SLIDING MODE FUNCTIONAL OBSERVERS FOR CLASSES OF LINEAR AND NON-LINEAR SYSTEMS

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Declaration

The work presented in this thesis was carried out in the School of Electrical, Electronic and Computer Engineering during the period, February 2006 to November 2008 in fulfillment of the requirements of the degree of Doctor of Philosophy.

I certify that the work embodied in this thesis is the result of original research unless otherwise stated in the text. The work presented here has not been previously submitted for a higher degree to any other University or Institution.

Navid Nikraz

November 2008

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Abstract

In many practical situations the physical state of a system cannot be determined by direct observation. The problem of constructing both state and functions of the states is currently one of the leading topics in Control Engineering. Since the re-emergence of the state-space methods to form a direct multivariable approach to linear control systems synthesis and design, a host of controllers now exist to meet the qualitative and quantitative criteria including system stability and optimality.

Sliding Mode Control (SMC) has been successfully applied to many practical control problems due to its attractive features such as invariance to matched uncertainties. The characteristic feature of the continuous-time SMC system is that sliding mode occurs on a prescribed manifold, where switching control is employed to maintain the state on the surface. When a sliding mode is realised, the system exhibits some superior robustness properties with respect to external matched uncertainties. Systems with sliding modes have proven to be an efficient tool to control complex high-order nonlinear dynamic plants operating under uncertainty conditions, a common problem for many processes that are in modern control. It is acknowledged that the notion of the sliding mode observer is inherently critical to the field of sliding mode control.

Sliding mode state observers have been the focus of research in the field of control for the past decade and their applications have extended to most areas in this field. The concept of the Sliding Mode Functional Observers is an area that is untouched by previous research. This thesis introduces the concept of the Sliding

Mode Functional Observer and demonstrates its vast potential in the field of control.

It is appreciated that sliding mode observers do have a very important place in the field of control, and in particular motor control. However, it has been discovered that currently one of the major fallbacks of sliding mode observers is their strict and sometimes unrealistic boundaries and conditions on systems. This has narrowed the scope to which they can effectively be applied to practical cases. The major objective of this thesis is to extend the application of sliding mode observers to incorporate sliding mode functional observers for systems that exhibit somewhat unique properties, that have made the practical applications relatively difficult in the past.

In this thesis the Sliding Mode Functional Observer is applied to a number critical areas in control; Neutral-Delay System, Systems with Unknown Inputs, Non-Linear Systems and Descriptor Systems. Firstly the problem of estimating a linear function of the states of a class of linear time-delay systems of the neutral-type using sliding mode functional observers is investigated. Next, conditions for designing a sliding mode functional observer when the system is subjected to unknown inputs are presented, reducing the existence conditions for the Utkin Observer and also reducing the well known Matching and Observability conditions required for the design of the observer.

The importance of non-linear systems in the Control Engineering environment is acknowledged and a design algorithm for the application of sliding mode functional observers for non-linear systems is presented. Conditions are derived for the solvability of the design matrices of the proposed observer and for the stability of its dynamics. Asymptotic stability conditions are then developed using Linear Matrix Inequality (LMI) formulation. Finally, the problem of estimating a linear function of the states of a system with unknown inputs using the sliding mode functional observer is investigated based on the descriptor system approach. The design procedure applies a reduced-order asymptotic observer to estimate a linear function of the

state and the unknown input. A design algorithm accompanies the four applications presented and simulations are presented in each case to validate the analysis.

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Chapter 1

Motivation

1.1 Introduction

In control theory, a state observer is a system that models a real system in order to provide an estimate of its internal state, given measurements of the input and output of the real system. The problem of observing the state vector of a deterministic linear time-invariant multivariable system has been the object of numerous studies ever since the original work of Luenberger [1], [2], [3] first appeared. The problem of constructing observers for systems, of whose states are not available for direct measurement is currently one of the major points of interest in the field of Control Engineering.

The primary consideration in the design of an observer is that the estimate of the state should be close to the actual value of the state of the observed system. According to [4] the state observation problem centres on the construction of an auxiliary dynamic system, known as the state reconstructor or observer, driven by the available system inputs and outputs. A block diagram of the open-loop system reconstruction process is presented in Fig 1.1. If, as is usually the case, the control strategy is of the linear state feedback type $u(t) = Fx(t)$, the observer can be regarded as forming a part of a linear feedback compensation scheme used to generate the desired control approximation $F\hat{x}(t)$. This closed-loop system configuration is depicted in Fig 1.2.

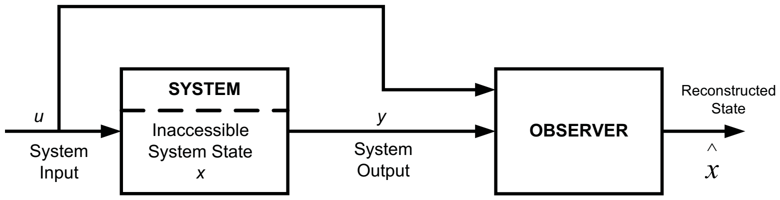

FIGURE 1.1: Open-Loop System Configuration

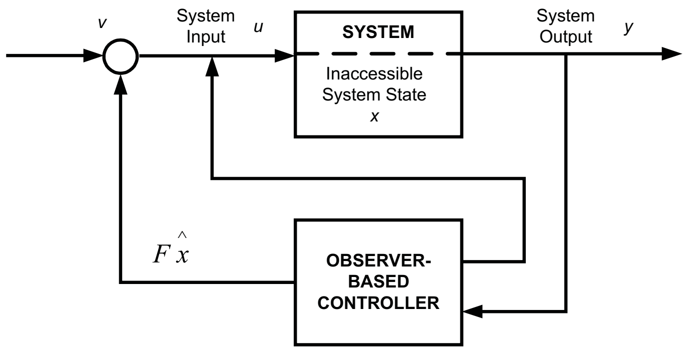

FIGURE 1.2: Closed-Loop Observer-based Control System

1.2 Historical Background

Control systems research has a long and distinguished tradition stretching back to nineteenth-century dynamics and stability theory. Its establishment as a major engineering discipline in the 1950s arose, essentially, from Second World War-driven work on frequency response methods by, amongst others, Nyquist, Bode and Wiener. The intervening 40 years have seen quite unparalleled developments in the underlying theory with applications ranging from the ubiquitous PID controller, widely encountered in the process industries, through to the high-performance fidelity controllers typical of aerospace applications. All of this has been increasingly underpinned by the rapid developments in the enabling of technology in computer software and hardware.

This view of model-based systems and control as a mature discipline masks relatively new and rapid developments in the general area of robust control. Currently an intense research effort is being directed to the development of high-performance controllers which, at least, are robust to specified classes of plant uncertainty. One measure of this effort is the fact that, after a relatively short period of work, "near world" tests on classes of robust controllers have been undertaken in the aerospace industry. Again, this work is supported by computing hardware and software developments, such as the toolboxes available within numerous commerically marketed controller design and simulation packages.

Since the re-emergence of state-space methods to form a direct multivariable approach to linear control systems synthesis and design, a host of controllers now exist to meet various qualitative and quantitative criteria including system stability and optimality. A common feature of these control schemes in the assumption that the system state vector is available for feedback control purposes. The fact that complex multivariate systems rarely satisfy this assumption necessitates either a radical revision of the state-space method, at the loss of its most favourable properties, or the reconstruction of the missing state variables.

1.3 Importance of Observers

Knowing the system state is necessary to solve many control theory problems; for example, stabilising a system using state feedback. In most practical cases, the physical state of the sytem cannot be determined by direct observation. Instead, indirect effects of the internal state are observed by way of the system outputs. A simple example is that of vehicles in a tunnel: the rates and velocities at which vehicles enter and leave the tunnel can be observed directly, but the exact state inside the tunnel can only be estimated. If the system is observable, is it possible to fully reconstruct the system state from its output measurements using the state observer?

In many practical situations only a few output quanitities are available. Application of theories which assume that the state vector is known is severely limited in these cases. If the entire state vector cannot be measured, as is typical in most complex systems, the traditional control law cannot be implemented. A number of strategies that can be applied to determine the state vectors will be discussed in Chapter 2.

The state variable representation has some conceptual advantages over the more conventional transfer function representation. The state vector contains enough information to completely summarize the past behaviour of the sytem, and the future behaviour is governed by a simple first-order differential equation. Thus the study of the system can be carried out in the field of matrix theory which is not only well developed, but has many notational and conceptual advantages over other methods. In addition to their practical utility, observers offer a unique theoretical fascination. The associated theory is intimately related to the fundamental linear system concepts of controllability, observability, dynamic response, and stability, and provides a simple setting in which all of these concepts interact [2].

1.3.1 Luenberger Observers

In position and velocity control Luenberger observers are most effective when the position sensor produces limited noise. Sensor noise is often a problem in motion-control systems. Noise in servo systems comes from two major sources: EMI generated by power converters and transmitted to the control section of the servo system, and resolution limitations in sensors, especially in the feedback sensor. EMI can be reduced through appropriate wiring practices and through the selection of components that limit noise generation such as those that comply with noise regulations. Noise from sensors is difficult to deal with. Luenberger observers often exacerbate sensor-noise problems. While some authors have described uses of observers to reduce noise, in many cases the observer will have the opposite effect. Lowering observer bandwidth will reduce noise susceptibility, but it also reduces the ability of the observer to improve the system. For example, reducing observer bandwidth reduces the accuracy of the observed disturbance signal. The availability of high-resolution feedback sensors raises the likelihood that an observer will substantially improve system performance.

The application of full-order Luenberger observers for a position sensorless operation of SRMs was initially introduced by [5]. SRMs are rapidly becoming more popular in industrial applications where variable speed is required because of their relatively simple construction, ease of maintenance, low cost and high efficiency. However, as already described the operation of the SRM is heavily dependent upon precise information about rotor position, measurement of which may be very expensive or prohibited because of the nature of the application. The use of observer theory to eliminate the problems of having to measure the rotor position was investigated by [5]. For this purpose, observer algorithms based upon the extended Luenberger observer are developed, which include plant non-linearities that naturally lead to an estimator with improved performance over existing estimation schemes.

Error Function

In the Luenberger observer design, one defining characterisic of the state estimation process is that of the error trajectory.

Let us assume that there is a $2nd$ Order system with two state variables $x_1(t)$ and $x_2(t)$. Subsequently the error functions can be defined simply as $e_1(t) = x_1(t) - \hat{x}_1(t)$ and $e_2(t) = x_2(t) - \hat{x}_2(t)$, where $\hat{x}_1$ and $\hat{x}_2$ represent the estimated values for x_1 and x_2 respectively. Figure 1.3 illustrates the error trajectory that will result when plotting $e_2(t)$ against $e_1(t)$. As can be seen, starting at an arbitrary point in the error space, the plot follows a spiral trajectory before reaching the origin.

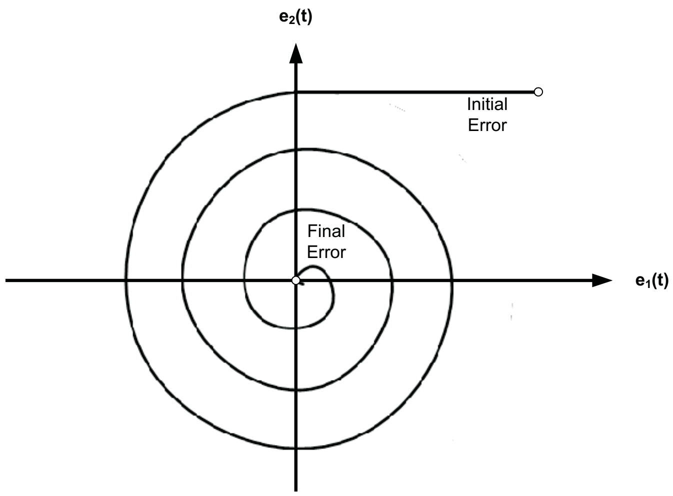

FIGURE 1.3: Luenberger Observer Error Trajectory

1.4 Sliding Mode Observers

The term "sliding mode control" first appeared in the context of variable-structure systems. Soon sliding modes became the principal operational mode for this class of control systems. Practically all design methods for variable-structure systems are based on deliberate introduction of sliding modes which have played, and are still playing, an exceptional role both in theoretical developments and in practical applications. Due to its order reduction property and its low sensitivity to disturbances and plant parameter variations, sliding mode control is an efficient tool to control complex high-order dynamic plants operating under uncertainty conditions which are common for many processes of modern technology.

The development of sliding mode control theory has revealed the true potential of the original research trends. Some methods have become conventional for feedback system design whereas others have proven less promising; new research directions have been initiated due to the appearance of new classes of control problems, new mathematical methods and new control principles. In the course of the entire history and automatic control theory, the intensity of investigation of systems with discontinuous control actions has been maintained at a high level. In systems with control as a discontinuous state function, so called sliding modes may arise. The control action switches at high frequency should sliding mode occur in the system. The study of sliding modes embraces a wide range of heterogeneous areas from pure mathematical problems to application aspects.

Sliding modes as a phenomenon may appear in a dynamic system by ordinary differential equations with discontinuous right-hand sides. The term *sliding mode* first appeared in the context of relay systems. It may happen that the control as a function of the system state switches at a high (theoretically infinite) frequency; this motion is called sliding mode. Systems with sliding modes have proven to be an efficient tool to control complex high-order nonlinear dynamic plants operating under uncertainty conditions, a common problem for many processes of modern technology. This explains the high level of research and publication activity in the area and the unremitting interest of practising engineers in sliding mode control during the last two decades.

Error Function

A similar analysis of error trajectory that was performed for Luenberger Observers will now be considered for the Sliding Mode Observer. Assuming the exact same system characteristics and equations as the Luenberger Observer, it can be seen from Figure 1.4 that the error trajectory does approach the origin, however does so in an oscillatory manner. The Sliding Mode Observer error equation $e_1(t)$ approaches zero in finite time, while $e_2(t)$ approaches zero as time approaches infinity.

1.5 Industrial Applications of Observers

As already stated, in most control situations, the state vector is not available for direct measurement. Control systems are used to regulate an enormous variety of machines, products, and processes. They control quantities such as motion, tem-

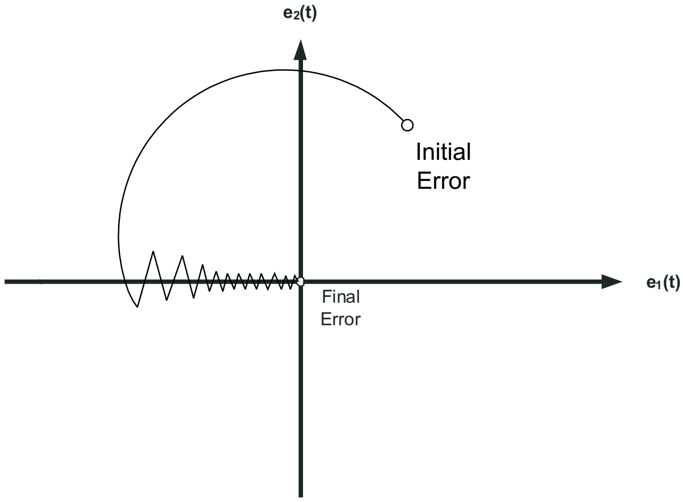

FIGURE 1.4: The Utkin Observer Error Trajectory

perature, heat flow, fluid flow, fluid pressure, tension, voltage, and current. Most concepts in control theory are based on having sensors to measure the quantity under control. In fact, control theory is often based on the assumption of the availability of near-perfect feedback signals. Unfortunately, such an assumption is often invalid. Physical sensors have shortcomings that can degrade a control system. There are at least four common problems caused by sensors. First, sensors are expensive. Sensor cost can substantially raise the total cost of a control system. In many cases, the sensors and their associated cabling are among the most expensive components in the system. Second, sensors and their associated wiring reduce the reliability of control systems. Third, some signals are impractical to measure. The objects being measured may be inaccessible for such reasons as harsh environments and relative motion between the controller and the sensor (for example, when trying to measure the temperature of a motor rotor). Fourth, sensors usually induce significant errors such as stochastic noise, cyclical errors, and limited responsiveness.

Observers can be used to augment or replace sensors in a control system. Observers are algorithms that combine sensed signals with other knowledge of the control system to produce observed signals. These observed signals can be more accurate, less expensive to produce, and more reliable than sensed signals. Observers offer designers an inviting alternative to adding new sensors or upgrading existing ones. Observers offer important advantages: they can remove sensors, which reduces cost and improves reliability, and improve the quality of signals that come from the sensors, allowing performance enhancement. However, observers have disadvantages: they can be complicated to implement and they expend computational resources. Also, because observers form software control loops, they can become unstable under certain conditions.

In some cases, the observer can be used to enhance system performance. It can be more accurate than sensors or can reduce the phase lag inherent in the sensor. Observers can also provide observed disturbance signals, which can be used to improve disturbance response. In other cases, observers can reduce system cost by augmenting the performance of a low-cost sensor so that the two together can provide performance equivalent to a higher cost sensor. In the extreme case, observers can eliminate a sensor altogether, reducing sensor cost and the associated wiring. For example, in a method called acceleration feedback acceleration is observed using a position sensor and thus eliminating the need for a separate acceleration sensor. Observer technology is not a panacea. Observers add complexity to the system and require computational resources. They may be less robust than physical sensors, especially when plant parameters change substantially during operation. Still, an observer applied with skill can bring substantial performance benefits and do so, in many cases, while reducing cost or increasing reliability.

1.5.1 Industrial Applications of Sliding Mode Observers

This section will primarily revolve around the application of the sliding mode technique, and more specifically its relevance to power electronic equipment and electrical drives. Interest in the sliding mode approach has emerged due to its potential for circumventing parameter variation effects under dynamic conditions with minimum implementation complexity. In electric-drive systems, the existence of parameter changes caused by, for instance, winding temperature variation, converter switching effect and saturation, is well recognized, though infrequently account for. In servo applications, siginificant parameter variations arise from often unknown loads; for example, in machine tool drives and robotics, the moment of inertia represents a variable parameter depending on the load of the tool or the payload. Among the distinctive features claimed for sliding mode control are order reduction, disturbance rejection, strong robustness and simple implementation by means of power converters. Hence sliding mode is attributed high potentials as a prospective control methodology for elctric drive systems. The experience gained so fair testifies to its efficiency and versatility. In fact, control of electric drives is one of the most challenging applications due to increasing interest in using electric servomechanisms in control systems, the advances of high-speeed switching circuitry, as well as insufficient linear control methodology for inherently nonlinear high-order multivariable systems such as AC motors.

The implementation of sliding mode control by means of the most common electric components has turned out to be simple enough [6]. Commerically available power converters enable several dozen kilowatts to be handled at frequencies of several hundred kilohertz. When using converters of this type, confining their function to pulse-width modulation seems unjustified, and it is reasonable to turn to algorithms with discontinuous control actions. Introduction of discontinuities is dictated by the very nature of power converters. It is shown in [6] that sliding mode control techniques are used flexibly to achieve desired control performance, not only in controller design but also in observer design and estimation processes.

Sliding Mode Observers for Switched Reluctance Motor Drives Switched reluctance motors (SRMs) have received considerable attention as an alternative to permanent magnet dc motors. For automotive applications, the switched reluctance motor avoids the problems associated with magnet bonding, corrosion and demagnetization. In addition, for systems requiring fast dynamic response it is often found that the torque to inertia ratio for the switched reluctance motor is higher than permanent magnet dc motors using ceramic ferrite or injection molded neodymium-iron-boron magnets. It is illustrated in [7] how the switched reluctance motor also holds promise for sensorless operation including position estimation at zero speed. Sensorless operation is important for automotive applications due to the need for minimum package size, high reliability and low cost for motor actuators. In [7] the sensorless operation of a switched reluctance motor for an advanced brake system is considered. In this configuration, an electric motor drives a hydraulic actuator which controls the vehicle brake pressure. During antilock braking conditions the electric motor modulates the brake system pressure.

This application combines the need for high reliability, small size, and fast dynamic response from the motor drive. Various methods of indirect (or sensorless) position estimation have been investigated for switched reluctance machines. A model based estimator was first proposed in [8] and very good results were obtained. In observer based state estimation schemes the dynamics of the motor are modeled in state space while a mathematical model runs in parallel with the physical machine. The model has the same inputs as the physical machine and the difference between its outputs and the measured outputs of the real machine are used to force the estimated variables to converge to the actual values. In the case of the SRM, terminal measurements of the phase currents and voltages are sufficient to develop the observer. The computational simplicity and robust stability properties of sliding mode controllers prompted the study of designing observers with sliding surfaces. The design of observers using sliding mode theory for nonlinear systems was first studied in [9]. Further study of sliding mode observers was presented by [10] and [11]. The sliding mode observer based rotor position estimation scheme for SRMs was first presented in [12]. The results presented in that work were based on computer simulations of a linear magnetic model for the SRM. The operation of a sliding mode observer with a floating-point digital signal processor was demonstrated in [13]. This paper extends the previous research by considering the discrete-time formulation of the observer, the nonlinearities associated with SRM operation, the effects of flux estimation errors, the use of velocity feedback; to modify the conduction angles of the motor during transient conditions, and hardware implementation using a fixed point DSP.

The sliding mode observer incorporates a state-space model of the SRM to estimate rotor position and velocity from the phase current and terminal voltage measurements. An error correction term is computed based on the difference of the motor flux computed from the mathematical model and that derived from motor measurements. A block diagram of the observer based motor drive is shown below

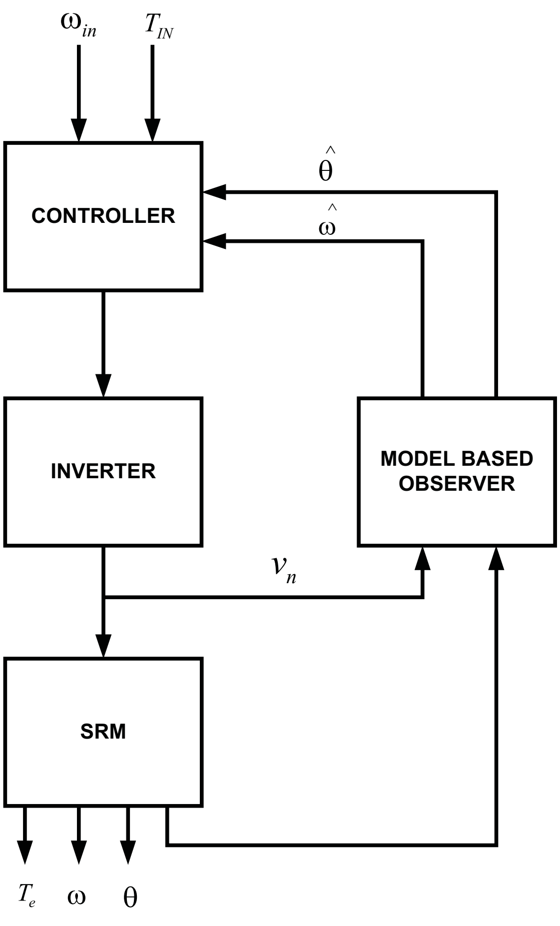

FIGURE 1.5: Block Diagram of the Observer Based Motor Drive

The machine model is defined in [7] by

$$\lambda = \lambda(i, \theta) \tag{1.1}$$

$$T = T(i, \theta) \tag{1.2}$$

The state space differential equations of the SRM to be solved are

$$\frac{d\lambda_n}{dt} = -R_n i_n(t) + v_n(t) \tag{1.3}$$

$$\frac{d\theta}{dt} = \omega(t) \tag{1.4}$$

$$\frac{d\omega}{dt} = -\frac{D}{J}\omega(t) + \frac{1}{J}\Sigma_n T_n(\theta_n, \lambda_n) - \frac{1}{J}T_L(t) \tag{1.5}$$

where R_n, i_n and v_n are the resistance, current and voltage of the n^{th} phase, respectively. It is assumed that each phase is magnetically uncoupled from every other phase. The rotor angular position and velocity are denoted by θ and ω, respectively. T_L is the hydraulic system load torque. For detailed computer simulation in MATLAB/SIMULINK refer to [7].

The sliding mode problem was defined in [7] by initially demonstrating that the phase flux can be obtained from the following equation using phase current i_n and phase voltage v_n measurements

$$\lambda_n(t) = \int_0^t (v_n(\xi) - i_n(\xi)R_n)d\xi \tag{1.6}$$

This method of estimating phase flux will accumulate integration errors if the motor is operating at a very low or zero speed. However, operating at non-zero speed will limit accumulated errors because the current, and hence the flux, of each phase will periodically go to zero. An observer may be constructed to estimate the unknowns in (1.3) and (1.4). Consider a second order sliding mode observer for the SRM of the form

$$\dot{\hat{\theta}} = \hat{\omega} + K_\theta sgn(e_f) \tag{1.7}$$

$$\dot{\hat{\omega}} = K_\omega sgn(e_f) \tag{1.8}$$

where $\hat{\Theta}$ and $\hat{\omega}$ are the estimated rotor position and velocity respectively, and e_f is an error function that is based on measured and estimated variables. To describe the observed error dynamics, the errors for the SRM observer can be defined as

$$e_\theta(t) = \theta(t) - \hat{\theta}(t) \tag{1.9}$$

$$e_\omega(t) = \omega(t) - \hat{\omega}(t) \tag{1.10}$$

For a detailed solution to the above equations refer to Appendix A. A block diagram of the sliding mode observer is shown below

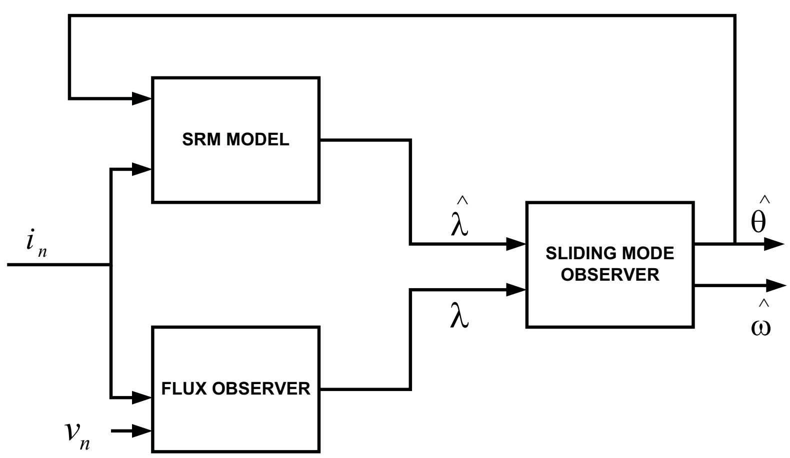

FIGURE 1.6: Block Diagram of the SRM Sliding Mode Observer

1.6 Challenges

The problem of finding a minimum (simplest) stabiliser is a long-standing unsolved classical problem in control theory. It is closely related to the problem of constructing a minimum functional observer. In the 1980s [4] obtained upper bounds for the dimension of functional observers under which a given linear functional for linear objects is exponentially recostructed at an arbitrary convergence rate of the estimation error. However, the reduction in the dimension of such an observer (possibly at the expense of a decrease in convergence rate) remained an open question.

The majority of work undertaken in the field of functional and sliding mode observers has revolved around deriving necessary and sufficient conditions for the existence of the observers and stabilizers of a given (minimal) order for linear objects. It has been shown that these problems are reduced to the intersection of a linear manifold and the stability domain in the parameter space of polynomials of a given order. Currently, the research is strictly limited to Sliding Mode Observers (Utkin Observers) and their applications to industry. This thesis aims to look at the theory and applications of Sliding Mode Functional Observers and to build a strong theoretical base, that focuses on a number of system variations, forming a very robust and extensive result.

The work in this thesis was posed with a number of challenges

1. While the concept of sliding mode estimatation for systems with known inputs has been quite thoroughly researched, the concept of sliding mode estimation for unknown input systems has been unaddressed. To compound this problem, the concept of sliding mode functional estimation for unknown inputs has currently been overlooked. One of the major challenges is to formulate a design algorithm that can in fact work for a large array of unknown input systems. The application of this theory to a system example will verify its validity.

2. Application of sliding mode functional observers to non linear systems. Initially the "output injection" approach was introduced, where essentially the aim was to find a coordinate transformation so that the state estimation error dynamics are linear in the new transformation and then linear techniques could be performed. However, the conditions that needed to be met for this approach to be conducted are quite restrictive and essentially only apply to a narrow band of systems. This thesis aims to formulate a design algorithm that will apply to a much wider range of non linear systems and incorporates a reduced order observer coupled with the Lyapunov function (and Lipschitz assumption) on the non-linear function. For design computational efficiency, an asymptotic stability condition is developed by using the linear matrix inequality (LMI) formulation.

3. It has been discovered that an underlying issue that faces the majority of the research in the sliding mode observer field is the application to real life systems. While the theories developed are quite rigorous and extensive, it appears that in the majority of cases, and especially when it comes to unknown input scenarios, the conditions posed by the theories only restrict the application to a very small range of systems. In the course of this thesis, a great deal of emphasis has been placed when formulating design algorithms to relax many of the constraints and conditions to make the theories as broad and applicable as possible. A great deal of research must hence be placed on predicting system dynamics and adapting the design algorithms to cater for these. This has somewhat diverted some of the attention to the analysis of some industrial processes e.g. Speed Sensorless Permanent Magnet Synchronous Machines to ensure that the control theories will provide the greatest value possible.

4. The primary objective of this thesis is to extend the theory that exists on Sliding Mode Observers, to Sliding Mode Functional Observers. The order

reduction properties that result from this are quite interesting and useful, however require some extensive mathematical formulation and application of complex control theory.

1.7 Overview of the Thesis

This thesis aims to introduce and develop the sliding mode functional observation technique, and propose new design algorithms for robust state and functional observation, both at a practical and theoretical level.

In developing design algorithms for sliding mode applications, knowledge of elementary and advanced state and functional observers and their characteristics is essential. For details relating to elementary control concepts (observability, controllability) refer to Appendix B.

- Chapter 2 provides a review of state observation, with particular emphasis placed on "Luenberger Observers" that are considered the basis for much of the current theory in this field. The review is extended to the "Unknown Input Observers" approach to the state estimation of dynamical systems. This technique has in fact resulted in new types of low-order observers for systems with unknown inputs. The introduction to Sliding Mode Observers is considered by reviewing the "Utkin Observer" and in particular to explore the possibilities of using sliding mode observers for robust state reconstruction. More importantly however, the introduction of Sliding Mode Functional Observers provides a good base for the proceeding chapters.

- Chapter 3 focuses on the development of sliding mode functional observers for time-delay systems that are considered of neutal type. This chapter provides an extensive discovery of a solution to the problem of determining low order sliding mode functional observers (of the same order as the dimension of the vector to be estimated) for neutral delay systems and provides a rigorous design prodedure for the determination of the observer parameters which can be derived based on the existence conditions that will be formulated.

- Chapter 4 addresses a very important issue in the field of state observation i.e. formulating an observer (functional observer) when some of the system inputs are in fact unknown. The theory here is essentially directed at sliding mode functional observers. In this chapter the proposed design algorithm is then applied to the sensorless control of Permanent Magnet Synchronous Machines.

- Chapter 5 proposes a design algorithm for sliding mode functional observers for a class of non linear system. The observer design is essentially based on Linear Matrix Inequalities. The transformation procedures that have been developed to enhance and extend the scope to which the design algorithm can be applied have been discussed and evaluated. Conditions for the solvability of the design matrices of the proposed observer and for the stability of its dynamics have also been addressed.

- Chapter 6 proposes a sliding mode functional observer algorithm that is directed at lower order observers that essentially do not include derivatives of the outputs. A design procedure for the determination of the observer parameters is presented and is based primarily on the existence conditions derived. The theory presented is demonstrated through a numerical example (7th Order System) that illustrates the design process developed.

- Chapter 7 presents the concluding remarks and possible future directions.

While it is essential to acknowledge the drawbacks of sliding mode controllers, this will not pose as a direct focus for this thesis. For a brief overview of one of the most common and major drawbacks of sliding mode controllers i.e. "The Chattering Problem" refer to Appendix C.

Chapter 2

Observers Review

2.1 Introduction

As described in Chapter 1, over the past three to four decades, there has been an increasing percentage of control systems literature written in the state variable point of view [1]- [14]. Modern optimal control laws almost invariably have control commands based ultimately upon the state variables. In addition, useful design tools such as pole placement are most easily implemented in a state feedback framework. As previously mentioned, a complete state vector is rarely available for use in state feedback, so it is necessary for the design to be modified. A useful approach is to use state estimates in place of actual state variables for use in the control law, these estimates will come from an observer or Kalman filter. While the focus of the research revolves around observers, the contribution provided from Kalman filtering is also handled to a degree of detail in this chapter.

This chapter is separated into two sections; traditional observers and advanced observers. The following fall under the category of traditional observers

- State Observers (Luenberger Observers)

- Functional Observers

- Unknown Input Observers
- Reduced Order Observers

The following are examples of Advanced observers that will be illustrated in this chapter

- Kalman Filters
- Sliding Mode Observers (Utkin Observers)
- Extended State Observers
- High Gain Observers

The chapter begins with an introduction into the the theories of state and functional observers through the state-space description of dynamical systems. Details of the fundamental equations for the Luenberger Observers are presented, with specific emphasis on Time Invariant Systems. It is appreciated that in most practical examples all of the system inputs are in fact not available for measurement, hence a detailed look at Unknown Input Observers is considered, both from a theoretical and from an implementation point of view. Further in the chapter, a look at Kalman Filters, Extended State and High Gain Observers provides a platform to illustrate the depth of research that has been performed in this field. A detailed comparison between Kalman Filtering, Extended State and High Gain Observers with Sliding Mode Observers provides a good distinction and demonstrates the importance of continued research into Sliding Mode Observers and their applications in Control Engineering. Finally, a brief introduction to the topic of Sliding Mode functional observers is given. This will underpin the majority of the work that is performed in the thesis.

Before, delving into the details of state and functional observation, an important distinction must be drawn between traditional and advanced observers. This is in fact quite simple. The major difference lies in the robustness and ability to reject disturbances. Traditional observers simply have the task of estimating the state or function of a system. However, in most cases (especially in practical cases) their is some form of disturbance to the system e.g. noise that can influence the state or functional observation process. The advanced observer techniques, while more complicated in their structure and implementation have the advantage of being able to reject system disturbances and effectively perform state or functional estimation.

2.2 Traditional Observers

2.2.1 State Observation

The following section provides an overview of state observation, in particular examining the basics of state and functional observers. Furthermore, an introduction into Kalman Filters and High-Gain observers provides a platform against which sliding mode observation can be compared in Section 2.3.5.

State Observers

Let us consider the state variable representation of a system as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

where, $x(t)$ is an $(n \times 1)$ state vector $u(t)$ is an $(m \times 1)$ input vector A is an $(n \times n)$ transition matrix B is an $(n \times m)$ distribution matrix

The state variable representation of systems has some conceptual advantages over the more conventional transfer function representation [15]. The state vector $x(t)$ contains enough information to completely summarize the past behaviour of the system, and the future behaviour is governed by the simple first-order differential equation. The properties of the system are determined by the constant matrices

A and B. Thus the study of the system can be carried out in the field of matrix theory which is not only well developed, but has many notational and conceptual advantages over other methods.

When faced with the problem of controlling a system, some scheme must be devised to choose the input vector $u(t)$ so that the system behaves in an acceptable manner. Since the state vector $x(t)$ contains all the essential information about the system, it is in fact reasonable to base the choice $u(t)$ solely on the values of $x(t)$ and perhaps also t (depending on whether the system is time-variant or time-invariant). In other words, u is determined by a relation of the form $u(t) = F[x(t)]$.

As described in Chapter 1, in most control situations, the state vector is not available for direct measurement. This means that it is not possible to evaluate the function $F[x(t)]$. It is shown in [15] how the available system inputs and outputs may be used to construct an estimate of the system state vector. The device which reconstructs the state vector is called an observer. The observer itself as a time-invariant linear system is driven by the inputs and outputs of the system it observes. It is further illustrated in [15] how the time constants of the observer can be chosen arbitrarily and that the number of dynamic elements required by the observer decreases as more output measurements become available.

Observation of the Entire State Vector It is described in [15] that almost any linear system will follow a free system which is driving it. In fact, state vectors of the two systems will be related by a constant linear transformation. The questions which naturally arises is; How does one guarantee that the transformation obtained will be invertible?

