

LOCALLY TRIANGULAR GRAPHS AND NORMAL QUOTIENTS OF THE n -CUBE

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ABSTRACT. For an integer $n \geq 2$, the triangular graph has vertex set the 2-subsets of $\{1, \dots, n\}$ and edge set the pairs of 2-subsets intersecting at one point. Such graphs are known to be halved graphs of bipartite rectagraphs, which are connected triangle-free graphs in which every 2-path lies in a unique quadrangle. We refine this result and provide a characterisation of connected locally triangular graphs as halved graphs of normal quotients of n -cubes. To do so, we study a parameter that generalises the concept of minimum distance for a binary linear code to arbitrary automorphism groups of the n -cube.

1. INTRODUCTION

For an integer $n \geq 2$, the *triangular graph* T_n has vertex set the 2-subsets of $\{1, \dots, n\}$ and edge set the pairs of 2-subsets intersecting at one point. A finite simple undirected graph Γ is *locally triangular* (respectively *locally T_n*) if for every vertex $u \in V\Gamma$, the graph induced by the neighbourhood $\Gamma(u)$ is isomorphic to a triangular graph (respectively T_n).

Various classifications of locally triangular graphs with symmetry exist, including those that are strongly regular [6], 1-homogeneous [5, Theorem 4.4], and locally rank 3 [1]. These characterisations only admit a few families of graphs, but there are many more examples of locally triangular graphs; for instance, any connected component of the distance 2 graph of a coset graph of a binary linear code with minimum distance at least 7 is locally triangular (cf. Sect. 2 for definitions).

Locally triangular graphs are closely related to *rectagraphs*, which are connected triangle-free graphs with the property that every path of length 2 lies in a unique quadrangle. Specifically, a halved graph of a bipartite rectagraph with $c_3 = 3$ is locally triangular, and conversely, every connected locally triangular graph is a halved graph of a bipartite rectagraph with $c_3 = 3$ by [4, Proposition 4.3.9] (cf. Sect. 2 for definitions). Rectagraphs were first named by Perkel [9] and have been studied by various authors including [1, 3, 8]. Bipartite rectagraphs also have links to geometry, for such graphs are the incidence graphs of semiplanes.

A large family of rectagraphs are quotients of the n -cube Q_n by [4, Proposition 4.3.6]. For a graph Γ and a partition \mathcal{B} of $V\Gamma$, the *quotient graph* $\Gamma_{\mathcal{B}}$ is the simple graph with vertex set \mathcal{B} , where distinct B_1 and B_2 in \mathcal{B} are adjacent whenever there exist $x_1 \in B_1$ and $x_2 \in B_2$ such that x_1 and x_2 are adjacent in Γ . If \mathcal{B} is the set of orbits of $K \leq \text{Aut}(\Gamma)$, then $\Gamma_{\mathcal{B}}$ is a *normal quotient* of Γ , and we write Γ_K for $\Gamma_{\mathcal{B}}$. Normal quotients are particularly well-behaved, for we retain some control of their automorphism groups, valencies and local actions.

Our first theorem improves the known characterisation of connected locally triangular graphs as halved graphs of bipartite rectagraphs. Note that every connected locally triangular graph is locally T_n for some n by [4, Proposition 4.3.9], so it suffices to consider connected locally T_n graphs. Precise definitions for terms in the following results, including the parameter d_K , are given in Sects. 2 and 3.

Theorem 1.1. *Let Γ be a graph. Let $n \geq 2$. The following are equivalent.*

- (i) Γ is a connected locally T_n graph.
- (ii) Γ is a halved graph of $(Q_n)_K$ for some $K \leq \text{Aut}(Q_n)$ such that K is even and $d_K \geq 7$.

Moreover, the group K in (ii) is unique up to conjugacy in $\text{Aut}(Q_n)$.

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In fact, K acts semiregularly on VQ_n and is therefore a 2-group. The even subgroups of $\text{Aut}(Q_n)$ are essentially those groups K for which $(Q_n)_K$ is bipartite (cf. Lemma 4.1), and the parameter d_K , first defined in [7], extends the concept of minimum distance for a binary linear code to arbitrary subgroups of $\text{Aut}(Q_n)$. Note that either $d_K \leq n$ or $d_K = \infty$. Note also that a connected locally T_n graph can be simultaneously a halved graph of a bipartite rectagraph and the distance 2 graph of a non-bipartite rectagraph (cf. Lemmas 4.1 and 5.1).

In order to prove Theorem 1.1, we first explore the relationship between the parameter d_K and the normal quotient $(Q_n)_K$. It is well known that the n -cube Q_n ($n \geq 1$) is a regular graph of valency n with $a_{i-1} = 0$ and $c_i = i$ for all i . In our next result, we see that the parameter d_K measures the extent to which $(Q_n)_K$ locally approximates the n -cube.

Theorem 1.2. *Let $K \leq \text{Aut}(Q_n)$ and ℓ a positive integer. The following are equivalent.*

- (i) $(Q_n)_K$ is regular of valency n with $a_{i-1} = 0$ and $c_i = i$ for $1 \leq i \leq \ell$.
- (ii) $d_K \geq 2\ell + 1$.

Rectagraphs are precisely the connected graphs with $a_1 = 0$ and $c_2 = 2$, and they are always regular by [4, Proposition 1.1.2]. As a corollary of Theorem 1.2, [4, Proposition 4.3.6] and [1, Proposition 3.4], we obtain a characterisation of a large family of rectagraphs as normal quotients of n -cubes, from which Theorem 1.1 quickly follows. This result was proved with the extra assumption of distance-regularity in [7, Lemma 4].

Corollary 1.3. *Let Π be a graph. The following are equivalent.*

- (i) Π is a rectagraph of valency n with $a_2 = 0$ and $c_3 = 3$.
- (ii) $\Pi \simeq (Q_n)_K$ for some $K \leq \text{Aut}(Q_n)$ such that $d_K \geq 7$.

