Superalgebras and brane actions

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Abstract

The Noether charge algebra of brane actions is typically modified by a topological anomalous term. The underlying cohomology of this anomalous term is investigated, and it is shown that the anomalous term possesses a gauge freedom. The result is that the anomalous term generates a parameterized family of topological charge algebras. When fermionic charges are taken to be nonvanishing, the known algebras underlying extended superspace formulations of the action appear in these families. This phenomena is investigated for minimal p-branes, Dp-branes and (p, q)-strings.

The algebras resulting from the D-brane actions are shown to allow the construction of extended superspace actions without worldvolume gauge fields. It is shown that the actions are κ-symmetric, and that the symmetry is generated by a right action. The global and local symmetry transformations of the Born-Infeld gauge field are thus shown to be described geometrically by left/right actions of the underlying extended supertranslation group.

An equivalence class construction is proposed for the description of compact fermionic dimensions. In this construction, open strings in extended superspace translate to closed strings in compact superspace, and fermionic topological charges may be realized by closed strings. The differential underlying the descent construction for Noether charge algebras is shown to be naturally described as a dual of the de Rham differential. The ghost fields used in the construction are shown to be described geometrically as a vielbein with respect to this differential.

Contents

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Chapter 1

Introduction

1.1 Background

One obstacle preventing the construction of a "theory of everything" is the problem of how to unify two essentially separate theories. General relativity is a classical theory that describes gravity and spacetime on macroscopic length scales. It is by now extremely well tested. On the other hand, the standard model of particle physics is a quantum field theory describing the properties of fundamental particles and their interactions. To date, it has also yielded perfect agreement with the observed world [1]. We may think of general relativity as a theory "of spacetime," and the standard model as a theory "on spacetime." Because of this difference, the two theories have proven difficult to unify into a single theory that explains all interactions on all length scales.

The basic assumption of string theory is that the fundamental objects of our universe are not necessarily point-like. For example, some fundamental objects may be described as strings. Whilst these strings are too small to be observed directly, their differing geometries and quantized resonances are indirectly observed as different fundamental particles (just as the different resonances of a guitar string correspond to different musical notes). String theories are quantum field theories in which the coordinates of the string in spacetime are classical fields. In many string theories, appearing amongst the quantized oscillations of the string is a particle which can be interpreted as the graviton. Although string theory is still evolving, it is widely considered to be the most promising candidate for the unification of gravity and particle physics.

Prior to the twentieth century, symmetries played only a minor role in physics. Post relativity, this changed dramatically. Symmetry groups began to be used as fundamental building blocks for physical theories. Such thinking was indeed crucial to the development of the standard model itself. As the quest for a unified theory began, "no-go" theorems emerged, showing that a nontrivial unification of gravity and quantum field theory through usual group theoretic methods is impossible. However, it also became evident that the fermionic degrees of freedom in field theory should be described classically by anticommuting variables [2, 3, 4]. This lead to the development of superalgebras, which evade the no-go theorems. The associated symmetry principle is supersymmetry [5, 6, 7, 8]. This symmetry mixes bosonic degrees of freedom with fermionic ones according to the supersymmetry algebra. Superstring theory is the generalization of bosonic string theory which includes the fermionic degrees of freedom.

String theory was initially proposed as a description of the strong interaction [9]. With the development of quantum chromodynamics, the theory subsided, but then reemerged after it was realized that string theories naturally contain gravitons in their spectra [10, 11, 12]. Realistic field theories can indeed be generated from superstring theory. However, the absence of Weyl anomalies requires that the strings propagate in spacetimes of dimension greater than four [13]. The prediction of superstring theory is ten dimensional spacetime. Since we certainly do not observe such extra dimensions, the assumption is that they are "compactified" into an extra manifold which is too small to observe directly. Although this complicates matters, it also provides an avenue with which string theory can connect with phenomenology. By using the *geometry* of this extra manifold, it is possible to reproduce many aspects of the standard model from string theory [14, 15].

Superstring theory can be formulated in two different ways. The original formulation is that of Neveu, Schwarz and Ramond, in which there is manifest worldvolume supersymmetry [16, 17, 14, 18, 19, 20]. However, spacetime supersymmetry is not manifest in this formulation; it emerges via the GSO projection upon quantization. In contrast, in the Green-Schwarz formalism [21], spacetime supersymmetry is manifest since the worldvolume is embedded in a superspace. This formalism naturally incorporates superspace geometry, and is the focus of this thesis.

It has emerged that superstring theory admits degrees of freedom with more than one spatial dimension. These extended objects, known as "branes," are manifested in several ways. Firstly, one can investigate solutions of the low energy effective action for the massless modes of string field theory. In this case, the branes appear as solitonic solutions [22]. At a more fundamental level, branes are deeply related to the nonperturbative structure of superstring theory and the intricate web of dualities that relate different superstring theories [23]. The work of this thesis focuses upon this latter description. In this case, the term "p-brane" refers to an extended object with p spatial dimensions embedded in the background superspace. Here and henceforth, we use the term "p-brane" to refer to the minimal case where the only degrees of freedom present are those of the embedding. A more generalized type is known as a Dp-brane (or D-brane), which is defined as a surface upon which the ends of a string may terminate [24] (the "D" in this case refers to the Dirichlet boundary conditions for the string terminations).

The action for a p-brane [25, 26] possesses a number of interesting geometrical properties. It consists of a kinetic term, which is supersymmetric, and a Wess-Zumino term, which is supersymmetric only up to a total derivative. Wess-Zumino terms can be classified in terms of Chevalley-Eilenberg cohomology [27]. The "quasi-invariance" of the Wess-Zumino term is a result of the fact that Wess-Zumino forms are always nontrivial cocycles. This has an interesting effect: in the presence of nontrivial superspace topology, it can cause the superalgebra of Noether charges to be modified by a topological "anomalous term" [28]. Using a construction involving ghost fields, this anomalous term is found to be derived from the WZ term through cohomological descent equations [29]. The fermionic superspace directions are conventionally assumed to be topologically trivial [30], which means that they cannot support topological charges. The anomalous term thus results in bosonic, "central" extensions[1] of standard superspace [28, 31].

Both general relativity and the standard model are gauge theories. That is, they contain more degrees of freedom than are physically observable. These extra degrees of freedom are nullified by the presence of the same number of gauge symmetries, which may be used to eliminate them. This apparently complicated situation yields greater physical insight than the alternative in which only physical degrees of freedom are present. For example, the diffeomorphism invariance of relativity reflects the fact that the choice of coordinate system used cannot have any physical consequence. With the introduction of extra degrees of freedom, the symmetries of a theory, both global and local, are typically enlarged. For example, in bosonic string theory, spacetime is described by the Poincaré group. In superstring theory, this is extended to the super-Poincaré group. The resulting additional local symmetry is κ-symmetry, and the additional global symmetry is supersymmetry. However, there are motivations to continue the extension process further.

Of particular relevance to this thesis is the fact that certain extended superspaces allow the construction of manifestly supersymmetric brane actions [32, 33, 34]. Due to the manifest supersymmetry, in this case the

[1] "Central" is a commonly occurring term in the literature which means (anti)commutation with supertranslation generators but not with Lorentz ones.

resulting Noether charge algebra has no anomalous term; it simply reproduces the underlying extended superalgebra. An interesting application of a similar principle occurs in the case of D-branes [24]. D-brane actions possess worldvolume gauge fields with highly nontrivial supersymmetry and κ-symmetry transformation properties [35, 36, 37, 38]. However, there exists an extended superspace formulation in which the worldvolume gauge fields can be replaced by ordinary fields on the background superspace [39, 34]. From this formulation one gains a geometric understanding of the supersymmetry transformations of the worldvolume gauge fields of the standard actions.

Everything described thus far has assumed vanishing of the fermionic topological terms. The physical realization of topological charges requires either winding behavior, or open strings with different values for coordinates at the endpoints. Calculations involving bosonic coordinates often do not find fermionic analogs, and vice versa. In part, this is due to the difficulty of using anything other than the de Witt ("all or nothing") topology [30] for fermionic variables. One alternative is to consider strings bridging a brane-antibrane system, with antiperiodic boundary conditions for fermionic coordinates [40, 41]. In certain cases, these fermionic topological terms are required. For example, in a pp-wave background, fermionic brane charges are required to ensure quantum consistency with Jacobi identities [42]. This background also admits zero modes in which these charges are nonvanishing.

Superbrane actions possess a local symmetry known as κ-symmetry [43], which can be used to gauge away half the fermionic coordinate fields. For the superstring in ten dimensions, this yields eight bosonic and eight fermionic degrees of freedom. This is consistent with the basic idea of supersymmetry, which is that there should be a fermionic partner for each bosonic degree of freedom. The κ-transformations, although quite complicated, have some interesting geometrical properties. Firstly, they form an on-shell algebra with the other gauge symmetries of the action (these being diffeomorphisms, and additionally in some formulations, Weyl transformations). Furthermore, κ-transformations are generated by an appropriately chosen right action of the underlying supertranslation group [44]. In the case of the extended superspace formulation of p-branes, the right action yields manifest κ-symmetry [45]. For certain geometries of a brane, the anomalous term of the Noether charge algebra becomes a projector of half rank, just like the projector which appears in κ-transformations [46]. The anomalous term therefore reflects the fact that only half the fermionic degrees of freedom are physical, consistent with supersymmetry. When the unphysical fields are gauged away, half the supersymmetries become nonlinearly realized. This is commonly known as PBRS, or partial breaking of rigid supersymmetry [47].

1.2 Thesis

The research to date poses some interesting questions regarding branes in extended superspaces. Firstly, although extended algebras exist which allow the construction of manifestly supersymmetric actions, there is no preferred choice of algebra. For example, in the case of the superstring, one may use the Green algebra [48] as in [32, 33], or an extended Green algebra [34]. We would thus like to understand the origin and relationship of these algebras. Secondly, it is clear that the trivial fermionic topology, used as a simplifying condition in anomalous term research, ends up revealing only half the picture. One of the main, new results of this thesis is to show that once fermionic topological terms are retained, the known algebras underlying extended superspace formulations of a brane action are generated as topological charge algebras of the standard superspace action. Furthermore, in the case where several algebras are admissible, we show how each one is selected by means of a gauge freedom in the anomalous term.

In the second chapter, the general construction is developed for p-branes. We show how the underlying cohomology leads to a gauge freedom in the anomalous term. By retaining the fermionic topological terms, the modified Noether charge algebra is shown to close into a parameterized family of superalgebras with extra fermionic generators. We find that the standard superstring action generates a one parameter family of topological charge algebras which contains the known extended superalgebras underlying the extended superspace actions. The content of this chapter has been published as [49].

The third chapter investigates the remaining p-branes (those with $p \geq 2$). As in the case of the superstring, superalgebras underlying the extended superspace formulation for p-branes are known [33, 34]. We construct a representation of the ideals of these algebras in terms of superspace forms, and find that the solution is unique. Using these forms it is shown that a representative of the anomalous term of the Noether charge algebra of the standard superspace action can be constructed. Thus, the known algebras of the extended superspace formulation are found to be generated by the standard action for all p-branes. It is demonstrated that the forms also represent Noether charges of the extended superspace formulation. The content of this chapter has been published as [50].

In the fourth chapter, the construction is generalized to the case of D-branes. By retaining fermionic topological charges, and using the gauge freedom of the anomalous term, we derive parameterized families of superalgebras associated with type IIA and type IIB D-branes. These families contain the known algebras used in extended superspace formulations of D-brane actions. The content of this chapter has been published as [51].

The fifth chapter is devoted to the extended superspace formulation of D-branes, and in particular, finding a geometrical description of the κ-symmetry. Using an established method [39, 34], the algebras obtained in the third chapter are used to construct D-brane actions without worldvolume gauge fields. The κ-symmetry of the actions is established as an explicitly determined right action of the underlying extended supergroup. Thus, a geometric understanding of the symmetries of the D-brane action is established in which the global and local symmetries are left and right group actions respectively.

In the sixth chapter, an equivalence class construction is proposed for the description of compact, fermionic dimensions. In this description, open strings in extended superspace translate to closed strings in compact superspace. Fermionic topological charges can then be realized by closed strings. The resulting topological charge algebra associated with the Green-Schwarz superstring is shown to be well defined. The geometry of the double complex underlying the topological charge algebras is further investigated. We find that the "ghost differential" used in the construction can be described as a simple "dual" of the de Rham differential. The ghost fields are shown to have a geometric origin in terms of the vielbein of this differential. We then investigate some implications of Čech cohomology, and a resulting "triple complex" is used to study the geometry of the Wess-Zumino form.

Chapter 2

Superalgebras from p-brane actions

2.1 Introduction

The action for a p-brane consists of the kinetic term and the WZ (Wess-Zumino) term [52, 25, 26]. The WZ term ensures that a local "κ symmetry" is present which means that only half the fermionic degrees of freedom are physical [53, 54, 43]. The Lagrangian is not manifestly invariant under the global action of the supertranslation group on the superspace in which the p-brane is embedded (it is not "left invariant") due to quasi-invariance (invariance up to a total derivative) of the WZ term. The WZ term is the pullback to the worldvolume of a $(p+1)$-form B which is a potential for a field strength H. Although H is left invariant, in standard superspace it is impossible to find a left invariant potential B. In terms of CE (Chevalley-Eilenberg) cohomology [55] this means that H is a nontrivial cocycle. In fact, H is characterized as the unique nontrivial CE $(p+2)$-cocycle of dimension $p+1$ [27]. There are two avenues of research that have resulted from this fact.

The first area of research concerns topological charge algebras. The Noether charges associated with left invariance of an action are phase space generators of the left group action ("left generators"). For manifestly left invariant actions, the algebra of Noether charges is the same as the underlying algebra of symmetries. This is the "minimal algebra" of Noether charges. However, Lagrangians are often quasi-invariant under the action of symmetry transformations. In this case the Noether charges need to be modified in order to ensure their conservation [56]. For p-brane actions, quasi-invariance of the WZ term under the action of the supertranslation group leads to an algebra of conserved charges which is an extension of the supertranslation algebra by a topological "anomalous term" [28]. Superspace is conventionally formulated using the trivial de Witt topology for the fermionic directions [30]. In this case, only the bosonic terms survive and the anomalous term takes the form of a projector which can be related to PBRS (partial breaking of rigid supersymmetry) [57, 46]. The algebra of constraints for the action is also modified in the presence of the WZ term [29]. The constraints, which are associated with local symmetries of the action, can be identified with generators of the right action of the supertranslation group ("right generators"). This leads to a modified algebra of right generators. The modified algebras of Noether charges and constraints can also be related to a construction involving ghost fields [29]. A BRST style "ghost differential" s acting on an infinite dimensional "loop superspace" is introduced. The anomalous term is then the result of solving cohomological descent equations. The construction can also be formulated in terms of the underlying superspace forms [58].

In a second line of research, fermionic extensions of the standard supertranslation algebra have been discovered [48], some of which allow manifestly left invariant WZ terms to be constructed for the p-brane action [32, 33, 34]. In such extensions (which are in general non-central) the fermionic generators can appear like fermionic analogs of the bosonic topological charges [59, 60]. The resulting manifestly invariant Lagrangians differ from the standard one only by a total derivative (and the extra superspace coordinates appear only in this derivative). In this case the Noether charges are not modified and they satisfy the (sign reflected) underlying extended supertranslation algebra. The Noether charges associated with extra coordinates are also topological in this formulation, and there exists a correspondence with the bosonic topological term of the standard formulation of the action [34].

The construction of topological anomalous terms has always allowed for nontrivial topology of the bosonic coordinates (otherwise all charges would vanish). However, the terms that would result from nontrivial fermionic topology have not been calculated. There are hints that incorporating fermionic charges in brane theory may yield interesting results. For example, the action of supersymmetries on bosonic charges produces fermionic charges [40]. The type of brane and the background superspace are important considerations in the question of realizing the fermionic charges. For example, when quantizing in the standard flat background superspace one may choose a trivial representation for the fermionic charges. If the string does not end on a brane, such charges could only arise via winding behavior in fermionic superspace directions. This cannot occur under the de Witt topology. In certain superspaces, nonvanishing fermionic charges are actually *required* for quantum consistency with Jacobi identities [42]. One possibility for realizing the fermionic charges is a string with antiperiodic boundary conditions, with the ends terminating on D-branes [41].

In this chapter we investigate p-brane superalgebras by further developing the double complex cohomology underlying the anomalous term of the Noether charge algebra. A number of new results follow. The differentials involved in the descent sequence are shown to be equivalent, which implies invertibility of the sequence. A gauge freedom is identified in the anomalous term, which makes the term well defined not as a form but as an entire *cohomology class*. In the standard, flat superspace background, this class is shown to be unique and nontrivial. It is demonstrated that it may be constructed on the basis of the same dimensionality and Lorentz invariance requirements used to construct H in [27]. The most important new result is that different representatives of the class result in the generation of a parameterized family of topological charge algebras, all of which are extensions of the super-Poincaré algebra by an ideal.

The construction is applied to the GS (Green-Schwarz) superstring. The topological charges are identified as extra generators of the Noether charge algebra. The resulting topological charge algebra is shown to be a one parameter family of extended superalgebras. When fermionic charges are retained, this family contains three extended algebras of interest. The first is an algebra developed by Green, which has a fermionic "central" extension [48]. The second is an algebra which extends the Green algebra by a noncentral bosonic generator. Both of these algebras have been used to construct string actions with manifest left invariance [32, 33, 34]. The third algebra, which is of the type considered in [40, 42], contains a charge which results from the action of supersymmetry on the bosonic charge. We show that the entire family of algebras derived in this way allows manifestly invariant actions to be constructed. It thus emerges naturally that if fermionic charges are retained, *all* the known extended algebras of the superstring appear in the family of topological charge algebras associated with the standard action. Furthermore, since this family of algebras cannot be obtained by simply rescaling known algebras, new superalgebras result from the process.

2.2 Preliminaries

2.2.1 p-branes

We start with a review of the required supergroup equations. Useful references on this material include [61, 48, 27, 29, 33, 34], with more comprehensive treatments in [30, 8]. The superalgebra of the supertranslation group is[1]

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \Gamma^a{}_{\alpha\beta}\mathcal{P}_a, \tag{2.1}$$

where Γ_a satisfy the Clifford algebra:

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}. \tag{2.2}$$

The corresponding group manifold can be parameterized

$$g(Z) = e^{x^a\mathcal{P}_a}e^{\theta^\alpha\mathcal{Q}_\alpha}, \tag{2.3}$$

where Z is the combined notation for coordinates

$$Z^A = \{x^a, \theta^\alpha\}.$$

Bosonic indices $(a, b, ...)$ take values in the range 0 to d, where d is the dimension of the background superspace. Fermionic coordinates θ^α are anti-commuting. The group (2.3) can be constructed as the coset space consisting of the super-Poincaré group modulo the Lorentz subgroup, however for our purposes this is an unnecessary complication. It is valid to assume that expressions are Lorentz invariant if upper indices are contracted with lower ones.

The left vielbein is defined by

$$\begin{aligned} L(Z) &= g^{-1}(Z)dg(Z) \\ &= dZ^M L_M{}^A(Z)T_A, \end{aligned} \tag{2.4}$$

where T_A represents the full set of superalgebra generators. The right vielbein is defined similarly:

$$\begin{aligned} R(Z) &= dg(Z)g^{-1}(Z) \\ &= dZ^M R_M{}^A(Z)T_A. \end{aligned} \tag{2.5}$$

The left group action is defined by

$$g(Z') = g(\epsilon)g(Z), \tag{2.6}$$

[1] The charge conjugation matrix will not be explicitly shown. It will only be used to raise/lower indices on gamma matrices, which have the standard position $\Gamma^\alpha{}_\beta$. $\Gamma_{\alpha\beta}$ is assumed to be symmetric. Majorana spinors are assumed throughout (thus, for example, $\overline{\theta}_\alpha = \theta^\beta C_{\beta\alpha}$). The right acting convention for the de Rham differential is used, and wedge product multiplication of forms is understood.

where ϵ^A is an infinitesimal constant. The corresponding superspace transformation is generated by the operators

$$Q_A = R_A{}^M \partial_M, \tag{2.7}$$

where $R_A{}^M$ are the inverse right vielbein components, defined by

$$R_A{}^M R_M{}^B = \delta_A{}^B. \tag{2.8}$$

Explicitly this yields

$$\begin{array}{ll} Q_\alpha x^m = -\frac{1}{2}(\Gamma^m\theta)_\alpha, & Q_\alpha \theta^\mu = \delta_\alpha{}^\mu, \\ Q_a x^m = \delta_a{}^m, & Q_a \theta^\mu = 0. \end{array} \tag{2.9}$$

Q_A are the generators of the left group action, and will be referred to as the "left generators." *The action of* Q_A *upon superspace forms is given by the Lie derivative with respect to the vector field* (2.7).

Forms that are invariant under the global left group action will be called "left invariant." The vielbein components L^A are left invariant by construction. Their explicit form is

$$\begin{aligned} L^a &= dx^a - \frac{1}{2} d\bar{\theta}\Gamma^a\theta \\ L^\alpha &= d\theta^\alpha. \end{aligned} \tag{2.10}$$

The right group action is defined by

$$g(Z') = g(Z)g(\epsilon). \tag{2.11}$$

The corresponding superspace transformation is generated by the operators

$$D_A = L_A{}^M \partial_M, \tag{2.12}$$

where $L_A{}^M$ are the inverse left vielbein components, defined by

$$L_A{}^M L_M{}^B = \delta_A{}^B. \tag{2.13}$$

D_A are generators of the right supertranslation group action, and will be referred to as the "right generators." They are also commonly known as "supercovariant derivatives" since they commute with the left generators as a result of the associativity of group multiplication. Unlike the Q_A they will not generate global symmetries of the brane action (however, they are related to the local κ-symmetry). The left and right vielbein and inverse vielbein components have been placed in appendix 7.1 for reference.

The NG (Nambu-Goto) action for a $p+1$ dimensional manifold embedded in the background superspace is

$$S = -\int d^{p+1}\sigma\sqrt{-g}. \tag{2.14}$$

The integral is over the $p+1$ dimensional "worldvolume," which has coordinates σ^i and is embedded in superspace. The worldvolume metric g_{ij} is defined using the pullback of the left vielbein components:

$$\begin{aligned} L_i{}^A &= \partial_i Z^M L_M{}^A \\ g_{ij} &= L_i{}^a L_j{}^b \eta_{ab}, \end{aligned} \tag{2.15}$$

and g denotes $\det g_{ij}$. A p-brane is the κ-symmetric generalization of the NG action. For simplicity we will use only flat superspaces in this work (i.e. the Lorentz connection 1-form is constrained to vanish). In this case, the p-brane action is [25, 26, 62]

$$S = -\int d^{p+1}\sigma\sqrt{-g} + \int B. \tag{2.16}$$

The first term is the "kinetic" term. The second term is the WZ term, which is the integral over the worldvolume of the pullback of a superspace form B. B is defined by the property [25, 26]

$$\begin{aligned} dB &= H \\ &\propto d\theta^\alpha d\theta^\beta L^{a_1}\ldots L^{a_p}(\Gamma_{a_1\ldots a_p})_{\alpha\beta}, \end{aligned} \tag{2.17}$$

where $\Gamma_{a_1\ldots a_p}$ denotes a product of the indicated gamma matrices. The proportionality constant in (2.17) depends on p and is determined by requiring κ symmetry of the action. There are certain identities required to ensure the consistency of this definition. Firstly, closure of H requires a Fierz identity [25, 26, 62]:

$$\Gamma^{[a_1\ldots a_p]}{}_{(\alpha\beta}\Gamma_{a_p\delta\epsilon)} = 0. \tag{2.18}$$

This condition on the gamma matrices can only be satisfied for certain combinations of p (spatial dimension of the brane) and d (superspace dimension) [62]. The allowed values of (p, d) (called the "minimal branescan") are such that

$$(\Gamma_{[a_1\ldots a_p]})_{\alpha\beta} = (\Gamma_{[a_1\ldots a_p]})_{\beta\alpha}. \tag{2.19}$$

This ensures that H can be nonzero. It turns out that H is the unique, closed, left invariant, superspace $(p+2)$-form of dimension $p+1$ [27]. In the following work, where p and d are left unspecified, it is understood that the analysis applies to any (p,d) pair belonging to the branescan. We may assume (2.18) and (2.19) are satisfied in this case.

The following notation for tensor components is used throughout. Indices M,N,L,P are used for the coordinate basis, while A,B,C,D are used to indicate components with respect to the left invariant basis (2.10). Indices i,j,k indicate worldvolume components. For example, for a superspace p-form T:

$$\begin{aligned} T &= dZ^{M_p}\dots dZ^{M_1}T_{M_1\dots M_p}\frac{1}{p!} \\ &= L^{A_p}\dots L^{A_1}T_{A_1\dots A_p}\frac{1}{p!}, \end{aligned} \tag{2.20}$$

and

$$T_{i_1,\dots i_p} \equiv \partial_{i_p}Z^{M_p}\dots\partial_{i_1}Z^{M_1}T_{M_1\dots M_p}. \tag{2.21}$$

2.2.2 Green algebra

The p-brane action (2.16) can also be used in extended superspace backgrounds [32]. In the general construction that follows we will not specify the background in order to allow for this possibility. In section 2.5 we will consider the GS superstring in both standard and extended superspaces. There are two known extended superspaces which allow the construction of manifestly left invariant superstring WZ terms. The first is described by a superalgebra that was introduced by Green [48]. It has a fermionic generator Σ^α that defines a central extension of the supertranslation group[2]:

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \Gamma^a{}_{\alpha\beta}P_a \\ [Q_\beta, P_a] &= \Gamma_{a\beta\gamma}\Sigma^\gamma. \end{aligned} \tag{2.22}$$

The corresponding group manifold can be parameterized[3]:

$$g(Z) = e^{x^aP_a}e^{\theta^\alpha Q_\alpha}e^{\phi_\beta\Sigma^\beta}, \tag{2.23}$$

where

$$Z^A = \{x^a, \theta^\alpha, \phi_\alpha\}.$$

Standard superspace is obtained by omitting the extra generator Σ^α (and its associated coordinate ϕ_α). The resulting left vielbein components are

$$L^a = dx^a - \frac{1}{2}d\bar{\theta}\Gamma^a\theta \tag{2.24}$$

[2] Of course, when Lorentz generators are included, Σ^α is no longer central.

[3] Parameterizations are not unique. In particular we note the Green algebra can alternatively be parameterized to yield a linear realization of the left group action [63].

$$
\begin{aligned}
L^\alpha &= d\theta^\alpha \\
L_\alpha &= d\phi_\alpha - dx^b(\Gamma_b\theta)_\alpha + \frac{1}{6}d\bar{\theta}\Gamma^b\theta(\Gamma_b\theta)_\alpha.
\end{aligned}
$$

The left and right vielbein and inverse vielbein components for this parametrization of the Green algebra may be found in appendix 7.2.

2.2.3 Extended Green algebra

Addition to the Green algebra of a noncentral bosonic generator Σ^a results in an extended Green algebra [33, 34]:

$$
\begin{aligned}
\{Q_\alpha, Q_\beta\} &= \Gamma^a{}_{\alpha\beta}P_a + \Gamma_{a\alpha\beta}\Sigma^a \qquad (2.25)\\
[Q_\beta, P_a] &= \Gamma_{a\beta\gamma}\Sigma^\gamma \\
[Q_\beta, \Sigma^a] &= \Gamma^a{}_{\beta\gamma}\Sigma^\gamma.
\end{aligned}
$$

The Green algebra results from the reduction

$$
\begin{aligned}
P'{}_a &= P_a + \eta_{ab}\Sigma^b \qquad (2.26)\\
\Sigma'^\alpha &= 2\Sigma^\alpha,
\end{aligned}
$$

where η_{ab} is the Minkowski metric. The extended Green algebra group manifold can be parameterized

$$
g(Z) = e^{x^a P_a} e^{y_b \Sigma^b} e^{\theta^\alpha Q_\alpha} e^{\phi_\beta \Sigma^\beta}, \qquad (2.27)
$$

with coordinates[4]:

$$
Z^A = (x^a, \theta^\alpha, y_a, \phi_\alpha).
$$

The left vielbein components are found to be

$$
\begin{aligned}
L^a &= dx^a - \frac{1}{2}d\bar{\theta}\Gamma^a\theta \qquad (2.28)\\
L^\alpha &= d\theta^\alpha \\
L_a &= dy_a - \frac{1}{2}d\bar{\theta}\Gamma_a\theta \\
L_\alpha &= d\phi_\alpha - dx^b(\Gamma_b\theta)_\alpha - dy_b(\Gamma^b\theta)_\alpha + \frac{1}{3}d\bar{\theta}\Gamma^b\theta(\Gamma_b\theta)_\alpha.
\end{aligned}
$$

The left/right vielbein and inverse vielbein components for this parametrization of the extended Green algebra may be found in appendix 7.3.

[4] Coordinate indices will not be raised/lowered in this chapter. In the notation being used $\{Z^a, Z^\alpha, Z_a, Z_\alpha\}$ are all independent coordinates.

2.3 Double complex for the p-brane

2.3.1 Cocycles from WZ terms

The exterior derivative d together with the space of differential forms constitutes the de Rham complex. The operator d is nilpotent (i.e. $d^2 = 0$) and can therefore be used to define cohomology classes. The n-th de Rham cohomology is the set of equivalence classes:

$$H_d^n = Z^n / B^n \tag{2.29}$$

where Z^n are the closed n-forms (i.e. those in the kernel of d) and B^n are the exact n-forms (those in the image of d). The de Rham complex can be extended into a double complex by the addition of a second nilpotent operator that commutes with d (see [64] for a comprehensive treatment). The operator used in this chapter is a "ghost differential" s. This operator was introduced in [29] acting on an infinite dimensional "loop superspace," while a finite dimensional analog was used in [58]. Here we give an axiomatic definition.

The introduction of a ghost partner e^A for each coordinate is required. The ghost fields have the opposite grading to coordinates:

$$\begin{aligned} [e^A, Z^M\} &= 0 \\ \{e^A, e^B] &= 0, \end{aligned} \tag{2.30}$$

where $[\ \ ,\ \ \}$ and $\{\ \ ,\ \ ]$ are the graded commutator/anticommutator:

$$\begin{aligned} [X_A, X_B\} &= -(-1)^{AB}[X_B, X_A\} \\ \{X_A, X_B] &= (-1)^{AB}\{X_B, X_A]. \end{aligned} \tag{2.31}$$

The e^A are independent of the fields Z^M, and hence satisfy $de^A = 0$. A general element of the double complex is a "ghost form valued differential form." The space of all such generalized forms of differential degree m and ghost degree n will be denoted by $\Omega^{m,n}$. The collection of these spaces will be denoted $\Omega^{*,*}$. Generalized forms $Y \in \Omega^{m,n}$ will be written using a comma to separate ghost indices from space indices:

$$Y = e^{B_n} \dots e^{B_1} L^{A_m} \dots L^{A_1} Y_{A_1 \dots A_m, B_1 \dots B_n} \frac{1}{m!n!}. \tag{2.32}$$

We then define the ghost differential by the following properties

- s is a right derivation[5]. That is, if X and Y are generalized forms and n is the ghost degree of Y then
$$s(XY) = Xs(Y) + (-1)^n s(X)Y. \tag{2.33}$$
- If X has ghost degree zero then
$$sX = e^A Q_A X, \tag{2.34}$$
where Q_A denotes a Lie derivative with respect to the vector field (2.7). Thus, s is related to the left action of the supertranslation group, which is a global symmetry of p-brane actions. In particular, note that left invariant superspace forms (of ghost degreee zero) are annihilated by s.
- $$se^A = \frac{1}{2} e^C e^B t_{BC}{}^A, \tag{2.35}$$
where $t_{BC}{}^A$ are the structure constants of the superalgebra associated with the background superspace[6].

One verifies that[7]
$$\begin{aligned} s^2 &= 0 \\ [s, d] &= 0. \end{aligned} \tag{2.36}$$
Hence s extends the de Rham complex into a double complex. s is similar to a BRST operator in that it requires the introduction of ghost fields; however unlike a BRST operator it has not been derived from constraints or gauge symmetries.

There is a total differential D that is naturally associated with the double complex:
$$\begin{aligned} D &= s + (-1)^{n+1} d \\ D^2 &= 0, \end{aligned} \tag{2.37}$$
where n is the ghost degree of the generalized form upon which D acts. The spaces Ω^l_D of the single complex upon which D acts are the sum along the anti-diagonal of the spaces of the double complex:

[5]Our conventions are such that d is also a right derivation (with respect to the differential degree).

[6]For standard p-brane actions, $t_{BC}{}^A$ are simply the structure constants of the supertranslation algebra (2.1). However, the analysis can also apply in the case of actions formulated on more general background spaces.

[7]To prove nilpotency of s one needs to use the Jacobi identity for the supertranslation algebra.

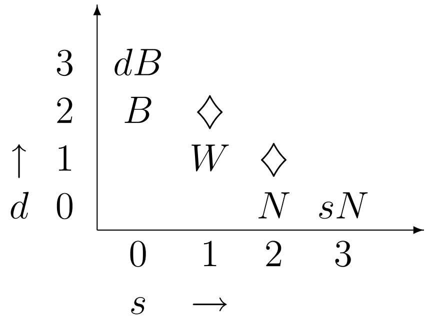

Figure 2.1: Descending sequence for the string

$$\Omega^l_D = \{\oplus\Omega^{m,n} : \qquad m + n = l\}. \tag{2.38}$$

The l-th cohomology of D is

$$H^l_D = Z^l_D / B^l_D, \tag{2.39}$$

where Z^l_D are the D closed generalized l-forms ("D cocycles"), and B^l_D are the generalized l-forms in the image of D ("D coboundaries"). The restriction of H^l_D to representatives within $\Omega^{m,l-m}$ will be denoted $H^{m,l-m}$. The representatives of $H^{m,0}$ can be used to define descending cohomology sequences. In particular, one may use the $(p+2)$-form H that defines the WZ term as the starting point for a set of descent equations [29].