One way to ensure that the transformation will be invertible will be to force it to become the identity transformation. This requirement guarantees that (after the intitial transient) the state vector of the observer will equal the state vector of the plant.

In the notation that will follow, vectors such as a are commonly column vectors, whereas row vectors are represented as transposes of column vectors, such as a'.

Assume that the plant has a single output y

$$\dot{x} = Axy = a'x \tag{2.2}$$

and that the corresponding observer is driven by y as its only input

$$\dot{z} = Bz + by \tag{2.3}$$

or

$$\dot{z} = Bz + ba'x \tag{2.4}$$

under these conditions $z = Tx$ where T satisfies

$$TA - BT = ba' \tag{2.5}$$

Forcing $T = I$ gives

$$B = A - ba' \tag{2.6}$$

which prescribes the observer in this case. In (2.6) A and a' are given as a part of the plant, hence choosing a vector b will prescribe B and the observer will be obtained.

The solution to the observer problem is illustrated in Fig 2.1.

The observer constructed above by requiring $T = I$ possesses a certain degree of mathematical simplicity. The state vector of the observer is equal to the state vector of the system being observed. Further examination will reveal a certain degree of redundancy in the observer system. The observer constructs the entire state vector when the output of the plant, which represents part of the state vector, is available by direct measurement. Intuitively, it seems that this redundancy might

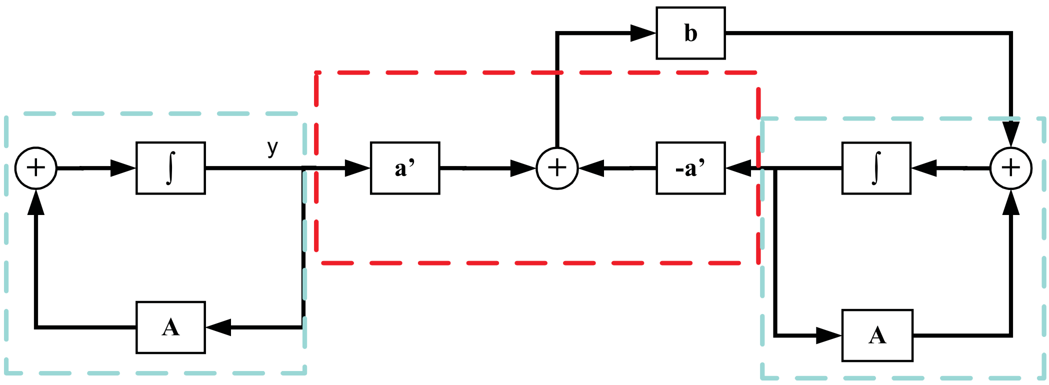

FIGURE 2.1: Observation of the Entire State Vector

be eliminated, thereby giving a simpler observer system. It is demonstrated in [15] that the redundancy can in fact be eliminated by reducing the dynamic order of the observer. It is possible, however to choose the pole locations of the observer in a fairly arbitrary manner. These so-called reduced order observers are explored later in this chapter in greater detail.

2.2.2 Functional Observers

The problem of the functional observer design was related to the constrained or unconstrained Sylvester equations [16], [17]. Generally to solve this problem, many authors have proposed to transform the initial system to an equivalent one (by using some regular transformations) of reduced-order and to design an observer for this system. In [3] a functional observer of dimension r is introduced, and a straightforward method for its design is derived.

Functional observers take advantage of a recurring theme in state feedback control. That is, frequently in feedback applications, only a linear combination or function of the state variables $Kx(t)$ is required, rather than a complete knowledge of the state vector. In the previous section, the focus was on the reconstruction of the whole state vector, highlighting a redundant feature. The question therefore arises as to whether a less complex observer can be constructed to produce a linear function of the states. This is possible and this section is concerned with the construction of a functional observer, an an investigation into one particular design algorithm that has been proposed. The primary aim of the literature is to produce observers that are of further reduced order and that are stable. In addition, it is interesting to note whether the designs place any restrictions (such as pole location) on the observer that may affect its performance.

A major result for this problem was presented by [1]. For the functional observer to be able to estimate any linear combination of states, it must itself have order of at least $v-1$ where v is known as the *observability index*. It is defined in [2] as the least positive integer for which the matrix

$$\begin{bmatrix} C' & A'C' & (A')^2 & ... & (A')^{v-1}C' \end{bmatrix} \tag{2.7}$$

has rank n. This gives rise to the following theorem.

Theorem 1

A single linear functional of the state of a linear system can be reconstructed by an observer with $v-1$ eigenvalues that may be chosen arbitrarily (where v is the observability index of the system).

For any completely observable system $v-1 \leq n-m$. Further, it is often the case that the order $v-1$ of a linear functional observer is less than the order $n-m$ of the reduced order observer. As a consequence, observing a linear function of the states may afford significant reduction in observer order compared to observing the entire state vector.

Functional State Reconstruction Problem

Let us define the system as follows

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.8}$$

$$y(t) = Cx(t) \tag{2.9}$$

where $\tilde{x}(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$. Without loss of generality we assume that the system is completely controllable, completely observable, and that B and C are of full rank. The objective of the functional observer is to reconstruct the typical linear feedback law of the form

$$z(t) = Lx(t) \tag{2.10}$$

where L is known and is of appropriate dimension. In order to reconstruct the state function we require an observer of the form

$$\dot{w}(t) = Nw(t) + Jy(t) + Hu(t) \tag{2.11}$$

$$\hat{z}(t) = Dw(t) + Ey(t) \tag{2.12}$$

where $w(t) \in \mathbb{R}^q$ and $\hat{z}(t) \in \mathbb{R}^r$. The functional observer matrices N, J, H, D and E are to be designed.

Figure 2.2 illustrates the role of the functional observer in reconstructing the control law to be directly fed back into the system. In contrast to observers discussed in the previous section, the compensator is housed in one observer block which, depending on the implementation may reduce wiring and associated connection noise. The diagram also illustrates the fact that the estimated linear combination need not form part of the control strategy, namely feedback control, but could be for any purpose with the same design method.

If we assume that any states which are unobservable can be eliminated by defining a lower dimensional observer state vector, then the order q of the observer in (2.11) and (2.12) should be less than or equal to the state observer defined in the previous section.

The observer $w(t)$ provides a reconstruction of the state variables, while $\hat{z}(t)$

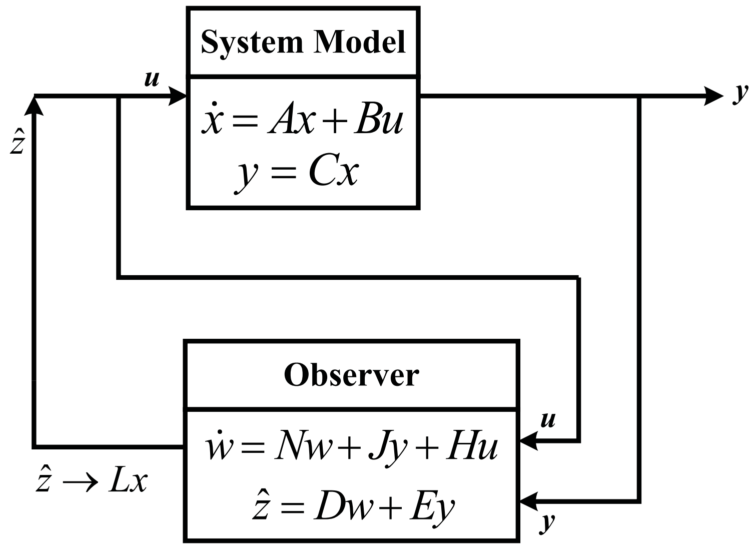

FIGURE 2.2: Feedback Control Law Lx Implemented by a Functional Observer

approximates $Lx(t)$. The output $\hat{z}(t)$ provides an asymptotic estimate of $Lx(t)$ if

$$\lim_{t\to\infty} [\hat{z}(t) - Lx(t)] = 0 \tag{2.13}$$

It stands to reason that if $z(t)$ estimates $Lx(t)$, then $w(t)$ estimates some linear combination of $x(t)$, call it $Px(t)$. This is formalized by [2] and gives rise to the following theorem

Theorem 2

The completely observable q^{th} order functional observer of (2.11) and (2.12) will estimate $Lx(t)$ if and only if the following conditions hold.

$$JC = PA - NP \tag{2.14a}$$

$$H = PB \tag{2.14b}$$

$$L = DP + EC \tag{2.14c}$$

and N is a stablity matrix. Although a formal proof of Theorem 2 will not be presented, it is necessary to verify and comment on the reasoning underlying the observer conditions. We will define the observer error in estimating states as

$$e(t) = w(t) - Px(t) \tag{2.15}$$

If we take the derivative of this, and then substitute the observer equation (2.11) and system equations we obtain

$$\begin{aligned} \dot{e}(t) &= \dot{w}(t) - P\dot{x}(t) && (2.16) \\ &= Nw(t) + JCx(t) + Hu(t) - PAx(t) - PBu(t) && (2.17) \end{aligned}$$

Applying conditions (2.14a) and (2.14b) yields

$$\begin{aligned} \dot{e}(t) &= Nw(t) + (PA - NP)x(t) + PBu(t) - PAx(t) - PBu(t) && (2.18) \\ &= Nw(t) - NPx(t) && (2.19) \\ &= Ne(t) && (2.20) \end{aligned}$$

The solution to this differential equation is an exponential of the form

$$e(t) = e^{Nt} \tag{2.21}$$

Clearly N controls the dynamics of the observer. Given that N must be a stability matrix we obtain the following result

$$\lim_{t \to \infty} e(t) = w(t) - Px(t) = 0 \tag{2.22}$$

Our main objective is to verify that the observer correctly estimates the control law $Lx(t)$ which is the equivalent to (2.13). From the substitution of the observer equation (2.12), the substitution of the system output and the application of (2.14c) we arrive at the following

$$\begin{aligned} e_z(t) &= \hat{z}(t) - Lx(t) & (2.23)\\ &= Dw(t) + ECx(t) - Lx(t) & (2.24)\\ &= D(w(t) - Px(t)) & (2.25) \end{aligned}$$

We expect this to asymptotically approach zero as a result of (2.22).

If we consider a closed loop feedback system with input $u(t) = Lx(t)$, which is estimated by the observer (2.12) in Figure 2.2, we can derive $\dot{x}$ and $\dot{e}$ by using condition (2.14c) as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) = Ax(t) - B(Dw(t) + Ey(t)) & (2.26)\\ &= Ax(t) + BDw(t) + BECx(t) & (2.27)\\ &= Ax(t) + BDw(t) + B(L - DP)x(t) & (2.28)\\ &= (A + BL)x(t) + (BD)e(t) & (2.29) \end{aligned}$$

$$\dot{e}(t) = Ne(t) \tag{2.30}$$

This yields a composite system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + BL & BD \\ 0 & N \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{2.31}$$

Aside from the difference in notation and the fact that the control law is $Lx(t)$ instead of $-Kx(t)$, they are in fact very similar. Matrix element N is require to be the stability matrix.

Considering the contraints presented in Theorem 2, the functional state reconstruction problem essentially revolved around designing the smallest possible r^{th} order observer with all observer matrices satisfying the conditions (2.14a), (2.14b) and (2.14c). Ideally, a small r should not compromise other favourable aspects of the observer, such as freedom of eigenvalue assignment and algorithm simplicity, so that it can easily be implemented in practice.

2.2.3 Darouach Functional Observer

A simple method for designing r^{th} order functional observers is presented in [3]. It is mentioned that the solution to the functional observer problem is related to the optimal unbiased functional filter problem, especially in satisfying condition (2.14a). The necessary and sufficient conditions for the existence and stability of the functional observer are presented in [3]. This culminates in a straight forward algorithm for the design of a minimal order observer based on the system parameters. The strength of this algorithm lies in its elegant implementation, which, unlike numerous other design approaches, does not require any ad hoc decisions. This design approach also clearly specifies the necessary and sufficient conditions for the existence of the observer.

Summary

We should first note that in Darouach's method the matrix D is an identity matrix.

Darouach's analysis commences with the observer condition (2.14a). This type of equation is known as the Sylvester equation, and its solution has been the subject of considerable research. The Sylvester equation is manipulated to an equivalent form

$$PA - NP - JC = 0 \tag{2.32}$$

$$(PA - NP - JC)\begin{bmatrix} L^+I - L^+L \end{bmatrix} = 0 \tag{2.33}$$

where L^+ denotes the Moore-Penrose generalized inverse of matrix L. If we then apply the modified condition (2.14c), where $P = L - EC$ and where we set $K =$

$J - NE$, we obtain the following

$$N = PAL^{+} - KCL^{+} \tag{2.34}$$

and

$$P\bar{A} = K\bar{C} \tag{2.35}$$

where $\bar{A} = A(I - L^{+}L)$ and $\bar{C} = C(I - L^{+}L)$

Two important conditions are presented in the section below, the first is to test for necessary and sufficient conditions for the observer, and if it does exist, a subequent condition that ensures matrix N is Hurwitz and the observer is stable.

Design Algorithm

In order to satisfy the conditions in Theorem 2, we present the following Lemma.

Lemma 1 [3]

$$Rank \begin{bmatrix} LA \\ CA \\ C \\ L \end{bmatrix} = Rank \begin{bmatrix} CA \\ C \\ L \end{bmatrix} \tag{2.36}$$

$$Rank \begin{bmatrix} sL - LA \\ CA \\ C \end{bmatrix} = Rank \begin{bmatrix} CA \\ C \\ L \end{bmatrix}, \tag{2.37}$$

$$s \in \mathbb{C}, \mathbb{R}(s) \geq 0$$

Whether Condition (2.36) is satisfied, can be determined by analyzing the ranks of the matrices on the LHS and RHS of (2.36). The author shows that in [3] that

Condition (2.37) is equivalent to the detectability of the pair (F, G), where

$$F = LAL^+ - LA(I - L^+L) \begin{bmatrix} CA(I - L^+L \\ C(I - L^+L \end{bmatrix}^+ \begin{bmatrix} CAL^+ \\ CL^+ \end{bmatrix} \quad (2.38)$$

$$G = (I - \begin{bmatrix} CA(I - L^+L) \\ C(I - L^+L) \end{bmatrix} \begin{bmatrix} CA(I - L^+L \\ C(I - L^+L) \end{bmatrix}^+) \begin{bmatrix} CAL^+ \\ CL^+ \end{bmatrix} \quad (2.39)$$

where L^+ denotes the Moore-Penrose generalized inverse of matrix L. Furthermore, if matrices J, H and E satisfy Theorem 2, a Hurwitz matrix N is given by

$$N = F - ZG \quad (2.40)$$

where matrix Z is obtained by any pole placement method so that $F - ZG$ is stable. Matrices E and K are obtained according to

$$\begin{bmatrix} E & K \end{bmatrix} = L\bar{A}\Sigma^+ + Z(I - \Sigma\Sigma^+) \quad (2.41)$$

where $\bar{A} = A - (I - L^+L)$, $\bar{C} = C - (I - L^+L)$ and $\Sigma = \begin{bmatrix} C\bar{A} \\ \bar{C} \end{bmatrix}$, and matrix J is obtained according to

$$J = K + NE \quad (2.42)$$

whilst matrix H is obtained according to

$$H = (L - EC)B \quad (2.43)$$

By using this algorithm we can compute all the observer parameters which provide a functional observer of the form

$$\dot{w}(t) = Nw(t) + Jy(t) + Hu(t) \tag{2.44}$$

$$\hat{z}(t) = w(t) + Ey(t) \tag{2.45}$$

The block diagram of the Darouach Functional Observer depicted in Figure 2.3 as three main areas, the blue represents the system dynamics, the red constructs some linear combination of the states $w(t)$, whilst the output of the summing function in the bottom left constructs the required state function.

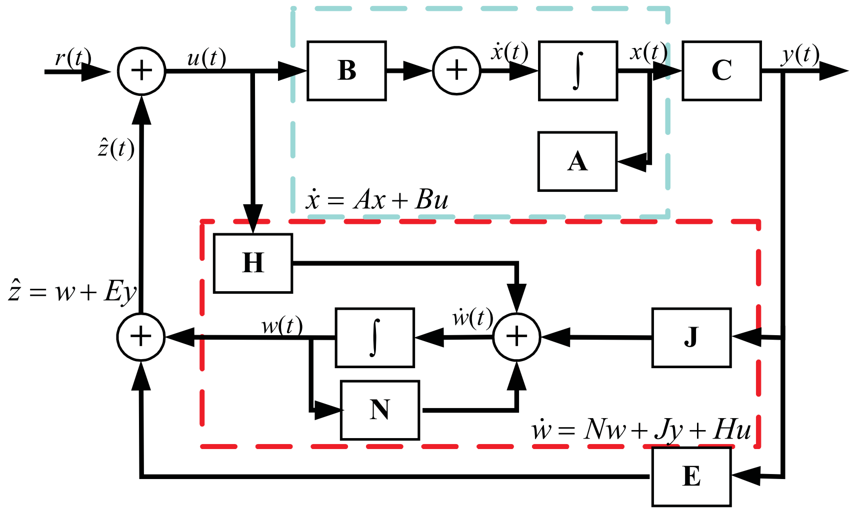

FIGURE 2.3: Exploded View of Darouach Functional Observer

2.2.4 Unknown Input Observers

Observers for systems that are characterised by having unknown inputs are considered below, both for linear and non-linear systems.

Linear Systems

The existence conditions for unknown input observers for LTI systems are well known and several methods for its design have been proposed in the literature. The problem is of considerable importance since in practice there are many situations where there are plant disturbances present, or some of the inputs to the system are inaccessible, and therefore conventional observers which assume the knowledge of all inputs cannot be used. An observer capable of estimating the state of a linear system with unknown inputs can also be of vital use when dealing with the problem of instrument fault detection, isolation, and accommodation, since in such systems most actuator failures can be generally modeled as unknown inputs to the system [18], [19].

Essentially, there have been two primary approaches to the problem of designing UIO's. A number of these attempts usch as [20], [21] assume some *a priori* information about the unmeasurable inputs. Specifically, [21] assumes a polynomial approximation to these inputs, and in [20], it is assumed that the unknown inputs can be modelled as the response of a known dynamical system represented by a constant coefficient differential equation. The next category of UIO studies relax the assumption of having information about the unkown inputs. This category therefore assumes no knowledge of the inaccessible inputs [22] - [23]. Among the earlier works is that of [22], which proposes a simple observer that is capable of reconstructing the entire state of a linear system with the presence of the unknown inputs. However, no systematic guideline for designing such an observer was provided in [22]. Since then, several authors have provided various techniques for designing such observers. In [24], the concept of multivariate systems inverse is used, whereas [25] provides a necessary condition for the existence of the UIO, and a design based on generalized inverse of matrices. Observers with relatively similar structures to that of [25] but simpler design techniques were proposed in [26] and [27]. Finally, [23] provides a necessary and sufficient condition for the design of such UIO's.

A new approach for the design of UIO's capable of estimating the state of linear dynmaical systems driven with both known and unknown inputs was presented in [28]. By carefully examining the dynamic system involved and simple algebraic manipulations, it was possible to rewrite new equations eliminating the unknown inputs from part of the system and put them into a form where it could be partitioned into two interconnected subsystems, one of which was directly driven by known inputs only. Therefore, it was made possible to use a conventional Luenberger observer with slight modifications for the purpose of estimating the state of the system. As a result, it was also possible to state similar necessary and sufficient conditions to that of a conventional observer for existence of a stable estimator and also arbitrary placement of the eigenvalues of the observer. This design approach stood out from the numerous algorithms proposed becasue of it greatly reduced computational complexity.

Non-Linear Systems

While the existence conditions for unknown input observers for LTI systems are well known and numerous design methodologies have been proposed in the literature (as described above) the results on non-linear UIO are scarce. A direct extension of the linear results to the non-linear case was originally considered by [29]. His approach was referred to as NUIO (Non-Linear UIO) and considered systems with non-linearities that are functions of inputs and outputs. However, this class of non-linear systems is rather limited and many physical system can no be modelled in this way. Another limitation is the difficulty of transforming a general non-linear system into the required form.

An alternative appraoch referred to as DDNO (Disturabance Decoupling Non-Linear Observer) was presented in a series of works by [30]- [31]. The class of non-linear systems considered by DDNO is more general and the basic idea is the use of non-linear state transformations to satisfy the decoupling condition. However, the existence conditions for these transformations are derived from the Frobenius Theorem and are rather restrictive. Another drawback of the DDNO is that the state transformation leads to another non-linear system for which an observer design is not a tractable problem.

In [32] the design problem for UIO is considered for the class of non-linear Lipschitz systems. It is demonstrated that with the necessary and sufficient conditions formulated, the Lipschitz UIO (LUIO) deisgn problem is equivalent to an H_∞ optimal control problem that satisfies all of the regularity assumptions. The formulation in [32] employs a new dynamic framework which is a generalization to the one that is used more extensively. The LUIO synthesis is caried out using H_∞ optimization and can therefore be done using commercially available software packages.

2.2.5 Reduced Order Observers

The full order observer is of order n, being equal to the number of states in the original system. Although simple, both conceptually and in its construction, there are some inherent redundancies in its design. Recall the objective proposed of reconstructing the state variables $x(t) \in \mathbb{R}^n$, and that the system outputs contain m linear combinations of the variables. Intuitively, the remaining $n - m$ states may be reconstructed by an observer of order $n - m$ which should provide the rest of the states. The following analysis will confirm this conjecture.

The design of a reduced-order observer is based on a partitioned form of the system dynamics

$$\begin{bmatrix} \dot{x}_m(t) \\ \dot{x}_u(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_m(t) \\ x_u(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \qquad (2.46a)$$

$$y(t) = \begin{bmatrix} I_m & 0 \end{bmatrix} \begin{bmatrix} x_m(t) \\ x_u(t) \end{bmatrix} \qquad (2.46b)$$

where x_m denotes states that are not measurable or unknown (note that from this point, in order to assist in presenting the derivation we have dropped the (t) to indicate time-varying). The identity matrix I_m in (2.46b) is an $m \times m$ matrix and the zero matrix is of dimensions $m \times (n-m)$ which gives

$$y = x_m \tag{2.47}$$

The system (2.46a) yields the following two equations

$$\dot{x}_m = A_{11}x_m + A_{12}x_u + b_1 u \tag{2.48a}$$

$$\dot{x}_u = A_{21}x_m + A_{22}x_u + b_2 u \tag{2.48b}$$

The first of these equations (2.48a) can be re-arranged to form an intermediate variable z

$$z = A_{12}x_b = \dot{x}_m - A_{11}x_m - B_1 u \tag{2.49}$$

The dynamics of the reduced-order observer are defined in the following

$$\dot{\hat{x}}_u = A_{22}\hat{x}_u + A_{21}x_m + B_u u + L(z - A_{12}\hat{x}_b) \tag{2.50}$$

The construction is similar to that of the full-order observer case, with the last term being a correction term. This correction term is based on the modelling error between the derivative of the actual measured state $\dot{x}_m$ and the estimated state $\dot{\hat{x}}_m$.

Notice that z in (2.49) requires knowledge of $\dot{x}_m$ which may be constructed by differentiating the signal. However, differentiating the signal will in fact severely degrade the signal quality if there is a small quantity of additive noise in the measurements. This difficulty can nonetheless be resolved by defining the reduced-order estimator states in terms of a new state vector as follows

$$z = \hat{x}_u = Ly \tag{2.51}$$

Taking the derivative of (2.51) and substituting (2.47), (2.50) and (2.51) gives

$$\begin{aligned} \dot{z} &= \dot{\hat{x}}_u - L\dot{x}_m && (2.52a) \\ &= (A_{22} - LA_{12})\hat{x}_u + (A_{21} - LA_{11})y + (B_2 - LB_1)u && (2.52b) \end{aligned}$$

Since we require the observer to be outputting $\hat{x}_u$, $\hat{x}_u$ should not appear in the expression. We therefore make use of (2.51) to give the following

$$\begin{aligned} \dot{z} &= (A_{22} - LA_{12})(z + Ly) + (A_{21} - LA_{11})y + (B_2 - LB_1)u && (2.53a) \\ &= (A_{22} - LA_{12})z + (A_{21} - LA_{11} + (A_{22} - LA_{12})L)y && (2.53b) \\ &\quad +(B_2 - LB_1)u && (2.53c) \\ &= Fz + Gy + Hu && (2.53d) \end{aligned}$$

where

$$\begin{aligned} F &= A_{22} - LA_{12} && (2.54a) \\ G &= A_{21} - LA_{11} + FL && (2.54b) \\ H &= B_2 - LB_1 && (2.54c) \end{aligned}$$

A schematic of the full feedback system with the reduced-order observer, represented by equations (2.46a), (2.46b) and (2.53d), is presented below in Figure 2.4. In light of the fact that $y(t) = x_m(t)$, it is important to note how $y(t)$ acts as inputs to the dynamic part of the observer in (2.50) as well as contributing directly to the state estimate $x_u(t)$ in (2.53d). This results in the estimate $x_u(t)$ being more susceptible to measurement errors in $y(t)$ than in the full-state observer case. The feedback loop is completed with the reduced-order observer providing the unkown states to augment known states. These are then made available to the controller. The dynamics of the

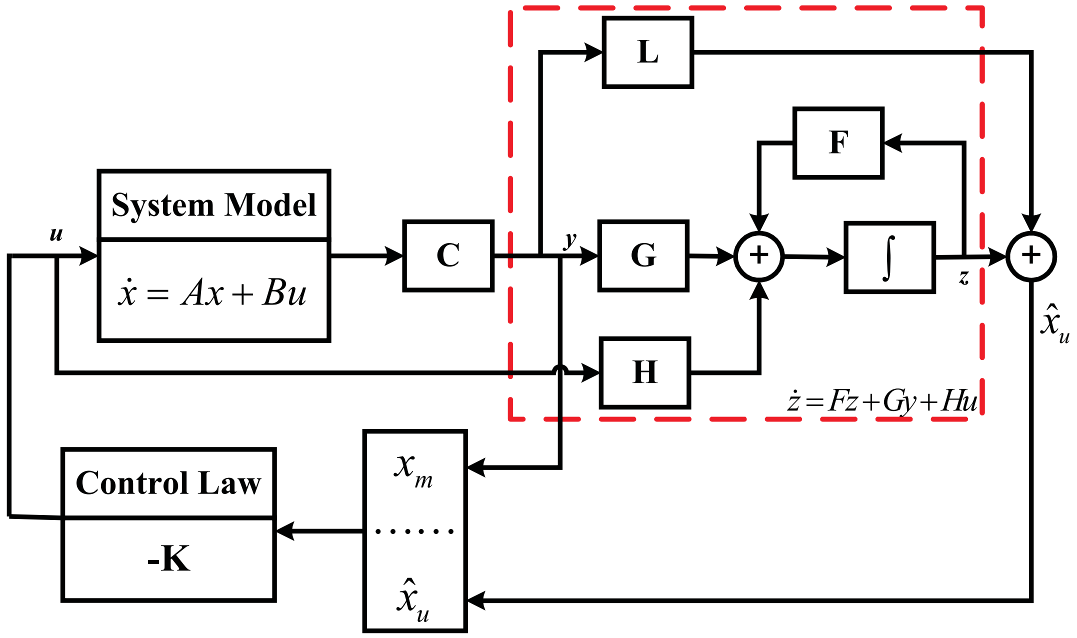

FIGURE 2.4: Schematic of a Reduced-Order Observer

error derivitive in terms of the error are defined as follows

$$\dot{e} = \dot{x}_u - \dot{\hat{x}}_u \tag{2.55a}$$

$$= A_{22}(x_u - \hat{x}_u) - LA_{12}(x_u - \hat{x}_u) \tag{2.55b}$$

$$= (A_{22} - LA_{12})e \tag{2.55c}$$

The result is remarkably similar to that in the full-state observer case. The matrix pair (A_{12}, A_{22}) must be completely observable. A useful lemma is presented by [2] which states that if (C, A) is completely observable, then so is (A_{12}, A_{22}). To drive the error asymptotically to zero, the same approach as the full-state observer is taken whereby the observer gain L is chosen such that the eigenvalues of $(A_{22} - LA_{12})$ lie in the left half complex plane. Theoretically, the eigenvalues could be moved arbitrarily towards negative infinity, which would yield rapid convergence. However, this tends to result in the observer acting as a differentiator, and becoming highly sensitive to noise.

An order reduction of m is not of much significance for a single output system, especially when considering the low cost of integrated circuits. However, for multiple output systems, a somewhat more substantial reduction in observer order is possible, which reduces observer complexity.

2.3 Advanced Observers

The so-called Advanced observers that will be described below all have the following properties in common

- Good ability for disturbance rejection
- High robustness
- Easy implementation

It must be noted that the Advanced observers chosen to be demonstrated below forn only a very minute portion of that currently in the literature.

2.3.1 Kalman Filters

The Kalman Filter is an efficient recursive filter that estimates the state of a dynamic system from a series of incomplete and noisy measurements. Together with the linear-quadratic regulator (LQR) the Kalman Filter solves the linear-quadratic-Gaussian control problem (LQG). The Kalman filter, the linear-quadratic regulator and the linear-quadratic-Gaussian controller are solutions to what probably are the most fundamental problems in control theory.

Kalman filters have found a vast array of applications. An example application would be providing accurate continuously-upadated information about the position and velocity of an object given only a sequence of observations about its position, each of which includes some error. It is used in a wide range of engineering applications from radar to computer vision. For example, in a radar application, where one is *interested in tracking a target*, information about the location, speed and acceleration of the target is measured with a great deal of corruption by noise at any time instant. The Kalman filter exploits the dynamics of the target, which governs its time evolution, to remove the effects of the noise and get a good estimate of the location of the target at the present time (filtering), at a future time (prediction), or at a time in the past (interpoloation or smoothing).

Kalman filters are based on linear dynamical systems discretised in the time domain. They are modelled on *Markov Chain* built on linear operators perturbed by Gaussian Noise. The state of the system is represented as a vector of real numbers. At each discrete time increment, a linear operator is applied to the state to generate the new state, with some noise mixed in, and optionally some information from the controls of the system if they are known. Then, another linear operator mixed with more noise generates the visible outputs from the hidden state. The Kalman Filter may be regarded as analogous to the hidden Markov Model, with the key difference that the hidden state variables take values in a continuous space (as opposed to a discrete space as in the hidden Markov Model). There is a strong duality between the equations of the Kalman Filter and those of the hidden Markov Model. A review of this is given in [33].

The Kalman filtering problem can be considered in a distributed control system setting where different components of the control system communicate over a wireless network. As shown in Figure 2.5, sensors, plants/actuators and controllers are located at different physical locations and thus require a communication network to exchange critical information for system control. In LQG control, the optimal controller consists of a Kalman filter and a state feedback controller. The Kalman filter uses the sensor measurements to compute the minimum mean square error estimate of the control system state and this state estimate is then used to compute the control command.

Wireless networks are playing increasingly important roles in distributed control

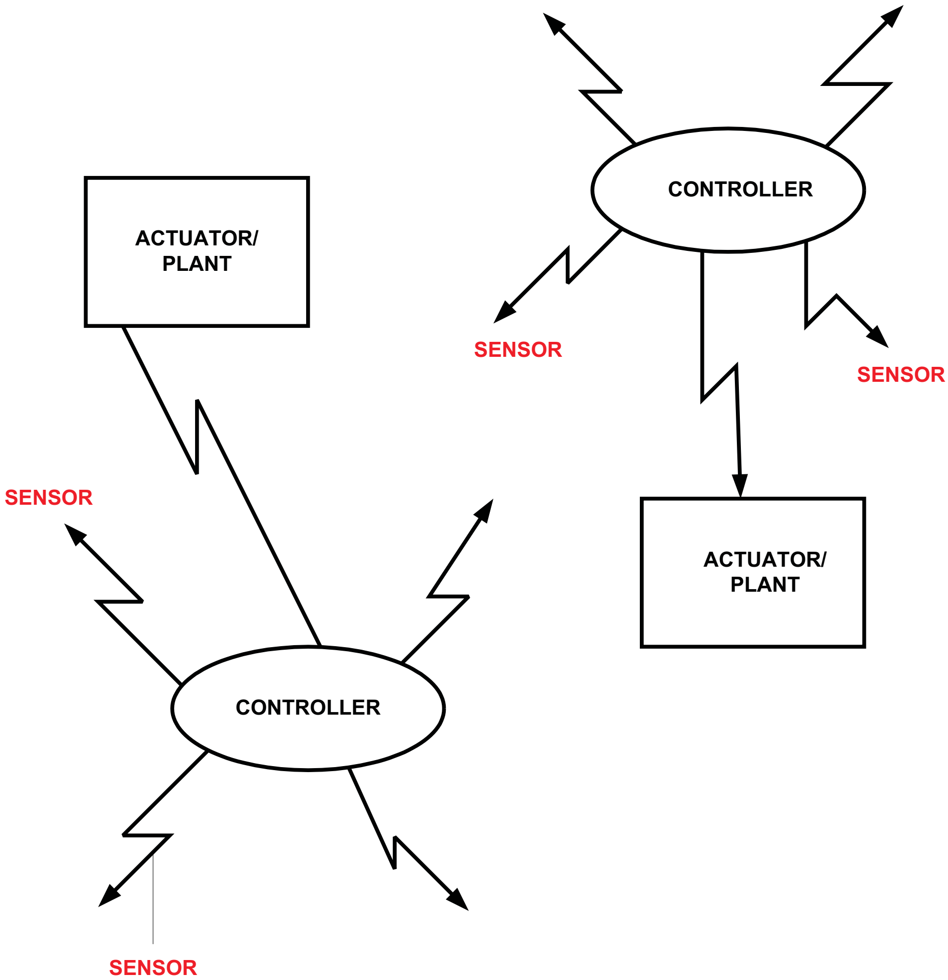

FIGURE 2.5: Distributed Control Systems

applications. Wireless technology allows fully mobile operation, fast deployment and flexible installation. However, wireless is a difficult channel due to the limited spectrum, time varying channel gains and interference. Packet losses are inevitable because of collisions and transmission errors. The Kalman filter is a well studied component in control theory with no information loss. According to [34] in a distributed control system, the Kalman filter collects sensor measurements from different sensors and each sensor encodes its own data into an individual packet. The Kalman filter must receive multiple packets to get all observation measurements. However, one or more of the packets can be lost during a given sample period. We are interested in how the Kalman filter updates the state estimate with partial observation losses and how the Kalman filter performs with these random observation losses. The Kalman filtering problem in the presence of packet losses has recently been studied in [35], [36], [37]. In [35], the statistical convergence properties of the Kalman filter are evaluated assuming the observation measurements are received in full or lost completely. To satisfy this assumption, all the sensor measurements need to be encoded in a single packet and the sensors should be colocated. A critical arrival rate of the observation is shown to exist and both an upper and a lower bound are computed. In [36], [37], the Kalman filter does not update if there is a packet loss. Thus, the sample period becomes random. The convergence properties are discussed under the random sampling, but the results are restricted to scalar systems. In [35], the results are generalized by allowing partial packet losses in the observation. Much of this research is along the same line as [35], but allowing partial observation losses introduces new dimensions to the problem. As in [35], the error covariance matrix iteration and the Kalman filter updates are stochastic and depend on the the random arrivals of the sensor measurements.

To understand the formulation of the Kalman Filter consider a discrete time linear time-invariant system defined by the system of equations

$$\dot{x}_{t+1} = Ax_t + w_t \tag{2.56}$$

$$y_t = Cx_t + v_t \tag{2.57}$$

where x_t is the system state, y_t is the measurement output, w_t is the system disturbance, and v_t is the measurement noise. The subscript t is the time index. Note that all boldface variables in this paper are vectors. Both w_t and v_t are assumed to be Gaussian random vectors and their covariance matrices are $Q = 0$ and $R > 0$, respectively.

Classical Kalman filtering theory assumes periodic measurement updates, i.e., no packet losses. The estimation error covariance matrices iterate according to the Algebraic Riccati Equation (ARE)

$$P_{t+1} = AP_tA' + Q - AP_tC'(CP_tC' + R)^{-1}CP_tA' \tag{2.58}$$

where P_t is the minimum mean square error covariance matrix of estimating x_t based on $y_0, , y_{t-1}$. When (A, Q) is controllable and (A, C) is observable, the ARE converges to a unique positive semidefinite matrix independent of the initial conditions.

