output price-path, and for each of four control strategies, with exogenous parameters set to base-case 'mid' values provided in Table 2.

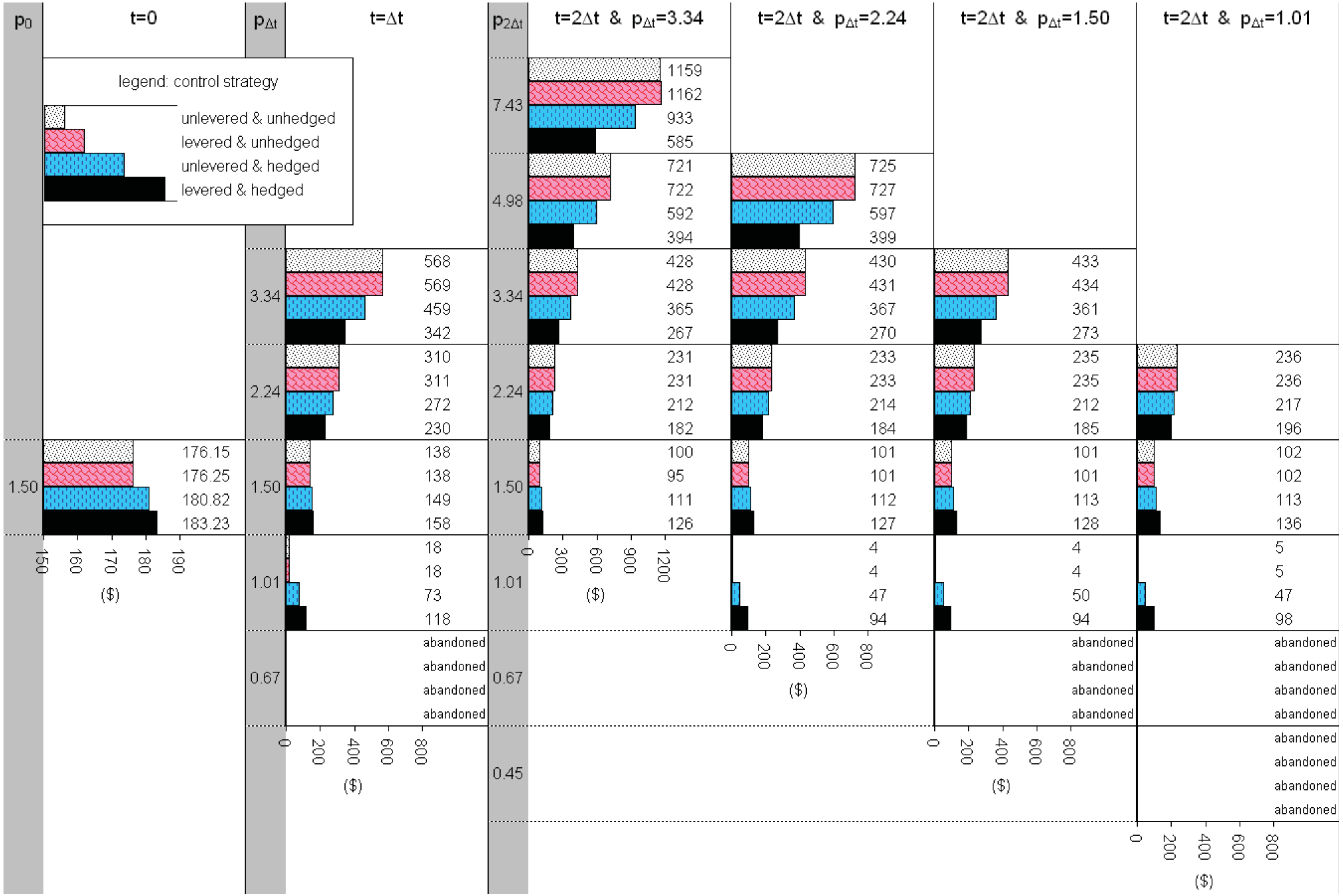


Figure 3b – Price-path-dependent expected deadweight cost-to-value ratio for base-case

Expected deadweight cost-to-value ratio ($\hat{E}_{t-\Delta t}[\Sigma DWC_t \mid p_t] / \hat{E}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t]$) for the value-maximised model company at the start of each controlled production period instantaneously before new control decisions are instigated, for each possible output price-path, and for each of four control strategies, with exogenous parameters set to base-case 'mid' values provided in Table 2.

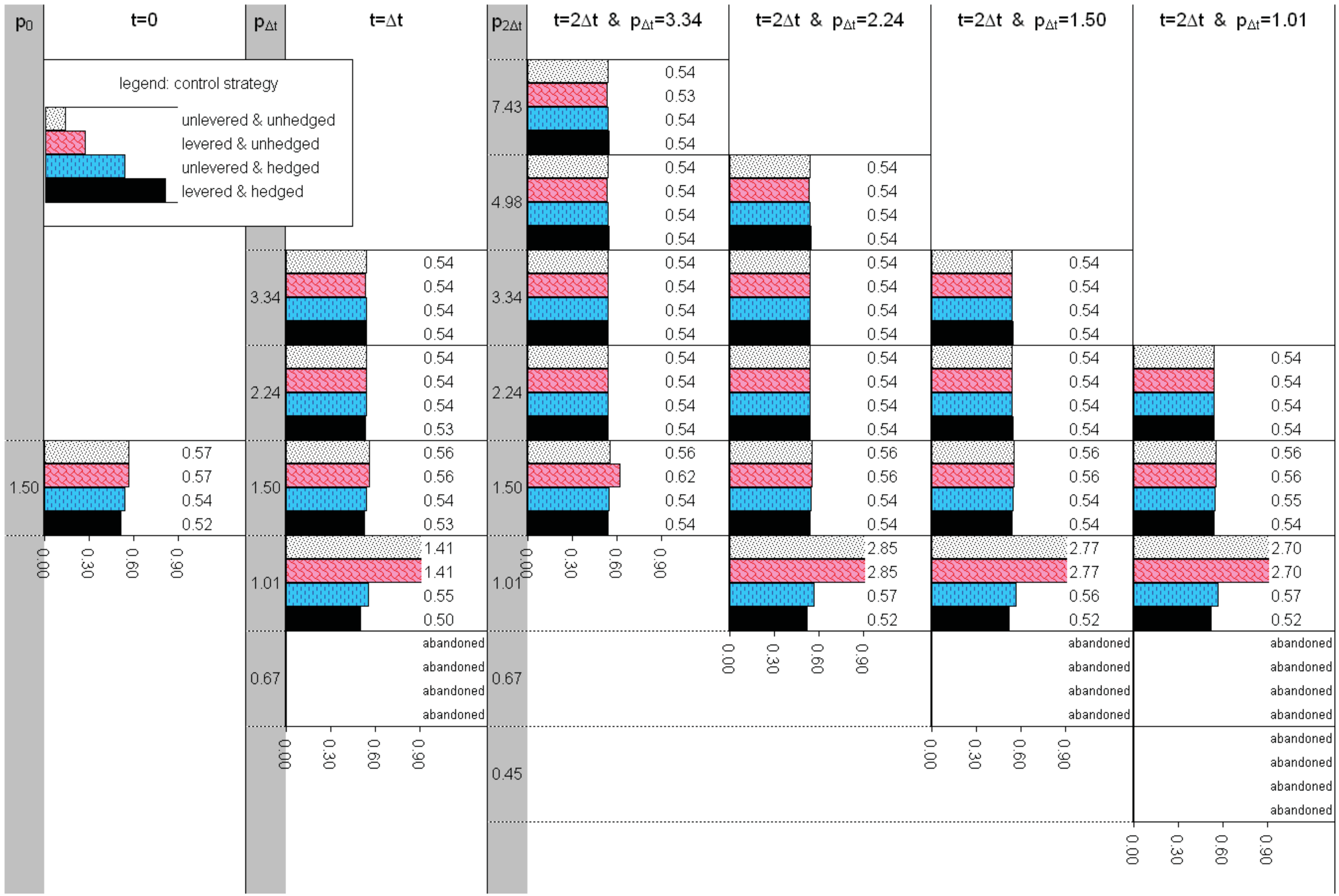


Figure 3c – Price-path-dependent optimal leverage for base-case

Optimal (value-maximising) leverage ($\ell_{t+} = D_{t+} / E_{t+}$) for the model company at the start of each controlled production period, for each possible output price-path, and for each of two control strategies, with exogenous parameters set to base-case 'mid' values provided in Table 2.

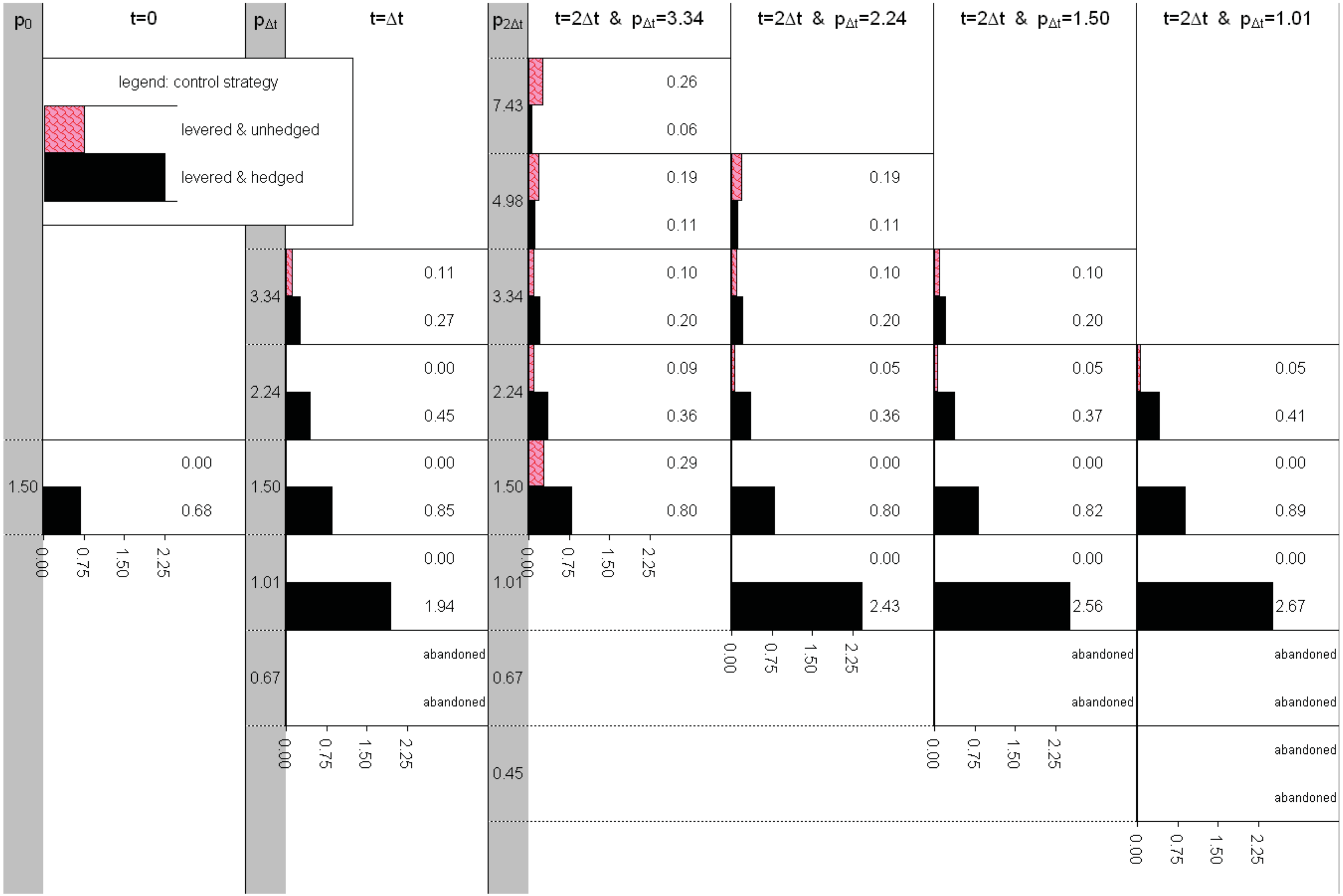


Figure 3d – Price-path-dependent optimal hedge-delta ratio for base-case

Optimal (value-maximising) hedge-delta ratio (h_{t+}) for the model company at the start of each controlled production period, for each possible output price-path, and for each of two control strategies, with exogenous parameters set to base-case 'mid' values provided in Table 2.

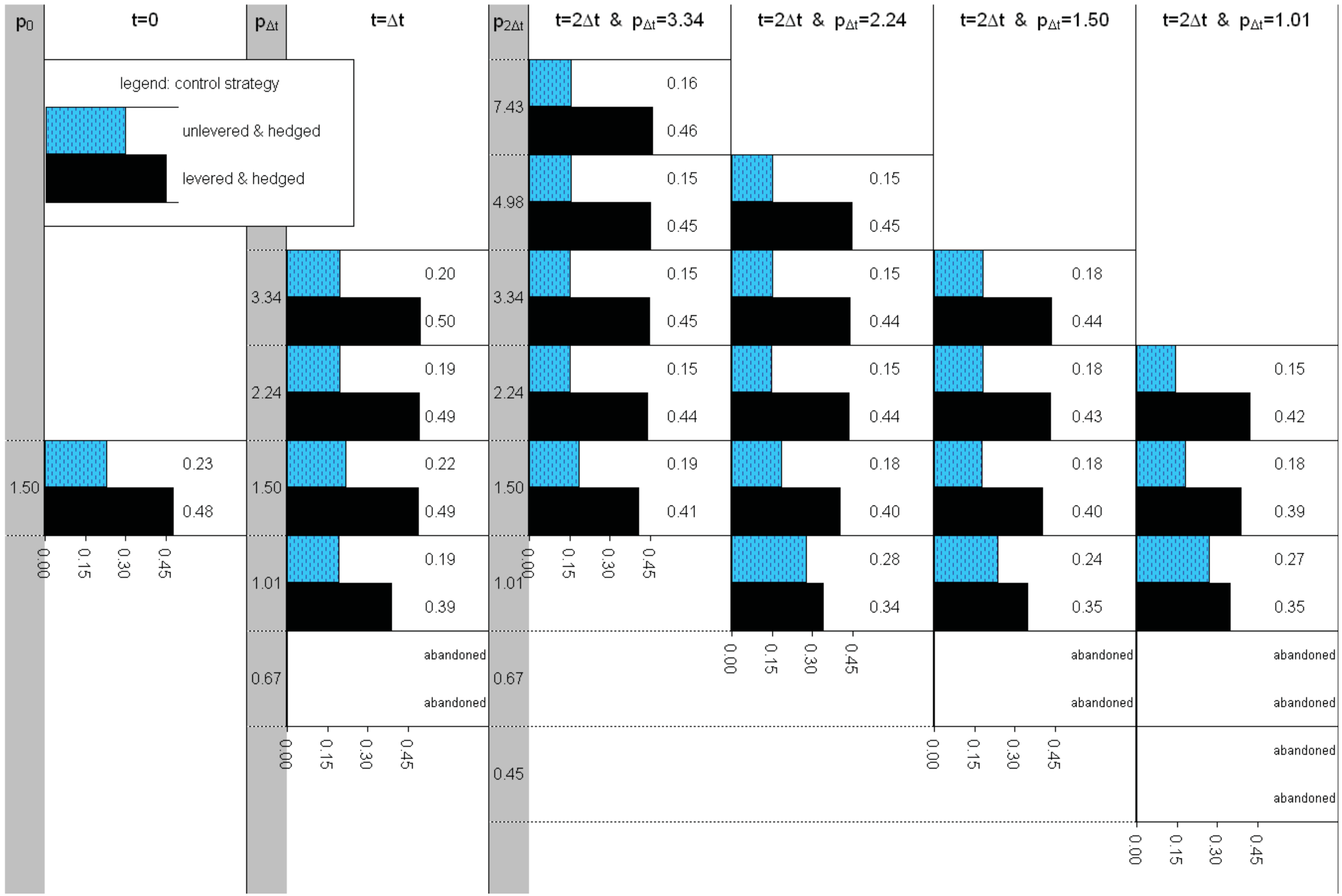


Figure 3e – Price-path-dependent relative limited liability risk for base-case

Relative limited liability risk ($-O_{t+}/E_{t+}$) for the value-maximised model company at the start of each controlled production period, for each possible output price-path, and for each of four control strategies, with exogenous parameters set to base-case 'mid' values provided in Table 2.

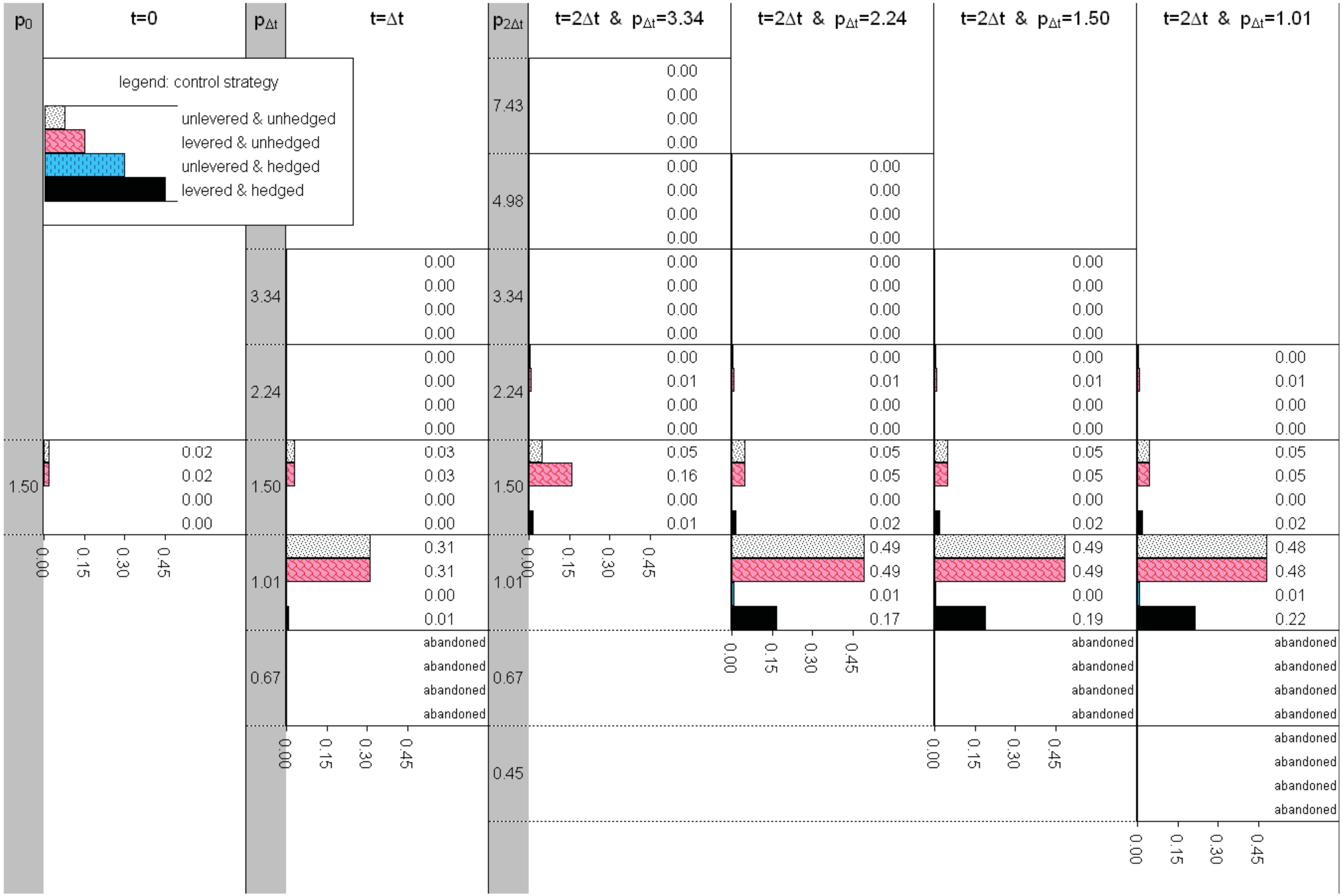


Figure 3f – Price-path-dependent relative equity value-at-risk for base-case

Equity's relative value-at-risk (υ_{t+}) for the value-maximised model company at the start of each controlled production period, for each possible output price-path, and for each of four control strategies, with exogenous parameters set to base-case 'mid' values provided in Table 2.

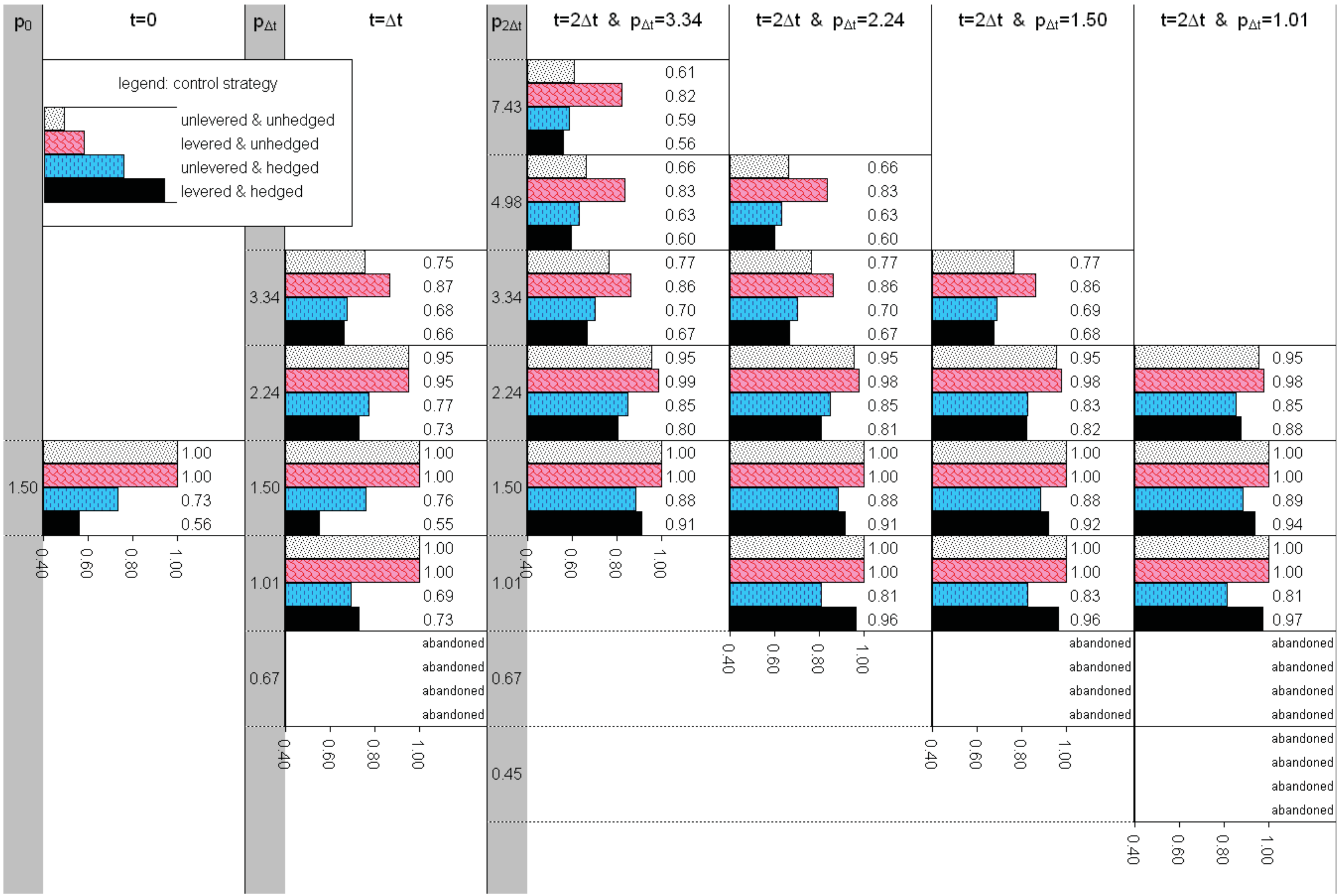


Figure 3g – Price-path-dependent output price beta of equity for base-case

Equity's output price beta ($\hat{\beta}_{t+}$) for the value-maximised model company at the start of each controlled production period, for each possible output price-path, and for each of four control strategies, with exogenous parameters set to base-case 'mid' values provided in Table 2.