To illustrate this, firstly note that H is a left invariant, closed form with ghost number zero. It is therefore closed under both s and d. Using the fact that s and d commute, $sH = 0$ implies that $dsB = 0$, and thus $sB = -dW$ for some $W \in \Omega^{p,1}$. This argument does not apply globally, but is valid on every coordinate patch[8]. The same logic that was applied to B can then be applied to W. This gives $sW = dN$ for some $N \in \Omega^{p-1,2}$. For the string, the last nonzero element of the sequence is $sN \in H^{0,3}$. For a p-brane, the sequence continues until we reach an element of $H^{0,p+2}$. The descending cohomology sequence can be graphically depicted using a "tic-tac-toe box" [64]. The string case is depicted in figure 2.1. The symbol $\diamondsuit$ indicates "zero with respect to the operator D." Precisely, for a p-brane, denote the "potentials" of the sequence by $B^{p+1-m,m}$ (we shall also use the B, W, N

[8]An expression like dB may represent something closed but not necessarily exact (with respect to de Rham cohomology). However, the de Rham triviality of such fields does not affect the (s,d) cohomology studied in this chapter. Aspects of de Rham cohomology relevant to this thesis are further investigated in chapter 6.

notation for the first three potentials in the p-brane case, e.g. $W = B^{p,1}$). Then each $\diamondsuit$ represents a relation

$$sB^{p-m+1,m} + (-1)^m dB^{p-m,m+1} = 0. \tag{2.40}$$

These are the "descent equations" (note that the first descent equation, not represented in the above, is $H = dB^{p+1,0}$).

The above definitions have been used so that the endpoints of the descent are linked via a coboundary of the D complex. For example, in the string case

$$-dB \oplus sN = D(B \oplus W \oplus N). \tag{2.41}$$

That is, $H = dB \in H^{3,0}$ is D cohomologous to $sN \in H^{0,3}$. We may write this as

$$H \simeq sN. \tag{2.42}$$

In general one finds that

$$H \simeq sB^{p+1-m,m} \qquad \forall m. \tag{2.43}$$

The D cocycle represented by H can therefore be alternatively represented by s acting on any of the potentials of the sequence.

We will call a nilpotent operator "exact" if its associated cohomology is trivial. For example, the de Rham differential d on an open set is exact; the cohomology H_d is trivial as a result of the Poincaré lemma. That is, given $Y \in H_d^m$, then for all $m \geq 1$ we can write $Y = dX$ for some $X \in \Omega_d^{m-1}$. Note that the "exactness" of an operator is dependent on the space upon which it acts. By definition, d is not exact (globally) on a manifold that possesses nontrivial de Rham cohomology. There are important consequences for D cohomology if we can show that the ghost differential s is exact on open sets.

Theorem 1 (exactness) *s is exact on open sets.*

To prove this we find a chain map for which the operators s and d become "dual" to each other. A chain map between two complexes is one that commutes with the differentials of the complexes. In our case, the required map Ψ must satisfy

$$\begin{aligned} \Psi(d)\Psi(Y) &= \Psi(dY) \\ \Psi(s)\Psi(Y) &= \Psi(sY) \end{aligned} \tag{2.44}$$

for any $Y \in \Omega^{*,*}$. Consider the following "check" chain map

$$\begin{aligned} \Psi : \Omega^{*,*} &\rightarrow \check{\Omega}^{*,*} \\ L^A &\rightarrow e^A \\ e^A &\rightarrow R^A. \end{aligned} \tag{2.45}$$

The map takes (m,n)-forms to (n,m)-forms. On $\check{\Omega}^{*,*}$ we have the operators $\check{s}$ and $\check{d}$ defined by

- $$\check{s} = d. \tag{2.46}$$
- $\check{d}$ is a right derivation.
- If X has ghost degree zero then
$$\check{d}X = e^A D_A X, \tag{2.47}$$
where D_A is a Lie derivative with respect to the vector field (2.12) associated with the global right action.
- $$\check{d}e^A = -\frac{1}{2} e^C e^B t_{BC}{}^A. \tag{2.48}$$

If we think of s as a generalized left differential, then $\check{d}$ is the analogous right differential. The check map is clearly invertible. Let Y be any s closed generalized form of ghost degree one or more over an open set. Then, using $\check{s} = d$ and the exactness of d on an open set, one shows that Y is an s coboundary:

$$\begin{aligned} sY &= 0 \\ \Rightarrow \check{s}\check{Y} &= 0 \\ \Rightarrow \check{Y} &= \check{s}\check{X} \\ \Rightarrow Y &= sX. \end{aligned} \tag{2.49}$$

Therefore s is exact on open sets since we have $H_s^m = H_d^m$.

In [27], CE cohomology was formulated as the restriction of de Rham cohomology to left invariant forms. Now, the $(p+2)$-form H is a D coboundary when it can be written $H = DB$. Equivalently, H is a D coboundary if a left invariant potential B can be found. This is precisely the definition of a trivial CE cocycle. A nontrivial D cocycle is one for which we must necessarily have $sB \neq 0$, which is equivalent to the definition of a nontrivial CE cocycle. CE cohomology is therefore the restriction of D cohomology to forms that have ghost degree zero. D cohomology is the natural extension of CE cohomology into the double complex $\Omega^{*,*}$. Since s is exact, we may reverse descending tic-tac-toe sequences into ascending ones, starting with any element of $H^{p+2-m,m}$ and finding an associated left invariant element of H_d^{p+2}. This establishes an isomorphism between D and CE cohomologies that would not exist if s were not exact.

2.3.2 Gauge freedom

Using the tic-tac-toe construction, the form $H \in H^{p+2,0}$ may be identified with any of the other representatives $sB^{p+1-m,m}$ of the p-brane D cocycle. This is a well defined map between the $H^{m,n}$ (which are sets of equivalence classes), but *not* between the forms themselves. In general there is gauge freedom for representatives. Although this freedom is associated with D coboundaries, there is no reason for these coboundaries to be exact in the de Rham sense. In this way we will see that the gauge freedom can affect the topological charge algebra.

We now explicitly derive the gauge transformations for the string. Consider the relation $H = dB$. Given H, this defines B only up to a closed form. Thus, given a solution B, the alternative solution $B' = B - d\psi$ is equally valid. We write this as

$$\Delta B = -d\psi. \tag{2.50}$$

In an extended superspace that allows a manifestly invariant WZ term, a transformation of this type is all that separates the standard WZ term from the invariant one [32, 33] (see section 2.5 for an explicit example). What then is the effect of the transformation (2.50) on W? Since the variation Δ commutes with s and d, we have

$$\begin{aligned} d\Delta W &= \Delta dW \\ &= -\Delta sB \\ &= ds\psi. \end{aligned} \tag{2.51}$$

The general solution may be written

$$\Delta W = s\psi + d\lambda, \tag{2.52}$$

where λ is a new gauge field. The gauge transformations of the field N are derived similarly. Directly from (2.52) we have

$$\begin{aligned} d\Delta N &= \Delta dN \\ &= \Delta sW \\ &= ds\lambda. \end{aligned} \tag{2.53}$$

This has the general solution

$$\Delta N = s\lambda + C, \tag{2.54}$$

where C is a d closed $(0,2)$-form. If one progressed in the other direction (an ascending sequence starting from sN) one would also find an s closed $(2,0)$-form gauge field C' for B. One can then write the gauge transformations in totality as

$$\Delta(B \oplus W \oplus N) = D(\psi \oplus \lambda) \oplus C \oplus C'. \tag{2.55}$$

One verifies that each potential has two gauge fields: one that is d closed and one that is s closed. These gauge transformations are additive. For example, the field W has two gauge transformations: one for ψ and one for λ, with ΔW given by (2.52). The gauge fields are independent (they are not required to satisfy descent equations like those that relate B, W and N). They may also affect more than one field. For example, ψ is a transformation that leaves dB invariant ($\Delta B = -d\psi$), and also sW invariant ($\Delta W = s\psi$). Although sB and dW are not gauge invariant, the ψ transformation is such that the descent equation $sB = -dW$ is true in all gauges. The construction ensures that in general, descent equations are preserved by the gauge transformations. The gauge transformations are identically the same as the D coboundaries as a result of the exactness of the operators s and d. The alternative representatives sB, sW and sN of the D cocycle defined by H are therefore well defined as elements of their respective $H^{m,n}$ sets.

2.4 Algebra modifications

2.4.1 The algebra of left generators

In the Noether charge construction there is a conserved charge associated with each global symmetry of the action, and these charges transform according to the underlying symmetry group. Topological extensions to these algebras can occur if the topology is such that "surface terms" contribute to the charge algebra [56]. The topological charge algebras considered here are those generalizing the Noether construction to the case of actions which are invariant up to a total derivative; the case with p-branes [28]. A more general treatment of this material may be found in [31]. We will review the construction in detail before further examining the anomalous term.

The action is formulated in terms of $(Z^M, \dot{Z}^M)$, which may be viewed as coordinates for the superspace tangent bundle. The Hamiltonian formulation of dynamics is cast in terms of coordinates Z^M and their associated conjugate momenta P_M, which together constitute the "phase space." The momenta are defined by

$$P_M = \frac{\partial L}{\partial \dot{Z}^M}. \tag{2.56}$$

The phase space can be viewed as coordinates for the superspace cotangent bundle. The Lagrangian then provides a map (a Legendre transform), defined by (2.56), from the tangent bundle to the cotangent bundle.

We use the following fundamental (graded) Poisson brackets on phase space

$$[P_M(\sigma), Z^N(\sigma')\} = \delta_M{}^N \delta(\overrightarrow{\sigma} - \overrightarrow{\sigma}'), \tag{2.57}$$

where it is assumed $\sigma'^0 = \sigma^0$ (i.e. equal time brackets). The Dirac delta function notation is shorthand for the product of the p delta functions on the spatial coordinates of the worldvolume. One can use (2.57) and the following Poisson bracket identities to evaluate general brackets

$$\begin{aligned} [U_A, U_B U_C\} &= [U_A, U_B\}U_C + (-1)^{AB} U_B [U_A, U_C\} \\ [U_A, U_B(V)\} &= [U_A, V^N\} \frac{\partial U_B}{\partial V^N}. \end{aligned} \tag{2.58}$$

The above relations can all be derived from an integral form of the Poisson bracket, which can be useful for certain proofs. The form we use is

$$[U_A, U_B\} = \int d^p\sigma \frac{\delta U_A}{\delta P_M(\sigma)} \frac{\delta U_B}{\delta Z^M(\sigma)} (-1)^{MA+M} - (-1)^{AB}[A \leftrightarrow B], \tag{2.59}$$

where here and henceforth, $\int d^p\sigma$ indicates an integral over the spatial section of the p-brane worldvolume. We define the following regularly used "bar map" by its action on superspace forms $Y \in \Omega^{m,n}$:

$$\overline{Y}^{m-p,n}(\sigma) = (-1)^{p(p+m+1)} i_{\partial_1} \ldots i_{\partial_p} Y^{m,n}(\sigma). \tag{2.60}$$

Here, i_V denotes interior derivation with respect to the vector V, and ∂_i is the i-th worldvolume tangent vector (note that the time derivative ∂_0 is not involved in the map). When $Y \in \Omega^{p,n}$ we will also indicate an integrated version of this map using the same symbol:

$$\overline{Y}^{0,n} = \int d^p\sigma \overline{Y}^{0,n}(\sigma). \tag{2.61}$$

Even though we may omit the argument in (2.60), it should be clear within context which of these is implied.

The Noether charges associated with a manifestly left invariant Lagrangian will be denoted $\overline{Q}_A$. One finds[9]:

$$\overline{Q}_A = \int d^p\sigma R_A{}^M P_M. \tag{2.62}$$

[9] "Bar" above Q_A or D_A is a definition, not an action of the map (2.60). The notation indicates that $\overline{Q}_A$ and $\overline{D}_A$ naturally act upon elements in the image of this map.

These charges are the phase space analog of the left generators (2.7). They satisfy the same algebra as the superalgebra underlying the background superspace, but with the sign reversed:

$$[\overline{Q}_A, \overline{Q}_B\} = -t_{AB}{}^C \overline{Q}_C. \tag{2.63}$$

This is the "minimal algebra." In general, the p-brane Lagrangian is not manifestly left invariant (i.e. it is only symmetric up to a total derivative) due to quasi-invariance of the WZ term. This results in modification of the Noether charge algebra by a topological anomalous term [28, 31], which results as follows. Define

$$w_A{}^i = -\frac{1}{p!}\tilde{\epsilon}^{i_p \ldots i_1 i} W_{i_1 \ldots i_p, A}, \tag{2.64}$$

where $\tilde{\epsilon}$ is the antisymmetric Levi-Civita symbol with $\tilde{\epsilon}^{0,1,\ldots p+1} = 1$. Since the variation of the WZ form is $Q_A B = -dW_A$, one verifies:

$$\begin{aligned} \delta_{Q_A}\mathcal{L} &= \delta_{Q_A}\mathcal{L}_{WZ} \\ &= \partial_i w_A{}^i. \end{aligned} \tag{2.65}$$

Now, upon using the EL (Euler-Lagrange) equations

$$\frac{\partial \mathcal{L}}{\partial Z^M} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i Z^M)} = 0, \tag{2.66}$$

we have identically

$$Q_A \mathcal{L} = \partial_i \left[Q_A Z^M \frac{\partial \mathcal{L}}{\partial(\partial_i Z^M)} \right]. \tag{2.67}$$

Hence, "on-shell" there are conserved currents

$$\begin{aligned} \tilde{q}_A{}^i &= R_A{}^M \frac{\partial \mathcal{L}}{\partial(\partial_i Z^M)} - w_A{}^i \\ \partial_i \tilde{q}_A{}^i &= 0. \end{aligned} \tag{2.68}$$

The associated conserved charges are [28, 31]

$$\widetilde{\overline{Q}}_A = \overline{Q}_A + \overline{W}_A, \tag{2.69}$$

which obey a modified version of the minimal algebra

$$[\widetilde{\overline{Q}}_A, \widetilde{\overline{Q}}_B\} = -t_{AB}{}^C \widetilde{\overline{Q}}_C + \overline{M}_{AB}, \tag{2.70}$$

with

$$\overline{M}_{AB} = [\overline{Q}_A, \overline{W}_B\} + [\overline{W}_A, \overline{Q}_B\} + t_{AB}{}^C \overline{W}_C. \tag{2.71}$$

This is the algebra of conserved charges.

Define the special representative $M = sW = dN$ of the p-brane D cocycle. One verifies that the definition of $\overline{M}$ given here is consistent with that obtained from (2.61) acting upon M (that is, the bracket and map operations commute). We refer to M or $\overline{M}$ as the "anomalous term." If we need to distinguish between the two, $\overline{M}$ will be referred to as the "topological anomalous term", and M as its "superspace representation." Since $\overline{M}$ is derived from a closed form, an associated current is conserved identically:

$$\begin{aligned} m^i_{AB} &= \tilde{\epsilon}^{i_p \ldots i_1 i} M_{i_1 \ldots i_p, AB} \frac{1}{p!} \\ \partial_i m^i_{AB} &= \tilde{\epsilon}^{i_p \ldots i_1 i} \partial_i \partial_{i_1} N_{i_2 \ldots i_p, AB} \frac{1}{(p-1)!} \\ &= 0. \end{aligned} \tag{2.72}$$

Let the spatial section of the worldvolume be embedded into superspace by the map Φ. One verifies that

$$\begin{aligned} \overline{M}_{AB} &= (-1)^p \int d^p\sigma \Phi^* M_{p \ldots 1, AB}(\sigma) \\ &= (-1)^p \int \Phi^* M_{AB}, \end{aligned} \tag{2.73}$$

Since the spatial section is assumed to be a closed manifold, $\overline{M}_{AB}$ is a topological integral of the closed p-form M_{AB}. The result of the integral will be determined by the topology of the spatial section, and the class of the associated de Rham cohomology to which M_{AB} belongs.

In explicit calculations of conserved charge algebras [28, 31, 65], fermionic topological terms have not been retained. For example, it is assumed that

$$\int d\sigma^1 \partial_1 \theta \tag{2.74}$$

vanishes, due to the trivial topology [30] of the fermionic directions. The topological integrals of closed forms with θ differentials and single valued coefficients must vanish in this case. With this simplifying condition, in the case of p-branes, the anomalous term is proportional to the pullback of the p-form [28]

$$\Gamma_{m_1 \ldots m_p \alpha\beta} dx^{m_1} \ldots dx^{m_p}. \tag{2.75}$$

There are motivations to investigate the fermionic topological terms. For example, in certain spaces more general than flat superspace, fermionic brane charges are *required* on the basis of quantum consistency with Jacobi identities [42]. Although many of the tools of differential topology do not apply to fermionic variables, the realization of fermionic charges can still be studied; for example, via antiperiodic boundary conditions for a string bridging two D-branes [40, 41]. Setting the physical interpretation aside, fermionic p-brane charges are mathematically consistent. Therefore, in this work we will retain the fermionic terms in order to see what features appear as a result.

There is no obvious reason to expect that it should be possible to incorporate the topological anomalous term $\overline{M}$ into the definition of an extended algebra. However, using its superspace representation M we now show that this is possible. In section 2.5 we will explicitly derive the extended algebras that result from the superstring anomalous term.

Theorem 2 (extension) *The anomalous term of the Noether charge algebra defines an extension of the supertranslation algebra by an ideal. The resulting extended superalgebra is solvable.*

First we need to show closure of the algebra. This requires that the anomalous term, and all brackets resulting from it, can be expressed using a finite number of new generators. To find the extended algebra one could investigate the Poisson brackets of $\widetilde{\overline{Q}}_A$ and $\overline{M}_{AB}$. However, the bar map introduces no time derivatives, or equivalently no dependence upon the phase space momenta. This results in an isomorphism, with charges, integrated forms and Poisson brackets on one side, and left generators, forms and Lie Brackets on the other. Recalling that Q_A acts on forms via Lie derivative with respect to the vector field (2.7), we have:

$$\begin{aligned} [\widetilde{\overline{Q}}_A, \widetilde{\overline{Q}}_B\}_{PB} &= \overline{[\widetilde{Q}_A, \widetilde{Q}_B\}_{LB}} \\ [\overline{Q}_A, \overline{M}_{BC}\}_{PB} &= \overline{Q_A M_{BC}} \\ [\overline{M}_{AB}, \overline{M}_{CD}\}_{PB} &= 0, \end{aligned} \tag{2.76}$$

where the modified left generators are defined as

$$\widetilde{Q}_A = Q_A + W_A. \tag{2.77}$$

We choose to work with the latter (RHS) since calculations are then simplified. The anomalous term is thus represented by the p-forms M_{AB}.

The minimal algebra is generated by the left generators Q_A of the double complex[10]. Let $\mathcal{G} = \{Q_A\}$ denote the minimal algebra, and $\widetilde{\mathcal{G}} = \{\widetilde{Q}_A, \Sigma_A\}$ denote the full algebra that is assumed to result by addition of the anomalous term. Now consider the action of the left generators on forms. Taking for example, the supersymmetry generators:

$$\begin{array}{ccccccc} & Q_\alpha & & Q_\alpha & & Q_\alpha & \\ x & \rightarrow & \theta & \rightarrow & \mathrm{const} & \rightarrow & 0 \\ & Q_\alpha & & Q_\alpha & & & \\ dx & \rightarrow & d\theta & \rightarrow & 0. & & \end{array}$$

Thus, if a form has coefficients with a polynomial structure then each action of Q_A brings it closer to annihilation. The requirements of Lorentz invariance and fixed dimensionality (see section 2.4.3) ensure that all valid forms have this polynomial structure. It follows that the anomalous term will be annihilated by the left generators in a finite number of steps.

There is then a stepwise process to define the extended algebra. At the first step we may factor out any Lorentz invariant tensors from M_{AB} (which become new structure constants). The remaining form is then written in terms of a minimal set of independent closed forms Σ_A, which become new generators of the algebra. The Σ_A commute with themselves and satisfy

$$[\widetilde{Q}_A, \Sigma_B\} = [Q_A, \Sigma_B\} \tag{2.78}$$

since W_A commutes with Σ_B. We then act again with the Q_A and introduce new generators to deal with any forms that cannot be written in terms of those generators previously defined. By the above annihilation argument it follows that this process is finite. That is, there will be a finite number of new generators. The resulting algebra has the structure

$$\begin{aligned} [\widetilde{Q}, \widetilde{Q}] &\subset \widetilde{Q} \oplus \Sigma \\ [\widetilde{Q}, \Sigma] &\subset \Sigma \end{aligned} \tag{2.79}$$

The second line shows that Σ is an ideal of the new algebra. The algebra $\widetilde{\mathcal{G}}$ is said to be *solvable* if

$$(\mathsf{Ad}_{\widetilde{\mathcal{G}}})^m(\widetilde{\mathcal{G}}) = 0 \tag{2.80}$$

for some finite integer m, where $\mathsf{Ad}_{\widetilde{\mathcal{G}}}$ is the adjoint action. The minimal algebra $\mathcal{G}$ is solvable. The annihilation argument shows that $\widetilde{\mathcal{G}}$ is also solvable, since the action of $\widetilde{\mathcal{G}}$ annihilates the new generators in a finite number of steps.

[10] Here we assume for brevity that $\mathcal{G}$ is the standard superalgebra. The same principles are also valid in the case where $\mathcal{G}$ is one of the extended superalgebras (e.g. those of section 2.2).

This shows that the new algebra closes, however to show that $\widetilde{\mathcal{G}}$ is a valid *superalgebra* we must also show that the super-Jacobi identities are satisfied. There are four cases to test. The first is

$$(-1)^{AC}[\widetilde{Q}_A, [\widetilde{Q}_B, \widetilde{Q}_C\}\} + \textsf{cycles}, \tag{2.81}$$

where "cycles" indicates the terms obtained from two repetitions of the cycling $A \to B \to C$. Using $M = sW$ one can show that this reduces to

$$(-1)^{AC} t_{BC}{}^{D} t_{AD}{}^{E} Q_E + \textsf{cycles}, \tag{2.82}$$

which vanishes since the original structure constants satisfy the Jacobi identity. The second case is

$$(-1)^{AC}[\widetilde{Q}_A, [\widetilde{Q}_B, \Sigma_C\}\} + \textsf{cycles}. \tag{2.83}$$

By (2.78) it is valid to replace $\widetilde{Q}$ by Q in the above expression since Σ is an ideal. The Jacobi identity is then identically satisfied since it reflects an action of the minimal algebra. The final two cases

$$\begin{aligned} &(-1)^{AC}[\widetilde{Q}_A, [\Sigma_B, \Sigma_C\}\} + \textsf{cycles} \\ &(-1)^{AC}[\Sigma_A, [\Sigma_B, \Sigma_C\}\} + \textsf{cycles} \end{aligned} \tag{2.84}$$

are trivially satisfied. The Jacobi identity therefore holds, and $\widetilde{\mathcal{G}}$ is an extended superalgebra.

2.4.2 The algebras of right generators and constraints

The p-brane action (2.16) yields constraint equations for the phase space variables. That is, equations of the form

$$C_M(Z, P) = 0 \tag{2.85}$$

for some functions C_M, which reduce to identities once the definitions (2.56) of momenta are used. The constraints are associated with a reduction of the phase space degrees of freedom. This is related to the presence of gauge symmetries (κ-transformations, diffeomeorphisms), which are associated with the first class subset of the constraints. However, our primary concern in this thesis is the investigation of algebras resulting from modified generators. For this purpose we do not need to classify or eliminate the constraints.

Evaluating $\frac{\partial L}{\partial \dot{x}^m}$ and $\frac{\partial L}{\partial \dot{\theta}^\mu}$ for the NG action one finds

$$\begin{aligned} P_m^{(NG)} &= -(-g)^{-\frac{1}{2}} g^{0i} L_i{}^a \eta_{am} \\ P_\mu^{(NG)} &= -\frac{1}{2}(\Gamma^n \theta)_\mu P_n^{(NG)}. \end{aligned} \tag{2.86}$$

One thus identifies the fermionic constraints of the NG action as

$$\begin{aligned} C_\alpha &= \delta_\alpha{}^\mu P_\mu^{(NG)} + \frac{1}{2}(\Gamma^n\theta)_\mu P_n^{(NG)} \\ &= L_\alpha{}^M P_M^{(NG)}. \end{aligned} \tag{2.87}$$

Now, the right generators for the phase space are

$$\overline{D}_A = L_A{}^M P_M, \tag{2.88}$$

which satisfy the minimal algebra:

$$[\overline{D}_A(\sigma), \overline{D}_B(\sigma')\} = \delta(\overrightarrow{\sigma} - \overrightarrow{\sigma}')t_{AB}{}^C \overline{D}_C(\sigma). \tag{2.89}$$

Thus, the C_α are simply the fermionic, right generators for the phase space, and they satisfy the same algebra.

$$\{C_\alpha(\sigma), C_\beta(\sigma')\} = \delta(\overrightarrow{\sigma} - \overrightarrow{\sigma}')t_{\alpha\beta}{}^A \overline{D}_A(\sigma). \tag{2.90}$$

Note that there is no constraint C_a: it is $\overline{D}_a$ which appears on the RHS.

Just as the Noether charges are modified in the presence of the WZ term, so are the constraints and their algebra [29]. If a WZ term is added to the NG action:

$$\mathcal{L}_{WZ} = \dot{Z}^M \overline{B}_M, \tag{2.91}$$

the new momenta (including those associated with new coordinates) are related to the NG action momenta $P_M^{(NG)}$ via

$$P_M = P_M^{(NG)} + \overline{B}_M. \tag{2.92}$$

We find that the constraints $\widetilde{C}_A$ in the presence of the WZ term can then be written

$$\widetilde{C}_A = L_A{}^M(P_M - \overline{B}_M), \qquad A \neq a. \tag{2.93}$$

Details of the calculation may be found in appendix 7.4. We will define a full set of modified right generators analogously:

$$\begin{aligned} \widetilde{\overline{D}}_A &= L_A{}^M P_M^{(NG)} \\ &= \overline{D}_A - \overline{B}_A. \end{aligned} \tag{2.94}$$

Again, the components of $\overline{B}$ contain no time derivatives. Thus, the modification to the right generators for the phase space contains no momentum dependence (just as in the left generator case). We assume that B is single valued. Then, as was found in [29], the modified algebra is defined by H:

$$[\widetilde{\overline{D}}_A(\sigma), \widetilde{\overline{D}}_B(\sigma')\} = \delta(\overrightarrow{\sigma} - \overrightarrow{\sigma}')[t_{AB}{}^C \widetilde{\overline{D}}_C - \overline{H}_{AB}](\sigma), \tag{2.95}$$

where

$$\delta(\overrightarrow{\sigma} - \overrightarrow{\sigma}')\overline{H}_{AB}(\sigma) = [\overline{D}_A(\sigma), \overline{B}_B(\sigma')\} + [\overline{B}_A(\sigma), \overline{D}_B(\sigma')\} - \delta(\overrightarrow{\sigma} - \overrightarrow{\sigma}')t_{AB}{}^C\overline{B}_C(\sigma) \quad (2.96)$$

is also as given by the bar map (4.51) applied to (2.17).

For the constraint surface to be well defined, the constraints must be invariant under the action of the Noether symmetries of the action. The constraints are therefore left invariant in the sense

$$[\overline{Q}_A, C_\beta(\sigma)\} \approx 0, \quad (2.97)$$

where $\approx$ means "equal on the constraint surface." For the NG action this is an example (in PB form) of the commutativity of the left and right actions. When the WZ term is added, this condition must continue to hold (i.e. upon replacing $\overline{Q}_A$ and C_A by their modified counterparts). In fact, assuming that W is single valued, one can use the descent equation $sB = -dW$ to show[11]

$$[\widetilde{\widetilde{Q}}_A, \widetilde{\widetilde{D}}_B(\sigma)\} = 0. \quad (2.98)$$

Since the constraints are a subset of the modified right generators, their left invariance is therefore guaranteed by the double complex cohomology. Furthermore, since the equation $sB = -dW$ is preserved by the gauge transformations, the left invariance of the constraints is independent of the gauge.

2.4.3 Cohomology of algebra modifications

We are now in a position to determine how gauge freedom affects the algebras of left/right generators. Before proceeding however, we need to establish some facts about the D cohomology of H. First let us review why the equation (2.17) defining H takes the form it does. H must have the following properties:

Properties of H

- H is closed.
- H is left invariant.
- dim $H = p + 1$.
- H is Lorentz invariant.

[11] Given $\widetilde{\widetilde{Q}}_A$, one could also use this as a definition of the $\widetilde{\widetilde{D}}_B$ [29].

In standard superspace, H is the *unique* $p+2$ form (up to a constant of proportionality) with these properties [27]. Furthermore, it is a nontrivial CE cocycle. In the double complex construction this implies that in standard superspace, H is the unique Lorentz invariant element of $H^{p+2,0}$ with dimension $p+1$. Since the last two properties in the list are preserved by the operators d and s, we conclude that Lorentz invariance and dimensionality $p+1$ must be a property of *all* elements of the double complex (including potentials and gauge transformations). The exactness of s means that the D cohomology of the single complex is equal to the de Rham cohomology of the first column of the double complex. Since we restrict ourselves to Lorentz invariant forms of dimension $p+1$, by the uniqueness of H, this cohomology is equal to the field of scalars we are using (the constant of proportionality multiplying H labels the class).

The uniqueness of H implies that the modification to the right generator algebra $\overline{H}$ is also unique. It is gauge invariant, and even independent of the underlying supertranslation algebra used (since the same definition of H is always used). Note however that the right generator algebra obtained in an extended background can be different to the right generator algebra obtained in standard superspace (even though the modification is the same) because then the *minimal* algebra that we start with is already different.

The left generator algebra is less straightforward. Note that due to nilpotency of the operators, moving twice in any one direction on the tic-tac-toe box gives zero. An interesting consequence of this is that the gauge freedom in the WZ term, resulting from ψ, C' in (2.55), has no effect upon the anomalous term M. However, using a different background superspace not only changes the minimal algebra but can also change the modification M (since the descent equations may have different solutions). We note that left invariant WZ terms can only be constructed in such extended superspace backgrounds.

The result of main interest is that the topological anomalous term $\overline{M}$ is not gauge invariant. Using (2.52)

$$\begin{aligned} \Delta M &= s\Delta W \\ &= sd\lambda. \end{aligned} \tag{2.99}$$

Although ΔM is a D coboundary, it need not be (de Rham) exact. $\Delta\overline{M}$ can therefore be nonzero in the presence of nontrivial topology (just as $\overline{M}$ can be). How much freedom do we have? At first it seems that we have full gauge freedom at our disposal, but in practice the requirements of Lorentz invariance and correct dimensionality are restrictive. In section 2.5 we will see that in the case of the string, these requirements on the gauge fields reduce the freedom in the anomalous term down to a single, global degree of freedom. A corresponding free constant parameterizes the family of algebras obtained from the process.

Since there is an orbit of gauge equivalent representatives, and there is no natural basis upon which to fix a gauge, one can no longer speak of "the" anomalous term if one defines it as a particular form or modified left generator algebra. In order that the anomalous term be a well defined object it must be defined as an entire D cohomology class $[M]$. We have already seen that the representatives M of this class are D cohomologous to H. Since s is exact, this correspondence is a *bijection* between the sets $H^{p+2,0}$ and $H^{p,2}$ to which H and M belong. That is, to each equivalence class $[H] \in H^{p+2,0}$ is associated a unique class $[M] \in H^{p,2}$ of the *same triviality*, and vice versa. The nature of the resulting class $[M]$ depends on the background space being used.

First consider standard superspace. Since the class $[H]$ is unique and nontrivial, $[M]$ must also be unique and nontrivial. The classes $[M]$ must be labelled by a single proportionality constant belonging to the field of scalars (just as the classes $[H]$ are). The difference between $[H]$ and $[M]$ is that $[H]$ consists of H only; there are no *coboundaries* in $H^{p+2,0}$. In general there *are* coboundaries in $H^{p,2}$; they are precisely the λ gauge transformations (and we will see that explicit, nonvanishing examples of such gauge transformations do exist). The D cocycle of the p-brane therefore has a set of equivalent representatives belonging to $\Omega^{p,2}$; it is this full set which makes the anomalous term a well defined object.

If an extended superspace is used then H is a D coboundary. One might argue that in this case the anomalous term should not even exist (since a manifestly left invariant WZ term is possible). However, from the cohomology point of view the anomalous term should consist of all possible modifications to the Noether charges that are consistent with charge conservation. In the double complex construction, charge conservation is guaranteed by the descent equations. The anomalous term $[M]$ therefore becomes the space of D coboundaries within $\Omega^{p,2}$. This is identically equal to the representatives ΔM resulting from the λ gauge transformations. Note that D coboundaries need not be exact; it is therefore possible to obtain nonzero topological integrals for $\overline{M}$ even in the case of a manifestly left invariant WZ term.

We summarize with the following:

Theorem 3 (cohomology) *The anomalous term is the restriction of $H^{p,2}$ to forms that are D cohomologous to H.*

Theorem 4 (uniqueness) *In the standard background, the anomalous term is the unique, Lorentz invariant, D nontrivial class of dimensionality $p+1$.*

From the second of these we conclude that in standard superspace it is possible to find the anomalous term without solving descent equations. If a single D nontrivial representative within $H^{p,2}$ can be found then the entire anomalous term will be generated by the λ gauge transformations. This class is unique (up to the constant of proportionality which labels the classes). In superspaces which allow manifestly left invariant Lagrangians, the anomalous term is the set of D coboundaries generated by the λ gauge transformations.

Note that the above arguments apply only to the superspace representation M of the anomalous term. The associated topological anomalous term $\overline{M}$ may vanish for topological reasons separate from D cohomology. For example, if we choose to compactify *no* dimensions, or if the brane does not "wrap", then topological integrals such as $\overline{M}$ must identically vanish. If we compactify only *some* dimensions then we may find that in standard superspace there do exist gauges in which the topological anomalous term vanishes, since a gauge transformation may shift the form M into a trivial sector of the cohomology of the spatial section. We will see an explicit example of this in section 2.5.

