

Finite totally k -closed groups

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Abstract

For a positive integer k , a group G is said to be totally k -closed if in each of its faithful permutation representations, say on a set Ω , G is the largest subgroup of $\text{Sym}(\Omega)$ which leaves invariant each of the G -orbits in the induced action on $\Omega \times \cdots \times \Omega = \Omega^k$. We prove that every abelian group G is totally $(n(G) + 1)$ -closed, but is not totally $n(G)$ -closed, where $n(G)$ is the number of invariant factors in the invariant factor decomposition of G . In particular, we prove that for each $k \geq 2$ and each prime p , there are infinitely many finite abelian p -groups which are totally k -closed but not totally $(k - 1)$ -closed. This result in the special case $k = 2$ is due to Abdollahi and Arezoomand. We pose several open questions about total k -closure.

1 Introduction

In 1969 Wielandt [8, Definition 5.3] introduced, for each positive integer k , the concept of the k -closure of a permutation group G on a set Ω . The k -closure $G^{(k),\Omega}$ of G is the set of all $g \in \text{Sym}(\Omega)$ (permutations of Ω) such that g leaves invariant each G -orbit in the induced G -action on ordered k -tuples from Ω . The k -closure $G^{(k),\Omega}$ is a subgroup of $\text{Sym}(\Omega)$ containing

G [8, Theorem 5.4], and a permutation group G is said to be *k-closed* if $G^{(k),\Omega} = G$. Different faithful permutation representations of the same group G may have quite different k -closures. For example, the symmetric group S_3 acts faithfully and intransitively on $\{1, 2, 3, 4, 5\}$ with orbits $\{1, 2, 3\}$ and $\{4, 5\}$, and in this action its 2-closure is $S_3 \times C_2$; while S_3 is 2-closed in its natural action on $\{1, 2, 3\}$.

In 2016, D. F. Holt¹ suggested a stronger concept independent of the permutation representation, and this was studied first by Abdollahi and Arezoomand in [1] in the case $k = 2$. For a positive integer k , a group G is said to be *totally k-closed* if $G^{(k),\Omega} = G$ whenever G is faithfully represented as a permutation group on Ω . The only totally 1-closed group is the trivial group consisting of a single element (see Remark 2.3), while Abdollahi and Arezoomand [1, Theorem 2] showed that a finite nilpotent group is totally 2-closed if and only if it is cyclic, or it is a direct product of a generalised quaternion group and a cyclic group of odd order. Here we consider larger values of k .

For a permutation group $G \leq \text{Sym}(\Omega)$, and $k \geq 2$, Wielandt [8, Theorem 5.8] proved that

$$G \leq G^{(k),\Omega} \leq G^{(k-1),\Omega}. \quad (1)$$

Thus if G is totally $(k - 1)$ -closed, then it is automatically totally k -closed. Moreover $G = G^{(k),\Omega}$ for sufficiently large k , since by [8, Theorem 5.12], this holds whenever there exist $k - 1$ points $\alpha_1, \dots, \alpha_{k-1} \in \Omega$ such that the only element of G fixing each α_i is the identity. The inclusion (1) does suggest that the family of totally k -closed groups might be larger than that of totally $(k - 1)$ -closed groups. We show that this is the case, even for abelian groups.

Theorem 1.1. *Let k be an integer with $k \geq 2$. Then, for each prime p , there are infinitely many finite abelian p -groups which are totally k -closed but not totally $(k - 1)$ -closed.*

The result of Abdollahi and Arezoomand shows that the finite totally 2-closed abelian groups are precisely the cyclic groups. It turns out, also for larger values of k , that the total k -closure property for abelian groups is linked with the numbers of cyclic direct factors in their direct decompositions. A study of these decompositions leads to useful bounds, from which we deduce Theorem 1.1.