The classical Kalman filtering problem to account for possible observation losses is investigated in [35]. The observation is either received in full or lost completely. When all the sensor measurements are encoded together and sent over the network in a single packet, the Kalman filter either receives the complete observation if the packet is correctly received or none of the observation if the packet is lost or substantially delayed. The packet delay and the probability of packet loss depend on network conditions such as the channel gain and the network traffic. In control applications, the sensor measurements are delay sensitive and old measurements are often discarded when new measurements are available. Thus, the packet loss is often defined to be delay dependent, that is, a packet is declared lost if it has not been received correctly after a certain time period. The random variable γ_t indicates whether the observation at time t is correctly received by the end of the t^{th} sample period. It is assumed that γ_t is i.i.d. Bernouli with $Pr(\gamma_t = 1) = \lambda$. This corresponds to the Binary Symmetric Channel (BSC) with the error probability $1 - \lambda$. For a fixed sampling rate and a given packet size, the throughput of the communication link (from the sensors to the Kalman filter) is the product of the sampling rate, the packet size and the probability λ that a packet is received correctly. For a discrete-time control system, the sampling rate is implicit. We sometimes refer to it as the link throughput since it is a scaled version of the throughput for a fixed sampling rate and a given packet size. The authors in [35] show that the error covariance matrix iterates according to the following stochastic equation

$$P_{t+1} = AP_tA' + Q - \Gamma_t AP_tC'(CP_tC' + R)^{-1}CP_tA' \tag{2.59}$$

Note this is a stochastic iteration due to the random observation losses while in the classical Kalman filter the iteration is deterministic. The convergence of the iteration thus depends on the sample path of γ_t. The authors show the existence of a critical value of λ_c such that $E[P_t]$ is bounded if $\lambda > \lambda_c$ and $E[P_t]$ goes to infinity as $t \to \infty$ if $\lambda < \lambda_c$. λ_c may not always be found explicitly but both an upper and lower bound can be computed. The steady state estimation error covariance matrix $lim_{t\to\infty}E[P_t]$ is bounded both from above and from below for sufficiently high observation arrival rate λ.

The results in [35] to allow partial observation losses are investigated in [38]. It is assumed that y_t has at least two elements and partitions y_t into multiple parts where each part can be lost independently. This corresponds to λ_t coming from multiple sensor data packets with some of the packets lost or delayed. For the sake of simplicity, it is assumed that the observation process y_t is divided into two parts $y_{1,t}$ and $y_{2,t}$, which are sent over two different channels using a shared wireless network. At each step, the Kalman filter may receive $y_{1,t}$ or $y_{2,t}$ alone, both, or neither. In the general case, the observation processes can be sent in more than two packets. All of the results presented in [38] can in fact be extended to this general case.

2.3.2 High Gain Observers

Variable Structure Systems utilize a high-speed switching control law to drive the plant state trajectories onto a specified and designer-chosen surface in the state space, called switching surface or sliding mode, and then maintain the plant state trajectories on this surface. The controller generated via VSS with sliding mode is non-linear and known as sliding mode control (described in section 2.5). Sliding mode control is somewhat related to the Lyapunov control. The outstanding feature of these controllers is their excellent robustness and invariance properties. The fundamental features of VSS with sliding mode and the Lyapunov control and their applications are given in [39]. Moreover, Lyaponov control and sliding mode control, which are commonly used for the systems have matching uncertainty, generate non-linear state feedback laws.

The fundamental property of high gain systems is given in [40] is their relationship with singularly perturbed systems. High-gain feedback systems have been studied in [41] and results have been obtained that demonstrate that the slow motions of high gain systems are the same as sliding motions in VSS.

The standard non-linear Luenberger-type observer contains inside its structure a copy of the system, plus a proportional correction given by the measurement error; this structure has several drawbacks. For example, the observable state components are estimated adequately, implying that the residual term turns out to be very small and has no practical effect on the observer structure and consequently, robustness against disturbances is not assured. In addition, this observer needs an accurate model of the system and therefore when large uncertainties exist in the mathematical model for an actual plant, Luenberger-type observers are not realisable. In order to avoid these drawbacks, the following modifications to the Luenberger observer structure are proposed in [42].

- In order to provide the observer with robust properties against disturbances, only an integral-type contribution of the measured error is considered.
- An uncertainty estimator is introduced in the observation methodology, in order to estimate the unknown terms of the non-linear vector.

In [42] an alternative form of integral-type observer has been discussed. It follows an alternative representation of the original system, in which the output disturbances (noise measurements) are transformed into a static disturbance. This was done to decouple the observer's gains from the output disturbances and thereby avoid noise amplification, which is a characteristic of standard proportional observers. The proposed methodology was compared to a standard Luenberger observer via a numerical simulation and it was observed that the integral-type observer (proposed methodology) demonstrates better performance than the classical methodology, for non-modelled dynamics and noisy measurements.

2.3.3 Utkin Observers

The design of Sliding Mode Observers, also referred to as *Utkin Observers* (after their founder) is presented in detail below.

The design procedure for a state observer with inputs as discontinuous functions of mismatches where motion preceding sliding mode and motion in the intersection of discontinuity surfaces may be handled independently. The observer is described in [6] by the following differential equations

$$\dot{\hat{x}} = Ax + Bu \tag{2.60}$$

where $x \in \mathbb{R}^n$,$u \in \mathbb{R}^m$ and $rank(B) = m$.

It is assumed that Matrix B may be represented in the form

$$B(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} \tag{2.61}$$

and $det(B_2) \neq 0$

The non-singular coordinate transformation

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = TAT^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + TT^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (2.62)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Tx \quad (2.63)$$

$$T = \begin{bmatrix} I_{n-m} & -B_1{B_2}^{-1} \\ 0 & {B_2}^{-1} \end{bmatrix} \quad (2.64)$$

reduces the system of equations (2.60) to the regular form consisting of two blocks

$$\dot{x_1} = A_{11}x_1 + A_{12}x_2 \quad (2.65)$$

$$\dot{x_2} = A_{21}x_1 + A_{22}x_2 + u \quad (2.66)$$

where $x_1 \in \mathbb{R}^{n-m}$, $x_2 \in \mathbb{R}^m$.

The observer is described by

$$\dot{\hat{x}} = A_{11}\hat{x}_1 + A_{12}\hat{y}_1 + B_1u + L_1v \quad (2.67)$$

$$\dot{\hat{y}} = A_{21}\hat{x}_1 + A_{22}\hat{y}_1 + B_2u - v \quad (2.68)$$

where $\hat{x}_1$ and $\hat{y}$ are the estimates of the system state,

$$v = Msgn(\hat{y} - y) \quad (2.69)$$

Where, $M = Const, M{>}0$.

The vector y is measured, hence $\hat{y} - y$ is available.

The discontinuous vector function $v \in \mathbb{R}^l$ is chosen such that sliding mode is enforced in the manifold $\bar{y} = \hat{y} - y = 0$ and the mismatch between the output vector y and its estimate $\hat{y}$ is essentially reduced to zero. A matrix L_1 must be found such that the mismatch $\bar{x} = \hat{x}_1 - x_1$ between x_1 and its estimate $\hat{x}_1$ decays at the desired rate. Equations with respect to $\bar{x}$ and $\bar{y}$ are presented below

$$\dot{\bar{x}} = A_{11}\bar{x}_1 + A_{12}\bar{y} + L_1 v \quad (2.70)$$

$$\dot{\bar{y}} = A_{21}\bar{x}_1 + A_{22}\bar{y} - v \quad (2.71)$$

$$v = Msgn(\bar{y}) \quad (2.72)$$

The sliding mode is enforced in the manifold $\bar{y} = 0$ if the matrix multiplying v in (2.71) is negative definite and M takes a high but finite value. For bounded initial conditions, sliding mode can be enforced in the manifold $\bar{y} = 0$. If follows from the equivalent control methods in literature that the solution v_{eq} to equation $\dot{\bar{y}}$ should be substituted into (2.70) with $\bar{y} = 0$ to derive the sliding mode equation

$$v_{eq} = A_{21}\bar{x}_1 \quad (2.73)$$

$$\dot{\bar{x}}_1 = (A_{11} + L_1 A_{21})\bar{x}_1 \quad (2.74)$$

Hence the desired rate of convergence of $\bar{x}_1$ to zero and convergence of $\hat{x}_1$ to x_1 can be provided by a proper choice of matrix L_1.

The observer with input as a discontinuous function of the mismatch (2.70), (2.71), (2.72) in sliding mode is equivalent to the reduced order observer. However, if the plant and observed signal are affected by noise (which is primarily the case), the non-linear observer may turn out to be preferable due to its filtering properties, which coincide with that of the Kalman Filter [33].

2.3.4 Extended State Observers

In [43] a novel non-linear observer, called an Extended State Observer(ESO), is introduced to accomplish the non-linear dynamic compensation problem. ESO was first proposed in [44] for on-line estimating the total dynamics, which lumps the internal non-linear dynamics and the external disturbance. ESO based control is not a model based approach and, therefore, can be applied to a wide variety of plants. Simplicity in the resulting control structure avoids computation burdens of the model-based aproaches. An overview of its design philosophy and potential applications can be found in [45]. Some initial results on the effectiveness of ESO for robot systems has been shown via the experiment on a two-linkrobot [46].

Compensation for the non-linear dynamics of the manipulator (defined in [43]) is of fundamental importance to achieve satisfactory performance of robot systems. In the contribution of [43] the extended state observer has been utilized for realizing the non-model based cancellation of the non-linear dynamics of the robot systems. The effectiveness of ESO based control is tested via the experiments on motion control of a robot finger.

ESO in Non-linear Control Strategy for Induction Motor Drives In [47] a new configuration called an Auto Disturbance Rejection Controller (ADRC) is used in the induction motor speed control. A key part of ADRC is ESO. By using ESO, ADRC can estimate accurately the derivative signals and accurate decoupling of induction motor is achieved too. In addition, the external disturbances and parameter variations could also be estimated and compensated by ADRC, so that the closed loop motor drive system under ADRC control does not entirely depend on the accurate mathematical model of induction motors. Therefore, the robustness and adaptability of the control system is significantly improved too. Simulation results provided in [47] demonstrated that the controller operates quite smootly and robustly under modelling uncertainty and external disturbance, and it can provide good dynamic performance such as small overshoot and fast transient time in the speed control.

2.3.5 Comparison Study of Advanced Observation Techniques

A comparison study of the performances and characteristics of three advanced state observers is presented in [48], including the high-gain observers, the sliding-mode observers and the extended state observers. These observers were originally proposed to address the dependence of the classical observers, such as the Kalman Filter and the Luenberger Observer, on the accurate mathematical representation of the plant. The results show that, over-all, the extended state observer is much more superior in dealing with dynamic uncertainties, disturbances and sensor noise. Several novel non-linear gain functions are proposed to address the difficulty in dealing with unknown initial conditions.

Since the original work by [1] and [2], the use of state observers proves to be useful in not only system monitoring and regulation but also detecting as well as identifying failures in dynamical systems. Since almost all observer designs are based on the mathematical model of the plant, the presence of disturbances, dynamic uncertainties, and non-linearities pose great challenges in practical applications. Toward this end, the high-performance robust observer design problem has been topic of considerable interest recently, and several advanced observer designs have been proposed, namely Sliding Mode, High-Gain and Extended-State observers. These have all been defined to some degree of detail earlier in this chapter.

A comparison of performances and characteristics of these observers is presented in [48]. The criterion for comparison is based on the observer tracking errors, both at steady state and during transients, and the robustness of the performance with respect to the uncertainties of plant. To further enhance the performance of non-linear extended state observers in the presence of unknown initial several non-linear gain functions are introduced.

In [48] an industrial motion control test bed is used for simulation purposes. The output measurement is corrupted by white noise to make the comparison realistic. The quality of observers is measured by the speed and accuracy of the states of the observer converging to those of the plant. To make the comparison fair, the parameters of the observers are adjusted so that their sensitivities to the measurement noise are roughly the same. The exact outputs of y and $\hat{y}$ are obtained directly from the simulation model of the plant to calculate the state estimation error. During the simulation, open-loop tests were performed, followed by closed-loop tests. The results of these tests will be briefly outlined below.

Open Loop Comparison Tests

In the case of open-loop tests, the input to the plant is a step function and the observers are evaluated according to their capability in tracking the step response. The tests were run in three conditions

- Nominal plant;
- Nominal plant plus coulomb friction;
- Nominal plant with 100% increase in inertia.

Figure 2.6 shows the position and velocity estimation errors for the nominal plant in terms of the tracking errors for y and $\hat{y}$. For all three observers perform well in steady state and have roughly the same accuracy and sensitivity to the noise. As expected, non-linear extended state observer takes longer to reach steady state, as expected, because it does not assume the knowledge of the plant dynamics.

Figure 2.7 illustrates the tracking errors for plants with added coulomb friction, clearly demonstrating that NESO and SMO are much more robust than HGO in the presence of disturbance.

Figure 2.8 illustrates the simulation results for the plant with a 100% increase of inertia. While NESO provides an overall superior performance, it is very closely

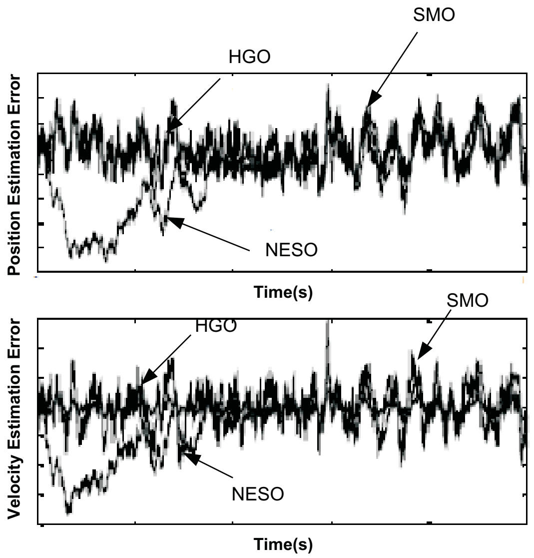

FIGURE 2.6: Estimation error of the nominal plant

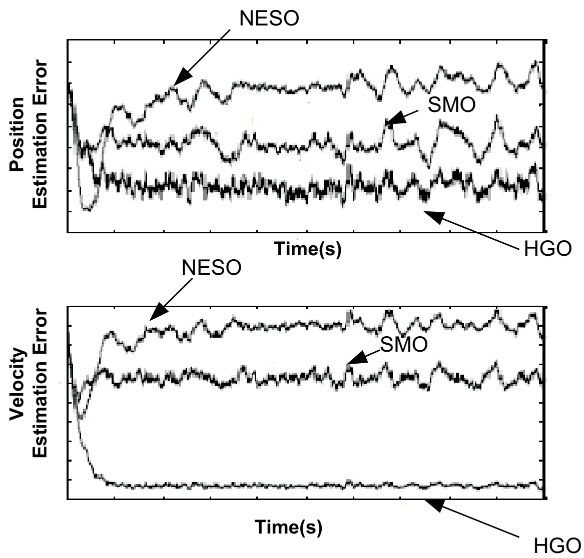

FIGURE 2.7: Estimation error of the plant with 0.5N-m coulomb friction

followed by SMO. All three observers are seen to follow very similar trajectories.

Simulations reveal that both SMO and NESO achieve better performance amongs the three observer types. However, if the plant has unknown initial conditions, it may produce significant transient estimation errors. This was dealt with in detail in [48], however is beyond the scope of this thesis.

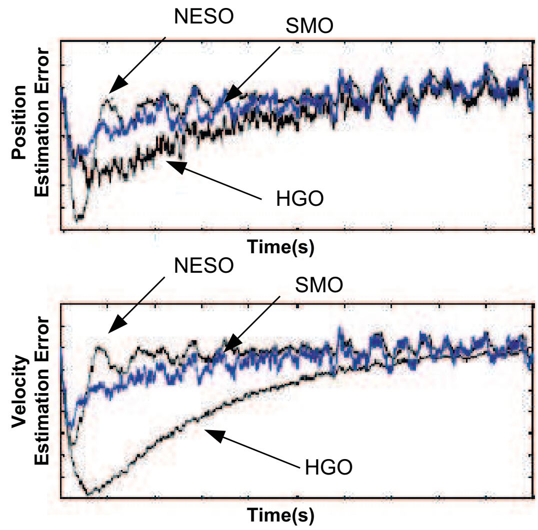

FIGURE 2.8: Estimated error with 100% change of inertia

Closed Loop Comparison

Based on their open loop performance, in [48] NESO and SMO are evaluated in a closed-loop feedback setting, such as that shown in Figure 2.9 for NESO.

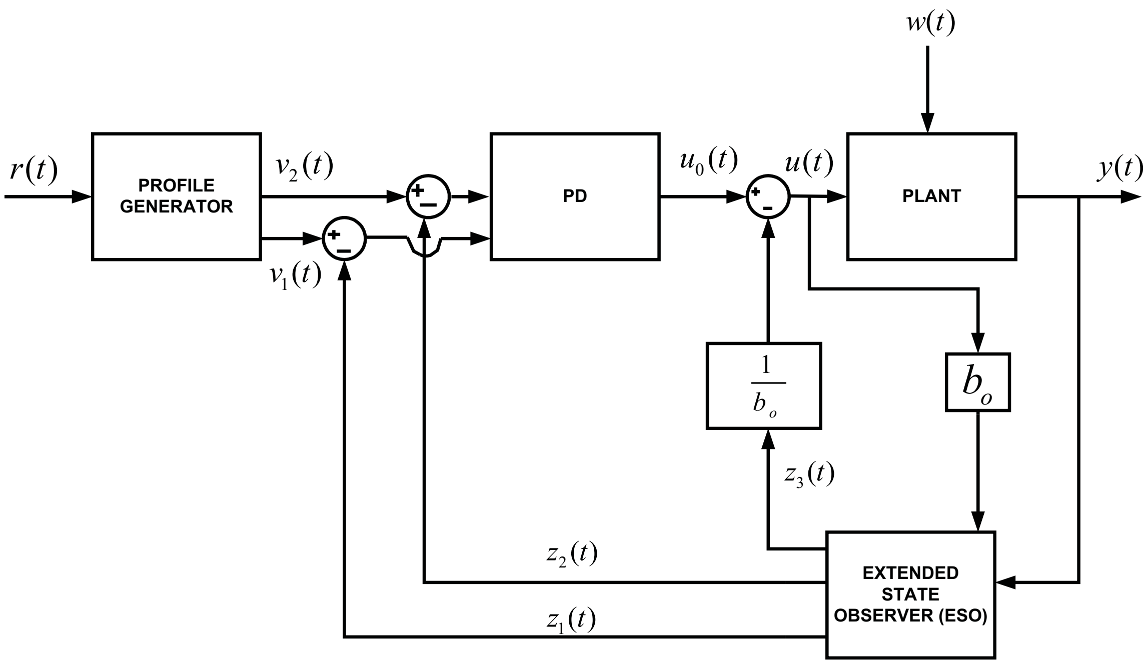

FIGURE 2.9: Observer Based State Feedback Control Configuration

The profile generator provides the desired state trajectory in both y and $\hat{y}$ using an industry standard trapezoidal profile. Based on the separation principle, the controller is designed independently, assuming all states are accessible in the control law. Simulation results are shown in Figure 2.10, 2.11 and 2.12. Both control systems appear to have similar output responses for the nominal plant. However, as soon as unknown friction or disturbances are introduced, the differences between NESO and SMO become apparent, indicating that NESO-based design has inherent robustness against the uncertainties.

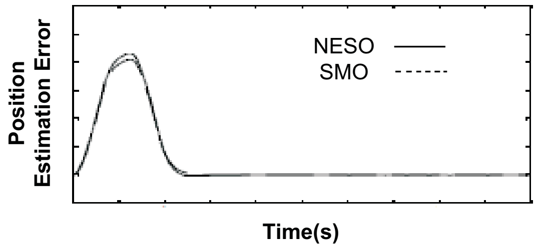

FIGURE 2.10: Nominal responses of the control systems

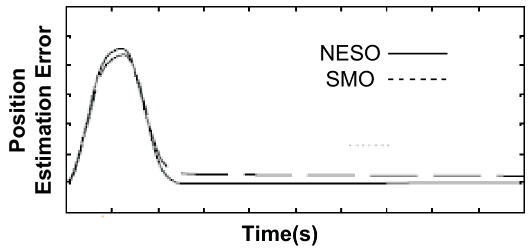

FIGURE 2.11: Simulation results with $0.5N - m$ coulomb frictions

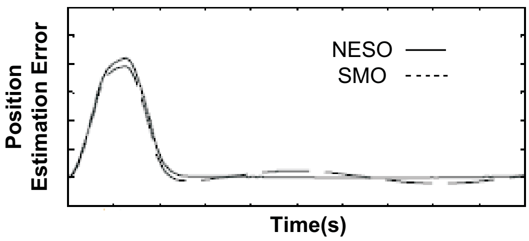

FIGURE 2.12: Simulation results with sinusoid disturbance

Conclusion

A comparison study of advanced observer designs, including the non-linear extended state observer, the high-gain observer and the sliding mode observer yields the following conclusions

- As a state estimator, NESO and SMO perform much better than high-gain observers. The robustness of the NESO and SMO to plant uncertainty and external disturbance is inherent in their structures. The chattering problem is the main drawback of the sliding-mode method in practical applications. For details on this, refer to Appendix C.

- As external disturbances to the systems were introduced the SMO and NESO appeared to perform slightly stronger than the HGO.

It can therefore be concluded that from the sample set of Advanced observers taken, Sliding Mode Observers performed quite favourably against the chosen criteria, making further research in this area quite attractive.

2.4 Sliding Mode Functional Observers

The main purpose of this section is to briefly introduce the problem of the Sliding Mode Functional Observer and to demonstrate the issues that are faced in solving related problems that are handled in this thesis.

Consider a linear time-invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.75}$$

$$y(t) = Cx(t) \tag{2.76}$$

$$z(t) = Lx(t) \tag{2.77}$$

where $x(t) \in \mathbb{R}^n$,$u(t) \in \mathbb{R}^m$, $v(t) \in \mathbb{R}^q$ and $y(t) \in \mathbb{R}^r$ are the state, known input, unknown input and the output vectors, respectively. $z(t) \in \mathbb{R}^p$ is the vector to be estimated. The pair (C, A) is detectable, (A, B) is controllable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. This is a common problem that is faced in most practical cases. It is safe to assume that in almost all industrial applications there will be some inputs that will not be known or are in fact unmeasurable. Furthermore, without loss of generality, it is assumed that $Rank(C) = r$, $Rank(L) = p$, $Rank \begin{bmatrix} L \\ C \end{bmatrix} = p + r - \tilde{p}, \tilde{p} \leq p$ and C takes the form $C = \begin{bmatrix} I_r & 0 \end{bmatrix}$ (otherwise the system can always be transformed into this form).

Consider the following sliding mode linear functional observer

$$\dot{w}(t) = Nw(t) + Jy(t) + Hu(t) + \Gamma \text{sgn}\,(Me(t)) \tag{2.78}$$

$$\hat{z}(t) = w(t) + Ey(t) \tag{2.79}$$

$$e(t) = z(t) - \hat{z}(t) \tag{2.80}$$

where $w(t) \in \mathbb{R}^p$, $M \in \mathbb{R}^{\tilde{p} \times p}$, $\Gamma \in \mathbb{R}^{p \times \tilde{p}}$, sgn$(\cdot)$ is the sign function and also the sliding surface is given by

$$Ke(t) = 0 \tag{2.81}$$

In view of the dimension of matrix K, the error vector $e(t)$ can be written as

$$e(t) = \begin{bmatrix} e_y(t) \\ e_1(t) \end{bmatrix} \tag{2.82}$$

where $e_y(t) \in \mathbb{R}^{\tilde{r}}$ and $e_1(t) \in \mathbb{R}^{p-\tilde{p}}$.

The first aim of this thesis is to design a sliding mode functional observer, that is to find N, J, H, E and a suitable Γ such that $e(t)$ slides along the surface $Ke(t) = 0$ and $e_1(t) \to 0$ as $t \to \infty$.

It is however acknowledged that in most practical (industrial) cases, the system cannot in fact be contained in the simple form of (2.75), (2.76) and (2.77). For example, as is the case in a great deal of industrial scenarios, one may be faced with a non-linear system. This will of course change the whole dynamic of the sliding mode functional observer and the design procedure that will result. This introduces the major impetus behind the thesis i.e. to extend and build upon the sliding mode functional observers application to practical scenarios with a greater range of cases that it will be compatible with.

2.5 Conclusion

This chapter has provided some theoretical and practical insight into Sliding Mode Observers and their relative importance both from a mathematical and implementation point of view. The SMO theory has been compared with that of the Kalman filter initially, however more importantly evaluated against more advanced observer techniques, namely Extended State and High-Gain Observers. While SMO are at the forefront of control theory currently, Section 2.7 demonstrates some of the advantages that are also proposed by extended state observers (over other advanced observers) through some simulations that illustrate the ESO's inherent robustness in the presence of uncertainties and disturbance. Section 7.2 of this thesis illustrates possible future work that can be done in lieu of these results.

The topic of Sliding Mode Functional Observers has been introduced in Section 2.4 and provides a base upon which the theory will be extended on in this thesis. As it stands, the theory in Section 2.4 essentially applies only to a narrow range of systems. The objective of this thesis is to broaden this range to include such systems as; neutral-delay, unknown input, non-linear and descriptor systems.

Chapter 3

Sliding Mode Functional Oservers for Time-Delay Systems of Neutral Type

This chapter illustrates the problem of estimating a linear function of the states of a class of linear time-delay systems of the neutral-type using sliding mode functional observer approach. Sliding mode functional observers proposed in this chapter are of low-order and do not include the derivatives of the outputs. New conditions for the existence of sliding mode functional observers are derived. A design procedure for the determination of the observer parameters can also be easily derived based on the existence conditions developed.

3.1 Introduction

In the control literature, there has been considerable attention focusing on the stabilization and state estimation of a class of time-delay systems commonly referred to as neutral systems [49] - [50]. As defined in [51], neutral systems are time-delay systems that have the same highest derivation order for some components of the state vector, $x(t)$, at both time t and past time(s) $t_s \leq t$. The presence of a retarded argument in the state derivatives increases mathematical complexity and makes the investigation of such equations more complicated than equations with delays only in the states [50] - [51]. It is commonly known that time-delay is often a source of instability in many dynamic systems, and thus considerable attention has been paid to the research on the stability ananlysis and control synthesis of time-delay systems.

The control and stabilization of neutral systems is often based on the assumption that the entire state vector is available for state feedback control. As discussed in [50], observer design for neutral-type delay systems has not yet been fully investigated in the literature and remains to be important. The observers derived in [50] for time delay systems of neutral type are full-order observers and include delayed state (estimated) derivatives.

Sliding mode functional observers on the other hand, estimate linear functions of the state vector of a system without estimating all of the individual states, while ensuring that sliding occurs on a manifold, where some function of the output prediction error is zero. Such functional estimates of the state vector are useful in feedback control system design because the control signal is often a linear combination of the states, and it is possible to utilize a sliding mode linear functional observer to directly estimate the feedback control signal. Although the theory for sliding mode State observers (also referred to as Utkin observers) are well established for linear systems [52] - [53], the concept of sliding mode linear functional observers for neutral-type delay systems has not yet been reported. In this chapter, some new results on designing sliding mode functional observers that can estimate a linear function of the states of a neutral-type delay system are presented.

This chapter is organized as follows: Section 3.2 provides a general outline of the problem to be solved. This section will provide a description to the neutral-delay class system of equations and the corresponding sliding mode linear functional observer structure that will be applied. Section 3.3 describes the conditions for the existence of the sliding mode functional observer and based on these conditions a design algorithm is presented. Section 3.4 verifies and demonstrates the theory presented with a simulation example. Finally, Section 3.5 presents the conclusions of the chapter.

3.2 Problem Statement

Consider the following class of linear delay systems of the neutral-type [50],

$$\tilde{E}\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{A}_d\tilde{x}(t-\tau) + \tilde{F}\dot{\tilde{x}}(t-\tau) + Bu(t), t \geq 0, \tag{3.1a}$$

$$\tilde{x}(t) = \tilde{\phi}(t), \forall t \in [-\tau, 0], \tag{3.1b}$$

$$y(t) = \tilde{C}\tilde{x}(t), \tag{3.1c}$$

$$z(t) = \tilde{L}\tilde{x}(t), \tag{3.1d}$$

where $\tilde{x}(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^r$ and $u(t) \in \mathbb{R}^m$ are respectively the state, measured output and input vectors. $\Phi(t)$ is a continuous vector-valued initial function and $\tau \geq 0$ is a known constant time delay. $z(t) \in \mathbb{R}^p$ is a functional state vector. Real constant matrices $\tilde{E} \in \mathbb{R}^{n\times n}, \tilde{A} \in \mathbb{R}^{n\times n}, \tilde{A}_d \in \mathbb{R}^{n\times n}, \tilde{F} \in \mathbb{R}^{n\times n}, \tilde{B} \in \mathbb{R}^{n\times m}$, $\tilde{C} \in \mathbb{R}^{r\times n}$ and $\tilde{L} \in \mathbb{R}^{p\times n}$ are known. Without loss of generality, it is assumed that $Rank(\tilde{E}) = q, q \leq n$, $Rank(\tilde{C}) = r$, $Rank(\tilde{L}) = p$ and $Rank \begin{bmatrix} \tilde{C} \\ \tilde{L} \end{bmatrix} = (r+p-\tilde{r}) \leq n$.

The aim of this chapter is to design a sliding mode functional observer capable of asymptotically estimating any given function of the state vector, $z(t) \in \mathbb{R}^p$. Let us consider the following sliding mode functional observer structure of order p for the system (3.1),

$$\begin{aligned}
\dot{\xi}(t) &= N\xi(t) + Jy(t) + J_d y(t-\tau) + Hu(t) \\
&\quad + \Gamma \mathrm{sgn}\left(Ke(t)\right), t > 0 && (3.2a) \\
\xi(t) &= \rho(t), \forall t \in [-\tau, 0], && (3.2b) \\
\hat{z}(t) &= \xi(t) + My(t) && (3.2c) \\
e(t) &= \hat{z}(t) - z(t), && (3.2d)
\end{aligned}$$

where $\xi(t) \in \mathbb{R}^p$, $K \in \mathbb{R}^{\tilde{r}\times p}$, $\Gamma \in \mathbb{R}^{p\times\tilde{r}}$, $\mathrm{sgn}(\cdot)$ is the sign function. $\rho(t)$ is a continuous vector-valued initial function and $\hat{z}(t)$ denotes the estimate of $z(t)$. The unknown matrices N, J, J_d, H and M are such that $N \in \mathbb{R}^{p\times p}$, $J \in \mathbb{R}^{p\times r}$, $J_d \in \mathbb{R}^{p\times r}$, $H \in \mathbb{R}^{p\times m}$ and $M \in \mathbb{R}^{p\times r}$. The sliding surface is given by

$$Ke(t) = 0. \tag{3.3}$$

In view of the dimension of matrix K, the error vector $e(t)$ can be written as

$$e(t) = \begin{bmatrix} e_y(t) \\ e_1(t) \end{bmatrix}, \tag{3.4}$$

where $e_y(t) \in \mathbb{R}^{\tilde{r}}$ and $e_1(t) \in \mathbb{R}^{r-\tilde{r}}$. The problem to be solved in this chapter is to design a sliding mode functional observer of the form (3.2), where matrices N, J, J_d, H, M and a suitable Γ are to be determined such that $e(t)$ slides along the surface $Ke(t) = e_y(t) = 0$ in finite time and $e_1(t) \to 0$ as $t \to \infty$.

3.3 Existence Conditions of the Observer

In order to deal with the term $\tilde{F}\dot{\tilde{x}}(t-\tau)$ in (3.1) and also for the convenience of design, let us transform the system (3.1) into the following descriptor form,

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-\tau) + Bu(t), t > 0, && (3.5a)\\ x(t) &= \phi(t), \forall t \in [-\tau, 0], && (3.5b)\\ y(t) &= Cx(t), && (3.5c)\\ z(t) &= Lx(t), && (3.5d) \end{aligned}$$

where $x(t) = \begin{bmatrix} \tilde{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} \in \mathbb{R}^{2n}$, $\Phi(t) = \begin{bmatrix} \tilde{\Phi}(t) \\ \dot{\tilde{\Phi}}(t) \end{bmatrix}$, $E = \begin{bmatrix} \tilde{E} & 0 \end{bmatrix} \in \mathbb{R}^{n\times 2n}$, $A = \begin{bmatrix} \tilde{A} & 0 \end{bmatrix} \in \mathbb{R}^{n\times 2n}$, $A_d = \begin{bmatrix} \tilde{A}_d & \tilde{F} \end{bmatrix} \in \mathbb{R}^{n\times 2n}$, $C = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix} \in \mathbb{R}^{r\times 2n}$ and $L = \begin{bmatrix} \tilde{L} & 0 \end{bmatrix} \in \mathbb{R}^{p\times 2n}$.

Now let $P \in \mathbb{R}^{p\times n}$ be a full-row rank matrix and define error vectors $\varepsilon(t) \in \mathbb{R}^p$ and $e(t) \in \mathbb{R}^p$ as

$$\begin{aligned} \varepsilon(t) &= \xi(t) - PEx(t) && (3.6a)\\ e(t) &= \hat{z}(t) - z(t). && (3.6b) \end{aligned}$$

From (3.6a) the following error dynamics equation is obtained

$$\begin{aligned} \dot{\varepsilon}(t) &= \dot{\xi}(t) - PE\dot{x}(t) \\ &= N\varepsilon(t) + (NPE + JC - PA)x(t) + (J_d C - PA_d)x(t-\tau) \\ &\quad + (H - PB)u(t) + \Gamma \mathrm{sgn}\left(Ke(t)\right). \qquad (3.7) \end{aligned}$$

From (3.6b), the error vector $e(t)$ can be expressed as

$$e(t) = \varepsilon(t) + (PE + MC - L)x(t). \qquad (3.8)$$

To define the sliding surface for the functional observer, let us first partition matrix L as follows;

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \tag{3.9}$$

where L_1 belong to the rows space of C. Since L_1 belong to the rows space of C, there always exists a full-row rank matrix $G \in \mathbb{R}^{\tilde{r} \times r}$, $\tilde{r} \leq r$ such that

$$L_1 = GC. \tag{3.10}$$

If $K \in \mathbb{R}^{\tilde{r} \times p}$ is chosen such that

$$K = \begin{bmatrix} I_{\tilde{r}} & 0_{\tilde{r} \times (p-\tilde{r})} \end{bmatrix} \tag{3.11}$$

and G according to (3.10), then the sliding surface (3.3) is

$$Ke(t) = e_y(t) = Gy(t) - K\hat{z}(t) = 0. \tag{3.12}$$

Considering the partitioning of $e(t)$ in (3.4), the error vector $\varepsilon(t)$ and matrix N can be partitioned as

$$\varepsilon(t) = \begin{bmatrix} \varepsilon_y(t) \\ \varepsilon_1(t) \end{bmatrix}, \tag{3.13a}$$

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}. \tag{3.13b}$$

If we now choose $\Gamma = \begin{bmatrix} \Gamma_1 \\ 0_{(p-\tilde{r}) \times \tilde{r}} \end{bmatrix}, \Gamma_1 \in \mathbb{R}^{\tilde{r} \times \tilde{r}}$, then the error dynamics described by (3.7) and (3.8) can be rewritten as

$$\begin{aligned}\begin{bmatrix}\dot{\varepsilon}_y(t)\\ \dot{\varepsilon}_1(t)\end{bmatrix} &= \begin{bmatrix}N_{11} & N_{12}\\ N_{21} & N_{22}\end{bmatrix}\begin{bmatrix}\varepsilon_y(t)\\ \varepsilon_1(t)\end{bmatrix}\\ &\quad +(NPE - JC - PA)x(t) + (J_dC - PA_d)x(t-\tau)\\ &\quad +(H - PB)u(t) + \Gamma\text{sgn}\left(Ke(t)\right). \end{aligned} \tag{3.14a}$$

The existence conditions for the sliding mode functional observer (3.2) are given in the following theorem, which ensures that $\hat{z}(t)$ converges asymptotically to $z(t)$.