We also prove the following (cf. [1, Proposition 3.4] and [7, Lemma 5]).

Theorem 1.4. *Let $K, L \leq \text{Aut}(Q_n)$ where $d_K \geq 5$. Then $(Q_n)_K$ and $(Q_n)_L$ are isomorphic if and only if K and L are conjugate in $\text{Aut}(Q_n)$. In particular, $\text{Aut}((Q_n)_K) = N_{\text{Aut}(Q_n)}(K)/K$.*

As a consequence of Theorem 1.1, we can determine the automorphism group of a connected locally T_n graph. In the following, E_n denotes the set of vectors in \mathbb{F}_2^n with even weight, and $E_n : S_n$ is an index 2 subgroup of $\text{Aut}(Q_n) = \mathbb{F}_2 \wr S_n$.

Theorem 1.5. *Let Γ be a halved graph of $(Q_n)_K$ where $K \leq \text{Aut}(Q_n)$ is even with $d_K \geq 7$ and $n \geq 5$. Then $\text{Aut}(\Gamma) = N_{E_n : S_n}(K)/K$.*

When C is a binary linear code in \mathbb{F}_2^n , the coset graph of C is the normal quotient $(Q_n)_C$, and if C has minimum distance at least 5, then $(Q_n)_C$ is a rectagraph. Moreover, as noted earlier, if C has minimum distance at least 7, then any connected component of the distance 2 graph of $(Q_n)_C$ is locally T_n . Every such rectagraph and locally T_n graph is vertex-transitive. However, in general there are groups $K \leq \text{Aut}(Q_n)$ for which neither $(Q_n)_K$ nor a corresponding locally T_n graph is vertex-transitive (cf. Examples 3.12 and 5.2); in fact, in the examples we give, the parameter d_K is nearly maximal.

Another property of a binary linear code C is that when $(Q_n)_C$ is bipartite and C has minimum distance at least 2, the halved graphs of $(Q_n)_C$ are isomorphic. Indeed, this is a consequence of the vertex-transitivity of $(Q_n)_C$. However, this is not true for arbitrary subgroups of $\text{Aut}(Q_n)$ (cf. Example 4.4). In light of Theorem 1.1, it would be interesting to classify those $K \leq \text{Aut}(Q_n)$ with $d_K \geq 2$ for which the halved graphs of a bipartite $(Q_n)_K$ are isomorphic; we prove that this holds for every such K when n is odd (cf. Corollary 4.3).

This paper is organised as follows. In Sect. 2, we give some notation and basic definitions. In Sect. 3, we first define and analyse the parameter d_K , and then we prove Theorem 1.2, Corollary 1.3 and Theorem 1.4. In Sect. 4, we consider some properties of the distance 2 graph of $(Q_n)_K$, and in Sect. 5, we prove Theorems 1.1 and 1.5.

2. NOTATION AND BASIC DEFINITIONS

All groups in this paper are finite, all actions are written on the right, and all graphs are finite and undirected with no multiple edges or loops. Basic graph theoretical terminology may be found in [4].

For groups G and H , we denote a semidirect product of G with H (with normal subgroup G) by $G : H$, and if H acts faithfully on $[n] := \{1, \dots, n\}$, then we denote the wreath product of G with H by $G \wr H$. If G acts on Ω and $\omega \in \Omega$, then G_ω denotes the pointwise stabiliser of ω in G and ω^G the orbit of G containing ω . We say that G is *transitive* if $\omega^G = \Omega$, and *semiregular* if $G_\omega = 1$ for all $\omega \in \Omega$. The symmetric group on n points is denoted by S_n .

Let Γ be a graph. We write $V\Gamma$ for the vertex set of Γ , $E\Gamma$ for the edge set of Γ , and $\text{Aut}(\Gamma)$ for the automorphism group of Γ . If $\text{Aut}(\Gamma)$ is transitive on $V\Gamma$, then Γ is *vertex-transitive*. The distance between $u, v \in V\Gamma$ is denoted by $d_\Gamma(u, v)$. For $u \in V\Gamma$ and $i \geq 0$, we define $\Gamma_i(u) := \{v \in V\Gamma : d_\Gamma(u, v) = i\}$ and $\Gamma(u) := \Gamma_1(u)$. For $u, v \in V\Gamma$ such that $d_\Gamma(u, v) = i$, let $c_i(u, v) := |\Gamma_{i-1}(u) \cap \Gamma(v)|$ and $a_i(u, v) := |\Gamma_i(u) \cap \Gamma(v)|$. We write c_i (respectively a_i) whenever $c_i(u, v)$ (respectively $a_i(u, v)$) does not depend on the choice of u and v . The complete graph on n vertices is denoted by K_n , and the complete multipartite graph with n parts of size a is denoted by $K_{n[a]}$.

The *distance 2 graph* Γ_2 of a graph Γ has vertex set $V\Gamma$, where two vertices are adjacent whenever their distance in Γ is 2. If Γ is connected but not bipartite, then Γ_2 is connected, and if Γ is connected and bipartite, then Γ_2 has exactly two connected components; these are called the *halved graphs* of Γ . The *bipartite double* $\Gamma.2$ of a graph Γ has vertex set $V\Gamma \times \mathbb{F}_2$, where vertices (u, x) and (v, y) are adjacent whenever u and v are adjacent in Γ and $x \neq y$.

For graphs Γ and Π , a surjective map $\pi : \Gamma \rightarrow \Pi$ is a *covering* if π induces a bijection from $\Gamma(x)$ onto $\Pi(x\pi)$ for all $x \in V\Gamma$. We denote by K^π the subgroup $\{g \in \text{Aut}(\Gamma) : g\pi = \pi\}$ of $\text{Aut}(\Gamma)$, where $g\pi$ denotes the composition of the functions g and π .