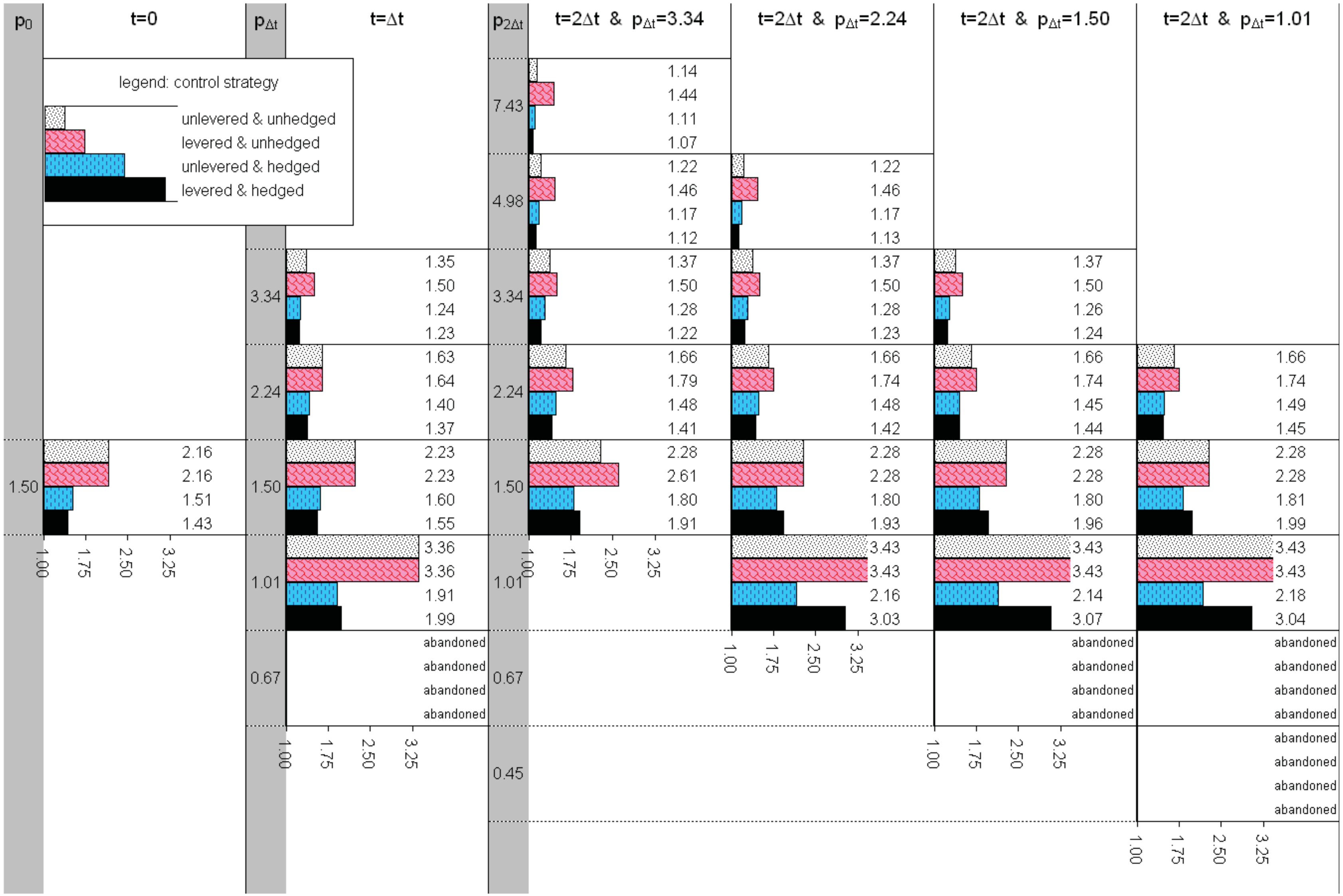


Figure 3h – Price-path-dependent ex-post probability of bankruptcy for base-case

Probability of bankruptcy for the preceding production period due to production quantity risk ($\Pr B_{t-}$) for the value-maximised model company at the start of each controlled production period, for each possible output price-path, and for each of four control strategies, with exogenous parameters set to base-case 'mid' values provided in Table 2.

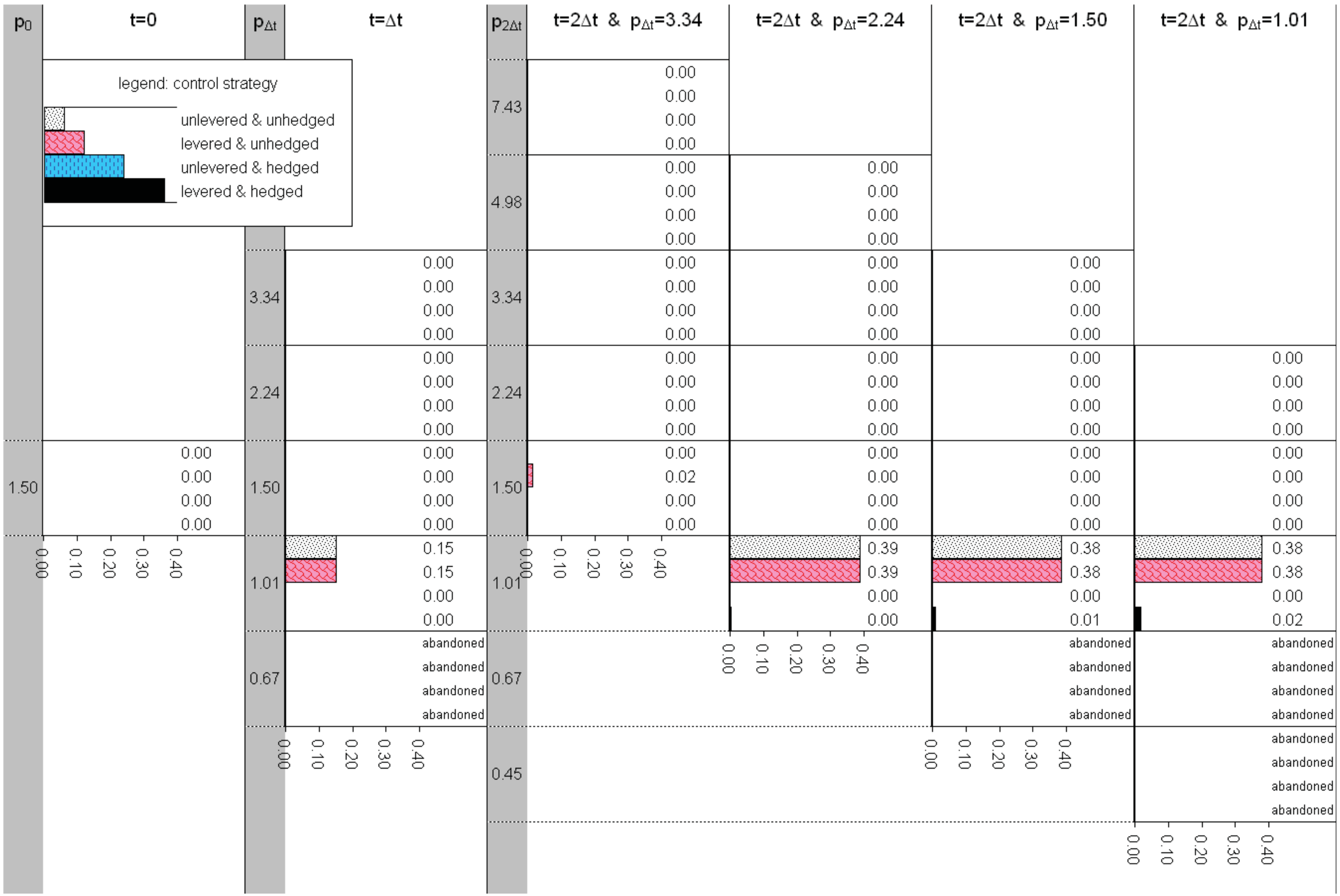
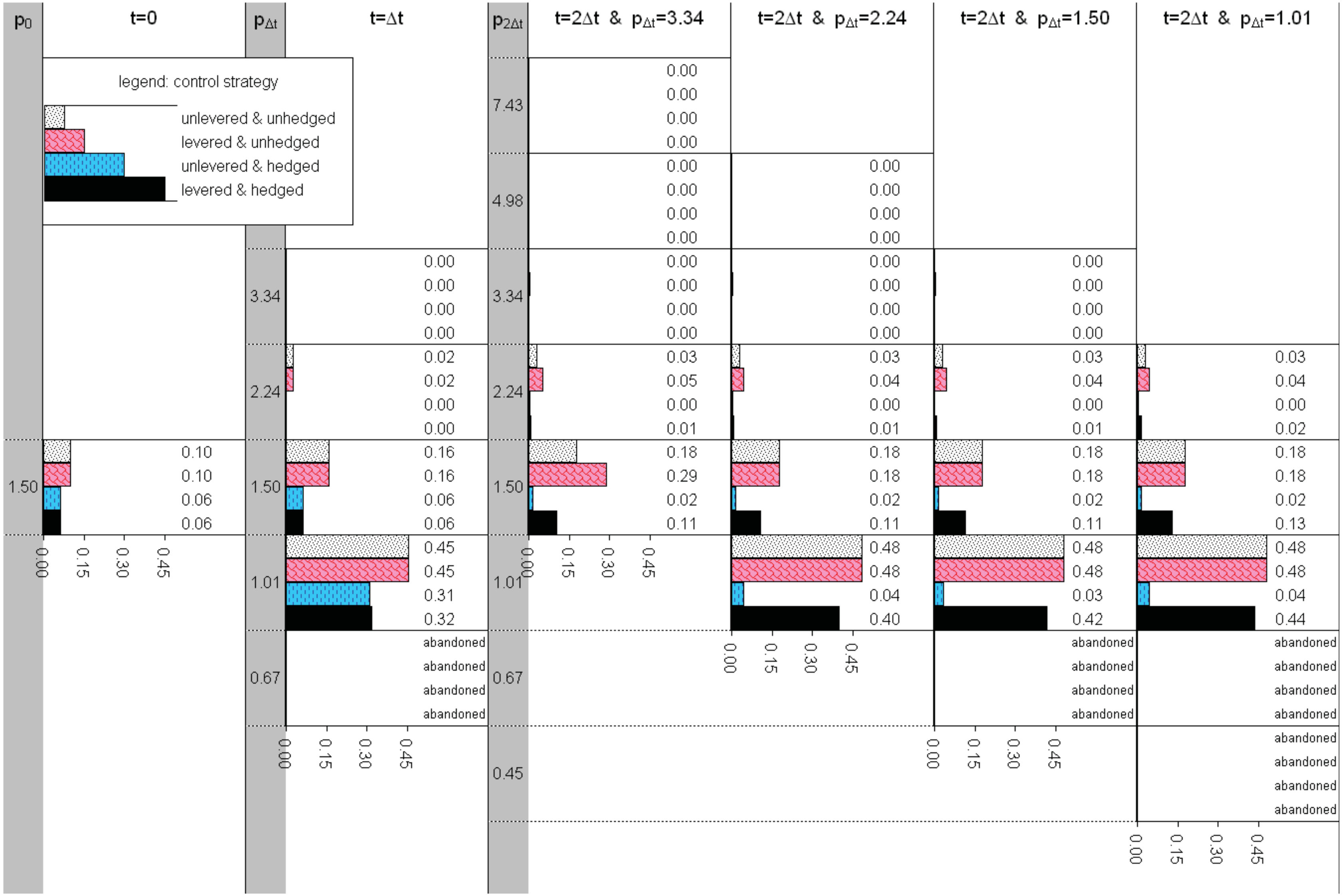


Figure 3i – Price-path-dependent ongoing probability of bankruptcy for base-case

Probability of bankruptcy for the ensuing production period ($\hat{\Pr} B_{t+}$) for the value-maximised model company at the start of each controlled production period, for each possible output price-path, and for each of four control strategies, with exogenous parameters set to base-case 'mid' values provided in Table 2.



p_0	t=0	$p_{\Delta t}$	t=Δt	$p_{2\Delta t}$	t=2Δt & $p_{\Delta t}$=3.34	t=2Δt & $p_{\Delta t}$=2.24	t=2Δt & $p_{\Delta t}$=1.50	t=2Δt & $p_{\Delta t}$=1.01
				7.43	0.00, 0.00, 0.00, 0.00			
				4.98	0.00, 0.00, 0.00, 0.00	0.00, 0.00, 0.00, 0.00		
		3.34	0.00, 0.00, 0.00, 0.00	3.34	0.00, 0.00, 0.00, 0.00	0.00, 0.00, 0.00, 0.00	0.00, 0.00, 0.00, 0.00	
		2.24	0.02, 0.02, 0.00, 0.00	2.24	0.03, 0.05, 0.00, 0.01	0.03, 0.04, 0.00, 0.01	0.03, 0.04, 0.00, 0.01	0.03, 0.04, 0.00, 0.02
1.50	0.10, 0.10, 0.06, 0.06	1.50	0.16, 0.16, 0.06, 0.06	1.50	0.18, 0.29, 0.02, 0.11	0.18, 0.18, 0.02, 0.11	0.18, 0.18, 0.02, 0.11	0.18, 0.18, 0.02, 0.13
		1.01	0.45, 0.45, 0.31, 0.32	1.01		0.48, 0.48, 0.04, 0.40	0.48, 0.48, 0.03, 0.42	0.48, 0.48, 0.04, 0.44
		0.67	abandoned, abandoned, abandoned, abandoned	0.67			abandoned, abandoned, abandoned, abandoned	abandoned, abandoned, abandoned, abandoned
				0.45				abandoned, abandoned, abandoned, abandoned

3.1. The leverage decision, without hedging

Continuing on from the work of Graham (1999), Graham (2000) cumulated the simulated expected tax benefit of each incremental dollar of debt finance held by each firm each year in his COMPUSTAT sample. With adjustment for personal taxes he found that the tax shield benefit of debt finance added more than 4% on average to the value of the firms. He further estimated that a typical firm could have more than doubled this increase in value by levering up to the point where the marginal tax benefit began to decline (the "kink"). Graham did not cumulate value-offsetting incremental disbenefits of debt, though he did argue that even extreme estimates of financial distress costs could not justify the observed conservative debt levels.[18] Nor was Graham satisfied that the debt conservatism could be justified by the risk of rare but disastrous events or by pecking order theory.

Graham's (2000) evidence of "money left on the table" stemmed from his finding that the median marginal corporate tax rate, after debt financing and after personal tax penalty, equalled 7.5% (as opposed to a value close to zero) for his entire sample of observations from 1980 to 1994. In contrast, for the same sample period, the earlier study of Graham (1999) found that the median marginal corporate tax rate, after debt financing and after personal tax penalty, was generally close to zero each year from 1986 to 1994 (and also for 1980 and 1981, though was considerably higher from 1982 to 1985).[19] This suggests that debt conservatism was not particularly prominent in the latter majority of years of the sample. Graham (2000) did not present annual values for the median marginal corporate tax rate, but instead presented annual values for the mean standard-

[18] Echoes of Miller's (1977) horse-and-rabbit stew.

[19] Graham (1999) estimated the median marginal corporate tax rate, after debt financing and after personal tax penalty, to have been 7.5% each year from 1982 to 1984, 6.1% in 1985, and generally close to zero for the remaining majority of years in the sample. Yet across the same years, Graham's (2000) entire sample had a median marginal corporate tax rate, after debt financing and after personal tax penalty, of 7.5%. Graham (1999) stated that his results were "consistent with the typical firm choosing optimal debt policy by equating the (near-zero) benefit of the last dollar of debt to the marginal cost", which is entirely opposite to the conclusion of Graham (2000). This discrepancy was not addressed by Graham (2000), but is presumably attributable to the different sample sizes: 65,429 observations for Graham (1999) versus 87,643 observations for Graham (2000).

ised additional interest expense that could have been born by the sample firms before suffering a declining marginal tax benefit of debt (which Graham termed "levering up to the kink"). These results do indicate that the degree of under-leverage declined over the latter years of the sample period, but it was still Graham's (2000) determination that a typical mid-1990s firm was foregoing a value benefit of more than 4% due to debt conservatism.

In contrast to Graham's (2000) evidence about the tax shield benefit of debt, this study finds the base-case value benefit of the *levered & unhedged* strategy relative to the *unlevered & unhedged* strategy to be a very modest gain of less than 0.1%.[20] As well as the tax shield benefit of debt (adjusted for personal taxes), this result incorporates the value-offsetting disbenefits of debt in the form of higher expected financial distress and bankruptcy costs. The model company's small value gain from leverage (without hedging) concurs with the results of Fama and French (1998), who regressed a company valuation measure against a raft of financial variables intended to capture all information about expected net cash-flows for their COMPUSTAT sample of firms each year from 1965 to 1992. It was anticipated that the debt variable would isolate the tax shield effect of debt if the other independent variables were sufficient to capture any non-tax shield implications for profitability engendered in leverage. In fact the coefficient of the debt variable was almost always found to be negative. Rather than claim this finding as evidence of there being no tax shield benefit of debt, Fama and French preferred to interpret it as being consistent with the managers versus investors information asymmetry arguments of Myers (1984) and others, entailing the contention that financing decisions are fundamentally associated with pre-tax cash-flow conditions as per pecking order theory. They concluded that leverage conveys adverse information about profitability that obscures any tax shield benefit.

This study's model company is set up to incorporate both trade-off and pecking order motivations for its optimal leverage decision. At the end of each production period the company depends on some combination of current-period revenue (facility to retain earnings is not specified), new debt finance and new equity finance to pay due liabili-

[20] For all exogenous parameter sensitivity analysis scenarios considered, the value benefit of the *levered & unhedged* strategy relative to the *unlevered & unhedged* strategy has a paltry range from zero (for the 'top' personal tax penalty of non-equity scenario) to 0.3% (for the 'low' production cost scenario).

ties.[21] Financial distress is incorporated as a deadweight cost that applies in event of the company resorting to new equity finance; this is readily conceivable as a cost of assuaging manager/equity-investor information asymmetry. The optimal trade-off leverage decision each period considers not only the immediate cost of resorting to new equity finance, but also gives balance to the risk and cost of having to seek new equity finance in the future. Consequently there can be trade-off value in maintaining pecking order borrowing capacity. Or it may be optimal to break pecking order and defer new debt finance for new equity finance, despite the deadweight cost, so as to avoid a more severe dependence on new equity finance in the future.

The model company's leverage behaviour for the *levered & unhedged* strategy indicates consistency with both trade-off and pecking order theories. Consistent with trade-off imperative, with rising output price the *levered & unhedged* strategy undertakes increasing leverage (see Figure 3c for the base-case scenario); but once leverage has been initiated (at p_{uu}), a large output price fall (to $p_{uu,dd}$) also leads to increased leverage, consistent with pecking order imperative. As depicted in Figure 3c, for the base-case scenario the *levered & unhedged* strategy commonly avoids taking on any leverage. Conversely the 'low' production cost scenario is more facilitative of leverage, making trade-off and pecking order dynamics more prominent as presented in Table 3.

For the 'low' production cost scenario, production cost per unit of expected production quantity is set to its 'low' value provided in Table 2 ($c = \$0.5$ /unit), while all other exogenous parameters are set to their base-case 'mid' values. For this scenario, the *levered & unhedged* strategy undertakes/maintains some level of leverage for nearly all output price outcomes and price-paths. Table 3 shows that while output price is flat or rising each period, leverage is generally also flat or rising, seemingly to a trade-off plan; but leverage also rises or is comparatively elevated following an output price fall (except for an extreme fall to $p_{2\Delta t} < c$), seemingly to a pecking order plan.

[21] The borrowing decision is exogenously controlled. If current revenue (internal finance) plus new debt finance is insufficient to pay all due liabilities, new equity endogenously finances the shortfall (if the company is a financially viable going concern). If current revenue plus new debt finance exceeds due liabilities, the excess is paid as a dividend to equity.

Table 3 – Leverage, debt, dividend and deadweight cost-to-value for the *levered & unhedged* strategy and <u>‘low’ production cost scenario</u>

Leverage ($\ell_{t+} = D_{t+} / E_{t+}$), absolute debt ($D_{t+}$), expected dividend ($\hat{\mathrm{E}}_{t-\Delta t}[G_t \mid p_t]$)* and expected deadweight cost-to-value ratio ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma DWC_t \mid p_t] / \hat{\mathrm{E}}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t]$) for the model company’s *levered & unhedged* control strategy, at the start of each controlled production period, and for each possible output price-path, with production cost per unit of expected production quantity set to its ‘low’ value ($c = 0.5$) and all other exogenous parameters set to base-case ‘mid’ values provided in Table 2.