Theorem 1: $e_1(t) \rightarrow 0$ as $t \rightarrow \infty$ and also $e(t)$ slide along the surface $Ke(t) = 0$, $t \geq t_s$ where $t_s \leq \left(\frac{\|Gy(0)-M\hat{z}(0)\|}{\eta}\right), \eta > 0$ for any $x(0), \hat{z}(0)$, and $u(t)$ if the following conditions hold:

$$N_{22},\ \textit{Hurwitz}, \tag{3.15}$$

$$NPE + JC - PA = 0, \tag{3.16}$$

$$J_dC - PA_d = 0, \tag{3.17}$$

$$H = PB, \tag{3.18}$$

$$PE + MC - L = 0, \tag{3.19}$$

$$e_y^T(t)\dot{e}_y(t) < -\eta\|e_y(t)\|. \tag{3.20}$$

Proof: If conditions (3.16), (3.17), (3.18) and (3.19) are satisfied, then by considering (3.8), the error dynamics (3.14a) of the observer can be rewritten as

$$\dot{e}_y(t) = N_{11}e_y(t) + N_{12}e_1(t) + \Gamma_1\text{sgn}(e_y(t)) \tag{3.21a}$$

$$\dot{e}_1(t) = N_{21}e_y(t) + N_{22}e_1(t). \tag{3.21b}$$

If (3.20) is satisfied, then for some $\Gamma_1 \in \mathbb{R}$ an ideal sliding motion will take place on the surface

$$S_0 = \{(e_1(t), e_y(t)) : e_y(t) = 0\} \tag{3.22}$$

and it follows that after some finite time t_s, for all subsequent time, $e_y(t) = 0$ [53]. The dynamics of $e_1(t) = 0$ then reduces to

$$\dot{e}_1(t) = N_{22}e_1(t). \tag{3.23}$$

If (3.15) is satisfied, then $e_1(t) \to 0$ as $t \to \infty$. As such, if (3.19) is satisfied, $e(t) \to 0$ as $t \to \infty$.

Remark 1. In order to derive the parameters of the sliding mode linear functional observer (3.2), the equations (3.16) - (3.19) must be solved to find out the unknown matrices N, P, J, J_d, H and M. The following theorem will provide the necessary and sufficient conditions for the solvability of matrix Equations (3.16) - (3.19) of *Theorem 1.*

Theorem 2: The matrix equations (3.16) - (3.19) are completely solvable, while ensuring (3.15), if and only if the following two conditions hold.

Condition 1.

$$Rank\begin{bmatrix} \tilde{C}\tilde{A} & 0 & 0 & \tilde{C} \\ \tilde{C} & 0 & 0 & 0 \\ (\tilde{E}-I_n)\tilde{A} & -\tilde{A}_d & -\tilde{F} & \tilde{E} \\ 0 & \tilde{C} & 0 & 0 \\ \tilde{L}_2\tilde{A} & 0 & 0 & \tilde{L}_2 \\ \tilde{L}_2 & 0 & 0 & 0 \end{bmatrix} = Rank\begin{bmatrix} \tilde{C}\tilde{A} & 0 & 0 & \tilde{C} \\ \tilde{C} & 0 & 0 & 0 \\ (\tilde{E}-I_n)\tilde{A} & -\tilde{A}_d & -\tilde{F} & \tilde{E} \\ 0 & \tilde{C} & 0 & 0 \\ \tilde{L}_2 & 0 & 0 & 0 \end{bmatrix}, \tag{3.24}$$

and,

Condition 2.

$$Rank\begin{bmatrix}(s\tilde{L}_2-\tilde{L}_2\tilde{A}) & 0 & 0 & -\tilde{L}_2\\ \tilde{C}\tilde{A} & 0 & 0 & \tilde{C}\\ \tilde{C} & 0 & 0 & 0\\ (\tilde{E}-I_n)\tilde{A} & -\tilde{A}_d & -\tilde{F} & \tilde{E}\\ 0 & \tilde{C} & 0 & 0\end{bmatrix} = Rank\begin{bmatrix}\tilde{C}\tilde{A} & 0 & 0 & \tilde{C}\\ \tilde{C} & 0 & 0 & 0\\ (\tilde{E}-I_n)\tilde{A} & -\tilde{A}_d & -\tilde{F} & \tilde{E}\\ 0 & \tilde{C} & 0 & 0\\ \tilde{L}_2 & 0 & 0 & 0\end{bmatrix}, \quad \forall s\in C, Re(s)\geq 0, \tag{3.25}$$

where $\tilde{L}_2$ is such that $L=\begin{bmatrix}L_1\\ L_2\end{bmatrix}=\begin{bmatrix}\tilde{L} & 0\end{bmatrix}=\begin{bmatrix}\tilde{L}_1 & 0\\ \tilde{L}_2 & 0\end{bmatrix}$.

Proof: Let $\bar{E}=E-I_u, I_u=\begin{bmatrix}I_n & 0_{(n\times n)}\end{bmatrix}$ and matrix A can then be expressed as,

$$A=\left(E-\bar{E}\right)A_u. \tag{3.26}$$

Substituting (3.19) and (3.26) into (3.16), the following equation is obtained;

$$NL=LA_u-\begin{bmatrix}M & J-NM & P\end{bmatrix}\begin{bmatrix}CA_u\\ C\\ \bar{E}A_u\end{bmatrix}. \tag{3.27}$$

Post multiply both sides of (3.27) by the following full-row rank matrix

$$S=\begin{bmatrix}L^+ & (I_{2n}-L^+L)\end{bmatrix}=\begin{bmatrix}S_1 & S_2\end{bmatrix}, \tag{3.28}$$

where L^+ denotes the generalized matrix inverse of L. This yields the following two equations;

$$N=LA_uS_1-\begin{bmatrix}M & J-NM & P\end{bmatrix}\begin{bmatrix}CA_u\\ C\\ \bar{E}A_u\end{bmatrix}S_1 \tag{3.29}$$

and

$$\begin{bmatrix} M & J-NM & P \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ \bar{E}A_u \end{bmatrix} S_2 = LA_uS_2. \tag{3.30}$$

To comply with the partitioning of the error vector $e(t)$ in (3.4), the equation (3.30) can be partitioned to yield two matrix equations as;

$$\begin{bmatrix} M_1 & T_1 & P_1 \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ \bar{E}A_u \end{bmatrix} S_2 = L_1A_uS_2 \tag{3.31}$$

$$\begin{bmatrix} M_2 & T_2 & P_2 \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ \bar{E}A_u \end{bmatrix} S_2 = L_2A_uS_2, \tag{3.32}$$

where $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$, $J = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$ and $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ with appropriate dimensions. And,

$$T_1 = J_1 - N_{11}M_1 - N_{12}M_2 \tag{3.33a}$$

$$T_2 = J_2 - N_{21}M_1 - N_{22}M_2. \tag{3.33b}$$

Applying these in (3.19) yields

$$P_1E + M_1C - L_1 = 0, \tag{3.34}$$

$$P_2E + M_2C - L_2 = 0. \tag{3.35}$$

With $J_d = \begin{bmatrix} J_{d1} \\ J_{d2} \end{bmatrix}$, (3.17) can be partitioned as

$$J_{1d}C - P_1 A_d = 0, \tag{3.36}$$

$$J_{2d}C - P_2 A_d = 0. \tag{3.37}$$

The sets of equations (3.31), (3.34), (3.36) and (3.32), (3.35), (3.37) can be written in augmented form respectively as

$$\begin{bmatrix} M_1 & T_1 & P_1 & J_{d1} \end{bmatrix} \Sigma = \Psi_1 \tag{3.38}$$

and

$$\begin{bmatrix} M_2 & T_2 & P_2 & J_{d2} \end{bmatrix} \Sigma = \Psi_2, \tag{3.39}$$

where

$$\Sigma = \begin{bmatrix} CA_uS_2 & 0 & C \\ CS_2 & 0 & 0 \\ \bar{E}A_uS_2 & -A_d & E \\ 0 & C & 0 \end{bmatrix} \in \mathbb{R}^{(3r+n)\times 6n} \tag{3.40a}$$

$$\Psi_1 = \begin{bmatrix} L_1A_uS_2 & 0 & L_1 \end{bmatrix} \in \mathbb{R}^{\tilde{r}\times 6n} \tag{3.40b}$$

$$\Psi_2 = \begin{bmatrix} L_2A_uS_2 & 0 & L_2 \end{bmatrix} \in \mathbb{R}^{(p-\tilde{r})\times 6n}. \tag{3.40c}$$

It is now clear that one must solve both (3.38) and (3.39) in order to find the complete solution for the matrix equations (3.16) - (3.19). Since, L_1 is linearly dependant on C according to (3.10), it is clear that $Rank \begin{bmatrix} \Psi_1 \\ \Sigma \end{bmatrix} = Rank \begin{bmatrix} \Sigma \end{bmatrix}$ and hence (3.38) is solvable. From (3.39) we can derive the necessary and sufficient condition for the existence of a solution of the unknown matrix $\begin{bmatrix} M_2 & T_2 & P_2 & J_{d2} \end{bmatrix}$. In (3.39), there exists a solution to the unknown matrix $\begin{bmatrix} M_2 & T_2 & P_2 & J_{d2} \end{bmatrix}$ if and only if $Rank \begin{bmatrix} \Sigma \\ \Psi_2 \end{bmatrix} = Rank \begin{bmatrix} \Sigma \end{bmatrix}$ i.e.,

$$Rank \begin{bmatrix} CA_uS_2 & 0 & C \\ CS_2 & 0 & 0 \\ \bar{E}A_uS_2 & -A_d & E \\ 0 & C & 0 \\ L_2A_uS_2 & 0 & L_2 \end{bmatrix} = Rank \begin{bmatrix} CA_uS_2 & 0 & C \\ CS_2 & 0 & 0 \\ \bar{E}A_uS_2 & -A_d & E \\ 0 & C & 0 \end{bmatrix}. \tag{3.41}$$

It is easy to show that the following condition

$$Rank \begin{bmatrix} CA_u & 0 & C \\ C & 0 & 0 \\ \bar{E}A_u & -A_d & E \\ 0 & C & 0 \\ L_2A_u & 0 & L_2 \\ L_2 & 0 & 0 \end{bmatrix} = Rank \begin{bmatrix} CA_u & 0 & C \\ C & 0 & 0 \\ \bar{E}A_u & -A_d & E \\ 0 & C & 0 \\ L_2 & 0 & 0 \end{bmatrix} \tag{3.42}$$

is equivalent to the condition (3.41). (NOTE: To show that (3.42) is equivalent to (3.41), post-multiply both sides of (3.42) by a full row-rank matrix $\left[\begin{smallmatrix} S_1 & S_2 & 0 & 0 \\ 0 & 0 & I_{2n} & 0 \\ 0 & 0 & 0 & I_{2n} \end{smallmatrix}\right]$). Then by using the substitutions $\bar{E} = E - I_u, E = \begin{bmatrix} \tilde{E} & 0 \end{bmatrix}$, $A = \begin{bmatrix} \tilde{A} & 0 \end{bmatrix}$, $A_d = \begin{bmatrix} \tilde{A}_d & \tilde{F} \end{bmatrix}$, $C = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}$ together with $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} \tilde{L} & 0 \end{bmatrix} = \begin{bmatrix} \tilde{L}_1 & 0 \\ \tilde{L}_2 & 0 \end{bmatrix}$ makes it possible to show that Condition 1 of *Theorem 2* is equivalent to the condition (3.42). To prove Condition 2 of *Theorem 2*, let us first partition S_1 in (3.29) as

$$S_1 = \begin{bmatrix} S_{11} & S_{22} \end{bmatrix}, \tag{3.43}$$

where $S_{11} \in \mathbb{R}^{2n\times\tilde{r}}$ and $S_{12} \in \mathbb{R}^{2n\times(p-\tilde{r})}$ to yield four matrix equations as;

$$N_{11} = L_1A_uS_{11} - \begin{bmatrix} M_1 & T_1 & P_1 & J_{d1} \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ \bar{E}A_u \\ 0 \end{bmatrix} S_{11} \tag{3.44}$$

$$N_{12} = L_1A_uS_{12} - \begin{bmatrix} M_1 & T_1 & P_1 & J_{d1} \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ \bar{E}A_u \\ 0 \end{bmatrix} S_{12} \tag{3.45}$$

$$N_{21} = L_2A_uS_{11} - \begin{bmatrix} M_2 & T_2 & P_2 & J_{d2} \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ \bar{E}A_u \\ 0 \end{bmatrix} S_{11} \tag{3.46}$$

$$N_{22} = L_2A_uS_{12} - \begin{bmatrix} M_2 & T_2 & P_2 & J_{d2} \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ \bar{E}A_u \\ 0 \end{bmatrix} S_{12}. \tag{3.47}$$

Upon the satisfaction of (3.24), a general solution to (3.39) is;

$$\begin{bmatrix} M_2 & T_2 & P_2 & J_{d2} \end{bmatrix} = \Psi_2\Sigma^+ + Z_2(I_{(n+3r)} - \Sigma\Sigma^+), \tag{3.48}$$

where $Z_2 \in \mathbb{R}^{(p-\tilde{r})\times(n+3r)}$ is any arbitrary matrix. Let us now substitute (3.48) into (3.47) to give

$$N_{22} = N'_{22} - Z_2N''_{22}, \tag{3.49}$$

where $N'_{22} = L_2 A_u S_{12} - \Psi_2 \Sigma^+ \Phi$, $N''_{22} = (I_{(n+3r)} - \Sigma\Sigma^+)\Phi$ and

$$\Phi = \begin{bmatrix} CA_u S_{12} \\ CS_{12} \\ \bar{E} A_u S_{12} \\ 0 \end{bmatrix}. \tag{3.50}$$

In (3.49), N'_{22} and N''_{22} are known matrices and matrix N_{22} is Hurwitz for some matrix Z_2, if and only if the pair (N'_{22}, N''_{22}) is detectable, i.e.,

$$Rank \begin{bmatrix} sI_p - N'_{22} \\ N''_{22} \end{bmatrix} = p, \forall s \in C, Re(s) \geq 0. \tag{3.51}$$

Now we will first show that the following condition is equivalent to the condition (3.51). i.e.

$$Rank \begin{bmatrix} (sL_2 - L_2 A_u) & 0 & -L_2 \\ CA_u & 0 & C \\ C & 0 & 0 \\ \bar{E} A_u & -A_d & E \\ 0 & C & 0 \end{bmatrix} = Rank \begin{bmatrix} CA_u & 0 & C \\ C & 0 & 0 \\ \bar{E} A_u & -A_d & E \\ 0 & C & 0 \\ L_2 & 0 & 0 \end{bmatrix},$$
$$\forall s \in C, Re(s) \geq 0. \tag{3.52}$$

First post-multiply RHS of (3.52) by a full row-rank matrix $\left[\begin{smallmatrix} S_1 & S_2 & 0 & 0 \\ 0 & 0 & I_{2n} & 0 \\ 0 & 0 & 0 & I_{2n} \end{smallmatrix}\right]$) to give

$$Rank \begin{bmatrix} CA_u & 0 & C \\ C & 0 & 0 \\ \bar{E} A_u & -A_d & E \\ 0 & C & 0 \\ L_2 & 0 & 0 \end{bmatrix} = p + rank(\Sigma). \tag{3.53}$$

The LHS of (3.52) can be expressed as follows;

$$
\begin{aligned}
& Rank\begin{bmatrix} (sL_2 - L_2A_u) & 0 & -L_2 \\ CA_u & 0 & C \\ C & 0 & 0 \\ \bar{E}A_u & -A_d & E \\ 0 & C & 0 \end{bmatrix} \\
= \quad & Rank\left\{\begin{bmatrix} (sL_2 - L_2A_u) & 0 & -L_2 \\ CA_u & 0 & C \\ C & 0 & 0 \\ \bar{E}A_u & -A_d & E \\ 0 & C & 0 \end{bmatrix}\begin{bmatrix} S_1 & S_2 & 0 & 0 \\ 0 & 0 & I_{2n} & 0 \\ 0 & 0 & 0 & I_{2n} \end{bmatrix}\right\} \\
= \quad & Rank\begin{bmatrix} (sI_p - L_2A_uS_{12}) & -\Psi_2 \\ \Phi & \Sigma \end{bmatrix} \\
= \quad & Rank\left\{\begin{bmatrix} I_p & \Psi_2\Sigma^+ \\ 0 & (I_{(n+3r)} - \Sigma\Sigma^+) \\ 0 & \Sigma\Sigma^+ \end{bmatrix}\begin{bmatrix} (sI_p - L_2A_uS_{12}) & -\Psi_2 \\ \Phi & \Sigma \end{bmatrix}\right\} \\
= \quad & Rank\begin{bmatrix} sI_p - N'_{22} & 0 \\ N''_{22} & 0 \\ \Sigma\Sigma^+\Phi & \Sigma \end{bmatrix} \\
= \quad & Rank\left\{\begin{bmatrix} sI_p - N'_{22} & 0 \\ N''_{22} & 0 \\ \Sigma\Sigma^+\Phi & \Sigma \end{bmatrix}\begin{bmatrix} I_p & 0 \\ -\Sigma^+\Phi & I_{6n} \end{bmatrix}\right\} \\
= \quad & Rank\begin{bmatrix} sI_p - N'_{22} \\ N''_{22} \end{bmatrix} + Rank[\Sigma], \qquad (3.54) \\
& \forall s \in C, Re(s) \geq 0.
\end{aligned}
$$

It is clear from (3.53) and (3.54) that (3.52) is equivalent to (3.51). Then by using the substitutions $\bar{E} = E - I_u, E = \begin{bmatrix} \tilde{E} & 0 \end{bmatrix}, A = \begin{bmatrix} \tilde{A} & 0 \end{bmatrix}, A_d = \begin{bmatrix} \tilde{A}_d & \tilde{F} \end{bmatrix}, C = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}$

together with $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} \tilde{L} & 0 \end{bmatrix} = \begin{bmatrix} \tilde{L}_1 & 0 \\ \tilde{L}_2 & 0 \end{bmatrix}$ makes it possible to show that Condition 2 of *Theorem 2* is equivalent to the condition (3.52) and therefore provides the necessary and sufficient condition for matrix N_{22} to be Hurwitz. This completes the proof of *Theorem 2*. Computation of N_{11}, N_{12}, N_{21} and N_{22} are further elaborated now. The general solution to (3.38) is given by

$$\begin{bmatrix} M_1 & T_1 & P_1 & J_{d1} \end{bmatrix} = \Psi_1\Sigma^+ + Z_1(I_{(n+3r)} - \Sigma\Sigma^+), \tag{3.55}$$

where Z_1 is any arbitrary matrix of appropriate dimension. Using (3.55) in (3.44) and (3.45), the following can be written;

$$N_{11} = F_{11} - Z_1G_1 \tag{3.56a}$$

$$N_{12} = F_{12} - Z_1G_2, \tag{3.56b}$$

where

$$F_{11} = L_1A_uS_{11} - \Psi_1\Sigma^+ \begin{bmatrix} CA_uS_{11} \\ CS_{11} \\ \bar{E}A_uS_{11} \\ 0 \end{bmatrix} \tag{3.57}$$

$$F_{12} = L_1A_uS_{12} - \Psi_1\Sigma^+ \begin{bmatrix} CA_uS_{12} \\ CS_{12} \\ \bar{E}A_uS_{12} \\ 0 \end{bmatrix} \tag{3.58}$$

$$G_1 = (I_{(n+3r)} - \Sigma\Sigma^+) \begin{bmatrix} CA_uS_{11} \\ CS_{11} \\ \bar{E}A_uS_{11} \\ 0 \end{bmatrix} \tag{3.59}$$

$$G_2 = (I_{(n+3r)} - \Sigma\Sigma^+) \begin{bmatrix} CA_uS_{12} \\ CS_{12} \\ \bar{E}A_uS_{12} \\ 0 \end{bmatrix}. \tag{3.60}$$

Upon the satisfaction of the Conditions 1 and 2 of *Theorem 2*, then from (3.49), we can easily find a matrix gain Z_2 and a stable matrix N_{22}. Once matrix Z_2 is obtained, the unknown matrix $\begin{bmatrix} M_2 & T_2 & P_2 & J_{d2} \end{bmatrix}$ is derived from (3.48). Using (3.48) in (3.46) and (3.47), the following can be written;

$$N_{21} = F_{21} - Z_2G_1 \tag{3.61a}$$

$$N_{22} = F_{22} - Z_2G_2, \tag{3.61b}$$

where

$$F_{21} = L_2A_uS_{11} - \Psi_2\Sigma^+ \begin{bmatrix} CA_uS_{11} \\ CS_{11} \\ \bar{E}A_uS_{11} \\ 0 \end{bmatrix} \tag{3.62a}$$

$$F_{22} = L_2A_uS_{12} - \Psi_2\Sigma^+ \begin{bmatrix} CA_uS_{12} \\ CS_{12} \\ \bar{E}A_uS_{12} \\ 0 \end{bmatrix}. \tag{3.62b}$$

The matrix N_{22} can be made Hurwitz if and only if the pair (F_{22}, G_2) is detectable. Now, from (3.18), (3.33a) and (3.33b), matrices H and J can be derived.

Design Algorithm

1. Partition L according to $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} GC \\ L_2 \end{bmatrix}$
2. Check if Condition 1 of *Theorem 2* is satisfied. If yes, continue, otherwise a sliding mode functional observer does not exist.
3. Compute F_{22} using (3.62b), G_2 using (3.60) and check if the pair (F_{22}, G_2) is detectable. If yes, continue, otherwise N_{22} cannot be made Hurwitz and consequently a stable sliding mode functional observer does not exist.
4. Using (3.61b) and any pole placement method obtain Z_2 to make N_{22} Hurwitz.
5. Use (3.61a) to compute F_{21}, compute G_1 using (3.59) and then use (3.61a) to compute N_{21}.
6. Choose any Z_1 and use (3.56a) and (3.56b) to compute N_{11} and N_{12} respectively.
7. Use (3.55) and (3.48) to compute M_1, T_1, P_1, J_{d1}, M_2, T_2, P_2 and J_{d2}.
8. Use (3.33a) and (3.33b) to compute J_1 and J_2 respectively.
9. Compute H according to (3.18).
10. Use any standard sliding mode technique to obtain Γ_1 to make error dynamics of (3.21a) stable.

3.4 Numerical Example

Let us consider a 2^{nd} order system with the system matrices $\tilde{E}$, $\tilde{A}$, $\tilde{A}_d$, $\tilde{F}$, B and $\tilde{C}$ given below:

$$\tilde{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \tilde{A} = \begin{bmatrix} -2 & 3 \\ 0.5 & -4 \end{bmatrix}, \ \tilde{A}_d = \begin{bmatrix} 1 & -0.5 \\ 0 & 0 \end{bmatrix}, \ \tilde{F} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $\tilde{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Let the function to be estimated given by

$$\tilde{L} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

For these system matrices, Condition 1 of *Theorem 2* is satisfied and hence a sliding mode functional observer exists. By computing F_{22} using (3.62b), G_2 using (3.60) it can be shown that $F_{22} = -4$ and $G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. Since (F_{22}, G_2) is detectable,N_{22} can be made Hurwitz and its pole is at $s = -4$. Using (3.61b) Z_2 and N_{22} can be obtained as $Z_2 = [0\,0\,0\,0\,0]$ and $N_{22} = [-4]$. Using (3.61a) and (3.59) we can find F_{21} and G_1 respectively as $F_{21} = [4.5]$ and $G_1 = [0\,1\,0\,0\,0]^T$. Using (3.61a) we can compute N_{21} as; $N_{21} = [4.5]$. The choice of Z_1 is arbitrary and we choose

$$Z_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can now compute N_{11} and N_{12} using (3.56a) and (3.56b) as $N_{11} = [0]$ and $N_{12} = [0]$.

Using (3.55) and (3.48) M, T, J_d and P are computed as $M = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, $T = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $J_d = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Using (3.33a) and (3.33b), J is computed to be $J = \begin{bmatrix} 0 & 0.5 \end{bmatrix}^T$. According to (3.18) H is computed as

$$H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The error dynamics of the observer can be written as follows (note that $e_y(t) \in \mathbb{R}$ and $e_1(t) \in \mathbb{R}$)

$$\begin{bmatrix} \dot{e}_y(t) \\ \dot{e}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 4.5 & -4 \end{bmatrix} \begin{bmatrix} e_y(t) \\ e_1(t) \end{bmatrix} + \begin{bmatrix} \Gamma_1 \text{sgn}(e_y(t)) \\ 0 \end{bmatrix}.$$

The above error dynamics is simulated to verify the convergence, which is shown in Fig 3.1, where $e_y(t) \in \mathbb{R}$ and $e_1(t) \in \mathbb{R}$. For $\Gamma_1 = -0.3$, sliding occur on the plane $e_y(t) = 0$ as

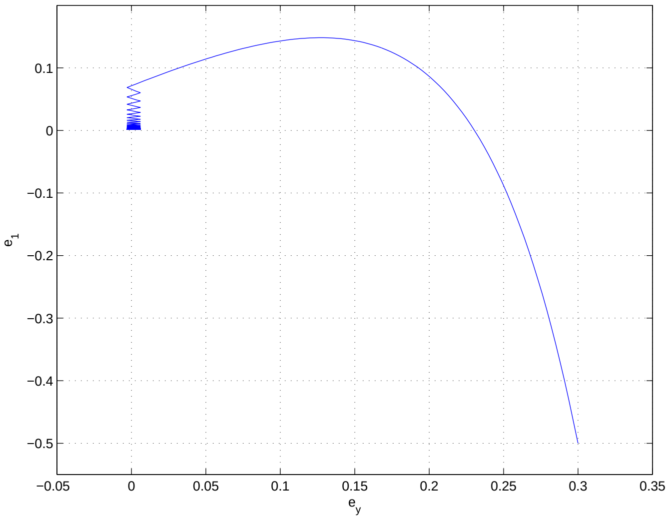

FIGURE 3.1: Error Trajectory for e_1, and e_y

in Fig 3.1 Here it is also noted that, while N_{22} is Hurwitz, N is not Hurwitz. Matrix N has eigenvalues at $\{0, -4\}$ yet a stable error response is achieved by the switching function in the sliding mode observer. As can be seen in Fig 3.1, the initial error in $e_y(t)$ is dragged towards the plane $e_y(t) = 0$ in finite time and $e(t)$ slides along the plane $e_y(t) = 0$ towards the origin.

3.5 Conclusion

This chapter has addressed the problem of estimating a linear function of the states of a neutral delay system using sliding mode functional observer approach. The observers proposed in this chapter have the advantages of having the order the same as the dimension of the vector to be estimated. Necessary and sufficient conditions for the existence of the sliding mode functional observer have been derived and proved. Based on the derived theory, a design procedure has been proposed and is verified by a numerical example.

Chapter 4

Sliding Mode Observers with Unknown Inputs

This chapter illustrates a method of designing a sliding mode linear functional observer for a system with unknown inputs. The existence conditions for the observer are presented. A structure and design algorithm for the sliding mode observer is proposed. The proposed algorithm is then applied for sensorless control of Permanent Magnet Synchronous Machines.

4.1 Introduction

Sliding mode observers differ from more traditional observers e.g. Luenberger observers, in that there is a non-linear term injected into the observer depending on the output estimation error. The concept of sliding mode was originally applied to control system design [54], [55] and later applied for estimating system states [55], [56], [57]. State estimators that utilize the concept of sliding mode are referred to as Utkin observers in the literature [56]. Sliding mode functional observers estimate linear functions of the state vector of a system without necessarily estimating all of the individual states while ensuring that sliding will occur on a manifold where some function of the output prediction error is zero. Since only required functions are estimated with functional observers, these observers have lower order than full order observers. The order of the functional observer can be as low as the number of functions estimated. Much of the work here has been motivated by the work of [52] and [58], where the importance of sliding modes is demonstrated in *Variable Structure Systems*. The practical applications of sliding mode observers in power and control have been illustrated in [59], [60], [61], [62] and [63] most importantly in the field of induction motor control. In this chapter the concept of sliding mode functional observers with unknown inputs is presented and applied to the sensorless control of Permanent Magnet Synchronous Motors (PMSM).

One of the most intriguing aspects of sliding modes as described in [56] is to switch between two distinctively different system structures (or components) such that a new type of system motion, sliding mode, exists in a manifold. This certain peculiar system characteristic which is a consequence of the switching function is claimed to result in superb system performance which includes insensitivity to parameter variations, and complete rejection of disturbances [56]. These properties of sliding mode are investigated in this chapter by simulating the proposed observer in sensorless control of PMSMs.

In literature, the state estimation problem for unknown input systems is well researched [64], [25], [65] however application of sliding mode concepts to estimate a linear function of the state vector has not yet been addressed. In this chapter, we present conditions for designing sliding mode functional observers when the system is subjected to unknown inputs. Under special circumstances, the existence conditions derived in this chapter reduces to the existence conditions for the Utkin Observer and also reduces to the well known matching and observability conditions required for the design of an unknown input state observer.

This chapter is organized as follows: Section 4.2 provides a general outline of the problem to be solved. This section will provide a description to the unknown input system of equations and the corresponding sliding mode linear functional observer structure that will be applied. Section 4.3 describes the conditions for the existence of the sliding mode functional observer and based on these conditions a design algorithm is presented. Section 4.4 provides the application of the theory to speed sensorless control of PMSMs. Finally, Section 4.5 presents the conclusions of the chapter.

4.2 Problem Statement

Consider a linear time-invariant system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dv(t) & (4.1)\\ y(t) &= Cx(t) & (4.2)\\ z(t) &= Lx(t) & (4.3) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$,$u(t) \in \mathbb{R}^m$, $v(t) \in \mathbb{R}^q$ and $y(t) \in \mathbb{R}^r$ are the state, known input, unknown input and the output vectors, respectively. $z(t) \in \mathbb{R}^p$ is the vector to be estimated. The pair (C, A) is detectable, (A, B) is controllable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times q}$ and $L \in \mathbb{R}^{r\times n}$ are known constant matrices. It is noted here that D represents the unknown input matrix. Without loss of generality, it is assumed that $Rank(C) = r$, $Rank(L) = p$, $Rank \begin{bmatrix} L \\ C \end{bmatrix} = p + r - \tilde{p}, \tilde{p} \leq p$ and C takes the form $C = \begin{bmatrix} I_r & 0 \end{bmatrix}$ (otherwise the system can always be transformed into this form).

Consider the following sliding mode linear functional observer

$$\begin{aligned} \dot{w}(t) &= Nw(t) + Jy(t) + Hu(t) + \Gamma \mathrm{sgn}\left(Me(t)\right) & (4.4)\\ \hat{z}(t) &= w(t) + Ey(t) & (4.5)\\ e(t) &= z(t) - \hat{z}(t) & (4.6) \end{aligned}$$

where $w(t) \in \mathbb{R}^p$, $M \in \mathbb{R}^{\tilde{p}\times p}$, $\Gamma \in \mathbb{R}^{p\times \tilde{p}}$, $\mathrm{sgn}(\cdot)$ is the sign function and also the sliding surface is given by

$$Ke(t) = 0 \tag{4.7}$$

In view of the dimension of matrix K, the error vector $e(t)$ can be written as

$$e(t) = \begin{bmatrix} e_y(t) \\ e_1(t) \end{bmatrix} \tag{4.8}$$

where $e_y(t) \in \mathbb{R}^{\tilde{r}}$ and $e_1(t) \in \mathbb{R}^{p-\tilde{p}}$.

The aim of this chapter is to design a sliding mode functional observer, that is to find N, J, H, E and a suitable Γ such that $e(t)$ slides along the surface $Ke(t) = 0$ and $e_1(t) \to 0$ as $t \to \infty$.

4.3 Existence Conditions of the Observer

To define the sliding surface the matrix L can be partitioned as

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \tag{4.9}$$

where L_1 belong to the rowspace of C. Since L_1 belong to the rowspace of C, there always exist a full row rank matrix $G \in \mathbb{R}^{\tilde{p}\times r}$, $\tilde{p} \leq r$ such that

$$L_1 = GC \tag{4.10}$$

If $K \in \mathbb{R}^{\tilde{p}\times p}$ is chosen such that

$$K = \begin{bmatrix} I_{\tilde{p}} & 0_{\tilde{p}\times(p-\tilde{p})} \end{bmatrix} \tag{4.11}$$

and G according to (4.10), then the sliding surface (4.7) is

$$Ke(t) = e_y(t) = Gy(t) - K\hat{z}(t) = 0 \tag{4.12}$$

Using (4.1)-(4.6) and (4.12) the error dynamics of the observer can be written as follows

$$\begin{aligned} \dot{e}(t) = Ne(t) + (PA - NP - JC)x(t) + (PB - H)u(t) \\ + PDv(t) - \Gamma\text{sgn}(e_y(t)) \end{aligned} \tag{4.13}$$

where $P = L - EC$. Considering the partitioning of $e(t)$ in (4.8), matrix N can be partitioned as follows

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{4.14}$$

where, $N_{11} \in \mathbb{R}^{\tilde{p}\times\tilde{p}}, N_{12} \in \mathbb{R}^{\tilde{p}\times(p-\tilde{p})}, N_{21} \in \mathbb{R}^{(p-\tilde{p})\times\tilde{p}}$ and $N_{22} \in \mathbb{R}^{(p-\tilde{p})\times(p-\tilde{p})}$. If we now choose $\Gamma = \begin{bmatrix} \Gamma_1 \\ 0_{(p-\tilde{p})\times\tilde{p}} \end{bmatrix}, \Gamma_1 \in \mathbb{R}^{\tilde{p}\times\tilde{p}}$ then the error dynamics (4.13) can be rewritten as

$$\begin{bmatrix} \dot{e}_y(t) \\ \dot{e}_1(t) \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} e_y(t) \\ e_1(t) \end{bmatrix} + (PA - NP - JC)x(t) + (PB - H)u(t) + PDv(t) - \begin{bmatrix} \Gamma_1 \text{sgn}(e_y(t)) \\ 0 \end{bmatrix} \tag{4.15}$$

The existence conditions of the observer (4.4)-(4.6) are given in the following Theorem.

Theorem 1: $e_1(t) \to 0$ as $t \to \infty$ and also $e(t)$ slide along the surface $Ke(t) = 0$, $t \geq t_s$ where $t_s \leq \left(\frac{\|Gy(0)-K\hat{z}(0)\|}{\eta}\right), \eta > 0$ for any $x(0), \hat{z}(0)$, and $u(t)$ if the following conditions hold:

$$N_{22},\ \textit{Hurwitz} \tag{4.16}$$

$$PA - NP - JC = 0 \tag{4.17}$$

$$H = PB \tag{4.18}$$

$$PD = 0 \tag{4.19}$$

$$e_y^T(t)\dot{e}_y(t) < -\eta\|e_y(t)\| \tag{4.20}$$

where, $\|.\|$ represents the *norm*

Proof: If Conditions (4.17) and (4.18) and (4.19) are satisfied then the error dynamics (4.15) of the observer is given by

$$\dot{e}_y(t) = N_{11}e_y(t) + N_{12}e_1(t) - \Gamma_1\text{sgn}(e_y(t)) \tag{4.21a}$$

$$\dot{e}_1(t) = N_{21}e_y(t) + N_{22}e_1(t) \tag{4.21b}$$

If (4.20) is satisfied then an ideal sliding motion will take place on the surface

$$S_0 = \{(e_1(t), e_y(t)) : e_y(t) = 0\} \tag{4.22}$$

It follows that after some finite time t_s, for all subsequent time, $e_y = 0$ [55]. The dynamics of $e_1(t)$ then reduces to

$$\dot{e}_1(t) = N_{22}e_1(t) \tag{4.23}$$

If (4.16) is satisfied then $e_1(t) \to 0$ as $t \to \infty$.