¹mathoverflow.net/questions/235114/2-closure-of-a-permutation-group

According to the fundamental theorem for finite abelian groups, each nontrivial finite abelian group G can be written as a direct product $G = H_1 \times \cdots \times H_n$, for some $n \geq 1$, such that each $H_i \cong \mathbb{Z}_{d_i}$, $d_1 > 1$, and $d_i | d_{i+1}$ for $1 \leq i < n$. The integer n and the d_i are uniquely determined by G , up to the order of the factors. The H_i are called the *invariant factors* of G , and we write $n(G) := n$ for the number of invariant factors. We also have the primary decomposition of G as $G = \prod_{p \in \pi(G)} G_p$, where $\pi(G)$ is the set of primes dividing $|G|$ and G_p is the (unique) Sylow p -subgroup of G . It is straightforward to see that $n(G) = \max_{p \in \pi(G)} n(G_p)$. Our main result is the following theorem, from which we deduce Theorem 1.1.

Theorem 1.2. *Let G be a finite abelian group with $|G| > 1$. Then G is totally $(n(G) + 1)$ -closed, but is not totally $n(G)$ -closed.*

The following auxiliary assertion on the k -closure of the direct product abelian permutation p -groups may be of independent interest. It is proved in Section 2, and is used in Section 3 to reduce the proof of Theorem 1.2 to the case of p -groups. For its statement it is convenient to use $\text{Syl}(G)$ to denote the set of all Sylow subgroups of a group G ; if G is abelian, $\text{Syl}(G)$ will consist of one Sylow p -subgroup for each prime $p \in \pi(G)$.

Theorem 1.3. *Let G be a finite abelian permutation group on a set Ω , and k an integer, $k \geq 2$. Then $G^{(k), \Omega} = \prod_{P \in \text{Syl}(G)} P^{(k), \Omega}$.*

The results in our short paper serve to raise a number of open questions, and we record a few here. The first relates to Theorem 1.2. It would be interesting to have a generalisation of the classification by Abdollahi and Arezoomand [1, Theorem 2] of nilpotent totally 2-closed nilpotent groups for larger values of k .

Problem 1. *For $k > 2$ determine all finite nilpotent groups G that are totally k -closed.*

As we noted above, the symmetric group S_3 is not totally 2-closed. Indeed, it was shown by Abdollahi, Arezoomand and Tracey [2, Theorem B] a finite soluble group is totally 2-closed if and only if it is nilpotent, hence known by [1, Theorem 2]. However it is not difficult to see that it is totally 3-closed, since in every faithful permutation representation of $G = S_3$ on a set Ω there must be a G -orbit of length 3 or 6, and the stabiliser in G of two points α, β from such an orbit is trivial. Hence by [8, Theorem 5.12],

$G = G^{(3),\Omega}$. As a first step it would be interesting to know which other non-nilpotent soluble groups are totally 3-closed.

Problem 2. *Determine the finite soluble groups that are totally 3-closed.*

For some time it was believed that all finite totally 2-closed groups would be soluble, and it was somewhat surprising to the authors of [3] to discover that exactly six of the sporadic simple groups are totally 2-closed, namely J_1, J_3, J_4, Ly, Th, M .

Problem 3. *Find all the the totally 3-closed sporadic simple groups. More generally, for each sporadic simple group G determine the least value of k such that G is totally k -closed.*

The classification of the finite nonabelian simple totally 2-closed groups is still not complete, and we refer the reader to [3] for details of the status of this problem and other open questions about total 2-closure.

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2 Preliminaries

In this section we give some background theory, and in particular we prove Theorem 1.3. First we state two results of Wielandt for convenience.

Theorem 2.1. [Wielandt, [8, Theorem 5.6]] *Let $G \leq \text{Sym}(\Omega)$, let $k \geq 1$, and let $x \in \text{Sym}(\Omega)$. Then $x \in G^{(k),\Omega}$ if and only if, for all $(\alpha_1, \dots, \alpha_k) \in \Omega^k$, there exists $g \in G$ such that $\alpha_i^x = \alpha_i^g$ for $i = 1, \dots, k$.*

Theorem 2.2. [Wielandt, [8, Theorem 5.12]] *Let $G \leq \text{Sym}(\Omega)$ and $k \geq 1$, and suppose that $\alpha_1, \dots, \alpha_k \in \Omega$ such that $G_{\alpha_1 \dots \alpha_k} = 1$. Then $G^{(k+1),\Omega} = G$.*

Next we discuss total 1-closure.