Let \mathbb{F}_2^n be the vector space of n -tuples over the field \mathbb{F}_2 . The *weight* $|u|$ of a vector $u \in \mathbb{F}_2^n$ is the number of non-zero coordinates in u , and the *Hamming distance* of $u, v \in \mathbb{F}_2^n$ is the number of coordinates at which u and v differ, or equivalently, $|u + v|$. For an m -subset $\{i_1, \dots, i_m\}$ of $[n]$, let e_{i_1, \dots, i_m} denote the vector of weight m in \mathbb{F}_2^n whose i_j -th coordinate is 1 for $1 \leq j \leq m$. The subspace of \mathbb{F}_2^n consisting of vectors with even weight is denoted by E_n .

For $n \geq 1$, the *n -cube* Q_n is the graph with vertex set \mathbb{F}_2^n , where two vectors are adjacent whenever their Hamming distance is 1. The n -cube is a connected regular bipartite graph of valency n with parts E_n and $\mathbb{F}_2^n \setminus E_n$. Its automorphism group is $\mathbb{F}_2 \wr S_n = \mathbb{F}_2^n : S_n$, where \mathbb{F}_2^n acts on VQ_n by translation and S_n acts by permuting coordinates. A subgroup K of $\text{Aut}(Q_n)$ is *even* if $K \leq E_n : S_n$. We write elements of $\mathbb{F}_2^n : S_n$ in the form (x, σ) where $x \in \mathbb{F}_2^n$ and $\sigma \in S_n$. A *binary linear code* C is a subspace of \mathbb{F}_2^n , or, equivalently, an additive subgroup of \mathbb{F}_2^n . The *minimum distance* of C is the minimum weight of the non-zero elements of C when $C \neq 0$, and ∞ otherwise. The *coset graph* of C is the normal quotient $(Q_n)_C$.

 3. THE MINIMUM DISTANCE OF $K \leq \text{Aut}(Q_n)$

Let $K \leq \text{Aut}(Q_n)$. As in [7], we define the *minimum distance of K* , denoted by d_K , as follows:

$$d_K := \begin{cases} \min\{d_{Q_n}(x, x^k) : x \in VQ_n, 1 \neq k \in K\} & \text{if } K \neq 1, \\ \infty & \text{otherwise.} \end{cases}$$

This definition generalises the concept of minimum distance for a binary linear code, for if $C \leq \mathbb{F}_2^n$, then $d_{Q_n}(x, x^c) = d_{Q_n}(x, x + c) = |c|$ for all $x \in VQ_n$ and $c \in C$.

Observe that $d_K \geq 1$ if and only if K is semiregular, which is true if and only if $(Q_n)_K$ has order $2^n/|K|$. Moreover, $d_K = d_{g^{-1}Kg}$ for all $g \in \text{Aut}(Q_n)$, and if $K \neq 1$, then $d_K \leq n$.

We wish to find a canonical set of representatives for the vertices at distance ℓ from vertex x^K in $(Q_n)_K$. This is not always possible, but we can say the following. Note that $(Q_n)_\ell(x) = \{x + e \in VQ_n : |e| = \ell\}$ for all $x \in VQ_n$ and $\ell \in [n]$.

Lemma 3.1. *Let $K \leq \text{Aut}(Q_n)$ and $\Pi := (Q_n)_K$. Then*

$$\Pi_\ell(x^K) \subseteq \{(x + e)^K \in V\Pi : |e| = \ell\}$$

for every positive integer ℓ and $x \in VQ_n$.

Proof. Fix $x \in VQ_n$. If $\Pi_1(x^K)$ is empty, then $\Pi_\ell(x^K)$ is empty for all $\ell \geq 1$, and we are done. Otherwise, let $y^K \in \Pi_1(x^K)$. Then y^{k_1} is adjacent to x^{k_2} for some $k_1, k_2 \in K$, so $y^{k_1 k_2^{-1}}$ is adjacent to x . Thus $y^{k_1 k_2^{-1}} = x + e_i$ for some i , and $y^K = (x + e_i)^K$. Suppose that $\ell > 1$. Again, if $\Pi_\ell(x^K)$ is empty, then we are done, so we assume otherwise. Let $y^K \in \Pi_\ell(x^K)$. There exists a path $(z_0^K, z_1^K, \dots, z_\ell^K)$ of length ℓ in Π where $z_0^K = x^K$ and $z_\ell^K = y^K$. Now $z_{\ell-1}^K \in \Pi_{\ell-1}(x^K)$, so we may assume by induction that $z_{\ell-1} = x + e_{i_1, \dots, i_{\ell-1}}$ for some $i_1 < \dots < i_{\ell-1} \in [n]$. Since $y^K \in \Pi(z_{\ell-1}^K)$, there exists $i_\ell \in [n]$ such that $y^K = (z_{\ell-1} + e_{i_\ell})^K$. If $i_\ell \in \{i_1, \dots, i_{\ell-1}\}$, then there is a path between y^K and x^K of length at most $\ell - 2$, a contradiction. Thus $y^K = (x + e_{i_1, \dots, i_{\ell-1}} + e_{i_\ell})^K$ where $|e_{i_1, \dots, i_{\ell-1}} + e_{i_\ell}| = \ell$. \square \square

Next we have an elementary observation concerning d_K .

Lemma 3.2. *Let $K \leq \text{Aut}(Q_n)$. If $x^K = y^K$ for distinct $x, y \in VQ_n$, then $|x + y| \geq d_K$.*

Proof. If $x^K = y^K$ for distinct $x, y \in VQ_n$, then there exists $1 \neq k \in K$ such that $x^k = y$, so $|x + y| = d_{Q_n}(x, x^k) \geq d_K$. \square \square

The following is a straightforward consequence of Lemma 3.2 and the definition of d_K .

Lemma 3.3. *Let $1 \neq K \leq \text{Aut}(Q_n)$. If $d_K \geq 3$, then $(Q_n)_K$ has a cycle of length d_K .*

By making some assumptions on d_K , we can improve Lemma 3.1.

Lemma 3.4. *Let $K \leq \text{Aut}(Q_n)$ and $\Pi := (Q_n)_K$. If $d_K \geq 2\ell$ for some positive integer ℓ , then*

$$\Pi_\ell(x^K) = \{(x + e)^K \in V\Pi : |e| = \ell\}$$

for all $x \in VQ_n$.