Considering the partitioning of $e(t)$ in (4.8), matrices P, J, E and H can be partitioned as $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$, $J = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$, $E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$ and $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ where $P_1 \in \mathbb{R}^{\tilde{p}\times n}$, $P_2 \in \mathbb{R}^{(p-\tilde{p})\times n}$, $J_1 \in \mathbb{R}^{\tilde{p}\times r}$, $J_2 \in \mathbb{R}^{(p-\tilde{p})\times r}$, $E_1 \in \mathbb{R}^{\tilde{p}\times r}$, $E_2 \in \mathbb{R}^{(p-\tilde{p})\times r}$, $H_1 \in \mathbb{R}^{\tilde{p}\times m}$ and $H_2 \in \mathbb{R}^{(p-\tilde{p})\times m}$. Using the definition of $P = L - EC$, the partitioned matrices P_1 and P_2 can be written as

$$\begin{aligned} P_1 &= L_1 - E_1C && (4.24a) \\ P_2 &= L_2 - E_2C && (4.24b) \end{aligned}$$

and using (4.18) H_1 and H_2 can be written as

$$\begin{aligned} H_1 &= P_1B && (4.25a) \\ H_2 &= P_2B && (4.25b) \end{aligned}$$

Furthermore, equation (4.17) can be rewritten as

$$\begin{aligned} P_1A - N_{11}P_1 - N_{12}P_2 - J_1C &= 0 && (4.26a) \\ P_2A - N_{21}P_1 - N_{22}P_2 - J_2C &= 0 && (4.26b) \end{aligned}$$

Equation (4.26a) is equivalent to

$$(P_1A - N_{11}P_1 - N_{12}P_2 - J_1C)\begin{bmatrix} L^+ & I - L^+L \end{bmatrix} = 0 \tag{4.27}$$

because $\begin{bmatrix} L^+ & I - L^+L \end{bmatrix}$ is a full row rank matrix. Let us now partition L^+ as follows

$$L^+ = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \tag{4.28}$$

where $M_1 \in \mathbb{R}^{n\times\tilde{p}}$, $M_2 \in \mathbb{R}^{n\times(p-\tilde{p})}$ and $[.]^+$ represents the *pseudoinverse*. Using (4.24a) and (4.28) and also the fact that $L_1M_1 = I_{\tilde{p}}$, $L_1M_2 = 0$, $L_2M_1 = 0$, $L_2M_2 = I_{(p-\tilde{p})}$, equation (4.27) leads to

$$N_{11} = \begin{bmatrix} L_1AM_1 \end{bmatrix} - \begin{bmatrix} E_1 & K_1 \end{bmatrix} \begin{bmatrix} CAM_1 \\ CM_1 \end{bmatrix} \tag{4.29a}$$

$$N_{12} = \begin{bmatrix} L_1AM_2 \end{bmatrix} - \begin{bmatrix} E_1 & K_1 \end{bmatrix} \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} \tag{4.29b}$$

$$\begin{bmatrix} E_1 & K_1 \end{bmatrix} \Sigma = \begin{bmatrix} L_1\bar{A} & L_1D \end{bmatrix} \tag{4.29c}$$

where

$$K_1 = J_1 - N_{11}E_1 - N_{12}E_2 \tag{4.30}$$

$$\Sigma = \begin{bmatrix} C\bar{A} & CD \\ \bar{C} & 0 \end{bmatrix} \tag{4.31}$$

$$\bar{A} = A(I - L^+L) \tag{4.32}$$

$$\bar{C} = C(I - L^+L) \tag{4.33}$$

It can be shown that (4.29c) can always be solved for E_1 and K_1 because $\begin{bmatrix} L_1\bar{A} & L_1D \end{bmatrix}(I - \Sigma^+\Sigma) = 0$ (i.e. $Rank \begin{bmatrix} L_1\bar{A} & L_1D \\ \Sigma \end{bmatrix} = Rank[\Sigma]$) and the general solution is given by

$$\begin{bmatrix} E_1 & K_1 \end{bmatrix} = \begin{bmatrix} L_1\bar{A} & L_1D \end{bmatrix}\Sigma^+ + Z_1(I - \Sigma\Sigma^+) \tag{4.34}$$

where Z_1 is any arbitrary matrix of appropriate dimension. Using (4.34) in (4.29a) and

(4.29b), the following can be written

$$N_{11} = F_{11} - Z_1 G_1 \tag{4.35a}$$

$$N_{12} = F_{12} - Z_1 G_2 \tag{4.35b}$$

where

$$F_{11} = [\ L_1 A M_1\ ] - [\ L_1 \bar{A} \quad L_1 D\ ]\Sigma^+ \begin{bmatrix} CAM_1 \\ CM_1 \end{bmatrix} \tag{4.36a}$$

$$F_{12} = [\ L_1 A M_2\ ] - [\ L_1 \bar{A} \quad L_1 D\ ]\Sigma^+ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} \tag{4.36b}$$

$$G_1 = (I - \Sigma\Sigma^+) \begin{bmatrix} CAM_1 \\ CM_1 \end{bmatrix} \tag{4.36c}$$

$$G_2 = (I - \Sigma\Sigma^+) \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} \tag{4.36d}$$

As with (4.26a), we can adopt the same procedure starting from equation (4.26b). Equation (4.26b) is equivalent to

$$(P_2 A - N_{21} P_1 - N_{22} P_2 - J_2 C)[\ L^+ \quad I - L^+ L\ ] = 0 \tag{4.37}$$

because $[\ L^+ \quad I - L^+ L\ ]$ is a full row rank matrix. Using (4.24b) and (4.28) and also the fact that $L_1 M_1 = I_{\tilde{p}}$, $L_1 M_2 = 0$, $L_2 M_1 = 0$, $L_2 M_2 = I_{(p-\tilde{p})}$, equation (4.37) leads to

$$N_{21} = [\ L_2 A M_1\ ] - [\ E_2 \quad K_2\ ] \begin{bmatrix} CAM_1 \\ CM_1 \end{bmatrix} \tag{4.38a}$$

$$N_{22} = [\ L_2 A M_2\ ] - [\ E_2 \quad K_2\ ] \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} \tag{4.38b}$$

$$\begin{bmatrix} E_2 & K_2 \end{bmatrix} \Sigma = [\ L_2 \bar{A} \quad L_2 D\ ] \tag{4.38c}$$

where

$$K_2 = J_2 - N_{21}E_1 - N_{22}E_2 \tag{4.39}$$

Using the properties of the generalized matrix inverse [66] (4.38c) has a solution for $\begin{bmatrix} E_2 & K_2 \end{bmatrix}$ iff

$$\begin{bmatrix} L_2\bar{A} & L_2D \end{bmatrix}(I - \Sigma^+\Sigma) = 0 \tag{4.40}$$

or equivalently,

$$Rank \begin{bmatrix} \begin{bmatrix} L_2\bar{A} & L_2D \end{bmatrix} \\ \Sigma \end{bmatrix} = Rank(\Sigma) \tag{4.41}$$

In the case that (4.38c) has a solution, the solution is given by

$$\begin{bmatrix} E_2 & K_2 \end{bmatrix} = \begin{bmatrix} L_2\bar{A} & L_2D \end{bmatrix}\Sigma^+ + Z_2(I - \Sigma\Sigma^+) \tag{4.42}$$

where Z_2 is any arbitrary matrix of appropriate dimension. Using (4.42) in (4.38a) and (4.38b), the following can be written

$$N_{21} = F_{21} - Z_2G_1 \tag{4.43a}$$

$$N_{22} = F_{22} - Z_2G_2 \tag{4.43b}$$

where

$$F_{21} = [\ L_2AM_1\ ] - [\ L_2\bar{A} \quad L_2D\ ]\Sigma^+ \begin{bmatrix} CAM_1 \\ CM_1 \end{bmatrix} \tag{4.44a}$$

$$F_{22} = [\ L_2AM_2\ ] - [\ L_2\bar{A} \quad L_2D\ ]\Sigma^+ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} \tag{4.44b}$$

Remark: The observer matrices, E_1, N_{11}, N_{12} J_1, H_1, can be computed using (4.34), (4.29a), (4.29b), (4.30) and (4.25a) respectively. The other observer matrices E_2, N_{21}, N_{22}, J_2, H_2 exist if and only if (4.41) is satisfied and can be computed using (4.42), (4.43a), (4.43b), (4.39) and (4.25b) respectively. The existence of the observer (4.4)-(4.6), i.e. finding solutions for all of the observer parameters, E_1, E_2, J_1, J_2, H_1, H_2, N_{11}, N_{12}, N_{21} and N_{22}, depends on the satisfaction of (4.41) which is also presented in the following theorem

Theorem 2: The observer (4.4)-(4.6) exists iff

$$Rank \begin{bmatrix} L_2A & L_2D \\ CA & CD \\ C & 0 \\ L_2 & 0 \end{bmatrix} = Rank \begin{bmatrix} CA & CD \\ C & 0 \\ L_2 & 0 \end{bmatrix} \tag{4.45}$$

Proof:

To prove (4.45) consider the *LHS* of (4.45)

$$\begin{aligned} Rank \begin{bmatrix} L_2A & L_2D \\ CA & CD \\ C & 0 \\ L_2 & 0 \end{bmatrix} &= Rank \begin{bmatrix} L_2A & L_2D \\ CA & CD \\ C & 0 \\ L_2 & 0 \end{bmatrix} \begin{bmatrix} M_2 & I - M_2L_2 & 0 \\ 0 & 0 & I_q \end{bmatrix} \\ &= Rank \begin{bmatrix} L_2AM_2 & L_2A(I - M_2L_2) & L_2D \\ CAM_2 & CA(I - M_2L_2) & CD \\ CM_2 & C(I - M_2L_2) & 0 \\ L_2M_2 & L_2(I - M_2L_2) & 0 \end{bmatrix} \\ &= Rank \begin{bmatrix} L_2AM_2 & L_2\bar{A} & L_2D \\ CAM_2 & C\bar{A} & CD \\ CM_2 & \bar{C} & 0 \\ I & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\therefore LHS = p + Rank \begin{bmatrix} \begin{bmatrix} L_2\bar{A} & L_2D \end{bmatrix} \\ \Sigma \end{bmatrix} \tag{4.46}$$

Now consider the *RHS* of (4.45)

$$\begin{aligned}
Rank\begin{bmatrix} CA & CD \\ C & 0 \\ L_2 & 0 \end{bmatrix} &= Rank\begin{bmatrix} CA & CD \\ C & 0 \\ L_2 & 0 \end{bmatrix}\begin{bmatrix} M_2 & I - M_2L_2 & 0 \\ 0 & 0 & I_q \end{bmatrix} \\
&= Rank\begin{bmatrix} CAM_2 & C\bar{A} & CD \\ CM_2 & \bar{C} & 0 \\ I & 0 & 0 \end{bmatrix} \\
\therefore RHS &= p + Rank(\Sigma) \qquad (4.47)
\end{aligned}$$

Given (4.46) and (4.47), (4.41) will be satisfied if and only if (4.45) holds. Theorem 2 ensures the solvability of the observer parameters N, J, H, and E. However, it does not ensure that the matrix N_{22} is Hurwitz. It can be seen from (4.43b) that N_{22} is stable if and only if the pair (F_{22}, G_2) is detectable. The necessary and sufficient condition for N_{22} to be Hurwitz is given in the following theorem.

Theorem 3: The matrix N_{22} is Hurwitz iff

$$Rank\begin{bmatrix} sL_2 - L_2A & -L_2D \\ CA & CD \\ C & 0 \end{bmatrix} = Rank\begin{bmatrix} CA & CD \\ C & 0 \\ L_2 & 0 \end{bmatrix} \qquad (4.48)$$

Proof:

Consider the *LHS* of (4.48)

$$\begin{aligned}
&= Rank\begin{bmatrix} sL_2 - L_2A & -L_2D \\ CA & CD \\ C & 0 \end{bmatrix} \\
&= Rank\begin{bmatrix} sL_2 - L_2A & -L_2D \\ CA & CD \\ C & 0 \end{bmatrix}\begin{bmatrix} M_2 & I - M_2L_2 & 0 \\ 0 & 0 & I_q \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= Rank \begin{bmatrix} sL_2M_2 - L_2AM_2 & -L_2\bar{A} & -L_2D \\ CAM_2 & C\bar{A} & CD \\ CM_2 & \bar{C} & 0 \end{bmatrix} \\
&= Rank \begin{bmatrix} sI - L_2AM_2 & \begin{bmatrix} -L_2\bar{A} & -L_2D \end{bmatrix} \\ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} & \Sigma \end{bmatrix} \\
&= Rank \begin{bmatrix} I & \begin{bmatrix} L_2\bar{A} & L_2D \end{bmatrix}\Sigma^+ \\ 0 & I - \Sigma\Sigma^+ \\ 0 & \Sigma\Sigma^+ \end{bmatrix} \begin{bmatrix} sI - L_2AM_2 & \begin{bmatrix} -L_2\bar{A} & -L_2D \end{bmatrix} \\ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} & \Sigma \end{bmatrix} \\
&= Rank \begin{bmatrix} sI - F_{22} & 0 \\ G_2 & 0 \\ \Sigma\Sigma^+ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} & \Sigma \end{bmatrix} \\
&= Rank \begin{bmatrix} sI - F_{22} & 0 \\ G_2 & 0 \\ \Sigma\Sigma^+ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} & \Sigma \end{bmatrix} \begin{bmatrix} I & 0 \\ \Sigma^+ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} & I \end{bmatrix} \\
&= Rank \begin{bmatrix} sI - F_{22} \\ G_2 \end{bmatrix} + Rank(\Sigma)
\end{aligned}
$$

$\therefore LHS = p + Rank(\Sigma), \forall s \in C : Re(s) \geq 0.$

From (4.43b) and (4.48) if follows that, if and only if (4.48) is satisfied (F_{22}, G_2) is detectable.

In the following we present the design algorithm for the sliding mode functional observer.

Design Algorithm

1. Partition L according to $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} GC \\ L_2 \end{bmatrix}$

2. Check if condition (4.45) is satisfied, if yes continue, otherwise a sliding mode functional observer doesn't exist.

3. Compute F_{22} using (4.44b), G_2 using (4.36d) and check if the pair (F_{22}, G_2) is detectable. If yes continue, otherwise N_{22} cannot be made Hurwitz and consequently a stable sliding mode functional observer doesn't exist.

4. Using (4.35b) and any poleplacement method obtain Z_2 to make N_{22} Hurwitz.

5. Use (4.44a) to compute F_{21}, compute G_1 using (4.36c) and then use (4.43a) to compute N_{21}.

6. Choose any Z_1 and use (4.35a) and (4.35b) to compute N_{11} and N_{12} respectively.

7. Use (4.34) and (4.42) compute E_1, K_1, E_2 and K_2.

8. Use (4.30) and (4.39) to compute J_1 and J_2 respectively.

9. Compute P_1 and P_2 using (4.24a) and (4.24b) respectively.

10. Compute H_1 and H_2 according to (4.24a) and (4.25b) respectively.

11. Use any standard sliding mode technique [55], [67] to obtain Γ_1 to make the error dynamics of (4.21a) stable.

The sliding mode functional observer can be simplified if $L = I_n$ and $D = 0_{n\times q}$. In this case both Theorem 2 and 3 are satisfied, and the observer structure (4.4)-(4.6) proposed in this chapter will in fact reduce to the Utkin Observer [55] and the design algorithm proposed in this chapter reduces to design algorithm of the Utkin Observer. If however, $L = I_n$ and $D \neq 0$ and sliding is not considered (i.e. $L_2 = L$) then the necessary and sufficient conditions proposed in this chapter reduce to the well known matching and observability conditions which are necessary and sufficient for the design of unknown input observers. In particular, Theorem 2 reduces to

$$Rank(CD) = Rank(D) \tag{4.51}$$

which is referred to as the *matching condition* as described in the literature [68] and Theorem 3 reduces to

$$Rank\begin{bmatrix} sI_n - A & D \\ C & 0 \end{bmatrix} = n + m, \forall s \in C : Re(s) \geq 0 \tag{4.52}$$

which is referred to as the *observability condition* for unknown input observers as seen in [25] and [65].

4.4 Application to Speed Sensorless Control of Permanent Magnet Synchronous Machines

Speed sensorless control of electrical machines is an area where the importance of lower order, robust estimation algorithms that are insensitive to machine parameter variations is highlighted [69]. In order to demonstrate the effectiveness of the Sliding Mode Functional Observer presented in this chapter, we now use its application for speed sensorless control of a surface mounted Permanent Magnet Synchronous Motor (PMSM). It must be mentioned here that there has been reported work on application of sliding mode technique for sensorless control of PMSMs earlier [70]. However, the sliding technique presented in this chapter is novel and hence its usage for sensorless control of PMSMs. With the observer topology presented in this chapter, there exists the added advantage of estimating the load torque or such other linear functions of the state variables in addition to the rotor speed, which is not a feature in most of the sensorless control algorithms found in literature [69]. Another motivation to use this Sliding Mode Functional Observer is its estimation strategy. Most of the speed sensorless control algorithms for PMSMs can be classified into two categories. i.e. Back emf based methods and high frequency injection type methods. It is well-known that the former is more applicable in surface mounted type PMSMs [71], while the latter can be used only with interior magnet type PMSMs because of the difference in direct and quadrature axis self inductances [72]. One major problem in back emf based methods is their poor performance in low speeds due to low signal to noise ratio in the sampled back emf [71]. The sliding technique proposed in this chapter is based on the quadrature axis current estimation error and therefore this method is capable of estimating the rotor speed even at low speed region. With this background, the formation of the standard dynamical model of the surface mounted PMSM into a system with unknown input will be demonstrated in the following sub-sections.

4.4.1 Dynamical Model of PMSM

The dynamical model of a Permanent Magnet Synchronous Motor (PMSM) in synchronous reference frame can be given as

$$\frac{di_d(t)}{dt} = -\frac{R_s}{L_d}i_d(t) + \frac{\omega L_q}{L_d}i_q(t) + \frac{1}{L_d}u_d(t) \tag{4.53a}$$

$$\frac{di_q(t)}{dt} = -\frac{R_s}{L_q}i_q(t) + \frac{\omega L_d}{L_q}i_d(t) - \frac{\omega\psi_m}{L_q} + \frac{1}{L_q}u_q(t) \tag{4.53b}$$

$$T_m = \frac{3}{2}n_p\left[\psi_m i_q(t) - (L_q - L_d)\, i_d(t) i_q(t)\right) \tag{4.53c}$$

$$\frac{d\omega}{dt} = \frac{1}{J_m}\left[T_m - T_l - B_m\omega\right], \tag{4.53d}$$

where $i_d(t)$ = Direct axis current, $i_q(t)$ = Quadrature axis current, R_s = Stator resistance, L_d = Direct axis self inductance, L_q = Quadrature axis self inductance, ω = Rotor electrical angular speed, $u_d(t)$ = Direct axis input voltage, $u_q(t)$ = Quadrature axis input voltage, T_m = Motor torque, n_p = Pole pair number, ψ_m = Flux due to permanent magnets in air gap, J_m = Rotor inertia, T_l = Load torque, and B_m = Torque due to damping [73].

Now we will first convert the dynamical equations of PMSM in (4.53a) into a linear state space form. Being time varying quantities under normal circumstances, the rotor electrical angular speed ω, motor torque T_m and load torque T_l can also be rewritten with time argument and the terms in (4.53a) can first be reorganized as:

$$\frac{di_d(t)}{dt} = -\frac{R_s}{L_d} i_d(t) + \frac{L_q}{L_d}\omega(t) i_q(t) + \frac{1}{L_d} u_d(t) \tag{4.54a}$$

$$\frac{di_q(t)}{dt} = -\frac{L_d}{L_q}\omega(t) i_d(t) - \frac{R_s}{L_q} i_q(t) - \frac{\psi_m}{L_q}\omega(t) + \frac{1}{L_q} u_q(t) \tag{4.54b}$$

$$0 = \frac{3}{2} n_p \psi_m i_q(t) - \frac{3}{2} n_p \psi_m \left(L_q - L_d\right) i_d(t) i_q(t) - T_m(t) \tag{4.54c}$$

$$\frac{d\omega(t)}{dt} = -\frac{B_m}{J_m}\omega\ (t) + \frac{1}{J_m} T_m(t) - \frac{1}{J_m} T_l(t), \tag{4.54d}$$

It can now be seen that the terms $+\frac{L_q}{L_d}\omega(t)i_q(t)$, $-\frac{L_d}{L_q}\omega(t)i_d(t)$, and $-\frac{3}{2}n_p\psi_m\left(L_q - L_d\right)i_d(t)i_q(t)$ in (4.54a), (4.54b) and (4.54c) introduce non-linear terms into the state space model. If we apply the well-known axis decoupling [73] by introducing a new control input defined by

$$u_d(t) = u'_d(t) - L_q \omega(t) i_q(t) \tag{4.55a}$$

$$u_q(t) = u'_q(t) + L_d \omega(t) i_d(t), \tag{4.55b}$$

nonlinear terms in (4.54a) and (4.54b) will vanish. In case of surface mounted PMSMs, where $L_d \simeq L_q$, nonlinear term in (4.54c) can be omitted [73]. These steps would result in

$$\frac{di_d(t)}{dt} = -\frac{R_s}{L_d} i_d(t) + \frac{1}{L_d} u'_d(t) \tag{4.56a}$$

$$\frac{di_q(t)}{dt} = -\frac{R_s}{L_q} i_q(t) - \frac{\psi_m}{L_q}\omega(t) + \frac{1}{L_q} u'_q(t) \tag{4.56b}$$

$$\frac{d\omega(t)}{dt} = -\frac{B_m}{J_m}\omega\ (t) + \frac{1}{J_m} T_m(t) - \frac{1}{J_m} T_l(t), \tag{4.56c}$$

$$0 = \frac{3}{2} n_p \psi_m i_q(t) - T_m(t) \tag{4.56d}$$

4.4.2 Conversion of PMSM model into a System with Unknown Input

In this approach $T_m(t)$ is eliminated from the (4.56a) by substituting for it using (4.56d). Then, if the load torque $T_l(t)$, which usually is treated as a disturbance in dynamical modeling of electrical machines, is considered as an unknown input, it can now be seen that the dynamical model of a surface mounted PMSM in synchronous reference frame can be given as:

$$\begin{bmatrix} \dot{i}_d(t) \\ \dot{i}_q(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_s}{L_d} & 0 & 0 \\ 0 & -\frac{R_s}{L_q} & -\frac{\psi_m}{L_q} \\ 0 & \frac{3}{2J_m} n_p \psi_m & -\frac{B_m}{J_m} \end{bmatrix} \begin{bmatrix} i_d(t) \\ i_q(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_d} & 0 \\ 0 & \frac{1}{L_q} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u'_d(t) \\ u'_q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{J_m} \end{bmatrix} T_l(t). \tag{4.57a}$$

If the problem under consideration is designing an observer for the estimation of rotor speed for implementation of sensorless control, then the typical output equation will be

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_d(t) \\ i_q(t) \\ \omega(t) \end{bmatrix}. \tag{4.58}$$

Now with the substitutions

$$A = \begin{bmatrix} -\frac{R_s}{L_d} & 0 & 0 \\ 0 & -\frac{R_s}{L_q} & -\frac{\psi_m}{L_q} \\ 0 & \frac{3}{2J_m} n_p \psi_m & -\frac{B_m}{J_m} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L_d} & 0 \\ 0 & \frac{1}{L_q} \\ 0 & 0 \end{bmatrix},$$

Parameter	*Value*
Rated voltage / VAC	230
Induced emf / $Vrms, l-l/1000rpm$	42
Rated current / A	16.3
Rated torque / Nm	10.5
Rated speed / rpm	3000
Number of poles	6
Rotor inertia / kgm^2	0.00167
Stator resistance / Ω_{l-l}	0.33
Inductance / mH_{l-l}	3.2

TABLE 4.1: Machine Parameter Values

$D = \begin{bmatrix} n_d \\ n_q \\ -\frac{1}{J_m} \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ together with $x(t) = \begin{bmatrix} i_d(t) \\ i_q(t) \\ \omega(t) \end{bmatrix}$, $u(t) = \begin{bmatrix} u'_d(t) \\ u'_q(t) \end{bmatrix}$
and $v(t) = T_l(t)$,

(4.57a) and (4.58) can be expressed in the form of equations (4.1), (4.2) and (4.3). n_d, n_q in D above represent the measurement noise variance and possible drifts introduced by the current sensors to the sampled $i_d(t)$ and $i_q(t)$ currents. The sign and magnitude of n_d and n_q must be determined based on the frequency domain analysis of sampled currents and the operating load torque range of the drive.

4.4.3 PMSM Test Rig

Two back-to-back connected PMSM machines (one to be operated as the drive, while the other to be controlled as the load for load tests) was used for the validation of performance of the observer. Each motor has the specifications given in Table 4.1. The design is done using the per-unit quantities calculated with respect to the base quantities given in Table 4.2. Thus the per-unit values of R_s, L_d, L_q, ψ_m, J_m and B_m are given in Table 4.3.

4.4.4 Design of Sliding Mode Functional Observer

From above per-unit values of the PMSM drive, system matrices A, B, C and D computed will be:

Primary Base Quantity	*Base value*
Voltage (U_{base})	$103V$
Current (I_{base})	$23.1A$
Electrical angular speed (ω_{base})	$942rad/s$
Time (t_{base})	$0.00106s/rad$

Table 4.2: Machine Base Values

Motor Parameter	*Per-unit value*
R_s	0.037
L_d	0.033
L_q	0.035
ψ_m	1
J_m	109.3
B_m	0.1256

Table 4.3: Machine Parameters in Per-unit

$$A = \begin{bmatrix} -0.1121 & 0 & 0 \\ 0 & -0.1057 & -2.8571 \\ 0 & 0.0412 & -0.0011 \end{bmatrix},$$

$$B = \begin{bmatrix} 3.0303 & 0 \\ 0 & 2.8571 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and with $n_d = n_q = 0.0001$, $D = \begin{bmatrix} 0.0001 \\ 0.0001 \\ -0.0091 \end{bmatrix}$. Let us assume that we want to estimate the rotor electrical speed. Hence the function to be estimated given by

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This means that the sliding mode is implemented using the estimated q-axis current $\hat{i}_q(t)$. It is well known that in controlling the surface mounted PMSMs, the d-axis current, which is the flux producing current is controlled at zero (since the air gap flux is provided by the permanent magnets). Therefore, in order to estimate the speed, sliding surface has to be created around the q-axis current. This will result in a reduced order observer which uses only one current information. To obtain the sliding surface for the observer, G is obtained according to (4.10)

$$G = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

and the sliding surface according to (4.12) is given below

$$Ke(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} y(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{z}(t).$$

It is easy to verify that condition (4.45) is satisfied. Now matrix F_{22} and G_2 can be obtained according to (4.44b) and (4.36d) and is given below

$$F_{22} = \begin{bmatrix} -396.0052 \end{bmatrix}$$
and $G_2 = \begin{bmatrix} 3.9109 & -3.9109 & 0.4385 & 0 \end{bmatrix}^T.$

Since the pair (F_{22}, G_2) is detectable, N_{22} can be made Hurwitz and its pole is chosen to be {-0.15} (justification for this pole placement will be given in the following section E). Using (4.43b) Z_2 and N_{22} can be obtained as follows

$$Z_2 = \begin{bmatrix} -50.2934 & 50.2934 & -5.6390 & 0 \end{bmatrix} \text{ and}$$
$$N_{22} = \begin{bmatrix} -0.15 \end{bmatrix}$$

Using (4.44a) and (4.36c) we can find F_{21} and G_1 respectively as follows

$$F_{21} = \begin{bmatrix} -4.2269 \end{bmatrix} \text{ and}$$
$$G_1 = \begin{bmatrix} 0.043 & -0.043 & 0.0048 & 1.0 \end{bmatrix}^T.$$

Using (4.38a) we can compute N_{21} as follows

$$N_{21} = \begin{bmatrix} 0.1231 \end{bmatrix}.$$

The choice of Z_1 is arbitrary and we choose

$$Z_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can now compute N_{11} and N_{12} using (4.35a) and (4.35b) as follows

$$N_{11} = \begin{bmatrix} -0.0430 \end{bmatrix} \text{ and } N_{12} = \begin{bmatrix} -3.9109 \end{bmatrix}.$$

Using (4.34) and (4.42) we can compute E and K as follows

$$E = \begin{bmatrix} 0.4969 & 0.5031 \\ -99.9811 & -0.0189 \end{bmatrix} \text{ and } K = \begin{bmatrix} 0.0557 & 0 \\ -11.2100 & 0 \end{bmatrix}.$$

Matrix P can be computed from (4.24a) and (4.24b), H from (4.25a) and (4.25b) and J from (4.30) and (4.39) as follows

$$P = \begin{bmatrix} -0.4969 & 0.4969 & 0 \\ 99.9811 & 0.0189 & 1.0 \end{bmatrix}$$

$$H = \begin{bmatrix} -1.6730 & 1.2906 \\ 336.6367 & 0.0491 \end{bmatrix}$$

and

$$J = \begin{bmatrix} 391.0480 & 0.0523 \\ 3.8483 & 0.0648 \end{bmatrix}.$$

The error dynamics of the observer can be written as follows (note that $e_y(t) \in \mathbb{R}$ and $e_1(t) \in \mathbb{R}$)

$$\begin{bmatrix} \dot{e}_y(t) \\ \dot{e}_1(t) \end{bmatrix} = \begin{bmatrix} -0.0430 & -3.9109 \\ 0.1231 & -0.1500 \end{bmatrix} \begin{bmatrix} e_y(t) \\ e_1(t) \end{bmatrix} - \begin{bmatrix} \Gamma_1 \text{sgn}(e_y(t)) \\ 0 \end{bmatrix}.$$

4.4.5 Simulation of Sliding Mode Functional Observer

In order to verify the performance of the Sliding Model Functional Observer designed, a simulation study was carried out. During this study, the PMSM model was controlled in closed loop using the measured d and q axis currents and estimated speed as feedbacks. The observer was used for estimation of the speed so that its convergence properties could be verified under different operating conditions. The Fig reffig:g1 below shows the complete simulation model in block diagram form. Simulations were carried out using $Matlab^{®}$ / $SIMULINK^{®}$ platform. In this section, the strength of the proposed sliding mode functional observer will be demonstrated by simulations showing; convergence properties with initial estimation errors, closed loop speed sensorless operation, estimation under a step load torque disturbance and estimation with inaccurate machine parameters (parameter sensitivity). First, some guidelines will be given on tuning the current controller, observer pole, sliding mode gain and the speed controller.

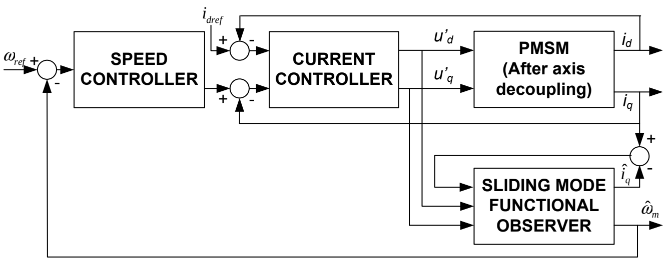

FIGURE 4.1: Complete simulation model in block diagram form

Guidelines for Tuning

In this type of a sensorless control algorithm, the estimation of the speed is dependant on the speed information contained in the sampled motor currents. Then the closed loop speed controller is dependant on the estimated speed by the observer. As such a rule of thumb in fixing the closed loop bandwidth of current controller (α), pole placement of sliding mode observer (p) and closed loop bandwidth of speed controller (α_s) must be according to in inequality $\alpha > p > \alpha_s$ [73]. It is recommended to make each pair of bandwidths at least a decade apart. As such, for these simulations the values chosen in per unit domain are: $\alpha = 2$, $p = -0.15$ and $\alpha_s = 0.02$. These settings lead to a closed loop rise time of d-axis and q-axis currents of approximately $1 - 2ms$ and a closed loop speed rise time of approximately $100ms$ with an overshoot of 10%. It must be noted here that these settings are without using any integrator anti-windup mechanism for the current and speed controllers and both control loops can be made faster by incorporating a suitable integrator anti-windup algorithm [73]. Fixing sliding mode gain (Γ_1) must be with respect to the bandwidth of the observer (i.e. pole placement) and it is recommended to have a value in the neighborhood of 1% of p. The value used in these simulations is $\Gamma_1 = 0.0001$.

Convergence Properties with Initial Estimation Errors

In sensorless control of PMSM, startup is always a challenging problem. The reason is not knowing the exact rotor position information. This is a separate research area and there is several published work on startup techniques of PMSMs [73]. As far as the performance of this particular algorithm is concerned, it is important to demonstrate its convergence capability with initial estimation errors in q-axis current and speed ($e_y(0)$ and $e_1(0)$ respectively). The Fig 4.2. shows the error response, when ($e_y(0) = 0.005$) and ($e_1(0) = 0.0055$). It can be seen that sliding occurs on the plane $e_y(t) = 0$. As can be seen in Fig 2., the initial error in $e_y(t)$ is dragged towards the plane $e_y(t) = 0$ in finite time and $e(t)$ slides along the plane $e_y(t) = 0$ towards the origin.

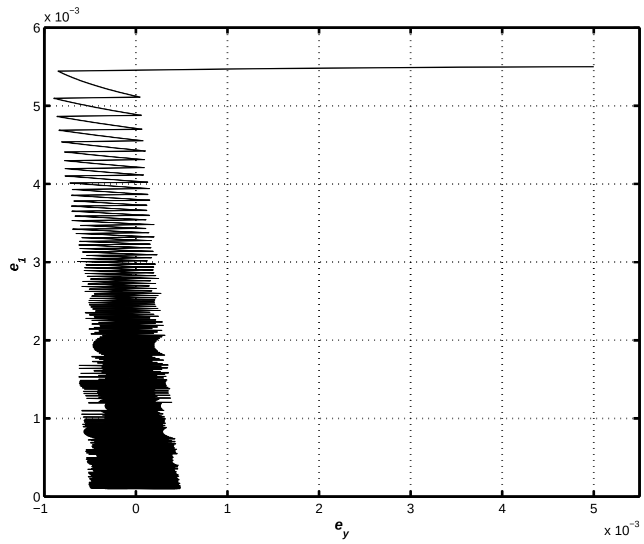

FIGURE 4.2: Error Trajectory for e_y and e_1

Closed Loop Speed Sensorless Operation

The speed control loop is closed by feeding back the estimated speed from the sliding mode observer and a step reference input of 1 pu is applied to the system at $t = 0$. The resulting q-axis current variation and the speed response is shown in Fig 4.3. Time variations of $e_y(t)$ and $e_1(t)$ during the transient period are shown in Fig 4.4 (only the first 200ms is shown).