To summarize, the p-brane has an associated D cocycle defined by the representative $H \in H^{p+2,0}$. The Noether charge algebra can be modified by a topological anomalous term deriving from cocycle representatives $M \in H^{p,2}$. The representatives are not unique due to the presence of λ gauge transformations of the cocycle. These transformations themselves represent topological integrals which can be nonzero. The anomalous term is well defined as a cohomology class, where elements related by λ gauge transformations are to be considered equivalent. Since each representative of the anomalous term defines an extended supertranslation algebra, each algebra in the family can be considered as being equivalent from a D cohomology point of view.

It is interesting to note that all the cocycle representatives of ghost degree two or less have physical interpretations:

- H measures the modification to the right generator algebra.
- sB measures the left variation of the WZ term.
- sW measures the modification to the left generator algebra.

One may ask if any other representatives are significant. The only one remaining in the case of the string is the ghost degree three element sN. Consider the following modified algebra[12]:

$$[Q_A, Q_B\} = -t_{AB}{}^C Q_C + N_{AB}. \tag{2.100}$$

[12] We present this for the sake of interest only since we have no physical interpretation for modified algebras resulting from N.

One finds that the Jacobi identity of this algebra is generated by sN:

$$(-1)^{AC} sN_{ABC} = (-1)^{AC} [Q_A, [Q_B, Q_C\}\} + \text{cycles}. \tag{2.101}$$

sN therefore determines whether or not N can define an extended superalgebra. Based on the cocycle triviality arguments we conclude that N defines extensions of extended backgrounds, but not of the standard background. Applying the same argument to M (and using $sM = 0$), we verify the claim of section 2.4.1 that M generates extensions of both standard and extended backgrounds.

We finally note that the argument [27] which shows that $H^{p+2,0}$ is unique in standard superspace implies the same for $H^{0,p+2}$. That is, the class containing $sB^{0,p+1}$ consists of one element. The components of $sB^{0,p+1}$ must also be proportional to those of H since the construction of a nontrivial representative in $H^{0,p+2}$ is the same mathematical task as the construction of a nontrivial representative in $H^{p+2,0}$.

2.5 Application to the GS superstring

To illustrate the above formalism we consider the case of the GS superstring. In this case, existence of the relevant Fierz identity restricts the superspace dimension to d=(3, 4, 6, 10) [62]. After presenting the action, the modified algebras of the left/right generators are found. The effect of the cocycle gauge transformations is then investigated.

2.5.1 Superstring actions

We wish to study the effects that the following may have upon the results:

- Extending the background superspace (in order to allow manifestly symmetric WZ terms to be used).
- Changing the WZ term.

For this purpose we use an action that has free parameters ("switches"). The action can be used in the standard superspace background and also on the two extended ones of section 2.2. The switches allow one of three WZ terms to be used, or alternatively no WZ term at all. The action is

$$\begin{aligned} S_{k,s,\bar{s}} \;=\; & -\int d^2\sigma\sqrt{-g}\,\Big[1 - \frac{k}{2}\epsilon^{ij}\Big(\bar{\theta}\Gamma_i\partial_j\theta - s\Big[1-\frac{\bar{s}}{2}\Big]\partial_i\theta^\mu\partial_j\phi_\mu \\ & -s\bar{s}\partial_i y_n \partial_j x^n\Big)\Big]. \end{aligned} \tag{2.102}$$

The switches k, s and $\overline{s}$ are restricted to the following values:

$k = \{-1, 0, 1\}$	controls the existence and sign of the WZ term.
$s = \{0, 1\}$	switches on a manifestly invariant WZ term.
$\overline{s} = \{0, 1\}$	controls the type of invariant WZ term.

$k = 0$ gives the NG action. For $k \neq 0$ we have three possibilities.

- $s = 0$ gives the standard WZ term on standard superspace. This results in the standard κ symmetric GS superstring action [52]. The corresponding Lagrangian is only left invariant up to a total derivative.

- $(s, \overline{s}) = (1, 0)$ gives a manifestly left invariant WZ term that exists on the superspace of the Green algebra [32, 33]. The resulting action can be brought to the form

$$S_{k,1,0} = -\int d^2\sigma\sqrt{-g}\Big[1 + \frac{k}{2}\epsilon^{ij}(L_i{}^{\alpha} L_{j\alpha})\Big], \tag{2.103}$$

  showing clearly the manifest left invariance. The WZ 2-form in this case is

$$B = \frac{k}{2} L^{\alpha} L_{\alpha}. \tag{2.104}$$

- $(s, \overline{s}) = (1, 1)$ gives another manifestly left invariant WZ term that exists on the superspace of the extended Green algebra [34]. In this case:

$$B = -\frac{k}{2} L^a L_a + \frac{k}{4} L^{\alpha} L_{\alpha}. \tag{2.105}$$

2.5.2 Constraint and right generator algebras

The action (2.102) yields the bosonic momentum

$$P_m = -(-g)^{\frac{1}{2}} g^{0i} L_i{}^a \eta_{am} - \frac{k}{2}\overline{\theta}\Gamma_m \partial_1 \theta. \tag{2.106}$$

The momenta other than P_m can be written in terms of P_m and Z^M. These equations are then written in the form of constraints on phase space[13]:

$$C_\mu = P_\mu + \frac{1}{2}(\Gamma^m\theta)_\mu P_m + \frac{k}{2} L_1^a (\Gamma_a\theta)_\mu + \frac{k}{4} s\overline{s}(\Gamma^n\theta)_\mu \partial_1 y_n \tag{2.107}$$

[13]These are the "modified" constraints of the general section. We have dropped the tilde since we are no longer considering the NG and GS actions separately.

$$
\begin{aligned}
& -\frac{sk}{2}\left[1-\frac{\overline{s}}{2}\right]\partial_1\phi_\mu \\
C^m &= P^m - \frac{k}{2}s\overline{s}\partial_1 x^m \\
C^\mu &= P^\mu - \frac{sk}{2}\left[1-\frac{\overline{s}}{2}\right]\partial_1\theta^\mu.
\end{aligned}
$$

The 1-form $\overline{B}$ is found to be

$$
\begin{aligned}
\overline{B}_m &= -\frac{k}{2}\overline{\theta}\Gamma_m\partial_1\theta - \frac{k}{2}s\overline{s}\partial_1 y_m \\
\overline{B}_\mu &= \frac{ks}{2}\left[1-\frac{\overline{s}}{2}\right]\partial_1\phi_\mu - \frac{k}{2}\partial_1 x^m(\Gamma_m\theta)_\mu \\
\overline{B}^m &= \frac{k}{2}s\overline{s}\partial_1 x^m \\
\overline{B}^\mu &= \frac{ks}{2}\left[1-\frac{\overline{s}}{2}\right]\partial_1\theta^\mu.
\end{aligned}
\tag{2.108}
$$

In standard superspace, C_μ coincide with the right generators for the phase space, but for the extended algebras it is the linear combinations of section 2.4.2 that generate the right action. These are

$$
C_A = L_A{}^M P_M - \overline{B}_A, \tag{2.109}
$$

where $\overline{B}_A = L_A{}^M\overline{B}_M$.

For $p = 1$, the Fierz identity is

$$
\Gamma^a{}_{(\alpha\beta}\Gamma_{a\delta)\epsilon} = 0. \tag{2.110}
$$

Using this, the Poisson bracket algebra of the constraints is found to be

$$
\begin{aligned}
\{C_\alpha(\sigma), C_\beta(\sigma')\} &= \delta(\overrightarrow{\sigma} - \overrightarrow{\sigma}')(\Gamma^a{}_{\alpha\beta}\widetilde{\overline{D}}_a + k\Gamma_{a\alpha\beta}L_1{}^a)(\sigma) \\
[C^a(\sigma), C_\beta(\sigma')] &= -\delta(\overrightarrow{\sigma} - \overrightarrow{\sigma}')\Gamma^a{}_{\beta\gamma}C^\gamma(\sigma),
\end{aligned}
\tag{2.111}
$$

with all other brackets vanishing. The second bracket is an example illustrating the fact that although the modification $H^a{}_\beta$ vanishes, the associated constraint bracket is nonzero for the extended Green algebra because the minimal algebra has a noncentral generator Σ^a. Note that the constraint C^a does not exist on standard or Green superspaces, and in these cases only the first bracket is present.

The algebra of right generators is slightly more general than (2.111) since there is a generator D_a that is not reflected as a constraint. Using the bar map (2.60) and the components of H:

$$
H_{c\beta\alpha} = k\Gamma_{c\beta\alpha} \tag{2.112}
$$

we obtain

$$\begin{aligned} \overline{H}_{\alpha\beta} &= -k\Gamma_{a\alpha\beta}L_1{}^a \\ \overline{H}_{a\beta} &= k(\Gamma_a\partial_1\theta)_\beta \end{aligned} \tag{2.113}$$

as the only nonzero components of the modification. The first of these is seen to agree with the first bracket of (2.111). The second is not present in the constraint case.

2.5.3 Left generator algebra

Standard superspace action

Let us find a representative of the anomalous term by solving the descent equations. First, using the Fierz identity one finds for the variation of the WZ form:

$$\begin{aligned} Q_\alpha B &= -\frac{k}{2}L^b(\Gamma_b d\theta)_\alpha \\ &= -\frac{k}{2}d\Big[(dx^b - \frac{1}{6}d\bar{\theta}\Gamma^b\theta)(\Gamma_b\theta)_\alpha\Big]. \end{aligned} \tag{2.114}$$

The bosonic symmetries are manifest (i.e. $Q_a B = 0$). Thus,

$$W = \frac{k}{2}e^\alpha(dx^b - \frac{1}{6}d\bar{\theta}\Gamma^b\theta)(\Gamma_b\theta)_\alpha \tag{2.115}$$

is a solution for the potential W. Evaluating $M = sW$ and using the Fierz identity we find that all θ dependence is lost:

$$M_{\alpha\beta} = kdx^m\Gamma_{m\alpha\beta}, \tag{2.116}$$

with all other components vanishing. Using the map (2.61) we then find $\overline{M}$

$$\overline{M}_{\alpha\beta} = -k\int d\sigma^1\partial_1 x^m\Gamma_{m\alpha\beta}. \tag{2.117}$$

This integral can be nonzero whenever the spatial section has nontrivial topology in the bosonic sector. It is equivalent to the previously known result [28] except that we have not needed to assume trivial fermionic topology. One of the new points is that (2.114) determines W only up to a gauge transformation (which we have called λ). The resulting anomalous term M is not gauge invariant under such transformations. In fact, we now show that if fermionic topology is trivial then the topological anomalous term (2.117) is gauge equivalent to zero.

The following gauge field satisfies the conditions of Lorentz invariance and dimensionality $p+1=2$:

$$\lambda = -ke^a x^b \eta_{ab}. \tag{2.118}$$

Let us find its effect upon the solutions (2.115) and (2.116) for W and M. Firstly:

$$\begin{aligned} \Delta W &= d\lambda \\ &= -ke^a dx^b \eta_{ab}. \end{aligned} \tag{2.119}$$

Using $\Delta M = sd\lambda$ we then find

$$\begin{aligned} \Delta M_{\alpha\beta} &= -kdx^m \Gamma_{m\alpha\beta} \\ \Delta M_{a\beta} &= -\frac{k}{2}(\Gamma_a d\theta)_\beta \\ \Delta M_{ab} &= 0. \end{aligned} \tag{2.120}$$

We see that $\Delta M_{\alpha\beta}$ is closed but not exact whenever dx^m is. Now, the de Rham nontriviality of dx^m is the condition for which the original representative (2.117) is nonzero. Therefore, in this case the gauge transformation $\Delta\overline{M}$ is nonzero whenever $\overline{M}$ itself is.

After the gauge transformation, the alternative representative M' is

$$M'_{a\beta} = -\frac{k}{2}(\Gamma_a d\theta)_\beta. \tag{2.121}$$

We have thus traded nonzero $M_{\alpha\beta}$ for nonzero $M_{a\beta}$. However, when converted to the topological anomalous term this becomes a topological θ integral of the type (2.74)

$$\overline{M}'_{a\beta} = \frac{k}{2}\int d\sigma^1 (\Gamma_a \partial_1 \theta)_\beta. \tag{2.122}$$

Therefore, even if the standard quasi-invariant Lagrangian is used, when fermionic topology is trivial, the topological charge algebra is gauge equivalent to the minimal algebra.

The case of main interest occurs when nontrivial fermionic topology is formally allowed so that the integral (2.122) is nonzero. Let us repeat the above procedure using instead the associated one parameter family of gauge transformations parameterized by a constant a:

$$\lambda = -ake^a x^b \eta_{ab}. \tag{2.123}$$

First we show that this is in fact the most general gauge transformation. There are two more possibilities for λ with the correct Lorentz and dimensionality properties. The first is

$$\lambda' = -\frac{ak}{2}x^a\bar{e}\Gamma_a\theta. \tag{2.124}$$

Defining $\Delta' W = d\lambda'$, one can verify that although ΔW differs from $\Delta' W$, the algebra modifications ΔM and $\Delta' M$ are the same. In the context of this chapter it is the algebra itself that is important, not any particular representation of its generators. The transformation (2.124) is therefore equivalent to (2.123). The only other possibilities appear to be gauge fields of the form

$$\lambda'' = \bar{e}\Gamma^{a_1\ldots a_b}\theta\bar{\theta}\Gamma_{a_1\ldots a_b}\theta, \tag{2.125}$$

where b is such that $\Gamma_{a_1\ldots a_b\alpha\beta}$ is antisymmetric. These transformations leave M invariant, and are hence redundant. We may therefore take (2.123) as the most general transformation. Applying this to the representative (2.117) one finds the equivalence class $[\overline{M}]$ of topological anomalous terms, with representatives parameterized by the gauge parameter a:

$$\begin{aligned}
[\overline{M}]_{\alpha\beta} &= -(1-a)k\Gamma_{m\alpha\beta}\int d\sigma^1\partial_1 x^m \\
[\overline{M}]_{a\beta} &= \frac{ak}{2}\int d\sigma^1(\Gamma_a\partial_1\theta)_\beta.
\end{aligned} \tag{2.126}$$

Now define the topological charges as new generators:

$$\begin{aligned}
\overline{\Sigma}^a &= \frac{k}{2}\int d\sigma^1\partial_1 x^a \\
\overline{\Sigma}^\gamma &= \frac{k}{2}\int d\sigma^1\partial_1\theta^\gamma
\end{aligned} \tag{2.127}$$

Note that $\overline{\Sigma}^a$ and $\overline{\Sigma}^\gamma$ are nonzero only when the associated superspace dimension is compact and the spatial section of the string wraps around it. Upon adding these to the set of conserved charges

$$\begin{aligned}
\overline{\widetilde{Q}}_\alpha &= R_\alpha{}^M P_M - \frac{k}{2}\int d\sigma^1(\partial_1 x^m - \frac{1}{6}\partial_1\bar{\theta}\Gamma^m\theta)(\Gamma_m\theta)_\alpha \\
\overline{\widetilde{P}}_a &= R_a{}^M P_M + ak\int d\sigma^1\partial_1 x^m\eta_{ma},
\end{aligned} \tag{2.128}$$

we then obtain the following algebra under Poisson bracket:

$$\begin{aligned}
\{\overline{\widetilde{Q}}_\alpha, \overline{\widetilde{Q}}_\beta\} &= -\Gamma^b{}_{\alpha\beta}\overline{\widetilde{P}}_b - 2(1-a)\Gamma_{b\alpha\beta}\overline{\Sigma}^b \\
[\overline{\widetilde{Q}}_\alpha, \overline{\widetilde{P}}_b] &= -a\Gamma_{b\alpha\gamma}\overline{\Sigma}^\gamma \\
[\overline{\widetilde{Q}}_\alpha, \overline{\Sigma}^b] &= -\frac{1}{2}\Gamma^b{}_{\alpha\gamma}\overline{\Sigma}^\gamma.
\end{aligned} \tag{2.129}$$

We will check that the Jacobi identity is satisfied. The only nontrivial possibility is

$$[\widetilde{\overline{Q}}_\alpha, \{\widetilde{\overline{Q}}_\beta, \widetilde{\overline{Q}}_\gamma\}] + \mathsf{cycles} = 3\Gamma^a{}_{(\alpha\beta}\Gamma_{a\gamma)\delta}\overline{\Sigma}^\delta, \tag{2.130}$$

which vanishes by the Fierz identity. We note three special cases:

- For $a = 1$ the extra generator $\overline{\Sigma}^a$ is redundant and may be excluded since it appears nowhere on the RHS of a bracket. We then recover the Green algebra[14].

- For $a = \frac{1}{2}$ we rescale $\overline{\Sigma}^\alpha$ with a factor of $\frac{1}{2}$ and recover the extended Green algebra.

- Turning off the gauge transformation altogether results in a variant in which $\widetilde{\overline{P}}_a$ is central. The structure of this algebra is of the type considered in [42]:

$$\begin{aligned} \{Q, Q\} &\sim P + P' \\ [Q, P'] &\sim \Sigma. \end{aligned} \tag{2.131}$$

An important point is that (2.129) cannot be obtained by simply rescaling the known algebras. It is therefore a generalization which yields new superalgebras.

Thus, the outcome of the construction is a family of superalgebras parameterized by a free constant. The anomalous term of the Noether charge algebra contains a gauge freedom, and the free constant of the algebra is the part of this freedom which is consistent with Lorentz invariance and dimensionality requirements. The algebras were constructed by identifying topological charges with extended superalgebra generators. One can then decompose the ideal arising from the topological anomalous term. The resulting family of algebras contains the three superalgebra extensions that have so far been associated with the string. We emphasize two new developments:

- Since representatives of the anomalous term are not gauge invariant, it is well defined only as an entire cohomology class. A free constant parameterizes the class.

- The result has been obtained by retaining the fermionic topological terms in conserved charge calculations.

[14]As in (2.63), a negative sign relative to the underlying superalgebra is expected.

Extended superspace actions

The motivation behind using an extended background superspace is to enable a manifestly left invariant WZ term to be used. The left invariant WZ form is generated by a ψ gauge transformation on the standard WZ form:

$\overline{s} = 0$:

$$\begin{aligned} \Delta B &= -d\psi \\ &= \frac{k}{2} d\theta^\mu d\phi_\mu . \end{aligned} \tag{2.132}$$

$\overline{s} = 1$:

$$\begin{aligned} \Delta B &= -d\psi \\ &= -\frac{k}{2} dx^m dy_m + \frac{k}{4} d\theta^\mu d\phi_\mu . \end{aligned} \tag{2.133}$$

The manifest left invariance of the $s = 1$ action allows us to choose vanishing components for W. $M = 0$ is then a representative of the anomalous term. As expected, M is therefore D trivial for the extended superspace actions as a result of manifest left invariance.

When using extended superspaces it is just as valid to use the standard WZ term as any other one (since they are gauge equivalent). Let us therefore consider using the standard action on an extended superspace. In this case the extra available coordinates still trivialize the anomalous term. For example, in the case of the Green algebra one can modify W from (2.115) using

$$\begin{aligned} \Delta W &= d\lambda \\ &= -\frac{k}{2} e^\alpha d\phi_\alpha . \end{aligned} \tag{2.134}$$

This completes W into an s closed form:

$$W = -\frac{k}{2} e^\alpha L_\alpha . \tag{2.135}$$

Therefore $M = 0$ in this gauge (even though W is non-zero). Thus, even when a quasi-invariant WZ term is used, the D cocycle is still trivialized by extending the superspace appropriately. This is consistent with the general observation made in section 2.4.3 that changing the WZ term does not affect the anomalous term; only changing the background will have an effect.

In the extended superspace case there are many more possibilities for the λ gauge transformations since one can form new λ fields using the extra coordinates. One might further extend the extended supertranslation algebras in this way. However, the number of possibilities for λ is considerable and the algebras obtained can be large. Since there is currently no direct physical application of such algebras (unlike the extensions of *standard* superspace considered in this chapter) we will not pursue this possibility here.

2.5.4 Cocycle trivialization

Here we will show that the whole family of topological charge algebras (2.129) associated with the standard GS superstring action can be used to construct extended superspace actions. First define two new vielbeins: L_α associated with Σ^α, and L_a associated with Σ^a. The Maurer-Cartan equations associated with (2.129) are then

$$\begin{aligned} dL^\beta &= 0 \\ dL^b &= -\frac{1}{2}d\bar{\theta}\Gamma^b d\theta \\ dL_\beta &= -aL^b(\Gamma_b d\theta)_\beta - \frac{1}{2}L_b(\Gamma^b d\theta)_\beta \\ dL_b &= -(1-a)d\bar{\theta}\Gamma_b d\theta. \end{aligned} \tag{2.136}$$

The WZ 2-form B can be constructed by forming all possible Lorentz invariant products of the vielbeins into 2-forms of dimension two. The requirement

$$\begin{aligned} dB &= H \\ &= \frac{k}{2}L^a d\bar{\theta}\Gamma_a d\theta \end{aligned} \tag{2.137}$$

then gives a set of equations that can be used to equate coefficients. We have two cases to consider:

- **Σ^a exists (general case)**
  One finds the solution
  $$B = -\frac{k}{2}L^a L_a + \frac{k}{2}L^\alpha L_\alpha. \tag{2.138}$$
  The coefficients are thus independent of the family parameter of the algebra.

- **Σ^a does not exist ($a = 1$ only)**
  In this case the solution is
  $$B = \frac{k}{2}L^\alpha L_\alpha. \tag{2.139}$$

Thus, all the superspaces associated with (2.129) can be used to construct manifestly left invariant superstring actions.

2.6 Comments

It was found that the entire family (2.129) of topological charge algebras of the GS superstring can be used to construct manifestly left invariant string actions. In the general p-brane case, applying the cocycle approach to the standard action also results in superalgebras which allow the construction of left invariant WZ forms. This is the focus of the next chapter.

We have restricted our attention to p-branes only for brevity. Similar principles to those of the p-brane WZ term also apply to the WZ terms of D-branes and M-branes [34]. The additional feature of these branes is the presence of worldvolume gauge fields. With modifications to allow for these fields, the cocycle construction can also be applied to these branes. Noether charge algebras resulting from D-brane and M-brane actions were derived in [57, 31, 65]. However, only the bosonic topological terms were retained in the anomalous term calculations. Work on D-brane charge algebras using the methods of this chapter is the focus of chapter four.

The "central" extension structure of the anomalous term arises in a particular choice of gauge (and when fermionic topological terms are discarded). This bosonic anomalous term relates to the PBRS construction, where the modified algebra is written in the form of a projector [46]. This represents the situation in supergravity field theory where half the supersymmetries are broken. The work of this chapter shows that allowing for λ gauge freedom results in an expanded definition of the anomalous term. It would be interesting to revisit the PBRS construction to determine whether the new possibilities for the anomalous term can be incorporated. Ideally one would like to find a generalization of PBRS which is λ covariant.

Chapter 3

Representations of p-brane topological charge algebras

3.1 Introduction

It is often the case that results for GS superstrings find analogs in their higher dimensional counterparts, the p-branes. As in the case of the superstring, extended supertranslation algebras associated with p-branes traditionally occur in two contexts. First, they can be derived as Noether charge algebras of the p-brane action. When only the bosonic topological terms are retained, this results in the well known "central" extensions of standard superspace [28]. Secondly, superalgebras with fermionic extensions have been constructed which allow the construction of manifestly super-Poincaré invariant actions [32, 33, 34]. In [49] we established that when fermionic topological terms are retained, in the case of the GS superstring, the second of these results from the first. Specifically, the family of topological charge algebras resulting from the standard Green-Schwarz superstring action allows the construction of manifestly super-Poincaré invariant superstring actions.

The main purpose of this chapter is to show that this correspondence continues for p-branes with $p \geq 2$. Since the results of [28] exclude not only fermionic charges, but also the fermionic corrections to the bosonic charges, the generalization where all terms are retained is required (the simplifications associated with trivial fermionic topology may be deduced at the end). We find this generalization, not by the descent method but by using uniqueness of the anomalous term. The charges are shown to be representations of the ideals of the extended algebras of [33, 34]. It follows that these extended algebras are indeed generated as topological charge algebras of the standard p-brane action. It emerges along the way that the topological charges uniquely satisfy the extended algebra, and also that the charges are the same in the standard and extended formulations of the action.

3.2 p-brane topological charge algebras

The family of topological charge algebras (2.129) of the superstring was found by solving descent equations. For higher values of p this approach becomes lengthy. In this chapter we will make use of the uniqueness of the anomalous term instead. We wish to find a Lorentz invariant, D cohomology nontrivial element:

$$M^{(p)} \in H^{p,2} \tag{3.1}$$

of dimensionality $p+1$, for each allowed value of p. By uniqueness of the class, this must then be a representative of the p-brane anomalous term. If required, the full class $[M]$ can be generated by applying the λ gauge transformations to this representative. There is no a priori obvious way to find $M^{(p)}$. However, we are motivated by the observation that the family of topological charge algebras of the string action (2.129) consisted of extended superalgebras that allow left invariant WZ forms to be constructed for the string action. This family contained three different types of algebra (when classified according to the generators present). Two of these algebras had been previously used to construct invariant actions: the Green algebra [48] used in [32], and also a four generator extension [33, 34]. An algebra which allows a left invariant WZ form to be constructed for each p-brane of higher dimension is also already known. The cases $p = 2, 3$ were given in [33]. In [34], an ansatz was presented to generate Maurer-Cartan equations for the required algebra for general values of p; however the minimal branescan dictates that p-branes exist only for $p \leq 5$ [26, 62].

In this chapter, the approach we will take to find $M^{(p)}$ somewhat reverses the process used in our paper [49]. We begin with the known extended algebra associated with a given value of p. We assume that this extended algebra is contained in the family of topological charge algebras generated by the standard superspace p-brane action. If this assumption is correct, the extended algebra must have a representation in which the generators of the ideal are represented by closed superspace forms. We will explicitly find these forms. A particular $(p, 2)$-form $M^{(p)}$ constructed from them will then be shown to be a representative of the anomalous term associated with the standard superspace p-brane action.

For reference, let us give the known extended algebras that allow left invariant WZ terms to be constructed. The algebras will be given in the operator-form convention for which we seek the representation (generators are negatives of those in the corresponding superalgebra underlying the extended superspace action).

3.2.1 $p=1$ superalgebra

[33, 34]:

$$\begin{aligned}
\left\{\widetilde{Q}_\alpha, \widetilde{Q}_\beta\right\} &= -\Gamma^a{}_{\alpha\beta}\widetilde{P}_a - \Gamma_{a\alpha\beta}\Sigma^a \\
\left[\widetilde{Q}_\alpha, \widetilde{P}_a\right] &= -\Gamma_{a\alpha\beta}\Sigma^\beta \\
\left[\widetilde{Q}_\alpha, \Sigma^a\right] &= -\Gamma^a{}_{\alpha\beta}\Sigma^\beta.
\end{aligned} \tag{3.2}$$

3.2.2 $p=2$ superalgebra

[33]:

$$\begin{aligned}
\left\{\widetilde{Q}_\alpha, \widetilde{Q}_\beta\right\} &= -\Gamma^a{}_{\alpha\beta}\widetilde{P}_a - \Gamma_{ab\alpha\beta}\Sigma^{ab} \\
\left[\widetilde{Q}_\alpha, \widetilde{P}_a\right] &= -\Gamma_{ab\alpha\beta}\Sigma^{b\beta} \\
\left[\widetilde{P}_a, \widetilde{P}_b\right] &= -\Gamma_{ab\alpha\beta}\Sigma^{\alpha\beta} \\
\left[\widetilde{Q}_\alpha, \Sigma^{ab}\right] &= -\Gamma^{[a}{}_{\alpha\beta}\Sigma^{b]\beta} \\
\left[\widetilde{P}_a, \Sigma^{bc}\right] &= -\frac{1}{2}\delta_a^{[b}\Gamma^{c]}{}_{\alpha\beta}\Sigma^{\alpha\beta} \\
\left\{\widetilde{Q}_\alpha, \Sigma^{a\beta}\right\} &= -\frac{1}{4}\Gamma^a{}_{\gamma\delta}\Sigma^{\gamma\delta}\delta_\alpha^\beta - 2\Gamma^a{}_{\alpha\gamma}\Sigma^{\gamma\beta}.
\end{aligned} \tag{3.3}$$

3.2.3 $p=3$ superalgebra

[33]:

$$\begin{aligned}
\left\{\widetilde{Q}_\alpha, \widetilde{Q}_\beta\right\} &= -\Gamma^a{}_{\alpha\beta}\widetilde{P}_a - \Gamma_{abc\alpha\beta}\Sigma^{abc} \\
\left[\widetilde{Q}_\alpha, \widetilde{P}_a\right] &= -\Gamma_{abc\alpha\beta}\Sigma^{bc\beta} \\
\left[\widetilde{P}_a, \widetilde{P}_b\right] &= -\Gamma_{abc\alpha\beta}\Sigma^{c\alpha\beta} \\
\left[\widetilde{Q}_\alpha, \Sigma^{abc}\right] &= -\Gamma^{[a}{}_{\alpha\beta}\Sigma^{bc]\beta} \\
\left[\widetilde{P}_a, \Sigma^{bcd}\right] &= -\frac{1}{2}\delta_a^{[b}\Gamma^c{}_{\alpha\beta}\Sigma^{d]\alpha\beta}
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
\left\{\widetilde{Q}_\alpha, \Sigma^{ab\beta}\right\} &= -\frac{1}{4}\Gamma^{[a}{}_{\gamma\delta}\Sigma^{b]\gamma\delta}\delta^\beta_\alpha - 2\Gamma^{[a}{}_{\alpha\gamma}\Sigma^{b]\gamma\beta} \\
\left[\widetilde{P}_a, \Sigma^{bc\alpha}\right] &= -\delta^{[b}_a\Gamma^{c]}{}_{\beta\gamma}\Sigma^{\beta\gamma\alpha} \\
\left[\widetilde{Q}_\alpha, \Sigma^{a\beta\gamma}\right] &= -\frac{1}{2}\Gamma^a{}_{\delta\epsilon}\Sigma^{\delta\epsilon(\beta}\delta^{\gamma)}_\alpha - \frac{5}{2}\Gamma^a{}_{\alpha\delta}\Sigma^{\delta\beta\gamma}.
\end{aligned}
$$

3.2.4 $p = 4$ superalgebra

Derived from an ansatz for Maurer-Cartan equations in [34]:

$$
\begin{aligned}
\left\{\widetilde{Q}_\alpha, \widetilde{Q}_\beta\right\} &= -\Gamma^a{}_{\alpha\beta}\widetilde{P}_a - \Gamma_{abcd\alpha\beta}\Sigma^{abcd} \qquad (3.5)\\
\left[\widetilde{Q}_\alpha, \widetilde{P}_a\right] &= -\Gamma_{abcd\alpha\beta}\Sigma^{bcd\beta} \\
\left[\widetilde{P}_a, \widetilde{P}_b\right] &= -\Gamma_{abcd\alpha\beta}\Sigma^{cd\alpha\beta} \\
\left[\widetilde{Q}_\alpha, \Sigma^{abcd}\right] &= -\Gamma^{[a}{}_{\alpha\beta}\Sigma^{bcd]\beta} \\
\left[\widetilde{P}_a, \Sigma^{bcde}\right] &= -\frac{1}{2}\delta^{[b}_a\Gamma^c{}_{\alpha\beta}\Sigma^{de]\alpha\beta} \\
\left\{\widetilde{Q}_\alpha, \Sigma^{abc\beta}\right\} &= -\frac{1}{4}\Gamma^{[a}{}_{\gamma\delta}\Sigma^{bc]\gamma\delta}\delta^\beta_\alpha - 2\Gamma^{[a}{}_{\alpha\gamma}\Sigma^{bc]\gamma\beta} \\
\left[\widetilde{P}_a, \Sigma^{bcd\alpha}\right] &= -\delta^{[b}_a\Gamma^c{}_{\beta\gamma}\Sigma^{d]\beta\gamma\alpha} \\
\left[\widetilde{Q}_\alpha, \Sigma^{ab\beta\gamma}\right] &= -\frac{1}{2}\Gamma^{[a}{}_{\delta\epsilon}\Sigma^{b]\delta\epsilon(\beta}\delta^{\gamma)}_\alpha - \frac{5}{2}\Gamma^{[a}{}_{\alpha\delta}\Sigma^{b]\delta\beta\gamma} \\
\left[\widetilde{P}_a, \Sigma^{bc\alpha\beta}\right] &= -\delta^{[b}_a\Gamma^{c]}{}_{\gamma\delta}\Sigma^{\gamma\delta\alpha\beta} \\
\left\{\widetilde{Q}_\alpha, \Sigma^{a\beta\gamma\delta}\right\} &= -\frac{3}{5}\Gamma^a{}_{\epsilon\sigma}\Sigma^{\epsilon\sigma(\beta\gamma}\delta^{\delta)}_\alpha - \frac{12}{5}\Gamma^a{}_{\alpha\epsilon}\Sigma^{\epsilon\beta\gamma\delta}.
\end{aligned}
$$