Proof. Fix $x \in VQ_n$. By Lemma 3.1, $\Pi_\ell(x^K) \subseteq \{(x + e)^K \in V\Pi : |e| = \ell\}$. In particular, we are done if $\ell > n$, so we assume that $\ell \leq n$, and let $y \in VQ_n$ be such that $|x + y| = \ell$. Now $d_\Pi(x^K, y^K) \leq \ell$. If $d_\Pi(x^K, y^K) < \ell$, then $y^K = z^K$ for some $z \in VQ_n$ where $|x + z| < \ell$ by Lemma 3.1, but $y \neq z$, so $2\ell > |x + y + x + z| = |y + z| \geq d_K$ by Lemma 3.2, a contradiction. Thus $y^K \in \Pi_\ell(x^K)$. \square \square

Although $\binom{n}{\ell}$ is an upper bound on $|\{(x + e)^K \in V\Pi : |e| = \ell\}|$, these need not be equal in general, even for $d_K \geq 2\ell$. Indeed, suppose that $d_K = 2\ell$ where ℓ is a positive integer. There exists $y \in VQ_n$ and $1 \neq k \in K$ such that $d_{Q_n}(y, y^k) = 2\ell$, so there exists $x \in (Q_n)_\ell(y) \cap (Q_n)_\ell(y^k)$, but this implies there exist distinct $e, f \in VQ_n$ such that $|e| = \ell = |f|$ and $(x + e)^K = y^K = (x + f)^K$.

Given $K \leq \text{Aut}(Q_n)$, there is a natural map $\pi : Q_n \rightarrow (Q_n)_K$ defined by $x \mapsto x^K$ for all $x \in VQ_n$. The following result implies Theorem 1.2 with $\ell = 1$.

Lemma 3.5. *Let $K \leq \text{Aut}(Q_n)$ and $\Pi := (Q_n)_K$. The following are equivalent.*

- (i) *The natural map $\pi : Q_n \rightarrow \Pi$ is a covering.*
- (ii) *Π is regular of valency n .*
- (iii) *$d_K \geq 3$.*

Proof. Clearly (i) implies (ii), and (iii) implies (i) by Lemmas 3.2 and 3.4, so it remains to prove (ii) implies (iii). Suppose that $d_K \leq 2$. If $x^K = (x + e_i)^K$ for some $x \in VQ_n$ and $i \in [n]$, then $|\Pi(x^K)| < n$, and if $x^K = (x + e_{i,j})^K$ for some $x \in VQ_n$ and distinct $i, j \in [n]$, then $|\Pi((x + e_i)^K)| < n$. Thus we may assume that $d_K = 0$, in which case there exist $x \in VQ_n$ and

$1 \neq k \in K$ such that $x^k = x$. Write $k = (y, \sigma)$. Clearly $\sigma \neq 1$, so σ moves some $i \in [n]$. Let $z := x + e_i$. Now

$$z^k = (x + e_i + y)^\sigma = x^k + e_{i^\sigma} = x + e_{i^\sigma} = z + e_{i, i^\sigma},$$

so $|\Pi((z + e_i)^K)| < n$. □

Next we consider the parameters a_i and c_i .

Lemma 3.6. *Let $K \leq \text{Aut}(Q_n)$ and $\Pi := (Q_n)_K$. Let ℓ be a positive integer.*

- (i) *If $d_K \geq 2\ell$, then $a_{\ell-1} = 0$.*
- (ii) *If $d_K \geq 2\ell + 1$, then $c_\ell = \ell$.*

Proof. (i) Suppose that $d_K \geq 2\ell$, and let $x^K, y^K, z^K \in V\Pi$ be such that $y^K \in \Pi_{\ell-1}(x^K)$ and $z^K \in \Pi_{\ell-1}(x^K) \cap \Pi(y^K)$. By Lemma 3.1, we may assume that $|x + y| = |x + z| = \ell - 1$ and $z^K = (y + e_i)^K$ for some $i \in [n]$. Note that $z \neq y + e_i$. Now $2(\ell - 1) + 1 \geq |z + y + e_i| \geq d_K$ by Lemma 3.2, a contradiction. Thus $a_{\ell-1} = 0$.

(ii) Suppose that $d_K \geq 2\ell + 1$. Let $x^K, y^K \in V\Pi$ be such that $y^K \in \Pi_\ell(x^K)$. By Lemma 3.1, we may assume that $y = x + e_{i_1, \dots, i_\ell}$ for some $i_1 < i_2 < \dots < i_\ell \in [n]$. By Lemmas 3.2 and 3.4, for $1 \leq j \leq \ell$, the vertices $(x + e_{i_1, \dots, i_\ell} + e_{i_j})^K$ are pairwise distinct and lie in $\Pi_{\ell-1}(x^K) \cap \Pi(y^K)$. Thus $|\Pi_{\ell-1}(x^K) \cap \Pi(y^K)| \geq \ell$. Let $z^K \in \Pi_{\ell-1}(x^K) \cap \Pi(y^K)$. By Lemma 3.1, we may assume that $|x + z| = \ell - 1$ and $z^K = (x + e_{i_1, \dots, i_\ell} + e_i)^K$ for some $i \in [n]$. If $z \neq x + e_{i_1, \dots, i_\ell} + e_i$, then $2\ell \geq |z + x + e_{i_1, \dots, i_\ell} + e_i| \geq d_K$ by Lemma 3.2, a contradiction. Thus $z = x + e_{i_1, \dots, i_\ell} + e_i$, so $i \in \{i_1, \dots, i_\ell\}$. Hence $c_\ell = \ell$. □

The following result is a simple counting exercise; we prove it here for completeness.