ℓ_{t+} (in bold); D_{t+} (in italics); $\hat{\mathrm{E}}_{t-\Delta t}[G_t \mid p_t]$; $\hat{\mathrm{E}}_{t-\Delta t}[\Sigma DWC_t \mid p_t] / \hat{\mathrm{E}}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t]$ (underlined)

p_t	$t = 0$	$t = \Delta t$	$t = 2\Delta t$ & $p_{\Delta t} = 3.34$	$t = 2\Delta t$ & $p_{\Delta t} = 2.24$	$t = 2\Delta t$ & $p_{\Delta t} = 1.50$	$t = 2\Delta t$ & $p_{\Delta t} = 1.01$	$t = 2\Delta t$ & $p_{\Delta t} = 0.67$
7.43			**0.32** *180* 567 <u>0.53</u>				
4.98			**0.29** *108* 311 <u>0.53</u>	**0.28** *109* 373 <u>0.53</u>			
3.34		**0.26** *98* 275 <u>0.53</u>	**0.24** *61* 143 <u>0.53</u>	**0.24** *61* 205 <u>0.53</u>	**0.24** *61* 245 <u>0.53</u>		
2.24		**0.18** *46* 144 <u>0.53</u>	**0.21** *35* 37 <u>0.54</u>	**0.17** *29* 93 <u>0.54</u>	**0.17** *29* 133 <u>0.54</u>	**0.17** *29* 141 <u>0.54</u>	
1.50	**0.08** *18* 84 <u>0.54</u>	**0.07** *12* 58 <u>0.54</u>	**0.43** *35* -21 <u>0.59</u>	**0.12** *13* 25 <u>0.54</u>	**0.07** *8* 60 <u>0.54</u>	**0.07** *8* 68 <u>0.54</u>	**0.07** *8* 66 <u>0.54</u>
1.01		**0.06** *6* 17 <u>0.54</u>		**0.26** *13* -13 <u>0.60</u>	**0.04** *2* 21 <u>0.54</u>	**0.02** *1* 27 <u>0.54</u>	**0.02** *1* 25 <u>0.54</u>
0.67		**0.19** *7* -9 <u>0.67</u>			**0.14** *4* -4 <u>0.65</u>	**0.05** *1* 3 <u>0.60</u>	**0.08** *2* 1 <u>0.61</u>
0.45						**0.00** *0* 0 <u>-0.43</u>	**0.00** *0* 0 <u>-0.46</u>
0.30							abandoned

* The actual dividend amount will depend on the production quantity outcome.

It is revealing to compare the financing decisions at a given price outcome for different price-paths. For example, for $p_t = 1.50$: the company initially takes on debt-to-equity leverage ($\ell_{t+} = D_{t+} / E_{t+}$) of 0.08 at $p_0 = 1.50$, and maintains leverage at a similar level of 0.07 for a flat price outcome at $p_m = 1.50$; a continuing flat price outcome at $p_{m,m} = 1.50$ or a price rebound to $p_{d,u} = 1.50$ or $p_{dd,uu} = 1.50$ also entail leverage maintained at 0.07; to get to $p_{u,d} = 1.50$, a moderate price rise with increased leverage is followed by a moderate price fall leading to a reduction of leverage, but to a comparatively elevated level at 0.12; and to get to $p_{uu,dd} = 1.50$, a large price rise leading to high trade-off leverage is followed by a large price fall leading to even higher pecking order leverage at 0.43. However pecking order impetus for increased leverage does not apply in event of an output price fall to an uneconomic level below the 'low' expected unit cost of production at $p_{d,dd} < c$ or $p_{dd,d} < c$; for these outcomes there is a small chance that the company can avoid bankruptcy if production quantity is higher than expected, in which case the company optimally does not choose to abandon so as to avoid bankruptcy costs, but nevertheless optimally eschews any debt finance due to prohibitive financial risk.

Trade-off and pecking order effects are also evident in absolute debt levels ($D_{t+} = y_{t,t+\Delta t} e^{-r\Delta t}$) given in Table 3, however the relative measure of debt-to-equity leverage makes pecking order leverage more clearly discernible. From Table 3 for example, at $p_{uu,dd} = 1.50$ the output price has had a large fall and debt-to-equity leverage has increased from 0.26 to 0.43, but absolute debt has been paid down from 98 to 35; nevertheless this level of absolute debt is very high compared to an absolute debt level of 8 associated with either a flat price outcome at $p_{m,m} = 1.50$ or a price rebound outcome at $p_{d,u} = 1.50$ or $p_{dd,uu} = 1.50$. Note that, for flat price outcomes, the absolute level of debt (and also total value) will fall over time as the company's underlying output resource is reduced by production.

The model company's pecking order financing decision after a large price fall is essentially an augmented trade-off decision that balances the financial distress cost associated with resorting to new equity finance immediately against the risk and financial distress cost of having to resort to new equity finance in the future. From Table 3 for example, after an output price fall to p_{dd}, $p_{uu,dd}$, $p_{u,dd}$ or $p_{m,dd}$, the company's pecking order decision combines relatively elevated leverage and also the expectation of some new equity finance as indicated by a negative value for the expected dividend ($\hat{\mathrm{E}}_{t-\Delta t}[G_t \mid p_t] < 0$; n.b. the actual dividend amount, negative or positive, depends on the production quantity outcome, which is left as an unobserved variable).

The model company's apparent trade-off leverage behaviour at healthy output price levels seems to disagree with the empirical evidence of Fama and French (2002), who found, indicative of pecking order, a negative relationship between the profitability and leverage of firms. However, Table 3 shows that the expected deadweight cost-to-value ratio ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma DWC_t \mid p_t] / \hat{\mathrm{E}}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t]$, serving to indicate profitability adversity) decreases only slightly with increasing output price and increasing trade-off leverage; but the ratio increases markedly following a large output price fall associated with increased pecking order leverage and expected new equity finance. This suggests that it is feasible for high leverage to be more strongly a signal of adverse profitability conditions, despite trade-off imperative for increased leverage when profitability conditions are favourable. Note that, at output price outcomes $p_{d,dd}$ and $p_{dd,d}$, the deadweight cost-to-value ratio is negative because bankruptcy is highly probable entailing an expected claim against non-equity's short comprehensive limited liability option (so that $\hat{\mathrm{E}}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t] < 0$).

The modelling approach of Titman and Tsyplakov (2004, 2007) indicated pecking order behaviour to be driven in large part by the financing preferences of equity-holders as distinct from debt-holders. When Titman and Tsyplakov's model firm was operated to maximise the value of equity, to avoid necessity for new equity finance and consequential wealth transfer to debt-holders, the firm maintained a conservative debt position consistent with pecking order; but when operated to maximise total value, the firm exhibited weaker pecking order behaviour and tended to make financing choices that moved it towards a (moving) debt target. The issue of equity-holder versus debt-holder financing preference conflict does not apply to this study's model company. The model company fairly compensates debt finance providers for default risk each control period concurrently with any financing adjustment or hedging decision. Thus equity-holder versus debt-holder wealth transfer is not subject to the company's control decisions, and maximisation of equity value is congruent with maximisation of total value. On this basis it is consistent that the company's financing behaviour for the *levered & unhedged* strategy is somewhat similar to that of Titman and Tsyplakov's model firm when operated to maximise total value. Titman and Tsyplakov's results do, however, attest to much higher levels of optimal leverage than obtained here, although this may be attributable to the fact that they did not incorporate a personal tax penalty in their application of debt as a tax shield.

Tempering their pecking order evidence of a negative relationship between the profitability and leverage of firms, Fama and French (2002) also found leverage to be mean-reverting over time. But they considered the mean reversion suspiciously slow and readily subject to explanations other than trade-off leverage targeting. Nevertheless trade-off behaviour is substantiated by other recent studies. For example, Flannery and Rangan (2006) and Kayhan and Titman (2007) found that, despite pecking order indications in the short-run, firms do tend to adjust towards trade-off leverage targets, although the two studies disagree on the rate of adjustment.

To provide a scale perspective to the leverage choices of this study's model company, Table 4 reports the mean and median leverage values for the empirical samples of several studies. With respect to the *levered & unhedged* strategy for both the 'low' production cost scenario and the base-case scenario, the model company's leverage varies for each control decision price-node from a minimum of zero to a maximum of 0.43, with a mean of 0.12 and a median of 0.08 (from the combined *levered & unhedged* leverage results given in Table 3 and Figure 3c, ignoring the actual probability of any price-node occurring and ignoring price-nodes at which the company has been abandoned). Tufano (1996) observed similar mean and median leverage for unhedged gold miners (see Table 4).

Table 4 – Empirical mean and median leverage values

Mean and median leverage values obtained by a variety of empirical studies.

			Leverage (debt-to-equity basis)	
Study	Sample description	Leverage measure	Mean	Median
Tufano (1996)	32 yearly observations from 1991 to 1993 of North American gold mining firms with no net gold price hedging.	Tufano reported the ratio of {book value of debt} to {market value of equity plus book value of preferred stock and debt}.*	0.15 $\left(=\frac{0.13}{1-0.13}\right)$	0.05 $\left(=\frac{0.05}{1-0.05}\right)$
’’	28 yearly observations from 1991 to 1993 of North American gold mining firms with extensive net gold price hedging.	’’	0.22 $\left(=\frac{0.18}{1-0.18}\right)$	0.16 $\left(=\frac{0.14}{1-0.14}\right)$
Graham (1999)	65,429 yearly observations of COMPUSTAT firms from 1980 to 1994.	Graham reported the ratio of {long-term and current debt} to {book value of total assets minus book value of equity plus market value of equity}.*	0.27 $\left(=\frac{0.213}{1-0.213}\right)$	0.22 $\left(=\frac{0.177}{1-0.177}\right)$
Haushalter (2000)	295 yearly observations from 1992 to 1994 of oil and gas producing firms.	Haushalter reported the ratio of {long-term and current debt} to {market value of total assets}.*	0.29 $\left(=\frac{0.226}{1-0.226}\right)$	0.26 $\left(=\frac{0.209}{1-0.209}\right)$
Hentschel & Kothari (2001)**	296 yearly observations from 1990 to 1993 of large US non-financial firms without derivative positions.	Ratio of {book value of liabilities} to {market value of equity}.	1.68	1.08
’’	394 yearly observations from 1990 to 1993 of large US non-financial firms with derivative positions.	’’	1.75	1.09
Adam (2002)	595 yearly observations from 1989 to 1999 of North American gold mining firms.	Ratio of {book value of long-term debt and preferred stock} to {book value of common equity}.	0.23	0.16
Flannery & Rangan (2006)	111,106 yearly observations of industrial COMPUSTAT firms from 1965 to 2001.	Flannery and Rangan reported the ratio of {book value of debt} to {book value of debt plus market value of common shares}.*	0.39 $\left(=\frac{0.278}{1-0.278}\right)$	0.29 $\left(=\frac{0.225}{1-0.225}\right)$

* The reported mean and median leverage values of these studies are here converted to a debt-to-equity ratio basis by dividing each value by its own complement relative to one.

** Hentschel and Kothari’s (2001) leverage values are comparatively high in part because they used book value of liabilities in their measure rather than a debt-specific subset of liabilities. Furthermore they limited their sample to very large firms (of which none were resource producers).

Discipline of debt

When the model company is subject to misappropriation of positive free cash-flow (at rate $a \geq 0$), the disciplinary effect of debt on free cash-flow misappropriation depends on whether new debt proceeds impact positive free cash-flow more or less than the commitment to repay the debt with interest in the future. This is cursorily assessed in two ways: comparison of the expected free cash-flow misappropriation deadweight cost between the *levered & unhedged* strategy and the *unlevered & unhedged* strategy, for the 'top' free cash-flow misappropriation scenario (indicating the consequence of optimal leverage for free cash-flow misappropriation); and comparison of optimal leverage for the *levered & unhedged* strategy, between the 'top' free cash-flow misappropriation scenario and the zero free cash-flow misappropriation (base-case) scenario (indicating the consequence of free cash-flow misappropriation for optimal leverage).

For both the 'top' free cash-flow misappropriation and zero free cash-flow misappropriation scenarios, the *levered & unhedged* strategy takes on initial leverage after a 'large-up' output price move to the p_{uu} price-node. Table 5 shows that, for the 'top' free cash-flow misappropriation scenario at p_{uu}, the expected free cash-flow misappropriation deadweight cost ($\hat{\mathrm{E}}_0[\Sigma MIS_{\Delta t} \mid p_{uu}]$) is fractionally higher for the *levered & unhedged* strategy than for the *unlevered & unhedged* strategy; the case is the same for the relative measure, expected free cash-flow misappropriation deadweight cost-to-value ratio ($\hat{\mathrm{E}}_0[\Sigma MIS_{\Delta t} \mid p_{uu}] / \hat{\mathrm{E}}_0[E_{\Delta t-} + (D_{\Delta t-} + O_{\Delta t-}) \mid p_{uu}]$). The implication is that optimal leverage does not necessarily discipline free cash-flow misappropriation. Table 5 also shows that, for the *levered & unhedged* strategy at p_{uu}, the zero free cash-flow misappropriation scenario entails fractionally higher leverage and absolute debt than the 'top' free cash-flow misappropriation scenario. The implication is that free cash-flow misappropriation does not necessarily motivate higher optimal leverage.

Table 5 – Consequences of leverage for free cash-flow misappropriation and vice versa

Leverage ($\ell_{t+} = D_{t+} / E_{t+}$), absolute debt ($D_{t+}$) and expected free cash-flow misappropriation deadweight cost-to-value ratio ($\hat{E}_{t-\Delta t}[\Sigma MIS_t \mid p_t] / \hat{E}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t]$) for the model company at 'large-up' output price-node p_{uu} at time $t = \Delta t$ for the *unlevered & unhedged* strategy and the *levered & unhedged* strategy; with the free cash-flow misappropriation rate set to its 'top' value ($a = 0.02$), and in turn to its base-case 'mid' value ($a = 0$), and all other exogenous parameters set to base-case 'mid' values provided in Table 2.

Leverage, absolute debt and expected free cash-flow misappropriation deadweight cost-to-value ratio at 'large-up' price-node $p_{\Delta t} = p_{uu}$		$\ell_{\Delta t+}$	$D_{\Delta t+}$	$\frac{\hat{E}_0[\Sigma MIS_{\Delta t} \mid p_{\Delta t}]}{\hat{E}_0[E_{\Delta t-} + (D_{\Delta t-} + O_{\Delta t-}) \mid p_{\Delta t}]}$
'Top' free cash-flow misappropriation scenario	*Unlevered & unhedged* strategy	0	0	$\frac{9.325}{558.8} = 0.01669$
	Levered & unhedged strategy	0.1065	38.68	$\frac{9.373}{559.9} = 0.01674$
Zero free cash-flow misappropriation (base-case) scenario	*Levered & unhedged* strategy	0.1073	39.45	$\frac{0}{569.3} = 0$

3.2. The hedging decision, with and without leverage

In their survey of research on the issue of whether or not financial risk management adds value to firms, Smithson and Simkins (2005) cited seven studies that found a positive empirical relationship between firm value and the use of derivatives; however, cautioning any inference of significant causality, they also noted Guay and Kothari's (2003) evidence indicating that firms' derivatives positions generally entail only small economic effects relative to entity-level risk exposures. Smithson and Simkins cited another three studies that specifically investigated commodity producers (two studies of gold producers and one of oil and gas producers) and which did not find any benefit for firm value or share price performance from hedging of the commodity price risk; and a footnote acknowledgement was given to the contrary evidence of Adam and Fernando (2006), who found that gold miners accrue economically significant cash-flow gains from their derivative transactions. Smithson and Simkins' summary seems suggestive of a dichotomy between commodity producers in particular and firms in general with respect to the value-adding benefits of hedging. Further suggestive of such dichotomy are the results of subsequent studies by Jin and Jorion (2007), who investigated gold miners, Bartram, Brown and Conrad (2007), who investigated non-financial firms in general, and MacKay and Moeller (2007), who investigated oil refiners (not producers).

This study's model company gains value from hedging, with and without leverage. For the base-case scenario, relative to the *levered & unhedged* strategy, the *unlevered & hedged* strategy and the *levered & hedged* strategy provide value benefits of 2.6% and 4.0% respectively (see Figure 3a at time $t = 0$). This is similar to the results of Leland's (1998) modelling approach, which combined leverage optimisation with a very simple 'hedging' decision entailing a choice between high or low risk for the firm's unlevered asset value. For various hedging risk levels, Leland's jointly value-maximising hedging and leverage strategy (which he termed 'ex-ante optimal') provided a value benefit of between 2.1% and 5.6% relative to a strategy of value-maximising leverage with no hedging.[22]

Leverage

The empirical evidence of Graham and Rogers (2002) indicated that firms are motivated to hedge to increase debt capacity and thereby access tax savings worth an estimated 1.1% of firm value on average. They also estimated hedging to be accountable for a 3% increase in debt-to-assets leverage on average, a result similar to the average leverage difference between non-financial firms with and without derivatives observed by Hentschel and Kothari (2001) (see Table 4), but much more moderate than the leverage effect implied by the results of Leland (1998) and this study. The optimal debt-to-assets (debt-to-equity) leverage for Leland's model firm increased from 43% (0.75) without hedging to between 52% (1.08) and 70% (2.33) with ex-ante optimal hedging and dif-

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[22] Leland (1998) also assessed an 'ex-post optimal' strategy entailing value-maximising leverage constrained by subsequential equity value-maximising hedging. He thus demonstrated that firms that cannot or do not commit to hedging in tandem with taking on leverage (i.e. given to an ex-post optimal strategy), as compared to firms that can and do commit (i.e. given to an ex-ante optimal strategy), face a considerably higher cost of debt but will only be moderately discouraged from borrowing and will still choose to hedge (albeit for a reduced range of future states) even though it benefits debt-holders. Indicated by the value difference between the ex-ante optimal and ex-post optimal strategies, the agency cost of debt due to the risk that a firm may renege on a commitment to hedge (equivalent to an asset substitution strategy as described by Jensen and Meckling (1976)) was estimated by Leland to be about 1% of total firm value.

ferent hedging risk levels. For this study's base-case scenario, the *levered & unhedged* strategy, with leverage (ℓ_{t+}) ranging from zero to 0.29 depending on price-node, usually entails markedly lower leverage than the *levered & hedged* strategy, for which leverage ranges from 0.06 to 2.67 depending on price-node (see Figure 3c). Tufano (1996) also reported a large difference in leverage between unhedged gold miners and extensively hedged gold miners (see Table 4).

A prominent result observable in Figure 3c is that pecking order apparently dominates the model company's leverage decision when made in conjunction with the hedging decision. For the *levered & hedged* strategy, leverage is clearly negatively related to output price outcome each period. Hedging initially ramps-up trade-off leverage. However, after a price rise, the maturing-hedge loss provides a tax shield substitute for debt, and the out-of-the-money live hedge portfolio poses financial risk which further diminishes the incentive for leverage; despite the improved operating profit environment being supportive of leverage, the net effect is reduced leverage (n.b. a hedge loss is more effective than debt expense as a tax shield because it does not attract a personal tax penalty). Conversely, after a price fall, the weak profit environment (combining very weak operating profit and far less than full hedging) creates incentive for roll-over debt financing, which is facilitated by the in-the-money live hedge portfolio; in combination with reduced company value, the effect is increased leverage. Hence trade-off and pecking order behaviours become somewhat aligned. The dampening effect of hedge losses on trade-off leverage is demonstrated by the result that, for the base-case scenario, the *levered & unhedged* strategy entails higher leverage than the *levered & hedged* strategy for strongly rising price outcomes $p_{uu,uu}$, $p_{uu,u}$ and $p_{u,uu}$.

To consider the viability of the high debt-to-equity leverage associated with very weak output price outcomes for the base-case *levered & hedged* strategy (see Figure 3c at $p_t = 1.01$), Table 6 indicates how the absolute debt level and limited liability option value vary with output price in conjunction with hedge portfolio value and operating value. For example, although debt-to-equity leverage (ℓ_{t+}) increases markedly from 0.68 at initial price-node p_0 to 1.94 at 'down' price-node p_d as shown by Figure 3c, the absolute debt level is little changed: Table 6 shows that absolute debt of $D_{0+} = \$60$ at p_0, becoming due face-value of $y_{0,\Delta t} = \$71$ at end-of-period one, is rolled over with new absolute debt of $D_{\Delta t+} = \$61$ at p_d together with a modest premium of

$-O_{\Delta t+} = \$0.20$ for the renewed comprehensive limited liability option. Furthermore, a very weak price outcome at end-of-period two (at price-node $p_{u,dd}$, $p_{m,d}$ or $p_{d,m}$) entails increased debt-to-equity leverage, but much reduced absolute debt, and also a much increased limited liability option premium. In event of the $p_{d,m}$ outcome for example: absolute debt of $D_{\Delta t+} = \$61$ together with a limited liability option premium of $-O_{\Delta t+} = \$0.20$ at p_d, becomes due face-value of $y_{\Delta t,2\Delta t} = \$72$ at end-of-period two and an expected claim against the limited liability option of $-\hat{\mathrm{E}}_{\Delta t}[O_{2\Delta t-} \mid p_{2\Delta t}] = \$0.33$ at $p_{d,m}$; given that bankruptcy is avoided (which is 98% probable as indicated by the converse of $\Pr B_{2\Delta t-}$ at $p_{d,m}$ in Figure 3h), the debt is only partly rolled over with new absolute debt of $D_{2\Delta t+} = \$44$ together with an expensive renewed limited liability option premium of $-O_{2\Delta t+} = \$3.59$ at $p_{d,m}$.

The model company's absolute debt level and limited liability option premium adjust each period to maximise the combined value of equity and non-equity. Consequently debt finance providers are always appropriately compensated for risk and agreeable to the company's leverage position. For the base-case *levered & hedged* strategy, at very weak output price outcomes the company is more valuable as a hedge portfolio than as an operating concern; thus it is the hedge portfolio that ensures the viability of the high leverage associated with very weak price outcomes. At price-nodes p_d, $p_{u,dd}$, $p_{m,d}$ and $p_{d,m}$, Table 6 shows that the maturing plus live hedge portfolio value ($\mathrm{tr}(\mathbf{x}_\eta^T \mathbf{X}_t)$) exceeds maturing debt face-value ($y_{t-\Delta t,t}$), which in turn exceeds current plus ongoing expected operating profits ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma OPP_t \mid p_t]$); similarly, live hedge portfolio value ($\mathrm{tr}(\mathbf{x}_\eta^T \mathbf{X}_t) - (\mathbf{e}_t^T \mathbf{x}_\eta)(\mathbf{e}_t^T \mathbf{X}_t)^T$) exceeds rolled over new debt (D_{t+}), which exceeds ongoing expected operating profits ($e^{-r\Delta t}\hat{\mathrm{E}}_t[\Sigma OPP_{t+\Delta t}]$).