3.2.5 $p = 5$ superalgebra

Derived from an ansatz for Maurer-Cartan equations in [34]:

$$
\begin{aligned}
\left\{\widetilde{Q}_\alpha, \widetilde{Q}_\beta\right\} &= -\Gamma^a{}_{\alpha\beta}\widetilde{P}_a - \Gamma_{abcde\alpha\beta}\Sigma^{abcde} \qquad (3.6)\\
\left[\widetilde{Q}_\alpha, \widetilde{P}_a\right] &= -\Gamma_{abcde\alpha\beta}\Sigma^{bcde\beta} \\
\left[\widetilde{P}_a, \widetilde{P}_b\right] &= -\Gamma_{abcde\alpha\beta}\Sigma^{cde\alpha\beta} \\
\left[\widetilde{Q}_\alpha, \Sigma^{abcde}\right] &= -\Gamma^{[a}{}_{\alpha\beta}\Sigma^{bcde]\beta} \\
\left[\widetilde{P}_a, \Sigma^{bcdef}\right] &= -\frac{1}{2}\delta^{[b}_a\Gamma^c{}_{\alpha\beta}\Sigma^{def]\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
\left\{\widetilde{Q}_\alpha, \Sigma^{abcd\beta}\right\} &= -\frac{1}{4}\Gamma^{[a}{}_{\gamma\delta}\Sigma^{bcd]\gamma\delta}\delta^\beta_\alpha - 2\Gamma^{[a}{}_{\alpha\gamma}\Sigma^{bcd]\gamma\beta} \\
\left[\widetilde{P}_a, \Sigma^{bcde\alpha}\right] &= -\delta^{[b}_a\Gamma^c{}_{\beta\gamma}\Sigma^{de]\beta\gamma\alpha} \\
\left[\widetilde{Q}_\alpha, \Sigma^{abc\beta\gamma}\right] &= -\frac{1}{2}\Gamma^{[a}{}_{\delta\epsilon}\Sigma^{bc]\delta\epsilon(\beta}\delta^{\gamma)}_\alpha - \frac{5}{2}\Gamma^{[a}{}_{\alpha\delta}\Sigma^{bc]\delta\beta\gamma} \\
\left[\widetilde{P}_a, \Sigma^{bcd\alpha\beta}\right] &= -\delta^{[b}_a\Gamma^c{}_{\gamma\delta}\Sigma^{d]\gamma\delta\alpha\beta} \\
\left\{\widetilde{Q}_\alpha, \Sigma^{ab\beta\gamma\delta}\right\} &= -\frac{3}{5}\Gamma^{[a}{}_{\epsilon\sigma}\Sigma^{b]\epsilon\sigma(\beta\gamma}\delta^{\delta)}_\alpha - \frac{12}{5}\Gamma^{[a}{}_{\alpha\epsilon}\Sigma^{b]\epsilon\beta\gamma\delta} \\
\left[\widetilde{P}_a, \Sigma^{bc\alpha\beta\gamma}\right] &= -\delta^{[b}_a\Gamma^{c]}{}_{\delta\epsilon}\Sigma^{\delta\epsilon\alpha\beta\gamma} \\
\left[\widetilde{Q}_\alpha, \Sigma^{a\beta\gamma\delta\epsilon}\right] &= -\frac{5}{6}\Gamma^a{}_{\sigma\rho}\Sigma^{\sigma\rho(\beta\gamma\delta}\delta^{\epsilon)}_\alpha - \frac{35}{12}\Gamma^a{}_{\alpha\sigma}\Sigma^{\sigma\beta\gamma\delta\epsilon}.
\end{aligned}
$$

Again we choose to work with the forms and left generators on the RHS of the isomorphism (2.76) instead of the Noether charges themselves. Thus, we wish to find closed forms $\Sigma^{A_1\ldots A_p}$ satisfying these algebras under the action of the modified left generators (2.77). If we can, then each extended algebra can be interpreted as the minimal algebra modified by an anomalous term $M^{(p)}$. These are the operator-form representations of the modified Noether charge algebra (2.70):

$$
\left[\widetilde{Q}_A, \widetilde{Q}_B\right\} = -t_{AB}{}^C\widetilde{Q}_C + M^{(p)}{}_{AB}. \tag{3.7}
$$

The components $M^{(p)}{}_{AB}$ are read as the modifications to the $[Q_A, Q_B\}$ brackets of the minimal algebra. For example, since the extended algebras all have

$$
\left\{\widetilde{Q}_\alpha, \widetilde{Q}_\beta\right\} = -\Gamma^a{}_{\alpha\beta}\widetilde{P}_a - \Gamma_{a_1\ldots a_p\alpha\beta}\Sigma^{a_1\ldots a_p} \tag{3.8}
$$

we require that

$$
M^{(p)}{}_{\alpha\beta} = -\Gamma_{a_1\ldots a_p\alpha\beta}\Sigma^{a_1\ldots a_p}. \tag{3.9}
$$

Reading similarly from the RHS of $\left[\widetilde{Q}_\alpha, \widetilde{P}_b\right]$ and $\left[\widetilde{P}_a, \widetilde{P}_b\right]$, and introducing once again the ghost fields (2.30), it follows that $M^{(p)}$ has the structure

- $p = 1$

$$
\begin{aligned}
M^{(1)} &= -\frac{1}{2}e^\beta e^\alpha\Gamma_{a\alpha\beta}\Sigma^a \\
&\quad -e^a e^\alpha\Gamma_{a\alpha\beta}\Sigma^\beta.
\end{aligned} \tag{3.10}
$$

- $p \geq 2$

$$
\begin{aligned}
M^{(p)} &= -\frac{1}{2}e^{\beta}e^{\alpha}\Gamma_{a_1\ldots a_p\alpha\beta}\Sigma^{a_1\ldots a_p} \\
&\quad -e^{a}e^{\alpha}\Gamma_{aa_1\ldots a_{p-1}\alpha\beta}\Sigma^{a_1\ldots a_{p-1}\beta} \\
&\quad -\frac{1}{2}e^{b}e^{a}\Gamma_{aba_1\ldots a_{p-2}\alpha\beta}\Sigma^{a_1\ldots a_{p-2}\alpha\beta}.
\end{aligned}
\tag{3.11}
$$

To find the required closed forms $\Sigma^{A_1\ldots A_p}$, one firstly observes that

$$
[\widetilde{Q}_A, \Sigma^{A_1\ldots A_p}\} = [Q_A, \Sigma^{A_1\ldots A_p}\}. \tag{3.12}
$$

The unmodified left generators are thus sufficient for our purposes and the explicit form of $\widetilde{Q}_A$ is not required. Secondly, the $\Sigma^{A_1\ldots A_p}$ must all have their "natural" dimension:

$$
\dim\,[\Sigma^{a_1\ldots a_m\alpha_1\ldots\alpha_n}] = m + \frac{n}{2}. \tag{3.13}
$$

This follows from the requirement $\dim M^{(p)} = p+1$, and the fact that Q_A reduces the dimension of a form by the dimension associated with its index. One finally notes that the generator $\Sigma^{\alpha_1\ldots\alpha_p}$ is "central." There is only one candidate for $\Sigma^{\alpha_1\ldots\alpha_p}$ satisfying the required properties:

$$
\Sigma^{\alpha_1\ldots\alpha_p} \propto d\theta^{\alpha_1}\ldots d\theta^{\alpha_p}. \tag{3.14}
$$

We shall fix the proportionality constant at unity since it serves only as an overall scaling for the extra generators. To find the remaining generators, one can first write the most general allowed form for $\Sigma^{a\alpha_1\ldots\alpha_{p-1}}$ using arbitrary coefficients. The coefficients are then found by requiring that the extended superalgebra be satisfied. The process is then continued for $\Sigma^{ab\alpha_1\ldots\alpha_{p-2}}$ and so on until the final generator $\Sigma^{a_1\ldots a_p}$ is found. The relevant Fierz identity is required to find the solutions, and its implementation is often more nontrivial than usual due to double symmetrizations which overlap only partially. In general, one finds that the requirement of satisfying the extended superalgebra results in more equations than coefficients present. A solution for such a system is only possible if a sufficient number of the equations are redundant. In fact, exactly the right number of redundant equations are present in order that the solution be unique. That is, the representation for each algebra is *unique*. Having obtained the solution, the redundant equations then provide a good consistency check. The results are:

3.2.6 $p = 1$ representation

Also derived as Noether charges of the extended superspace action [34]:

$$\begin{aligned} \Sigma^\alpha &= d\theta^\alpha \\ \Sigma^a &= 2dx^a. \end{aligned} \tag{3.15}$$

3.2.7 $p = 2$ representation

Also derived as Noether charges of the extended superspace action [34]:

$$\begin{aligned} \Sigma^{\alpha\beta} &= d\Big[d\theta^\alpha\theta^\beta\Big] \\ \Sigma^{a\beta} &= d\Big[\frac{9}{2}dx^a\theta^\beta + \frac{1}{4}\overline{\theta}\Gamma^a d\theta\theta^\beta\Big] \\ \Sigma^{ab} &= d\Big[5x^a dx^b + \frac{1}{2}x^{[a}\overline{\theta}\Gamma^{b]}d\theta\Big]. \end{aligned} \tag{3.16}$$

3.2.8 $p = 3$ representation

$$\begin{aligned} \Sigma^{\alpha\beta\gamma} &= d\Big[d\theta^\alpha d\theta^\beta\theta^\gamma\Big] \\ \Sigma^{a\beta\gamma} &= d\Big[6dx^a d\theta^\beta\theta^\gamma + \frac{1}{2}\overline{\theta}\Gamma^a d\theta d\theta^{(\beta}\theta^{\gamma)}\Big] \\ \Sigma^{ab\beta} &= d\Big[-\frac{29}{2}dx^a dx^b\theta^\beta - \frac{3}{2}dx^{[a}\overline{\theta}\Gamma^{b]}d\theta\theta^\beta - x^{[a}\overline{\theta}\Gamma^{b]}d\theta d\theta^\beta \\ &\quad -\frac{1}{8}\overline{\theta}\Gamma^a d\theta\overline{\theta}\Gamma^b d\theta\theta^\beta\Big] \\ \Sigma^{abc} &= d\Big[-\frac{35}{3}x^a dx^b dx^c - 3x^{[a}dx^b\overline{\theta}\Gamma^{c]}d\theta - \frac{1}{4}x^{[a}\overline{\theta}\Gamma^b d\theta\overline{\theta}\Gamma^{c]}d\theta\Big]. \end{aligned} \tag{3.17}$$

3.2.9 $p = 4$ representation

$$\begin{aligned} \Sigma^{\alpha\beta\gamma\delta} &= d\Big[d\theta^\alpha d\theta^\beta d\theta^\gamma\theta^\delta\Big] \\ \Sigma^{a\beta\gamma\delta} &= d\Big[6dx^a d\theta^\beta d\theta^\gamma\theta^\delta + \frac{3}{5}\overline{\theta}\Gamma^a d\theta d\theta^{(\beta}d\theta^\gamma\theta^{\delta)}\Big] \\ \Sigma^{ab\beta\gamma} &= d\Big[-19dx^a dx^b d\theta^\beta\theta^\gamma - 3dx^{[a}\overline{\theta}\Gamma^{b]}d\theta d\theta^{(\beta}\theta^{\gamma)} + x^{[a}\overline{\theta}\Gamma^{b]}d\theta d\theta^\beta d\theta^\gamma \\ &\quad -\frac{1}{4}\overline{\theta}\Gamma^a d\theta\overline{\theta}\Gamma^b d\theta d\theta^{(\beta}\theta^{\gamma)}\Big] \end{aligned} \tag{3.18}$$

$$\Sigma^{abc\beta} = d\Big[-\frac{65}{2}dx^a dx^b dx^c \theta^\beta - \frac{19}{4}dx^{[a}dx^b\bar\theta\Gamma^{c]}d\theta\theta^\beta + 6x^{[a}dx^b\bar\theta\Gamma^{c]}d\theta d\theta^\beta$$
$$-\frac{7}{8}dx^{[a}\bar\theta\Gamma^b d\theta\bar\theta\Gamma^{c]}d\theta\theta^\beta + \frac{1}{2}x^{[a}\bar\theta\Gamma^b d\theta\bar\theta\Gamma^{c]}d\theta d\theta^\beta$$
$$-16\bar\theta\Gamma^a d\theta\bar\theta\Gamma^b d\theta\bar\theta\Gamma^c d\theta\theta^\beta\Big]$$
$$\Sigma^{abcd} = d\Big[-21x^a dx^b dx^c dx^d - \frac{19}{2}x^{[a}dx^b dx^c\bar\theta\Gamma^{d]}d\theta - \frac{7}{4}x^{[a}dx^b\bar\theta\Gamma^c d\theta\bar\theta\Gamma^{d]}d\theta$$
$$-\frac{1}{8}x^{[a}\bar\theta\Gamma^b d\theta\bar\theta\Gamma^c d\theta\bar\theta\Gamma^{d]}d\theta\Big].$$

3.2.10 $p = 5$ representation

$$\Sigma^{\alpha\beta\gamma\delta\epsilon} = d\Big[d\theta^\alpha d\theta^\beta d\theta^\gamma d\theta^\delta \theta^\epsilon\Big] \tag{3.19}$$
$$\Sigma^{a\beta\gamma\delta\epsilon} = d\Big[\frac{15}{2}dx^a d\theta^\beta d\theta^\gamma d\theta^\delta\theta^\epsilon + \frac{5}{6}\bar\theta\Gamma^a d\theta d\theta^{(\beta}d\theta^\gamma d\theta^\delta\theta^{\epsilon)}\Big]$$
$$\Sigma^{ab\beta\gamma\delta} = d\Big[-\frac{47}{2}dx^a dx^b d\theta^\beta d\theta^\gamma\theta^\delta - \frac{9}{2}dx^{[a}\bar\theta\Gamma^{b]}d\theta d\theta^{(\beta}d\theta^\gamma\theta^{\delta)}$$
$$-x^{[a}\bar\theta\Gamma^{b]}d\theta d\theta^\beta d\theta^\delta\theta^\delta - \frac{3}{8}\bar\theta\Gamma^a d\theta\bar\theta\Gamma^b d\theta d\theta^{(\beta}d\theta^\gamma\theta^{\delta)}\Big]$$
$$\Sigma^{abc\beta\gamma} = d\Big[-52dx^a dx^b dx^c d\theta^\beta\theta^\gamma - \frac{47}{4}dx^{[a}dx^b\bar\theta\Gamma^{c]}d\theta d\theta^{(\beta}\theta^{\gamma)}$$
$$-\frac{15}{2}x^{[a}dx^b\bar\theta\Gamma^{c]}d\theta d\theta^\beta\delta\theta^\gamma - \frac{17}{8}dx^{[a}\bar\theta\Gamma^b d\theta\bar\theta\Gamma^{c]}d\theta d\theta^{(\beta}\theta^{\gamma)}$$
$$-\frac{5}{8}x^{[a}\bar\theta\Gamma^b d\theta\bar\theta\Gamma^{c]}d\theta d\theta^\beta d\theta^\gamma - \frac{7}{48}\bar\theta\Gamma^a d\theta\bar\theta\Gamma^b d\theta\bar\theta\Gamma^c d\theta d\theta^{(\beta}\theta^{\gamma)}\Big]$$
$$\Sigma^{abcd\beta} = d\Big[\frac{281}{4}dx^a dx^b dx^c dx^d\theta^\beta + 13dx^{[a}dx^b dx^c\bar\theta\Gamma^{d]}d\theta\theta^\beta$$
$$+\frac{47}{2}x^{[a}dx^b dx^c\bar\theta\Gamma^{d]}d\theta d\theta^\beta + \frac{31}{8}dx^{[a}dx^b\bar\theta\Gamma^c d\theta\bar\theta\Gamma^{d]}d\theta\theta^\beta$$
$$+\frac{17}{4}x^{[a}dx^b\bar\theta\Gamma^c d\theta\bar\theta\Gamma^{d]}d\theta d\theta^\beta + \frac{7}{12}dx^{[a}\bar\theta\Gamma^b d\theta\bar\theta\Gamma^c d\theta\bar\theta\Gamma^{d]}d\theta\theta^\beta$$
$$+\frac{7}{24}x^{[a}\bar\theta\Gamma^b d\theta\bar\theta\Gamma^c d\theta\bar\theta\Gamma^{d]}d\theta d\theta^\beta + \frac{7}{192}\bar\theta\Gamma^a d\theta\bar\theta\Gamma^b d\theta\bar\theta\Gamma^c d\theta\bar\theta\Gamma^d d\theta\theta^\beta\Big]$$
$$\Sigma^{abcde} = d\Big[\frac{77}{2}x^a dx^b dx^c dx^d dx^e + 26x^{[a}dx^b dx^c dx^d\bar\theta\Gamma^{e]}d\theta$$
$$+\frac{31}{4}x^{[a}dx^b dx^c\bar\theta\Gamma^d d\theta\bar\theta\Gamma^{e]}d\theta + \frac{7}{6}x^{[a}dx^b\bar\theta\Gamma^c d\theta\bar\theta\Gamma^d d\theta\bar\theta\Gamma^{e]}d\theta$$
$$+\frac{7}{96}x^{[a}\bar\theta\Gamma^b d\theta\bar\theta\Gamma^c d\theta\bar\theta\Gamma^d d\theta\bar\theta\Gamma^{e]}d\theta\Big].$$

Having found a representation of the $\Sigma^{A_1 \dots A_p}$, we now need to check the validity of the ansatz (3.10) and (3.11) for the corresponding anomalous term representatives. Firstly, one verifies using the relevant Fierz identity that $sM^{(p)} = 0$. $M^{(p)}$ is also identically d closed since the $\Sigma^{A_1 \dots A_p}$ are closed forms. We therefore have $M^{(p)} \in H^{p,2}$. Because [M] is the unique, D nontrivial class, *any* nontrivial representative of $H^{p,2}$ is a representative of $[M]$. It therefore suffices to show that $M^{(p)}$ is D nontrivial. The coboundaries of $H^{p,2}$ are identically equal to the gauge transformations. Hence, if there exists a gauge field $\lambda \in \Omega^{p-1,1}$ such that

$$M^{(p)} = sd\lambda \tag{3.20}$$

then $M^{(p)}$ is trivial (since then $M^{(p)} = Dd\lambda$). Otherwise it is nontrivial.

In the case of the superstring, we explicitly demonstrated that $M^{(1)}$ is D cohomologous to H [49]. The nontriviality of $M^{(1)}$ then follows from that of H. For $p \geq 2$ one notes that $M^{(p)}$ is constructed using the structure constants $\Gamma_{a_1 \dots a_p \alpha\beta}$, $\Gamma_{a\alpha\beta}$ and η_{ab}. In attempting to solve (3.20) one therefore needs to consider only those λ gauge fields constructed using these structure constants. We believe the following to be a complete set of such fields:

$$\begin{aligned}
\lambda^{(i)} &= x^a dx^{a_1} \dots dx^{a_i} \bar{\theta}\Gamma^{a_{i+1}} d\theta \dots \bar{\theta}\Gamma^{a_{p-1}} d\theta \bar{e}\Gamma_{aa_1 \dots a_{p-1}}\theta, \\
&\qquad 0 \leq i \leq p-1. \\
\lambda'^{(i)} &= \bar{e}\Gamma^a\theta x^b dx^{a_1} \dots dx^{a_i} \bar{\theta}\Gamma^{a_{i+1}} d\theta \dots \bar{\theta}\Gamma^{a_{p-2}} d\theta \bar{\theta}\Gamma_{aba_1 \dots a_{p-2}} d\theta, \\
&\qquad 0 \leq i \leq p-2. \\
\lambda''^{(i)} &= e^a x^b dx^{a_1} \dots dx^{a_i} \bar{\theta}\Gamma^{a_{i+1}} d\theta \dots \bar{\theta}\Gamma^{a_{p-2}} d\theta \bar{\theta}\Gamma_{aba_1 \dots a_{p-2}} d\theta, \\
&\qquad 0 \leq i \leq p-2.
\end{aligned} \tag{3.21}$$

In equation (3.20), it suffices to consider the terms of highest order in x^m. One then needs to consider a linear combination of only $\lambda^{(p-1)}$ and $\lambda''^{(p-2)}$. One finds that a solution for the coefficients does not exist for any value of p. Provided that the set (3.21) is complete, $M^{(p)}$ is therefore nontrivial, and is thus a representative for the anomalous term associated with the standard superspace p-brane action. The charges (3.15) through (3.19) (and their associated anomalous terms) generalize the results of [28] to the case where fermionic topological terms are retained. Note that for $p \geq 3$ there are additional charges not present in the anomalous term; these are simply those which close the extended algebra (they result from the action of Q_A on the anomalous term). We conclude that the algebras (3.2) through (3.6) are indeed generated as topological charge algebras of the standard p-brane action.

3.3 Extended superspace actions

The extended superalgebras (3.2) through (3.6) can be used to construct left invariant potentials B for the field strength H [32, 33, 34]. The corresponding extended superspace p-brane action is the same as (2.16), but where B is now a left invariant potential. In this case, $W = 0$ solves the descent equations, and the corresponding Noether charge algebra is the minimal algebra. In [34], Noether charges associated with the extra coordinates of the $p = 1, 2$ extended superspace actions were found, and equations (3.15, 3.16) are proportional to the forms given there. Although these results were obtained in different contexts[1], they should agree. In each case the forms transform according to the same extended superalgebra, and we claim that based upon this transformation property alone the solution is unique. The conserved charges are therefore the same in both the standard and extended superspace formulation of the action. Let us now demonstrate this in detail.

Separate the generators of the extended supertranslation algebras into standard/extended parts as $T_A = \{T_{\widetilde{A}}, T_{\check{A}}\}$, with

$$\begin{aligned} T_{\widetilde{A}} &= \left\{-\widetilde{Q}_\alpha, -\widetilde{P}_a\right\} \\ T_{\check{A}} &= \{-\Sigma_{\check{A}}\} \\ &= \left\{-\Sigma^{A_1\ldots A_p}\right\}. \end{aligned} \tag{3.22}$$

The extra generators $T_{\check{A}}$ form an ideal. It follows that the standard coordinates do not transform under the left/right group actions generated by $T_{\check{A}}$. The inverse vielbeins therefore satisfy

$$\begin{aligned} R_{\check{A}}{}^{\widetilde{M}} &= 0 \\ L_{\check{A}}{}^{\widetilde{M}} &= 0. \end{aligned} \tag{3.23}$$

Now, the momenta of the action can be written [29, 49]

$$P_M = P_M^{(NG)} + (i_{\partial_1} \ldots i_{\partial_p} B)_M, \tag{3.24}$$

[1] In the previous section we constructed topological charges of the standard superspace action and showed that they generate the extended algebras (3.2) through (3.6). We may contrast this with the work of [34], where the extended algebras were used from the outset to construct invariant extended superspace actions. The resulting Noether charges associated with extra coordinates were then found for the cases $p = 1, 2$. It was noted there that the bosonic topological term of these charges agrees with that obtained from the anomalous term of the standard superspace formulation [28]. Showing that this correspondence also holds for the fermionic topological terms is a new result which is the main purpose of this section.

where i is the interior derivation and ∂_i is the i-th worldvolume tangent vector. $P_M^{(NG)}$ are the conjugate momenta for the NG action, which vanish for the extra coordinates:

$$P_{\check{M}}^{(NG)} = 0. \tag{3.25}$$

It follows that for the extended superspace action, the Noether charge associated with the generator $T_{\check{A}}$ is (as also derived in [34])

$$\begin{aligned} \overline{Q}_{\check{A}} &= \int d^p\sigma R_{\check{A}}{}^M (i_{\partial_1} \dots i_{\partial_p} B)_M \\ &= \overline{(i_{V_{\check{A}}} B)}, \end{aligned} \tag{3.26}$$

where

$$V_{\check{A}} = R_{\check{A}}{}^M \partial_M \tag{3.27}$$

is the left invariant vector field associated with $T_{\check{A}}$. Since the Noether charges satisfy the extended superalgebras (3.2) through (3.6) under Poisson brackets, it follows that the forms $i_{V_{\check{A}}} B$ must satisfy the same algebra under the action of $Q_{\tilde{A}}$. We claim that forms satisfying this transformation property have the unique solutions (3.15) through (3.19). Therefore, for an appropriate normalization of the action we must have

$$i_{V_{\check{A}}} B = \Sigma_{\check{A}}, \tag{3.28}$$

and the Noether charges

$$\overline{Q}_{\check{A}} = \overline{\Sigma}_{\check{A}}. \tag{3.29}$$

Interestingly enough, this argument has explicitly determined some Noether charges for a p-brane action without needing the explicit structure of the WZ term. It is only required that the extended background must *admit* a left invariant WZ form. That such WZ forms do indeed exist was shown for $p \leq 3$ by explicitly constructing the required potential B [33, 34].

The conserved charges $\Sigma^{A_1 \dots A_p}$ are thus the same in both the standard and extended superspace formulations of the action. In the former they are anomalous terms of the Noether charge algebra, while in the latter they are the Noether charges themselves. This result extends that of [34] to establish correspondence between fermionic as well as bosonic terms, and also for all allowed values of p.

3.4 Comments

The representations for $\Sigma^{A_1 \dots A_p}$ appear to be a basis for the p-forms. It seems possible to invert each representation to write

$$dx^{m_1} \dots dx^{m_i} d\theta^{\mu_1} \dots d\theta^{\mu_{p-i}} \leftrightarrow \{\Sigma^{a_1 \dots a_j \alpha_1 \dots \alpha_{p-j}}, \quad j \leq i\}. \tag{3.30}$$

For example, for $p = 2$:

$$
\begin{aligned}
d\theta^\alpha d\theta^\beta &= \Sigma^{\alpha\beta} \\
dx^a d\theta^\alpha &= \frac{2}{9}\Sigma^{a\alpha} + \frac{1}{18}\theta^\alpha \Gamma^a{}_{\beta\gamma}\Sigma^{\beta\gamma} - \frac{1}{18}(\Gamma^a\theta)_\beta \Sigma^{\alpha\beta} \\
dx^a dx^b &= -\frac{1}{5}\Sigma^{ab} + \frac{1}{45}(\Gamma^{[a}\theta)_\alpha \Sigma^{b]\alpha} - \frac{1}{10}x^{[a}\Gamma^{b]}{}_{\alpha\beta}\Sigma^{\alpha\beta} \\
&\quad -\frac{1}{180}(\Gamma^a\theta)_\alpha(\Gamma^b\theta)_\beta\Sigma^{\alpha\beta}.
\end{aligned}
\tag{3.31}
$$

This constitutes a change of basis for the p-forms, which in this case is not inherited in the usual way from a vielbein.

The topological anomalous term $\overline{M}^{(p)}$ is a topological integral of its form representation $M^{(p)}$. If the fermionic topology is taken to be trivial, then the only contribution to $\overline{M}^{(p)}$ comes from the $(dx)^p$ term of $\Sigma^{a_1 \dots a_p}$. This is the "central" anomalous term found in [28]. The corresponding extended algebra can be related to partial breaking of supersymmetry [57, 46]. We note that this central extension is not present in all gauges. Using the gauge transformation generated by $\lambda^{(p-1)}$ one finds that the fully modified Noether charge algebra in the presence of trivial fermionic topology is

$$
\left\{\widetilde{\overline{Q}}_\alpha, \widetilde{\overline{Q}}_\beta\right\} = -\Gamma^a{}_{\alpha\beta}\widetilde{\overline{P}}_a - E\Gamma_{a_1\dots a_p\alpha\beta}\int dx^{a_1}\dots dx^{a_p}, \tag{3.32}
$$

where the integral is over the spatial section of the brane, and E is a free constant resulting from the λ gauge freedom. The familiar bosonic extension of the p-brane Noether charge algebra is thus the result of a specific choice of gauge. In another gauge one obtains the minimal algebra[2].

A precursor to the $p = 2$ algebra (3.3) was an algebra that results from setting $\Sigma^{\alpha\beta} = 0$ in (3.3) [66]. This algebra does not appear in the family of topological charge algebras generated by the standard action. One may see this by noting that $\Sigma^{a\beta}$ becomes "central" in this algebra. Since the only left invariant possibilities for a form representing this generator are not closed, this cannot be a topological charge algebra. This might also have been expected on the basis that this contracted algebra does not allow the construction of a left invariant WZ form [33] (topological charge algebras of

[2] A free multiplicative constant also results from an optional tension parameter normalizing the action [28]. In this case one obtains the minimal algebra only in the limiting case of zero tension (the action used here vanishes at the limit). Tension and gauge parameters have completely different effects when fermionic topological terms are retained; in that case there may be multiple anomalous terms and the gauge parameters are not global scale factors.

the standard action appear to be such that they *do* allow the construction of such WZ forms [49]). Although $\Sigma^{\alpha\beta}$ appears to be a necessary generator in topological charge algebras, it's possible for the associated *anomalous term* M_{ab} to vanish (and commuting translations are thus restored: $[P_a, P_b] = 0$). For example, to obtain such algebras for $p = 2$, one first applies the gauge transformation generated by $\frac{1}{2}\lambda''^{(0)}$ from (3.21), which sets $M_{ab} = 0$. All remaining gauge transformations then preserve this property.

One may ask if there are any new algebras of interest generated as topological charge algebras of the standard action. Upon investigating the set of $p = 2$ gauge transformations (3.21) we found that new superalgebras were generated which allowed the construction of left invariant WZ forms. However, they seem to require the introduction of more generators than are present in (3.3). Upon constructing the left invariant WZ form, one then finds that free parameters remain. This is because the space has been extended more than is necessary; one might say that the associated superspace is not "minimally extended." In the case $p = 1$, we found that the entire family of topological charge algebras yielded minimally extended superspaces [49]. However, for $p \geq 2$ it appears that (3.3) through (3.6) may be the unique, minimally extended topological charge algebras generated by the standard p-brane action.

Chapter 4

Superalgebras from D-brane actions

4.1 Introduction

Various types of branes are classified according to the CE (Chevalley-Eilenberg) cohomology [55] of their field strengths. For p-branes, the field strength of the WZ (Wess-Zumino) form is nontrivial [27]. A similar classification also occurs for D-branes [34, 67], and the CE nontriviality of these brane field strengths has some interesting consequences.

For p-branes, the Noether charge algebra becomes extended by a topological anomalous term [28], and for D-branes, the same mechanism also applies to the WZ term. However, for branes with worldvolume gauge fields there is a second modification to the algebra that results from the transformation properties of the gauge field [57, 31]. For D-branes, this modification is due to the presence of the BI (Born-Infeld) worldvolume gauge field. In the case of the D-membrane, the modifications to the Noether charge algebra associated with bosonic topology were explicitly found [31]. Representative solutions were found for the remaining cases, and the associated bosonic topological charges given [68, 69, 70]. As with p-branes, the appearance of anomalous terms in the D-brane Noether charge algebra can be described using descent equations and ghost fields [58].

One can also construct actions for (p, q)-strings, D-branes and string-brane systems in extended superspaces [39, 71, 34, 67, 41]. The basic idea is the same as for p-branes, except that now there are two cocycles. Trivialization of the cocycle associated with the WZ term allows manifestly super-Poincaré invariant actions to be constructed. Trivialization of the cocycle associated with the NS-NS (Neveu-Schwarz) 2-form potential allows the construction of actions without BI gauge fields.

We have approached p-brane topological charge algebras from the point of view of a single cocycle associated with the p-brane [49]. The WZ field strength and the anomalous term are described as two different representatives of this cocycle. Due to gauge transformations of the cocycle, the anomalous term is a full cohomology class which generates a family of topological charge algebras. Upon retaining the terms associated with fermionic topology, the algebras used in extended superspace formulations of p-branes appear in the family of topological charge algebras associated with the standard action [49, 50].

In this chapter, we generalize this work to the case of D-branes. We show that the anomalous terms in the Noether charge algebra generate extensions of the standard supertranslation algebra by two disjoint, commuting ideals. Explicit representatives of both anomalous terms, including the fermionic topological terms, are found for the (p, q)-strings and the D-membrane. For the string, gauge freedom is used to generate a family of topological charge algebras which is invariant under type IIB $SO(2)$ rotations. A topological charge algebra for (p, q)-strings is then deduced. For the membrane, the topological charge algebras associated with the NS-NS potential are derived. These subalgebras (those associated with the NS-NS potential) are common to all type IIB and type IIA D-branes. In both cases, the family of topological charge algebras contains known algebras [39, 71, 34, 67, 41] which allow the construction of extended superspace actions.

4.2 D-branes

4.2.1 Standard actions

We will work with the standard, flat, background superspaces in d=10. The backgrounds are defined by the chirality of the spinor (i.e. fermionic) coordinates. Weyl spinors are eigenspinors of the idempotent chirality matrix:

$$\Gamma_{11} = \Gamma_0 \dots \Gamma_9. \tag{4.1}$$

Since Γ_{11} is traceless, the eigenvalues are ± 1 in equal numbers. Majorana spinors satisfy $\overline{\theta}_\alpha = \theta^\beta C_{\beta\alpha}$, where $C_{\beta\alpha}$ is the antisymmetric charge conjugation matrix. Type IIA superspace consists of a single Majorana spinor (or equivalently, two Majorana-Weyl spinors of opposite chirality). Type IIB superspace consists of two Majorana-Weyl spinors of the same chirality. For type IIB superspace it will be assumed that spinor indices are accompanied by a suppressed index $I = (1, 2)$ which identifies the spinor. The Pauli matrices $(\sigma_i)_{IJ}$ act upon these indices. Indices on Pauli matrices are raised and lowered with the Kronecker delta, while indices on gamma matrices are raised and lowered from the left by the charge conjugation matrix. $\Gamma^a{}_{\alpha\beta}$ is assumed to be symmetric. The de Rham differential acts from the right, and wedge product multiplication of forms is understood. The underlying supertranslation algebra (2.1) and worldvolume metric g_{ij} are the same as for the ordinary p-brane case.

Super-Dirichlet-p-branes (Dp-branes) are κ-symmetric, $p+1$ dimensional manifolds ("worldvolumes") embedded in the background superspace. Dp-branes in type IIA superspace exist only for p even, while those in type IIB superspace exist only for p odd. Actions for D-branes have been developed in both flat and more general backgrounds [35, 36, 37, 38]. We now present the action with the conventions adopted in this chapter.

Let the worldvolume be once again parameterized by coordinates σ^i. The action consists of two terms:

$$S = S_{DBI} + S_{WZ}. \tag{4.2}$$

The DBI (Dirac-Born-Infeld) term is

$$S_{DBI} = -\int d^{p+1}\sigma \sqrt{-\mathsf{det}(g_{ij} + F_{ij})}. \tag{4.3}$$

F is a 2-form[1]:

$$F = B - dA. \tag{4.4}$$

$A = d\sigma^i A_i$ is the BI worldvolume gauge field, which is a 1-form defined only on the worldvolume. The NS-NS potential B is a superspace 2-form defined by

$$dB = H, \tag{4.5}$$

where H is the left invariant, NS-NS 3-form field strength. For type IIA superspace, H is

$$H = \frac{1}{2} L^a d\bar{\theta} \Gamma_{11} \Gamma_a d\theta, \tag{4.6}$$

while for type IIB:

$$H = -\frac{1}{2} L^a d\bar{\theta} \Gamma_a \sigma_3 d\theta. \tag{4.7}$$

[1]It suits us to have $dF = H$. Hence the difference in sign convention with respect to some prior literature.