Lemma 3.7. *Let Π be a regular graph of valency n , and let ℓ be a positive integer. If $a_{i-1} = 0$ and $c_i = i$ for $1 \leq i \leq \ell$, then $|\Pi_\ell(u)| = \binom{n}{\ell}$ for all $u \in V\Pi$.*

Proof. Let $u \in V\Pi$. Let $m := \min(\ell, n)$. Now $\Pi_i(u)$ is non-empty for $0 \leq i \leq m$, for if not, then there exists $0 \leq j < m$ such that $\Pi_j(u) \neq \emptyset$ and $\Pi_{j+1}(u) = \emptyset$, but $a_j = 0$ and $c_j = j$, so a vertex in $\Pi_j(u)$ has $n - j$ neighbours in $\Pi_{j+1}(u)$, a contradiction. In particular, if $\ell > n$, then $\Pi_n(u) \neq \emptyset$ and $a_n = 0$ and $c_n = n$, so $\Pi_{n+1}(u) = \emptyset$ and $|\Pi_\ell(u)| = 0 = \binom{n}{\ell}$.

Thus we may assume that $\ell \leq n$. Now $\Pi_i(u)$ is non-empty for $1 \leq i \leq \ell$. Since $a_{\ell-1} = 0$ and $c_{\ell-1} = \ell - 1$, every vertex in $\Pi_{\ell-1}(u)$ has $n - \ell + 1$ neighbours in $\Pi_\ell(u)$. Thus the number of edges between $\Pi_{\ell-1}(u)$ and $\Pi_\ell(u)$ is $(n - \ell + 1)|\Pi_{\ell-1}(u)|$. On the other hand, since $c_\ell = \ell$, the number of edges between $\Pi_{\ell-1}(u)$ and $\Pi_\ell(u)$ is $\ell|\Pi_\ell(u)|$. By induction, $|\Pi_{\ell-1}(u)| = \binom{n}{\ell-1}$, so $|\Pi_\ell(u)| = \binom{n}{\ell}$. □

Theorem 1.2. Let $\Pi := (Q_n)_K$. Suppose that Π is regular of valency n with $a_{i-1} = 0$ and $c_i = i$ for $1 \leq i \leq \ell$. Then

$$(*) \quad |\Pi_i(x^K)| = \binom{n}{i}$$

for all $x \in VQ_n$ and $1 \leq i \leq \ell$ by Lemma 3.7, so

$$(\dagger) \quad \Pi_i(x^K) = \{(x + e)^K \in V\Pi : |e| = i\}$$

for all $x \in VQ_n$ and $1 \leq i \leq \ell$ by Lemma 3.1. Moreover, $d_K \geq 3$ by Lemma 3.5, so K is semiregular and Π has $2^n/|K|$ vertices. If $K = 1$, then $d_K = \infty$, as desired, so we may assume that $K \neq 1$. In particular, $\ell < n$, or else Π has 2^n vertices by (*), in which case $K = 1$.

Let $x \in VQ_n$ and $1 \neq k \in K$. Note that $x \neq x^k$. Now $d_{Q_n}(x, x^k) > \ell$ by (\dagger), so we may write $d_{Q_n}(x, x^k) = j + \ell$ for some positive integer j . Suppose for a contradiction that $j \leq \ell$. There exists $y \in VQ_n$ such that $d_{Q_n}(x, y) = j$ and $d_{Q_n}(y, x^k) = \ell$, so $y^K \in \Pi_j(x^K) \cap \Pi_\ell(x^K)$ by (\dagger). Thus $j = \ell$. Let $x_1 := y + x$ and $x_2 := y + x^k$. Now x_1 and x_2 are distinct vectors of weight ℓ , while $(y + x_1)^K = (y + x_2)^K$, so $|\{(y + e)^K \in V\Pi : |e| = \ell\}| < \binom{n}{\ell}$, contradicting (*) and (\dagger). Hence $d_K \geq 2\ell + 1$.

Conversely, suppose that $d_K \geq 2\ell + 1$. Then Π is regular of valency n by Lemma 3.5, and $a_{i-1} = 0$ and $c_i = i$ for $1 \leq i \leq \ell$ by Lemma 3.6. \square \square

The assumption that $(Q_n)_K$ has valency n cannot be removed from the statement of Theorem 1.2, as the following example shows.

Example 3.8. For each positive $m < n$, there exists a subgroup K of $\text{Aut}(Q_n)$ with $d_K = 2$ for which $(Q_n)_K$ is regular of valency m with $a_{i-1} = 0$ and $c_i = i$ for all $i \geq 1$. Define K to be the set of vectors in E_n whose first $m - 1$ coordinates are 0. Let $\pi : Q_n \rightarrow (Q_n)_K$ be the natural map. Viewing Q_m as the subgraph of Q_n induced by $\mathbb{F}_2^m \times 0^{n-m}$, the restriction $\pi : Q_m \rightarrow (Q_n)_K$ is a graph isomorphism, and Q_m has the desired properties.

Corollary 1.3. Let Π be a rectagraph of valency n with $a_2 = 0$ and $c_3 = 3$. By [1, Lemma 3.1] (cf. [4, Lemma 4.3.5 and Proposition 4.3.6]), there exists a covering $\pi : Q_n \rightarrow \Pi$. Let $K := K^\pi = \{g \in \text{Aut}(Q_n) : g\pi = \pi\}$. Then $\Pi \simeq (Q_n)_K$ by [1, Proposition 3.4], and $d_K \geq 7$ by Theorem 1.2. The converse also follows from Theorem 1.2. \square \square

For $K \leq \text{Aut}(Q_n)$, the normaliser $N_{\text{Aut}(Q_n)}(K)$ acts naturally on $V(Q_n)_K$ by

$$(x^K)^g := (x^g)^K$$

for all $x \in VQ_n$ and $g \in N_{\text{Aut}(Q_n)}(K)$. For $d_K \geq 5$, in which case $(Q_n)_K$ is a rectagraph, every automorphism of $(Q_n)_K$ arises in this way by [1, Proposition 3.4] (cf. [7, Lemma 5]). We generalise this result (and its proof) in order to prove that isomorphisms of normal quotients arise only from conjugate groups, thereby establishing Theorem 1.4.