Table 6 – Debt, limited liability, hedge portfolio and operating values for the base-case *levered & hedged* strategy

Juxtaposition of: maturing debt face-value ($y_{t-\Delta t,t}$) and new debt (D_{t+}); maturing short comprehensive limited liability option expected payoff ($\hat{\mathrm{E}}_{t-\Delta t}[O_{t-} \mid p_t]$) and new short comprehensive limited liability option value (O_{t+}); maturing plus live hedge portfolio value ($\mathrm{tr}(\mathbf{x}_\eta^T \mathbf{X}_t)$) and live hedge portfolio value ($[\mathrm{tr}(\mathbf{x}_\eta^T \mathbf{X}_t) - (\mathbf{e}_t^T \mathbf{x}_\eta)(\mathbf{e}_t^T \mathbf{X}_t)^T]$); and current plus ongoing expected operating profits ($\hat{\mathrm{E}}_{t-\Delta t}[\Sigma OPP_t \mid p_t]$) and ongoing expected operating profits ($e^{-r\Delta t}\hat{\mathrm{E}}_t[\Sigma OPP_{t+\Delta t}]$), for the model company's *levered & hedged* control strategy at the start of each controlled production period, and for each possible output price-path, with exogenous parameters set to base-case 'mid' values provided in Table 2.

$y_{t-\Delta t,t}$, D_{t+} (in bold); $\hat{\mathrm{E}}_{t-\Delta t}[O_{t-} \mid p_t]$, O_{t+} (in italics); $\mathrm{tr}(\mathbf{x}_\eta^T \mathbf{X}_t)$, $[\mathrm{tr}(\mathbf{x}_\eta^T \mathbf{X}_t) - (\mathbf{e}_t^T \mathbf{x}_\eta)(\mathbf{e}_t^T \mathbf{X}_t)^T]$;
$\hat{\mathrm{E}}_{t-\Delta t}[\Sigma OPP_t \mid p_t]$, $e^{-r\Delta t}\hat{\mathrm{E}}_t[\Sigma OPP_{t+\Delta t}]$ (underlined)

p_t	$t = 0$	$t = \Delta t$	$t = 2\Delta t$ & $p_{\Delta t} = 3.34$	$t = 2\Delta t$ & $p_{\Delta t} = 2.24$	$t = 2\Delta t$ & $p_{\Delta t} = 1.50$	$t = 2\Delta t$ & $p_{\Delta t} = 1.01$
7.43			**61** , **22** *0.00* , *-0.01* -880 , -467 <u>1783</u> , <u>1072</u>			
4.98			**61** , **26** *0.00* , *-0.01* -500 , -265 <u>1109</u> , <u>678</u>	**62** , **27** *0.00* , *-0.01* -500 , -265 <u>1116</u> , <u>685</u>		
3.34		**71** , **52** *0.00* , *0.00* -349 , -246 <u>874</u> , <u>626</u>	**61** , **30** *0.00* , *-0.02* -246 , -130 <u>658</u> , <u>410</u>	**62** , **30** *0.00* , *-0.01* -246 , -130 <u>662</u> , <u>414</u>	**65** , **31** *0.00* , *-0.01* -246 , -130 <u>667</u> , <u>418</u>	
2.24		**71** , **53** *0.00* , *-0.01* -126 , -89 <u>479</u> , <u>350</u>	**61** , **32** *0.00* , *-0.07* -75 , -40 <u>356</u> , <u>228</u>	**62** , **32** *0.00* , *-0.07* -75 , -40 <u>359</u> , <u>230</u>	**65** , **33** *0.00* , *-0.08* -75 , -40 <u>361</u> , <u>232</u>	**72** , **38** *0.00* , *-0.12* -61 , -33 <u>364</u> , <u>235</u>
1.50	**-** , **60** - , *0* - , 0 <u>279</u> , <u>228</u>	**71** , **55** *0.00* , *-0.04* 24 , 17 <u>218</u> , <u>167</u>	**61** , **36** *-0.01* , *-0.67* 39 , 21 <u>156</u> , <u>105</u>	**62** , **36** *-0.01* , *-0.69* 39 , 21 <u>157</u> , <u>106</u>	**65** , **37** *-0.03* , *-0.78* 39 , 21 <u>158</u> , <u>107</u>	**72** , **41** *-0.01* , *-0.92* 50 , 26 <u>159</u> , <u>108</u>
1.01		**71** , **61** *0.00* , *-0.20* 124 , 87 <u>53</u> , <u>52</u>		**62** , **42** *-0.06* , *-2.87* 116 , 61 <u>27</u> , <u>26</u>	**65** , **43** *-0.15* , *-3.17* 116 , 61 <u>27</u> , <u>26</u>	**72** , **44** *-0.33* , *-3.59* 122 , 61 <u>27</u> , <u>27</u>
0.67		**71** *0.00* 191 <u>-32</u> abandoned			**65** *0.00* 167 <u>-32</u> abandoned	**72** *-0.38* 149 <u>-32</u> abandoned
0.45						**72** *-0.10* 166 <u>-53</u> abandoned

Hedging

The model company's hedging behaviour for the base-case scenario is shown in Figure 3d. The *levered & hedged* strategy maintains a hedge-delta ratio generally much higher than the *unlevered & hedged* strategy. For both hedging strategies, Table 7 disaggregates the overall hedge-delta ratio at time $t = 0$ into hedging term structure components with maturities at $t = \Delta t$, $t = 2\Delta t$ and $t = 3\Delta t$. For the *unlevered & hedged* strategy the company establishes a term structure of reducing short forward positions at $t = 0$, and increases the overall short forward position for most flat or falling output price outcomes. For the *levered & hedged* strategy, the company establishes a term structure of increasing short forward positions at $t = 0$, but in response to a weak output price outcome ($p_{\Delta t} = 1.01$ or $p_{2\Delta t} \leq 1.50$) the company partly reverses its overall hedge position with long forwards or short put options. This somewhat counterintuitive result corresponds with empirical evidence from Graham and Rogers (2002) who found that firms reduce hedging in response to accumulated operating losses, and Brown, Crabb and Haushalter (2006) who found a tendency for gold miners to reduce hedging with falling gold price. For the base-case scenario, depending on price-node the hedge-delta ratio varies between 0.15 and 0.28 for the *unlevered & hedged* strategy, and between 0.34 and 0.50 for the *levered & hedged* strategy. For comparison, Tufano's (1996) sample of gold miners had an average (minimum, median, maximum) hedge-delta of 0.26 (zero, 0.23, 0.86).

By model set-up, together with every control decision, debt-holders and other non-equity stakeholders are fairly compensated for their limited liability risk. Hence the apparent asset substitution decision to reduce hedging for weak output price outcomes does not manifest wealth transfer from non-equity to equity. Instead it is driven by cost/benefit asymmetry between upside and downside output price outcomes. At outset (time $t = 0$) the model company's jointly optimal hedging and leverage decision boosts the levels of both hedging and leverage (compared to the company being optimally hedged and unlevered, and optimally levered and unhedged respectively). Then as long as output price is generally rising (from $p_0 = 1.50$ to $p_{\Delta t} \geq 1.50$ and $p_{2\Delta t} \geq 2.24$), the hedging level is largely maintained, while leverage generally falls due to maturing-hedge losses substituting for debt as a tax shield. For conversely weak price outcomes ($p_{\Delta t} = 1.01$, $p_{2\Delta t} \leq 1.50$), the company's ongoing operational value is low, but the hedge portfolio is deep in-the-money and facilitative of increased leverage, driven by pecking order roll-over of hedging-boosted debt. Low ongoing operational value imposes a net cost/benefit asymmetry with respect to subsequent output price outcome, because an output price fall to below unit production cost will lead to abandonment/bankruptcy regardless of hedge portfolio value,[23] but an output price rise will warrant continued operation and production. However the operational benefit of a subsequent output price rise is vulnerable to the financial risk posed by high leverage (in the presence of unhedgeable production quantity risk) and the fact that the hedge portfolio would lose value; that is, an output price rise actually entails adversely high risk of bankruptcy and consequential foregone profitable production. To counter this, some reduction of the overall hedge position increases the likelihood that the company will avoid bankruptcy in event of an output price rise and thereby benefit from ongoing profitable production. Consequently the facilitative relationship from hedging to leverage gets completely undone.

The jointly optimal hedging and leverage decision at time $t = 0$ assumes jointly optimal hedging and leverage decisions at all future times. After a 'down' price move to the p_d price-node for example, the jointly optimal hedging and leverage decision entails hedge reduction; alternatively consider that the model company may switch to an optimal leverage-only decision and leave the ongoing live hedge position unadjusted. Tables 8 and 9 indicate how these two alternative decisions affect the company's bankruptcy risk, optimal leverage and valuation. Referring to Table 8, compared to leaving the hedge position unadjusted, the hedge reduction decision at p_d leads to reduced probability of bankruptcy at subsequent flat or rising output price outcomes (at price-nodes $p_{d,m}$, $p_{d,u}$ or $p_{d,uu}$), whereas abandonment occurs regardless for falling price outcomes (at $p_{d,d}$ or $p_{d,dd}$). Consequently the hedge reduction decision at p_d offers higher probability of accessing ongoing profitable production, which increases company value and thereby reduces the probability of bankruptcy at p_d also.

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[23] Abandonment allows the in-the-money hedge portfolio to be cashed-in instead of used to subsidise uneconomic production. The precise output price below which abandonment is optimal is approximate to but less than the expected unit production cost ($c = \$1$ /unit for the base-case scenario), depending on the balance of costs associated with abandonment (i.e. bankruptcy) versus continued production at an uneconomic price level.

Referring to Table 9 the first point of note is the undoing of hedging to leverage facilitation. Compared to leaving the hedge position unadjusted, the hedge reduction decision at p_d ($h_{\Delta t+}$ equal to 0.39 versus 0.49) is associated with higher optimal leverage ($\ell_{\Delta t+}$ equal to 1.94 versus 1.75). Secondly, the valuation benefit due to net reduction in total expected deadweight costs (equal to 0.35)[24] is more than completely offset by the additional expected hedge transaction costs (equal to 0.36). But the valuation benefit due to a net increase in expected operating profit (equal to 0.26) means that there is an overall total value benefit of 0.25. That is, the hedge reduction lowers the expected production opportunity loss associated with the occurrence of bankruptcy at output price levels that offer a positive margin over expected per unit production cost. This opportunity loss represents an indirect bankruptcy cost arising from the fact that the model company does not receive any liquidation payment for lost ongoing production value.[25] However recall that equity will always provide new finance to stave off bankruptcy if ongoing equity value is at least equal to the free cash-flow shortfall plus financial distress cost. If new equity finance is not forthcoming, then the production opportunity loss represents a bankruptcy penalty endogenously scaled to the extent of the company's financial non-viability.

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[24] Table 9 provides the break-down of total expected deadweight costs. Compared to leaving the hedge position unadjusted, the hedge reduction decision and associated higher tax-shielding leverage entails lower expected deadweight cost due to corporate taxation, but higher deadweight cost due to the personal tax penalty of non-equity. Also, by locking-in part of the in-the-money hedge portfolio value, the hedge reduction decision reduces dependence on new equity finance and thus entails lower expected deadweight cost of financial distress, but the associated higher leverage entails higher expected deadweight cost of abandonment/bankruptcy.

[25] Effectively the ongoing production value of the company is liquidated at a 100% bankruptcy discount. A lower bankruptcy discount would directly lessen the impetus for hedge reversal at weak output price outcomes, but should also increase optimal leverage which would feedback an increased impetus for hedge reversal at weak output price outcomes. Future extension of the model warrants accommodation of exogenous choice for this bankruptcy discount for liquidated ongoing production value.

Table 7 – Term structure of initial hedge-delta ratio

Disaggregation of the hedge-delta ratio at time $t = 0$ (h_{0+})* into hedging term structure components with maturities at $t = \Delta t$, $t = 2\Delta t$ and $t = 3\Delta t$ for the *unlevered & hedged* strategy and the *levered & hedged* strategy, with exogenous parameters set to base-case 'mid' values provided in Table 2.

		Hedge-delta ratio disaggregation with term structure maturities at:				
		$t = \Delta t$	&	$t = 2\Delta t$	&	$t = 3\Delta t$
Unlevered & hedged strategy						
Initial hedge-delta ratio: (h_{0+} = 92/400 = **0.23**)	=	(39/400 = **0.10**)	+	(29/400 = **0.07**)	+	(24/400 = **0.06**)
Levered & hedged strategy						
Initial hedge-delta ratio: (h_{0+} = 191/400 = **0.48**)	=	(56/400 = **0.14**)	+	(63/400 = **0.16**)	+	(71/400 = **0.18**)

* The hedging term structure components of h_{0+} (see Equation (2.2)) with respective maturities at times $t = \Delta t$, $t = 2\Delta t$ and $t = 3\Delta t$ are:

$$\frac{-\frac{\Delta}{\Delta p_0} x_{0,\Delta t} X_{0:0,\Delta t}}{R}; \quad \frac{-\frac{\Delta}{\Delta p_0} x_{0,2\Delta t} X_{0:0,2\Delta t}}{R}; \text{ and } \frac{-\frac{\Delta}{\Delta p_0} x_{0,3\Delta t} X_{0:0,3\Delta t}}{R}.$$

Table 8 – Bankruptcy risk effect of optimal hedge reduction for a weak price outcome

Bankruptcy risk for the model company with and without optimal hedge reduction after a 'down' output price move to the p_d price-node at time $t = \Delta t$ for the *levered & hedged* strategy with exogenous parameters set to base-case 'mid' values provided in Table 2.

Probability of bankruptcy for the preceding production period due to production quantity risk ($\Pr B_{t-}$)	With optimal hedge reduction at 'down' price-node $p_{\Delta t} = p_d$	Without any hedge reduction or adjustment at 'down' price-node $p_{\Delta t} = p_d$
At 'down' price-node $p_{\Delta t} = p_d$	0.00011	0.00014
At five possible subsequent price-nodes:		
'down, large-down' price-node $p_{2\Delta t} = p_{d,dd}$	abandoned	abandoned
'down, down' price-node $p_{2\Delta t} = p_{d,d}$	abandoned	abandoned
'down, middle' price-node $p_{2\Delta t} = p_{d,m}$	0.01882	0.03010
'down, up' price-node $p_{2\Delta t} = p_{d,u}$	0.00018	0.00094
'down, large-up' price-node $p_{2\Delta t} = p_{d,uu}$	0.00000	0.00001

Table 9 – Value effect of optimal hedge reduction for a weak price outcome

Hedge-delta ratio, optimal leverage and valuation components for the model company with and without optimal hedge reduction after a 'down' output price move to the p_d price-node at time $t = \Delta t$ for the *levered & hedged* strategy with exogenous parameters set to base-case 'mid' values provided in Table 2.

Hedge-delta ratio, optimal leverage and valuation components at 'down' price-node $p_{\Delta t} = p_d$		With optimal hedge reduction	Without any hedge reduction or adjustment	Difference
Hedge-delta ratio with and without optimal hedge reduction subsequent to initial hedge position	$h_{\Delta t+}$	0.39	0.49	-0.10
Optimal leverage	$\ell_{\Delta t+} = D_{\Delta t+} / E_{\Delta t+}$	1.94	1.75	0.19
Current and ongoing expected deadweight costs respectively due to: corporate taxation; the personal tax penalty of non-equity; financial distress; and bankruptcy	$\hat{E}_0 \left[\Sigma TAX_{\Delta t} \mid p_{\Delta t} \right]$	41.81	42.64	-0.83
	$\hat{E}_0 \left[\Sigma PTP_{\Delta t} \mid p_{\Delta t} \right]$	5.47	5.24	0.24
	$\hat{E}_0 \left[\Sigma FND_{\Delta t} \mid p_{\Delta t} \right]$	0.61	0.82	-0.21
	$\hat{E}_0 \left[\Sigma BNK_{\Delta t} \mid p_{\Delta t} \right]$	10.71	10.26	0.45
Total current and ongoing expected deadweight costs	$\hat{E}_0 \left[\Sigma DWC_{\Delta t} \mid p_{\Delta t} \right]$	58.60	58.95	**-0.35**
Current and ongoing expected earnings before interest and tax	$\hat{E}_0 \left[\Sigma EBIT_{\Delta t} \mid p_{\Delta t} \right]$	176.33	176.43	-0.10
Current and ongoing expected operating profit	$\hat{E}_0 \left[\Sigma OPP_{\Delta t} \mid p_{\Delta t} \right]$	52.85	52.59	**0.26**
Backed-out difference in current and ongoing expected hedge transaction costs: difference in $\hat{E}_0 \left[\Sigma HDG_{\Delta t} \mid p_{\Delta t} \right]$ = difference in ($\hat{E}_0 \left[\Sigma OPP_{\Delta t} \mid p_{\Delta t} \right] - \hat{E}_0 \left[\Sigma EBIT_{\Delta t} \mid p_{\Delta t} \right]$)				**0.36**
Total value benefit of optimal hedge reduction: difference in $\hat{E}_0 [E_{\Delta t-} + (D_{\Delta t-} + O_{\Delta t-}) \mid p_{\Delta t}]$ = difference in ($\hat{E}_0 \left[\Sigma EBIT_{\Delta t} \mid p_{\Delta t} \right] - \hat{E}_0 \left[\Sigma DWC_{\Delta t} \mid p_{\Delta t} \right]$)				**0.25**

3.3. Financial risk

Studies that assess whether hedging is empirically associated with reduced financial risk, after controlling for leverage and other risk factors, are principally assessing whether firms actually hedge or instead speculate with derivative securities.[26] This study finds no occasion for optimal hedging to entail a speculative stance. Nevertheless, this study distinguishes hedging's risk-reducing process from its value-adding purpose. Accepting that hedging's value-adding purpose is inextricably dependent on interaction with risk-increasing leverage, does optimal (value-maximising) hedging, in conjunction with optimal leverage, reduce financial risk? Given the base-case financial risk results presented by Figures 3e to 3i and sensitivity analysis results not presented in detail, the answer is, generally, yes. Perhaps fortuitously for investors not familiar with a Modigliani and Miller (1958) based irrelevance argument, this is a concordance of hedging's value-adding purpose (in conjunction with leverage) and its risk-reducing process.

The base-case financial risk results given by Figures 3e to 3i present evidence that is highly consistent across each of the five financial risk measures. The two hedging strategies with and without leverage always entail less financial risk (or equal financial risk at a minimum value of zero for some measures) than the two unhedged strategies with and without leverage. Furthermore, while output price is generally rising (from $p_0 = 1.50$ to $p_{\Delta t} \geq 1.50$ to $p_{2\Delta t} \geq 2.24$), the *levered & hedged* strategy is generally less risky or no more risky than the *unlevered & hedged* strategy; but is more risky for weak output price outcomes ($p_{\Delta t} = 1.01$, $p_{2\Delta t} \leq 1.50$), for which the *levered & hedged* strategy is associated with high leverage and hedge reducing (i.e. risk-seeking) behaviour. The *levered & hedged* strategy's hedge reduction at weak output price outcomes is a prominent result which, in an empirical setting, could be mistaken for selective hedging or asset substitution.

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[26] Guay (1999) and Bartram, Brown and Conrad (2007) found firms' use of derivatives to be risk reducing. Hentschel and Kothari (2001) and Guay and Kothari (2003) found little effect on risk from derivatives use. Geczy, Minton and Schrand's (2007) survey revealed a large proportion of firms undertake speculation to exploit perceived information or costs advantages, but not to increase risk. Adam and Fernando (2006) and Brown, Crabb and Haushalter (2006) found that gold miners selectively hedge, but for little economic benefit.

Financial risk measures

With respect to the base-case results for relative limited liability risk ($-O_{t+}/E_{t+}$), relative value-at-risk of equity (υ_{t+}), beta of equity with respect to output price ($\hat{\beta}_{t+}$), and risk-neutral ongoing probability of bankruptcy ($\hat{\Pr} B_{t+}$) financial risk measures (see Figures 3e, 3f, 3g and 3i), the intuitive results are:

- for all hedging and leverage strategies, $-O_{t+}/E_{t+}$, $\hat{\beta}_{t+}$ and $\hat{\Pr} B_{t+}$ are negatively related to the output price level each production period (except that $-O_{t+}/E_{t+}$ is nearly always zero for the *unlevered & hedged* strategy);

- optimal hedging with or without optimal leverage reduces $-O_{t+}/E_{t+}$, υ_{t+}, $\hat{\beta}_{t+}$ and $\hat{\Pr} B_{t+}$, except when already equal to zero (i.e. compare the *unlevered & hedged* strategy with the *unlevered & unhedged* strategy, and compare the *levered & hedged* strategy with the *levered & unhedged* strategy); and

- optimal leverage without hedging increases $-O_{t+}/E_{t+}$, υ_{t+}, $\hat{\beta}_{t+}$ and $\hat{\Pr} B_{t+}$ (i.e. compare the *levered & unhedged* strategy with the *unlevered & unhedged* strategy for price-nodes at which the *levered & unhedged* strategy is levered).

Not so clearly intuitive is that, given optimal hedging, optimal leverage may or may not increase financial risk. Compared to the *unlevered & hedged* strategy: for healthy output price outcomes the *levered & hedged* strategy is far more highly hedged, consequently financial risk is lower; but for weak price outcomes the *levered & hedged* strategy entails high leverage and hedge reduction, consequently financial risk is higher.

The results for equity's beta with respect to output price ($\hat{\beta}_{t+}$, i.e. rate of return sensitivity) can be compared with the evidence of Tufano (1998b). Equity's output price beta varies between 1.07 and 3.43 across all output price outcomes and control strategies for the base-case scenario (see Figure 3g), and is clearly negatively related to the output price level each production period. Always the *levered & unhedged* strategy has the highest beta (although for price-nodes at which the *levered & unhedged* strategy is unlevered, the strategy's beta is equal highest with that of the *unlevered & unhedged* strategy). At healthy price outcomes ($p_t \geq 2.24$) for which the *levered & hedged* strategy has low leverage, the *levered & hedged* strategy has lower beta than the *unlevered*

& hedged strategy due to a much higher level of hedging. These results are consistent with the empirical evidence of Tufano, whose sample of gold miners had a mean (first quartile, median, third quartile) three-monthly gold price beta equal to 2.21 (1.13, 2.09, 3.13). Tufano found the miners' gold price betas to be negatively related to their extent of hedging, positively related to their financial leverage, and negatively related to the gold price level.