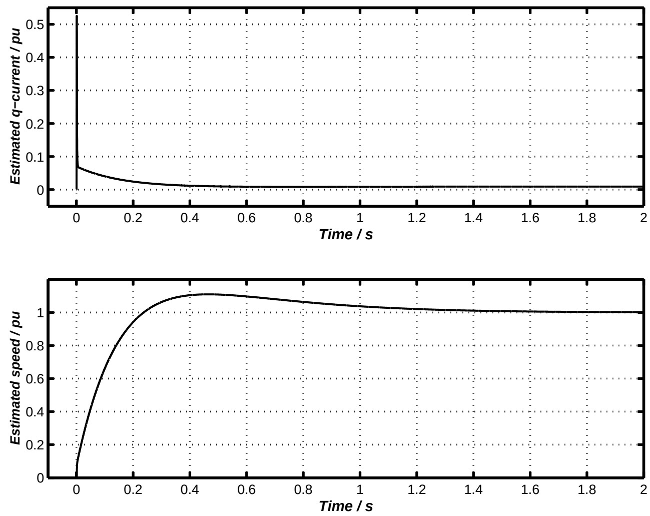

FIGURE 4.3: q-axis current variation and the speed response for a step

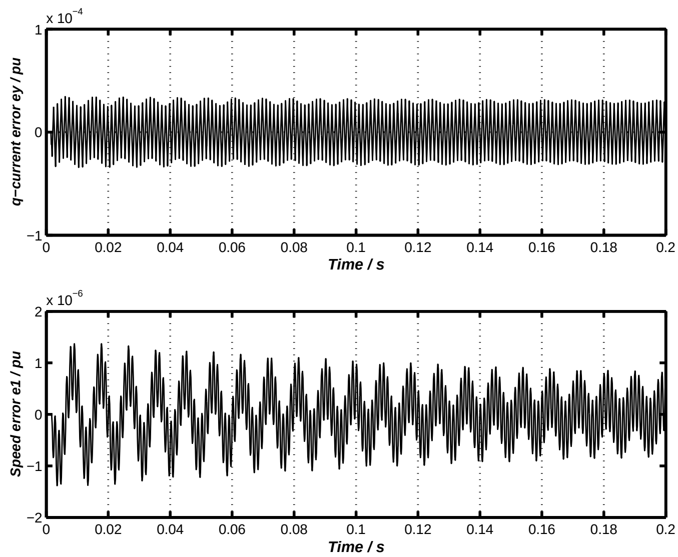

FIGURE 4.4: Time variations of $e_y(t)$ and $e_1(t)$ during the transient period

Estimation Under a Step Load Torque Disturbance

After the initial 1 pu speed step is applied and the PMSM has reached closer to the steady state, a step load torque of 0.2 pu is applied at $t = 1.5$ s. Since Proportional Integral control is used in the speed loop, motor speed gradually recovers after this load disturbance back to the set speed 0.1 pu. In Fig 4.5 shown are the q-axis current variation and the speed response. Time variations of $e_y(t)$ and $e_1(t)$ during the speed fluctuation are shown in Fig 4.6. It must be noted here that the saw-tooth nature of $e_y(t)$ is due to

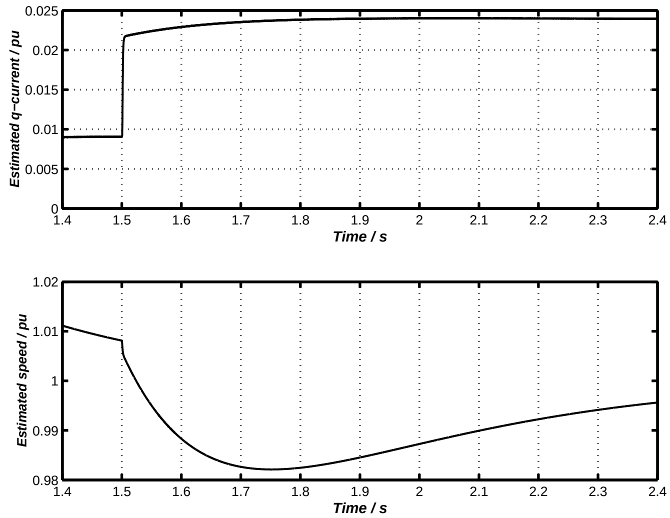

FIGURE 4.5: q-axis current and the speed response for a step load torque

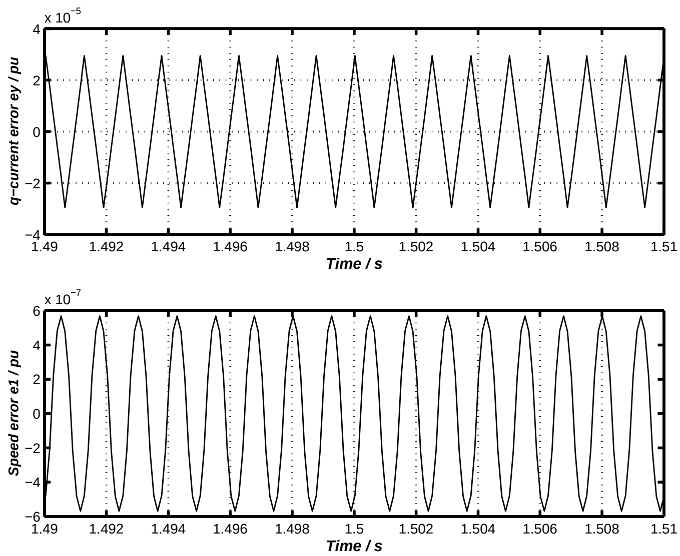

FIGURE 4.6: Time variations of $e_y(t)$ and $e_1(t)$ during the speed fluctuation

the sliding mode term in the error dynamics. Though some oscillatory behavior can be seen in $e_1(t)$, it must be noted that it is of negligible peak-to-peak amplitude. Both $e_y(t)$ and $e_1(t)$ do not show any significant change due to the load torque disturbance. This further establishes the robustness of the proposed observer.

Estimation with Inaccurate Machine Parameters (Parameter Sensitivity)

Sensitivity to changing machine parameters is considered as one of the biggest challenges in developing reliable speed sensorless control algorithms for electrical machines [69], [73]. The d and q axes inductances and the stator resistance tend to change with the internal temperature of the machine. Aging can also cause changes in these quantities. As such, there can always be a difference between the actual machine parameters and the machine parameter values used in the PMSM model that is incorporated in designing the sliding mode observer. In order to verify the robustness of the proposed observer against machine parameter variations, $\pm 10\%$ deviations are introduced to L_d, L_q and R_s used in the observer model with respect to those used in the simulated PMSM model. The Fig 4.7 show the q-axis current variation and the speed response, when a step change in speed reference is applied. Time variations of $e_y(t)$ and $e_1(t)$ during the transient are shown in Fig 4.8 (only the first $200ms$ is shown). It can be seen that the estimated q-axis current

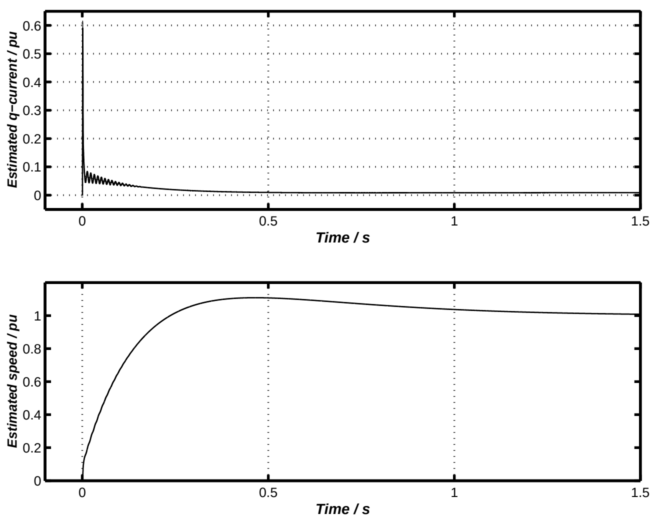

FIGURE 4.7: Estimated q-axis current and speed response with inaccurate machine parameters

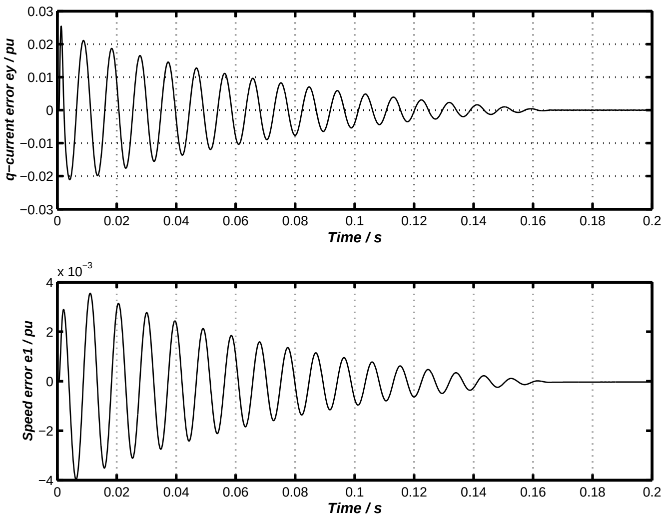

FIGURE 4.8: Time variations of $e_y(t)$ and $e_1(t)$ with inaccurate machine parameters

heavily oscillates before showing the typical sliding mode switching after about $160ms$. These oscillations do not make a significant impact on the closed loop control as measured q-axis current is used for feedback control. It can also be seen that the initial oscillations seen in the estimated speed is of negligible amplitude and hence, they do not make an impact when estimated speed is used for feedback control. Thus it can be seen that the proposed observer shows robust behavior even under parameter variations.

4.5 Conclusion

In this chapter the problem of designing a Sliding Mode Functional Observer for a system with unknown inputs has been addressed. The conditions for the existence of such an observer are illustrated and proven. Sensitivity to changing machine parameters, which is considered as one of the biggest challenges in developing reliable speed sensorless control algorithms for electrical machines has been analysed. An eleven step design algorithm for the design of the sliding mode functional observer has been proposed. This theory has been applied for speed sensorless control of surface mounted PMSM.

Chapter 5

Sliding Mode Observers for Nonlinear Systems

This chapter deals with the design of sliding mode functional observers for a class of nonlinear systems. Necessary and sufficient conditions are derived for the existence of the sliding mode functional observers. An observer design procedure based on Linear Matrix Inequality is given.

5.1 Introduction

State observation is a very important concept for linear and non linear systems. The state observers design problem for nonlinear systems has received widespread attention in literature. The contributions made towards solving this problem can be broadly classified into two approaches. The first is known as the "output injection" approach. Here, the aim is to find a coordinate transformation so that the state estimation error dynamics are linear in the new coordinates and then linear techniques can be performed [74]- [75]. In [76]- [77], necessary and sufficient conditions under which a nonlinear system can be transformed into an observer canonical form have been established. In [75], the class of nonlinear systems that can be transformed into a linear observable form has been identified. Overall, for this approach, the conditions for achieving the desired coordinate transformation are difficult to satisfy and the approach is applicable to a rather restrictive class of nonlinear systems. Significant research efforts have been directed towards developing transformation procedures that involve larger classes of nonlinear systems and canonical forms [75]- [78].

In the second approach, methods have been developed to design state observers for nonlinear systems without the need of the state transformation. State observer design using high-gain Luenberger-like observer for triangular nonlinear systems have been developed [79]- [80]. For this method, the nonlinear system is brought into an observable form and a sufficiently large constant gain dominates the nonlinearity in the error dynamics equations. Dynamic output feedback stabilization using high-gain observers has also been studied for fully-linearizable systems [81] and for systems with "input-to-state stable" inverse dynamics [82]. The design of observers for nonlinear systems, where the estimation error decays irrespective of the input, has been reviewed and generalized in [83]. For the class of systems which are driven by nonlinear functions which are Lipschitz in nature, some fundamental insights into the design of observers and existence conditions of full-order observers have been reported in [84]- [85]. Zhu and Han [86] showed that a $(n - p)$ reduced-order state observer, where n is the system order and p is the number of outputs, can be designed under the same existence conditions as for a full-order state observer [85].

All the above mentioned methods involve the design of full-order or reduced-order observers to estimate the entire state vector of nonlinear systems. There are applications where the use of sliding mode observers can be very useful. Non linear control theory has been applied to the control of induction motors by many authors [87–90]. Sliding mode control theory, due to its order reduction, disturbance rejection, strong robustness, and simple implementation by means of power converter, is described by [91] as one of the prospective control methodologies for electrical machines.

One of the most intriguing aspects of sliding modes as described in [92] is to switch between two distinctively different systems structures (or components) such that a new type of system motion, sliding mode, exists in a manifold. The concept of sliding mode control was originally applied to control system design [93], [94] and later applied for estimating system states [94], [92], [57]. State estimators that utilise the concept of sliding mode are referred to as Utkin Observers in the literature [92].

This chapter aims to develop a sliding mode observer for non-linear systems. The chapter is arranged as follows: Section 5.2 provides a general outline of the problem to be solved. This section will provide a description to the sliding mode observer form. Section 5.3 desribes conditions that are derived for the solvability of the design matrices of the proposed observer and for the stability of its dynamics. For design computational efficiency, an asymptotic stability condition is then developed using LMI formulation. Finally, Section 5.4 presents some conclusions for the chapter.

5.2 Problem Statement

Consider the following class of nonlinear systems described by

$$\dot{x}(t) = Ax(t) + f(x,u) + Bu(t) \tag{5.1a}$$

$$y(t) = Cx(t) \tag{5.1b}$$

$$z(t) = Lx(t), \tag{5.1c}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$ are the state, input and the output vectors, respectively. $f(x,u) \in \mathbb{R}^n$ is a real nonlinear vector function. $z(t) \in \mathbb{R}^p$ is the vector to be estimated. Matrices A, B, C and L are real constant and of appropriate dimensions. We assume that the pair (A, B) is controllable and also without loss of generality, it is assumed that $rank(C) = r$, $rank(L) = p$ and $rank \begin{bmatrix} C \\ L \end{bmatrix} = (p + r) \leq n$.

The problem to be addressed in this chapter is the design of an rth-order observer to estimate the partial state vector $z(t)$. Let us consider the following reduced-order observer for the system (5.1).

$$\dot{\omega}(t) = N\omega(t) + Jy(t) + Pf\left(\begin{bmatrix} y(t) \\ \hat{z}(t) \end{bmatrix}, u\right) + PBu(t) + \Gamma \text{sgn}\left(Ke(t)\right) \tag{5.2a}$$

$$\hat{z}(t) = \omega(t) + Ey(t), \tag{5.2b}$$

where $\omega(t) \in \mathbb{R}^r$, matrices N, J, P, E and nonlinear function $f\left(\begin{bmatrix} y(t) \\ \hat{z}(t) \end{bmatrix}, u\right) \in \mathbb{R}^n$ are to be determined such that $\hat{z}(t)$ converges asymptotically to $z(t)$. $K \in \mathbb{R}^{\tilde{p}\times p}$, $\Gamma \in \mathbb{R}^{p\times\tilde{p}}$, sgn$(\cdot)$ is the sign function and also the sliding surface is given by

$$Ke(t) = 0 \tag{5.3}$$

In view of the dimension of matrix K, the error vector $e(t)$ can be written as

$$e(t) = \begin{bmatrix} e_y(t) \\ e_1(t) \end{bmatrix} \tag{5.4}$$

where $e_y(t) \in \mathbb{R}^{\tilde{p}}$ and $e_1(t) \in \mathbb{R}^{p-\tilde{p}}$.

5.3 Existence Conditions of the Observer

To define the sliding surface for the functional observer let us first partition the matrix L as follows

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \tag{5.5}$$

where L_1 belong to the rowspace of C. Since L_1 belong to the rowspace of C, there always exist a full row rank matrix $G \in \mathbb{R}^{\tilde{p}\times r}$, $\tilde{p} \leq p$ such that

$$L_1 = GC \tag{5.6}$$

If $K \in \mathbb{R}^{\tilde{p}\times p}$ is chosen such that

$$K = \begin{bmatrix} I_{\tilde{p}} & 0_{\tilde{p}\times(p-\tilde{p})} \end{bmatrix} \tag{5.7}$$

and G according to (5.6), then the sliding surface 5.2 is

$$Ke(t) = e_y(t) = Gy(t) - K\hat{z}(t) = 0 \tag{5.8}$$

Using (5.1) to (5.6) and (5.8) the error dynamics of the observer can be written as follows

$$\dot{e}(t) = Ne(t) + (PA - NP - JC)x(t) - \Gamma \mathrm{sgn}(e_y(t)) + P(f\left(\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, u\right) - f\left(\begin{bmatrix} y(t) \\ \hat{z}(t) \end{bmatrix}, u\right)) \tag{5.9}$$

where $P = L - EC$. Considering the partitioning of $e(t)$ in (5.2), matrix N can be partitioned as follows

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{5.10}$$

If we now choose $\Gamma = \begin{bmatrix} \Gamma_1 \\ 0_{(p-\tilde{p})\times\tilde{p}} \end{bmatrix}, \Gamma_1 \in \mathbb{R}^{\tilde{p}\times\tilde{p}}$ then the error dynamics (5.10) can be rewritten as

$$\begin{bmatrix} \dot{e}_y(t) \\ \dot{e}_1(t) \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} e_y(t) \\ e_1(t) \end{bmatrix} + (PA - NP - JC)x(t) - \begin{bmatrix} \Gamma_1 \mathrm{sgn}(e_y(t)) \\ 0 \end{bmatrix} + P(f\left(\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, u\right) - f\left(\begin{bmatrix} y(t) \\ \hat{z}(t) \end{bmatrix}, u\right)) \tag{5.11}$$

Convergence of Observer (2): Let $P \in \mathbb{R}^{p\times n}$ be a full-row rank matrix and let us define the error vector $e(t) \in \mathbb{R}^p$ as

$$e(t) = z(t) - \hat{z}(t). \tag{5.12}$$

Theorem 1: *$\hat{z}(t)$ in (5.2) is an asymptotic estimate of $z(t)$ for the decomposition of the nonlinearity as in (5.3) for all $x(0)$, $\hat{z}(0)$, $u(t)$ and all possible set of nonlinear functions $f(x, u) \in \mathbb{R}^n$ if and only if the following conditions hold.*

Condition 1: $\dot{e}_1(t) = N_{22}e_1(t) + P_q(\tilde{f}) \to 0$ as $t \to \infty$.

Condition 2: $e_y^T(t)\dot{e}_y(t) < -\eta\|e_y(t)\|$,

Condition 3: $\begin{cases} PA - NP - JC = 0 \\ P - L + EC = 0. \end{cases}$

Proof (Sufficiency): If Condition 3 of Theorem 1 is satisfied then the error dynamics (5.3) of the observer is given by

$$\dot{e}_y(t) = N_{11}e_y + N_{12}e_1 - \Gamma_1 \mathrm{sgn}(e_y(t)) + P_p(\tilde{f}) \quad (5.13a)$$

$$\dot{e}_1(t) = N_{21}e_y + N_{22}e_1 + P_q(\tilde{f}), \quad (5.13b)$$

where,

$$P(\tilde{f}) = P(f\left(\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, u\right) - f\left(\begin{bmatrix} y(t) \\ \hat{z}(t) \end{bmatrix}, u\right)) \quad (5.14)$$

$P(\tilde{f})$ can be split into the following,

$$P(\tilde{f}) = \begin{bmatrix} P_p(\tilde{f}) \\ P_q(\tilde{f}) \end{bmatrix} \quad (5.15)$$

where, $P_p(\tilde{f}) \in \mathbb{R}^{\tilde{p}\times\tilde{p}}$ and $P_q(\tilde{f}) \in \mathbb{R}^{(p-\tilde{p})\times\tilde{p}}$

If Condition 2 of Theorem 1 is satisfied then an ideal sliding motion will take place on the surface

$$S_0 = \{(e_1(t), e_y(t)) : e_y(t) = 0\} \quad (5.16)$$

It follows that after some finite time t_s, for all subsequent time, e_y. The dynamics of $e_1(t)$ then reduces to

$$\dot{e}_1(t) = N_{22}e_1(t) + P_q(\tilde{f}) \quad (5.17)$$

Condition 1 specifies that $e_1(t) \to 0$ as $t \to 0$. This condition is necessary for the convergence of the observer (5.2).

Necessity: If Condition 1 is not satisfied then even for $u(t) = 0$, $x(0) = 0$ and $f(x, u) = 0$ we have $e_1(t) \nrightarrow 0$ as $t \to \infty$. Now that the necessity of Condition 1 is established, we have to establish the necessity of Condition 2 and Condition 3. If Condition 2 is not satisfied, then $Ke(t) \neq 0, t \geq t_s$. Finally, if Condition 3 is not satisfied it is possible to choose $x(t)$ using $u(t)$ to make $e_1(t) \neq 0$. This completes the proof of Theorem 1.

The following theorem provides a procedure for determining matrices, N, P, J and E

so that conditions 1-3 of Theorem 1 are satsfied.

Theorem 2: *The estimation error* $e(t) = (z(t) - \hat{z}(t))$ *of observer (5.2) converges asymptotically to zero if the following conditions are satisfied*

Condition A:

$$rank \begin{bmatrix} L_2 A \\ CA \\ C \\ L_2 \end{bmatrix} = rank \begin{bmatrix} CA \\ C \\ L_2 \end{bmatrix} \tag{5.18a}$$

$$rank \begin{bmatrix} sL_2 - L_2 A \\ CA \\ C \end{bmatrix} = rank \begin{bmatrix} CA \\ C \\ L_2 \end{bmatrix} \tag{5.18b}$$

Condition B: The nonlinear function $f(\xi, u) \in \mathbb{R}^n$ *is Lipschitz in its first argument with a Lipschitz constant* γ*, i.e.*

$$||f(\xi, u) - f(\hat{\xi}, u)|| \leq \gamma ||\xi - \hat{\xi}||, \forall u, \tag{5.19}$$

where $\xi(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \in \mathbb{R}^{(r+p)}$*,* γ *is a positive real scalar and* $||.||$ *denotes the norm symbol.*

Condition C: There exist matrices $R = R^T > 0$*,* T *and positive scalars* β_1 *and* β_2 *such that the following linear matrix inequality is satisfied*

$$\begin{bmatrix} RF_{22} + G_2^T R - TG_2 - G_2^T T^T + \gamma^2(\beta_1 + \beta_2) I_r & RP_1 & TP_2 \\ P_1^T R & -\beta_1 I_n & 0 \\ P_2^T T^T & 0 & -\beta_2 I_n \end{bmatrix} < 0, \tag{5.20}$$

where,

$$F_{11} = L_1AM_1 - \Psi\Omega^+ \begin{bmatrix} CAM_1 \\ CM_1 \end{bmatrix}, F_{12} = L_1AM_2 - \Psi\Omega^+ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix}, \quad (5.21)$$

$$F_{21} = L_2AM_1 - \Psi\Omega^+ \begin{bmatrix} CAM_1 \\ CM_1 \end{bmatrix}, F_{22} = L_2AM_2 - \Psi\Omega^+ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix}, \quad (5.22)$$

$$G_1 = (I_{2p} - \Omega\Omega^+) \begin{bmatrix} CAM_1 \\ CM_1 \end{bmatrix}, G_2 = (I_{2p} - \Omega\Omega^+) \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} \quad (5.23)$$

$$P_1 = L_2 - \Psi\Sigma^+ \begin{bmatrix} C \\ 0 \end{bmatrix}, P_2 = (I_{2p} - \Sigma\Sigma^+) \begin{bmatrix} C \\ 0 \end{bmatrix}, \quad (5.24)$$

$$\Sigma = \begin{bmatrix} CA(I_n - M_2L_2) \\ C(I_n - M_2L_2) \end{bmatrix}, \Psi = \begin{bmatrix} L_2A(I_n - M_2L_2) \end{bmatrix}, \quad (5.25)$$

(L^+ denotes the generalized matrix inverse of L) and γ is the Lipschitz constant defined in (5.19). L^+ is partitioned as follows

$$L^+ = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \quad (5.26)$$

Furthermore, matrices N, P, E and J of the observer (5.2) are then determined as

$$N_{22} = F_{22} - R^{-1}TG_2, \quad (5.27)$$

$$P_q = P_1 - R^{-1}TP_2, \quad (5.28)$$

$$E_2 = \left\{\Psi\Sigma^+ + R^{-1}G(I_{2r} - \Sigma\Sigma^+)\right\} \begin{bmatrix} I_r \\ 0_{r\times r} \end{bmatrix}, \quad (5.29)$$

$$J_2 = K_2 + N_{22}E_2 \quad (5.30)$$

where $K_2 = \left\{\Psi\Sigma^+ + R^{-1}T(I_{2r} - \Sigma\Sigma^+)\right\} \begin{bmatrix} 0_{r\times r} \\ I_r \end{bmatrix}.$

Proof: Let us substitute $P_2 = L_2 - E_2 C$ into Conditions 2 and 3 of Theorem 1 to give

$$N_{22} L_2 = L_2 A - \begin{bmatrix} E_2 & K_2 \end{bmatrix} \begin{bmatrix} CA \\ C \end{bmatrix}, \tag{5.31}$$

$$K_2 = J_2 - N_{22} E_2, \tag{5.32}$$

Post-multiply both sides of (5.31) by a full row-rank matrix $\begin{bmatrix} M_2 & (I_n - M_2 L_2) \end{bmatrix}$ to obtain N_{22}. N_{11}, N_{12} and N_{21} follow a similar procedure

$$N_{11} = L_1 A M_1 - \begin{bmatrix} E_1 & K_1 \end{bmatrix} \begin{bmatrix} CAM_1 \\ CM_1 \end{bmatrix}, \tag{5.33}$$

$$N_{12} = L_1 A M_2 - \begin{bmatrix} E_1 & K_1 \end{bmatrix} \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix}, \tag{5.34}$$

$$N_{21} = L_2 A M_1 - \begin{bmatrix} E_2 & K_2 \end{bmatrix} \begin{bmatrix} CAM_1 \\ CM_1 \end{bmatrix}, \tag{5.35}$$

$$N_{22} = L_2 A M_2 - \begin{bmatrix} E_2 & K_2 \end{bmatrix} \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix}, \tag{5.36}$$

$$\begin{bmatrix} E_2 & K_2 \end{bmatrix} \begin{bmatrix} CA(I_n - M_2 L_2) \\ C(I_n - M_2 L_2) \end{bmatrix} = L_2 A (I_n - M_2 L_2). \tag{5.37}$$

Now, (5.37) can be written in an augmented matrix equation as

$$\begin{bmatrix} E_2 & K_2 \end{bmatrix} \Omega = \Psi, \tag{5.38}$$

where Σ and Ψ are as defined in equation (5.25). It is clear from the above equations that the knowledge of $\begin{bmatrix} E_2 & K_2 \end{bmatrix}$ is necessary and sufficient for the determination of matrices N_{22}, P_2 and J_2. From (5.38), a solution for $\begin{bmatrix} E_2 & K_2 \end{bmatrix}$ exists if and only if the following condition holds [95]

$$rank \begin{bmatrix} \Sigma \\ \Psi \end{bmatrix} = rank(\Sigma), \ i.e.$$

$$rank \begin{bmatrix} CA(I_n - M_2L_2) \\ C(I_n - M_2L_2) \\ L_2A(I_n - M_2L_2) \end{bmatrix} = rank \begin{bmatrix} CA(I_n - M_2L_2) \\ C(I_n - M_2L_2) \end{bmatrix}. \tag{5.39}$$

Post-multiply both sides of (5.18b) by a full row-rank matrix $\begin{bmatrix} M_2 & (I_n - M_2L_2) \end{bmatrix}$, it is easy to show that (5.18b) is equivalent to the right hand side of (5.39). Accordingly, (5.38) has the following solution [95]

$$\begin{bmatrix} E_2 & K_2 \end{bmatrix} = \Psi\Sigma^+ + Z_2(I_{2r} - \Sigma\Sigma^+), \tag{5.40}$$

where $Z_2 \in \mathbb{R}^{p\times 2r}$ is an arbitrary matrix. From (5.35) and (5.36), matrices $N_{11}, N_{12}, N_{21}, N_{22}$ and H_q can be expressed as

$$N \begin{cases} N_{11} = F_{11} - Z_1G_1, \\ N_{12} = F_{12} - Z_1G_2, \\ N_{21} = F_{21} - Z_2G_1, \\ N_{22} = F_{22} - Z_2G_2, \end{cases} \tag{5.41}$$

$$P_q = P_1 - Z_2P_2 \tag{5.42}$$

where F_{11}, F_{12}, F_{21}, F_{22}, P_1 and P_2 are as defined in equations (5.23) and (5.24).

Incorporating (5.41) and (5.42) into the observer error system given in Condition 1 of Theorem 1 gives

$$\dot{e}_1(t) = (F_{22} - Z_2G_2)e_1(t) + (P_1 - Z_2P_2)\tilde{f}_1, \tag{5.43}$$

To ensure that $e_1(t)$ in (5.43) converges asymptotically to zero, it is first necessary that matrix $N_{22} = (F_{22} - Z_2G_2)$ is Hurwitz. Accordingly, N_{22} is Hurwitz if and only if the pair (F_{22}, G_2) is detectable, i.e.

$$rank \begin{bmatrix} sI_r - F_{22} \\ G_2 \end{bmatrix} = p, \forall s \in \mathbb{C}, \Re(s) \geq 0. \tag{5.44}$$

Now, the left-hand side of (5.18b) is equivalent to

$$\begin{aligned}
rank \begin{bmatrix} sL_2 - L_2A \\ CA \\ C \end{bmatrix} &= rank \begin{bmatrix} sL_2 - L_2A \\ CA \\ C \end{bmatrix} \begin{bmatrix} M_2 & (I_n - M_2L_2) \end{bmatrix} \\
= \quad rank & \begin{bmatrix} sI_r - L_2AM_2 & -L_2\bar{A} \\ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} & \Sigma \end{bmatrix} \qquad (5.45) \\
= \quad rank & \begin{bmatrix} I_r & L\bar{A}\Sigma^+ \\ 0 & (I_{2p} - \Sigma\Sigma^+) \\ 0 & \Sigma\Sigma^+ \end{bmatrix} \begin{bmatrix} sL_2 - L_2AM_2 & -L\bar{A} \\ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} & \Sigma \end{bmatrix} \\
= \quad rank & \begin{bmatrix} sI_r - F_{22} & 0 \\ G_2 & 0 \\ \Sigma\Sigma + \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} & \Sigma \end{bmatrix} \begin{bmatrix} I_r & 0 \\ -\Sigma^+ \begin{bmatrix} CAM_2 \\ CM_2 \end{bmatrix} & I_{n+d} \end{bmatrix} \\
= \quad rank & \begin{bmatrix} sI_r - F_{22} \\ G_2 \end{bmatrix} + rank(\Omega). \qquad (5.46)
\end{aligned}$$

It is easy to show that the right-hand side of (5.18b) is equivalent to

$$rank \begin{bmatrix} CA \\ C \\ L_2 \end{bmatrix} = p + rank(\Omega). \tag{5.47}$$

Therefore (5.44) is satisfied if (5.18b) holds and hence matrix N_{22} is Hurwitz.

Thus, in the absence of the nonlinear function $f(\xi, u)$ (i.e. $\tilde{f} = 0$), condition (5.18b) is the necessary and sufficient condition for the determination of a matrix Z_2 such that $e_1(t)$ in (5.43) converges asymptotically to zero. When $\tilde{f} \neq 0$, then (5.18b) is not sufficient to ensure asymptotic convergence of (5.43). The stability of the nonlinear differential equation of the type (5.43) has been extensively studied in the literature and various sufficient conditions have been proposed to ensure its stability. One of the most commonly used approaches in the literature is the use of a Lyapunov function coupled with the Lipschitz assumption on the nonlinear function $\tilde{f}$ (see, for example [84]- [86], [96]). In this chapter, a similar line of approach as in [84]- [86], [96] is adopted and a sufficient condition to ensure asymptotic convergence of (5.43) is subsequently derived. Also, for the design computational efficiency, an asymptotic stability condition is developed by using the linear matrix inequality (LMI) formulation. To proceed, let us assume that the nonlinear function $\tilde{f}$ satisfies the Lipschitz assumption as stated in Condition B of Theorem 2 and let us now consider the following Lyapunov function

$$V(e_1(t), t) = e_1^T(t) P e(t), \tag{5.48}$$

where $P = P^T > 0$. Taking its time derivative gives

$$\begin{aligned} \dot{V}(e_1, t) = & \, e_1^T(t) \left[R(F_{22} - Z_2 G_2) + (F_{22} - Z_2 G_2)^T R \right] e_1(t) \\ & + e_1^T(t) R(P_1 - Z_2 P_2) \tilde{f} + \tilde{f}^T (P_1 - Z_2 P_2)^T P e_1(t). \end{aligned} \tag{5.49}$$

Using the well-known matrix inequality $\pm(z^T y + y^T z) \leq \beta y^T y + \frac{1}{\beta} z^T z$ (where β is any positive scalar, z and y are vectors of appropriate dimensions) and subject to the satisfaction of the Condition B of Theorem 2, (5.49) can be expressed as

$$\dot{V}(e_1, t) \leq e_1^T(t) \Theta e_1(t), \tag{5.50}$$

where

$$\Theta = R(F_{22} - Z_2G_2) + (F_{22} - Z_2G_2)^T R + \frac{1}{\beta_1} R P_1 P_1^T R + \frac{1}{\beta_2} R Z_2 P_2 P_2^T Z_2^T R + \gamma^2(\beta_1 + \beta_2) I_r. \tag{5.51}$$

From (5.50) and (5.51), the LMI (5.20) is obtained by using the Schur decomposition and by letting $T = RZ_2$. This completes the proof of Theorem 2.

5.4 Conclusion

This chapter has presented a method for designing a sliding mode observer for a class of nonlinear systems. Necessary and sufficient conditions have been derived for the existence of the sliding mode observer. An essential requirement posed is that the non-linear function be of Lipschitz nature. An observer design procedure based on Linear Matrix Inequality has been given.

Chapter 6

The Descriptor System Approach

This chapter aims to address the problem of estimating a linear function of the states of a system with unknown inputs using the sliding mode functional observer approach. The sliding mode functional observers proposed in this chapter are of low-order and do not include the derivatives of the outputs. New conditions for the existence of sliding mode functional observers are derived. A design procedure for the determination of the observer parameters can also be easily derived based on the derived existence conditions.

6.1 Introduction

Sliding mode functional observers estimate linear functions of the state vector of a system without estimating all the individual states, while ensuring that sliding occurs on a manifold, where some function of the output prediction error is zero. Such functional estimates of the state vector are useful in feedback control system design because the control signal is often a linear combination of the states, and it is possible to utilize a sliding mode linear functional observer to directly estimate the feedback control signal. Although the theory for sliding mode state observers (also referred to as Utkin observers) are well established for linear systems [52]- [55], the concept of sliding mode linear functional observers for descriptor systems has not yet been reported. In this chapter, we present some new results on designing sliding mode functional observers that can estimate a linear function of the states of a system with unknown inputs.

This chapter is arranged as follows: Section 6.2 provides a description of the problem to be solved. Section 6.3 gives a general outline of the problem to be solved. This section will provide a description to the sliding mode observer form for the descriptor type system. Conditions that are derived for the solvability of the design matrices of the proposed observer and for the stability of its dynamics are presented, followed by a design procedure for the observer. Section 6.4 illustrates the design procedure by a numerical example. Finally, Section 6.5 presents the conclusions for the chapter.

6.2 Problem Statement

Let us consider the linear system with unknown inputs as described by

$$\dot{\omega}(t) = \bar{A}\omega(t) + Bu(t) + Dd(t) \tag{6.1a}$$

$$y(t) = \bar{C}\omega(t) + Wd(t) \tag{6.1b}$$

where $\omega(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^r$, $u(t) \in \mathbb{R}^m$ and $d(t) \in \mathbb{R}^q$ are the state, measured output, input and unknown input respectively. Constant matrices $\bar{A} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{n\times q}$, $\bar{C} \in \mathbb{R}^{r\times n}$ and $W \in \mathbb{R}^{r\times q}$ are known. It is assumed that $Rank(\bar{C}) = r$ and $r \geq q$.

For the above system (6.1), the problem of estimating the state and unknown input has been a subject of widespread interest in the literature (see, for example [97]- [98] and the references therein). Unlike the existing work in the literature [97]- [98], the aim of this chapter is to design a reduced-order asymptotic observer to estimate a linear function of the state, $x(t)$, and the unknown input, $d(t)$. The approach adopted in this chapter is based on the Descriptor System Approach [99]. Let us define an augmented state vector, $x(t) = \begin{bmatrix} \omega(t) \\ d(t) \end{bmatrix}$. Accordingly the system (6.1) can be rewritten in descriptor form as

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{6.2a}$$

$$y(t) = Cx(t), \tag{6.2b}$$

$$z(t) = Lx(t), \tag{6.2c}$$

where $x(t) \in \mathbb{R}^{n+q}$, $y(t) \in \mathbb{R}^r$ and $u(t) \in \mathbb{R}^m$ are the descriptor variable (state), measurement (output) and control (input) vectors respectively. Matrices $E \in \mathbb{R}^{n\times(n+q)}$, $A \in \mathbb{R}^{n\times(n+q)}$, and $C \in \mathbb{R}^{r\times(n+q)}$, are such that, $E = \begin{bmatrix} I_n & 0_{n\times q} \end{bmatrix}$, $A = \begin{bmatrix} \bar{A} & D \end{bmatrix}$ and $C = \begin{bmatrix} \bar{C} & W \end{bmatrix}$. $z(t) \in \mathbb{R}^p$ is the vector to be estimated and it follows that $L \in \mathbb{R}^{p\times(n+q)}$. The pair (C, A) is detectable, (A, B) is controllable. It is assumed that *Rank*$(C) = r, r \geq q$ and the system (E, A, C) satisfies the well-known minimum phase condition. Without loss of generality, it is assumed that *Rank*$(C) = r$, *Rank*$(L) = p$, *Rank* $\begin{bmatrix} C \\ L \end{bmatrix} = (r + p - \tilde{r}) \leq (n + q - \tilde{r})$, $\tilde{r} \leq r$ and C takes the form $C = \begin{bmatrix} I_r & 0 \end{bmatrix}$ (otherwise the system can always be transformed into this form).