Proposition 3.9. *Let $K, L \leq \text{Aut}(Q_n)$. Let $\pi_K : Q_n \rightarrow (Q_n)_K$ and $\pi_L : Q_n \rightarrow (Q_n)_L$ denote the natural maps.*

- (i) *If $d_K \geq 5$ and $\varphi : (Q_n)_K \rightarrow (Q_n)_L$ is a graph isomorphism, then there exists $g \in \text{Aut}(Q_n)$ such that $g^{-1}Kg = L$ and the diagram below commutes.*
- (ii) *If $g \in \text{Aut}(Q_n)$ and $g^{-1}Kg = L$, then there exists a graph isomorphism $\varphi : (Q_n)_K \rightarrow (Q_n)_L$ such that the diagram below commutes.*

$$\begin{array}{ccc} Q_n & \xrightarrow{g} & Q_n \\ \pi_K \downarrow & & \downarrow \pi_L \\ (Q_n)_K & \xrightarrow{\varphi} & (Q_n)_L \end{array}$$

Proof. (i) Suppose that $\varphi : (Q_n)_K \rightarrow (Q_n)_L$ is a graph isomorphism. By Theorem 1.2, $(Q_n)_K$ is a rectagraph of valency n , so $(Q_n)_L$ is a rectagraph of valency n , in which case Theorem 1.2 implies that $d_L \geq 5$. Hence π_K and π_L are coverings by Lemma 3.5. Let $\theta := \pi_L \varphi^{-1}$. Now $\theta : Q_n \rightarrow (Q_n)_K$ is a covering, so there exists $y \in VQ_n$ such that $y\theta = 0^K$, and θ induces a bijection from $Q_n(y)$ onto $(Q_n)_K(0^K) = \{e_i^K : 1 \leq i \leq n\}$ by Lemma 3.4. There exists a covering $g : Q_n \rightarrow Q_n$ for which $0^g = y$ and $e_i^g = e_i^K \theta^{-1}$ for all $i \leq n$ by [1, Lemma 3.1]. Now π_K and $g\theta$ agree on $\{0\} \cup Q_n(0)$, so $\pi_K = g\theta$ by [1, Lemma 3.2]. Thus $\pi_K \varphi = g\pi_L$, and the diagram commutes. Since g is surjective, it must be injective, so $g \in \text{Aut}(Q_n)$. By [1, Lemma 2.4], $K = K^{\pi_K}$ and $L = L^{\pi_L}$. Hence for $k \in K$,

$$(g^{-1}kg)\pi_L = g^{-1}k\pi_K\varphi = g^{-1}\pi_K\varphi = g^{-1}g\pi_L = \pi_L,$$

so $g^{-1}Kg \leq L$; since $2^n/|K| = 2^n/|L|$, we conclude that $g^{-1}Kg = L$, as desired.

(ii) Suppose that $L = g^{-1}Kg$ for some $g \in \text{Aut}(Q_n)$. Define $\varphi : (Q_n)_K \rightarrow (Q_n)_L$ by $x^K \mapsto (x^g)^L$ for all $x \in VQ_n$. Clearly the diagram commutes, and φ is a graph isomorphism. \square \square

We remark that Proposition 3.9(i) cannot be improved to include $d_K \leq 4$. Observe that if $K, L \leq \text{Aut}(Q_n)$ such that $(Q_n)_K \simeq (Q_n)_L$ but K and L are not conjugate in $\text{Aut}(Q_n)$, then

$(d_K, d_L) \in \{0, 1, 2\} \times \{0, 1, 2\}$ or $\{(3, 3), (4, 4)\}$ by Lemmas 3.3, 3.5 and 3.6(i). When $n = 6$, examples of such K and L exist for each possible (d_K, d_L) by [2].

Proposition 3.9 implies that rectagraphs arising from binary linear codes are fundamentally different to those arising from subgroups of $\text{Aut}(Q_n)$ not contained in \mathbb{F}_2^n , for if C is a binary linear code in \mathbb{F}_2^n , then $g^{-1}Cg \leq \mathbb{F}_2^n$ for all $g \in \text{Aut}(Q_n)$.

Theorem 1.4. This follows from Proposition 3.9 and [1, Lemma 2.4]. \square \square

We finish this section with some examples. Recall that if $d_K = \infty$, then $(Q_n)_K$ is the n -cube, and if d_K is finite, then $d_K \leq n$. The groups with largest possible finite minimum distance are described in the following.

Example 3.10. Let $K \leq \text{Aut}(Q_n)$ where $n \geq 4$. If $d_K = n - 1$ or n , then $K = \{0, x\}$ where $x \in VQ_n$ and $|x| = n - 1$ or n respectively. In the latter case, $(Q_n)_K$ is the folded n -cube. In both cases, K is a binary linear code, so $(Q_n)_K$ is vertex-transitive.

Next we consider the minimum distance of subgroups of $\text{Aut}(Q_n)$ of order 2.

Example 3.11. Let $K \leq \text{Aut}(Q_n)$ where $|K| = 2$. Let (x, σ) be the involution in K . Write $x = (x_1, \dots, x_n)$, and let $\text{fix}(\sigma)$ be the set of fixed points of σ . Then $d_K = |\{i \in \text{fix}(\sigma) : x_i = 1\}|$.

In Example 3.10, we saw that the groups $K \leq \text{Aut}(Q_n)$ with largest possible minimum distance are binary linear codes, and so the corresponding graphs are vertex-transitive. Now we see that there exist $K \leq \text{Aut}(Q_n)$ such that d_K is large but $(Q_n)_K$ is not vertex-transitive.