Equity's relative value-at-risk measure (υ_{t+}) indicates the fraction of equity value that will be lost with a 'large-down' output price move (for which the risk-neutral probability of occurrence is 0.0623 for all scenarios except the 'low' output price volatility scenario). The primary difference in results for this particular financial risk measure is that, for the two hedging strategies with and without leverage, υ_{t+} is not always negatively related to the output price level each production period (see Figure 3f). This is related to υ_{t+} pre-supposing an adverse price outcome, and the payoff to hedging being negatively related to the output price level. For example, the base-case *unlevered & hedged* strategy has lower υ_{t+} at price-node p_d than at p_m because, although a 'large-down' price move from either price-node will result in abandonment, a 'large-down' price move from p_d will lead to a larger maturing hedge payoff plus live hedge liquidation payoff. The situation for the *levered & hedged* strategy is also affected by hedge reduction for weak price outcomes, which tends to re-establish a negative relationship between υ_{t+} and the output price level.

The ex-post probability of bankruptcy measure ($\Pr B_{t-}$) indicates the probability of bankruptcy at a price-node due to production quantity risk for the preceding production period, and hence gives the probability that operations will not be ongoing at a price-node. For the base-case scenario, Figure 3h shows that, for the first two controlled production periods, production quantity risk primarily poses notable bankruptcy threat only for the two unhedged strategies with and without leverage and only at very weak output price outcomes ($p_t = 1.01$) close to the unit production cost.

3.4. Sensitivity analysis

For most of the sensitivity analysis results presented in Table 10, the valuation premiums associated with joint hedging and leverage generally correspond with the base-case scenario: that is, the valuation premium of the *levered & hedged* strategy is about 3.5% to 5% relative to the *unlevered & unhedged* and *levered & unhedged* strategies, and about 0.8% to 1.7% relative to the *unlevered & hedged* strategy. Deviations from this generalisation are highlighted in bold in Table 10. The optimal hedging and leverage dynamics and consequences for relative financial risk for the sensitivity analysis scenarios are also generally qualitatively consistent with the base-case scenario.

For the 'low' output price volatility scenario, hedging of output price risk has reduced cash-flow impact; consequently the value benefit of hedging is much reduced: going from the *levered & unhedged* strategy to the *levered & hedged* strategy the added value is about 1% (versus 4% for the base-case). For the 'top' production quantity uncertainty scenario, unhedgeable financial risk is higher and can only be checked with reduced leverage; consequently the value benefit of leverage is much reduced: going from the *unlevered & hedged* strategy to the *levered & hedged* strategy the added value is about 0.4% (versus 1.3% for the base-case).

For the 'low' production cost scenario, higher operating profit margin reduces the risk of financial distress and bankruptcy and reduces the benefit of hedging; consequently the value benefit of hedging is much reduced: going from the *levered & unhedged* strategy to the *levered & hedged* strategy the added value is 1.4% (versus 4% for the base-case). This situation is reversed for the 'top' production cost scenario so that the value benefit of hedging is much increased: going from the *levered & unhedged* strategy to the *levered & hedged* strategy the added value is 36%.

The 'top' personal tax penalty of non-equity scenario is the most particular exception to the generalisation represented by the base-case results. For this scenario there is no effective tax shield benefit from debt and the comprehensive limited liability option. Pecking order motivation for leverage remains, but the value benefit of leverage drops to near zero: going from the *unlevered & hedged* strategy to the *levered & hedged* strategy the added value is only 0.06%. For the *levered & hedged* strategy, although the 'top' personal tax penalty scenario results in much lower leverage at each output price-node compared to the base-case, the same pattern applies: leverage is negatively related to output price outcome each period. However a distinction arises with respect to the hedging pattern, such that the ‘top’ personal tax penalty scenario is the only scenario not to entail any hedge reduction in response to weak output price outcomes. Two elements contribute to this distinction: lower leverage reduces the financial risk incentive to reduce hedging at weak price levels to promote bankruptcy avoidance in event of a subsequent price rise; and the tax-efficiency of hedge reduction is reduced because there will be no effective tax shield from the increased expense for the comprehensive limited liability option.

For some sensitivity analysis scenarios, hedge reduction for weak output price outcomes is also evident for the *unlevered & hedged* strategy (not just the *levered & hedged* strategy). In particular, the ‘top’ production quantity uncertainty scenario entails sufficient unhedgeable financial risk that leverage is not necessary to induce hedge reduction at weak price levels to promote bankruptcy avoidance in event of a subsequent price rise. Also the ‘low’ hedge transaction cost scenario facilitates hedge adjustment, leading to hedge reduction for weak price outcomes for the *unlevered & hedged* strategy.

Table 10 – Valuation premium of joint hedging and leverage for each sensitivity analysis scenario

Total company value ($\hat{\mathrm{E}}_{-\Delta t}[E_{0-} + O_{0-} \mid p_0]$) premium for the *levered & hedged* strategy relative to the unhedged or unlevered strategy alternatives for each exogenous parameter sensitivity analysis variation from base-case detailed in Table 2.

Exogenous parameter sensitivity analysis scenario (all exogenous parameters are set equal to base-case 'mid' values except for specified individual variation)		Total company value ($\hat{\mathrm{E}}_{-\Delta t}[E_{0-} + O_{0-} \mid p_0]$) premium for the *levered & hedged* strategy relative to strategy alternative: *unlevered & unhedged* (valuation premium of optimal joint hedging & leverage)	*levered & unhedged* (value contribution of optimal hedging to optimal leverage)	*unlevered & hedged* (value contribution of optimal leverage to optimal hedging)
Base-case		4.02% (= 183.2 / 176.1 – 1)	3.96% (= 183.2 / 176.3 – 1)	1.33% (= 183.2 / 180.8 – 1)
'Low' output price volatility	$\sigma_p = 0.1$ /year	**1.14%** (= 177.9 / 175.9 – 1)	**0.98%** (= 177.9 / 176.2 – 1)	0.84% (= 177.9 / 176.4 – 1)
'Top' production quantity uncertainty	$\sigma_q = 0.2$ /year	4.46% (= 180.2 / 172.5 – 1)	4.44% (= 180.2 / 172.6 – 1)	**0.38%** (= 180.2 / 179.6 – 1)
'Low' output price and production quantity correlation	$\rho = 0$	4.05% (= 182.5 / 175.4 – 1)	3.98% (= 182.5 / 175.5 – 1)	1.26% (= 182.5 / 180.2 – 1)
'Top' output price and production quantity correlation	$\rho = 0.3$	3.93% (= 184.7 / 177.8 – 1)	3.87% (= 184.7 / 177.9 – 1)	1.51% (= 184.7 / 182.0 – 1)
'Low' production cost per unit of expected production quantity	$c = \$0.5$ /unit	**1.67%** (= 304.1 / 299.1 – 1)	**1.40%** (= 304.1 / 299.9 – 1)	1.61% (= 304.1 / 299.3 – 1)
'Top' production cost per unit of expected production quantity	$c = \$1.5$ /unit	**36.6%** (= 67.8 / 49.7 – 1)	**36.5%** (= 67.8 / 49.7 – 1)	1.09% (= 67.8 / 67.1 – 1)
'Top' personal tax penalty of non-equity	$\alpha - A_\alpha = 0.35$	3.56% (= 180.9 / 174.7 – 1)	3.56% (= 180.9 / 174.7 – 1)	**0.06%** (= 180.9 / 180.8 – 1)
'Top' financial distress cost rate	$\gamma = 1$	4.78% (= 182.3 / 174.0 – 1)	4.74% (= 182.3 / 174.0 – 1)	0.93% (= 182.3 / 180.6 – 1)
'Top' free cash-flow misappropriation rate	$a = 0.02$	4.09% (= 180.2 / 173.1 – 1)	4.02% (= 180.2 / 173.2 – 1)	1.28% (= 180.2 / 177.9 – 1)
'Low' hedge transaction cost rate	$\varepsilon = 0$	4.71% (= 184.4 / 176.1 – 1)	4.65% (= 184.4 / 176.3 – 1)	1.69% (= 184.4 / 181.4 – 1)

4. CONCLUSION

To investigate both the value and risk impacts of a joint hedging and leverage decision, this study utilises a multi-period model for a company subject to respectively hedgeable and unhedgeable production output price and quantity risk variables, endogenously derived deadweight costs, and the tandem availability of risky leverage and flexible hedging control variables. For sensible base-case exogenous parameters, it is found that the optimal (value-maximising) joint hedging and leverage strategy increases company value by about 4.0% compared to the unhedged optimal leverage strategy, by about 1.3% compared to the unlevered optimal hedge strategy, and by about 4.0% compared to the company being unlevered and unhedged. Also found is that optimal leverage is usually much higher in conjunction with optimal hedging than with no hedging, but the relationship is not purely a matter of higher hedging facilitating higher leverage.

At outset a jointly optimal hedging and leverage strategy markedly boosts the levels of both hedging and leverage, compared to optimal hedging without leverage, and optimal leverage without hedging respectively. Then as long as the price for production output is at least rising modestly, the hedging level is largely maintained, while leverage tends to fall due to maturing-hedge losses substituting for debt as a tax shield. For conversely weak output price outcomes close to the unit cost of production, the ongoing operational value of the company is low, but the hedge portfolio is deep in-the-money and facilitative of increased leverage, driven by pecking order roll-over of hedging-boosted debt. Low ongoing operational value imposes a net cost/benefit asymmetry with respect to subsequent output price outcome, because an output price fall (below unit production cost) will lead to abandonment/bankruptcy regardless of hedge portfolio value, but an output price rise will warrant continued operation and production. However the operational benefit of a subsequent output price rise is vulnerable to the financial risk posed by high leverage (in the presence of unhedgeable production quantity risk) and the fact that the hedge portfolio would lose value; that is, an output price rise actually entails adversely high risk of bankruptcy and consequential foregone profitable production. To counter this, some reversal of the overall hedge position increases the likelihood that the company will avoid bankruptcy in event of an output price rise and thereby benefit from ongoing profitable production. The value-maximising optimality of this risk-seeking behaviour arises in conjunction with fair compensation to debt-holders and other non-equity stakeholders for limited liability risk.

For base-case exogenous parameters and most sensitivity analysis scenarios, optimal hedging with or without optimal leverage always entails less financial risk (or equal financial risk at a minimum value of zero for some measures) than the unhedged strategies with or without leverage. Furthermore, while output price is generally rising, jointly optimal hedging and leverage is generally less risky or no more risky than optimal hedging without leverage; but is more risky for weak output price outcomes, for which jointly optimal hedging and leverage is associated with high leverage and hedge reducing (i.e. risk-seeking) behaviour.

Reflections and directions

The model company presented in this thesis culminates several permutations. Paramount throughout this process has been the model's taxation framework. Two crucial developments pertain to incorporation of a personal tax penalty for debt income. Originally the corporate tax rate was simply adjusted from a full rate to a lower, 'effective' corporate tax rate. The disadvantage of this was that non-debt tax shields (e.g. hedge losses) were also penalised in their effectiveness. Hence the revised model left the full corporate tax rate unadjusted and deducted the personal tax penalty from the proceeds of newly issued debt. However, because the comprehensive limited liability option does not distinguish between debt and non-debt bearers of limited liability risk, the personal tax penalty was initially charged only for the risk-free value of debt income (i.e. ignoring debt's share of the income from equity's purchase of the comprehensive limited liability option and thus undercharging for the personal tax penalty). It was subsequently decided that non-debt limited liability risk bearers (i.e. non-debt non-equity stakeholders such as company employees) would have a tax position more closely aligned with debt than with equity, which warranted the personal tax penalty be charged for the entire value of the comprehensive limited liability option. This meant that limited liability risk became more expensive for the company; a notable consequence being that the hedge reduction behaviour at weak output price outcomes became considerably more moderate.

The analysis presented restricted the leverage choice to single-period debt. Although the model allows for multi-period debt, the principle of this is undermined by the periodic charge for the comprehensive limited liability option (i.e. the limited liability premium for multi-period debt would be paid by the company as floating instalments over the life of the debt). The initial model set-up transacted the comprehensive limited option only at outset (time $t = 0$) and then with any subsequent issue of new debt rather than automatically every period. However this was considered to unsatisfactorily represent the flexibility of non-debt non-equity stakeholders to periodically renegotiate compensation for limited liability risk. Consequently the finalised model incorporates automatic re-contracting of the comprehensive limited liability option every period.

Subsequent model development would benefit from separation of the comprehensive limited liability option into non-debt and debt components; with the non-debt limited liability option transacted each period, but the debt limited liability option transacted only with the issue of new debt. This would introduce the requirement to assign claim priorities in event of bankruptcy, adding to model complexity, particularly if different issues of multi-period debt are to be given different bankruptcy claim priorities. Nevertheless this would provide a meaningful future extension. The value of risky debt would be distinctly observable (separate from the broader non-equity value defined in this study), as well as the dynamics of the limited liability risk premiums attributable to different debt issues.

Two earlier comments on this study referred to a potential role for a spot position amongst the model company's control decisions.[27] Short-selling or stock-piling the production resource is a hybrid hedging and leverage decision (just as hedging is a hybrid spot and leverage decision and leverage is a hybrid hedging and spot decision). Preference for a spot position in some combination with or alternately to separate hedging and leverage positions will depend on the comparative exposure to transaction costs, the personal tax penalty of non-equity and corporate taxation.

For simplicity of conceptualisation, the model company has been presented as the producer of a single commodity. To represent a diversified commodity producer, the model's price and quantity risk variables could be characterised with respect to an appropriate 'basket' of commodities. Or to represent a producer of exports, the price and quantity variables should be respectively characterised as the foreign exchange rate and foreign currency revenue (with production costs preferably calculated with respect to the actual quantity variable rather than expected quantity). The modelling approach can

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[27] I thank Bruce Grundy and Ken Clements for this pertinent observation.

also be readily adjusted to represent a company with hedgeable production costs and unhedgeable revenue.

Two stand-out results of this study are that the optimal joint hedging and leverage decision demonstrates: firstly, a negative relationship between leverage and output price outcome, which in an empirical setting could be mistaken for evidence against the trade-off theory of leverage; and secondly, hedge reduction at weak output price outcomes, which in an empirical setting could be mistaken for selective hedging or asset substitution. For future research the model company could be used to synthesise corporate hedging and leverage time-series data for regression analysis, in the hope of gaining insight into the statistical results and non-results and consequential conclusions of past empirical analyses.[28]

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[28] I thank Tom Smith for this suggestion.

APPENDIX A. THE MODEL

The intention is to define the model company's equity and non-equity[29] stakes as complex, interdependent, American-style options on production and hedging cash-flows. Thereby an option-pricing approach can be used to value equity and non-equity and to also assess the effects of hedging and leverage control decisions. The model company faces two sources of uncertainty: at time t, being the end of a discrete production period of duration Δt years, the production output for the period is an uncertain quantity of q_t units, and the price obtainable for the production output is an uncertain amount of p_t dollars per unit. The bivariate price and quantity process is assumed to be Markovian. Hedging can only be contracted with respect to price uncertainty (i.e. the price risk is hedgeable and the quantity risk is not). The company is assumed to have uniform production periods, and periodic expected production quantity is assumed to be independent of previous unexpected production quantity deviations. This allows the company to be specified as blind to the history of the stochastic component of production quantities, which advantageously allows equity and non-equity, conditional on price and control variables, to have generalisable valuation solutions with respect to quantity uncertainty.

Price and quantity random variable specification

Defining $s \geq 0$ to be continuous time, the risk-neutral production output price process (p_s) follows the stochastic differential equation:

$$d\ln p_s = \left(r - \delta - \frac{\sigma_p^2}{2} \right) ds + \sigma_p dW_s^p$$

where: r is the annual, continuously compounding risk-free interest rate; δ is the annual, continuously compounding convenience yield of the production output; σ_p^2 is the annual variance rate for the output price; and W_s^p is a standard Brownian motion.

[29] The non-equity stake refers to the combined position of debt plus the short comprehensive limited liability option provided to equity by all non-equity stakeholders in the company.

Now defining $t=\{0,\Delta t,2\Delta t,...\}$ to be discrete time, the production output quantity is observed in discrete time (i.e. at the end of each production period) so that:

$$\ln q_t=\mu_t^{\ln q}+\sigma_q\psi_t^{(q)}\sqrt{\Delta t}\ ,\quad \text{for } t=\{\Delta t,2\Delta t,...\}$$

where: σ_q^2 is the annual variance rate for the production quantity; $\left(\psi_t^{(q)}\right)_{t=\{\Delta t,2\Delta t,...\}}$ is a series of independent, standard (i.e. zero mean, unit variance) normal random variables; and $\mu_t^{\ln q}$ is the risk-neutral expected value of $\ln q_t$ ($\hat{\mathrm{E}}_{t-\Delta t}[\ln q_t]$, where $\hat{\mathrm{E}}_{t-\Delta t}[\cdot]$ is the risk-neutral expectation operator). Periodic $\mu_t^{\ln q}$ is predictable and specified to move with the absolute output price level as follows:

$$\mu_t^{\ln q}=\hat{\mathrm{E}}_{t-\Delta t}[\ln q_t]=\ln\overline{q}_0+\rho\left(\frac{\sigma_q}{\sigma_p}\right)\left[\ln\left(\frac{p_{t-\Delta t}}{p_0}\right)-\left(r-\delta-\frac{\sigma_p^2}{2}\right)(t-\Delta t)\right],$$

$$\text{for } t=\{\Delta t,2\Delta t,...\}$$

where: ρ is the correlation between periodic output price and production quantity; p_0 is the initial output price; and $\mu_{\Delta t}^{\ln q}=\hat{\mathrm{E}}_0[\ln q_{\Delta t}]=\ln\overline{q}_0$ (i.e. $\overline{q}_0$ is the initial period's median production quantity). The intuition here is that the expected production quantity equals a time-zero base level plus an adjustment (dependent on ρ) for the output price level deviation from time-zero price net of the expected price drift; that is, the expected value of q_t grows with unexpected growth of $p_{t-\Delta t}$. The intention is that a positive value can be specified for ρ to represent a price-taking company that adjusts expected production to follow step with output price movements.

In discrete time, the risk-neutral bivariate process for production output price and quantity is:

$$\begin{bmatrix}\ln p_t\\ \ln q_t\end{bmatrix}=\begin{bmatrix}\ln p_{t-\Delta t}+\left(r-\delta-\frac{\sigma_p^2}{2}\right)\Delta t\\ \ln\overline{q}_0+\rho\left(\frac{\sigma_q}{\sigma_p}\right)\left[\ln\left(\frac{p_{t-\Delta t}}{p_0}\right)-\left(r-\delta-\frac{\sigma_p^2}{2}\right)(t-\Delta t)\right]\end{bmatrix}+\begin{bmatrix}\sigma_p\psi_t^{(p)}\sqrt{\Delta t}\\ \sigma_q\psi_t^{(q)}\sqrt{\Delta t}\end{bmatrix},$$

$$\text{for } t=\{\Delta t,2\Delta t,...\}$$

where $\left(\psi_t^{(p)},\psi_t^{(q)}\right)_{t=\{\Delta t,2\Delta t,\ldots\}}$ is a serially independent and identically normally distributed series of bivariate random variables such that:

$$\begin{bmatrix}\psi_t^{(p)}\\ \psi_t^{(q)}\end{bmatrix}\sim \mathrm{N}\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right),\quad \text{for } t=\{\Delta t,2\Delta t,\ldots\}.$$

The 'in-period' correlation between $\psi_t^{(p)}$ and $\psi_t^{(q)}$ effectively assumes that the model company can make in-period production quantity responses instantaneously with respect to output price fluctuations, to a limit specified by the scale of ρ. This can be conceptualised in terms of the company's production responsiveness, and some existence of short-term (in-period) predictability/momentum for the output price (which, however, requires relaxation of the assumed diffusion process, but otherwise does not impact other aspects of the model set-up).

For the sake of model implementation, the periodic lognormal price distribution is approximated from a binomial process. The control decisions and price risk can then be incorporated into the valuation problem via a price-tree, specifically with non-recombining nodes so as to enable path-dependent control behaviour. Define $(n+1)$ to be the number of possible future price realisations emanating from a price-node for a single production period (i.e. price realisations numbered from 0 to n). Hence the risk-neutral process for p_t is approximated by:

$$p_t=p_{t-\Delta t}\left[e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/n)+\sigma_p\sqrt{\Delta t/n}}\right]^{\mathrm{P}_t}\left[e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/n)-\sigma_p\sqrt{\Delta t/n}}\right]^{(n-\mathrm{P}_t)},$$

where P_t is the following binomial random variable:

$$\mathrm{P}_t\sim \mathrm{BIN}\left(n,\frac{e^{(r-\delta)(\Delta t/n)}-e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/n)-\sigma_p\sqrt{\Delta t/n}}}{e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/n)+\sigma_p\sqrt{\Delta t/n}}-e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/n)-\sigma_p\sqrt{\Delta t/n}}}\right).$$

The conditional distribution and the conditional expectation of q_t are:

$$\left(\ln q_t \mid p_t\right)_{t-\Delta t}\sim \mathrm{N}\left(\ln\overline{q}_0+\rho\left(\frac{\sigma_q}{\sigma_p}\right)\left[\ln\left(\frac{p_t}{p_0}\right)-\left(r-\delta-\frac{\sigma_p^2}{2}\right)t\right],\sigma_q^2(1-\rho^2)\Delta t\right),$$

$$\hat{\mathrm{E}}_{t-\Delta t}\left[q_t \mid p_t\right]=\overline{q}_0\left(\frac{p_t}{p_0}\right)^{\rho\left(\frac{\sigma_q}{\sigma_p}\right)}\exp\left\{-\rho\left(\frac{\sigma_q}{\sigma_p}\right)\left(r-\delta-\frac{\sigma_p^2}{2}\right)t+(1-\rho^2)\frac{\sigma_q^2}{2}\Delta t\right\} \tag{A.1}$$

where the subscript $t-\Delta t$ in the representation $(\ln q_t \mid p_t)_{t-\Delta t}$ is used to indicate that there is an entire period's uncertainty (from time $t-\Delta t$ to t) in the probability distribution for $(\ln q_t \mid p_t)$.