It is a characteristic feature of super-p-branes of various types that closure of field strengths requires "Fierz identities" for products of gamma matrices. Closure of H requires a "standard" identity [36]. For type IIA superspace this can be written

$$\Gamma^a{}_{(\alpha\beta}(\Gamma_{11}\Gamma_a)_{\gamma\delta)} = 0, \tag{4.8}$$

while for type IIB:

$$\Gamma^a{}_{(\alpha\beta}(\Gamma_a\sigma_3)_{\gamma\delta)} = 0. \tag{4.9}$$

The second term in the action is the WZ term:

$$S_{WZ} = \int b. \tag{4.10}$$

It is defined by the formal sum of forms:

$$b = \breve{b}e^F. \tag{4.11}$$

The form of degree $p+1$ is selected from this sum and the integral is then performed over the worldvolume of the brane. In general we will denote the form of a specific degree in a formal sum by a number in brackets. For example:

$$\breve{b} = \oplus\breve{b}^{(n)}. \tag{4.12}$$

The R-R (Ramond) potentials $\breve{b}^{(n)}$ are defined by

$$R = d\breve{b} + \breve{b}H. \tag{4.13}$$

The R-R field strengths $R^{(n)}$ are left invariant superspace forms:

$$R^{(n)} = (-1)^p d\bar{\theta} S^{(n-2)} d\theta, \tag{4.14}$$

where for type IIA superspace the $S^{(n)}$ are given by

$$S^{(n)} = \frac{1}{2n!} L^{a_1} \ldots L^{a_n} \Gamma_{a_1 \ldots a_n} {\Gamma_{11}}^{[\frac{n}{2}+1]}, \tag{4.15}$$

while for type IIB:

$$S^{(n)} = \frac{1}{2n!} L^{a_1} \ldots L^{a_n} \Gamma_{a_1 \ldots a_n} {\sigma_3}^{[\frac{n+1}{2}+1]} \sigma_1. \tag{4.16}$$

It follows from (4.13) that the total field strength for the WZ term is the degree $p+2$ piece of

$$\begin{aligned} h &= db \\ &= Re^F. \end{aligned} \tag{4.17}$$

Closure of h is equivalent to some more general Fierz identities. For type IIA superspace these are

$$\begin{aligned} (m-1)(\Gamma_{11}{}^{\frac{m}{2}}\Gamma_{[a_1 \ldots a_{m-2}})_{(\alpha\beta}(\Gamma_{11}\Gamma_{a_{m-1}]})_{\gamma\delta)} & \\ -\Gamma^{a_m}{}_{(\alpha\beta}(\Gamma_{11}{}^{\frac{m+2}{2}}\Gamma_{a_1 \ldots a_m})_{\gamma\delta)} &= 0, \end{aligned} \tag{4.18}$$

while for type IIB:

$$\begin{aligned} (m-1)(\Gamma_{[a_1 \ldots a_{m-2}}\sigma_3{}^{\frac{m+1}{2}}\sigma_1)_{(\alpha\beta}(\Gamma_{a_{m-1}]}\sigma_3)_{\gamma\delta)} & \\ +\Gamma^{a_m}{}_{(\alpha\beta}(\Gamma_{a_1 \ldots a_m}\sigma_3{}^{\frac{m+3}{2}}\sigma_1)_{\gamma\delta)} &= 0. \end{aligned} \tag{4.19}$$

Most of these can be shown to hold by repeated use of the $m=2$ identity [36, 37].

Left invariance of the action requires that the BI gauge field must transform under the left action of the supertranslation group. This transformation is determined by the requirement that the potential F must be left invariant. Denote the generators of the left group action once again by Q_A. Since $[d, Q_A] = 0$, it is required

$$dQ_A A = Q_A B. \tag{4.20}$$

From the left invariance of H it follows that

$$Q_A B = -dW_A \tag{4.21}$$

for some set of 1-forms W_A. Hence:

$$Q_A A_i = -(W_A)_i \tag{4.22}$$

is the required transformation of the BI gauge field [31]. Furthermore, since H is CE nontrivial, there does not exist a potential B such that $Q_A B = 0$ for all Q_A [34, 67].

4.2.2 Manifestly left invariant action

The variation of the WZ term of the standard action under the left group action is analogous to (4.21); from the left invariance of h it follows that the variation of the WZ term is a total derivative:

$$Q_A b = -dw_A. \tag{4.23}$$

Since h is CE nontrivial, there does not exist a potential b such that $Q_A b = 0$ for all Q_A [34, 67]. As a result, the standard Lagrangian is not manifestly left invariant.

A manifestly left invariant formulation for D-branes which we will not explicitly describe here is the "scale invariant" approach [58]. For the purposes of this chapter we find it more convenient to define a simple, manifestly left invariant generalization of the standard action. First introduce an additional worldvolume p-form gauge field:

$$a = d\sigma^{i_p} \dots d\sigma^{i_1} a_{i_1 \dots i_p} \frac{1}{p!} \tag{4.24}$$

satisfying

$$Q_A a_{i_1 \dots i_p} = -(w_A)_{i_1 \dots i_p}. \tag{4.25}$$

One then uses the alternative action:

$$\begin{aligned} S &= -\int d^{p+1}\sigma \sqrt{-\mathsf{det}(g_{ij} + F_{ij})} + \int f \\ f &= b - da. \end{aligned} \tag{4.26}$$

Unlike the components of the BI gauge field, the fields $a_{i_1 \dots i_p}$ are not physical degrees of freedom since they appear trivially (in a total derivative) in the action.

4.2.3 Type IIB $SO(2)$ rotations

There are various dualities relating different D-brane actions [72]. If one includes nonvanishing background scalars (dilaton and axion) in the action, the dualities can be explicitly studied. This is a rather indirect issue for the purposes of this chapter. However, in section 4.4 we will find it useful to consider rotations of the type IIB superstring actions. Classically there is an $SL(2, \mathbb{R})$ duality, but quantum considerations restrict this to $SL(2, \mathbb{Z})$. There is then an $SL(2, \mathbb{Z})$ multiplet of (p, q)-strings [73, 72, 74, 75, 76]. Although the background scalars transform inhomogeneously under $SL(2, \mathbb{R})$, one may consistently set them to zero if one considers only the $SO(2)$ automorphism subgroup. The Pauli matrix σ_2 can be taken as the generator for these automorphisms, and the standard type IIB superspace action corresponds to a particular choice of $SO(2)$ frame [36]. We wish to investigate how these frame rotations affect the results. The automorphisms can be implemented via rotations of the Pauli matrices [37]. However, for studying the properties of the Noether charge algebra it is useful to have an implementation in terms of *field transformations* instead. Such possibilities were considered in [68]. We take

$$\begin{aligned} x_\phi &= x \\ \theta_\phi &= e^{i\phi\sigma_2}\theta. \end{aligned} \tag{4.27}$$

The worldvolume metric is invariant under these transformations. The worldvolume gauge field A_ϕ is defined as usual by its transformation properties (in particular, the left invariance of F must be preserved). The set of type IIB D-brane actions S_ϕ with a free angular parameter ϕ is then

$$S_\phi[Z, A] = S[Z_\phi, A_\phi]. \tag{4.28}$$

4.3 D-brane cohomology

Most of the cohomology from p-brane actions follows through to the case of D-branes. We again write generalized (m, n)-forms as in (2.32), denote their spaces by $\Omega^{m,n}$, and the collection of such spaces $\Omega^{*,*}$. The total differential D is again given by (2.37), and the associated restriction to $\Omega^{m,l-m}$ denoted $H^{m,l-m} \subset H^l_D$. The D cocycle of the p-brane is associated with the CE (Chevalley-Eilenberg) nontrivial (p+2)-form field strength of the WZ term. The D-brane has two such field strengths: the NS-NS 3-form H and the WZ $(p+2)$-form h. Thus, there are two separate D cocycles associated with the D-brane: the "NS-NS cocycle" and the "WZ cocycle."

First consider the NS-NS field strength $H = dB$. This is a nontrivial element of the CE cohomology in both the IIA and IIB cases [34, 67]. The D cocycle associated with H exists in the "NS-NS double complex." All elements of this complex are required to be Lorentz invariant, generalized forms of dimension two. The different representatives of the NS-NS cocycle are generated by the descent equations [58]

$$\begin{aligned} H &= dB \\ sB &= -dW \\ sW &= dN. \end{aligned} \tag{4.29}$$

These equations are graphically depicted in the "tic-tac-toe box" [64] of figure 4.1. The different representatives of the NS-NS cocycle are found on the LHS of these equations. Just as in the p-brane case, there is gauge freedom for the cocycle [49]. The gauge fields for the NS-NS cocycle that are of interest to us are $\Psi \in \Omega^{1,0}$ and $\Lambda \in \Omega^{0,1}$. The corresponding transformations can be summarized as

$$\Delta(B \oplus W \oplus N) = D(\Psi \oplus \Lambda). \tag{4.30}$$

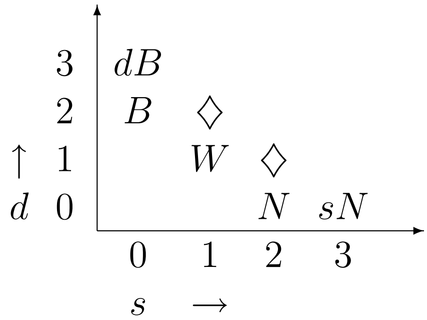

Figure 4.1: Descending sequence for the NS-NS field strength

Explicitly this gives

$$\begin{aligned} \Delta B &= -d\Psi \\ \Delta W &= s\Psi + d\Lambda \\ \Delta N &= s\Lambda. \end{aligned} \tag{4.31}$$

The algebra of conserved charges of the D-brane action contains an anomalous term due to the transformation properties of the BI gauge field [57, 31]. Let (P_M, P^i) denote the momenta conjugate to (Z^M, A_i). The minimal charges of the action are

$$\begin{aligned} \overline{Q}_A &= \int d^p\sigma \Big[Q_A Z^M P_M + Q_A A_i P^i \Big] \\ &= \int d^p\sigma \Big[R_A{}^M P_M - (W_A)_i P^i \Big]. \end{aligned} \tag{4.32}$$

Introduce the fundamental (graded) Poisson brackets for the phase space:

$$\begin{aligned} \big[P_M(\sigma), Z^N(\sigma') \big\} &= \delta_M{}^N \delta(\overrightarrow{\sigma} - \overrightarrow{\sigma}') \\ \big[P^i(\sigma), A_j(\sigma') \big\} &= \delta^i{}_j \delta(\overrightarrow{\sigma} - \overrightarrow{\sigma}'), \end{aligned} \tag{4.33}$$

where again it is assumed $\sigma'^0 = \sigma^0$ (i.e. equal time brackets). Let us denote the $H^{1,2}$ cocycle representative by

$$M = sW = dN. \tag{4.34}$$

One then obtains the "minimal algebra" under Poisson bracket [31]:

$$\big[\overline{Q}_A, \overline{Q}_B \big\} = -t_{AB}{}^C \overline{Q}_C - \int d^p\sigma (M_{AB})_i P^i. \tag{4.35}$$

For convenience we define a "hat map" for elements $Y \in \Omega^{1,n}$ of the NS-NS double complex:

$$\hat{Y} \;=\; -\int d^p\sigma Y_i P^i, \tag{4.36}$$

so that the algebra (4.35) is

$$\left[\overline{Q}_A, \overline{Q}_B\right\} \;=\; -t_{AB}{}^C\overline{Q}_C + \hat{M}_{AB}. \tag{4.37}$$

The minimal algebra is therefore already a modification by $\hat{M}_{AB}$ of the standard supertranslation algebra due to the presence of the BI gauge field. This modification will be referred to as the "NS-NS anomalous term" (since it descends from the NS-NS field strength H).

The BI gauge field appears in the action only through its field strength. This leads to constraints on the conjugate momenta P^i [31]. Firstly, since $\frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0$, there is the primary constraint:

$$P^0 \;=\; 0. \tag{4.38}$$

Denote the spatial worldvolume coordinates by σ^I. The Euler-Lagrange equation for A_0 then yields the secondary "Gauss law" constraint:

$$\partial_I P^I \;=\; 0. \tag{4.39}$$

Now applying these constraints, and using $M = dN$, gives

$$\hat{M}_{AB} \;=\; -\int d^p\sigma \partial_I (N_{AB} P^I). \tag{4.40}$$

The NS-NS anomalous term therefore consists of topological integrals, just as the p-brane anomalous term does. Note that once the constraints are imposed, the minimal charges lose their status as generators of the left group action. Therefore, the constraints should be applied only after the topological charge algebra has been evaluated.

Just as in the case of the p-brane, the minimal charges (4.32) are generally non-conserved, and this is due to quasi-invariance of the WZ term [28, 31]. The second modification to the Noether charge algebra derives from the WZ field strength. Since descent equations resulting from the WZ term involve the left invariant potential F, we must first define [58]

$$sF = 0. \tag{4.41}$$

Due to (4.29) this is equivalent to

$$sA = -W. \tag{4.42}$$

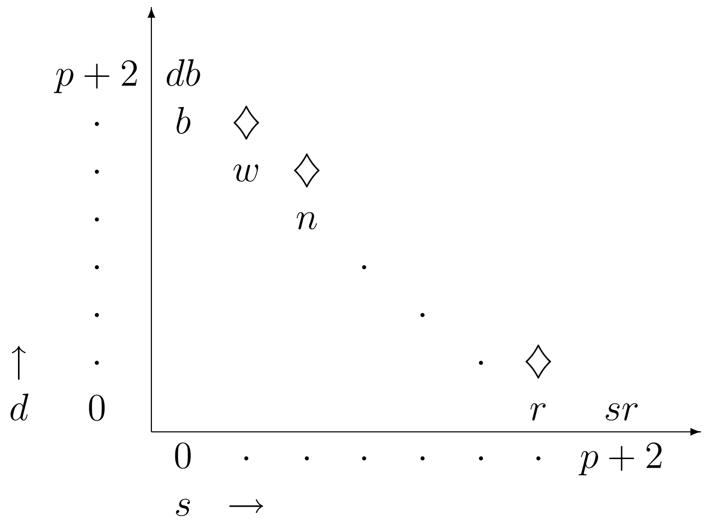

Figure 4.2: Descending sequence for the WZ field strength

Since B is D nontrivial, sW is nonzero. Thus, there is no solution to (4.41) which is compatible with the nilpotency of s. This is a reflection of the fact that extending the degrees of freedom according to

$$Z^M \rightarrow \{Z^M, A_i\}, \tag{4.43}$$

as in the standard D-brane action (4.2), *does not* lead to a geometrical description of the left transformations of the BI gauge field. A formulation of the action which *does* achieve this purpose is the extended superspace formulation [39, 34]. In that case, equation (4.41) remains the same, but the BI gauge field is eliminated in favor of additional superspace degrees of freedom. Equation (4.42) is then present only as a relation involving the composite field A for which the RHS vanishes and there is no conflict with nilpotency of s. In the standard formulation of the action, the BI gauge field occurs *only as a part of the potential* F. In the following, we may thus consistently use (4.41) or (4.42) with the understanding that the field content of A (an extension of the background) is *to be determined*.

The first three descent equations for the WZ cocycle are

$$\begin{aligned} h &= db \\ sb &= -dw \\ sw &= dn. \end{aligned} \tag{4.44}$$

The sequence ends with the potential $r \in \Omega^{0,p+1}$, and the associated cocycle representative $sr \in H^{0,p+2}$. This has been depicted in the tic-tac-toe box of figure 4.2. The exponential e^F in the WZ term is preserved by the operators d and s. All fields of the sequence are therefore formal sums containing this factor. Defining

$$w = \breve{w} e^F \tag{4.45}$$

(and similarly for other fields) the descent equation $sb = -dw$ is then equivalent to [58]

$$s\breve{b} = -\breve{w}H - d\breve{w}. \tag{4.46}$$

The $H^{p,2}$ cocycle representative is then

$$\begin{aligned} m &= \breve{m} e^F \\ &= s\breve{w} e^F. \end{aligned} \tag{4.47}$$

This leads to the algebra of conserved charges as follows. The variation of the WZ term is a total derivative:

$$Q_A \mathcal{L}_{WZ} = -\partial_i w_A{}^i, \tag{4.48}$$

where

$$w_A{}^i = \frac{1}{p!} \tilde{\epsilon}^{i_p \ldots i_1 i} w_{i_1 \ldots i_p, A}. \tag{4.49}$$

The conserved currents associated with this quasi-invariance are then

$$\begin{aligned} Q_A{}^i &= Q_A Z^M \frac{\partial \mathcal{L}}{\partial(\partial_i Z^M)} + Q_A A_j \frac{\partial \mathcal{L}}{\partial(\partial_i A_j)} + w_A{}^i \\ \partial_i Q_A{}^i &= 0. \end{aligned} \tag{4.50}$$

Once again let the spatial section of the worldvolume be a closed manifold embedded in superspace by the map Φ. For an action on (p, n)-forms Y, the bar map (2.61) reduces to:

$$\overline{Y} = (-1)^p \int \Phi^* Y. \tag{4.51}$$

The conserved charges of the currents (4.50) are then "modified Noether charges":

$$\widetilde{\overline{Q}}_A = \overline{Q}_A + \overline{w}_A. \tag{4.52}$$

The $\widetilde{\widetilde{Q}}_A$ obey a modified version of the minimal algebra [31]:

$$\left[\widetilde{\widetilde{Q}}_A, \widetilde{\widetilde{Q}}_B\right\} \quad = \quad -t_{AB}{}^C \widetilde{\widetilde{Q}}_C + \hat{M}_{AB} + \overline{m}_{AB}, \tag{4.53}$$

with

$$\overline{m}_{AB} = \left[\overline{Q}_A, \overline{w}_B\right\} + \left[\overline{w}_A, \overline{Q}_B\right\} + t_{AB}{}^C \overline{w}_C. \tag{4.54}$$

We refer to $\overline{m}$ as the "WZ anomalous term" (since it descends from the WZ field strength h). Just as in the p-brane case, the components $\overline{m}_{AB}$ are topological integrals since $m = dn$ is a closed form.

Let us investigate what happens if we use the manifestly left invariant action (4.26) instead of the standard one. In this case there will be no contribution to the topological charge algebra from quasi-invariance of the WZ term. However, the mechanism outlined for the BI gauge field contribution now applies to the p-form gauge field [31, 58]. Since the worldvolume is $p+1$ dimensional, before constraints are taken into account, the p-form gauge field has $p+1$ independent components. We will conveniently take[2]

$$a^i \quad = \quad \frac{1}{p!} \widetilde{\epsilon}^{i_p \ldots i_1 i} a_{i_1 \ldots i_p} \tag{4.55}$$

as the independent components. The left transformation of a^i follows from (4.25) and (4.49)

$$Q_A a^i \quad = \quad -w_A{}^i. \tag{4.56}$$

Define the momenta conjugate to a^i:

$$p_i \quad = \quad \frac{\partial}{\partial(\partial_0 a^i)} \mathcal{L}. \tag{4.57}$$

The conserved charges are then the Noether charges:

$$\begin{aligned} \overline{Q}_A \quad &= \quad \int d^p\sigma [Q_A Z^M P_M + Q_A A_i P^i + Q_A a^i p_i] \qquad (4.58) \\ &= \quad \int d^p\sigma [R_A{}^M P_M - (W_A)_i P^i - w_A{}^i p_i]. \end{aligned}$$

This yields the Noether charge algebra:

$$\left[\overline{Q}_A, \overline{Q}_B\right\} \quad = \quad -t_{AB}{}^C \overline{Q}_C - \int d^p\sigma \Big[(M_{AB})_i P^i + m_{AB}{}^i p_i \Big], \tag{4.59}$$

[2]Note this is the same "Hodge dual like" map used in (4.49).

with ${m_{AB}}^i$ defined in the same way as (4.49, 4.55). These charges are once again topological in nature as a result of constraints for the momenta conjugate to the p-form gauge field. These constraints, which arise in the same way as those for the BI gauge field, are found to be

$$\begin{aligned} \partial_I p_0 &= 0 \\ p_I &= 0. \end{aligned} \tag{4.60}$$

In fact, since the p-form gauge field enters the action (4.26) trivially, we can simply evaluate the momenta to obtain

$$\begin{aligned} p_0 &= -1 \\ p_I &= 0. \end{aligned} \tag{4.61}$$

Using this in (4.59), we then recover exactly the topological charge algebra (4.53) of the standard action, but with $\overline{\widetilde{Q}}_A$ replaced by $\overline{Q}_A$. That is, the conserved charges are now strict Noether charges instead of "modified" ones. Thus, whether one uses the standard action (4.2) or the manifestly invariant one (4.26), the algebra of conserved charges is the same. This is a slightly different result than one would obtain from the scale invariant formulation [58]. In that case, the p-form momenta p_0 is not fixed to a specific value as in (4.61), so it becomes a constant multiplying the associated anomalous term. In the present case, once momenta are substituted, the two anomalous terms are identical. The same observations also clearly apply to the analogous formulations of ordinary p-brane actions [77].

The topological charge algebra of the D-brane action is generated by taking Poisson brackets between $\overline{\widetilde{Q}}_A$ and all terms on the RHS of (4.53). This requires the calculation of Poisson brackets between $\overline{\widetilde{Q}}_A, \hat{M}_{AB}, \overline{m}_{AB}$, and any resulting new terms. For p-branes, using the isomorphism (2.76), we preferred to work with the symmetry generators and forms instead of Poisson brackets. The anomalous term was then shown to generate an extension of the supertranslation algebra by an ideal. We now show that this procedure also applies to the anomalous terms of the D-brane Noether charge algebra.

For the WZ anomalous term (4.47), the only difference from the case of the p-brane is the presence of factors of F. However, since F is left invariant, only the variations of $\breve{m}$ contribute to the algebra. For the NS-NS anomalous term, the additional feature is the presence of the momenta P^i conjugate to the components of the BI gauge field. At first this seems to complicate matters since both the conserved charges $\overline{\widetilde{Q}}_A$ and the WZ anomalous term $\overline{m}$ have dependence upon A_i. This could in principle generate "cross terms" that do not arise in the case of the p-brane (because there are no momenta in the p-brane anomalous term). However, it turns out that these cross terms vanish. Firstly, A_i appears only through its field strength, in products of

$$F_{ij} \ = \ B_{ij} - 2\partial_{[i}A_{j]}. \tag{4.62}$$

Using the Poisson bracket

$$[F_{ij}(\sigma), P^k(\sigma')] \ = \ -2\delta_{[i}{}^k\partial_{j]}\delta(\sigma - \sigma'), \tag{4.63}$$

one then finds

$$\begin{aligned} [F_{ij}(\sigma), \hat{M}_{AB}] \ &= \ -2\partial_{[i}\partial_{j]}N_{AB}(\sigma) \\ &= \ 0. \end{aligned} \tag{4.64}$$

If M_{AB} is split into closed forms representing superalgebra generators, the same calculation also holds for each generator. We thus have

$$[\overline{m}_{AB}, \hat{M}_{CD}\} \ = \ 0. \tag{4.65}$$

It also follows that the action of the conserved charges on the anomalous terms is equivalent to the action of the minimal charges:

$$\begin{aligned} \Big[\widetilde{\overline{Q}}_A, \hat{M}_{CD}\Big\} \ &= \ \Big[\overline{Q}_A, \hat{M}_{CD}\Big\} \\ \Big[\widetilde{\overline{Q}}_A, \overline{m}_{CD}\Big\} \ &= \ \Big[\overline{Q}_A, \overline{m}_{CD}\Big\}. \end{aligned} \tag{4.66}$$

The result is that we may use the double complex to find the topological charge algebra. Define the "modified left generators:"

$$\widetilde{Q}_A \ = \ Q_A + w_A. \tag{4.67}$$

We assign to Q_A the minimal algebra:

$$[Q_A, Q_B\} \ = \ -t_{AB}{}^C Q_C + M_{AB} \tag{4.68}$$

so that

$$\Big[\widetilde{Q}_A, \widetilde{Q}_B\Big\} \ = \ -t_{AB}{}^C \widetilde{Q}_C + M_{AB} + m_{AB}. \tag{4.69}$$

We then find the topological charge algebra by acting with $\widetilde{Q}_A$ on M_{AB} and m_{AB}. It follows from (4.65) that the family of algebras resulting from each anomalous term may be considered separately. We summarize with:

Theorem 5 (extension) *The anomalous terms of the D-brane Noether charge algebra define extensions of the standard supertranslation algebra by two disjoint ideals. The first derives from the equivalence class $[M] \in H^{1,2}$ of representatives for the NS-NS cocycle, the second from the class $[m] \in H^{p,2}$ of representatives for the WZ cocycle. The generators of both ideals commute amongst themselves and with each other.*

4.4 Application to (p,q)-strings

4.4.1 D-strings

Let us investigate a combination of the manifestly left invariant action (4.26) and the rotated action (4.28). The SO(2) rotated fields are given by[3]

$$\begin{aligned}
{x_\phi}^a &= x^a \\
{\theta_\phi}^{\alpha I} &= (e^{i\phi\sigma_2})^I{}_J \theta^{\alpha J} \\
{e_\phi}^a &= e^a \\
{e_\phi}^{\alpha I} &= (e^{i\phi\sigma_2})^I{}_J e^{\alpha J} \\
\begin{bmatrix} A_\phi \\ a_\phi \end{bmatrix} &= \begin{bmatrix} \cos(2\phi) & -\sin(2\phi) \\ \sin(2\phi) & \cos(2\phi) \end{bmatrix} \begin{bmatrix} A \\ a \end{bmatrix}.
\end{aligned} \tag{4.70}$$

The string case is special in that the field strengths (H,h) transform as an $SO(2)$ vector doublet. The potentials and worldvolume gauge fields of the double complex will be chosen such that they respect this transformation property. That is, only solutions to the descent equations which transform as vector doublets under (4.70) will be considered. The defining properties of the relevant doublets are

- (B,b):

$$\begin{aligned}
dB &= H \\
&= \frac{1}{2} L^a d\bar{\theta}\Gamma_a \sigma_1 d\theta \\
db &= h \\
&= \frac{1}{2} L^a d\bar{\theta}\Gamma_a \sigma_3 d\theta.
\end{aligned} \tag{4.71}$$

- (W,w):

$$\begin{aligned}
sB &= -dW \\
sb &= -dw.
\end{aligned} \tag{4.72}$$

- (A,a):

$$\begin{aligned}
sA &= -W \\
sa &= -w.
\end{aligned} \tag{4.73}$$

[3] Note that ${e_\phi}^{\alpha I}$ are chiral ghost fields while $(e^{i\phi\sigma_2})^I{}_J$ is an exponential.

Define left invariant potentials in the usual way:

$$\begin{aligned} F &= B - dA \\ f &= b - da. \end{aligned} \tag{4.74}$$

The set of $SO(2)$ dual actions is then given by (4.28), with

$$S = -\int d^2\sigma\sqrt{-\mathsf{det}(g_{ij} + F_{ij})} + \int f. \tag{4.75}$$

which differs from that of [68] by the inclusion of the p-form gauge field and the gauge field rotations. Because of the equivalence of the NS-NS and R-R sectors, all strings in the orbit are viewed as being of the same generalized type. In fact, up to a normalization constant, the actions S_0 and $S_{\frac{\pi}{4}}$ as defined by (4.28) describe the $(1, q)$ and $(p, 1)$ elements of the (p, q)-strings that are related through the $SL(2, \mathbb{Z})$ duality[4] [73, 72, 74, 75, 76, 78].

Construction of the anomalous term follows along the lines of (4.59), except that no "Hodge dual like" fields are required since *both* worldvolume gauge fields are 1-forms. After constraints are imposed, their conjugate momenta are constants. Define (P^i, p^i) as the doublet of momenta conjugate to (A_i, a_i) respectively. For convenience we define "hat" and "check" maps by their action on $(1, n)$-forms Y, y:

$$\begin{aligned} \hat{Y} &= -\int d\sigma^1 Y_i P^i \\ \check{y} &= -\int d\sigma^1 y_i p^i. \end{aligned} \tag{4.76}$$

Since the cocycle potentials (W, w) form an $SO(2)$ doublet, and the momenta (P^i, p^i) transform contragradiently, the Noether charges are $SO(2)$ invariant:

$$\overline{Q}_A = \int d\sigma^1 (R_A{}^M P_M) + \hat{W}_A + \check{w}_A. \tag{4.77}$$

Since the Lagrangian is manifestly left invariant, the fully modified charge algebra is then the algebra of Noether charges:

$$[\overline{Q}_A, \overline{Q}_B\} = -t_{AB}{}^C \overline{Q}_C + \hat{M}_{AB} + \check{m}_{AB}, \tag{4.78}$$

which is also $SO(2)$ invariant.

Let us now solve the descent equations to find the anomalous term representatives. Solutions can be obtained by taking linear combinations of all

4 The "fundamental" string used here has a DBI kinetic term rather than Nambu-Goto. All actions in the $SO(2)$ orbit of the action (4.75) are D-strings in a generalized sense.

possible terms, and then equating coefficients in the equation. One requires the type II, string Fierz identities:

$$\begin{aligned}\Gamma_{a(\alpha\beta}\Gamma^a\sigma_{1\gamma\delta)} &= 0 \\ \Gamma_{a(\alpha\beta}\Gamma^a\sigma_{3\gamma\delta)} &= 0.\end{aligned} \tag{4.79}$$

The first two equations $dB = H$ and $db = h$ are found to be solved by

$$\begin{aligned}B &= \frac{1}{2}\left[dx^a - \frac{1}{4}d\bar{\theta}\Gamma^a\theta\right]d\bar{\theta}\Gamma_a\sigma_1\theta \\ b &= \frac{1}{2}\left[dx^a - \frac{1}{4}d\bar{\theta}\Gamma^a\theta\right]d\bar{\theta}\Gamma_a\sigma_3\theta.\end{aligned} \tag{4.80}$$

The next descent equations $sB = -dW$ and $sb = -dw$ then have the solutions:

$$\begin{aligned}W &= -\frac{1}{2}dx^a\bar{\theta}\Gamma_a\sigma_1 e + \frac{1}{24}d\bar{\theta}\Gamma^a\theta\bar{\theta}\Gamma_a\sigma_1 e + \frac{1}{24}\bar{\theta}\Gamma^a e d\bar{\theta}\Gamma_a\sigma_1\theta \\ w &= -\frac{1}{2}dx^a\bar{\theta}\Gamma_a\sigma_3 e + \frac{1}{24}d\bar{\theta}\Gamma^a\theta\bar{\theta}\Gamma_a\sigma_3 e + \frac{1}{24}\bar{\theta}\Gamma^a e d\bar{\theta}\Gamma_a\sigma_3\theta,\end{aligned} \tag{4.81}$$

where e refers to the e^α ghosts. We comment that solutions b and w for type IIB D-branes with higher values of p could be deduced from [69]. We now obtain the anomalous term representatives $M = sW$ and $m = sw$:

$$\begin{aligned}M &= \frac{1}{2}dx^a\bar{e}\Gamma_a\sigma_1 e + \frac{1}{8}d\left[\bar{e}\Gamma^a\theta\bar{e}\Gamma_a\sigma_1\theta\right] \\ m &= \frac{1}{2}dx^a\bar{e}\Gamma_a\sigma_3 e + \frac{1}{8}d\left[\bar{e}\Gamma^a\theta\bar{e}\Gamma_a\sigma_3\theta\right].\end{aligned} \tag{4.82}$$

Let us now calculate the extended algebras resulting from these representatives. First we need to identify the gauge transformations. These are generated by Lorentz invariant fields in $\Omega^{0,1}$ of dimension two. Define some "rotated Pauli matrices" as[5]

$$\begin{aligned}\sigma_1^\varphi &= \cos(2\varphi)\sigma_1 - \sin(2\varphi)\sigma_3 \\ \sigma_3^\varphi &= \sin(2\varphi)\sigma_1 + \cos(2\varphi)\sigma_3.\end{aligned} \tag{4.83}$$

By requiring the gauge fields to form a vector doublet (Λ, λ):

$$\begin{aligned}\Lambda &= -Ex^a\bar{e}\Gamma_a\sigma_1^\varphi\theta \\ \lambda &= -Ex^a\bar{e}\Gamma_a\sigma_3^\varphi\theta,\end{aligned} \tag{4.84}$$

[5]The angle φ is unrelated to ϕ used to rotate the action.

the anomalous term remains $SO(2)$ invariant. E and φ are free constants which become polar coordinates for the equivalence class of the anomalous term. The gauge transformations generated by (4.84) are

$$
\begin{aligned}
\Delta M &= sd\Lambda \\
&= -Edx^a\bar{e}\Gamma_a\sigma_1^\varphi e - \frac{1}{2}Ed\Big[\bar{e}\Gamma^a\theta\bar{e}\Gamma_a\sigma_1^\varphi\theta\Big] + Ee^a\bar{e}\Gamma_a\sigma_1^\varphi d\theta \\
\Delta m &= sd\lambda \\
&= -Edx^a\bar{e}\Gamma_a\sigma_3^\varphi e - \frac{1}{2}Ed\Big[\bar{e}\Gamma^a\theta\bar{e}\Gamma_a\sigma_3^\varphi\theta\Big] + Ee^a\bar{e}\Gamma_a\sigma_3^\varphi d\theta.
\end{aligned}
\tag{4.85}
$$

The equivalence classes $[M]$ and $[m]$ are obtained by applying these transformations to the representatives from (4.82). This gives

$$
\begin{aligned}
[M]_{\alpha\beta} &= (1-2E)dx^a(\Gamma_a\sigma_1^\varphi)_{\alpha\beta} \\
&\quad + \left[E - \frac{1}{4}\right] d\Big[(\Gamma^a\theta)_{(\alpha}(\Gamma_a\sigma_1^\varphi\theta)_{\beta)}\Big] \\
[M]_{a\beta} &= -E(\Gamma_a\sigma_1^\varphi d\theta)_\beta \\
[m]_{\alpha\beta} &= (1-2E)dx^a(\Gamma_a\sigma_3^\varphi)_{\alpha\beta} + \left[E - \frac{1}{4}\right] d\Big[(\Gamma^a\theta)_{(\alpha}(\Gamma_a\sigma_3^\varphi\theta)_{\beta)}\Big] \\
[m]_{a\beta} &= -E(\Gamma_a\sigma_3^\varphi d\theta)_\beta.
\end{aligned}
\tag{4.86}
$$

One then notes that extended superalgebras are generated from $[M]$ and $[m]$ if the following new generators are defined:

$$
\begin{aligned}
\Sigma^a &= -2dx^a \\
\Sigma^\alpha &= -d\theta^\alpha \\
\Sigma^\varphi_{1\,\alpha\beta} &= -d\Big[(\Gamma^a\theta)_{(\alpha}(\Gamma_a\sigma_1^\varphi\theta)_{\beta)}\Big] \\
\Sigma^\varphi_{3\,\alpha\beta} &= -d\Big[(\Gamma^a\theta)_{(\alpha}(\Gamma_a\sigma_3^\varphi\theta)_{\beta)}\Big].
\end{aligned}
\tag{4.87}
$$

The new, resulting family of topological charge algebras is

$$
\begin{aligned}
\Big\{\overline{Q}_\alpha, \overline{Q}_\beta\Big\} &= -\Gamma^a{}_{\alpha\beta}\overline{P}_a \\
&\quad + \left[E - \frac{1}{2}\right]\Big[(\Gamma_a\sigma_1^\varphi)_{\alpha\beta}\hat{\Sigma}^a + (\Gamma_a\sigma_3^\varphi)_{\alpha\beta}\check{\Sigma}^a\Big] \\
&\quad - \left[E - \frac{1}{4}\right]\Big[\hat{\Sigma}^\varphi_{1\,\alpha\beta} + \check{\Sigma}^\varphi_{3\,\alpha\beta}\Big] \\
\Big[\overline{Q}_\alpha, \overline{P}_b\Big] &= -E\Big[(\Gamma_b\sigma_1^\varphi)_{\alpha\beta}\hat{\Sigma}^\beta + (\Gamma_b\sigma_3^\varphi)_{\alpha\beta}\check{\Sigma}^\beta\Big] \\
\Big[\overline{Q}_\alpha, \hat{\Sigma}^b\Big] &= -\Gamma^b{}_{\alpha\beta}\hat{\Sigma}^\beta
\end{aligned}
\tag{4.88}
$$

$$
\begin{aligned}
\left[\overline{Q}_\alpha, \check{\Sigma}^b\right] &= -\Gamma^b{}_{\alpha\beta}\check{\Sigma}^\beta \\
\left[\overline{Q}_\alpha, \hat{\Sigma}^\varphi_{1\,\beta\gamma}\right] &= \left[\Gamma^a{}_{\alpha(\beta}(\Gamma_a\sigma^\varphi_1)_{\gamma)\delta} - \Gamma^a{}_{\delta(\beta}(\Gamma_a\sigma^\varphi_1)_{\gamma)\alpha}\right]\hat{\Sigma}^\delta \\
\left[\overline{Q}_\alpha, \check{\Sigma}^\varphi_{3\,\beta\gamma}\right] &= \left[\Gamma^a{}_{\alpha(\beta}(\Gamma_a\sigma^\varphi_3)_{\gamma)\delta} - \Gamma^a{}_{\delta(\beta}(\Gamma_a\sigma^\varphi_3)_{\gamma)\alpha}\right]\check{\Sigma}^\delta.
\end{aligned}
$$

For $E = \frac{1}{4}$ the last two lines of the algebra may be omitted, and the resulting algebra is equivalent to one used in [71]. There it was used in the construction of a manifestly invariant string action.

