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INVOLUTION GRAPHS WHERE THE PRODUCT OF TWO ADJACENT VERTICES HAS ORDER THREE

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Abstract

An S_3-involution graph for a group G is a graph with vertex set a union of conjugacy classes of involutions of G such that two involutions are adjacent if they generate an S_3-subgroup in a particular set of conjugacy classes. We investigate such graphs in general and also for the case where $G = \mathrm{PSL}(2, q)$.

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1. Introduction

In [11] an interesting tower of graphs was discovered and investigated that is associated with the subgroup chain

$$A_5 < \mathrm{PSL}(2, 11) < M_{11} < M_{12}.$$

The smallest graph in the tower is the line graph of the Petersen graph, while the largest graph is the Johnson graph $J(12, 4)$. (The *Johnson graph* $J(n, k)$ is the graph with vertices the k-subsets of an n-set such that two k-subsets are adjacent if their intersection has size $k - 1$.) The graphs associated with $\mathrm{PSL}(2, 11)$ and M_{11} are related to the Witt designs on 11 and 12 points. A uniform description of the graphs in the tower was achieved via involutions and S_3-subgroups of the groups in the subgroup chain. This leads to the following definition.

DEFINITION 1.1. Let G be a group with a nonempty set X of involutions closed under conjugation and a nonempty set $\mathcal{S}$ of S_3-subgroups also closed under conjugation. The S_3-*involution graph* $\Gamma(G, X, \mathcal{S})$ of G with respect to X and $\mathcal{S}$ is the graph with vertices the elements of X such that two vertices x, y are adjacent if and only if $\langle x, y\rangle \in \mathcal{S}$. In order to avoid degeneracies, we always require that X is the set of all involutions contained in elements of $\mathcal{S}$.

The tower of graphs is then given by a series of S_3-involution graphs for A_5, PSL(2, 11), M_{11} and M_{12}, where, for each group, both X and $\mathcal{S}$ are single conjugacy classes. The existence of this tower suggests the following natural problem.

PROBLEM 1.2. Investigate S_3-involution graphs for other families of groups.

This paper is intended as an initial investigation of S_3-involution graphs in general and we also investigate the S_3-involution graphs arising from the simple groups PSL(2, q), for $q \geq 4$.

In Section 2, after describing some examples of S_3-involution graphs, we investigate automorphisms, connectivity and triangles. Given an S_3-involution graph $\Gamma(G, X, \mathcal{S})$, the three involutions of each $S \in \mathcal{S}$ give rise to a triangle in the graph. The following gives a sufficient condition for these to be the only triangles. See Section 2.3 for a discussion about the converse.

THEOREM 1.3. *Let G be a finite group with conjugacy class X of involutions and union of conjugacy classes $\mathcal{S}$ of S_3-subgroups. If G has no subgroups of the form $C_3^2 \rtimes C_2$ or $C_p^2 \rtimes S_3$ for some prime p, then the only triangles of $\Gamma(G, X, \mathcal{S})$ are those given by subgroups in $\mathcal{S}$.*

In Section 3 we analyse the S_3-involution graphs for PSL(2, q). In particular we determine the full automorphism groups (Theorems 3.9 and 3.11) and show that there is a duality with the graph induced on S_3-triangles if and only if $q = 11$ and 13 (Theorem 3.8). We also give the following determination of the size of the largest cliques.

THEOREM 1.4. *Let $G = \mathrm{PSL}(2, q)$ for $q \geq 4$, let X be the unique conjugacy class of involutions in G and let $\mathcal{S}$ be a conjugacy class of S_3-subgroups. The size of the largest clique is 3^e if $q = 9^e$, 4 if $q = 25^e$ and 3 otherwise.*

The definition of an S_3-involution graph is reminiscent of Fischer's 3-*transposition groups*, that is, groups generated by a conjugacy class X of involutions such that any pair of noncommuting elements of X generates an S_3. The elements of X are called 3-*transpositions*. Fischer's investigation of such groups [13, 14] led to the discovery of three new sporadic simple groups. If G is a 3-transposition group with class X of 3-transpositions and $\mathcal{S}$ is the set of all S_3-subgroups generated by a pair of noncommuting 3-transpositions, the S_3-involution graph $\Gamma(G, X, \mathcal{S})$ is called the *diagram* of X and was used in [10, 13, 16] in the study of 3-transposition groups. In fact, a 3-transposition group with class X of 3-transpositions is a quotient of the Coxeter group with Coxeter diagram the diagram of X. We are interested in S_3-involution graphs for arbitrary groups. Indeed, the groups A_5, PSL(2, 11), M_{11} and M_{12} are not 3-transposition groups.