Here we are interested in designing a sliding mode functional observer to estimate $z(t) \in \mathbb{R}^p$. Let us consider the following sliding mode functional observer of order p for the system (6.2)

$$
\begin{aligned}
\dot{\xi}(t) &= N\xi(t) + Jy(t) + Hu(t) + \Gamma \mathrm{sgn}\left(Ke(t)\right) && (6.3a)\\
\hat{z}(t) &= \xi(t) + My(t) && (6.3b)\\
e(t) &= \hat{z}(t) - z(t), && (6.3c)
\end{aligned}
$$

where $\xi(t) \in \mathbb{R}^p$, $K \in \mathbb{R}^{\tilde{r}\times p}$, $\Gamma \in \mathbb{R}^{p\times\tilde{r}}$, sgn$(\cdot)$ is the sign function and $\hat{z}(t)$ denotes the estimate of $z(t)$. The unknown matrices N, J, H and M are such that $N \in \mathbb{R}^{p\times p}$, $J \in \mathbb{R}^{p\times r}$, $H \in \mathbb{R}^{p\times m}$, and $M \in \mathbb{R}^{p\times r}$. The sliding surface is given by

$$Ke(t) = 0. \tag{6.4}$$

In view of the dimension of matrix K, the error vector $e(t)$ can be written as

$$e(t) = \begin{bmatrix} e_y(t) \\ e_1(t) \end{bmatrix}, \tag{6.5}$$

where $e_y(t) \in \mathbb{R}^{\tilde{r}}$ and $e_1(t) \in \mathbb{R}^{r-\tilde{r}}$.

The problem to be solved in this chapter is to design a sliding mode functional observer of the form (6.3), where matrices N, J, H, M and a suitable Γ are to be determined such that $e(t)$ slides along the surface $Ke(t) = 0$ and $e_1(t) \to 0$ as $t \to \infty$ in finite time.

6.3 Existence Conditions of the Observer

Let $P \in \mathbb{R}^{p\times n}$ be a full-row rank matrix and define error vectors $\varepsilon(t) \in \mathbb{R}^p$ and $e(t) \in \mathbb{R}^p$ as

$$\begin{aligned} \varepsilon(t) &= \xi(t) - PEx(t) && (6.6a) \\ e(t) &= \hat{z}(t) - z(t). && (6.6b) \end{aligned}$$

From (6.6a) the following error dynamics equation is obtained

$$\begin{aligned} \dot{\varepsilon}(t) &= \dot{\xi}(t) - PE\dot{x}(t) \\ &= N\varepsilon(t) + (NPE + JC - PA)x(t) \\ &\quad +(H - PB)u(t) + \Gamma \text{sgn}\,(Ke(t))\,. \end{aligned} \quad (6.7)$$

From 6.6b, the error vector $e(t)$ can be expressed as

$$e(t) = \varepsilon(t) + (PE + MC - L)x(t). \quad (6.8)$$

To define the sliding surface for the functional observer, let us first partition matrix L as follows

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad (6.9)$$

where L_1 belong to the rows space of C. Since L_1 belong to the rows space of C, there always exists a full-row rank matrix $G \in \mathbb{R}^{\tilde{r}\times r}$, $\tilde{r} \leq r$ such that

$$L_1 = GC. \quad (6.10)$$

If $K \in \mathbb{R}^{\tilde{r}\times p}$ is chosen such that

$$K = \begin{bmatrix} I_{\tilde{r}} & 0_{\tilde{r}\times(p-\tilde{r})} \end{bmatrix} \tag{6.11}$$

and G according to (6.10), then the sliding surface (6.4) is

$$Ke(t) = e_y(t) = Gy(t) - M\hat{z}(t) = 0. \tag{6.12}$$

Considering the partitioning of $e(t)$ in (6.5), the error vector $\varepsilon(t)$ and matrix N can be partitioned as

$$\varepsilon(t) = \begin{bmatrix} \varepsilon_y(t) \\ \varepsilon_1(t) \end{bmatrix}, \tag{6.13a}$$

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}. \tag{6.13b}$$

If we now choose $\Gamma = \begin{bmatrix} \Gamma_1 \\ 0_{(p-\tilde{r})\times\tilde{r}} \end{bmatrix}$, $\Gamma_1 \in \mathbb{R}^{\tilde{r}\times\tilde{r}}$, then the error dynamics described by (6.7) and (6.8) can be rewritten as

$$\begin{aligned} \begin{bmatrix} \dot{\varepsilon}_y(t) \\ \dot{\varepsilon}_1(t) \end{bmatrix} = & \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_y(t) \\ \varepsilon_1(t) \end{bmatrix} \\ & +(NPE - JC - PA)x(t) \\ & +(H - PB)u(t) + \Gamma \mathrm{sgn}\left(Ke(t)\right). \end{aligned} \tag{6.14}$$

The existence conditions for the sliding mode functional observer (6.3a) are given in the following theorem, which ensures that $\hat{z}(t)$ converges asymptotically to $z(t)$.

Theorem 1: $e_1(t) \to 0$ as $t \to \infty$ and also $e(t)$ slide along the surface $Ke(t) = 0$, $t \geq t_s$

where $t_s \leq \left(\frac{\|Gy(0)-M\hat{z}(0)\|}{\eta}\right), \eta > 0$ for any $x(0), \hat{z}(0)$, and $u(t)$ if and only if

$$N_{22},\ Hurwitz \tag{6.15}$$

$$NPE + JC - PA = 0, \tag{6.16}$$

$$H = PB, \tag{6.17}$$

$$PE + MC - L = 0, \tag{6.18}$$

$$e_y^T(t)\dot{e}_y(t) < -\eta\|e_y(t)\|. \tag{6.19}$$

Proof (*Sufficiency*): If conditions (6.16), (6.17) and (6.18) are satisfied, then by considering (6.8), the error dynamics (6.14) of the observer can be rewritten as

$$\dot{e}_y(t) = N_{11}e_y(t) + N_{12}e_1(t) + \Gamma_1 \text{sgn}(e_y(t)) \tag{6.20a}$$

$$\dot{e}_1(t) = N_{21}e_y(t) + N_{22}e_1(t) \tag{6.20b}$$

If (6.14) is satisfied, then for some $\Gamma_1 \in \mathbb{R}$ an ideal sliding motion will take place on the surface

$$S_0 = \{(e_1(t), e_y(t)) : e_y(t) = 0\} \tag{6.21}$$

and it follows that after some finite time t_s, for all subsequent time, $e_y(t) = 0$ [55]. The dynamics of $e_1(t) = 0$ then reduces to

$$\dot{e}_1(t) = N_{22}e_1(t). \tag{6.22}$$

If (6.15) is satisfied, then $e_1(t) \to 0$ as $t \to \infty$. As such, if (6.18) is satisfied, $e(t) \to 0$ as $t \to \infty$.

(*Necessity*): If Condition (6.17) is not satisfied, then it is possible to choose a $u(t)$ to make $e(t) \neq 0$. If Condition (6.16) is not satisfied then it is possible to choose a $x(t)$ using $u(t)$ to make $e(t) \neq 0$. If equation (6.15) is not satisfied, then even for $u(t) = 0$ and $x(0) = 0$, $e_1(t) \nrightarrow 0$ as $t \to \infty$. If condition (6.19) is not satisfied then $Ke(t) \neq 0, \forall t \geq t_s$.

Remark 1. In order to derive the parameters of the sliding mode linear functional observer (6.3a), the equations (6.16) - (6.18) must be solved to find out the unknown matrices N,

P, J, H and M. The following theorem will provide the necessary and sufficient conditions for the solvability of matrix Equations (6.16) - (6.18) of *Theorem 1*.

Theorem 2: The matrix equations (6.16) - (6.18) are completely solvable if and only if the following two conditions hold.

Condition 1.

$$Rank\begin{bmatrix} L_2A_u & L_2 \\ CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} = Rank\begin{bmatrix} CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix}, \tag{6.23}$$

where A_u is such that $A = EA_u$.

Condition 2.

$$Rank\begin{bmatrix} (sL_2 - L_2A_u) & -L_2 \\ CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} = Rank\begin{bmatrix} CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix},$$

$$\forall s \in C, Re(s) \geq 0. \tag{6.24}$$

Proof: Since $Rank\begin{bmatrix} A^T \\ E^T \end{bmatrix} = Rank\begin{bmatrix} E^T \end{bmatrix}$, it is always possible to find a matrix $A_u \in \mathbb{R}^{(n+q)\times(n+q)}$ such that

$$A = EA_u. \tag{6.25}$$

Substituting (6.18) and (6.25) into (6.16), the following equation is obtained;

$$NL = LA_u - \begin{bmatrix} M & J - NM \end{bmatrix}\begin{bmatrix} CA_u \\ C \end{bmatrix}. \tag{6.26}$$

Post multiply both sides of (6.26) by the following full-row rank matrix

$$S = \begin{bmatrix} L^+ & (I_{(n+q)} - L^+L \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \end{bmatrix}, \tag{6.27}$$

where L^+ denotes the generalized matrix inverse of L. This yields the following two equations

$$N = LA_uS_1 - \begin{bmatrix} M & J - NM \end{bmatrix} \begin{bmatrix} CA_u \\ C \end{bmatrix} S_1 \tag{6.28}$$

and

$$\begin{bmatrix} M & J - NM \end{bmatrix} \begin{bmatrix} CA_u \\ C \end{bmatrix} S_2 = LA_uS_2. \tag{6.29}$$

To comply with the partitioning of the error vector $e(t)$ in (6.25), the equation (6.29) can be partitioned to yield two matrix equations as;

$$\begin{bmatrix} M_1 & J_1 - N_{11}M_1 - N_{12}M_2 \end{bmatrix} \begin{bmatrix} CA_u \\ C \end{bmatrix} S_2 = L_1A_uS_2 \tag{6.30}$$

$$\begin{bmatrix} M_2 & J_2 - N_{21}M_1 - N_{22}M_2 \end{bmatrix} \begin{bmatrix} CA_u \\ C \end{bmatrix} S_2 = L_2A_uS_2, \tag{6.31}$$

where $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ and $J = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$ with appropriate dimensions. By appropriately partitioning $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ and applying it in (6.18) yields

$$P_1E + M_1C - L_1 = 0, \tag{6.32}$$

$$P_2E + M_2C - L_2 = 0. \tag{6.33}$$

The pairs of equations (6.30), (6.32) and (6.31), (6.33) can be written in augmented form as;

$$\begin{bmatrix} M_1 & T_1 & P_1 \end{bmatrix} \Sigma = \Psi_1 \tag{6.34}$$

and

$$\begin{bmatrix} M_2 & T_2 & P_2 \end{bmatrix} \Sigma = \Psi_2, \tag{6.35}$$

where

$$\Sigma = \begin{bmatrix} CA_uS_2 & C \\ CS_2 & 0 \\ 0 & E \end{bmatrix} \tag{6.36a}$$

$$\Psi_1 = \begin{bmatrix} L_1A_uS_2 & L_1 \end{bmatrix} \tag{6.36b}$$

$$T_1 = J_1 - N_{11}M_1 - N_{12}M_2 \tag{6.36c}$$

$$\Psi_2 = \begin{bmatrix} L_2A_uS_2 & L_2 \end{bmatrix} \tag{6.36d}$$

$$T_2 = J_2 - N_{21}M_1 - N_{22}M_2. \tag{6.36e}$$

S_1 in (6.28) can be partitioned as

$$S_1 = \begin{bmatrix} S_{11} & S_{22} \end{bmatrix}, \tag{6.37}$$

where $S_{11} \in \mathbb{R}^{(n+q)\times\tilde{r}}$ and $S_{12} \in \mathbb{R}^{(n+q)\times(r-\tilde{r})}$ to yield four matrix equations as;

$$N_{11} = L_1A_uS_{11} - \begin{bmatrix} M_1 & T_1 & P_1 \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ 0 \end{bmatrix} S_{11} \tag{6.38}$$

$$N_{12} = L_1A_uS_{12} - \begin{bmatrix} M_1 & T_1 & P_1 \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ 0 \end{bmatrix} S_{12} \tag{6.39}$$

$$N_{21} = L_2 A_u S_{11} - \begin{bmatrix} M_2 & T_2 & P_2 \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ 0 \end{bmatrix} S_{11} \tag{6.40}$$

$$N_{22} = L_2 A_u S_{12} - \begin{bmatrix} M_2 & T_2 & P_2 \end{bmatrix} \begin{bmatrix} CA_u \\ C \\ 0 \end{bmatrix} S_{12}. \tag{6.41}$$

It can be shown that equation (6.34) can always be solved for $\begin{bmatrix} M_1 & T_1 & P_1 \end{bmatrix}$ because $L_1 A_u S_1 (I - \Sigma^+ \Sigma) = 0$ (i.e.$Rank \begin{bmatrix} L_1 A_u S_1 \\ \Sigma \end{bmatrix} = Rank \begin{bmatrix} \Sigma \end{bmatrix}$) and the general solution is given by

$$\begin{bmatrix} M_1 & T_1 & P_1 \end{bmatrix} = L_1 A_u S_1 \Sigma^+ + Z_1 (I - \Sigma \Sigma^+), \tag{6.42}$$

where Z_1 is any arbitrary matrix of appropriate dimension. Using (6.42) in (6.38) and (6.39), the following can be written;

$$N_{11} = F_{11} - Z_1 G_1 \tag{6.43a}$$

$$N_{12} = F_{12} - Z_1 G_2, \tag{6.43b}$$

where

$$F_{11} = L_1 A_u S_{11} - L_1 A_u S_2 \Sigma^+ \begin{bmatrix} CA_u S_{11} \\ CS_{11} \\ 0 \end{bmatrix} \tag{6.44}$$

$$F_{12} = L_1 A_u S_{12} - L_1 A_u S_2 \Sigma^+ \begin{bmatrix} CA_u S_{12} \\ CS_{12} \\ 0 \end{bmatrix} \tag{6.45}$$

$$G_1 = (I - \Sigma \Sigma^+) \begin{bmatrix} CA_u S_{11} \\ CS_{11} \\ 0 \end{bmatrix} \tag{6.46}$$

$$G_2 = (I - \Sigma\Sigma^+)\begin{bmatrix} CA_uS_{12} \\ CS_{12} \\ 0 \end{bmatrix}. \quad (6.47)$$

From (6.35) we can derive the necessary and sufficient condition for the existence of a solution of the unknown matrix $\begin{bmatrix} M_2 & T_2 & P_2 \end{bmatrix}$. Then by substituting the solution (i.e. $\begin{bmatrix} M_2 & T_2 & P_2 \end{bmatrix}$) into (6.41), the necessary and sufficient condition for ensuring that matrix N_{22} Hurwitz can be derived. In (6.35), there exists a solution to the unknown matrix $\begin{bmatrix} M_2 & T_2 & P_2 \end{bmatrix}$ if and only if $Rank\begin{bmatrix} \Psi_2 \\ \Sigma \end{bmatrix} = Rank\begin{bmatrix} \Sigma \end{bmatrix}$ i.e.,

$$Rank\begin{bmatrix} L_2A_uS_2 & L_2 \\ CA_uS_2 & C \\ CS_2 & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} = Rank\begin{bmatrix} CA_uS_2 & C \\ CS_2 & 0 \\ 0 & E \end{bmatrix}. \quad (6.48)$$

It is easy to show that Condition 1 of *Theorem 2* is equivalent to the condition (6.48) (NOTE: To show that (6.23) is equivalent to (6.48), post multiply both sides of (6.23) by a full row-rank matrix $\begin{bmatrix} S_1 & S_2 & 0 \\ 0 & 0 & I_{(n+q)} \end{bmatrix}$). Therefore upon the satisfaction of (6.23), a general solution to (6.35) is;

$$\begin{bmatrix} M_2 & T_2 & P_2 \end{bmatrix} = \Psi_2\Sigma^+ + Z_2(I_{(n+2r)} - \Sigma\Sigma^+), \quad (6.49)$$

where $Z_2 \in \mathbb{R}^{p\times(n+2r)}$ is any arbitrary matrix. Let us now substitute (6.49) into (6.41) to give

$$N_{22} = N'_{22} - Z_2N''_{22}, \quad (6.50)$$

where $N'_{22} = L_2 A_u S_{12} - \Psi_2 \Sigma^+ \Phi$, $N''_{22} = (I_{(n+2r)} - \Sigma\Sigma^+)\Phi$ and

$$\Phi = \begin{bmatrix} CA_u S_{12} \\ CS_{12} \\ 0 \end{bmatrix}. \tag{6.51}$$

In (6.50), N'_{22} and N''_{22} are known matrices and matrix N_{22} is Hurwitz for some matrix Z_2, if and only if the pair (N'_{22},N''_{22}) is detectable, i.e.,

$$Rank \begin{bmatrix} (sI_p - N'_{22} \\ N''_{22} \end{bmatrix} = p, \forall s \in C, Re(s) \geq 0. \tag{6.52}$$

In the following, we will show that the Condition 2 of the *Theorem 2* is equivalent to the condition (6.52) and therefore ensuring that matrix N_{22} is Hurwitz. First post multiply the RHS of (6.24) by a full row-rank matrix $\begin{bmatrix} S_1 & S_2 & 0 \\ 0 & 0 & I_{(n+q)} \end{bmatrix}$ to give

$$\begin{aligned} Rank & \begin{bmatrix} CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} \\ = & \; Rank \begin{bmatrix} CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} \begin{bmatrix} S_1 & S_2 & 0 \\ 0 & 0 & I_{(n+q)} \end{bmatrix}, \\ = & \; p + Rank(\Sigma). \end{aligned} \tag{6.53}$$

Now the LHS of (6.24) can be expressed as follows;

$$
\begin{aligned}
& Rank\begin{bmatrix} (sL_2 - L_2A_u) & -L_2 \\ CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} \\
= \; & Rank\left\{\begin{bmatrix} (sL_2 - L_2A_u) & -L_2 \\ CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix}\begin{bmatrix} S_1 & S_2 & 0 \\ 0 & 0 & I_{(n+q)} \end{bmatrix}\right\} \\
= \; & Rank\begin{bmatrix} (sI_p - L_2A_uS_{12}) & -\Psi_2 \\ \Phi & \Sigma \end{bmatrix} \\
= \; & Rank\left\{\begin{bmatrix} I_p & \Psi_2\Sigma^+ \\ 0 & (I_{(n+2r)} - \Sigma\Sigma^+) \\ 0 & \Sigma\Sigma^+ \end{bmatrix}\begin{bmatrix} (sI_p - L_2A_uS_{12}) & -\Psi_2 \\ \Phi & \Sigma \end{bmatrix}\right\} \\
= \; & Rank\begin{bmatrix} sI_p - N'_{22} & 0 \\ N''_{22} & 0 \\ \Sigma\Sigma^+\Phi & \Sigma \end{bmatrix} \\
= \; & Rank\left\{\begin{bmatrix} sI_p - N'_{22} & 0 \\ N''_{22} & 0 \\ \Sigma\Sigma^+\Phi & \Sigma \end{bmatrix}\begin{bmatrix} I_p & 0 \\ -\Sigma^+\Phi & I_{2(n+q)} \end{bmatrix}\right\} \qquad (6.54) \\
= \; & Rank\begin{bmatrix} sI_p - N'_{22} \\ N''_{22} \end{bmatrix} + Rank[\Sigma], \\
& \forall s \in C, Re(s) \geq 0.
\end{aligned}
$$

It is clear from (6.53) and (6.55) that Condition (6.24) is equivalent to (6.52) and therefore provides the necessary and sufficient condition for matrix N_{22} to be Hurwitz. Upon the satisfaction of the Conditions 1 and 2 of *Theorem 2*, then from (6.50), we can easily find a matrix gain Z_2 and a stable matrix N_{22}. Once matrix Z_2 is obtained, the unknown matrix $\begin{bmatrix} M_2 & T_2 & P_2 \end{bmatrix}$ is derived from (6.49). From (6.17), (6.36c) and (6.36e), matrices H and J can be derived. Computation of N_{21} and N_{22} are further elaborated below. In the same way as equation (6.42) was derived, general solution for (6.35) can be given as

$$\begin{bmatrix} M_2 & T_2 & P_2 \end{bmatrix} = L_2 A_u S_2 \Sigma^+ + Z_2(I - \Sigma\Sigma^+), \tag{6.55}$$

where Z_2 is any arbitrary matrix of appropriate dimension. Equation (6.55) can be solved if and only if *Theorem 2* is satisfied. Using (6.55) in (6.38) and (6.39), the following can be written

$$N_{21} = F_{21} - Z_2 G_1 \tag{6.56a}$$

$$N_{22} = F_{22} - Z_2 G_2 \tag{6.56b}$$

where

$$F_{21} = L_2 A_u S_{11} - L_2 A_u S_2 \Sigma^+ \begin{bmatrix} C A_u S_{11} \\ C S_{11} \\ 0 \end{bmatrix} \tag{6.57a}$$

$$F_{22} = L_2 A_u S_{12} - L_2 A_u S_2 \Sigma^+ \begin{bmatrix} C A_u S_{12} \\ C S_{12} \\ 0 \end{bmatrix}. \tag{6.57b}$$

The matrix N_{22} can be made Hurwitz if and only if the pair (F_{22}, G_2) is detectable.

Design Algorithm

1. Partition L according to $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} GC \\ L_2 \end{bmatrix}$
2. Check if Condition 1 and 2 of *Theorem 2* is satisfied. If yes, continue, otherwise a

sliding mode functional observer does not exist.

3. Compute F_{22} using (6.57b), G_2 using (6.47) and check if the pair (F_{22}, G_2) is detectable. If yes, continue, otherwise N_{22} cannot be made Hurwitz and consequently a stable sliding mode functional observer does not exist.

4. Using (6.56b) and any pole placement method obtain Z_2 to make N_{22} Hurwitz.

5. Use (6.57a) to compute F_{21}, compute G_1 using (6.46) and then use (6.56a) to compute N_{21}.

6. Choose any Z_1 and use (6.43a) and (6.43b) to compute N_{11} and N_{12} respectively.

7. Use (6.42) and (6.55) to compute M_1, T_1, M_2 and T_2.

8. Use (6.36c) and (6.36e) to compute J_1 and J_2 respectively.

9. Compute H according to (6.17).

10. Use any standard sliding mode technique to obtain Γ_1 to make error dynamics of (6.20a) stable.

A numerical example that follows the above design procedure is given in the following Section. Now we will consider two special cases, where Condition 1 and 2 of *Theorem 2* can be further simplified. *Special Case 1*: $L = I_{(n+q)}$(i.e. $Rank(L) = p = n + q$) and clearly, $L_1 = \begin{bmatrix} I_r & 0 \end{bmatrix} \in \mathbb{R}^{r\times(n+q)}$ and $L_2 = \begin{bmatrix} 0 & I_{(n+q-r)} \end{bmatrix} \in \mathbb{R}^{(n+q-r)\times(n+q)}$ (i.e. $\tilde{r} = r$).
Corollary 1: When $L = I_{(n+q)}$, then the matrix equations (6.16) and (6.18) are completely solvable if and only if the following two conditions hold:

Condition 1.

Matrix W is full-column rank, i.e.

$$Rank(W) = q. \tag{6.58}$$

Condition 2.

$$Rank(sI_{n+q} - A_u) = Rank(W) - q. \tag{6.59}$$

Proof: Let us now consider the Condition 1 of *Theorem 2.* Incorporating above in LHS together with the fact that C can take the form $C = \begin{bmatrix} I_r & 0 \end{bmatrix}$ will result in

$$\begin{array}{rcl} Rank\begin{bmatrix} L_2A_u & L_2 \\ CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} & = & Rank\begin{bmatrix} \begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix} A_u & \begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix} \\ \begin{bmatrix} I_r & 0 \end{bmatrix} A_u & \begin{bmatrix} I_r & 0 \end{bmatrix} \\ \begin{bmatrix} I_r & 0 \end{bmatrix} & 0 \\ 0 & E \\ \begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix} & 0 \end{bmatrix} \\ & = & Rank\begin{bmatrix} I_{(n+q)}A_u & I_{(n+q)} \\ I_{(n+q)} & 0 \\ 0 & E \end{bmatrix} \\ & = & 2(n+q). \end{array}$$

RHS will result in

$$\begin{array}{rcl} rclRank\begin{bmatrix} CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} & = & Rank\begin{bmatrix} CA_u & C \\ \begin{bmatrix} I_r & 0 \end{bmatrix} & 0 \\ 0 & E \\ \begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix} & 0 \end{bmatrix} \\ & = & Rank\begin{bmatrix} CA_u & C \\ 0 & E \\ I_{(n+q)} & 0 \end{bmatrix} \\ & = & Rank\begin{bmatrix} C \\ E \end{bmatrix} + (n+q). \end{array} \tag{6.61}$$

When E and C in (6.61) are replaced from $E = \begin{bmatrix} I_n & 0_{n\times q} \end{bmatrix}$ and $C = \begin{bmatrix} \bar{C} & W \end{bmatrix}$ we get:

$$\begin{aligned} Rank\begin{bmatrix} C \\ E \end{bmatrix} &= Rank\begin{bmatrix} \bar{C} & W \\ I_n & 0_{n\times q} \end{bmatrix} \\ &= (2n+q) + Rank(W). \end{aligned}$$

Therefore the Condition 1 of *Theorem 2* boils down to (6.58). Let us now consider the Condition 2 of *Theorem 2*. Incorporating above in LHS together with the fact that C can take the form $C = \begin{bmatrix} I_r & 0 \end{bmatrix}$ will result in

$$\begin{aligned} & Rank\begin{bmatrix} (sL_2 - L_2A_u) & -L_2 \\ CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} \\ = \; & Rank\begin{bmatrix} \left(s\begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix} - \begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix}A_u\right) & -\begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix} \\ \begin{bmatrix} I_r & 0 \end{bmatrix}A_u & \begin{bmatrix} I_r & 0 \end{bmatrix} \\ \begin{bmatrix} I_r & 0 \end{bmatrix} & 0 \\ 0 & E \\ \begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix} & 0 \end{bmatrix} \\ = \; & Rank\begin{bmatrix} \left(s\begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix} - \begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix}A_u\right) & -\begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix} \\ \left(s\begin{bmatrix} I_r & 0 \end{bmatrix} - \begin{bmatrix} I_r & 0 \end{bmatrix}A_u\right) & -\begin{bmatrix} I_r & 0 \end{bmatrix} \\ \begin{bmatrix} I_r & 0 \end{bmatrix} & 0 \\ \begin{bmatrix} 0 & I_{n+q-r} \end{bmatrix} & 0 \\ 0 & E \end{bmatrix} \\ = \; & Rank\begin{bmatrix} (sI_{n+q} - A_u) & -I_{n+q} \\ I_{n+q} & 0 \\ 0 & E \end{bmatrix} \end{aligned}$$

$$= \quad Rank \begin{bmatrix} (sI_{n+q} - A_u) & -I_{n+q} \\ 0 & E \end{bmatrix} + (n+q). \tag{6.63}$$

When E is replaced from $E = \begin{bmatrix} I_n & 0_{n \times q} \end{bmatrix}$,

$$\begin{aligned} & Rank \begin{bmatrix} (sI_{n+q} - A_u) & -I_{n+q} \\ 0 & E \end{bmatrix} + (n+q) \\ = \quad & Rank \begin{bmatrix} (sI_{n+q} - A_u) & -I_{n+q} \\ 0 & \begin{bmatrix} I_n & 0_{n \times q} \end{bmatrix} \end{bmatrix} + (n+q) \\ = \quad & Rank(sI_{n+q} - A_u) + 2(n+q). \end{aligned}$$

It was shown earlier that RHS reduces to $(2n+q) + Rank(W)$. Therefore, the Condition 2 of *Theorem 2* can be rewritten as (6.59).

Remark 2. It must be noted here that when $L = I_{(n+q)}$, the resulting observer becomes a full order state observer using sliding mode technique.

Special Case 2: $Rank \begin{bmatrix} C \\ L \end{bmatrix} = (n+q)$ (i.e. $Rank \begin{bmatrix} C \\ L \end{bmatrix} = (r+p-\tilde{r}) = (n+q-\tilde{r})$ and hence $(r+p) = (n+q)$).

Special Case 1: $L = I_{(n+q)}$(i.e. $Rank(L) = p = n+q$) and clearly, $L_1 = \begin{bmatrix} I_r & 0 \end{bmatrix} \in \mathbb{R}^{r \times (n+q)}$ and $L_2 = \begin{bmatrix} 0 & I_{(n+q-r)} \end{bmatrix} \in \mathbb{R}^{(n+q-r) \times (n+q)}$ (i.e. $\tilde{r} = r$).

Corollary 1: When $L = I_{(n+q)}$, then the matrix equations (6.16) and (6.18) are completely solvable if and only if the following two conditions hold:

Condition 1.

Matrix W is full-column rank, i.e.

$$Rank(W) = q. \tag{6.65}$$

Condition 2.

$$Rank\left(s\begin{bmatrix} L_2 \\ C \end{bmatrix} - \begin{bmatrix} L_2 \\ C \end{bmatrix} A_u\right) = Rank(W) - q. \qquad (6.66)$$

Proof: Let us now consider the Condition 1 of *Theorem 2* and incorporate above in LHS. When above condition is true, it follows that $Rank\begin{bmatrix} C \\ L_2 \end{bmatrix} = (n+q)$. This leads to

$$\begin{aligned} Rank\begin{bmatrix} L_2A_u & L_2 \\ CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} &= Rank\begin{bmatrix} L_2A_u & L_2 \\ CA_u & C \\ C & 0 \\ L_2 & 0 \\ 0 & E \end{bmatrix} \\ &= 2(n+q). \end{aligned}$$

RHS will result in

$$\begin{aligned} Rank\begin{bmatrix} CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} &= Rank\begin{bmatrix} CA_u & C \\ 0 & E \\ C & 0 \\ L_2 & 0 \end{bmatrix} \\ &= Rank\begin{bmatrix} C \\ E \end{bmatrix} + (n+q). \end{aligned}$$

With the same substitutions made to (6.61) it can be shown that the Condition 1 of *Theorem 2* again boils down to (6.65). Let us now consider the Condition 2 of *Theorem 2*. Incorporating above in LHS. When above condition is true, it follows that $Rank\begin{bmatrix} C \\ L_2 \end{bmatrix} =$

$(n+q)$. This leads to

$$
\begin{aligned}
& Rank \begin{bmatrix} (sL_2 - L_2A_u) & -L_2 \\ CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} \\
= \; & Rank \begin{bmatrix} (sL_2 - L_2A_u) & -L_2 \\ (sC - CA_u) & -C \\ L_2 & 0 \\ C & 0 \\ 0 & E \end{bmatrix} \\
= \; & Rank \begin{bmatrix} (s\begin{bmatrix} L_2 \\ C \end{bmatrix} - \begin{bmatrix} L_2 \\ C \end{bmatrix} A_u) & -\begin{bmatrix} L_2 \\ C \end{bmatrix} \\ \begin{bmatrix} L_2 \\ C \end{bmatrix} & 0 \\ 0 & E \end{bmatrix} \\
= \; & Rank \begin{bmatrix} (s\begin{bmatrix} L_2 \\ C \end{bmatrix} - \begin{bmatrix} L_2 \\ C \end{bmatrix} A_u) - \begin{bmatrix} L_2 \\ C \end{bmatrix} \\ 0 \qquad\qquad E \end{bmatrix} + (n+q) \\
= \; & Rank \left(s\begin{bmatrix} L_2 \\ C \end{bmatrix} - \begin{bmatrix} L_2 \\ C \end{bmatrix} A_u \right) + 2(n+q). \qquad (6.69)
\end{aligned}
$$

(NOTE: Replace E from $E = \begin{bmatrix} I_n & 0_{n\times q} \end{bmatrix}$). It was shown earlier that RHS reduces to $(2n+q) + Rank(W)$. Therefore, the Condition 2 of *Theorem 2* can be rewritten as (6.66)

Special Case 3: The matrix W has full-column rank (i.e. $Rank(W) = q, q \leq r$) and $Rank \begin{bmatrix} C \\ L \end{bmatrix} = (r + p - \tilde{r}) \leq (n + q - \tilde{r})$. When matrix W is full-column rank, we have $Rank \begin{bmatrix} E \\ C \end{bmatrix} = (n+q)$. Note that this is a well-known assumption that has been used in the design of Luenberger-type observers for descriptor systems [100]. Since W is a full-column rank matrix, there always exists a nonsingular matrix $Q \in \mathbb{R}^{r\times r}$ such that

$$QW = \begin{bmatrix} W_1 \\ GW_1 \end{bmatrix}, \tag{6.70}$$

where $W_1 \in \mathbb{R}^{q\times q}$, $Rank(W_1) = q$ and $G \in \mathbb{R}^{(r-q)\times q}$. Let us also define the following:

$$Q\bar{C} = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix}, \tag{6.71a}$$

$$\begin{bmatrix} I_n & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} E \\ C \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \tag{6.71b}$$

$$\begin{bmatrix} I_n & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} 0 \\ CA_u \end{bmatrix} = \begin{bmatrix} Q_{u1} \\ Q_{u2} \end{bmatrix}, \tag{6.71c}$$

where matrices $\bar{C}_1 \in \mathbb{R}^{q\times n}$, $\bar{C}_2 \in \mathbb{R}^{(r-q)\times n}$, $Q_1 = \begin{bmatrix} E \\ C_1 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ \bar{C}_1 & W_1 \end{bmatrix} \in \mathbb{R}^{(n+q)\times(n+q)}$, $Rank(Q_1) = (n+q)$, $Q_2 = \begin{bmatrix} \bar{C}_2 & GW_1 \end{bmatrix} \in \mathbb{R}^{(r-q)\times(n+q)}$, $Q_{u1} \in \mathbb{R}^{(n+q)\times(n+q)}$ and $Q_{u2} \in \mathbb{R}^{(r-q)\times(n+q)}$ are all known and real constant. Accordingly Condition 1 and 2 of Theorem 2 can be rewritten as in the *Corollary 3* below.