Remark 2.3. Suppose that G is a finite totally 1-closed group. Consider the regular representation of G on $\Omega = G$. Since G is transitive on Ω it follows from Theorem 2.1 that $G^{(1),\Omega} = \text{Sym}(\Omega)$. Thus, since G is totally 1-closed, it follows that $\text{Sym}(\Omega) = G$ is regular, and hence $|G| \leq 2$. However, if $G = C_2$, then in the representation $G = \langle (12)(34) \rangle \leq \text{Sym}(\Omega)$ on $\Omega = \{1, 2, 3, 4\}$ we have $G^{(1),\Omega} = \langle (12), (34) \rangle \neq G$. Hence $G = 1$ is the only possibility.

For a prime $p \mid n$, the largest p -power divisor of n is denoted by n_p ; if π is a set prime divisors of n , then $n_\pi := \prod_{p \in \pi} n_p$ denotes the π -part of n . Recall that, for a finite group G , $\pi(G)$ is the set of prime divisors of $|G|$. For $p \in \pi(G)$, we denote by $\text{Syl}_p(G)$ the set of Sylow p -subgroups of G . For a subgroup $G \leq \text{Sym}(\Omega)$ we denote by $\text{Orb}(G)$ the set of G -orbits in Ω .

The proof of Theorem 1.3 is developed using ideas from [5]. First we present separately two lemmas as they are general results about finite nilpotent groups.

Lemma 2.4. *Let G be a finite nilpotent permutation group, let $p \in \pi(G)$, k be a positive integer, and $P \in \text{Syl}_p(G)$. Let $\Delta_1, \dots, \Delta_k \in \text{Orb}(P)$, $\Delta = \bigcup_{i=1}^k \Delta_i$, and L be the subgroup of G consisting of all elements fixing each Δ_i setwise. Then $L^\Delta = P^\Delta$.*

Proof. By the definition of L , the subgroup $P \leq L$, and hence $P^\Delta \leq L^\Delta$. We now prove the converse. Since G is nilpotent, we have $G = P \times H$, where H is the Hall p' -subgroup of G . Let $g \in L$, so $g = xy$ for some (unique) $x \in P$ and $y \in H$. Since $P \leq L$, we have $y = x^{-1}g \in L$.

We claim that $y^\Delta = 1$, or equivalently, that $y^{\Delta_i} = 1_{\Delta_i}$ for each $i = 1, \dots, k$. Since $y \in H \leq C_G(P)$ it follows that, for each i , y^{Δ_i} belongs to the centralizer Z_i of the transitive group $P^{\Delta_i} \leq \text{Sym}(\Delta_i)$, which is semiregular by [7, Theorem 3.2]. In particular $|Z_i|$ divides $|\Delta_i|$ which is a p -power, so Z_i is a p -group. Consequently, y^{Δ_i} is a p -element. Since $y \in H$ and $|H|$ is coprime to p , this implies that $y^{\Delta_i} = 1$, for each i , and hence that $y^\Delta = 1$, proving the claim. Thus, $g^\Delta = (xy)^\Delta = x^\Delta y^\Delta = x^\Delta \in P^\Delta$, as required. \square

Lemma 2.5. [5, Lemma 2.4] *Let $G \leq \text{Sym}(\Omega)$, where $n = |\Omega|$ and $\pi \subseteq \pi(G)$. Suppose that G is transitive and nilpotent, and let H be a Hall π -subgroup of G . Then*

- (1) *the size of every H -orbit is equal to n_π , and*
- (2) *G acts on $\text{Orb}(H)$; moreover, the kernel of this action is equal to H .*

Proof of Theorem 1.3

Let G be a finite abelian permutation group on a set Ω , and let $k \geq 2$. Then by [8, Theorem 5.8] and [8, Exercise 5.26], $G^{(k),\Omega}$ is abelian, and $\pi(G^{(k),\Omega}) = \pi(G)$. Let $p \in \pi(G)$, and let P and Q be the (unique) Sylow p -subgroups of G and $G^{(k),\Omega}$ respectively.