Furthermore, for later reference, from Black (1976) the risk-neutral conditional expected payoffs for a vanilla put option, an asset-or-nothing digital put option and a cash-or-nothing digital put option contracted on q_t with strike χ (respectively designated as $\mathrm{VP}_{q_t<\chi}$, $\mathrm{ADP}_{q_t<\chi}$ and $\mathrm{CDP}_{q_t<\chi}$) are:

$$\begin{aligned}\mathrm{VP}_{q_t<\chi}&=\hat{\mathrm{E}}_{t-\Delta t}\left[(\chi-q_t)\mathrm{I}_{q_t<\chi}\mid p_t\right]\\&=\chi\Phi\left\{\frac{\ln\chi-\hat{\mathrm{E}}_{t-\Delta t}\left[\ln q_t\mid p_t\right]}{\sigma_q\sqrt{\left(1-\rho^2\right)\Delta t}}\right\}-\hat{\mathrm{E}}_{t-\Delta t}\left[q_t\mid p_t\right]\Phi\left\{\frac{\ln\chi-\hat{\mathrm{E}}_{t-\Delta t}\left[\ln q_t\mid p_t\right]-\sigma_q^2\left(1-\rho^2\right)\Delta t}{\sigma_q\sqrt{\left(1-\rho^2\right)\Delta t}}\right\},\end{aligned}$$

$$\mathrm{ADP}_{q_t<\chi}=\hat{\mathrm{E}}_{t-\Delta t}\left[q_t\,\mathrm{I}_{q_t<\chi}\mid p_t\right]=\hat{\mathrm{E}}_{t-\Delta t}\left[q_t\mid p_t\right]\Phi\left\{\frac{\ln\chi-\hat{\mathrm{E}}_{t-\Delta t}\left[\ln q_t\mid p_t\right]-\sigma_q^2\left(1-\rho^2\right)\Delta t}{\sigma_q\sqrt{\left(1-\rho^2\right)\Delta t}}\right\},$$

$$\mathrm{CDP}_{q_t<\chi}=\hat{\mathrm{E}}_{t-\Delta t}\left[\mathrm{I}_{q_t<\chi}\mid p_t\right]=\Phi\left\{\frac{\ln\chi-\hat{\mathrm{E}}_{t-\Delta t}\left[\ln q_t\mid p_t\right]}{\sigma_q\sqrt{\left(1-\rho^2\right)\Delta t}}\right\}$$

where: the indicator function, $\mathrm{I}_{logical\ statement}$, equals one if the *logical statement* is true and zero otherwise; and $\Phi\{\cdot\}$ is the standard normal cumulative distribution function.

Control variables

Define positive integer N to be the number of production periods for which control decisions will be made: N may cover the company's total production life or, if an estimate can be made for ongoing company value at some intermediate stage of its total life, N may then be the number of production periods up to this intermediate stage. Time is denominated by these discrete production periods $\left(t\in\{0,\Delta t,2\Delta t,...,N\Delta t\}\right)$.

Given the positive integer approximation of $(n+1)$ possible price realisations for a production period, the total number of non-recombining price-paths is therefore equal to $(n+1)^N$. Define $\eta \in \{1, 2, ..., (n+1)^N\}$ to signify a specific price-path.

The company has control over hedging and leverage. The hedging control decision at each start-of-period entails application of a hedging term structure comprising a series of hedge quantities ($x_{t,t+\kappa\Delta t}$ production units) for hedge contracts with initial terms to maturity ($\kappa\Delta t$) increasing from Δt years (i.e. one period) to a maximum of $(N\Delta t - t)$ years (i.e. $(N\Delta t - t)/\Delta t$ periods); similarly the leverage control decision at each start-of-period entails application of a debt term structure in the form of a series of zero-coupon bonds designated by their dollar face-values ($y_{t,t+\kappa\Delta t} \geq 0$) also with initial terms to maturity ($\kappa\Delta t$) increasing from Δt years to $(N\Delta t - t)$ years. For a given price-path (η), the entire matrices of hedging and leverage control variables are respectively:

$$\mathbf{x}_\eta = \begin{bmatrix} x_{0,\Delta t} & 0 & 0 & \cdots & 0 \\ x_{0,2\Delta t} & x_{\Delta t,2\Delta t} & 0 & \cdots & 0 \\ x_{0,3\Delta t} & x_{\Delta t,3\Delta t} & x_{2\Delta t,3\Delta t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{0,N\Delta t} & x_{\Delta t,N\Delta t} & x_{2\Delta t,N\Delta t} & \cdots & x_{(N-1)\Delta t,N\Delta t} \end{bmatrix},$$

$$\mathbf{y}_\eta = \begin{bmatrix} y_{0,\Delta t} & 0 & 0 & \cdots & 0 \\ y_{0,2\Delta t} & y_{\Delta t,2\Delta t} & 0 & \cdots & 0 \\ y_{0,3\Delta t} & y_{\Delta t,3\Delta t} & y_{2\Delta t,3\Delta t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{0,N\Delta t} & y_{\Delta t,N\Delta t} & y_{2\Delta t,N\Delta t} & \cdots & y_{(N-1)\Delta t,N\Delta t} \end{bmatrix}.$$

The columns of $\mathbf{x}_\eta$ and $\mathbf{y}_\eta$ represent each start-of-period's hedging and leverage choices for $t = 0$ to $t = (N-1)\Delta t$. With each increment in time, the number of remaining production periods reduces by one, and accordingly the number of term structure elements in each of the hedging and leverage control decisions also reduces by one. The number of non-recombining price-nodes, for each of which a unique hedging and leverage control decision can be made, amounts to $(n+1)^{t/\Delta t}$ at each new start-of-period from $t = 0$ to $t = (N-1)\Delta t$. Hence the total number of hedging and leverage elements that can potentially be chosen uniquely amounts to $2\sum_{i=0}^{N-1}(n+1)^i(N-i)$.

Now define $\mathbf{X}_t$ to be the matrix of net values ($X_{t:\tau,\tau+\kappa\Delta t}$ in dollars per production unit) of 'live' and maturing hedge contracts; $X_{t:\tau,\tau+\kappa\Delta t}$ is undefined (set to zero) if specified time t is earlier than hedge contract initiation at time τ or later than hedge contract maturity at time $(\tau+\kappa\Delta t)$ (i.e. $X_{t:\tau,\tau+\kappa\Delta t}=0$ if $t<\tau$ or $t>(\tau+\kappa\Delta t)$):

$$\mathbf{X}_t=\begin{bmatrix} X_{t:0,\Delta t} & 0 & 0 & \cdots & 0 \\ X_{t:0,2\Delta t} & X_{t:\Delta t,2\Delta t} & 0 & \cdots & 0 \\ X_{t:0,3\Delta t} & X_{t:\Delta t,3\Delta t} & X_{t:2\Delta t,3\Delta t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{t:0,N\Delta t} & X_{t:\Delta t,N\Delta t} & X_{t:2\Delta t,N\Delta t} & \cdots & X_{t:(N-1)\Delta t,N\Delta t} \end{bmatrix}.$$

The hedge contracts are specified to have zero up-front cost (i.e. the applicable up-front hedge contract premium, if any, is compounded forward at the risk-free interest rate and included as a deduction in the net hedge payoff calculation at contract maturity). Hence the net value of each hedge contract at initiation ($X_{t:t,t+\kappa\Delta t}$) is zero. If the hedge contracts mature in favour of the company ($X_{t:t-\kappa\Delta t,t}>0$), they contribute to end-of-period revenue. If the hedge contracts mature out of the company's favour ($X_{t:t-\kappa\Delta t,t}<0$), they become an end-of-period liability.

The model company is able to combine a short forward hedge with a put option hedge. In addition to control of hedge quantity (described by matrix $\mathbf{x}_\eta$), at each start-of-period the company can also choose a term structure for: the fraction of total hedge quantity committed to short forwards as opposed to put options ($0\le w_{t,t+\kappa\Delta t}\le 1$, i.e. $w_{t,t+\kappa\Delta t}x_{t,t+\kappa\Delta t}$ production units are hedged with short forwards and $(1-w_{t,t+\kappa\Delta t})x_{t,t+\kappa\Delta t}$ units are hedged with put options); and the strike price of the put options ($z_{t,t+\kappa\Delta t}>0$). The entire matrices of these additional hedging control variables are:

$$\mathbf{w}_\eta=\begin{bmatrix} w_{0,\Delta t} & 0 & 0 & \cdots & 0 \\ w_{0,2\Delta t} & w_{\Delta t,2\Delta t} & 0 & \cdots & 0 \\ w_{0,3\Delta t} & w_{\Delta t,3\Delta t} & w_{2\Delta t,3\Delta t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{0,N\Delta t} & w_{\Delta t,N\Delta t} & w_{2\Delta t,N\Delta t} & \cdots & w_{(N-1)\Delta t,N\Delta t} \end{bmatrix},$$

$$\mathbf{z}_\eta = \begin{bmatrix} z_{0,\Delta t} & 0 & 0 & \cdots & 0 \\ z_{0,2\Delta t} & z_{\Delta t,2\Delta t} & 0 & \cdots & 0 \\ z_{0,3\Delta t} & z_{\Delta t,3\Delta t} & z_{2\Delta t,3\Delta t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{0,N\Delta t} & z_{\Delta t,N\Delta t} & z_{2\Delta t,N\Delta t} & \cdots & z_{(N-1)\Delta t,N\Delta t} \end{bmatrix}.$$

The net hedge payoff of any particular hedge term structure element is:

$$\begin{aligned} x_{t-\kappa\Delta t,t} X_{t:t-\kappa\Delta t,t} = {} & x_{t-\kappa\Delta t,t} w_{t-\kappa\Delta t,t} \left[(1-\varepsilon) p_{t-\kappa\Delta t} e^{(r-\delta)\kappa\Delta t} - p_t \right] \\ & + x_{t-\kappa\Delta t,t} (1 - w_{t-\kappa\Delta t,t}) \max\left[0, z_{t-\kappa\Delta t,t} - p_t \right] \\ & - x_{t-\kappa\Delta t,t} (1 - w_{t-\kappa\Delta t,t}) \left(\hat{\mathrm{E}}_{t-\kappa\Delta t} \left[\max\left[0, z_{t-\kappa\Delta t,t} - p_t \right] \right] + \varepsilon z_{t-\kappa\Delta t,t} \right) \end{aligned}$$

where $0 \le |\varepsilon| < 1$ is the transaction cost rate for the hedge contracts. The transaction cost rate should be positive ($0 \le \varepsilon < 1$) for 'positive'-hedge positions (i.e. combined short forward and long put positions meaning that $x_{t-\kappa\Delta t,t} > 0$), and negative ($-1 < \varepsilon \le 0$) for 'negative/anti'-hedge positions (i.e. combined long forward and short put positions meaning that $x_{t-\kappa\Delta t,t} < 0$). The hedge transaction cost rate is charged against the forward price for forward contracts and, for equivalent scale effect, against the strike price for put options. Note that the put option premium is deferred for payment at hedge maturity and incorporated into the net hedge payoff.

The value of a live hedge position ($\tau < t < \tau + \kappa\Delta t$) is:

$$\begin{aligned} x_{\tau,\tau+\kappa\Delta t} X_{t:\tau,\tau+\kappa\Delta t} = {} & e^{-r(\tau+\kappa\Delta t-t)} x_{\tau,\tau+\kappa\Delta t} w_{\tau,\tau+\kappa\Delta t} \left[(1-\varepsilon) p_\tau e^{(r-\delta)\kappa\Delta t} - \hat{\mathrm{E}}_t [p_{\tau+\kappa\Delta t}] \right] \\ & + e^{-r(\tau+\kappa\Delta t-t)} x_{\tau,\tau+\kappa\Delta t} (1 - w_{\tau,\tau+\kappa\Delta t}) \hat{\mathrm{E}}_t \left[\max\left[0, z_{\tau,\tau+\kappa\Delta t} - p_{\tau+\kappa\Delta t} \right] \right] \\ & - e^{-r(\tau+\kappa\Delta t-t)} x_{\tau,\tau+\kappa\Delta t} (1 - w_{\tau,\tau+\kappa\Delta t}) \left(\hat{\mathrm{E}}_\tau \left[\max\left[0, z_{\tau,\tau+\kappa\Delta t} - p_{\tau+\kappa\Delta t} \right] \right] + \varepsilon z_{\tau,\tau+\kappa\Delta t} \right). \end{aligned}$$

(A.2)

The hedge transaction cost is due at hedge maturity, and the liability for the hedge transaction cost is assumed to not come into effect until instantaneously after hedge initiation, therefore precisely at initiation ($t = \tau$) the hedge position has zero value (i.e. $x_{t,t+\kappa\Delta t} X_{t:t,t+\kappa\Delta t} = 0$).

Operating earnings and cash-flows

Define $c \geq 0$ to be the cost of production in dollars per unit of expected production conditional on price outcome. Total production costs $(c\,\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t])$ thereby vary with the expected production quantity response to price outcome, but do not vary with that part of quantity risk that is orthogonal to price outcome. The intention is that the model company is able to have a predictable production response to price behaviour as specified by the correlation coefficient (ρ), and that total production costs adjust for this predictable response but are otherwise fixed.

Earnings before interest and tax ($EBIT_t$) is comprised of operating profit ($p_t q_t - c\,\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t]$), plus total net payoff of maturing hedge contracts ($(\mathbf{e}_t^T \mathbf{x}_\eta)(\mathbf{e}_t^T \mathbf{X}_t)^T$):

$$EBIT_t = p_t q_t - c\,\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t] + \left(\mathbf{e}_t^T \mathbf{x}_\eta\right)\left(\mathbf{e}_t^T \mathbf{X}_t\right)^T \qquad \text{(A.3)}$$

where $\mathbf{e}_t$ is a vector of length N made up of zeros except for a one at the $t/\Delta t$ vector element, and the superscript T signifies a vector or matrix transpose.

Free cash-flow before tax (FBT_t) comprises $EBIT_t$, plus proceeds of new debt (Y_t), minus payment of total face-value of maturing debt ($(\mathbf{e}_t^T \mathbf{y}_\eta)\mathbf{1}$, where $\mathbf{1}$ is a vector of length N made up of ones). Introducing the function $f(p_t)$ to represent total due pre-tax liabilities ($c\,\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t] + (\mathbf{e}_t^T \mathbf{y}_\eta)\mathbf{1}$), less net hedge payoff ($(\mathbf{e}_t^T \mathbf{x}_\eta)(\mathbf{e}_t^T \mathbf{X}_t)^T$), less new debt finance (Y_t); then FBT_t is:

$$FBT_t = p_t q_t - \left[c\,\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t] + \left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1} - \left(\mathbf{e}_t^T \mathbf{x}_\eta\right)\left(\mathbf{e}_t^T \mathbf{X}_t\right)^T - Y_t\right] = p_t q_t - f(p_t)\,. \qquad \text{(A.4)}$$

It is assumed that the expense for new debt finance (i.e. total face-value minus total initial value of newly issued debt, $[\mathbf{1}^T(\mathbf{y}_\eta \mathbf{e}_{t+\Delta t}) - Y_t]$) can be immediately tax-deducted. Hence earnings before tax (EBT_t) is:

$$\begin{aligned} EBT_t &= p_t q_t - c\,\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t] + \left(\mathbf{e}_t^T \mathbf{x}_\eta\right)\left(\mathbf{e}_t^T \mathbf{X}_t\right)^T - \left[\mathbf{1}^T\left(\mathbf{y}_\eta \mathbf{e}_{t+\Delta t}\right) - Y_t\right] \\ &= FBT_t + \left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1} - \mathbf{1}^T\left(\mathbf{y}_\eta \mathbf{e}_{t+\Delta t}\right) \\ &= p_t q_t - f(p_t) - \left[\mathbf{1}^T\left(\mathbf{y}_\eta \mathbf{e}_{t+\Delta t}\right) - \left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}\right]. \end{aligned}$$

Positive EBT_t is assumed subject to a tax rate of $0 \le \alpha < 1$, while negative EBT_t is assumed subject to a tax rate of $\lambda\alpha$, where $0 \le \lambda \le 1$ represents the claimability of a tax refund in event of a negative EBT_t result.[30] Therefore free cash-flow (F_t) comprises FBT_t minus (plus) tax payable (refundable) on positive (negative) EBT_t. Define the function $d_t(\mathbf{y}_\eta)$ to equal the increment in total debt face-value, $[\mathbf{1}^T(\mathbf{y}_\eta \mathbf{e}_{t+\Delta t}) - (\mathbf{e}_t^T \mathbf{y}_\eta)\mathbf{1}]$ (i.e. total face-value of newly issued debt minus total face-value of maturing debt). The expression for F_t is:

$$
\begin{aligned}
F_t &= FBT_t - \alpha EBT_t + (1-\lambda)\alpha EBT_t \, \mathrm{I}_{EBT_t < 0} \\
&= FBT_t - \alpha\left\{FBT_t - \left[\mathbf{1}^T\left(\mathbf{y}_\eta \mathbf{e}_{t+\Delta t}\right) - \left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}\right]\right\} \\
&\qquad + (1-\lambda)\alpha\left\{FBT_t - \left[\mathbf{1}^T\left(\mathbf{y}_\eta \mathbf{e}_{t+\Delta t}\right) - \left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}\right]\right\} \mathrm{I}_{FBT_t < \left[\mathbf{1}^T\left(\mathbf{y}_\eta \mathbf{e}_{t+\Delta t}\right) - \left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}\right]} \\
&= (1-\alpha)\left[p_t q_t - f(p_t)\right] + \alpha d_t(\mathbf{y}_\eta) + (1-\lambda)\alpha\left\{p_t q_t - f(p_t) - d_t(\mathbf{y}_\eta)\right\} \mathrm{I}_{q_t < \frac{f(p_t) + d_t(\mathbf{y}_\eta)}{p_t}} .
\end{aligned}
$$

Cash-flows to equity

Solvency is defined to be the condition where free cash-flow is zero or positive ($F_t \ge 0$). It is assumed that a fraction ($0 \le a \le 1$) of positive free cash-flow is misappropriated by management and the remainder is paid out as a dividend to equity. Financial distress is defined to be the condition of a free cash-flow shortfall ($F_t < 0$) necessitating new equity finance via a 'negative dividend' (which is considered to be a rights issue); an additional fraction ($\gamma \ge 0$) of the free cash-flow shortfall is required from equity finance providers to cover financial distress costs and the transaction costs of the equity issue. Bankruptcy is defined to occur under condition of negative free cash-flow such that the required equity-raising ($-(1+\gamma)F_t$) is larger than the ongoing equity value (being the present value of future cash-flows to equity given that bankruptcy has not occurred). Using $E_{t+} \ge 0$ to signify the ongoing equity value, the dividend payment (G_t) is determined by:

[30] The model does not specifically allow for the carry-forward or carry-back of tax losses. Instead the parameter λ is used to represent the company's 'average' ability to claim a tax refund in event of a loss.

$$G_t = \begin{cases} (1-a)F_t & , \quad if \quad F_t \geq 0 \\ (1+\gamma)F_t & , \quad if \quad -E_{t+}/(1+\gamma) \leq F_t < 0 \\ 0 & , \quad if \quad F_t < -E_{t+}/(1+\gamma) \end{cases}$$ (A.5)

$$= (1-a)F_t + (a+\gamma)F_t \, \mathrm{I}_{F_t<0} - (1+\gamma)F_t \, \mathrm{I}_{F_t < \frac{-E_{t+}}{(1+\gamma)}} .$$

Noting, for some value J, that:

$$\mathrm{I}_{F_t<J} = \mathrm{I}_{q_t < \frac{(1-\alpha)f(p_t)+J-\alpha d_t(\mathbf{y}_\eta)}{(1-\alpha)p_t}} - \mathrm{I}_{q_t<\min\left[\frac{(1-\alpha)f(p_t)+J-\alpha d_t(\mathbf{y}_\eta)}{(1-\alpha)p_t}, \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}\right]} + \mathrm{I}_{q_t<\min\left[\frac{(1-\alpha\lambda)f(p_t)+J-\alpha\lambda d_t(\mathbf{y}_\eta)}{(1-\alpha\lambda)p_t}, \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}\right]} .$$

Then substituting for F_t in equation (A.5) gives:

$$\begin{aligned}
G_t &= (1-a)(1-\alpha)\left[p_t q_t - f(p_t)\right] + (1-a)\alpha d_t(\mathbf{y}_\eta) \\
&-(1-a)(1-\lambda)\alpha p_t \left\{\frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t} - q_t\right\} \mathrm{I}_{q_t<\frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}} \\
&-(a+\gamma)(1-\alpha)p_t \left\{\frac{(1-\alpha)f(p_t)-\alpha d_t(\mathbf{y}_\eta)}{(1-\alpha)p_t} - q_t\right\} \mathrm{I}_{q_t<\frac{(1-\alpha)f(p_t)-\alpha d_t(\mathbf{y}_\eta)}{(1-\alpha)p_t}} \\
&+(1+\gamma)(1-\alpha)p_t \left\{\frac{(1-\alpha)f(p_t)-\alpha d_t(\mathbf{y}_\eta)}{(1-\alpha)p_t} - q_t\right\} \mathrm{I}_{q_t<\frac{(1+\gamma)(1-\alpha)f(p_t)-E_{t+}-\alpha(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha)p_t}} \\
&+(a+\gamma)(1-\alpha)p_t \left\{\frac{(1-\alpha)f(p_t)-\alpha d_t(\mathbf{y}_\eta)}{(1-\alpha)p_t} - q_t\right\} \mathrm{I}_{q_t<\min\left[\frac{(1-\alpha)f(p_t)-\alpha d_t(\mathbf{y}_\eta)}{(1-\alpha)p_t}, \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}\right]} \\
&-(1+\gamma)(1-\alpha)p_t \left\{\frac{(1-\alpha)f(p_t)-\alpha d_t(\mathbf{y}_\eta)}{(1-\alpha)p_t} - q_t\right\} \mathrm{I}_{q_t<\min\left[\frac{(1+\gamma)(1-\alpha)f(p_t)-E_{t+}-\alpha(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha)p_t}, \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}\right]} \\
&-(a+\gamma)(1-\alpha\lambda)p_t \left\{\frac{(1-\alpha\lambda)f(p_t)-\alpha\lambda d_t(\mathbf{y}_\eta)}{(1-\alpha\lambda)p_t} - q_t\right\} \mathrm{I}_{q_t<\min\left[\frac{(1-\alpha\lambda)f(p_t)-\alpha\lambda d_t(\mathbf{y}_\eta)}{(1-\alpha\lambda)p_t}, \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}\right]} \\
&+(1+\gamma)(1-\alpha\lambda)p_t \left\{\frac{(1-\alpha\lambda)f(p_t)-\alpha\lambda d_t(\mathbf{y}_\eta)}{(1-\alpha\lambda)p_t} - q_t\right\} \mathrm{I}_{q_t<\min\left[\frac{(1+\gamma)(1-\alpha\lambda)f(p_t)-E_{t+}-\alpha\lambda(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha\lambda)p_t}, \frac{f(p_t)+d_t(\mathbf{y}_\eta)}{p_t}\right]} .
\end{aligned}$$ (A.6)

Note that $f(p_t)$ and E_{t+} are dependent on p_t but not q_t (although q_t does have a role in determining whether bankruptcy occurs and thus whether E_{t+} is accessed or not). This means that at each period the expected dividend payment can be solved conditional on p_t. To do so it can be seen that the dividend payment comprises a series of put option payoffs dependent on q_t: two standard/vanilla put options; five asset-or-nothing digital put options; and five cash-or-nothing digital put options.