Example 3.12. Suppose that $n \geq 8$. Choose $x = (x_1, \dots, x_n) \in \mathbb{F}_2^n$ with $|x| = n - 1$ or n , and let $\sigma := (i j)$ where $x_i = 1 = x_j$. Let $K := \{0, (x, \sigma)\}$. Then $K \leq \text{Aut}(Q_n)$ and $d_K = n - 3$ or $n - 2$ respectively. Let $N := N_{\text{Aut}(Q_n)}(K) = \{(y, \tau) \in \text{Aut}(Q_n) : y^\sigma = y, x^\tau = x, \sigma\tau = \tau\sigma\}$. Then $e_i^K \notin (0^K)^N$, so $(Q_n)_K$ is not vertex-transitive by Theorem 1.4.

4. DISTANCE 2 GRAPH OF $(Q_n)_K$

It is well known that for a binary linear code C with $d_C \geq 2$, the graph $(Q_n)_C$ is bipartite precisely when $C \leq E_n$. Moreover, when C is not even, the bipartite double $(Q_n)_{C.2}$ is isomorphic to $(Q_n)_{C \cap E_n}$, and for $d_C \geq 4$, every halved graph of $(Q_n)_{C.2}$ is isomorphic to the distance 2 graph of $(Q_n)_C$. These results generalise to subgroups K of $\text{Aut}(Q_n)$ with $d_K \geq 2$. Note that if K is such a group and $L := K \cap (E_n : S_n)$, then L is an even subgroup of $\text{Aut}(Q_n)$ with $d_L \geq 2$.

Lemma 4.1. Let $K \leq \text{Aut}(Q_n)$ where $d_K \geq 2$, and let $\Pi := (Q_n)_K$.

- (i) Π is bipartite with parts $\{x^K \in V\Pi : |x| \equiv i \pmod{2}\}$ for $i = 0, 1$ if and only if K is even.
- (ii) If K is not even, then $\Pi.2 \simeq (Q_n)_L$ where $L := K \cap (E_n : S_n)$.
- (iii) If K is not even and $d_K \geq 4$, then every halved graph of $\Pi.2$ is isomorphic to Π_2 .

Proof. (i) Let $B_i := \{x^K \in V\Pi : |x| \equiv i \pmod{2}\}$ for $i = 0, 1$. Suppose that K is even. If $x \in VQ_n$ and $k \in K$, then $|x| \equiv |x^k| \pmod{2}$, so B_0 and B_1 partition $V\Pi$. If x^K and y^K are adjacent and $x^K \in B_0$, then $y^K \in B_1$ by Lemma 3.1. Thus Π is bipartite with parts B_0 and B_1 .

Now suppose that K is not even. Let $r := \min\{d_{Q_n}(x, x^k) : x \in VQ_n, k \in K \setminus (E_n : S_n)\}$. Choose $x \in VQ_n$ and $k \in K \setminus (E_n : S_n)$ such that $d_{Q_n}(x, x^k) = r$. There exists a path (x_0, x_1, \dots, x_r) in Q_n such that $x_0 = x$ and $x_r = x^k$. If $x_i^K = x_j^K$ for some $0 \leq i < j < r$, then $x_i^\ell = x_j^\ell$ for some $\ell \in K$, so $d(x_i, x_i^\ell) = j - i < r$. Thus $\ell \in E_n : S_n$. Since $d_{Q_n}(x, x_i) = i$ and $d_{Q_n}(x_i, x^{k\ell^{-1}}) = d_{Q_n}(x_i^\ell, x^k) = r - j$, it follows that $d_{Q_n}(x, x^{k\ell^{-1}}) \leq r + i - j < r$. But $k\ell^{-1} \in K \setminus (E_n : S_n)$, a contradiction. Since r is odd and $r \geq d_K \geq 2$, it follows that (x_0^K, \dots, x_r^K) is an odd cycle, so Π is not bipartite.

(ii) Define a map $\varphi : (Q_n)_L \rightarrow \Pi.2$ by $x^L \mapsto (x^K, |x| \pmod{2})$ for all $x \in VQ_n$. Using Lemma 3.4, it is routine to verify that φ is a graph isomorphism.

(iii) A halved graph Γ of Π_2 has vertex set $\{(x^K, i) : x \in VQ_n\}$ for some $i \in \mathbb{F}_2$, so there is a bijection $\varphi : \Gamma \rightarrow \Pi_2$ defined by $(x^K, i) \mapsto x^K$ for all $x \in VQ_n$. Since Π has no triangles by Lemma 3.6, (x^K, i) is adjacent to (y^K, i) in Γ if and only if $d_\Pi(x^K, y^K) = 2$, so φ is a graph isomorphism. \square

It is also well known that for a binary linear code C with $d_C \geq 2$, the halved graphs of $(Q_n)_C$ are isomorphic. However, as we will see shortly, this is not always true for arbitrary subgroups K of $\text{Aut}(Q_n)$ with $d_K \geq 2$. By Lemma 4.1(ii) and (iii), it is true when $K = M \cap (E_n : S_n)$ for some non-even subgroup M of $\text{Aut}(Q_n)$ with $d_M \geq 4$. Of course, such a group M normalises K , and we can generalise this observation as follows.

Proposition 4.2. *Let $K \leq \text{Aut}(Q_n)$ be even where $d_K \geq 2$. If $N_{\text{Aut}(Q_n)}(K)$ is not even, then the halved graphs of $(Q_n)_K$ are isomorphic.*

Proof. Let Γ and Σ be the halved graphs of $(Q_n)_K$, and let $g \in N_{\text{Aut}(Q_n)}(K)$ be such that $g \notin E_n : S_n$. By Proposition 3.9(ii), the image of g in $N_{\text{Aut}(Q_n)}(K)/K$ is an automorphism of $(Q_n)_K$, and $V\Gamma^g = V\Sigma$ by Lemma 4.1(i), so Γ and Σ are isomorphic. \square

Translating by the vector $(1, \dots, 1)$ is central in $\text{Aut}(Q_n)$, so for odd n , the group $N_{\text{Aut}(Q_n)}(K)$ is not even, and we obtain the following immediate consequence of Proposition 4.2.