Claim 1. $P \leq P^{(k),\Omega} \leq Q$, and $\text{Orb}(P) = \text{Orb}(Q)$.

Proof of Claim 1. By [8, Theorem 5.8] and [8, Exercise 5.28], the group $P^{(k),\Omega}$ is a p -group, and hence $P \leq P^{(k),\Omega} \leq Q$. It remains to prove that each P -orbit is a Q -orbit. Let Δ be a P -orbit, and let Γ be the G -orbit containing Δ . By (1), $G \leq G^{(k),\Omega} \leq G^{(1),\Omega}$, and hence $G^{(k),\Omega}$ has the same orbits as G in Ω . Thus Γ is also a $G^{(k),\Omega}$ -orbit, and hence the Q -orbit Δ' containing Δ satisfies $\Delta \subseteq \Delta' \subseteq \Gamma$. The induced permutation groups G^Γ and $(G^{(k),\Omega})^\Gamma$ are both transitive and abelian, so applying Lemma 2.5 to each of these groups with Hall subgroups P^Γ, Q^Γ , respectively, yields $|\Delta| = |\Gamma|_p = |\Delta'|$. Thus $\Delta = \Delta'$, and Claim 1 is proved. \square

Claim 2. $P^{(k),\Omega} = Q$.

Proof of Claim 2. Let $(\alpha_1, \dots, \alpha_k) \in \Omega^k$, and $g \in Q$. By Theorem 2.1, there exists $h \in G$ such that

$$(\alpha_1, \dots, \alpha_k)^g = (\alpha_1, \dots, \alpha_k)^h.$$

For each $i = 1 \dots k$, let Δ_i be the Q -orbit containing α_i . Then by Claim 1, each Δ_i is also a P -orbit. Since $P \trianglelefteq G$, the group G permutes the P -orbits, and for each i , since $\alpha_i^h = \alpha_i^g \in \Delta_i$, it follows that h fixes each Δ_i setwise. Thus h lies in the subgroup L of Lemma 2.4, and setting $\Delta = \bigcup_{i=1}^k \Delta_i$, it follows from Lemma 2.4 that $h^\Delta = u^\Delta$ for some $u \in P$. Thus

$$(\alpha_1, \dots, \alpha_k)^g = (\alpha_1, \dots, \alpha_k)^h = (\alpha_1, \dots, \alpha_k)^u.$$

Since such an element $u \in P$ exists for each k -tuple of points and each $g \in Q$, it follows from Theorem 2.1 that $g \in P^{(k),\Omega}$. Thus $Q \leq P^{(k),\Omega}$, and the reverse inclusion holds by Claim 1. \square

Now we complete the proof of Theorem 1.3. Since $G^{(k),\Omega}$ is abelian, $G^{(k),\Omega}$ is the direct product of its Sylow subgroups. Further, for each $p \in \pi(G)$ it follows from Claim 2 that the unique Sylow p -subgroup of $G^{(k),\Omega}$ is $P^{(k),\Omega}$, where P is the unique Sylow p -subgroup of G . \square

3 Proof of the main results

Recall the definition of $n(G)$ given in Section 1 for a finite abelian group G . We also set $N(G) := \sum_{p \in \pi(G)} n(G_p)$. If $G \leq \text{Sym}(\Omega)$ then the *base size* $b(G, \Omega)$ of G is the smallest integer b for which there exist $\alpha_1, \dots, \alpha_b \in \Omega$ such that $G_{\alpha_1 \dots \alpha_b} = 1$. Such a set $\alpha_1, \dots, \alpha_b$ is called a *base* of G . Note that, by Theorem 2.2, $G = G^{(b+1), \Omega}$, where $b = b(G, \Omega)$.