Given a 3-transposition group G with a class X of 3-transpositions and $\mathcal{S}$ the set of all S_3-subgroups generated by pairs of elements of X, one can construct a partial linear space known as a *Fischer space*, whose points are the elements of X and lines are the sets of three involutions contained in an S_3 in $\mathcal{S}$. Moreover, Fischer spaces are precisely the partial linear spaces such that each plane is either an affine plane over GF(3) or a dual affine plane over GF(2) (see [9, 10]). The S_3-involution graph $\Gamma(G, X, \mathcal{S})$ is the collinearity graph of the Fischer space.

Another graph with vertex set a conjugacy class of 3-transpositions was also used in the investigation of 3-tranposition groups. Given a group G and conjugacy class X of involutions, the *commuting involution graph* $\mathcal{C}(G, X)$ is the graph with vertices the elements of X such that two vertices are adjacent if they commute. If G is a 3-transposition group such that X is a class of 3-transpositions and $\mathcal{S}$ is the set of all S_3-subgroups generated by pairs of elements of X, then $\mathcal{C}(G, X)$ is the complement of $\Gamma(G, X, \mathcal{S})$. Commuting involution graphs for groups other than 3-transposition groups have recently been studied in [2–4].

2. General theory

We begin this section with a few simple examples.

EXAMPLE 2.1. Let $G = S_n$, X the conjugacy class of transpositions and $\mathcal{S}$ the conjugacy class of S_3-subgroups generated by two transpositions. Note that X is a class of 3-transpositions. The map from X to the set of 2-subsets of an n-set that maps each transposition x to the set of two points moved by x yields a one-to-one correspondence between X and the vertex set of $J(n, 2)$. Moreover, two transpositions generate an S_3 if and only if their 2-cycles have a unique point in common. Thus $\Gamma(G, X, \mathcal{S}) \cong J(n, 2)$.

EXAMPLE 2.2. Let V be a $(2n)$-dimensional vector space over GF(2) equipped with an alternating form $(.,.)$. Let $G = \mathrm{Sp}(2n, 2)$ be the group of all linear transformations of V that preserve $(.,.)$ and let X be the set of all transvections contained in G, that is all maps

$$\begin{array}{rccl} t_v : & V & \to & V \\ & x & \mapsto & x + (x, v)v, \end{array}$$

where v is a nonzero vector of V. Calculations show that if $(v, u) = 0$ then $t_v t_u$ has order two, otherwise $t_u t_v$ has order three and $t_u^{t_v} = t_{u+v}$. Thus X is a class of 3-transpositions for G. Letting $\mathcal{S}$ be the set of all S_3-subgroups of G generated by pairs of noncommuting elements of X, it follows that $\Gamma(G, X, \mathcal{S})$ is isomorphic to the graph with vertices the nonzero vectors of V such that two vertices u, v are adjacent if and only if $(v, u) = 1$, that is if and only if they lie in a hyperbolic line. Moreover, the third vector of that line corresponds to the third involution of $\langle t_u, t_v \rangle$. Hence the Fischer space of G is the partial linear space whose points are the nonzero vectors of V and whose lines are the hyperbolic lines of V. This space is called a symplectic copolar space over GF(2) and is denoted by $\overline{W(2n-1, 2)}$ or $\overline{\mathrm{Sp}(2n, 2)}$.

Similar graphs can be constructed from the other classical groups which are also 3-transposition groups.

EXAMPLE 2.3. Let $G = M_{11}$. Then G has a unique conjugacy class X of involutions and two classes of S_3-subgroups. Now $|X| = 165$ and there exists a bijection from X to the set of 3-subsets of an 11-set where each involution is mapped to its set of fixed points. Also G has two conjugacy classes $\mathcal{S}_1$, $\mathcal{S}_2$ of S_3-subgroups with $|\mathcal{S}_1| = 220$ and $|\mathcal{S}_2| = 660$.

The graph $\Gamma(G, X, \mathcal{S}_1)$ is the graph associated with M_{11} in the tower investigated in [11]. By [11, Theorem 1.1] it has valency 8 and M_{11} as its full automorphism group. Moreover, $\Gamma(G, X, \mathcal{S}_1)$ is isomorphic to the graph with vertex set the set of 3-subsets of an 11-set such that two vertices are adjacent if they are disjoint and the complement of their union is a pentad in the Witt design on 11 points associated with M_{11}.

Each S_3-subgroup in the class $\mathcal{S}_2$ fixes two points of an 11-set. Thus if two involutions are adjacent in $\Gamma(G, X, \mathcal{S}_2)$ then they have two fixed points in common, that is, their sets of fixed points are adjacent in $J(11, 3)$. Each involution of G is contained in twelve S_3-subgroups of $\mathcal{S}_2$ and so is adjacent to 24 involutions in $\Gamma(G, X, \mathcal{S}_2)$. This is the valency of $J(11, 3)$ and so $\Gamma(G, X, \mathcal{S}_2) \cong J(11, 3)$.