The Jacobi identity for the algebra is satisfied due to properties of the cocycle [49]. Indeed, one verifies that the only nontrivial Jacobi identity is given by

$$
\begin{aligned}
[\overline{Q}_\alpha, \{\overline{Q}_\beta, \overline{Q}_\gamma\}] + \textsf{cycles} &= \frac{3}{2}\Big[\Gamma^b{}_{(\alpha\beta}(\Gamma_b\sigma^\varphi_1)_{\gamma\delta)}\hat{\Sigma}^\delta \\
&\quad +\Gamma^b{}_{(\alpha\beta}(\Gamma_b\sigma^\varphi_3)_{\gamma\delta)}\check{\Sigma}^\delta\Big],
\end{aligned}
\tag{4.89}
$$

which vanishes by the Fierz identities.

Only half the fermionic coordinates of the action are physical degrees of freedom due to the presence of κ-symmetry. A simple condition one can use to fix κ-symmetry is $\theta_1 = 0$ [36]. In this case H vanishes. It is then simplest to fix the associated potential B and worldvolume gauge field A to be vanishing as well. For simplicity, we will then consider only the "unbroken" supersymmetries (those preserving $\theta_1 = 0$ without the need for gauge transformations). Under these conditions, the $\Sigma_{\alpha\beta}$ charges vanish, as do all hatted fields and $\check{\Sigma}^{\alpha 1}$. The free angular parameter ϕ can then be scaled away into $\check{\Sigma}^a$ and $\check{\Sigma}^{\alpha 2}$, and the algebra reduces to

$$
\begin{aligned}
\left\{\overline{Q}_{\alpha 2}, \overline{Q}_{\beta 2}\right\} &= -\Gamma^a{}_{\alpha\beta}\overline{P}_a + \left[E - \frac{1}{2}\right]\Gamma_{a\alpha\beta}\check{\Sigma}^a \\
\left[\overline{Q}_{\alpha 2}, \overline{P}_b\right] &= -E\Gamma_{b\alpha\beta}\check{\Sigma}^{\beta 2} \\
\left[\overline{Q}_{\alpha 2}, \check{\Sigma}^b\right] &= -\Gamma^b{}_{\alpha\beta}\check{\Sigma}^{\beta 2}.
\end{aligned}
\tag{4.90}
$$

Due to the gauge condition $\theta_1 = 0$ there is no further equivalence class freedom, so this parameterized family is in its most general form. Upon rescaling, it is equivalent to the topological charge algebra derived in [49] of the Green-Schwarz superstring action. This is not surprising since, with the gauge fixing conditions, the $\varphi = 0$ action (4.75) becomes equivalent [68] to the standard Green-Schwarz superstring action [52]. The only difference is the presence of the p-form gauge field in the WZ term, but as in (4.59) this gauge field has no effect upon the topological charge algebra. The $SO(2)$ rotation φ now interpolates between Green-Schwarz and Born-Infeld forms of the action, and this also has no effect upon the charge algebra. The effect that nonlinearly realized supersymmetries of the gauge fixed action have upon the charge algebra is a more complicated problem that we will not address here.

4.4.2 (p,q)-strings

To describe (p,q)-strings, the action (4.75) needs modification in order to obtain the required expression for the tension [73]. We will not explicitly give the required action here (see [75, 76] for a "duality covariant" formulation). Instead, let us simply note the following properties of the action for a (J,j)-string:

- The action is manifestly left invariant, and is constructed from the left invariant potentials (F,f).
- After constraints are imposed, the momenta (P^i,p^i) conjugate to (A_i,a_i) are

$$\begin{aligned}(P^0,p^0) &= (0,0) \\ (P^1,p^1) &= (J,j),\end{aligned} \tag{4.91}$$

  where (J,j) are two integers.

This is sufficient information for us to give a topological charge algebra for the (J,j)-string. The descent equations once again lead to the representatives (4.82) for the anomalous terms (M,m). The simplest gauge for the resulting algebra is obtained by setting $(E,\varphi)=(\frac{1}{4},0)$ in (4.88). In this case one can remove $\overline{\Sigma}^{\varphi}_{1\,\alpha\beta}$ and $\overline{\Sigma}^{\varphi}_{3\,\alpha\beta}$ from the algebra since they do not appear in the anomalous term. Now impose the constraints (4.91), and factor out the constant momenta from the integrals (4.76). The algebra is then

$$\begin{aligned}\left\{\overline{Q}_\alpha,\overline{Q}_\beta\right\} &= -\Gamma^a{}_{\alpha\beta}\overline{P}_a-\frac{1}{4}\Big[J(\Gamma_a\sigma_1)_{\alpha\beta}\overline{\Sigma}^a+j(\Gamma_a\sigma_3)_{\alpha\beta}\overline{\Sigma}'^a\Big] \\ \left[\overline{Q}_\alpha,\overline{P}_b\right] &= -\frac{1}{4}\Big[J(\Gamma_b\sigma_1)_{\alpha\beta}\overline{\Sigma}^\beta+j(\Gamma_b\sigma_3)_{\alpha\beta}\overline{\Sigma}'^\beta\Big] \\ \left[\overline{Q}_\alpha,\overline{\Sigma}^b\right] &= -\Gamma^b{}_{\alpha\beta}\overline{\Sigma}^\beta \\ \left[\overline{Q}_\alpha,\overline{\Sigma}'^b\right] &= -\Gamma^b{}_{\alpha\beta}\overline{\Sigma}'^\beta.\end{aligned} \tag{4.92}$$

In the above, we have kept the charges

$$\begin{aligned} \overline{\Sigma}^a &= \overline{\Sigma}'^a = 2\int d\sigma^1 \partial_1 x^a \\ \overline{\Sigma}^\alpha &= \overline{\Sigma}'^\alpha = \int d\sigma^1 \partial_1 \theta^\alpha \end{aligned} \tag{4.93}$$

distinct, since the general construction allows this. The Jacobi identity

$$\begin{aligned} [\overline{Q}_\alpha, \{\overline{Q}_\beta, \overline{Q}_\gamma\}] + \mathsf{cycles} &= \frac{3}{2}\Big[J\Gamma^b{}_{(\alpha\beta}(\Gamma_b\sigma_1)_{\gamma\delta)}\overline{\Sigma}^\delta \\ &\quad + j\Gamma^b{}_{(\alpha\beta}(\Gamma_b\sigma_3)_{\gamma\delta)}\overline{\Sigma}'^\delta\Big] \end{aligned} \tag{4.94}$$

vanishes by the Fierz identities.

Certain of the algebras contained in this family have seen use in the construction of extended superspace actions. The cases $(J, j) = (0, 1)$ and $(J, j) = (1, 0)$ correspond to algebras used in [39, 67], while $(J, j) = (1, 1)$ corresponds to an algebra used in [71]. These algebras can be used to construct left invariant potentials F and f on the associated extended superspaces. This allows extended superspace actions for strings and type IIB D-branes to be constructed. The generation of such algebras as topological charge algebras is a new result, as are the free parameters resulting from the cocycle freedom.

In our papers [49, 50], the family of topological charge algebras of standard p-brane actions were shown to contain the known algebras that allow the construction of left invariant WZ forms. The appearance of known algebras associated with D-branes in (4.92) generalizes this result. We make the observation that topological charge algebras generated by a brane cocycle appear to be those which trivialize that cocycle. As a result, these algebras then allow the construction of extended superspace actions.

Note that fermionic winding charges are formally retained and used to close the algebra. Such charges are generated, for example, by open strings with different values for fermionic coordinates at the endpoints [40], or by strings bridging a brane-antibrane system [41]. Further motivation is provided by the fact that fermionic brane charges are necessary in certain backgrounds to ensure quantum consistency with Jacobi identities [42]. In flat backgrounds, the fermionic topological charges have usually been taken to vanish due to the trivial topology associated with fermionic coordinates [30]. In that case, the bosonic charges become "central" and the algebra (4.92) reduces to

$$\left\{\overline{Q}_\alpha, \overline{Q}_\beta\right\} = -\Gamma^a{}_{\alpha\beta}\overline{P}_a - \frac{1}{4}\Big[J(\Gamma_a\sigma_1)_{\alpha\beta}\overline{\Sigma}^a + j(\Gamma_a\sigma_3)_{\alpha\beta}\overline{\Sigma}'^a\Big]. \tag{4.95}$$

This type of algebra can be related to partial breaking of rigid supersymmetry [47] via the consideration of particular extended geometries of the brane [57, 46].

Since $\overline{\Sigma}^A$ and $\overline{\Sigma}'^A$ are physically the same charges, a reduced form of the algebra (4.92) can be written where these generators are identified. This is

$$\begin{aligned} \left\{\overline{Q}_\alpha, \overline{Q}_\beta\right\} &= -\Gamma^a{}_{\alpha\beta}\overline{P}_a - \frac{1}{4}\Big[J(\Gamma_a\sigma_1)_{\alpha\beta} + j(\Gamma_a\sigma_3)_{\alpha\beta}\Big]\overline{\Sigma}^a \\ \left[\overline{Q}_\alpha, \overline{P}_b\right] &= -\frac{1}{4}\Big[J(\Gamma_b\sigma_1)_{\alpha\beta} + j(\Gamma_b\sigma_3)_{\alpha\beta}\Big]\overline{\Sigma}^\beta \\ \left[\overline{Q}_\alpha, \overline{\Sigma}^b\right] &= -\Gamma^b{}_{\alpha\beta}\overline{\Sigma}^\beta. \end{aligned} \tag{4.96}$$

Whilst the momenta (J, j) can be viewed as scale factors in (4.92), this is no longer the case in (4.96). The Jacobi identity

$$\begin{aligned} \left[\overline{Q}_\alpha, \{\overline{Q}_\beta, \overline{Q}_\gamma\}\right] + \text{cycles} &= \frac{3}{2}\Big[J\Gamma^b{}_{(\alpha\beta}(\Gamma_b\sigma_1)_{\gamma\delta)} \\ &\quad + j\Gamma^b{}_{(\alpha\beta}(\Gamma_b\sigma_3)_{\gamma\delta)}\Big]\overline{\Sigma}^\delta \end{aligned} \tag{4.97}$$

again vanishes.

4.5 Application to the D-membrane

Let us solve the descent equations for the D2-brane in order to find representatives for the two anomalous terms of the Noether charge algebra. The Fierz identities for the membrane are required:

$$\begin{aligned} \Gamma^a{}_{(\alpha\beta}(\Gamma_{11}\Gamma_a)_{\gamma\delta)} &= 0 \\ \Gamma_{11(\alpha\beta}(\Gamma_{11}\Gamma_a)_{\gamma\delta)} - \Gamma^b{}_{(\alpha\beta}\Gamma_{ab\gamma\delta)} &= 0. \end{aligned} \tag{4.98}$$

We begin with the NS-NS sequence. The solution for B is found to be

$$B = \frac{1}{2}\left[dx^a - \frac{1}{4}d\bar{\theta}\Gamma^a\theta\right]d\bar{\theta}\Gamma_{11}\Gamma_a\theta. \tag{4.99}$$

The equation $sB = -dW$ is then solved by[6]

$$\begin{aligned} W &= -\frac{1}{2}dx^a\bar{\theta}\Gamma_{11}\Gamma_a e \\ &\quad + \frac{1}{24}d\bar{\theta}\Gamma^a\theta\bar{\theta}\Gamma_{11}\Gamma_a e + \frac{1}{24}\bar{\theta}\Gamma^a e d\bar{\theta}\Gamma_{11}\Gamma_a\theta. \end{aligned} \tag{4.100}$$

[6] An analogous solution (without ghost fields) appears in [31].

This yields the representative $M = sW$ for the NS-NS anomalous term:

$$M = \frac{1}{2}dx^a\bar{e}\Gamma_{11}\Gamma_a e + \frac{1}{8}d\Big[\bar{e}\Gamma^a\theta\bar{e}\Gamma_{11}\Gamma_a\theta\Big]. \tag{4.101}$$

We now turn to the WZ cocycle. Representatives for b and w for type IIA D-branes may be found in [70]. We will find these quantities for the D2-brane. The equation for $\breve{b}^{(1)}$ that follows from (4.13) is

$$d\breve{b}^{(1)} = R^{(2)}, \tag{4.102}$$

which is easily solved by

$$\breve{b}^{(1)} = \frac{1}{2}d\bar{\theta}\Gamma_{11}\theta. \tag{4.103}$$

The equation for $\breve{b}^{(3)}$ is then

$$d\breve{b}^{(3)} = R^{(4)} - \breve{b}^{(1)}H. \tag{4.104}$$

This is solved by

$$\begin{aligned}\breve{b}^{(3)} = {} & \frac{1}{4}dx^a dx^b d\bar{\theta}\Gamma_{ab}\theta \\ & + dx^a\Big[-\frac{1}{8}d\bar{\theta}\Gamma^b\theta d\bar{\theta}\Gamma_{ab}\theta + \frac{1}{8}d\bar{\theta}\Gamma_{11}\theta d\bar{\theta}\Gamma_{11}\Gamma_a\theta\Big] \\ & + d\bar{\theta}\Gamma^a\theta\Big[\frac{1}{48}d\bar{\theta}\Gamma^b\theta d\bar{\theta}\Gamma_{ab}\theta - \frac{1}{24}d\bar{\theta}\Gamma_{11}\theta d\bar{\theta}\Gamma_{11}\Gamma_a\theta\Big].\end{aligned} \tag{4.105}$$

From (4.46) we determine the equation for $\breve{w}^{(0)}$:

$$d\breve{w}^{(0)} = -s\breve{b}^{(1)}, \tag{4.106}$$

which is easily solved by

$$\breve{w}^{(0)} = -\frac{1}{2}\bar{\theta}\Gamma_{11}e. \tag{4.107}$$

The equation for $\breve{w}^{(2)}$ is then

$$d\breve{w}^{(2)} = -s\breve{b}^{(3)} - \breve{w}^{(0)}H. \tag{4.108}$$

This is solved by

$$\breve{w}^{(2)} = -\frac{1}{4}dx^a dx^b\bar{\theta}\Gamma_{ab}e \tag{4.109}$$

$$
\begin{aligned}
&+\frac{1}{24}dx^a\Big[\bar\theta\Gamma^b ed\bar\theta\Gamma_{ab}\theta + d\bar\theta\Gamma^b\theta\bar\theta\Gamma_{ab}e + 5\bar\theta\Gamma_{11}ed\bar\theta\Gamma_{11}\Gamma_a\theta \\
&-d\bar\theta\Gamma_{11}\theta\bar\theta\Gamma_{11}\Gamma_a e\Big] \\
&+\frac{1}{240}\Big[-d\bar\theta\Gamma^a\theta d\bar\theta\Gamma^b\theta\bar\theta\Gamma_{ab}e + \bar\theta\Gamma^a ed\bar\theta\Gamma^b\theta d\bar\theta\Gamma_{ab}\theta \\
&+2d\bar\theta\Gamma^a\theta d\bar\theta\Gamma_{11}\theta\bar\theta\Gamma_{11}\Gamma_a e - 14d\bar\theta\Gamma^a\theta\bar\theta\Gamma_{11}ed\bar\theta\Gamma_{11}\Gamma_a\theta \\
&-\bar\theta\Gamma^a ed\bar\theta\Gamma_{11}\theta d\bar\theta\Gamma_{11}\Gamma_a\theta\Big].
\end{aligned}
$$

We then finally obtain the forms:

$$
\begin{aligned}
\breve{m}^{(0)} &= \frac{1}{2}\bar e\Gamma_{11}e && (4.110)\\
\breve{m}^{(2)} &= -\frac{1}{4}dx^a dx^b\bar e\Gamma_{ab}e \\
&+dx^a\Big[\frac{1}{24}\bar\theta\Gamma^b ed\bar\theta\Gamma_{ab}e - \frac{1}{24}\bar e\Gamma^b ed\bar\theta\Gamma_{ab}\theta - \frac{1}{24}d\bar\theta\Gamma^b\theta e\Gamma_{ab}e \\
&-\frac{7}{24}d\bar\theta\Gamma^b e\bar\theta\Gamma_{ab}e + \frac{5}{24}\bar\theta\Gamma_{11}ed\bar\theta\Gamma_{11}\Gamma_a e - \frac{5}{24}\bar e\Gamma_{11}ed\bar\theta\Gamma_{11}\Gamma_a\theta \\
&+\frac{1}{24}d\bar\theta\Gamma_{11}\theta\bar e\Gamma_{11}\Gamma_a e + \frac{1}{24}d\bar\theta\Gamma_{11}e\bar\theta\Gamma_{11}\Gamma_a e\Big] \\
&+\frac{1}{240}d\bar\theta\Gamma^a\theta d\bar\theta\Gamma^b\theta\bar e\Gamma_{ab}e - \frac{1}{80}d\bar\theta\Gamma^a\theta d\bar\theta\Gamma^b e\bar\theta\Gamma_{ab}e \\
&+\frac{1}{240}\bar\theta\Gamma^a ed\bar\theta\Gamma^b\theta\bar d\theta\Gamma_{ab}e - \frac{1}{60}\bar\theta\Gamma^a ed\bar\theta\Gamma^b e\bar d\theta\Gamma_{ab}\theta \\
&-\frac{1}{240}\bar e\Gamma^a ed\bar\theta\Gamma^b\theta\bar d\theta\Gamma_{ab}\theta - \frac{1}{120}d\bar\theta\Gamma^a\theta d\bar\theta\Gamma_{11}\theta\bar e\Gamma_{11}\Gamma_a e \\
&-\frac{1}{120}d\bar\theta\Gamma^a\theta d\bar\theta\Gamma_{11}e\bar\theta\Gamma_{11}\Gamma_a e + \frac{1}{80}d\bar\theta\Gamma^a ed\bar\theta\Gamma_{11}\theta\bar\theta\Gamma_{11}\Gamma_a e \\
&-\frac{7}{120}d\bar\theta\Gamma^a\theta\bar\theta\Gamma_{11}ed\bar\theta\Gamma_{11}\Gamma_a e + \frac{7}{120}d\bar\theta\Gamma^a\theta\bar e\Gamma_{11}ed\bar\theta\Gamma_{11}\Gamma_a\theta \\
&-\frac{11}{240}d\bar\theta\Gamma^a e\bar\theta\Gamma_{11}ed\bar\theta\Gamma_{11}\Gamma_a\theta - \frac{1}{240}\bar\theta\Gamma^a ed\bar\theta\Gamma_{11}\theta d\bar\theta\Gamma_{11}\Gamma_a e \\
&-\frac{1}{240}\bar\theta\Gamma^a ed\bar\theta\Gamma_{11}ed\bar\theta\Gamma_{11}\Gamma_a\theta + \frac{1}{240}\bar e\Gamma^a ed\bar\theta\Gamma_{11}\theta d\bar\theta\Gamma_{11}\Gamma_a\theta.
\end{aligned}
$$

The corresponding representative of the WZ anomalous term is given by $\overline{m}$, where

$$
m = \breve{m}^{(0)}F + \breve{m}^{(2)}. \tag{4.111}
$$

The first term contains a topological integral of the field strength of the BI gauge field, while the first term of $\breve{m}^{(2)}$ is a familiar bosonic term:

$$
dx^a dx^b\bar e\Gamma_{ab}e \tag{4.112}
$$

that also exists in the case of ordinary p-branes [28]. These two terms, plus the first term of (4.101), generate the three central extensions of the standard supertranslation algebra that are associated with bosonic topology [31]. The remaining terms are the ones associated with fermionic topology which generalize the solutions of [31, 70]. Since the number of fermionic terms is quite large, we will not explicitly calculate the family of algebras associated with the WZ anomalous term (4.111) in this work.

Let us now calculate the extended algebras resulting from the NS-NS anomalous term. Two Lorentz invariant Λ gauge fields with the correct dimensionality are

$$\begin{aligned} \Lambda_1 &= -x^a \bar{e}\Gamma_a\theta \\ \Lambda_2 &= -x^a \bar{e}\Gamma_{11}\Gamma_a\theta. \end{aligned} \tag{4.113}$$

A third possibility

$$\Lambda_3 = -2e^a x^b \eta_{ab} \tag{4.114}$$

is equivalent to Λ_1 since $sd\Lambda_3 = sd\Lambda_1$. Some other possibilities that we will not use here are given in appendix 7.5. The gauge transformations generated are

$$\begin{aligned} \Delta_1 M &= sd\Lambda_1 \\ &= -dx^a \bar{e}\Gamma_a e + e^a \bar{e}\Gamma_a d\theta \\ \Delta_2 M &= sd\Lambda_2 \\ &= -dx^a \bar{e}\Gamma_{11}\Gamma_a e - \frac{1}{2} d\Big[\bar{e}\Gamma^a\theta\bar{e}\Gamma_{11}\Gamma_a\theta\Big] + e^a\bar{e}\Gamma_{11}\Gamma_a d\theta. \end{aligned} \tag{4.115}$$

The full class [M] for the anomalous term is then obtained by applying these transformations to the representative (4.101):

$$[M] = M + E_1\Delta_1 M + E_2\Delta_2 M, \tag{4.116}$$

where E_1 and E_2 are free constants which parameterize the class. This gives

$$\begin{aligned} [M]_{\alpha\beta} &= (1-2E_2)dx^a(\Gamma_{11}\Gamma_a)_{\alpha\beta} - 2E_1 dx^a \Gamma_{a\alpha\beta} \\ &\quad + \Big[E_2 - \frac{1}{4}\Big] d\Big[(\Gamma^a\theta)_{(\alpha}(\Gamma_{11}\Gamma_a\theta)_{\beta)}\Big] \\ [M]_{a\beta} &= -E_1(\Gamma_a d\theta)_\beta - E_2(\Gamma_{11}\Gamma_a d\theta)_\beta. \end{aligned} \tag{4.117}$$

One then notes that $[M]$ generates a family of extended superalgebras if three new generators are defined[7]:

$$\Sigma^a = -2dx^a \tag{4.118}$$

[7] A term analogous to $\Sigma_{\alpha\beta}$ was also obtained in [70]. However, not all fermionic terms were retained, leading to the vanishing of certain charges. In particular, the $[Q, P]$ and $[P, P]$ anomalous terms (which are fermionic on dimensional grounds) were not obtained.

$$\Sigma^\alpha = -d\theta^\alpha$$
$$\Sigma_{\alpha\beta} = -d\Big[(\Gamma^a\theta)_{(\alpha}(\Gamma_{11}\Gamma_a\theta)_{\beta)}\Big].$$

The resulting topological charge algebra is then

$$\begin{aligned}
\left\{\tilde{\tilde{Q}}_\alpha, \tilde{\tilde{Q}}_\beta\right\} &= -\Gamma^a{}_{\alpha\beta}\tilde{\tilde{P}}_a \\
&\quad + \left[\left[E_2 - \frac{1}{2}\right](\Gamma_{11}\Gamma_a)_{\alpha\beta} + E_1\Gamma_{a\alpha\beta}\right]\hat{\Sigma}^a \\
&\quad - \left[E_2 - \frac{1}{4}\right]\hat{\Sigma}_{\alpha\beta} \\
\left[\tilde{\tilde{Q}}_\alpha, \tilde{\tilde{P}}_b\right] &= -\Big[E_1\Gamma_{b\alpha\beta} + E_2(\Gamma_{11}\Gamma_b)_{\alpha\beta}\Big]\hat{\Sigma}^\beta \\
\left[\tilde{\tilde{Q}}_\alpha, \hat{\Sigma}^b\right] &= -\Gamma^b{}_{\alpha\beta}\hat{\Sigma}^\beta \\
\left[\tilde{\tilde{Q}}_\alpha, \hat{\Sigma}_{\beta\gamma}\right] &= \Big[\Gamma^a{}_{\alpha(\beta}(\Gamma_{11}\Gamma_a)_{\gamma)\delta} - \Gamma^a{}_{\delta(\beta}(\Gamma_{11}\Gamma_a)_{\gamma)\alpha}\Big]\hat{\Sigma}^\delta.
\end{aligned} \tag{4.119}$$

The Jacobi identity for the algebra is again satisfied due to properties of the cocycle. One verifies that the only nontrivial possibility is

$$\left[\tilde{\tilde{Q}}_\alpha, \left\{\tilde{\tilde{Q}}_\beta, \tilde{\tilde{Q}}_\gamma\right\}\right] + \text{cycles} = \frac{3}{2}\Gamma^b{}_{(\alpha\beta}(\Gamma_{11}\Gamma_b)_{\gamma\delta)}\hat{\Sigma}^\delta, \tag{4.120}$$

which vanishes by the standard Fierz identity. The algebra (4.119) is not restricted to the membrane. The DBI term of the Dp-brane action is the same for all values of p (worldvolume embedding aside). The NS cocycle thus generates the same algebra for all standard, type IIA D-brane actions with $p \geq 2$. Similarly, the subalgebra of (4.88) which contains only the NS charges is the same for all type IIB D-branes.

An algebra within the family (4.119) has also been used in the context of trivializing cocycles. In the special case $E_2 = \frac{1}{4}$, $\Sigma_{\alpha\beta}$ is not present in the anomalous term and can be excluded from the algebra. The gauge $E_1 = 0$ then yields an algebra which corresponds to one used in the construction of extended, type IIA superspace actions for strings, D-branes and string-brane systems [39, 34, 41][8]. We note that in both the type IIA and IIB cases, the free parameters in the algebras do not correspond to rescalings of the previously known algebras, so the algebras found here are new. The conserved charge algebras of standard superspace D-brane actions thus generate new candidates for algebras underlying extended superspace action formulations.

[8] A redefinition $\Sigma^\alpha \to \Gamma_{11}\Sigma^\alpha$ is required to establish the correspondence with [39, 41].

4.6 Comments

For simplicity, we have only considered actions without the background scalars. If these scalars are included, the action is invariant when they take their vacuum values [75]. Representatives for the required anomalous terms then result from solving the same descent equations. The process thus depends only upon the field strengths (i.e. nontrivial cocycles) involved, and the background scalars do not contribute directly to the topological charge algebra. However, they may contribute indirectly through the consideration of dualities (for example, as a restriction on the gauge fields, as in section 4.4.1). We note here that an algebra featuring background scalars was considered in [71]. This type of algebra might be expected to arise as a topological charge algebra if the scalars (belonging to the coset $SL(2,\mathbb{R})/SO(2)$) are identified with coordinates of the duality group.

Chapter 5

D-branes and geometrical kappa-symmetry

5.1 Introduction

In the standard formulation of D-brane actions, the BI gauge field transforms under the left action of the supertranslation group. These transformations are postulated to compensate for those of the NS-NS potential in order to generate the global symmetry. However, since the BI gauge field is not part of the background, this is a somewhat mysterious feature; it suggests that this field should be part of some larger representation which includes the standard background. This is indeed the case, as demonstrated by the existence of D-brane actions on extended spaces [39, 71, 34, 67, 41]. In these cases, the BI gauge field becomes an ordinary super-form on the extended background. In chapter 4, we demonstrated that examples of these extended backgrounds are generated by the standard D-brane action as topological charge algebras [51].

A defining feature of both p-branes and D-branes is the presence of κ-symmetry. This local symmetry allows half the fermionic degrees of freedom to be gauged away, which equates the number of physical bosonic and fermionic degrees of freedom. For p-branes, κ-symmetry is associated with fermionic supercovariant derivatives [29], and is naturally implemented in quite general backgrounds via a right action of the supertranslation group [44]. This continues to be the case in the extended superspace formulation of the p-brane action [45]. In the standard formulation of D-brane actions, the BI gauge field has a nontrivial κ-transformation for which there is no understanding in terms of group geometry.

In this chapter we will show that in the extended superspace formulation of D-brane actions, κ-transformations are generated by a right action. First, we derive a set of sufficient conditions to prove this. The family of D-brane topological charge algebras found in [51] is then investigated. The extended algebras associated with the NS-NS potential are shown to allow the construction of D-brane actions without BI gauge fields. κ-symmetry of the actions is established by solving for the required right action. It follows that both the global and local transformations of the BI gauge field are described geometrically as left/right actions of the underlying, extended supergroup.

5.2 κ-symmetry of the standard action

Let us review the mechanism for κ-symmetry of the standard D-brane action (4.2) as given in [36, 37, 38]. Let V_κ denote the infinitesimal vector field associated with a κ transformation. This field is assumed to take the form

$$\begin{aligned} V_\kappa{}^M &= \epsilon^\alpha L_\alpha{}^M \\ \bar{\epsilon}_\alpha &= (\bar{\kappa} P_-)_\alpha, \end{aligned} \tag{5.1}$$

where $\kappa^\alpha(\sigma)$ are local worldvolume parameters and P_- is a projection operator to be determined. κ-symmetry of the action requires that

$$\delta_\kappa F = i_{V_\kappa} H, \tag{5.2}$$

where i denotes the interior derivation. The κ variation of a superspace form Y is determined by the Lie derivative

$$\begin{aligned} \delta_\kappa Y &= \mathcal{L}_{V_\kappa} Y \\ &= i_{V_\kappa} dY + d i_{V_\kappa} Y. \end{aligned} \tag{5.3}$$

Equation (5.2) then determines the κ transformation of the BI gauge field up to a total derivative:

$$\delta_\kappa A_i = (i_{V_\kappa} B)_i. \tag{5.4}$$

Using (5.2) one finds

$$\delta_\kappa b = i_{V_\kappa} R e^F + d(i_{V_\kappa} \breve{b} e^F). \tag{5.5}$$

We now summarize the proof of κ-symmetry given in [36] using the conventions adopted in this chapter. For type IIA superspace, define the matrix valued forms

$$\begin{aligned} \rho &= 2e^F \sum_{n \text{ odd}} S^{(n)} \\ T &= 2e^F \sum_{n \text{ even}} S^{(n)}, \end{aligned} \tag{5.6}$$

while for type IIB:

$$\begin{aligned} \rho &= 2e^F \sum_{n \text{ even}} S^{(n)} \\ T &= 2e^F \sum_{n \text{ odd}} S^{(n)}. \end{aligned} \tag{5.7}$$

T is related to the WZ field strength via

$$h = \frac{(-1)^p}{2} d\bar{\theta} T d\theta. \tag{5.8}$$

Then define the matrix valued densities:

$$\begin{aligned} \tilde{\rho} &= \frac{1}{(p+1)!} \tilde{\epsilon}^{i_{p+1} \ldots i_1} \rho_{i_1 \ldots i_{p+1}} \\ T^i &= \frac{1}{p!} \tilde{\epsilon}^{i_p \ldots i_1 i} T_{i_1 \ldots i_p}, \end{aligned} \tag{5.9}$$

where $\tilde{\epsilon}$ is the antisymmetric Levi-Civita symbol. The associated matrix

$$\gamma = -\frac{\tilde{\rho}}{\mathcal{L}_{DBI}} \tag{5.10}$$

can be shown to be idempotent [36]:

$$\gamma^2 = 1. \tag{5.11}$$

The κ variation of the Lagrangian is then found to be

$$\delta_\kappa \mathcal{L} = -\bar{\epsilon}(1+\gamma) T^i \partial_i \theta. \tag{5.12}$$

Setting

$$P_- = \frac{1}{2}(1-\gamma) \tag{5.13}$$

then gives the required κ-symmetry. Since γ is traceless, the ± 1 eigenvalues are present in equal numbers. This allows half the θ^α to be gauged away using the local parameter in (5.1).