Lemma 3.1. *Let G be a finite abelian group and suppose that G has a faithful permutation representation on a finite set Ω . Then $b(G, \Omega) \leq N(G)$, and equality holds for some Ω .*

Proof. Let $G = \prod_{p \in \pi(G)} G_p$ with $\pi(G)$ the set of primes dividing $|G|$, and G_p the Sylow p -subgroup of G , for $p \in \pi(G)$. Then, by the definition of $N(G)$ and the $n(G_p)$, G has a direct decomposition $G = H_1 \times \dots \times H_n$, with each H_i nontrivial and cyclic of prime power order, and $n = N(G)$. For each i , H_i acts regularly on $\Omega_i := H_i$ by (right) multiplication, and G acts faithfully on $\Omega := \cup_{i=1}^n \Omega_i$ (where H_j acts trivially on Ω_i for $i \neq j$). Thus the G -orbits in Ω are the sets Ω_i , and for each i the subgroup H_i acts nontrivially only on the orbit Ω_i . Thus each base must contain a point from each of the G -orbits. It follows that the base size equals $N(G)$ for this faithful permutation representation of G .

Now consider an arbitrary faithful permutation representation of G , that is, suppose that $G \leq \text{Sym}(\Omega)$. We prove by induction on $N(G)$ that G has base size at most $N(G)$. Now $H_1 = \langle h_1 \rangle \cong \mathbb{Z}_{p^a}$, for some prime p and positive integer a , and as G acts faithfully on Ω there exists $\alpha \in \Omega$ which is not fixed by $h_1^{p^a-1}$. This implies that $G_\alpha \cap H_1 = 1$. If $N(G) = 1$ then $G = H_1$ is a cyclic p -group, and $G_\alpha = 1$, so $\{\alpha\}$ is a base. Assume now that $N(G) \geq 2$ and that the assertion holds for groups X with $N(X) < N(G)$. Since $G_\alpha \cap H_1 = 1$, we have $G_\alpha \cong (G_\alpha H_1)/H_1 \leq G/H_1 \cong \prod_{i=2}^n H_i$ so $N(G_\alpha) \leq n-1 = N(G)-1$, and hence by induction, G_α has a base $\alpha_1, \dots, \alpha_s$ in $\Omega \setminus \{\alpha\}$ with $s \leq N(G) - 1$. Then $\alpha_1, \dots, \alpha_s, \alpha$ is a base for G in Ω , and the result follows by induction. \square

We now prove Theorem 1.2 in the case of p -groups. The second part of the lemma is proved using a construction developed from ideas in the book of Chen and Ponomarenko [4, Proposition 2.2.26]. An element $\tau \in \text{Sym}(\Omega)$ is called a *cycle* if it is not the identity and has exactly one cycle of length greater than 1 in its disjoint cycle representation; the length of this cycle is

denoted $|\tau|$. Two cycles are said to be *independent* if the sets of points they move are disjoint.

Lemma 3.2. *Let G be a finite abelian p -group with $|G| > 1$. Then G is totally $(n(G) + 1)$ -closed, but is not totally $n(G)$ -closed.*

Proof. Since G is an abelian p -group, $N(G) = n(G)$. By Lemma 3.1, if G is faithfully represented as a subgroup of $\text{Sym}(\Omega)$, then $b := b(G, \Omega) \leq n(G)$, and by Theorem 2.2, $G = G^{(b+1), \Omega}$. It follows from (1) that $G = G^{(n(G)+1), \Omega}$. Since this holds for all faithful permutation representations of G , G is totally $(n(G) + 1)$ -closed.

As discussed in Section 1, $G \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \dots \times \mathbb{Z}_{d_n}$, with $d_1 > 1$, $d_i | d_{i+1}$ for $1 \leq i < n$, and $n = n(G)$. Let Ω be a set of size $d_1 + \sum_{i=1}^n d_i$, and let $\tau_0, \tau_1, \dots, \tau_n \in \text{Sym}(\Omega)$ be pairwise independent cycles on Ω such that $|\tau_0| = d_1$, and $|\tau_i| = d_i$ for $i = 1 \dots n$. Let $H_1 = \langle \tau_0 \tau_1 \rangle$ and $H_i = \langle \tau_0^{-1} \tau_i \rangle$ for $i = 2 \dots n$, and let

$$H = \langle H_1, \dots, H_n \rangle.$$

We claim that $H \cong G$. Indeed, the groups H_i commute, and an easy proof by induction on n shows that

$$H_i \cap \langle H_1, \dots, H_{i-1}, H_{i+1}, \dots, H_n \rangle = 1, \text{ for } i = 1 \dots n.$$

Thus $H = H_1 \times \dots \times H_n$, with $H_i \cong \mathbb{Z}_{d_i}$ for $i = 1 \dots n$, proving the claim.