Corollary 4.3. *Let $K \leq \text{Aut}(Q_n)$ be even where $d_K \geq 2$. If n is odd, then the halved graphs of $(Q_n)_K$ are isomorphic.*

Next we give an example of a normal quotient whose halved graphs are not isomorphic.

Example 4.4. Let $x := e_{1,2,3,4}$ and $y := e_{1,3,6,8}$. Let $\sigma := (15)(26)(37)(48)$ and $\tau := (12)(34)(56)(78)$. Let $K := \langle (x, \sigma), (y, \tau) \rangle \leq \text{Aut}(Q_8)$. Note that K is isomorphic to the quaternion group. Now K is even and $d_K = 4$, so $\Pi := (Q_n)_K$ is bipartite. But $|\Pi_2(0^K)| = 13$ while $|\Pi_2(e_1^K)| = 14$, so the halved graphs of Π are not isomorphic by Lemma 4.1(i).

If K is an even subgroup of $\text{Aut}(Q_n)$ such that $d_K \geq 5$, then the halved graphs of $(Q_n)_K$ are regular of valency $\binom{n}{2}$ by Lemmas 3.6 and 3.7, in which case the failure we observed in Example 4.4 cannot occur. It would be interesting to determine in general when the halved graphs of a bipartite normal quotient are isomorphic, especially for those subgroups with minimum distance at least 5. We can at least say the following.

Example 4.5. Let $K \leq \text{Aut}(Q_n)$ where K is even, $|K| = 2$ and $d_K \geq 2$. Let (x, σ) be the involution in K . Then $i^\sigma = i$ for some $i \in [n]$ by Example 3.11, so $e_i \in N_{\text{Aut}(Q_n)}(K)$. Thus the halved graphs of $(Q_n)_K$ are isomorphic by Proposition 4.2.

5. LOCALLY T_n GRAPHS

In this section, we prove Theorems 1.1 and 1.5. First we give a generalisation of one direction of Theorem 1.1.

Lemma 5.1. *Let $K \leq \text{Aut}(Q_n)$ and $\Pi := (Q_n)_K$ where $n \geq 2$. If $d_K \geq 7$, then any connected component of Π_2 is locally T_n .*

Proof. Let Γ be a connected component of Π_2 , and let $x^K \in V\Gamma$. By Lemma 3.4, there is a surjective map φ from T_n to the subgraph of Γ induced by $\Gamma(x^K)$ defined by $\{i, j\} \mapsto (x + e_{i,j})^K$ for all $\{i, j\} \in VT_n$, and this map is injective by Lemma 3.2. Let $\{i, j\} \in VT_n$. By Lemma 3.4,

$$\Gamma((x + e_{i,j})^K) = \{(x + e_{i,j} + e_{\ell,m})^K \in V\Gamma : \{\ell, m\} \in VT_n\}.$$

If $(x + e_{i',j'})^K = (x + e_{i,j} + e_{\ell,m})^K$ for some $\{i', j'\}, \{\ell, m\} \in VT_n$ where $|\{i, j\} \cap \{\ell, m\}| = 0$, then $6 \geq |e_{i',j'} + e_{i,j} + e_{\ell,m}| \geq d_K$ by Lemma 3.2, a contradiction. Thus

$$\Gamma(x^K) \cap \Gamma((x + e_{i,j})^K) = \{(x + e_{\ell,m})^K \in V\Gamma : \{\ell, m\} \in VT_n, |\{i, j\} \cap \{\ell, m\}| = 1\},$$

and we conclude that φ is a graph isomorphism. \square

Theorem 1.1. Suppose that Γ is connected and locally T_n . Then Γ is a halved graph of some bipartite rectagraph Π of valency n with $c_3 = 3$ by [4, Proposition 4.3.9], so we may apply Corollary 1.3. Also, K is even by Lemma 4.1(i).

Conversely, suppose that Γ is a halved graph of $(Q_n)_K$ for some $K \leq \text{Aut}(Q_n)$ where K is even and $d_K \geq 7$. Then Γ is locally T_n by Lemma 5.1. The group K is unique up to conjugacy in $\text{Aut}(Q_n)$ by Theorem 1.4. □ □

Theorem 1.5. Let $\Pi := (Q_n)_K$ and $N := N_{\text{Aut}(Q_n)}(K)$. Now $\text{Aut}(\Pi) = N/K$ by Theorem 1.4. Since Γ is locally T_n and $n \geq 5$, $\text{Aut}(\Gamma)$ is the setwise stabiliser of $V\Gamma$ in $\text{Aut}(\Pi)$ by [1, Lemma 4.2]. Recall that $V\Gamma = \{x^K \in V\Pi : |x| \equiv i \pmod{2}\}$ where $i = 0$ or 1 by Lemma 4.1(i). Thus $\text{Aut}(\Gamma) = N_{E_n:S_n}(K)/K$. □ □

Hence the automorphism group of a connected locally T_n graph is essentially known for $n \geq 5$. This group can also be determined for $n \leq 4$. Suppose that Γ is a connected locally T_n graph where $n \leq 4$. Then Γ is locally K_1 for $n = 2$, K_3 for $n = 3$, or $K_{3[2]}$ for $n = 4$, so Γ is K_2 , K_4 or $K_{4[2]}$ respectively (i.e., Γ is the halved n -cube). Hence $\text{Aut}(\Gamma)$ is S_2 , S_4 or $S_2 \wr S_4$ respectively. Note that $\text{Aut}(\Gamma)$ is only isomorphic to $N_{E_n:S_n}(K)/K$ when $n = 3$.

We finish by observing that there are examples of locally T_n graphs that are not vertex-transitive.

Example 5.2. Let K and N be as in Example 3.12 where n is chosen so that $(Q_n)_K$ is bipartite and $d_K \geq 7$. Let Γ be the halved graph of $(Q_n)_K$ with vertex set $\{x^K \in V(Q_n)_K : |x| \text{ is even}\}$. If $\ell \in [n] \setminus \{i, j\}$, then $e_{i,\ell}^K \notin (0^K)^N$, so Γ is not vertex-transitive by Theorem 1.5.

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