5.3 Extended superspaces

The supertranslation transformation properties of closed forms can be classified using CE group cohomology [27]. A closed, left invariant form $Y = dX$

on the background superspace is CE trivial if there exists a potential X such that

$$Q_A X = 0. \tag{5.14}$$

Otherwise Y is CE nontrivial. CE nontrivial forms will be termed "cocycles," and CE trivial forms "coboundaries."

Note that even for standard D-brane actions, one has the left invariant 3-form $H = dF$, with $Q_A F = 0$. Since F is not a form on the background superspace, CE cohomology does not apply. However, this preempts the philosophy for D-brane actions on extended superspaces. One wishes to find an extended superspace which allows the construction of a left invariant potential for H [39, 71, 34, 67]:

$$\begin{aligned} dF' &= H \\ Q_A F' &= 0. \end{aligned} \tag{5.15}$$

Since F' is now a form on the extended background superspace, this represents a "trivialization of the cocycle." Denote the standard D-brane action (4.2) as $S[x, \theta, F]$. The extended superspace D-brane action is then simply

$$S'[x, \theta, \ldots] = S[x, \theta, F'], \tag{5.16}$$

where $\ldots$ indicates the extra superspace coordinates required to define F'. In this way, the BI gauge field is replaced by a form involving extra superspace coordinates. This demonstrates a strength of the extended superspace formulation: it describes the left group transformation property (4.22) of the BI gauge field *geometrically*.

As already noted, a defining property of D-brane actions is the existence of local κ-symmetry. For p-branes, it is a general property of κ-symmetry that it is generated by a right action of the underlying supertranslation group [44]. Now consider the standard D-brane action. In this case, the κ-transformations (5.1) of the superspace background are generated by the right action

$$\begin{aligned} g(Z + V_\kappa) &= g(Z)g(\epsilon) \\ \epsilon^A &= \{0, \epsilon^\alpha\}, \end{aligned} \tag{5.17}$$

where ϵ^α is defined in (5.1). This exhibits another property of κ transformations: the parameter ϵ^a of the right action (corresponding to the superalgebra generator P_a) vanishes. These properties go hand in hand with the structure of the field strengths H and h. The κ variations of F and b in (5.2) and (5.5) are determined by the structure of $i_{V_\kappa}H$ and $i_{V_\kappa}R^{(n)}$ respectively. Both H and $R^{(n)}$ have the structure

$$Y = L^{a_1}\dots L^{a_m}d\bar{\theta}\Gamma_{a_1\dots a_m}Sd\theta, \tag{5.18}$$

where S is either Γ_{11} or a Pauli matrix. This yields the interior derivations

$$i_{V_\kappa}Y = -2L^{a_1}\dots L^{a_m}\bar{\epsilon}\Gamma_{a_1\dots a_m}Sd\theta \tag{5.19}$$

provided that V_κ results from a right action with the parameters (5.17). The resulting structure of $\delta_\kappa F_{ij}$ differs from $\delta_\kappa g_{ij}$ only in the index symmetry, which then generates the required variation of the kinetic term.

For p-branes, manifest κ-symmetry of the extended superspace Lagrangian is generated by a right action [45]. We will use a similar mechanism to establish κ-symmetry for extended superspace D-brane actions. In the case of p-branes (with the exception of the scale invariant approach), the extra superspace coordinates appear in the action only through a total derivative. The mechanism is optional in this case since the variation of the Lagrangian will be a total derivative irrespective of the way κ-transformations of extra coordinates are defined. However, in the case of D-branes the mechanism *must* be used since extra coordinates appear in the DBI term.

Let Z^M now denote both the standard and extra coordinates of the extended superspace:

$$\begin{aligned} Z^M &= \{Z^{\widetilde{M}}, Z^{\check{M}}\} \\ &= \{x^m, \theta^\mu, Z^{\check{M}}\}. \end{aligned} \tag{5.20}$$

We first note that topological charge algebras are extensions of the standard superalgebra by an ideal, and that the algebras known to allow construction of extended superspace actions appear in the family of topological charge algebras [49, 50, 51]. We therefore assume that the superalgebra describing the background superspace is an extension of the standard supertranslation algebra by an ideal. The following properties follow:

- The standard coordinate blocks $L_{\widetilde{M}}{}^{\widetilde{A}}$ and $L_{\widetilde{A}}{}^{\widetilde{M}}$ retain their original structure.
- $L_{\check{A}}{}^{\widetilde{M}} = 0$.

This leads us to seek a new infinitesimal vector field V'_κ generated by the right action

$$\begin{aligned} V'^{M}_{\kappa} &= \epsilon^A L_A{}^M \\ \epsilon^A &= \{0, \epsilon^\alpha, \epsilon^{\check{A}}\}, \end{aligned} \tag{5.21}$$

where ϵ^α is the same as in (5.1), and $\epsilon^{\check{A}}$ are to be determined. Firstly, the κ variation of the standard coordinates is then unchanged. Thus:

$$\delta'_\kappa g_{ij} \quad = \quad \delta_\kappa g_{ij}. \tag{5.22}$$

Secondly, although the proof of κ-symmetry for the standard D-brane action depends on the explicit structure of $\delta_\kappa F$, it does not depend on that of F itself (see, for example [36]). This allows us to use the κ-symmetry mechanism of the standard action in proving κ-symmetry of the extended action. If the parameters ϵ^A of the right action are chosen such that

$$d i_{V'_\kappa} F' = 0, \tag{5.23}$$

then one again obtains the transformation $\delta'_\kappa F' = i_{V'_\kappa} H$. Since H and $R^{(n)}$ are constructed only from $d\theta^\alpha$ and L^a, the extra parameters in (5.21) do not affect the relevant interior derivations. We thus have $i_{V'_\kappa} H = i_{V_\kappa} H$, and therefore $\delta'_\kappa F' = \delta_\kappa F$. We also have $i_{V'_\kappa} R^{(n)} = i_{V_\kappa} R^{(n)}$, from which it follows that the variation $\delta'_\kappa b'$ of the new WZ form is equal to the variation $\delta_\kappa b$ of the standard one up to a total derivative. This derivative is ignored in the same way as the second term of (5.5). We then have $\delta'_\kappa \mathcal{L}'$ equal to the standard variation $\delta_\kappa \mathcal{L}$ from (5.12) up to a total derivative. Equations (5.15), (5.21) and (5.23) are therefore sufficient conditions to establish κ-symmetry of the extended superspace D-brane actions.

We now show explicitly that κ-symmetry exists in extended superspace formulations of D-brane actions, and that it is generated by a right action.

5.4 Extended type IIB superspace actions

5.4.1 Cocycle trivialization

We wish to find extended type IIB superspaces that admit left invariant solutions for F' of the equation

$$\begin{aligned} dF' &= H \\ &= -\frac{1}{2} L^a d\bar{\theta} \Gamma_a \sigma_3 d\theta. \end{aligned} \tag{5.24}$$

Such superspaces allow the construction of D-brane actions without BI gauge fields via the action (5.16). We consider the family of topological charge algebras of the D-strings derived in chapter 4. The subalgebra of this family associated with the NS-NS potential of type IIB D-branes is[1]

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= \Gamma^a{}_{\alpha\beta} P_a + \left[E - \frac{1}{2}\right](\Gamma_a\sigma_3)_{\alpha\beta}\Sigma^a \\
&\quad - \left[E - \frac{1}{4}\right]\Sigma_{\alpha\beta} \\
[Q_\alpha, P_b] &= -E(\Gamma_b\sigma_3)_{\alpha\beta}\Sigma^\beta \\
[Q_\alpha, \Sigma^b] &= \Gamma^b{}_{\alpha\beta}\Sigma^\beta \\
[Q_\alpha, \Sigma_{\beta\gamma}] &= -\left[\Gamma^a{}_{\alpha(\beta}(\Gamma_a\sigma_3)_{\gamma)\delta} - \Gamma^a{}_{\delta(\beta}(\Gamma_a\sigma_3)_{\gamma)\alpha}\right]\Sigma^\delta,
\end{aligned} \tag{5.25}$$

where E is a free constant. The generators will be associated with the following left vielbein components:

$$\{P_a, Q_\alpha, \Sigma^a, \Sigma^\alpha, \Sigma_{\alpha\beta}\} \;\rightarrow\; \{L^a, L^\alpha, L_a, L_\alpha, L^{\alpha\beta}\}. \tag{5.26}$$

The Maurer-Cartan equations following from (5.25) are

$$\begin{aligned}
dL^\alpha &= 0 \\
dL^a &= -\frac{1}{2} d\bar{\theta}\Gamma^a d\theta \\
dL_a &= -\frac{1}{2}\left[E - \frac{1}{2}\right] d\bar{\theta}\Gamma_a\sigma_3 d\theta \\
dL_\alpha &= EL^b(\Gamma_b\sigma_3 d\theta)_\alpha - L_b(\Gamma^b d\theta)_\alpha \\
&\quad + L^{\beta\gamma}\left[(\Gamma^b d\theta)_\beta(\Gamma_b\sigma_3)_{\gamma\alpha} - \Gamma^b{}_{\alpha\beta}(\Gamma_b\sigma_3 d\theta)_\gamma\right] \\
dL^{\alpha\beta} &= \frac{1}{2}\left[E - \frac{1}{4}\right] d\theta^\alpha d\theta^\beta.
\end{aligned} \tag{5.27}$$

The required potential F' on the extended superspace can be found by forming all possible super-Poincaré invariant 2-forms of dimension two, and then equating coefficients in the equation $dF' = H$. Since the vielbein components are already left invariant, it suffices to consider all possible Lorentz invariant bilinear combinations. It is useful to find the general solution of the equation

$$dF' = -KL^a d\bar{\theta}\Gamma_a\sigma_3 d\theta, \tag{5.28}$$

with K an arbitrary constant (the action used here corresponds to $K = \frac{1}{2}$). The solution is found to exist for all $E \neq \frac{1}{4}$, and contains a free parameter

[1] The free phase angle in the algebra of our paper [51] is set to vanish (corresponding to the $SO(2)$ frame of the standard action used here) so that a minimal number of extra generators are required. We have also rescaled $(Q_\alpha, P_a) \rightarrow (-Q_\alpha, -P_a)$ relative to [51].

C:

$$\begin{aligned}
F' &= C_1 L^a L_a + C_2 L^\alpha L_\alpha + C_3 L^a L^{\alpha\beta}(\Gamma_a\sigma_3)_{\alpha\beta} \\
&\quad + C_4 L_a L^{\alpha\beta}\Gamma^a{}_{\alpha\beta} + C_5 L^{\alpha\beta}\Gamma^a{}_{\alpha\beta}L^{\gamma\delta}(\Gamma_a\sigma_3)_{\gamma\delta} \\
C_1 &= -4K - \frac{C}{4}(4E-1) \\
C_2 &= 2K \\
C_3 &= -\frac{C}{2}(2E-1) \\
C_4 &= C \\
C_5 &= -\frac{2C(2E-1)}{4E-1}.
\end{aligned} \tag{5.29}$$

The presence of the free parameter indicates that the solution has the form

$$F' = -4KL^a L_a + 2KL^\alpha L_\alpha + CF'_{\mathsf{hom}}, \tag{5.30}$$

where $dF'_{\mathsf{hom}} = 0$. This type of solution is familiar for differential equations; the first two terms are the unique particular solution, while the last is the solution of the associated homogenous equation. F'_{hom} is a Lorentz invariant 2-cocycle which is explicitly obtained by setting $K = 0$ in the solution (5.29) for F'.

There are two special cases of the algebra (5.25) where generators become redundant. The first is for $E = \frac{1}{4}$ (the singular case of the general solution (5.29)). In this case $\Sigma^{\alpha\beta}$ appears nowhere on the RHS of a bracket and may therefore be excluded from the algebra entirely. Upon rescaling, this algebra is equivalent to one considered in the literature, for which the unique solution for F' is [39, 71, 67]

$$F' = -4KL^a L_a + 2KL^\alpha L_\alpha. \tag{5.31}$$

The second special case is for $E = \frac{1}{2}$, which allows Σ^a to be excluded from the algebra. In this case one finds the unique solution

$$F' = 2KL^\alpha L_\alpha. \tag{5.32}$$

We conclude that topological charge algebras associated with the NS-NS potential of the standard, type IIB action can be used to construct D-brane actions on extended, type IIB superspaces. In the general case where all generators are present, the existence of a Lorentz invariant 2-cocycle of dimension two indicates that the superspace is extended more than is necessary for cocycle trivialization. The algebras in the two special cases $E = \{\frac{1}{4}, \frac{1}{2}\}$ are the desirable, “minimal extension” cases.

5.4.2 κ-symmetry

To show that the new F' potentials yield D-brane actions (5.16) which are κ-symmetric we need to find a right action satisfying the conditions (5.21) and (5.23). Let us first consider the general case where all the generators are present. Associate the parameters ϵ^A of the right action with the generators of the algebra (5.25) as

$$\{P_a, Q_\alpha, \Sigma^a, \Sigma^\alpha, \Sigma_{\alpha\beta}\} \quad \rightarrow \{ \quad \epsilon^a, \epsilon^\alpha, \epsilon_a, \epsilon_\alpha, \epsilon^{\alpha\beta}\}. \tag{5.33}$$

From (5.21), we already know that $\epsilon^a = 0$, while ϵ^α is given by (5.1). To find the remaining parameters one needs to use properties of the matrices Γ^a and σ_i under trace[2] in d=10:

$$\begin{aligned}(\Gamma_a\sigma_3\Gamma_b\sigma_3)^\alpha{}_\alpha &= 64\eta_{ab} \\ (\Gamma_a\sigma_3\Gamma_b)^\alpha{}_\alpha &= 0.\end{aligned} \tag{5.34}$$

By requiring $di_{V_\kappa}F' = 0$ one then finds a solution for the parameters $\epsilon^A = \{0, \epsilon^\alpha, \epsilon_a, 0, \epsilon^{\alpha\beta}\}$, with

$$\begin{aligned}\epsilon_a &= \left(\frac{C_2}{C_1 + \frac{C_3C_4}{C_5}}\right)\eta_{ab}L_i{}^b g^{ij}\epsilon^\alpha L_{j\alpha} \\ \epsilon^{\alpha\beta} &= -\left(\frac{C_4}{64C_5}\right)\epsilon_a(\Gamma^a\sigma_3)^{\alpha\beta}.\end{aligned} \tag{5.35}$$

In the case $E = \frac{1}{4}$ (with $\Sigma^{\alpha\beta}$ absent), the solution for F' is given by (5.31). In this case one finds the solution for the parameters $\epsilon^A = \{0, \epsilon^\alpha, \epsilon_a, 0\}$, with

$$\epsilon_a = \left(\frac{C_2}{C_1}\right)\eta_{ab}L_i{}^b g^{ij}\epsilon^\alpha L_{j\alpha}. \tag{5.36}$$

This is analogous to the solution for manifest κ-symmetry of the minimal GS superstring action considered in [45].

The mechanism used in the previous two cases relies on the existence of a matrix $K_a{}^i$ such that

$$L_i{}^a K_a{}^j = \delta_i{}^j. \tag{5.37}$$

Indeed, the results were obtained by setting

$$K_a{}^j = \eta_{ab}L_k{}^b g^{kj}. \tag{5.38}$$

[2]Since we are working with compound IIB indices, $\Gamma^\alpha{}_\beta$ is actually shorthand for $\Gamma^{\alpha I}{}_{\beta J} = \Gamma^\alpha{}_\beta \delta^I{}_J$. This yields twice the ordinary trace of gamma matrices.

In the case $E = \frac{1}{2}$ (with Σ^a absent), the solution for F' is given by (5.32). In this case, a similar mechanism works only if there exists a matrix $K_\alpha{}^i$ such that

$$L_i{}^\alpha K_\alpha{}^j = \delta_i^j. \tag{5.39}$$

Such a solution does not exist due to the noninvertibility of fermionic variables [30]. Thus, the right action mechanism for κ-symmetry considered here does not appear to apply when Σ^a is absent from the algebra.

Of the three solutions (5.29), (5.31) and (5.32) for F', the last two make use of algebras which are minimal extensions of the standard background, while the first two admit κ-symmetry via the right action mechanism considered here. The extended superalgebra leading to (5.31) is thus a good choice for extended superspace, type IIB D-brane actions.

5.5 Extended type IIA superspace actions

5.5.1 Cocycle trivialization

The calculations proceed in the same way as for the type IIB case. We wish to find extended type IIA superspaces which admit left invariant solutions of the equation

$$dF' \;=\; KL^a d\bar{\theta}\Gamma_{11}\Gamma_a d\theta \tag{5.40}$$

(the action used here again corresponds to $K = \frac{1}{2}$). These superspaces will then allow D-brane actions without BI gauge fields to be defined via the action (5.16). We consider the family of topological charge algebras associated with the NS-NS potential of the D-membrane, derived in chapter 4. This family is

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= \Gamma^a{}_{\alpha\beta}P_a - \left[\left[E_2 - \frac{1}{2}\right](\Gamma_{11}\Gamma_a)_{\alpha\beta} + E_1\Gamma_{a\alpha\beta}\right]\Sigma^a \qquad (5.41)\\
&\quad + \left[E_2 - \frac{1}{4}\right]\Sigma_{\alpha\beta}\\
[Q_\alpha, P_b] &= \left[E_1\Gamma_{b\alpha\beta} + E_2(\Gamma_{11}\Gamma_b)_{\alpha\beta}\right]\Sigma^\beta\\
[Q_\alpha, \Sigma^b] &= \Gamma^b{}_{\alpha\beta}\Sigma^\beta\\
[Q_\alpha, \Sigma_{\beta\gamma}] &= -\left[\Gamma^a{}_{\alpha(\beta}(\Gamma_{11}\Gamma_a)_{\gamma)\delta} - \Gamma^a{}_{\delta(\beta}(\Gamma_{11}\Gamma_a)_{\gamma)\alpha}\right]\Sigma^\delta,
\end{aligned}$$

where E_1 and E_2 are free constants. Associate the generators with the left vielbein components as

$$\{P_a, Q_\alpha, \Sigma^a, \Sigma^\alpha, \Sigma_{\alpha\beta}\} \;\rightarrow\; \{L^a, L^\alpha, L_a, L_\alpha, L^{\alpha\beta}\}. \tag{5.42}$$

The resulting Maurer-Cartan equations are

$$\begin{aligned}
dL^\alpha &= 0 \qquad (5.43)\\
dL^a &= -\frac{1}{2}d\bar{\theta}\Gamma^a d\theta\\
dL_a &= \frac{1}{2}\left[E_2-\frac{1}{2}\right]d\bar{\theta}\Gamma_{11}\Gamma_a d\theta+\frac{1}{2}E_1 d\bar{\theta}\Gamma_a d\theta\\
dL_\alpha &= -E_1 L^b(\Gamma_b d\theta)_\alpha - E_2 L^b(\Gamma_{11}\Gamma_b d\theta)_\alpha - L_b(\Gamma^b d\theta)_\alpha\\
&\quad +L^{\beta\gamma}\left[(\Gamma^b d\theta)_\beta(\Gamma_{11}\Gamma_b)_{\gamma\alpha}-\Gamma^b{}_{\alpha\beta}(\Gamma_{11}\Gamma_b d\theta)_\gamma\right]\\
dL^{\alpha\beta} &= -\frac{1}{2}\left[E_2-\frac{1}{4}\right]d\theta^\alpha d\theta^\beta.
\end{aligned}$$

The general solution exists for all $E_2 \neq \frac{1}{4}$, and again contains a free parameter C:

$$\begin{aligned}
F' &= C_1 L^a L_a + C_2 L^\alpha L_\alpha + C_3 L^a L^{\alpha\beta}\Gamma_{a\alpha\beta} + C_4 L^a L^{\alpha\beta}(\Gamma_{11}\Gamma_a)_{\alpha\beta} \qquad (5.44)\\
&\quad +C_5 L_a L^{\alpha\beta}\Gamma^a{}_{\alpha\beta} + C_6 L^{\alpha\beta}\Gamma^a{}_{\alpha\beta}L^{\gamma\delta}(\Gamma_{11}\Gamma_a)_{\gamma\delta}\\
C_1 &= -4K+\frac{C}{4}(4E_2-1)\\
C_2 &= 2K\\
C_3 &= E_1 C\\
C_4 &= \frac{C}{2}(2E_2-1)\\
C_5 &= C\\
C_6 &= -\frac{2C(2E_2-1)}{4E_2-1}.
\end{aligned}$$

Again, the free parameter C shows the presence of a Lorentz invariant 2-cocycle of dimension two on the extended superspace. This cocycle is explicitly given by the solution (5.44) for F' with $K = 0$.

The first special case of the algebra (5.41) is $E_2 = \frac{1}{4}$, which allows $\Sigma^{\alpha\beta}$ to be excluded. In the gauge $E_1 = 0$, this algebra corresponds to one used in [34]. However, the unique solution for F' exists for *all values* of E_1, and is found to be

$$F' = -4KL^a L_a + 2KL^\alpha L_\alpha. \qquad (5.45)$$

The second special case is when $E_1 = 0$ and $E_2 = \frac{1}{2}$, in which case Σ^a may be excluded. The unique solution is then

$$F' = 2KL^\alpha L_\alpha. \qquad (5.46)$$

We conclude that topological charge algebras associated with the NS-NS potential of the standard, type IIA D-brane action can be used to define actions on their associated superspaces. The two special cases yield extensions of the standard supertranslation algebra which are "minimal" in the sense of cocycle trivialization.

5.5.2 κ-symmetry

The calculations for the κ-symmetries proceed similarly to the type IIB case. One now needs the trace properties

$$\begin{aligned} (\Gamma_{11}\Gamma_a\Gamma_{11}\Gamma_b)^\alpha{}_\alpha &= -32\eta_{ab} \\ (\Gamma_{11}\Gamma_a\Gamma_b)^\alpha{}_\alpha &= 0. \end{aligned} \tag{5.47}$$

Associate the parameters ϵ^A of the right action with the generators of the superalgebra as

$$\{P_a, Q_\alpha, \Sigma^a, \Sigma^\alpha, \Sigma_{\alpha\beta}\} \quad \rightarrow \{ \quad \epsilon^a, \epsilon^\alpha, \epsilon_a, \epsilon_\alpha, \epsilon^{\alpha\beta}\}. \tag{5.48}$$

In the general case where all the generators are present one finds the solution $\epsilon^A = \{0, \epsilon^\alpha, \epsilon_a, 0, \epsilon^{\alpha\beta}\}$, with

$$\begin{aligned} \epsilon_a &= \left(\frac{C_2}{C_1 + \frac{C_4 C_5}{C_6}}\right) \eta_{ab} L_i{}^b g^{ij} \epsilon^\alpha L_{j\alpha} \\ \epsilon^{\alpha\beta} &= -\left(\frac{C_5}{32 C_6}\right) \epsilon_a (\Gamma_{11}\Gamma^a)^{\alpha\beta}. \end{aligned} \tag{5.49}$$

When $\Sigma^{\alpha\beta}$ is absent ($E_2 = \frac{1}{4}$), one again obtains $\epsilon^A = \{0, \epsilon^\alpha, \epsilon_a, 0\}$, with

$$\epsilon_a = \left(\frac{C_2}{C_1}\right) \eta_{ab} L_i{}^b g^{ij} \epsilon^\alpha L_{j\alpha}. \tag{5.50}$$

The case where Σ^a is absent ($E_1 = 0$ and $E_2 = \frac{1}{2}$) again does not admit κ-symmetry via the mechanism considered here.

Thus, in the type IIA case there is a *one parameter family* of algebras $(E_1, E_2) = (E_1, \frac{1}{4})$ characterized as "minimal extensions" which also admit κ-symmetry via a right action.

5.6 Comments

New superalgebras resulting from the topological charge algebras of the standard D-brane action were shown to allow the construction of D-brane actions on extended superspaces. In all but one discrete case, these actions were shown to admit κ-symmetry via an appropriately chosen right group action.

There is some correspondence between *manifest* symmetries of Green-Schwarz superstring actions and (non-manifest) symmetries of the D-brane action. In the case of the superstring, one constructs a WZ 2-form on an extended background superspace. One requires that this form be super-Poincaré invariant, and also be such that it admits manifest κ-symmetry of the action. Although the type II superalgebras are somewhat different in structure, solutions of the same structure appear for the F field of the D-branes. These solutions yield κ-symmetric actions without BI gauge fields.

For extended superspace D-brane actions, the correspondence of κ-symmetry with a right action shows that supercovariant derivatives are involved. We can thus expect the same structure found for the standard p-brane action to emerge [29]. The constraints associated with θ^α should once again be supercovariant derivatives on the extended superspace (note that this is not the case for the standard D-brane action due to the gauge field contributions). The commutator of these constraints should be a half rank matrix which corresponds to the projector (5.13) via a Legendre transformation.

Concerning gauge symmetries and the extended superspace formulation, we point out that κ-symmetry is not the full story. The "price" one has paid for an action without worldvolume gauge fields is the introduction of extra superspace degrees of freedom. If the formulation in terms of the two actions are to be equivalent, they must possess the same number of degrees of freedom. There must then exist gauge symmetries which allow the extra superspace degrees of freedom to be reduced to those of the BI gauge field. This issue has been considered in [34, 79].

Chapter 6

Superspace geometry and supersymmetry

6.1 Introduction

It has been a recurring theme in our work that the consideration of fermionic topological charges leads to some interesting results in the context of Noether charge algebras. Prior calculations of Noether charge algebras [28, 31, 65] have omitted fermionic topological charges due to the trivial topology that is traditionally assigned to fermionic dimensions [30]. However, in certain cases, the fermionic topological terms are *required*. For example, in a pp-wave background, fermionic brane charges are required to ensure quantum consistency with Jacobi identities [42], and this background admits zero modes in which these charges are nonvanishing. We might also ask an obvious question: if the fermionic directions are not compact, why do we not observe them directly as extended fermionic dimensions? We thus find motivation to seek to explain how fermionic charges might arise.

One has a clear geometric understanding of why Poincaré invariance is present in field theories: it is a consequence of the inability to prefer one inertial reference frame over another. Supersymmetry, by comparison, is not geometric in nature and thus appears quite abstract. What is its meaning? In this chapter we will argue that supersymmetry is a consequence of a particular choice of compact submanifold for the background superspace of the theory. In this description, the compactified dimensions are represented by an equivalence class of coordinates defined by transition functions which are super-Poincaré transformations. The parameters describing the lengths of the compactified dimensions are free, fundamental constants. We argue that the resulting compact, fermionic geometry is consistent with well defined equations of motion.

In considering the topological charge algebras, we must define the geometry of the "ghost fields" which arise in the descent construction [29, 58, 49, 50, 51]. In this construction, a second differential operator is introduced in a rather asymmetrical way to the de Rham differential. However, these two differentials are not so different: in chapter 2 we found that their actions are isomorphic via a chain map. This indicates that a duality exists between the two differentials. In this chapter we make this duality explicit. In the process, the ghost fields are described as a right vielbein, so that their geometry is well defined.

We argue that the equivalence class description provides a geometric basis for the existence of well defined, nonvanishing, fermionic topological charges. The resulting topological charge algebra is shown to be well defined. The picture is one in which fundamental particles, both bosonic *and* fermionic, might be described as different windings of the compactified superspace dimensions.

6.2 Geometric interpretation of supersymmetry

For superstrings, nonvanishing topological charges are realized either via open strings, or via closed strings with "winding" behavior. However, the theory we are considering contains only closed strings. In this and other such theories the fermionic charges have always been taken to vanish due to the trivial de Witt topology [30] adopted for the fermionic dimensions. However, the previous chapters have formally shown that fermionic topological charges have equal significance to bosonic ones, and that the full picture becomes apparent only when both are retained. Furthermore, in certain backgrounds, nonvanishing fermionic brane charges are required to ensure quantum consistency with Jacobi identities [42]. Yet, we do not apparently observe anything corresponding to the infinitely extended fermionic dimensions described by the de Witt topology. For these reasons we would like a mechanism by which strings can wrap compact, fermionic dimensions. Here we will argue that such a possibility exists.

The simplest way to describe a bosonic torus is via a single set of coordinate differences ("transition functions")

$$\delta x^m = c^m \tag{6.1}$$

between charts. We thus have an equivalence class, where points differing by coordinates c^m are identified. This is the "multi-valued coordinate" approach (in contrast to the "good cover" approach of manifold theory). The operator δ is the "sheet difference operator." Note that, for example, dx^m is well defined on the bosonic torus since $\delta dx^m = 0$. This is analogous to separating revolutions of the complex plane into different Riemann sheets:

$$\begin{aligned} \delta\phi &= 2\pi \\ \delta e^{i\phi} &= 0. \end{aligned} \tag{6.2}$$

All superstring theories with extended spacetime manifolds have in common that the four observable bosonic dimensions are extended, while the remaining bosonic dimensions are compact. Fermionic degrees of freedom are traditionally described by the de Witt topology [30], which describes an infinitely extended fibre of trivial topology. This topology does not allow compactification, yet we do not observe any extended fermionic spacetime dimensions either. The only choice in this case is to treat fermionic dimensions as an abstract degree of freedom which is not directly observable as part of the spacetime continuum. It would certainly be preferable to establish a compact description of the fermionic dimensions instead. In this chapter we will propose such a description via reinterpretation of the supersymmetry transformations, and make some investigations into its mathematical consistency.

Consider then, a compact superspace with a set of transition functions

$$\begin{aligned} \delta\theta^\mu &= \rho^\mu \\ \delta x^m &= c^m - \frac{1}{2}\overline{\rho}\Gamma^m\theta \end{aligned} \tag{6.3}$$

that define coordinate differences between sheets. Note that since $\overline{\rho}\Gamma^a\rho = 0$, the coordinate differences themselves are well defined: $\delta^2 Z^M = 0$. The coordinate differences take the same form as left supertranslations:

$$\begin{aligned} \delta_C Z^M &= C^A R_A{}^M \\ C^A &= (c^a, \rho^\alpha). \end{aligned} \tag{6.4}$$

We define an equivalence class by identifying all coordinates differing by applications of the transformations (6.3) with a single point on the compact submanifold. Since we cannot prefer one Riemann sheet over another, the only option is to allow *all*, and to see if the theory is still well defined.

The equations are indicative of a torus-like configuration (e.g. θ is defined modulo ρ); the only difference being that the bosonic directions pick up a fermionic term in the transition between sheets. The lengths of each compact dimension shall be defined by some fundamental constants ${C_0}^A = ({c_0}^a, {\rho_0}^\alpha)$, and are identically given by the RHS of (6.4) with C^A replaced by ${C_0}^A$. We then require C^A to be *quantized* to independent integer multiples of ${C_0}^A$. These integers, which are the number of times the string wraps the associated dimension, are the *winding numbers*.

The parameters C^A are associated with the *string itself*: they describe the winding of the string around the background. This is reasonable, since each string is governed by its own action, and in that action we always consider the *pullback* of the background superspace geometry onto the string worldvolume. Thus, unless a string wraps a certain dimension, the topology of that dimension is trivial as far as the action is concerned. Using the torus analogy, we may unfold the dimensions wrapped by the string into a "cuboid" with dimensions (6.3), and the string will traverse each dimension of this cuboid exactly once. What we have just described is a *geometric* interpretation of supersymmetry as the inability to prefer one coordinate sheet of the closed string worldvolume over another.

In [40], a solution to linearized fermionic equations of motion for an open D-string was found. Fermionic "topological" charges were then viewed as arising from open strings taking different fermionic values at the endpoints. In the present, compact description of the fermionic directions we would simply identify the endpoint values of the open string. Such a solution thus translates into a closed string wrapping around a compact, fermionic direction.

The equations of motion for the string are explicitly super-Poincaré invariant. Thus, a string winding around a given dimension will not be "broken" due to potentially differing dynamics at two identified coordinate points. The total winding number of a system of strings is thus *conserved*, and a candidate for a conserved quantity in particle physics. In the case of the bosonic torus, one can consider processes where a closed string bifurcates into two closed strings, and where the sum of the winding numbers is conserved [20]. The present description extends this possibility to the fermionic directions as well: if one has initially a string described by C_1, and finally strings described by C_2 and C_3, then $C_1 = C_2 + C_3$.

6.3 Balanced differential pair

Topological anomalous terms of Noether charge algebras can been derived via the descent construction [29, 58, 49], which requires the introduction of a "ghost field" for each coordinate. Since we are dealing with topologically nontrivial superspaces, we need to know the geometry of these ghosts. For example, if e^A are the ghosts and δ is the sheet difference operator, then what is δe^A? With the definitions of chapter 2 there is no obvious answer to this question. Here we will redefine the s differential so that the duality between s and d becomes manifest. Note that equation (2.35) and the chain map (2.45) suggest that the ghosts might be identified with a right vielbein. We will show that the following definition for s not only leads to the properties described in chapter 2, but also eliminates the ghost fields by identifying them as a right vielbein of s.