Now we will use Theorem 2.1 to show that $\tau_0 \in H^{(n), \Omega}$. Let $(\alpha_1, \dots, \alpha_n) \in \Omega^n$, and for $i = 0, \dots, n$, let Δ_i denote the set of points of Ω moved by τ_i , so that $\{\Delta_0, \dots, \Delta_n\}$ is the set of H -orbits in Ω . Since H has $n + 1$ nontrivial orbits, there exists $k \in \{0, 1, \dots, n\}$ such that $\Delta_k \cap \{\alpha_1, \dots, \alpha_n\} = \emptyset$. Define a permutation τ as follows:

$$\tau = \begin{cases} 1, & \text{if } k = 0, \\ \tau_0 \tau_k^{-1}, & \text{if } 1 \leq k \leq n. \end{cases}$$

By definition, $\tau \in H$. If $\tau = 1$, then both τ and τ_0 fix each of the α_i so $(\alpha_1, \dots, \alpha_n)^{\tau_0} = (\alpha_1, \dots, \alpha_n)^\tau$. On the other hand, if $\tau = \tau_0 \tau_k^{-1}$ for some k , then τ and τ_0 induce the same permutation on $\Omega \setminus \Delta_k$, and again we have $(\alpha_1, \dots, \alpha_n)^{\tau_0} = (\alpha_1, \dots, \alpha_n)^\tau$. Thus, by Theorem 2.1, $\tau_0 \in H^{(n), \Omega}$. By the construction, $\tau_0 \notin H$, and hence $H \neq H^{(n), \Omega}$. Thus G is not totally n -closed. \square

Remark 3.3. Theorem 1.1 follows from Lemma 3.2 since, for each integer $k \geq 2$, there are infinitely many finite abelian p -groups with k invariant factors.

Finally we prove Theorem 1.2 for an arbitrary finite abelian group G with $|G| > 1$. Suppose that G is faithfully represented on a set Ω . Since $n(G) = \max_{p \in \pi(G)} n(G_p)$, every Sylow subgroup G_p of G is $(n(G) + 1)$ -closed by Lemma 3.2, and hence, by Theorem 1.3, we have $G^{(n(G)+1), \Omega} = G$. Thus G is totally $(n(G) + 1)$ -closed.

Set $n := n(G)$. If $n = 1$ then, since $|G| > 1$, it follows from Remark 2.3 that G is not totally 1-closed. Thus we may assume that $n \geq 2$. Now $n(G) = \max_{p \in \pi(G)} n(G_p)$, and hence we have $n = n(G_q)$ for some $q \in \pi(G)$. By Lemma 3.2, G_q is not totally n -closed, so there exists a set Ω_q such that G_q acts faithfully on Ω_q and $G^{(n), \Omega_q} \neq G_q$. There is nothing further to prove if $G = G_q$ so we may assume that $|\pi(G)| \geq 2$. For each $p \in \pi(G) \setminus \{q\}$, let $\Omega_p = G_p$, and consider G_p acting regularly on Ω_p by right multiplication. Thus G acts faithfully on $\Omega := \cup_{p \in \pi(G)} \Omega_p$. Since $n \geq 2$, it follows from Theorem 1.3 that

$$G^{(n), \Omega} = \prod_{p \in \pi(G)} (G_p)^{(n), \Omega_p} = (G_q)^{(n), \Omega_q} \times \prod_{\substack{p \in \pi(G) \\ p \neq q}} (G_p)^{(n), \Omega_p},$$

which is not equal to G , because $(G_q)^{(n), \Omega_q} > G_q$ and for every $p \in \pi(G), p \neq q$ the group G_p is n -closed as a regular group. Thus, G is not totally n -closed, and the proof of Theorem 1.2 is complete.

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