To simplify notation, define χ_1, χ_2, χ_3, χ_4 and χ_5:

$$\chi_1 = \frac{f(p_t) + d_t(\mathbf{y}_\eta)}{p_t}, \; \chi_2 = \frac{(1-\alpha)f(p_t) - \alpha d_t(\mathbf{y}_\eta)}{(1-\alpha)p_t},$$

$$\chi_3 = \frac{(1+\gamma)(1-\alpha)f(p_t) - E_{t+} - \alpha(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha)p_t}, \; \chi_4 = \frac{(1-\alpha\lambda)f(p_t) - \alpha\lambda d_t(\mathbf{y}_\eta)}{(1-\alpha\lambda)p_t},$$

$$\chi_5 = \frac{(1+\gamma)(1-\alpha\lambda)f(p_t) - E_{t+} - \alpha\lambda(1+\gamma)d_t(\mathbf{y}_\eta)}{(1+\gamma)(1-\alpha\lambda)p_t}.$$

Recall that $\mathrm{VP}_{q_t<\chi}$, $\mathrm{ADP}_{q_t<\chi}$ and $\mathrm{CDP}_{q_t<\chi}$ signify the risk-neutral conditional expected payoffs respectively for a vanilla put option, an asset-or-nothing digital put option and a cash-or-nothing digital put option contracted on q_t with strike χ. Hence:

$$\begin{aligned}
\hat{\mathrm{E}}_{t-\Delta t}[G_t \mid p_t] &= (1-a)(1-\alpha)\Big[p_t \hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t] - f(p_t)\Big] + (1-a)\alpha d_t(\mathbf{y}_\eta) \\
&\quad - (1-a)(1-\lambda)\alpha p_t \mathrm{VP}_{q_t<\chi_1} - (a+\gamma)(1-\alpha)p_t\Big\{\mathrm{VP}_{q_t<\chi_2} - \chi_2 \mathrm{CDP}_{q_t<\min[\chi_2,\chi_1]} + \mathrm{ADP}_{q_t<\min[\chi_2,\chi_1]}\Big\} \\
&\quad - (a+\gamma)(1-\alpha\lambda)p_t\Big\{\chi_4 \mathrm{CDP}_{q_t<\min[\chi_4,\chi_1]} - \mathrm{ADP}_{q_t<\min[\chi_4,\chi_1]}\Big\} \\
&\quad + (1+\gamma)(1-\alpha)p_t\Big\{\chi_2 \mathrm{CDP}_{q_t<\chi_3} - \mathrm{ADP}_{q_t<\chi_3} - \chi_2 \mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]} + \mathrm{ADP}_{q_t<\min[\chi_3,\chi_1]}\Big\} \\
&\quad + (1+\gamma)(1-\alpha\lambda)p_t\Big\{\chi_4 \mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]} - \mathrm{ADP}_{q_t<\min[\chi_5,\chi_1]}\Big\}.
\end{aligned} \tag{A.7}$$

In the event of bankruptcy ($F_t < -E_{t+}/(1+\gamma)$), the company is liquidated for a cash-flow (L_t) equal to $EBIT_t$, plus the market value of the company's live (non-maturing) hedge contracts, minus the total face-value of outstanding debt ($\sum_{i=t/\Delta t}^{N}(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta)\mathbf{k}_t$, where $\mathbf{k}_t$ is an array of length N made up of ones for the first array position through to the $t/\Delta t$ array position and zeros otherwise) increased by a bankruptcy cost factor (b). Introducing the function $g(p_t)$ to represent the company's liquidation liability for production costs ($c\,\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t]$) and for debt ($(1+b)\sum_{i=t/\Delta t}^{N}(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta)\mathbf{k}_t$), less both the net hedge payoff of maturing contracts and the market value of live hedge contracts ($\mathrm{tr}(\mathbf{x}_\eta^T \mathbf{X}_t)$), then the liquidation cash-flow term is:

$$L_t = p_t q_t - \left(c\,\hat{\mathrm{E}}_{t-\Delta t}\left[q_t \mid p_t\right] + (1+b)\sum_{i=t/\Delta t}^{N}\left(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta\right)\mathbf{k}_t - \mathrm{tr}\left(\mathbf{x}_\eta^T \mathbf{X}_t\right)\right) = p_t q_t - g(p_t). \qquad \text{(A.8)}$$

The distribution of L_t to non-equity stakeholders is capped at the total face-value of outstanding debt. Any remainder is paid to equity as a taxable liquidating dividend (GL_t):

$$\begin{aligned} GL_t &= \begin{Bmatrix} (1-\alpha)L_t \;, & if \;\; L_t \geq 0 \;\; and \;\; F_t < -E_{t+}/(1+\gamma) \\ 0 \;, & otherwise \end{Bmatrix} \\ &= (1-\alpha)\left\{p_t q_t - g(p_t)\right\}\left\{1 - \mathrm{I}_{q_t < \frac{g(p_t)}{p_t}}\right\}\mathrm{I}_{F_t < \frac{-E_{t+}}{(1+\gamma)}}. \end{aligned} \qquad \text{(A.9)}$$

Therefore:

$$\begin{aligned} \hat{\mathrm{E}}_{t-\Delta t}[GL_t \mid p_t] &= (1-\alpha)p_t\left\{\mathrm{ADP}_{q_t<\chi_3} - \mathrm{ADP}_{q_t<\min[\chi_3,\chi_1]} + \mathrm{ADP}_{q_t<\min[\chi_5,\chi_1]}\right\} \\ &\quad -(1-\alpha)g(p_t)\left\{\mathrm{CDP}_{q_t<\chi_3} - \mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]} + \mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\right\} \\ &\quad -(1-\alpha)p_t\left\{\mathrm{ADP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_3\right]} - \mathrm{ADP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_3,\chi_1\right]} + \mathrm{ADP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_5,\chi_1\right]}\right\} \\ &\quad +(1-\alpha)g(p_t)\left\{\mathrm{CDP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_3\right]} - \mathrm{CDP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_3,\chi_1\right]} + \mathrm{CDP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_5,\chi_1\right]}\right\}. \end{aligned} \qquad \text{(A.10)}$$

The total value attributable to equity immediately prior to any dividend payment at time t (E_{t-}) is:

$$E_{t-} = G_t + GL_t + E_{t+}\left\{1 - \mathrm{I}_{F_t < \frac{-E_{t+}}{(1+\gamma)}}\right\}.$$

Noting that the probability of the bankruptcy condition ($F_t < -E_{t+}/(1+\gamma)$) occurring at a price-node is:

$$\hat{\mathrm{E}}_{t-\Delta t}\left[\mathrm{I}_{F_t<\frac{-E_{t+}}{(1+\gamma)}} \mid p_t\right]=\hat{\mathrm{E}}_{t-\Delta t}\left[\mathrm{I}_{q_t<\chi_3}-\mathrm{I}_{q_t<\min[\chi_3,\chi_1]}+\mathrm{I}_{q_t<\min[\chi_5,\chi_1]} \mid p_t\right]$$
$$=\left\{\mathrm{CDP}_{q_t<\chi_3}-\mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]}+\mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\right\}. \tag{A.11}$$

Then:

$$\hat{\mathrm{E}}_{t-\Delta t}[E_{t-} \mid p_t]=\hat{\mathrm{E}}_{t-\Delta t}[G_t \mid p_t]+\hat{\mathrm{E}}_{t-\Delta t}[GL_t \mid p_t]$$
$$+E_{t+}\left\{1-\mathrm{CDP}_{q_t<\chi_3}+\mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]}-\mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\right\}. \tag{A.12}$$

Cash-flows to debt and against the limited liability option

The model company's non-equity stakeholders are the debt finance providers, the hedge contract providers, the providers of production labour and equipment, and the sources of the bankruptcy deadweight cost; together they also provide equity with its limited liability option. The periodic combined cash-flow to debt and against the limited liability option (H_t) equals: the total due face-value of maturing debt, under condition of solvency or financial distress (i.e. non-bankruptcy); or, in the event of bankruptcy (which entails termination of all outstanding debt), the total face-value of all outstanding debt less any liquidation cash-flow shortfall. That is, given bankruptcy, a positive liquidation cash-flow means that all non-equity claims can be satisfied in full, but a negative liquidation cash-flow represents a claim by equity against its limited liability option. Therefore:

$$H_t=\left\{\begin{array}{lll}\left(\mathbf{e}_t^T\mathbf{y}_\eta\right)\mathbf{1} & , \ if & F_t\geq -E_{t+}/(1+\gamma)\\ \sum_{i=t/\Delta t}^{N}\left(\mathbf{e}_{i\Delta t}^T\mathbf{y}_\eta\right)\mathbf{k}_t & , \ if & L_t\geq 0 \ and \ F_t<-E_{t+}/(1+\gamma)\\ L_t+\sum_{i=t/\Delta t}^{N}\left(\mathbf{e}_{i\Delta t}^T\mathbf{y}_\eta\right)\mathbf{k}_t & , \ if & L_t<0 \ and \ F_t<-E_{t+}/(1+\gamma)\end{array}\right\}$$
$$=\left(\mathbf{e}_t^T\mathbf{y}_\eta\right)\mathbf{1}+\left\{\sum_{i=t/\Delta t}^{N}\left(\mathbf{e}_{i\Delta t}^T\mathbf{y}_\eta\right)\mathbf{k}_t-\left(\mathbf{e}_t^T\mathbf{y}_\eta\right)\mathbf{1}\right\}\mathrm{I}_{F_t<\frac{-E_{t+}}{(1+\gamma)}}+L_t\,\mathrm{I}_{q_t<\frac{g(p_t)}{p_t}}\,\mathrm{I}_{F_t<\frac{-E_{t+}}{(1+\gamma)}}. \tag{A.13}$$

Therefore:

$$\hat{\mathrm{E}}_{t-\Delta t}[H_t \mid p_t] = \left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}$$
$$+\left\{\sum_{i=t/\Delta t}^{N}\left(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta\right)\mathbf{k}_t - \left(\mathbf{e}_t^T \mathbf{y}_\eta\right)\mathbf{1}\right\}\left\{\mathrm{CDP}_{q_t<\chi_3} - \mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]} + \mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\right\}$$
$$+ p_t\left\{\mathrm{ADP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_3\right]} - \mathrm{ADP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_3,\chi_1\right]} + \mathrm{ADP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_5,\chi_1\right]}\right\} \quad \text{(A.14)}$$
$$- g(p_t)\left\{\mathrm{CDP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_3\right]} - \mathrm{CDP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_3,\chi_1\right]} + \mathrm{CDP}_{q_t<\min\left[\frac{g(p_t)}{p_t},\chi_5,\chi_1\right]}\right\}.$$

Consider that risky debt is a combination of risk-free debt plus a short position in a limited liability option that has some individualised exercise priority amongst all the various limited liability options provided by non-equity stakeholders to equity. This model is not concerned with separating out the specific limited liability exposure of debt from other non-equity stakes. Instead the value of equity's comprehensive limited liability option underwritten by all of the non-equity stakes is of interest. To obtain this value it is necessary to separate out the risk-free debt valuation from the total value of debt and the short limited liability option. Towards this purpose, define D_{t-}, D_t and D_{t+} to each be the risk-free value of outstanding debt at time t but respectively at the immediate instances: before settlement of maturing debt and before issue of any new debt; after settlement of maturing debt and before issue of any new debt; and after settlement of maturing debt and after issue of any new debt. Therefore:

$$D_{t-} = y_{0,t} + y_{\Delta t,t} + \ldots + y_{t-\Delta t,t} + \frac{y_{0,t+\Delta t} + y_{\Delta t,t+\Delta t} + \ldots + y_{t-\Delta t,t+\Delta t}}{e^{r\Delta t}}$$
$$+ \frac{y_{0,t+2\Delta t} + y_{\Delta t,t+2\Delta t} + \ldots + y_{t-\Delta t,t+2\Delta t}}{e^{2r\Delta t}} + \ldots + \frac{y_{0,N\Delta t} + y_{\Delta t,N\Delta t} + \ldots + y_{t-\Delta t,N\Delta t}}{e^{(N\Delta t-t)r}}$$
$$= \sum_{i=t/\Delta t}^{N} \frac{\left(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta\right)\mathbf{k}_t}{e^{(i-t/\Delta t)r\Delta t}},$$

$$D_t = \frac{y_{0,t+\Delta t} + y_{\Delta t,t+\Delta t} + \ldots + y_{t-\Delta t,t+\Delta t}}{e^{r\Delta t}}$$
$$+ \frac{y_{0,t+2\Delta t} + y_{\Delta t,t+2\Delta t} + \ldots + y_{t-\Delta t,t+2\Delta t}}{e^{2r\Delta t}} + \ldots + \frac{y_{0,N\Delta t} + y_{\Delta t,N\Delta t} + \ldots + y_{t-\Delta t,N\Delta t}}{e^{(N\Delta t-t)r}}$$
$$= \sum_{i=1+t/\Delta t}^{N} \frac{\left(\mathbf{e}_{i\Delta t}^T \mathbf{y}_\eta\right)\mathbf{k}_t}{e^{(i-t/\Delta t)r\Delta t}},$$

$$D_{t+} = \frac{y_{0,t+\Delta t} + y_{\Delta t,t+\Delta t} + \ldots + y_{t-\Delta t,t+\Delta t} + y_{t,t+\Delta t}}{e^{r\Delta t}}$$
$$+ \frac{y_{0,t+2\Delta t} + y_{\Delta t,t+2\Delta t} + \ldots + y_{t-\Delta t,t+2\Delta t} + y_{t,t+2\Delta t}}{e^{2r\Delta t}} + \ldots + \frac{y_{0,N\Delta t} + y_{\Delta t,N\Delta t} + \ldots + y_{t-\Delta t,N\Delta t} + y_{t,N\Delta t}}{e^{(N\Delta t - t)r}}$$
$$= \sum_{i=1+t/\Delta t}^{N} \frac{\left(\mathbf{e}_{i\Delta t}^{T}\mathbf{y}_{\eta}\right)\mathbf{k}_{t+\Delta t}}{e^{(i-t/\Delta t)r\Delta t}}.$$

It is assumed that equity must 'purchase' from non-equity a new comprehensive limited liability option each period: $O_{t-} \leq 0$ symbolises the short payoff value of the limited liability option expiring at time t; and $O_{t+} \leq 0$ symbolises the short initial value of the limited liability option newly contracted at time t for the ensuing period. Equity's periodic expense for the limited liability option is incorporated in the model as an offset against the risk-free valuation of new debt proceeds. The risk-free return on debt plus receipt for the limited liability option represents taxable income to non-equity, for which a personal tax penalty is charged. Hence the risky measure of new debt proceeds (Y_t) is:

$$Y_t = O_{t+} + \left(\frac{y_{t,t+\Delta t}}{e^{r\Delta t}} + \frac{y_{t,t+2\Delta t}}{e^{2r\Delta t}} + \ldots + \frac{y_{t,N\Delta t}}{e^{(N\Delta t - t)r}} \right) - \left[\mathbf{1}^{T} \left(\mathbf{y}_{\eta}\mathbf{e}_{t+\Delta t} \right) - Y_t \right] (\alpha - A_{\alpha})$$
$$= \frac{O_{t+} + \left(\sum_{i=1+t/\Delta t}^{N} \frac{y_{t,i\Delta t}}{e^{(i-t/\Delta t)r\Delta t}} \right) - \mathbf{1}^{T} \left(\mathbf{y}_{\eta}\mathbf{e}_{t+\Delta t} \right) (\alpha - A_{\alpha})}{1 - (\alpha - A_{\alpha})} \qquad \text{(A.15)}$$

where: $(\alpha - A_{\alpha})$ is the personal tax penalty rate acting against the corporate tax shield benefit of non-equity (i.e. α is the corporate tax rate and A_{α} is the effective rate of combined personal and corporate tax shielded by distributing non-equity income instead of equity income to the marginal stakeholder); and $[\mathbf{1}^{T}(\mathbf{y}_{\eta}\mathbf{e}_{t+\Delta t}) - Y_t]$ is the total face-value of newly issued debt minus the risky measure of new debt proceeds, which represents non-equity's taxable income from the new debt and limited liability option. It is assumed that non-equity requires up-front compensation for its personal tax penalty; this is consistent with the previous assumption that the model company can immediately tax-deduct the expense for new debt finance.

Normally the cost of the limited liability option associated with risky zero-coupon debt is deducted from the up-front financing proceeds. For this model set-up the cost of the debt-specific limited liability option is paid in variable instalments at the start of each period of the debt's life (which for the case of single-period debt is equivalent to a single up-front deduction from financing proceeds). Additionally the company must pay up-front each period for the limited liability exposure of other non-equity stakeholders. This payment will vary from period to period and can be conceptualised as a 'sign-on' expense for labour, equipment and service suppliers plus a risk-premium transaction cost for hedge contracts.[31] The cost of the comprehensive limited liability option for any period may be such that the risky new debt proceeds measure is zero or negative; this will naturally be the case if no new debt is actually being issued $(\mathbf{1}^T(\mathbf{y}_\eta \mathbf{e}_{t+\Delta t}) = 0)$. Nevertheless the effect is that the comprehensive limited liability option is incorporated into the model company as a periodic and variable tax deductible expense.

The total value attributable to debt and non-equity's expiring short limited liability option at time t immediately prior to any payment to non-equity and before any issue of new debt or contracting of a new limited liability option $(D_{t-} + O_{t-})$ is:

$$D_{t-} + O_{t-} = H_t + D_t \left\{ 1 - \mathrm{I}_{F_t < \frac{-E_{t+}}{(1+\gamma)}} \right\}.$$

Therefore:

$$\hat{\mathrm{E}}_{t-\Delta t}[D_{t-} + O_{t-} \mid p_t] = \hat{\mathrm{E}}_{t-\Delta t}[H_t \mid p_t] + D_t \left\{ 1 - \mathrm{CDP}_{q_t < \chi_3} + \mathrm{CDP}_{q_t < \min[\chi_3, \chi_1]} - \mathrm{CDP}_{q_t < \min[\chi_5, \chi_1]} \right\}.$$

(A.16)

Valuation method

Assume that at time $t = N\Delta t$ the company either ceases operations or becomes an all-equity company. Then the ongoing debt value and short comprehensive limited liability option value are zero and no new debt is issued $(\mathbf{1}^T(\mathbf{y}_\eta \mathbf{e}_{N\Delta t+\Delta t}) = D_{N\Delta t+} = O_{N\Delta t+} = Y_{N\Delta t} = 0)$. The ongoing equity value ($E_{N\Delta t+}$) is zero if the company ceases operations or otherwise can be set equal to some estimate (e.g. from a basic present value analysis). It

[31] The hedge contract providers are assumed to not pose any default risk to the company.

is then possible to solve for $\hat{\mathrm{E}}_{(N-1)\Delta t}[E_{N\Delta t-} \mid p_{N\Delta t}]$ and $\hat{\mathrm{E}}_{(N-1)\Delta t}[D_{N\Delta t-} + O_{N\Delta t-} \mid p_{N\Delta t}]$ for all possible $p_{N\Delta t}$. Solutions obtain for equity and for debt plus short comprehensive limited liability option values at each price-node by continuing backwards through the price-tree with risk-neutral valuation such that:

$$E_{(t-\Delta t)+} = e^{-r\Delta t}\,\hat{\mathrm{E}}_{t-\Delta t}[E_{t-}]\,, \tag{A.17}$$

$$D_{(t-\Delta t)+} + O_{(t-\Delta t)+} = e^{-r\Delta t}\,\hat{\mathrm{E}}_{t-\Delta t}[D_{t-} + O_{t-}]\,. \tag{A.18}$$

Break-down of valuation

The model company's total value at each price-node ($\hat{\mathrm{E}}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t]$) can be broken down into constituents attributable to: earnings before interest and tax ($EBIT_t$), which is the culminating production period's in-hand value from operations including hedging; the value of future operations (V_{t+}) as a going concern or in liquidation; and current deadweight costs (DWC_t) due to corporate taxation (TAX_t), the personal tax penalty of non-equity (PTP_t), free cash-flow misappropriation (MIS_t), financial distress (FND_t) or bankruptcy (BNK_t):

$$\hat{\mathrm{E}}_{t-\Delta t}[E_{t-} + (D_{t-} + O_{t-}) \mid p_t] = \hat{\mathrm{E}}_{t-\Delta t}[EBIT_t \mid p_t] + \hat{\mathrm{E}}_{t-\Delta t}[V_{t+} \mid p_t] - \hat{\mathrm{E}}_{t-\Delta t}\left[DWC_t \mid p_t\right].$$

From equation (A.3):

$$\hat{\mathrm{E}}_{t-\Delta t}[EBIT_t \mid p_t] = (p_t - c)\,\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t] + \left(\mathbf{e}_t^T \mathbf{x}_\eta\right)\left(\mathbf{e}_t^T \mathbf{X}_t\right)^T. \tag{A.19}$$

The company's value of future operations as a going concern is $[E_{t+} + (D_{t+} + O_{t+})]$, and in liquidation it is the market value of live hedge contracts $[\mathrm{tr}(\mathbf{x}_\eta^T \mathbf{X}_t) - (\mathbf{e}_t^T \mathbf{x}_\eta)(\mathbf{e}_t^T \mathbf{X}_t)^T]$:

$$\begin{aligned}\hat{\mathrm{E}}_{t-\Delta t}[V_{t+} \mid p_t] = {} & \left[E_{t+} + (D_{t+} + O_{t+})\right]\left\{1 - \mathrm{CDP}_{q_t<\chi_3} + \mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]} - \mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\right\} \\ & + \left[\mathrm{tr}\left(\mathbf{x}_\eta^T \mathbf{X}_t\right) - \left(\mathbf{e}_t^T \mathbf{x}_\eta\right)\left(\mathbf{e}_t^T \mathbf{X}_t\right)^T\right]\left\{\mathrm{CDP}_{q_t<\chi_3} - \mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]} + \mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\right\}.\end{aligned}$$

Note that deadweight hedge transaction costs are incorporated in the formulations of $(\mathbf{e}_t^T\mathbf{x}_\eta)(\mathbf{e}_t^T\mathbf{X}_t)^T$ and $[\mathrm{tr}(\mathbf{x}_\eta^T\mathbf{X}_t)-(\mathbf{e}_t^T\mathbf{x}_\eta)(\mathbf{e}_t^T\mathbf{X}_t)^T]$ in $\hat{\mathrm{E}}_{t-\Delta t}[EBIT_t \mid p_t]$ and $\hat{\mathrm{E}}_{t-\Delta t}[V_{t+} \mid p_t]$, separate from $\hat{\mathrm{E}}_{t-\Delta t}[DWC_t \mid p_t]$.