Corollary 3: When $Rank\begin{bmatrix} E \\ C \end{bmatrix} = (n+q)$, then the matrix equations (6.16) and (6.18) are completely solvable if and only if the following two conditions hold:

Condition 1.

$$Rank\begin{bmatrix} Q_{u2} - Q_2(Q_1)^{-1}Q_{u1} \\ L_2\{A_u - (Q_1)^{-1}Q_{u1}\} \\ C \\ L_2 \end{bmatrix} = Rank\begin{bmatrix} Q_{u2} - Q_2(Q_1)^{-1}Q_{u1} \\ C \\ L_2 \end{bmatrix}, \tag{6.72}$$

Condition 2.

$$Rank\begin{bmatrix} sL_2 - L_2\{A_u - (Q_1)^{-1}Q_{u1}\} \\ Q_{u2} - Q_2(Q_1)^{-1}Q_{u1} \\ C \end{bmatrix} = Rank\begin{bmatrix} Q_{u2} - Q_2(Q_1)^{-1}Q_{u1} \\ C \\ L_2 \end{bmatrix},$$

$$\forall s \in C, Re(s) \geq 0.$$

Proof: When $Rank\begin{bmatrix} E \\ C \end{bmatrix} = (n+q)$, using (6.71b) and (6.71c), RHS of Condition 1 of *Theorem 2* can be expressed as

$$\begin{aligned} & Rank\begin{bmatrix} CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix} \\ = \; & Rank\left\{\begin{bmatrix} I_n & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I_{(r+p)} \end{bmatrix}\begin{bmatrix} 0 & E \\ CA_u & C \\ C & 0 \\ L_2 & 0 \end{bmatrix}\right\} \\ = \; & Rank\left\{\begin{bmatrix} Q_{u1} & Q_1 \\ Q_{u2} & Q_2 \\ C & 0 \\ L_2 & 0 \end{bmatrix}\begin{bmatrix} I_{(n+q)} & 0 \\ -(Q_1)^{-1}Q_{u1} & I_{(n+q)} \end{bmatrix}\right\} \\ = \; & Rank\begin{bmatrix} 0 & Q_1 \\ Q_{u2} - (Q_1)^{-1}Q_{u1} & Q_2 \\ C & 0 \\ L_2 & 0 \end{bmatrix} \end{aligned}$$

$$= \quad (n+q) + Rank \begin{bmatrix} Q_{u2} - (Q_1)^{-1}Q_{u1} \\ \\ C \\ L_2 \end{bmatrix} \tag{6.73}$$

Similarly LHS of Condition 1 of *Theorem 2* can be expressed as

$$Rank \begin{bmatrix} L_2A_u & L_2 \\ CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix}$$

$$= \quad Rank \left\{ \begin{bmatrix} I_n & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I_{(r+p)} \end{bmatrix} \begin{bmatrix} 0 & E \\ CA_u & C \\ L_2A_u & L_2 \\ C & 0 \\ L_2 & 0 \end{bmatrix} \right\} \tag{6.74}$$

$$= \quad Rank \left\{ \begin{bmatrix} Q_{u1} & Q_1 \\ Q_{u2} & Q_2 \\ L_2A_u & L_2 \\ C & 0 \\ L_2 & 0 \end{bmatrix} \begin{bmatrix} I_{(n+q)} & 0 \\ -(Q_1)^{-1}Q_{u1} & I_{(n+q)} \end{bmatrix} \right\}$$

$$= \quad Rank \begin{bmatrix} 0 & Q_1 \\ Q_{u2} - (Q_1)^{-1}Q_{u1} & Q_2 \\ \\ L_2\{A_u - (Q_1)^{-1}Q_{u1}\} & L_2 \\ C & 0 \\ L_2 & 0 \end{bmatrix}$$

$$= \quad (n+q) + Rank \begin{bmatrix} Q_{u2} - (Q_1)^{-1}Q_{u1} \\ L_2\{A_u - (Q_1)^{-1}Q_{u1}\} \\ C \\ L_2 \end{bmatrix} \tag{6.75}$$

Therefore, by combining (6.73) and (6.75) the Condition 1 of *Theorem 2* can be rewritten as (6.72). Now it is easy to show that the LHS of Condition 2 of *Theorem 2* is equivalent to

$$Rank \begin{bmatrix} (sL_2 - L_2A_u) & -L_2 \\ CA_u & C \\ C & 0 \\ 0 & E \\ L_2 & 0 \end{bmatrix}$$

$$= \quad Rank \begin{bmatrix} sL_2 - L_2\{A_u - (Q_1)^{-1}Q_{u1}\} \\ Q_{u2} - Q_2(Q_1)^{-1}Q_{u1} \\ C \end{bmatrix},$$

$$\forall s \in C, Re(s) \geq 0.$$

Therefore, by combining (6.73) and (6.76a) the Condition 2 of *Theorem 2* can be rewritten as (6.73). This completes the proof of *Corollary 3*.

6.4 Numerical Example

Let us consider a 7^{th} order system with the system matrices A, B, C and E given below

$$A = \begin{bmatrix} -2 & 4 & -1 & 7 & 0 & -1 & -3 \\ 1 & -5 & 0 & -3 & -1 & -2 & -1 \\ -1 & -7 & 0 & -8 & -1 & 1 & 3 \\ 0 & -6 & -2 & -4 & -2 & -2 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } E = \begin{bmatrix} 0 & -1 & 1 & -2 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 & 0 & 1 & 1 \\ 1 & 0 & -2 & 3 & 2 & 0 & -1 \\ -3 & -2 & -2 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

Let the function to be estimated given by

$$L = \begin{bmatrix} 1 & 3 & 5 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

To obtain the sliding surface for the observer, G is obtained according to (6.10)

$$G = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$$

and the sliding surface according to (6.12) is given below

$$Ke(t) = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} y(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{z}(t).$$

It is easy to verify that conditions (6.23) and (6.24) are satisfied. Now matrix F_{22} and G_2 can be obtained according to (6.57b) and (6.47) and is given below;

$$F_{22} = [-1] \text{ and } G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Since the pair (F_{22}, G_2) is detectable, N_{22} can be made Hurwitz and its pole is at $s = -1$. Using (6.56b) Z_2 and N_{22} can be obtained as $Z_2 = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$ and $N_{22} = [-1]$. Using (6.57a) and (6.46) we can find F_{21} and G_1 respectively as $F_{21} = [-0.5714]$ and $G_1 = [0\ 0\ 0\ 0.0286\ 0.0857\ 0.1429\ 0\ 0\ 0\ 0]^T$. Using (6.56a) we can compute N_{21} as; $N_{21} = [-0.5714]$. The choice of Z_1 is arbitrary and we choose

$$Z_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can now compute N_{11} and N_{12} using (6.43a) and (6.43b) as $N_{11} = [0]$ and $N_{12} = [0]$.

Using (6.36c) and (6.36e), J is computed to be

$J = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -4 & -2 \end{bmatrix}$. According to (6.17) H is computed as
$H = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

The error dynamics of the observer can be written as follows (note that $e_y(t) \in \mathbb{R}$ and $e_1(t) \in \mathbb{R}$)

$$\begin{bmatrix} \dot{e}_y(t) \\ \dot{e}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -0.5714 & -1 \end{bmatrix} \begin{bmatrix} e_y(t) \\ e_1(t) \end{bmatrix} + \begin{bmatrix} \Gamma_1 \text{sgn}(e_y(t)) \\ 0 \end{bmatrix}.$$

The above error dynamics is simulated to verify the convergence, which is shown in Fig 6.1,

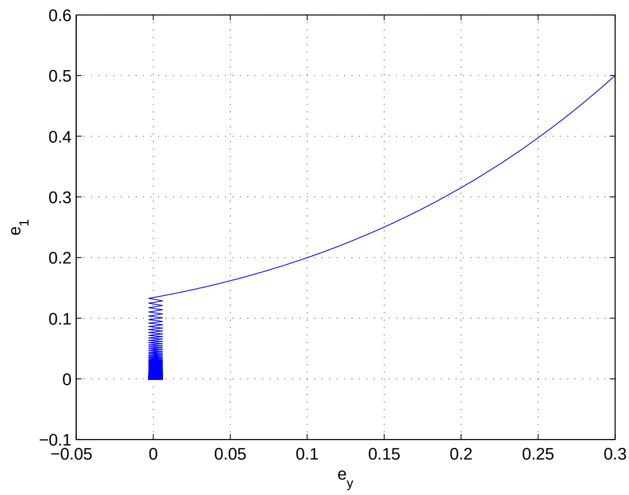

FIGURE 6.1: Error Trajectory for e_1, and e_y

$e_y(t) \in \mathbb{R}$ and $e_1(t) \in \mathbb{R}$. For $\Gamma_1 = -0.3$, sliding occur on the plane $e_y(t) = 0$ as in Fig 6.1. Here it is also noted that, while N_{22} is Hurwitz, N is not Hurwitz. Matrix N has eigenvalues at $\{0, -1\}$ yet a stable error response is achieved by the switching function in the sliding mode observer. As can be seen in Fig 6.1, the initial error in $e_y(t)$ is dragged towards the plane $e_y(t) = 0$ in finite time and $e(t)$ slides along the plane $e_y(t) = 0$ towards the origin.

6.5 Conclusion

This chapter has addressed the problem of estimating a linear function of the states of a system with unknown inputs using the Descriptor system sliding mode functional observer approach. The observers proposed in this chapter have the advantages of having the order the same as the dimension of the vector to be estimated. Necessary and sufficient conditions for the existence of the sliding mode functional observer have been derived and proved. Three special cases where the above necessary and sufficient conditions can further be reduced have also been presented and proved. Based on the derived theory, a design procedure has been proposed and is verified by a numerical example.

Chapter 7

Conclusions and Future Directions

In this chapter, the main outcomes of the analysis and evaluation of the sliding mode observers are discussed. Particular emphasis will also be placed on presenting a practical viewpoint on the results that have been demonstrated throughout the thesis. The key conclusions that have resulted from the thesis are summarized.

7.1 Major Findings

The current body of knowledge has been built upon through

- Understanding and development of the sliding mode functional observer design for neutral-delay systems.
- Illustrating the effect of having unknown system inputs to the sliding mode functional observer design. This is a scenario that affects almost all practical systems.
- Application of the sliding mode functional observers to non-linear systems. Again, non-linear systems are a very integral part of Control Engineering, and contribute a large representation of real world systems.
- Application of the sliding mode functional observer to unknown input systems, through the descriptor system approach.

The outcomes from this research are summarized as follows

In Chapter 3, the problem of estimating a linear function of the states of a class of linear time-delay systems of the neutral type using sliding mode functional observers was illustrated. The sliding mode functional observers in this chapter were of low-order and did not include derivatives of the outputs. This led to a very high robustness and relatively low complexity in the final design. A design algorithm was formulated that essentially involved Hurwitz analysis, pole-placement and the use of standard sliding mode techniques. By transforming the original system into descriptor form and deriving the existence conditions, the observers proposed have the advantages of having the order the same as the dimension of the vector to be esimated.

In Chapter 4, a method of designing a sliding mode linear functional observer for a system with unknown inputs was illustrated. The conditions for designing this observer were specified. The theoretical results in this chapter were accompanied by a practical example that illustrated the application of such observers to real life control scenarios, namely the application to the speed sensorless control of Permanent Magnet Synchronous Motors. The convergence properties of these Motors were illustrated, and simulation results were specified and analysed. The proposed observer was seen to exhibit extremely robust behaviour even under quite strong parameter variations.

Chapter 5 dealt with the design of sliding mode functional observers for a class of non-linear systems. The necessary and sufficient conditions are derived for the existence of the observer. Linear Matrix Inequalities were implemented into the design to create a degree of computational efficiency, and develop an asymptotic stability condition. While the results were strictly directed towards non-linear systems that are Lipschitz in nature, the practical applications of the observer are still quite extensive. In Section 7.2 possible future extensions to this work are proposed.

Chapter 6 addressed the problem of estimating a linear function of the states of a system with unknown inputs with a sliding mode functional observer. The approach adopted in this chapter was based on the descriptor representation of linear systems. Three special cases where the necessary and sufficient conditions derived in the design procedure were also investigated. A numerical example illustrated the error dynamics of such an observer and these dynamics were simulated to verify the convergence properties that were claimed in the design.

7.2 Future Direction

The field of sliding mode observers in control is still a relatively new topic of interest, and there is still a vast amount of work that remains to be done in the field.

One of the major outcomes of the thesis was the acknowledgement of the stringent requirements that are placed on systems. This leads to one prominent shortfall of sliding mode observers i.e. lack of significance in a practical environment. For example, in Chapter 5, the theory developed and associated design algorithm were based around the non-linear system being Lipschitz. In some cases this is quite an unrealistic assumption and narrows down the range of systems that are applicable to the model.

The possible future research goals that can arise from this thesis are summarized below

- Extension of the theories developed to incorporate Extended State Observers. As illustrated in Section 2.3.5 ESOs have been verified to demonstrate quite impressive performance characteristics when compared to other state observers even Sliding Mode Observers in some cases. Simulation results provided in [47] demonstrated that the controller resulting from the ESO (in induction motor control specifically) operates quite smootly and robustly under modelling uncertainty and external disturbance, and it can provide good dynamic performance such as small overshoot and fast transient time in the speed control. The field of ESOs is still considered to be largely untouched and there is definitely a great deal of potential there for industrial applications.

- As mentioned above, in the field of Sliding Mode Observers for Non-Linear systems there will be a significant improvement to the design developed if the theory extends beyond that of requiring the Lipschitz assumption to be met, and to incorporate all non-linear systems. This will create a much more robust theoretical base that will be more suitable to industrial applications that in some cases will not satisfy this assumption.

- Perhaps the most obvious extension to the current work is to re-formulate all of the theories in the discrete-time domain. In this thesis, next to all of the theories developed were in the continuous time domain. This provides a good theoretical base, however when it comes to practical applications, especially in the modern world, one cannot ignore the importance of digital systems, and the application of observers to these are no exception.

- This thesis aimed to primarily focus on the development of theories in relation to sliding mode observers, and hence there was very little emphasis placed on performance evaluation. It is quite interesting to see how sliding mode observers (for non-linear, descriptor, neutral-delay and unknown input systems) compare in performance to, for example, standard Luenberger observers. MATLAB/SIMULINK provide excellent tools for both practical and hypothetical control system analysis.

- Perhaps the most important development in this field is in the industrial applications. State, functional and sliding mode observers have such a great potential to reduce system complexity and enhance industrial control processes. Currently the use of these observers in industry is quite limited because of their lack of practicality, however if more research focus is placed on their applications, there is a great deal of value that can be gained from this research topic.

Appendix A

A.1 SRM Observer Equation Solution

Differentiating both sides of (1.9) yields

$$\frac{de_\theta}{dt} = \frac{d\theta}{dt} - \frac{d\hat{\theta}}{dt} \tag{A.1}$$

Substituting in (1.7) produces

$$\frac{de_\theta}{dt} = \omega(t) - \hat{\omega}(t) - K_\theta sgn(e_f) \tag{A.2}$$

and using the velocity error definition in (1.10) yields the position error dynamics

$$\frac{de_\theta}{dt} = e_\omega - K_\theta sgn(e_f) \tag{A.3}$$

To derive the velocity error dynamics, begin with

$$\frac{de_\omega}{dt} = \frac{d\omega}{dt} - d\hat{\omega}dt \tag{A.4}$$

Substituting into (1.5) yields

$$\frac{de_\omega}{dt} = -\frac{D}{J}\omega(t) + \frac{1}{J}\Sigma_n T_n(\theta_n, \lambda_n) - \frac{1}{J}T_L(t) - K_\omega sgn(e_f) \tag{A.5}$$

If it is assumed that K_ω may be selected to be large enough such that the first two terms may be neglected, the velocity error dynamics become

$$\frac{de_\omega}{dt} = -K_\omega sgn(e_f) \tag{A.6}$$

Thus, (A.4) and (A.6) describe the convergence properties of the observer. Once the sliding surface e_θ is reached the error dynamics become

$$\frac{de_\theta}{dt} = 0 \tag{A.7}$$

$$\frac{de_\omega}{dt} = -\frac{K_\omega}{K_\theta} e_\omega \tag{A.8}$$

There are many possible error functions which will stabilise the error dynamics. In general, the error function must compare a variable dependent upon the estimated position and a measurable machine variable. Assuming θ is increasing from the unaligned position, the error function was chosen as

$$e_f = \sum_{n=1}^{N} f_n^{'}(\hat{\theta})(\lambda_n - \hat{\lambda}_n) \tag{A.9}$$

with

$${f_n}' = \frac{df_n}{d\theta} \tag{A.10}$$

with $\theta = \hat{\theta}$

The error term includes a flux estimate based on the estimate rotor position

$$\hat{\lambda}_n(t) = \hat{\lambda}_n(i_n(t), \hat{\theta}(t)) \tag{A.11}$$

The function $f_n\theta$ is selected such that the error function (A.9) forces the position estimate to converge to a motoring condition. This may be accomplished by defining the following

$$f_n\theta = cos(\theta) \tag{A.12}$$

.

Appendix B

There are two properties of linear state-space descriptions that are often necessary assumptions about a system for controller and observer construction. In order to construct an observer we require that the system be minimal, that is the system must be controllable and observable. Essentially these terms imply that for a system of equations it is possible to control the state of each variable by altering the inputs (controllable), and that each state variable can be extracted by measuring the output of the system. The following definitions appear in [4].

Consider,

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{(B.1)}$$

$$y(t) = Cx(t) \quad \text{(B.2)}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^r$ and $u(t) \in \mathbb{R}^m$

Definition B 2.1

The state of a continuous-time linear system (B.1) and (B.2) is said to be *reachable(from the zero state)* at time t if there exists a $\tau \leq t$ and an input and an input $u \in \mathbb{R}^m$ which transfers the zero state at time τ to the state x at time t.

Definition B 2.2 The continuous-time linear system (B.1) and (B.2) is said to be *controllable (to the zero state)* if, given any initial state $x(\tau)$, there exists a $t \geq \tau$ and a $u(t) \in \mathbb{R}^m$ such that $x(t) = 0$.

Definition B 2.3 Let $y(t; \tau, x, u)$ denote the output response fo the linear system (B.1) and (B.2) to the intial state $x(\tau)$. Then the (present) state $x(\tau)$ of the linear system is *unobservable* if the (future) output

$$y(t : \tau, x, 0) = 0 \tag{B.3}$$

for all $t \geq \tau$.

The proofs of accompanying these definitions are given in [101].

B.1 Minimal Realisations

A system is minimal if and only if it is fully state controllable and fully observable. Minimal systems are invariant to similarity transformations, so neither controllability or observability of a minimal system is affected by similarity transformations. However not all the systems are minimal. It would therefore be useful to have techniques in which non-controllable and/or non-observable systems can be represented and also have methods for these non-controllable/non-observable state variables to be separated out. One possible way in which this can be achieved is via similarity transformations. Much of this discussion on realisations can be attributed to [102].

B.1.1 Similarity Transformations

The state-space equations used to model a system are non-unique. Therefore it is possible to transform a given system into another system of a different form whilst still maintaining the same input/output behaviour.

A non-singular transformation matrix T_1 can be chosen to transform the state vector as shown

$$x(t) = T_1 \hat{x}(t) \tag{B.4}$$

By taking the derivative of (B.4) and substituting into the state-space equations (B.1) and (B.2) we get

$$\dot{\hat{x}}(t) = (T_1 A T_1{}^{-1})\hat{x}(t) + (T_1{}^{-1} B)u(t) \tag{B.5}$$

$$y(t) = (C T_1)\hat{x}(t) + (D)u(t) \tag{B.6}$$

The two realisations $\{A, B, C, D\}$ and $\{T_1^{-1}AT_1, T_1^{-1}B, CT_1, D\}$ are called similar realisations. Alternatively, if the transformation matrix is defined as $\hat{x}(t) = T_2 x(t)$, the similar realisation is $\{T_2 A T_2^{-1}, T_2 B, C T_2^{-1}, D\}$.

B.1.2 Representation of Non-Controllable Realisations

Let A, B, C be such that if

$$Rank\, C(A, B) = r < n \tag{B.7}$$

Then there is always a transformation T such that the realisation

$$\{\bar{A} = T^{-1}AT, \quad \bar{B} = T^{-1}B, \quad \bar{C} = CT\} \tag{B.8}$$

has the form

$$\bar{A} = \begin{bmatrix} \bar{A}_c & \bar{A}_{c\bar{c}} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \tag{B.9a}$$

$$\bar{B} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} \tag{B.9b}$$

$$\bar{C} = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \tag{B.9c}$$

where the transformation matrix T of rank n has the following structure

$$T = \begin{bmatrix} B & AB & ... & A^{r-1} & v_1 & v_2 & ... & v_{n-r} \end{bmatrix} \tag{B.10}$$

where the last $n - r$ column vectors $v_1, v_2, ...v_{n-r}$ are chosen such that they are linearly independent of the first r columns of the transformation T. This realisation has the following two useful properties

1. The $r \times r$ subsystem $\{\bar{A}_c, \bar{B}_c, \bar{C}_c\}$ is controllable

2. The subsystem has the same transfer function as the original system.

B.1.3 Representation of Non-Observable Realisations

A dual statement can be made about the non-observable realisation. Thus if

$$Rank\, O(C, A) = r < n \tag{B.11}$$

we can find a nonsingular matrix T such that the realisation

$$\{\bar{A} = T^{-1}AT, \quad \bar{B} = T^{-1}B, \quad \bar{C} = CT\} \tag{B.12}$$

has the form

$$\bar{A} = \begin{bmatrix} \bar{A}_o & 0 \\ 0 & \bar{A}_{oo} \end{bmatrix} \tag{B.13a}$$

$$\bar{B} = \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} \tag{B.13b}$$

$$\bar{C} = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \tag{B.13c}$$

where the transformation matrix T of $Rank = n$ has the structure

$$T = \begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{r-1} \\ v_1 \\ v_2 \\ . \\ . \\ v_{n-r} \end{bmatrix} \tag{B.14}$$

This realisation has the following two useful properties

1. The $r \times r$ subsystem $\{\bar{A}_o, \bar{B}_o, \bar{C}_o\}$ is observable

2. The subsystem has the same transfer function as the original system.

The above results suggest that it is possible to produce a minimal realisation from any given realisation.

Appendix C

The Chattering Problem

Almost ever since sliding mode ideas have been put forward, the audible noise some sliding mode controllers exhibit have irritated control engineers and has often led to their resentment, even rejection of the technique. The phenomenon is best known as chattering. Two main causes have been identified.

- The fast dynamics in the control loop which were neglected in the system model, are often excited by the fast switching of sliding mode controllers [6].
- Digital implementations in microcontrollers with fixed sampling rates may lead to discretization chatter [6].

C.1 Problem Analysis

The term "chattering" describes the phenomenon of finite-frequency, finite-amplitude oscillations appearing in many sliding mode implementations. These oscillations are caused by the high-frequency switching of a sliding mode controller exciting unmodelled dynamics in the closed loop. "Unmodelled Dynamics" may refer to sensors and actuators neglected in the principal modelling process since they are generally significantly faster than the main system dynamics. However, since *ideal* sliding mode systems are infinitely fast, all system dynamics should be considered in the control design.

According to [6], fortuntely preventing chattering usually does not require a detailed model of all system components. Rather, a sliding mode controller may first be designed under idealized assumptions of no unmodelled dynamics. The solution of the chattering problem is of great importance when exploiting the benefits of a sliding mode controller in a real-life system. Without proper treatment in the control design, chattering may be a major obstacle to the implementation of sliding mode in a wide range of applications. Note that the switching action itself, as the core of a continuous-time sliding mode system, is *not* called chattering itself, in the ideal case, the switching is intended and its frequency tends to infinity.

C.1.1 Example System: Model

A simple first-order plant with second order “unmodelled” actuator dynamics is used as an example for illustration purposes in [6]. The model of the first-order system with state and output $x(t)$ is given by

$$\dot{x}(t) = ax(t) + d(x,t) + bw(t) \tag{C.1}$$

where $a^{-1} \leq a \leq a^{+}$ and $0 < b^{-1} \leq b \leq b^{+}$ are unknown parameters within known bounds, $w(t)$ is the control variable and disturbance $d(t)$ is assumed to be uniformly bounded for all operating conditions (x,t) as $\|d(x,t)\| \leq d^{+}$. Control variable $w(t)$ is the output of an “unmodelled” actuator with stable dynamics dominated by second-order function

$$w(t) = \frac{\omega^2}{p^2 + 2\omega p + \omega^2} u(t) \tag{C.2}$$

$$= \frac{1}{\epsilon p + 1^2} u(t) \tag{C.3}$$

where $u(t)$ is the actual control input to the plant (C.1) and p denotes the Laplace variable. In equations (C.2) and (C.3) a mixed representation of time domain and frequency (Laplace) domain functions is used for ease of presenation, although it is not formally correct. For example, it is understood in (C.2) that time-domain control variable $w(t)$ is the output of the lowpass filter described by the inverse of the Laplace transfer function in p with time-domain input $u(t)$. In equation (C.2) $\omega > 0$ is the unknown actuator bandwidth with $\omega >> a$ in equation (C.1). A small time constant $\epsilon = 1/\omega > 0$ was substituted to symbolize that the actuator dynamics are assumed to be significantly faster than the system dynamics (C.1). The goal of control is to make the state and output $x(t)$ of the system (C.1) track a desired trajectory $x_d(t)$ with a known amplitude bound as $\|x_d(t)\| \leq x_d{}^+$ and a known bound on the rate of change $\|\dot{x}_d(t)\| \leq v_d{}^+$. The parameters for the simulation examples provided below are $a = 0.5, b = 1, d(t) = 0.2sin(10t) + 0.3cos(20t) \leq 0.5, \omega = 50$ and thus $\epsilon = 0.02$, with a limit on available control resources of $\|u(t)\| \leq 2.01$ and a desired trajectory $x_d(t) = sint$, i.e. $x_d(t)^+ = 1$ and $v_d(t)^+ = 1$. Note that with $a < 0$, the example plant (C.1) is *unstable*.

C.1.2 Example System: Ideal Sliding Mode

Standard sliding mode control design for the ideal plant (C.1) i.e. neglecting actuator dynamics (C.2) and (C.3) by setting $w(t) = u(t)$ defines the sliding variable as

$$s(t) = x_d(t) - x(t) \tag{C.4}$$

and the associated sliding mode controller as

$$w(t) = Msign(s(t)) \tag{C.5}$$

Stability of the closed-loop system and tracking of desired $x_d(t)$ are manifested by examination of the Lyapunov function candidate. The candidate is

$$V(t) = \frac{1}{2b} s^2(t) \tag{C.6}$$

Differentiation of (C.6) along the system trajectories (C.1) under control (C.5) and without the actuator dynamics (C.2) yields

$$\dot{V}(t) = \frac{1}{b} s(t)\dot{s}(t) \tag{C.7}$$

$$= g(x, x_d, t)s(t) - M\|s(t)\| \tag{C.8}$$

where the term

$$g(x, x_d, t) = \frac{\dot{x}_d(t) - ax(t) - d(t)}{b} \tag{C.9}$$

has an upper bound

$$\|g(x, x_d, t)\| \leq g^+ = \frac{v_d^+ + a^+ x_d^+ + d^+}{b^-} \tag{C.10}$$

under the assumption that $x(t) \approx x_d(t)$. For $M \geq g^+ + \frac{\alpha}{\sqrt{2b^-}}$ with scalar $\alpha > 0$, substitution of the control law (C.5) into (C.7) leads to

$$\dot{V}(t) \leq -\alpha V^{1/2}(t) \tag{C.11}$$

which testifies to convergence to $s(t) = 0$ within finite time. The solution of (C.11) for an arbitrary intial condition $V(0) > 0$ yields

$$V(t) = -\frac{\alpha}{2}t + V^{1/2}(0)^{2} \tag{C.12}$$

which implies that $V(t)$ is identical to zero after finite time $t_{sm} \geq \frac{2}{\alpha}V^{1/2}(0)$. Reaching time t_{sm} is a conservative estimate of the maximum time necessary to reach $s(t) = 0$. In practice, sliding mode often occurs much earlier [6]. Subsequently, the system is invariably confined to the manifold $s(t) = 0$ in (C.4), despite parametric uncertainty in a and b and unknown disturbance $d(x, t)$. A block diagram of the ideal sliding mode system is shown in Figure C.1. The behaviour of the plant (C.1) in sliding mode control (C.5) can be

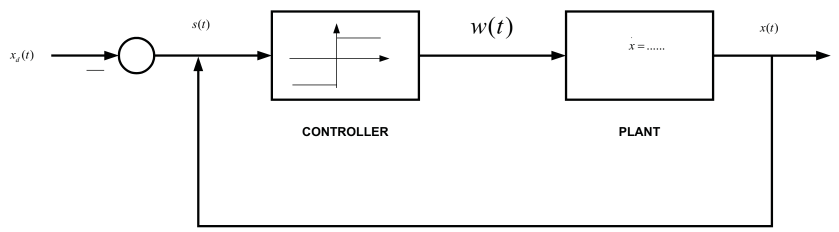

FIGURE C.1: Block Diagram of ideal sliding mode control loop. A discontinuous controller forces the output $x(t)$ of the plant to exactly track the desired trajectory $x_d(t)$. No chattering occurs since the control loop is free of unmodelled dynamics

examined using the equivalent control method [6]. Since $s(t)$ is invariantly identical to zero after reaching the sliding manifold, $\dot{s}(t)$ can be formally set to zero. Solving

$$\dot{s}(t) = \dot{x}_d(t) - \dot{x}(t) \tag{C.13}$$

$$= b(g(x,t) - w(t)) \tag{C.14}$$

for the continuous equivalent control yields

$$w_{eq}(t) = g(x,t) \tag{C.15}$$

which can be viewed as an average of the discontinuous control $w(t)$ in (C.5). Applying equivalent control $w_{eq}(t)$ to plant (C.1) would result in exactly the same motion trajectory as applying discontinuous control $w(t)$ (C.5), but this is not possible since $g(x,t)$ contains unknown terms. Substitution of $w_{eq}(t)$ into (C.1) validates the exact tracking performance in sliding mode with $x_d(t) = x(t)$. For the simulation of (C.1) under control (C.5) with $M = 2.01$ in Figure C.2, initial condition $x(0) = 1$ was chosen. After reaching the sliding manifold $s(t) = 0$ at $t \approx 0.45s$, system trajectory $x(t)$ coincides exactly with the desired $x_d(t)$, and control $w(t)$ is switched at very high frequency, creating a solidly black area. For illustration, Figure C.2(b) shows equivalent control $w_{eq}(t)$ in (C.15) as a white dashed line in this black area. Setting the parameter bounds to $a^- = a^+ = a = 0.5$ and $b^- = b^+ = b = 1.0$ results in $g^+ = 2$ which leads to slow convergence to $s(t) = 0$ due to small $\alpha \approx 0.014$.

C.1.3 Example System: Causes of Chattering

In a practical application, unmodelled dynamics in the closed-loop actuator (C.2) often prevent ideal sliding mode from occurring and cause, fast, finite-amplitude oscillations. Figure C.3 shows a block diagram of the closed control loop including the previously neglected actuator dynamics. In systems with *continuous* motion equations, fast motion components like those of actuators for large ω in (C.2) decay rapidly provided that they are stable (as is the case for $\omega > 0$). The slow motion component of the plant (C.1) thus continuously depends on the steady-state solution of (C.2). In other words, as an approximation, and continuous control design may very well neglect the actuator dynamics. In the case of (C.2) $w = u$ leads to Figures C.1 and C.2 as predicted. In systems with

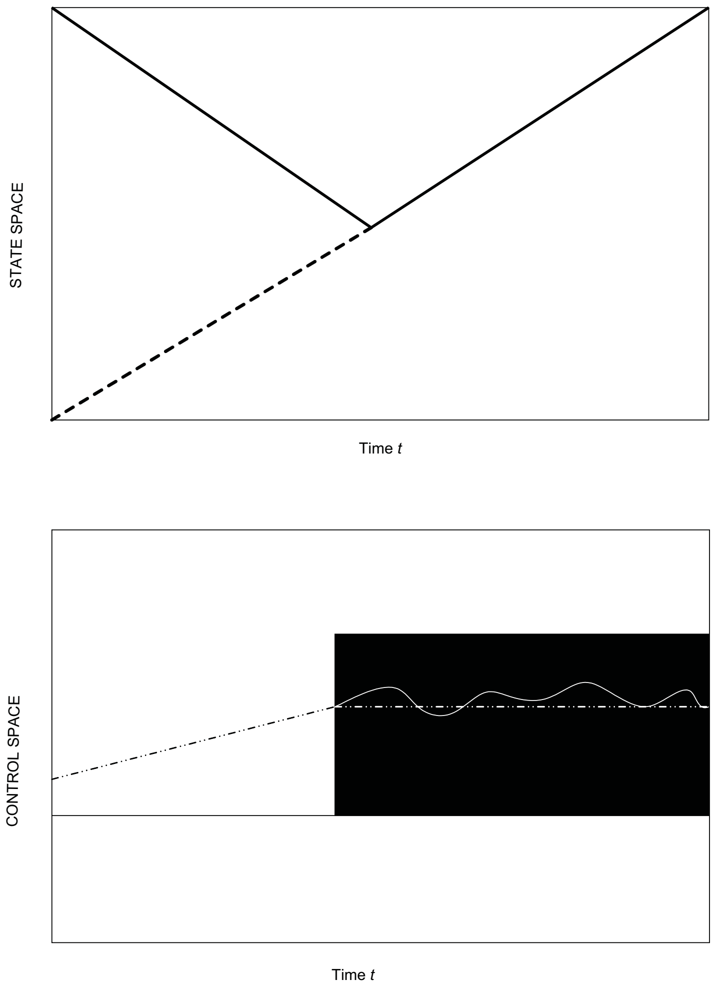

FIGURE C.2: Ideal sliding mode in first-order system. State $x(t)$ converges to the desired $x_d(t)$ in finite time, i.e. $s(t) = 0$ after $t \approx 0.45s$. Thereafter, control $u(t)$ switches with infinite frequency and shows as a black area. Equivalent control u_{eq} is drawn as a dashed line (a) output and desired output, (b) control inputs and sliding variable

discontinuities the solution to the motion equations depends on the small time constants of fast components as well. But unlike in systems with continuous control, discontinuities in the control excite the unmodelled dynamics, leading to oscillations in the state vector. This phenomenon is known as "chattering" in the control literature. These oscillations are known to result in low control accuracy, high heat loss in electrical power circuits and high wear of moving mechanical parts. Figure C.4 shows the chattering behaviour of system

(C.15) under control (C.5), but with actuator dynamics in the loop of Figure C.3. Figure C.4(a) depicts output $x(t)$ oscillating around the desired $x_d(t)$ after $t \approx 0.5s$. In Figure C.4(b) control $u(t)$ switches at finite frequency (solid line), whereas output $w(t)$ of the actuator (dotted line) clearly cannot follow the steps in control commant $u(t)$. Note that an increase of the actuator bandwidth would increase the frequency of the square-wave behaviour of $u(t)$, but would not be able to eliminate the oscillations. In fact, when a mechanical device is performing oscillations similar to Figure C.4, audible noise often results in high frequency, which led to the name "chattering". Chattering is extremely harmful to the mechanical system components.

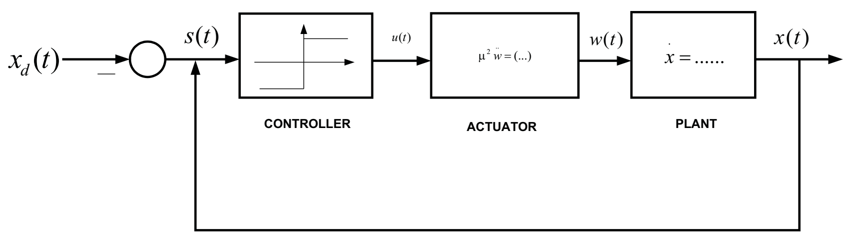

FIGURE C.3: Control loop with actuator dynamics in ideal control design. Sliding mode does not occur since the actuator dynamics are excited by the fast switching of the discontinuous controller, leading to chattering in the loop

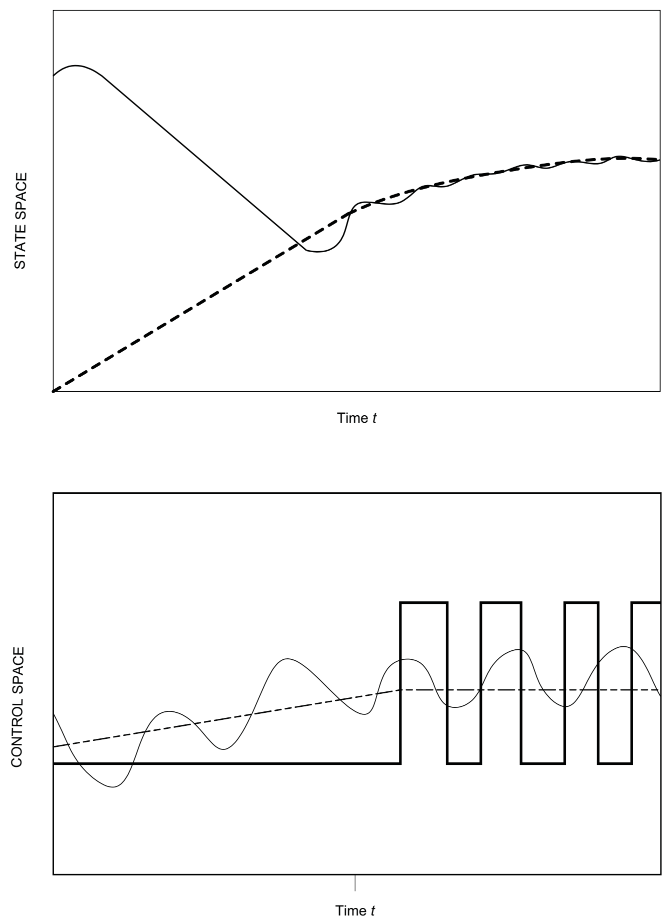

FIGURE C.4: Chattering in first-order system with second-order actuator dynamics, under discontinuous control. After switches in control $u(t)$, actuator ouptut, $w(t)$ lags behind, leading to oscillatory system trajectories. (a) output and desired output, (b) control inputs and sliding variable

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