First, redefine s as *a copy of the de Rham differential*. The associated fundamental coordinate differentials are sZ^M, where Z^M are the superspace coordinates. s and sZ^M exactly the same properties as d and dZ^M. For example, s is a right derivation satisfying

$$\begin{aligned} s^2 &= 0 \\ \left\{sZ^M, sZ^N\right] &= 0 \\ sf(Z) &= sZ^M \partial_M f(Z), \end{aligned} \tag{6.5}$$

where $f(Z)$ is a function, and in the second line a graded anticommutator is used. Like d, s is exact. These properties are sufficient to define the action of s on s-forms. We can also introduce the additional vielbeins

$$\begin{aligned} l &= g^{-1} sg \\ r &= sgg^{-1}, \end{aligned} \tag{6.6}$$

with g as defined in (2.3). These have the components

$$\begin{aligned} l^A &= sZ^M L_M{}^A \\ r^A &= sZ^M R_M{}^A, \end{aligned}$$

with L and R as defined in (2.4) and (2.5). The interplay between this differential and the de Rham differential is defined by two more fundamental properties. The first is the interplay of the gradings:

$$\begin{aligned} [s, d] &= 0 \\ \left[sZ^M, dZ^N\right\} &= 0. \end{aligned} \tag{6.7}$$

The final property

$$sL^A = 0 \tag{6.8}$$

is required to determine the action of s on d forms. Thus, with respect to d forms, s is a differential acting along the orbits of the left group action.

Equation (6.8) also defines the action of d on s forms, which can be seen as follows. Expanding (6.8) we find

$$sdZ^M = L^A r^B Q_B L_A{}^M, \tag{6.9}$$

where Q_A and D_A are as defined in (2.7) and (2.12). The following is a group identity:

$$D_B R_A{}^M - (-1)^{AB} Q_A L_B{}^M = 0 \tag{6.10}$$

which may be seen as follows. Since the L^A are left invariant by construction, under Lie derivative they satisfy $\mathcal{L}_U L^A = 0$ for all left invariant vector fields U. Similarly, $\mathcal{L}_V R^A = 0$ for all right invariant vector fields V. Expanding either one of these identities in a suitable basis yields equation (6.10). Using this in (6.9) we find

$$dsZ^M = r^A L^B D_B R_A{}^M, \tag{6.11}$$

or

$$dr^A \;=\; 0. \tag{6.12}$$

Thus, with respect to s forms, d is a differential acting along the orbits of the right group action. Equations (6.8) and (6.12) are equivalent: either may be considered a postulate and the other follows.

We now have sufficient information to evaluate the differentials of "generalized forms." An (m,n)-form F will be written using indices M, N, L for the coordinate bases, and A, B, C for the invariant bases:

$$\begin{aligned} F &= sZ^{N_n} \ldots sZ^{N_1} dZ^{M_m} \ldots dZ^{M_1} F_{M_1 \ldots M_m, N_1 \ldots N_n} \frac{1}{m!n!} \\ &= r^{B_n} \ldots r^{B_1} L^{A_m} \ldots L^{A_1} F_{A_1 \ldots A_m, B_1 \ldots B_n} \frac{1}{m!n!}. \end{aligned} \tag{6.13}$$

The differentials of the vielbeins may be obtained from the given rules. For reference, we provide them here:

$$\begin{aligned} dL^A &= -\tfrac{1}{2} L^C L^B t_{BC}{}^A, & sL^A &= 0 \\ dr^A &= 0, & sr^A &= \tfrac{1}{2} r^C r^B t_{BC}{}^A \\ dl^A &= -l^C L^B t_{BC}{}^A, & sl^A &= -\tfrac{1}{2} l^C l^B t_{BC}{}^A \\ dR^A &= \tfrac{1}{2} R^C R^B t_{BC}{}^A, & sR^A &= -r^C R^B t_{BC}{}^A \end{aligned} \tag{6.14}$$

In the above, $t_{BC}{}^A$ are the structure constants of the underlying supertranslation algebra. The first two lines are the properties we have previously used to evaluate differentials. *We thus identify the ghost fields with the right s-vielbein* r^A. Sheet differences such as δr^A are thus defined.

6.4 Topological charge algebra

Topological charge algebras are the result of descent sequences involving fields which are *not* super-Poincaré invariant. The consequences of introducing the superspace equivalence classes of section 6.2, defined by super-Poincaré transformations, therefore requires examination. Let us investigate the topological charge algebra (2.129) of the GS superstring action. We will consider the ambiguity induced in the anomalous term by summing the contributions made by each potential of the descent sequence.

First let us consider the ambiguity in the anomalous term due to that of the WZ term. We have

$$\begin{aligned} \delta B &= C^A Q_A B \\ &= -C^A dW_A. \end{aligned} \tag{6.15}$$

However, this is a *gauge transformation* of the form $\Delta B = -d\Psi$. We have shown in chapter 1 that such gauge transformations do not affect the topological charge algebra. Thus there is no change in M due to δB.

Let us now consider the ambiguity in the anomalous term due to δW, or equivalently, due to δM itself. Using the result (2.126) and the identification of ghosts with r^A, this algebra is defined by a family of anomalous terms M with a free parameter E:

$$M = (E-1)s\bar{\theta}\Gamma_a s\theta dx^a - (E+1)r^a s\bar{\theta}\Gamma_a d\theta. \tag{6.16}$$

Since

$$\begin{aligned} \delta x^a &= -\frac{1}{2}\bar{\rho}\Gamma^a\theta \\ \delta r^a &= -\bar{\rho}\Gamma^a s\theta, \end{aligned} \tag{6.17}$$

we have

$$\delta M = -Es\bar{\theta}\Gamma^a s\theta\bar{\rho}\Gamma_a d\theta. \tag{6.18}$$

When the ambiguity due to sheet differences is included, the full family of topological charge algebras is

$$[Q_A, Q_B\} = -t_{AB}{}^C + M_{AB} + \delta M_{AB}, \tag{6.19}$$

which is explicitly

$$\begin{aligned} [Q_\alpha, Q_\beta\} &= -\Gamma^a{}_{\alpha\beta}P_a + 2(E-1)\Gamma_{a\alpha\beta}dx^a - 2E\Gamma_{a\alpha\beta}\bar{\rho}\Gamma^a d\theta \\ [Q_\alpha, P_b\} &= -(E+1)(\Gamma_b d\theta)_\alpha. \end{aligned} \tag{6.20}$$

However, the ρ dependent term is central and may be redefined away. For example, defining

$$\begin{aligned} P'_a &= P_a - 2\overline{\rho}\Gamma^a d\theta \\ \Sigma^a &= dx^a - \overline{\rho}\Gamma^a d\theta \\ \Sigma^\alpha &= d\theta, \end{aligned} \tag{6.21}$$

we obtain the algebra

$$\begin{aligned} [Q_\alpha, Q_\beta\} &= -\Gamma^a{}_{\alpha\beta}P'_a + 2(E-1)\Gamma_{a\alpha\beta}\Sigma^a \\ [Q_\alpha, P'_b\} &= -(E+1)\Gamma_{b\alpha\beta}\Sigma^\beta \\ [Q_\alpha, E^b\} &= -\frac{1}{2}\Gamma^b{}_{\alpha\beta}\Sigma^\beta, \end{aligned} \tag{6.22}$$

which yields (2.129) with the identification $a \to \frac{1}{2}(E+1)$.

Now consider the topological charges themselves:

$$\overline{\Sigma}^A = \int d\sigma \Sigma^A. \tag{6.23}$$

The sheet differences of the corresponding forms are

$$\begin{aligned} \delta\Sigma^\alpha &= 0 \\ \delta\Sigma^a &= -\frac{1}{2}\overline{\rho}\Gamma^a d\theta. \end{aligned} \tag{6.24}$$

However, recalling that ρ *defines* the integral

$$\int d\sigma d\theta^\alpha = \rho^\alpha, \tag{6.25}$$

we then have

$$\begin{aligned} \delta\overline{\Sigma}^a &= -\frac{1}{2}\overline{\rho}\Gamma^a \rho \\ &= 0. \end{aligned} \tag{6.26}$$

Thus, the charges are well defined. We conclude that the topological charge algebra of the string is well defined in the presence of multiple coordinate sheets.

6.5 Triple complex and p-branes

Here we will develop some new general observations about p-brane cohomology, especially those relating to the nontrivial topology of the background superspace. In chapter 2 we sidestepped the issue by assuming that forms were globally defined. However, in cases of nontrivial topology this is not always so. The situation is analogous to the "magnetic monopole," where the magnetic field strength is well defined, but one cannot define its potential globally without singularities. In the case of compact background superspaces, such an obstruction can also apply to the p-brane WZ term [80]. In order to deal with such cases we first need a brief review of Čech cohomology.

Let U_I be a set of open charts forming an atlas for the background superspace. Indicate chart intersections with multiple indices:

$$\begin{aligned} U_{IJ} &= U_I \cap U_J \\ U_{IJK} &= U_I \cap U_J \cap U_K. \end{aligned} \tag{6.27}$$

Formal sums of U_I are referred to as "0-chains", sums of U_{IJ} as "1-chains," sums of U_{IJK} as "2-chains" and so on. A field defined on a k-chain is referred to as a k-cochain. The chart indices on a k-cochain $Y_{I_k \ldots I_0}$ are assumed to be antisymmetric in order to eliminate overcounting. The fundamental differential of Čech cohomology is the difference operator δ, which maps k-cochains to $(k+1)$-cochains. The action of δ is defined by

$$\delta Y_{I_{k+1} \ldots I_0} = \sum_{a=1}^{k} (-1)^a Y_{I_{k+1} \ldots \hat{I}_a \ldots I_0}, \tag{6.28}$$

where $\hat{I}_a$ indicates that I_a is omitted. This action can be shown to be nilpotent [64], and it thus defines a differential.

The Čech differential and the de Rham differential commute identically (this is equivalent to the fact that d commutes with the action of a pullback). Since s and d are equivalent operators when considered in isolation, they should each have the same interaction with the Čech differential. That is, s and δ should commute identically. Thus, sZ^M and dZ^M represent two copies of the same cotangent space, with no distinction between them except in the action of one differential upon the other.

We now have three mutually commuting operators:

$$[s, d] = [s, \delta] = [d, \delta] = 0. \tag{6.29}$$

All three operators have trivial cohomology over the space of cochains when considered in isolation. For example, if we write a cochain Y using the r^A basis, then the de Rham differential sees something equivalent to an ordinary d form. Suppose Y is d closed and has d degree greater than one. Since a chain is simply a collection of open sets, the Poincaré lemma then tells us that

$Y = dX$ for some cochain X. This is what we mean by trivial cohomology over the space of cochains. Note that this does not imply trivial *de Rham* cohomology, since it may not be possible to write X as a globally defined form. Since s and d are equivalent, the same argument shows that s also has trivial cohomology over the space of cochains. The cohomology of δ also turns out to be trivial in the same sense (for a proof, see [64]). Such *exact* operators work well in pairs because they preserve triviality with respect to a total differential constructed from them. These properties lead us to define a "triple complex" with the following total differential

$$\begin{aligned} \mathcal{D} &= \delta + (-1)^l s + (-1)^{l+n+1} d \\ \mathcal{D}^2 &= 0, \end{aligned} \tag{6.30}$$

where l is the δ degree, and n is the s degree of the cochain upon which $\mathcal{D}$ acts. There exist several consistent possibilities for the sign choices in $\mathcal{D}$. These may be discovered by considering a (D, δ) system, where D is the (d, s) differential considered in chapter 2. However, D is *not* an exact operator, even over trivial spaces (for example $DH = 0$ yet there is no B in standard superspace such that $H = DB$). There is no (D, δ) descent system because the Poincaré lemma does not apply to D. So the triple complex just defined is not equivalent to a double complex.

This triple complex has a natural application to p-branes since the field strength H is triply closed:

$$sH = dH = \delta H = 0. \tag{6.31}$$

The first two follow from the explicit structure of H (2.17). The third is equivalent to the requirement that field strengths are in general observable, and must therefore be globally defined forms. When considering possibilities for compact superspaces, we must check that the structure (2.17) does indeed yield a globally defined H.

There are two sets of descent equations following from the closure of H. Let us first consider those following from $\delta H = 0$. In general, the WZ form is not globally defined, but is a 0-cochain $B = \oplus B_I$ which defines "local WZ terms" [80]. Using $[d, \delta] = 0$, one finds that on U_{IJ}

$$\begin{aligned} \delta B_{IJ} &= B_I - B_J \\ &= -d\Psi_{IJ}, \end{aligned} \tag{6.32}$$

for some 1-cochain $\Psi = \oplus \Psi_{IJ}$. We choose the sign in the second line such that the relevant term in $\mathcal{D}(B \oplus \Psi)$ vanishes. Equation (6.32) is one step of a (d, δ) descent sequence. For the string, the remaining equation is

$$\delta \Psi_{IJK} = dK_{IJK}, \tag{6.33}$$

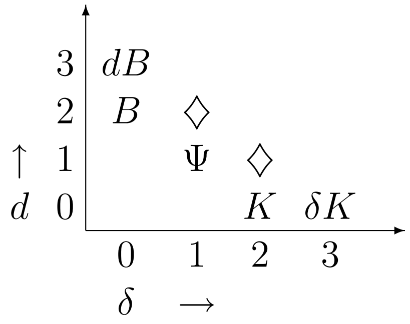

Figure 6.1: (d, δ) tic-tac-toe box for the string

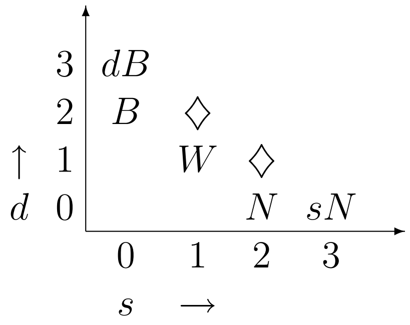

Figure 6.2: (d, s) tic-tac-toe box for the string

for some 2-cochain K. This sequence can be graphically represented by a "tic-tac-toe box" [64], as shown in figure 6.1. The sequence ends with δK. Since δK is d closed, it is a δ closed 3-cochain of constants: a Čech 3-cocycle. Nontrivial forms in de Rham cohomology correspond (via the descent sequence) to nontrivial Čech cocycles, and vice versa. This is the isomorphism between de Rham cohomology and Čech cohomology, which has some interesting consequences [80].

Next we consider the descent equations resulting from $sH = 0$. For the string, these are [29, 58, 49]

$$\begin{aligned} sB &= -dW \\ sW &= dN. \end{aligned} \tag{6.34}$$

This sequence is depicted in figure 6.2. At this point we would like to know whether δsN vanishes. If it does, there will be a third (s, δ) tic-tac-toe sequence. Firstly, let us point out that the structure of sN is unique and known [49]:

$$sN \propto s\theta^\alpha s\theta^\beta r^{a_1} \dots r^{a_p} (\Gamma_{a_1 \dots a_p})_{\alpha\beta}. \tag{6.35}$$

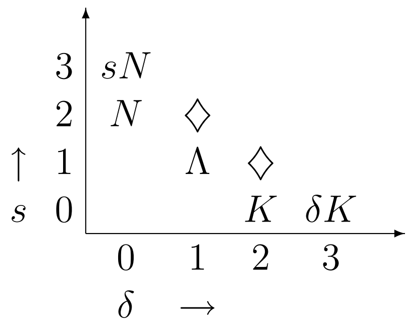

Figure 6.3: (s, δ) tic-tac-toe box for the string

This follows from the fact that H is unique [27], and from properties of (s, d) cohomology.

Since the (s, δ) double complex is equivalent to the (d, δ) double complex, we can consider the "s analog" of de Rham cohomology. To start with we need sN to be globally defined. If no fermionic directions are compactified then forms with a $d\theta^\alpha$ or $s\theta^\alpha$ are de Rham trivial. In this case, de Rham cohomology sees r^a as sx^a, and L^a as dx^a, which means that sN is then just H (up to a sign), with d replaced everywhere by s. Since H is globally defined by assumption, then so is sN. For example, toric compactification of the bosonic directions is consistent with globally defined H and sN. It also physically realizes the bosonic topological charges [28].

6.6 Commuting sequences

In the case of the superstring, further simplification is possible. Making use of the $p = 1$ Fierz identity, the (s, d) descent equations (6.34) for the string have solutions

$$
\begin{aligned}
B &= -\frac{1}{2} L^a \bar{\theta} \Gamma_a d\theta \\
W &= r^a L^b \eta_{ab} - \frac{1}{6} s\bar{\theta} \Gamma^a \theta \bar{\theta} \Gamma_a d\theta \\
N &= -\frac{1}{2} r^a \bar{\theta} \Gamma_a s\theta.
\end{aligned}
\tag{6.36}
$$

The terminating elements of the sequence are

$$
\begin{aligned}
H = dB &= \frac{1}{2} L^a d\bar{\theta} \Gamma_a d\theta \\
sN &= \frac{1}{2} r^a s\bar{\theta} \Gamma_a s\theta
\end{aligned}
\tag{6.37}
$$

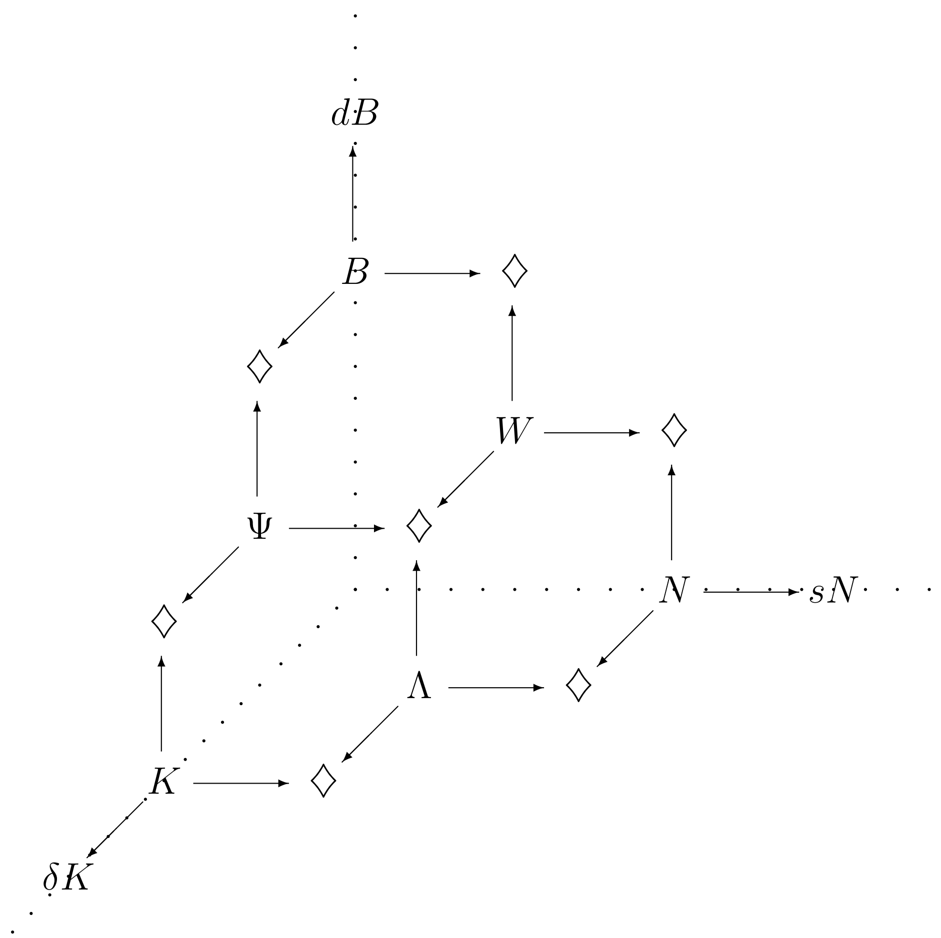

Figure 6.4: (d, δ, s) sequence for the string

$$= \frac{1}{2} l^a s\bar{\theta} \Gamma_a s\theta,$$

where the Fierz identity is used in the last step. Thus, since sN has the same structure as H, sN is globally defined whenever H is. It is also characterized by the *same element* of de Rham cohomology as H. This implies that the descending (s, δ) sequence shown in figure 6.3 terminates with the same element δK. That is, the three sequences *commute*. We can present these three sequences in totality using the three dimensional diagram shown in figure 6.4. The equation represented by $\diamondsuit$ in the center of the diagram is found as follows. Firstly, by expressing $s\delta B = \delta s B$ in terms of Ψ and W, we find $s\Psi - \delta W = dF$, with F an arbitrary $(1,1)$-form valued 1-cochain. Applying the same argument to $d\delta N$ and dsK, then requiring that the diagram commutes, we obtain

$$s\Psi - \delta W + d\Lambda = 0. \tag{6.38}$$

The entire diagram of figure 6.4 may be expressed as

$$-dB \oplus sN \oplus \delta K = \mathcal{D}(B \oplus W \oplus N \oplus \Psi \oplus \Lambda \oplus K). \tag{6.39}$$

In other words, equation (2.42) is now modified by the Čech term δK:

$$H \simeq sN \oplus \delta K. \tag{6.40}$$

The geometric meaning of δK is the charge associated with the field strength H that is enclosed by the p-brane worldvolume.

For every element of the triple complex with Čech degree $l \geq 1$ there is an associated gauge field of Čech degree $l-1$. The action of the gauge fields is defined by

$$\Delta(B \oplus W \oplus N \oplus \Psi \oplus \Lambda \oplus K) = \mathcal{D}(\psi \oplus \lambda \oplus k), \tag{6.41}$$

where ψ, λ, k are cochains of (d, s, δ) degree $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ respectively. The gauge fields ψ, λ defined previously, become 0-cochains. For example:

$$\Delta\Psi_{uv} = \delta\psi_{uv} + dk_{uv}, \tag{6.42}$$

where ψ_u is the same field as in (2.50), now expressed as a 0-cochain.

Chapter 7

Appendices

7.1 Standard vielbein components

7.1.1 $L_M{}^A$ components

$$
\begin{array}{ll}
L_m{}^a = \delta_m{}^a, & L_m{}^\alpha = 0 \\
L_\mu{}^a = -\frac{1}{2}(\Gamma^a\theta)_\mu, & L_\mu{}^\alpha = \delta_\mu{}^\alpha
\end{array}
$$

7.1.2 $L_A{}^M$ components

$$
\begin{array}{ll}
L_a{}^m = \delta_a{}^m, & L_a{}^\mu = 0 \\
L_\alpha{}^m = \frac{1}{2}(\Gamma^m\theta)_\alpha, & L_\alpha{}^\mu = \delta_\alpha{}^\mu
\end{array}
$$

7.1.3 $R_M{}^A$ components

$$
\begin{array}{ll}
R_m{}^a = \delta_m{}^a, & R_m{}^\alpha = 0 \\
R_\mu{}^a = \frac{1}{2}(\Gamma^a\theta)_\mu, & R_\mu{}^\alpha = \delta_\mu{}^\alpha
\end{array}
$$

7.1.4 $R_A{}^M$ components

$$
\begin{array}{ll}
R_a{}^m = \delta_a{}^m, & R_a{}^\mu = 0 \\
R_\alpha{}^m = -\frac{1}{2}(\Gamma^m\theta)_\alpha, & R_\alpha{}^\mu = \delta_\alpha{}^\mu
\end{array}
$$

7.2 Green algebra vielbein components

7.2.1 $L_M{}^A$ components

$$
\begin{array}{lll}
L_m{}^a = \delta_m{}^a, & L_m{}^\alpha = 0, & L_{m\alpha} = -(\Gamma_m\theta)_\alpha \\
L_\mu{}^a = -\frac{1}{2}(\Gamma^a\theta)_\mu, & L_\mu{}^\alpha = \delta_\mu{}^\alpha, & L_{\mu\alpha} = \frac{1}{6}(\Gamma^b\theta)_\mu(\Gamma_b\theta)_\alpha \\
L^{\mu a} = 0, & L^{\mu\alpha} = 0, & L^\mu{}_\alpha = \delta^\mu{}_\alpha
\end{array}
$$

7.2.2 $L_A{}^M$ components

$$
\begin{array}{lll}
L_a{}^m = \delta_a{}^m, & L_a{}^\mu = 0, & L_{a\mu} = (\Gamma_a\theta)_\mu \\
L_\alpha{}^m = \frac{1}{2}(\Gamma^m\theta)_\alpha, & L_\alpha{}^\mu = \delta_\alpha{}^\mu, & L_{\alpha\mu} = \frac{1}{3}(\Gamma^b\theta)_\alpha(\Gamma_b\theta)_\mu \\
L^{\alpha m} = 0, & L^{\alpha\mu} = 0, & L^\alpha{}_\mu = \delta^\alpha{}_\mu
\end{array}
$$

7.2.3 $R_M{}^A$ components

$$
\begin{array}{lll}
R_m{}^a = \delta_m{}^a, & R_m{}^\alpha = 0, & R_{m\alpha} = 0 \\
R_\mu{}^a = \frac{1}{2}(\Gamma^a\theta)_\mu, & R_\mu{}^\alpha = \delta_\mu{}^\alpha, & R_{\mu\alpha} = -x^b\Gamma_{b\mu\alpha} + \frac{1}{6}(\Gamma^b\theta)_\mu(\Gamma_b\theta)_\alpha \\
R^{\mu a} = 0, & R^{\mu\alpha} = 0, & R^\mu{}_\alpha = \delta^\mu{}_\alpha
\end{array}
$$

7.2.4 $R_A{}^M$ components

$$
\begin{array}{lll}
R_a{}^m = \delta_a{}^m, & R_a{}^\mu = 0, & R_{a\mu} = 0 \\
R_\alpha{}^m = -\frac{1}{2}(\Gamma^m\theta)_\alpha, & R_\alpha{}^\mu = \delta_\alpha{}^\mu, & R_{\alpha\mu} = x^b\Gamma_{b\alpha\mu} - \frac{1}{6}(\Gamma^b\theta)_\alpha(\Gamma_b\theta)_\mu \\
R^{\alpha m} = 0, & R^{\alpha\mu} = 0, & R^\alpha{}_\mu = \delta^\alpha{}_\mu
\end{array}
$$

7.3 Extended Green algebra vielbein components

7.3.1 $L_M{}^A$ components

$$
\begin{array}{llll}
L_m{}^a = \delta_m^a, & L_m{}^\alpha = 0, & L_{ma} = 0, & L_{m\alpha} = -(\Gamma_m\theta)_\alpha \\
L_\mu{}^a = -\frac{1}{2}(\Gamma^a\theta)_\mu, & L_\mu{}^\alpha = \delta_\mu{}^\alpha, & L_{\mu a} = -\frac{1}{2}(\Gamma_a\theta)_\mu, & L_{\mu\alpha} = \frac{1}{3}(\Gamma^b\theta)_\mu(\Gamma_b\theta)_\alpha \\
L^{ma} = 0, & L^{m\alpha} = 0, & L^m{}_a = \delta^m{}_a, & L^m{}_\alpha = -(\Gamma^m\theta)_\alpha \\
L^{\mu a} = 0, & L^{\mu\alpha} = 0, & L^\mu{}_a = 0, & L^\mu{}_\alpha = \delta^\mu{}_\alpha
\end{array}
$$

7.3.2 $L_A{}^M$ components

$$
\begin{array}{llll}
L_a{}^m = \delta_a{}^m, & L_a{}^\mu = 0, & L_{am} = 0, & L_{a\mu} = (\Gamma_a\theta)_\mu \\
L_\alpha{}^m = \frac{1}{2}(\Gamma^m\theta)_\alpha, & L_\alpha{}^\mu = \delta_\alpha{}^\mu, & L_{\alpha m} = \frac{1}{2}(\Gamma_m\theta)_\alpha, & L_{\alpha\mu} = \frac{2}{3}(\Gamma^b\theta)_\alpha(\Gamma_b\theta)_\mu \\
L^{am} = 0, & L^{a\mu} = 0, & L^a{}_m = \delta^a{}_m, & L^a{}_\mu = (\Gamma^a\theta)_\mu \\
L^{\alpha m} = 0, & L^{\alpha\mu} = 0, & L^\alpha{}_m = 0, & L^\alpha{}_\mu = \delta^\alpha{}_\mu
\end{array}
$$

7.3.3 $R_M{}^A$ components

$$
\begin{array}{llll}
R_m{}^a = \delta_m{}^a, & R_m{}^\alpha = 0, & R_{ma} = 0, & R_{m\alpha} = 0 \\
R_\mu{}^a = \frac{1}{2}(\Gamma^a\theta)_\mu, & R_\mu{}^\alpha = \delta_\mu{}^\alpha, & R_{\mu a} = \frac{1}{2}(\Gamma_a\theta)_\mu, & R_{\mu\alpha} = -x^b\Gamma_{b\mu\alpha} - y_b\Gamma^b{}_{\mu\alpha} \\
 & & & +\frac{1}{3}(\Gamma^b\theta)_\mu(\Gamma_b\theta)_\alpha \\
R^{ma} = 0, & R^{m\alpha} = 0, & R^m{}_a = \delta^m{}_a, & R^m{}_\alpha = 0 \\
R^{\mu a} = 0, & R^{\mu\alpha} = 0, & R^\mu{}_a = 0, & R^\mu{}_\alpha = \delta^\mu{}_\alpha
\end{array}
$$

7.3.4 $R_A{}^M$ components

$$
\begin{array}{llll}
R_a{}^m = \delta_a{}^m, & R_a{}^\mu = 0, & R_{am} = 0, & R_{a\mu} = 0 \\
R_\alpha{}^m = -\frac{1}{2}(\Gamma^m\theta)_\alpha, & R_\alpha{}^\mu = \delta_\alpha{}^\mu, & R_{\alpha m} = -\frac{1}{2}(\Gamma_m\theta)_\alpha, & R_{\alpha\mu} = +x^b\Gamma_{b\alpha\mu} + y_b\Gamma^b{}_{\alpha\mu} \\
 & & & -\frac{1}{3}(\Gamma^b\theta)_\alpha(\Gamma_b\theta)_\mu \\
R^{am} = 0, & R^{a\mu} = 0, & R^a{}_m = \delta^a{}_m, & R^a{}_\mu = 0 \\
R^{\alpha m} = 0, & R^{\alpha\mu} = 0, & R^\alpha{}_m = 0, & R^\alpha{}_\mu = \delta^\alpha{}_\mu
\end{array}
$$

7.4 Constraints for the p-brane action

Here we show that the constraints in the presence of the WZ term take the simple form (2.93) in both standard and extended backgrounds. The structure of the definitions of momenta may be written as the vanishing of functions $\widetilde{C}_M$ (one for each coordinate):

$$\widetilde{C}_M = P_M - P_M^{(NG)} - \overline{B}_M, \tag{7.1}$$

where the terms $P_M^{(NG)}$ are the functions of $(Z, \dot{Z})$ obtained as momenta from the NG action. However, $P_M^{(NG)}$ are nonzero only for the standard superspace coordinates, and they are related by

$$
\begin{aligned}
P_\mu^{(NG)} &= -\frac{1}{2}(\Gamma^m\theta)_\mu P_m^{(NG)} \\
\Rightarrow L_\alpha{}^M P_M^{(NG)} &= 0.
\end{aligned} \tag{7.2}
$$

For $M \neq m$, the $\widetilde{C}_M$ are constraints. Consider then the linear combinations:

$$L_A{}^M \widetilde{C}_M, \qquad M \neq m. \tag{7.3}$$

One can generate new sets of constraints by taking such linear combinations as long as the constraint surface so defined remains unchanged. This will be true provided that we maintain a “linearly independent” combination of the original constraints (which are all independent in the sense of intersecting surfaces). The linear combinations (7.3) will then be constraints of the form

$$\widetilde{C}_A = L_A{}^M(P_M - \overline{B}_M), \qquad A \neq a \tag{7.4}$$

provided that

$$L_A{}^M P_M^{(NG)} = 0. \tag{7.5}$$

Denote the extra generators of the superalgebra by $T_{\check{A}}$. These generators are assumed to form an ideal. It follows that the standard coordinates do not transform under the left/right group actions generated by $T_{\breve{A}}$. From this it follows that the components of the inverse vielbeins satisfy

$$\begin{aligned} L_{\breve{A}}{}^m &= L_{\breve{A}}{}^\mu = 0 \\ R_{\breve{A}}{}^m &= R_{\breve{A}}{}^\mu = 0. \end{aligned} \tag{7.6}$$

Using this and (7.2), the required result (7.5) follows. The constraints can therefore be written in the form of equation (2.93).

7.5 Additional gauge fields

The gauge transformations (4.84) and (4.113) are not the only ones consistent with dimensionality and Lorentz invariance. For example, in the IIA case one can also consider the gauge fields

$$\begin{aligned} \Lambda^{(b)} &= \bar{e}\Gamma^{a_1\ldots a_b}\theta\bar{\theta}\Gamma_{11}\Gamma_{a_1\ldots a_b}\theta \\ \Lambda'^{(b)} &= \bar{e}\Gamma_{11}\Gamma^{a_1\ldots a_b}\theta\bar{\theta}\Gamma_{a_1\ldots a_b}\theta, \end{aligned} \tag{7.7}$$

where in $\Lambda^{(b)}$, b is such that $\Gamma_{11}\Gamma_{a_1\ldots a_b}$ is antisymmetric, while in $\Lambda'^{(b)}$, b is such that $\Gamma_{a_1\ldots a_b}$ is antisymmetric. The minimal Green-Schwarz superstring appears to be special in that this type of gauge transformation does not contribute to the topological charge algebra [49]. In the present type IIA example, extra terms are contributed to (4.119). However, there are no extra generators required. Define $\Delta^b M = sd\Lambda^{(b)}$ and $\Delta'^b M = sd\Lambda'^{(b)}$. For $E_2 \neq \frac{1}{4}$ one can then set

$$\Sigma'_{\alpha\beta} = \Sigma_{\alpha\beta} - \left[\frac{1}{E_2 - \frac{1}{4}}\right](E_b\Delta^b M_{\alpha\beta} + E'_b\Delta'^b M_{\alpha\beta}). \tag{7.8}$$

The only alteration to the algebra then occurs as additional terms on the RHS of $[Q_\alpha, \Sigma_{\beta\gamma}]$. These additional terms do not appear to contribute to calculations involving trivialization of the cocycle (a point we will not illustrate here). Since this is currently the main application of the algebras, we chose not to make use of such gauge transformations.

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