Incorporating equations (A.12) and (A.16), the backed-out formulation for $\hat{\mathrm{E}}_{t-\Delta t}[DWC_t \mid p_t]$ comprises: the value from $EBIT_t$; plus net cash-flow from new debt and to the new comprehensive limited liability option on a pre-personal tax penalty basis and under condition of non-bankruptcy; plus market (liquidation) value of live hedge contracts under condition of bankruptcy; minus net cash-flow to equity and debt and from the expiring comprehensive limited liability option:

$$\begin{aligned}\hat{\mathrm{E}}_{t-\Delta t}[DWC_t \mid p_t] &= (p_t-c)\,\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t]+\left(\mathbf{e}_t^T\mathbf{x}_\eta\right)\left(\mathbf{e}_t^T\mathbf{X}_t\right)^T \\ &+\left(O_{t+}+\sum_{i=(t+\Delta t)/\Delta t}^{N}\frac{y_{t,i\Delta t}}{e^{(i-t/\Delta t)r\Delta t}}\right)\left\{1-\mathrm{CDP}_{q_t<\chi_3}+\mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]}-\mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\right\} \\ &+\left[\mathrm{tr}\left(\mathbf{x}_\eta^T\mathbf{X}_t\right)-\left(\mathbf{e}_t^T\mathbf{x}_\eta\right)\left(\mathbf{e}_t^T\mathbf{X}_t\right)^T\right]\left\{\mathrm{CDP}_{q_t<\chi_3}-\mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]}+\mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\right\} \\ &-\hat{\mathrm{E}}_{t-\Delta t}[G_t \mid p_t]-\hat{\mathrm{E}}_{t-\Delta t}[GL_t \mid p_t]-\hat{\mathrm{E}}_{t-\Delta t}[H_t \mid p_t].\end{aligned} \qquad \text{(A.20)}$$

Alternatively the constituents of DWC_t can be individually formulated. TAX_t is equal to the difference between free cash-flow before tax (FBT_t) and free cash-flow (F_t) under condition of non-bankruptcy, or the tax deducted from the liquidating dividend (GL_t, which can only be non-zero under condition of bankruptcy):

$$TAX_t=\left(FBT_t-F_t\right)\left(1-\mathrm{I}_{F_t<\frac{-E_{t+}}{(1+\gamma)}}\right)+\frac{\alpha}{(1-\alpha)}GL_t\,.$$

Hence:

$$\begin{aligned}\hat{\mathrm{E}}_{t-\Delta t}[TAX_t \mid p_t] &= \alpha\left[p_t\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t]-f(p_t)-d_t(\mathbf{y}_\eta)\right]+(1-\lambda)\alpha p_t\,\mathrm{VP}_{q_t<\chi_1} \\ &+\alpha\left[f(p_t)+d_t(\mathbf{y}_\eta)\right]\left\{\mathrm{CDP}_{q_t<\chi_3}-\mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]}+\lambda\,\mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\right\} \\ &-\alpha p_t\left\{\mathrm{ADP}_{q_t<\chi_3}-\mathrm{ADP}_{q_t<\min[\chi_3,\chi_1]}+\lambda\,\mathrm{ADP}_{q_t<\min[\chi_5,\chi_1]}\right\}+\frac{\alpha}{(1-\alpha)}\hat{\mathrm{E}}_{t-\Delta t}[GL_t \mid p_t].\end{aligned} \qquad \text{(A.21)}$$

PTP_t is equal to the difference between the value of non-equity's new debt plus short limited liability option ($D_{t+} - D_t + O_{t+}$) and risky new debt proceeds (Y_t) under condition of non-bankruptcy. Hence:

$$\hat{\mathrm{E}}_{t-\Delta t}[PTP_t \mid p_t] = \frac{(\alpha - A_\alpha)}{1-(\alpha - A_\alpha)}\left[\sum_{i=1+t/\Delta t}^{N}\left(y_{t,i\Delta t} - \frac{y_{t,i\Delta t}}{e^{(i-t/\Delta t)r\Delta t}}\right) - O_{t+}\right]\left\{1 - \mathrm{CDP}_{q_t<\chi_3} + \mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]} - \mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\right\}$$

(A.22)

MIS_t is equal to the free cash-flow misappropriation rate (a) multiplied by positive free cash-flow, which can be represented by a rearrangement of equation (A.5):

$$MIS_t = aF_t\left(1 - \mathrm{I}_{F_t<0}\right) = \frac{a(1+\gamma)}{(a+\gamma)}F_t\left(1 - \mathrm{I}_{F_t<\frac{-E_{t+}}{(1+\gamma)}}\right) - \frac{a}{(a+\gamma)}G_t.$$

Hence:

$$\begin{aligned}\hat{\mathrm{E}}_{t-\Delta t}[MIS_t \mid p_t] &= \frac{a(1+\gamma)}{(a+\gamma)}\Big[(1-\alpha)\Big[p_t\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t] - f(p_t)\Big] + \alpha d_t(\mathbf{y}_\eta) - \alpha(1-\lambda)p_t\,\mathrm{VP}_{q_t<\chi_1}\Big]\\ &+ \frac{a(1+\gamma)}{(a+\gamma)}\Big[(1-\alpha)f(p_t) - \alpha d_t(\mathbf{y}_\eta)\Big]\Big\{\mathrm{CDP}_{q_t<\chi_3} - \mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]} + \mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\Big\}\\ &- \frac{a(1+\gamma)}{(a+\gamma)}(1-\alpha)p_t\Big\{\mathrm{ADP}_{q_t<\chi_3} - \mathrm{ADP}_{q_t<\min[\chi_3,\chi_1]} + \mathrm{ADP}_{q_t<\min[\chi_5,\chi_1]}\Big\}\\ &+ \frac{a(1+\gamma)}{(a+\gamma)}\alpha(1-\lambda)\Big\{\Big[f(p_t) + d_t(\mathbf{y}_\eta)\Big]\mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]} - p_t\,\mathrm{ADP}_{q_t<\min[\chi_5,\chi_1]}\Big\}\\ &- \frac{a}{(a+\gamma)}\hat{\mathrm{E}}_{t-\Delta t}[G_t \mid p_t].\end{aligned}$$

(A.23)

FND_t can be obtained by subtracting MIS_t from the difference between free cash-flow (F_t) under condition of non-bankruptcy and the dividend (G_t, which can only be non-zero under condition of non-bankruptcy):

$$FND_t = F_t\left(1 - \mathrm{I}_{F_t<\frac{-E_{t+}}{(1+\gamma)}}\right) - G_t - MIS_t = \frac{\gamma(1-a)}{(a+\gamma)}F_t\left(1 - \mathrm{I}_{F_t<\frac{-E_{t+}}{(1+\gamma)}}\right) - \frac{\gamma}{(a+\gamma)}G_t.$$

Hence:

$$\begin{aligned}\hat{\mathrm{E}}_{t-\Delta t}[FND_t \mid p_t] &= \frac{\gamma(1-a)}{(a+\gamma)}\Big[(1-\alpha)\Big[p_t\hat{\mathrm{E}}_{t-\Delta t}[q_t \mid p_t]-f(p_t)\Big]+\alpha d_t(\mathbf{y}_\eta)-\alpha(1-\lambda)p_t\,\mathrm{VP}_{q_t<\chi_1}\Big] \\ &+\frac{\gamma(1-a)}{(a+\gamma)}\Big[(1-\alpha)f(p_t)-\alpha d_t(\mathbf{y}_\eta)\Big]\Big\{\mathrm{CDP}_{q_t<\chi_3}-\mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]}+\mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\Big\} \\ &-\frac{\gamma(1-a)}{(a+\gamma)}(1-\alpha)p_t\Big\{\mathrm{ADP}_{q_t<\chi_3}-\mathrm{ADP}_{q_t<\min[\chi_3,\chi_1]}+\mathrm{ADP}_{q_t<\min[\chi_5,\chi_1]}\Big\} \\ &+\frac{\gamma(1-a)}{(a+\gamma)}\alpha(1-\lambda)\Big\{\Big[f(p_t)+d_t(\mathbf{y}_\eta)\Big]\mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}-p_t\,\mathrm{ADP}_{q_t<\min[\chi_5,\chi_1]}\Big\} \\ &-\frac{\gamma}{(a+\gamma)}\hat{\mathrm{E}}_{t-\Delta t}[G_t \mid p_t].\end{aligned}$$

(A.24)

BNK_t is equal to the bankruptcy cost rate (b) multiplied by the total face-value of maturing and outstanding debt under condition of bankruptcy (which excludes newly issued debt). Hence:

$$\hat{\mathrm{E}}_{t-\Delta t}[BNK_t \mid p_t] = b\left(\sum_{i=t/\Delta t}^{N}\left(\mathbf{e}_{i\Delta t}^T\mathbf{y}_\eta\right)\mathbf{k}_t\right)\Big\{\mathrm{CDP}_{q_t<\chi_3}-\mathrm{CDP}_{q_t<\min[\chi_3,\chi_1]}+\mathrm{CDP}_{q_t<\min[\chi_5,\chi_1]}\Big\}. \quad \text{(A.25)}$$

APPENDIX B. LEVERAGE, HEDGING AND FINANCIAL RISK MEASURES

The chosen measure of leverage (ℓ_{t+}) for the model company is the risk-free valuation of newly issued debt (D_{t+}) divided by ongoing equity value (E_{t+}):

$$\ell_{t+} = D_{t+} / E_{t+} .$$

The chosen measure of hedging is based on Tufano's (1996) delta-percentage technique. The value at time t of an individual hedge position entered at time $\tau \leq t$ and maturing at time $(\tau + \kappa\Delta t) > t$ is $x_{\tau,\tau+\kappa\Delta t} X_{t:\tau,\tau+\kappa\Delta t}$ (see equation (A.2) in Appendix A for the formulation). Assuming geometric Brownian motion for the underlying price process, the delta of the individual hedge position (i.e. the sensitivity of the value of the hedge position with respect to the underlying price) is:

$$\frac{\partial}{\partial p_t} x_{\tau,\tau+\kappa\Delta t} X_{t:\tau,\tau+\kappa\Delta t}$$

$$= -e^{-\delta(\tau+\kappa\Delta t-t)} x_{\tau,\tau+\kappa\Delta t} \left[w_{\tau,\tau+\kappa\Delta t} + (1 - w_{\tau,\tau+\kappa\Delta t}) \Phi \left\{ \frac{-\ln\left(\frac{p_t}{z_{\tau,\tau+\kappa\Delta t}}\right) - \left(r - \delta + \frac{\sigma_p^2}{2}\right)(\tau + \kappa\Delta t - t)}{\sigma_p \sqrt{\tau + \kappa\Delta t - t}} \right\} \right]$$

where: p_t is the unit price of the underlying production output being hedged; σ_p^2 is the annual variance rate for the output price; r is the annual, continuously compounding risk-free interest rate; δ is the annual, continuously compounding convenience yield of the production output; $x_{\tau,\tau+\kappa\Delta t}$ is the units of hedge quantity; $0 \leq w_{\tau,\tau+\kappa\Delta t} \leq 1$ is the ratio choice for hedge quantity committed to short forwards as opposed to put options; $z_{\tau,\tau+\kappa\Delta t}$ is the strike price of the put options; and $\Phi\{\cdot\}$ is the standard normal cumulative distribution function.

Given that a binomial price process is assumed for model implementation, the corresponding discrete delta measure for the individual hedge position is:

$$
\begin{aligned}
&\frac{\Delta}{\Delta p_t} x_{\tau,t+T} X_{t:\tau,t+T} \\
&\quad = -e^{-\delta(T-\iota)} x_{\tau,t+T} w_{\tau,t+T} \\
&\qquad + x_{\tau,t+T}(1-w_{\tau,t+T}) \frac{\sum_{j=0}^{(T/\iota-1)} \max\left[0, \frac{z_{\tau,t+T}}{p_t e^{\left(r-\delta-\sigma_p^2/2\right)T}} - e^{(2j-T/\iota+2)\sigma_p\sqrt{\iota}}\right] \binom{T/\iota-1}{j} m^j (1-m)^{(T/\iota-1-j)}}{e^{\left(\delta+\sigma_p^2/2\right)(T-\iota)} \left(e^{\sigma_p\sqrt{\iota}} - e^{-\sigma_p\sqrt{\iota}}\right)} \\
&\qquad - x_{\tau,t+T}(1-w_{\tau,t+T}) \frac{\sum_{j=0}^{(T/\iota-1)} \max\left[0, \frac{z_{\tau,t+T}}{p_t e^{\left(r-\delta-\sigma_p^2/2\right)T}} - e^{(2j-T/\iota)\sigma_p\sqrt{\iota}}\right] \binom{T/\iota-1}{j} m^j (1-m)^{(T/\iota-1-j)}}{e^{\left(\delta+\sigma_p^2/2\right)(T-\iota)} \left(e^{\sigma_p\sqrt{\iota}} - e^{-\sigma_p\sqrt{\iota}}\right)}
\end{aligned}
\tag{B.1}
$$

where: $T = (\tau + \kappa\Delta t - t)$ is the time remaining until hedge contract maturity such that $\tau, t, \tau + \kappa\Delta t \in \{0, \Delta t, 2\Delta t, ..., N\Delta t\}$, $\tau \le t < \tau + \kappa\Delta t$, and positive integer N is the number of controlled production periods; $\iota = \Delta t / n$ is the discrete time-step of the binomial process being as that a single production period has duration Δt years and n binomial time-steps; and $m = (e^{\iota\sigma_p^2/2} - e^{-\sigma_p\sqrt{\iota}})/(e^{\sigma_p\sqrt{\iota}} - e^{-\sigma_p\sqrt{\iota}})$.

The hedging measure (h_{t+}) is named the hedge-delta ratio and is defined to be the negative of, the ratio of, the delta of the total portfolio of new and live hedge positions, to the total expected quantity of future production output:

$$
h_{0+} = \frac{-\sum_{j=1}^{N} \frac{\Delta}{\Delta p_0} x_{0,j\Delta t} X_{0:0,j\Delta t}}{R}, \quad h_{t+} = \frac{-\sum_{i=0}^{t/\Delta t} \sum_{j=t/\Delta t+1}^{N} \frac{\Delta}{\Delta p_t} x_{i\Delta t, j\Delta t} X_{t:i\Delta t, j\Delta t}}{R - \sum_{j=1}^{t/\Delta t} \hat{\mathrm{E}}_{(j-1)\Delta t}[q_{j\Delta t} \mid p_{j\Delta t}]}
$$

for $t \in \{\Delta t, 2\Delta t, ..., (N-1)\Delta t\}$, and where R is the initial total expected quantity of future production output.

Five measures of financial risk are defined. The value of equity's comprehensive limited liability option relative to ongoing equity value ($-O_{t+} / E_{t+}$) is used as the first measure of the model company's financial risk. The second measure is equity's relative value-at-risk from an extreme drop in output price for a production period (υ_{t+}):

$$\upsilon_{t+} = 1 - \frac{\hat{\mathrm{E}}_t\left[E_{(t+\Delta t)-} \mid p_{t+\Delta t} = p_t e^{\left(r-\delta-\sigma_p^2/2\right)\Delta t - \sigma_p\sqrt{n\Delta t}} \right]}{E_{t+}}$$

where: $\hat{\mathrm{E}}_t[\cdot]$ is the risk-neutral expectation operator; and $E_{(t+\Delta t)-}$ is the total equity value (dividend plus ongoing equity value) at the end of the ensuing period.

As further measures of financial risk, two probability of bankruptcy ($\Pr B_{t-}$, $\hat{\Pr} B_{t+}$) calculations and an output price beta ($\hat{\beta}_{t+}$) calculation are defined. $\Pr B_{t-}$ is the probability of bankruptcy for the preceding production period due to production quantity risk:

$$\Pr B_{t-} = \hat{\mathrm{E}}_{t-\Delta t}\left[\mathrm{I}_{F_t < \frac{-E_{t+}}{(1+\gamma)}} \mid p_t \right] \tag{B.2}$$

where: the indicator function, $\mathrm{I}_{logical\ statement}$, equals one if the *logical statement* is true and zero otherwise; and $F_t < -E_{t+}/(1+\gamma)$ is the free cash-flow condition for instigation of bankruptcy (see equation (A.11) in Appendix A).

$\hat{\Pr} B_{t+}$ is the risk-neutral probability of bankruptcy for the ensuing production period:

$$\hat{\Pr} B_{t+} = \hat{\mathrm{E}}_t\left[\mathrm{I}_{F_{t+\Delta t} < \frac{-E_{(t+\Delta t)+}}{(1+\gamma)}} \right]. \tag{B.3}$$

$\hat{\beta}_{t+}$ is the risk-neutral sensitivity of equity's rate of return to the underlying output price rate of return for the ensuing production period (i.e. equity's output price beta):

$$\hat{\beta}_{t+} = \frac{\hat{\mathrm{COV}}_t\left[\left(\frac{p_{t+\Delta t}}{p_t e^{-\delta\Delta t}} - 1 \right), \left(\frac{\hat{\mathrm{E}}_t[E_{(t+\Delta t)-} \mid p_{t+\Delta t}]}{E_{t+}} - 1 \right) \right]}{\hat{\mathrm{VAR}}_t\left[\left(\frac{p_{t+\Delta t}}{p_t e^{-\delta\Delta t}} - 1 \right) \right]}$$
$$= \frac{\hat{\mathrm{E}}_t\left[\left(\frac{p_{t+\Delta t}}{p_t e^{-\delta\Delta t}} - 1 \right)\left(\frac{\hat{\mathrm{E}}_t[E_{(t+\Delta t)-} \mid p_{t+\Delta t}]}{E_{t+}} - 1 \right) \right] - \hat{\mathrm{E}}_t\left[\frac{p_{t+\Delta t}}{p_t e^{-\delta\Delta t}} - 1 \right]\hat{\mathrm{E}}_t\left[\frac{\hat{\mathrm{E}}_t[E_{(t+\Delta t)-} \mid p_{t+\Delta t}]}{E_{t+}} - 1 \right]}{\hat{\mathrm{E}}_t\left[\left(\frac{p_{t+\Delta t}}{p_t e^{-\delta\Delta t}} - 1 \right)^2 \right] - \hat{\mathrm{E}}_t\left[\frac{p_{t+\Delta t}}{p_t e^{-\delta\Delta t}} - 1 \right]^2}. \tag{B.4}$$

APPENDIX C. REPRESENTATIVE MINE PRODUCTION DATA

For the purpose of obtaining a representative value for production quantity uncertainty (σ_q) for a resource producer, three gold mining companies headquartered in Western Australia were chosen for investigation: Resolute Mining Limited, Equigold NL and Sons of Gwalia Ltd.[32] From these companies' annual reports, gold production data for various years from 2000 to 2005 was obtained for each company for eight specific mines/mining regions variously located in Africa (Resolute's Golden Pride and Obotan mines), Queensland (Resolute's Ravenswood mine and Equigold's Mount Rawdon mine) and Western Australia (Equigold's Kirkalocka mine, and Sons of Gwalia's Marvel Loch mine and Leonora and Southern Cross mining regions).

Table C1 indicates an average/median annual mine production of about 140,000 ounces of gold. The standard deviation of the relative difference (i.e. log difference) between forecast and realised annual production quantity ($\ln(q_T / q_0)$) is used to provide a representative measure of production quantity uncertainty (σ_q) for a resource producer. For the entire sample of mines/mining regions the annual production quantity uncertainty is 17%, but it is notable that most of this uncertainty is attributable to Resolute's operations. When the highest and lowest production quantity deviations from forecast are excluded from the sample, annual production quantity uncertainty is calculated to be 11%. Also, while the entire sample indicates very little forecast bias for production quantity, this is not uniformly the case for the operations of each separate company.

[32] Pertaining to this study, it is interesting to note that in Sons of Gwalia's 2002/03 annual report it was stated that "hedging has been, and remains a cornerstone of profitability for the Company". Nevertheless, Sons of Gwalia went into administration in August 2004 largely as a result of a hedging program that was subsequently reported to have been severely mismanaged for several years (Australian Financial Review, 18 August 2005).

Table C1 – Representative mine production data

Gold mine production data for various years from 2000 to 2005 for eight gold mines/mining regions operated by three Western Australian companies.

Company	Mine/mining region	Year ended	Start-of-year production quantity forecast, q_0 (ounces)	Realised production quantity, q_T (ounces)	$\ln(q_T / q_0)$
Resolute Mining Limited	Golden Pride	30-Jun-01	180,000	216,567	0.18
	Golden Pride	30-Jun-02	180,000	148,702	-0.19
	Golden Pride	30-Jun-03	170,000	150,997	-0.12
	Golden Pride	30-Jun-04	155,000	169,151	0.09
	Golden Pride	30-Jun-05	140,000	149,866	0.07
	Obotan	30-Jun-01	120,000	127,670	0.06
	Obotan	30-Jun-02	80,000	97,761	0.20
	Obotan*	30-Jun-03	30,000	49,149	0.49
	Ravenswood	30-Jun-05	200,000	165,522	-0.19
Equigold NL	Mt Rawdon	30-Jun-02	73,000	72,945	0.00
	Mt Rawdon	30-Jun-03	79,000	81,381	0.03
	Mt Rawdon	30-Jun-04	97,500	89,102	-0.09
	Mt Rawdon	30-Jun-05	95,000	94,394	-0.01
	Kirkalocka	30-Jun-04	67,500	67,454	0.00
	Kirkalocka	30-Jun-05	66,000	59,367	-0.11
Sons of Gwalia Ltd	Leonora region	30-Jun-00	175,000	132,046	-0.28
	Leonora region	30-Jun-01	190,000	195,875	0.03
	Leonora region	30-Jun-03	230,000	237,036	0.03
	Marvel Loch	30-Jun-00	142,500	142,528	0.00
	Southern Cross region	30-Jun-03	225,000	217,019	-0.04
		Mean	134,775	133,227	0.01
		Median	141,250	137,287	0.00
		Standard deviation	58,629	56,119	0.17

* Mining operations at Obotan finished in December 2002 due to the completion of processing of all known economic mineable reserves.

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