EXAMPLE 2.4. Let $G = \mathrm{AGL}(1, 3^n)$ for some positive integer n. Then G is the group of all maps

$$\begin{array}{rccc} t_{a,b}: & \mathrm{GF}(3^n) & \to & \mathrm{GF}(3^n) \\ & x & \mapsto & ax + b \end{array}$$

for any $a, b \in \mathrm{GF}(3^n)$ with $a \neq 0$. Let $X = \{t_{-1,b} \mid b \in \mathrm{GF}(3^n)\}$, the unique conjugacy class of involutions of G. Let x be the involution $t_{-1,0}$. Note that any involution $t_{-1,b} = t_{1,-b}t_{-1,0}$ is of the form hx, where h is an element of order 3. Now G contains $(3^n - 1)/(3 - 1)$ subgroups of order three and each element of order 3 is inverted by any involution in X. Thus G has $3^n(3^n - 1)/6$ subgroups isomorphic to S_3. Moreover, G acts transitively by conjugation on the set of subgroups of order three, while, given $h, h_1, h_2 \in G$ of order 3, the S_3-subgroups $\langle h, h_1x\rangle$ and $\langle h, h_2x\rangle$ are conjugate under the element $h_1^{-1}h_2$. Thus G has a unique conjugacy class $\mathcal{S}$ of S_3-subgroups. Moreover, given two distinct involutions $x_1, x_2 \in X$, we have that x_1x_2 has order three. Hence $\Gamma(G, X, \mathcal{S})$ is the complete graph K_{3^n} on 3^n vertices.

In fact the following theorem shows that the only complete graphs that occur as S_3-involution graphs are those on 3^n vertices.

THEOREM 2.5. *Let G be a finite group with X a conjugacy class of involutions and $\mathcal{S}$ a union of conjugacy classes of S_3-subgroups. If $\Gamma(G, X, \mathcal{S})$ is the complete graph on X, then $|X| = 3^n$ for some positive integer n. Moreover, for each positive integer n there exists a group G with an S_3-involution graph isomorphic to K_{3^n}.*

PROOF. Suppose $\Gamma(G, X, \mathcal{S})$ is a complete graph. Then for all $x, y \in X$, xy has order three. Thus $\langle X\rangle$ is a 3-transposition group and by [1, (8.6)], $\langle X\rangle = N \rtimes \langle x\rangle$ for $x \in X$ and N a 3-subgroup. Thus $|x^{\langle X\rangle}|$ is a power of 3. Moreover, by Sylow's theorem,

$X = x^{\langle X\rangle}$ and so the first part follows. Example 2.4 provides the required examples for the second part. □

Our next two examples arise from subgroups of $\mathrm{AGL}(1, 3^n)$.

EXAMPLE 2.6. Let n be an even positive integer and G be the index-two subgroup of $\mathrm{AGL}(1, 3^n)$ given by all maps

$$\begin{array}{rccc} t_{a,b}: & \mathrm{GF}(3^n) & \to & \mathrm{GF}(3^n) \\ & x & \mapsto & ax+b \end{array}$$

for any $a, b \in \mathrm{GF}(3^n)$ with a a nonzero square. This group is isomorphic to $C_3^n \rtimes C_{(3^n-1)/2}$ and is a point stabilizer in $\mathrm{PSL}(2, 3^n)$. Then G still has a unique conjugacy class $X = \{t_{-1,b} \mid b \in \mathrm{GF}(3^n)\}$ of involutions. However, G has two conjugacy classes of S_3-subgroups: those for which the elements of order three are of the form $t_{1,a}$ with a a nonzero square; and those for which the elements of order three are of the form $t_{1,a}$ with a a nonsquare. Let $\mathcal{S}_1$ be the first class. Then $\langle t_{-1,b_1}, t_{-1,b_2}\rangle \in \mathcal{S}_1$ if and only if $b_2 - b_1$ is a nonzero square. Since X is in one-to-one correspondence with the elements of $\mathrm{GF}(3^n)$, it follows that $\Gamma(G, X, \mathcal{S}_1)$ is isomorphic to the graph with vertices the elements of $\mathrm{GF}(3^n)$ such that two elements are adjacent if and only if their difference is a square. (Note that $3^n \equiv 1 \bmod 4$ and so this relation is symmetric.) Thus $\Gamma(G, X, \mathcal{S}_1)$ is the Paley graph of $\mathrm{GF}(3^n)$.

EXAMPLE 2.7. Let $G = C_3^2 \rtimes C_2$, where elements of order two in G invert each element of order three. Then G has a unique conjugacy class X of involutions and four conjugacy classes $\mathcal{S}_i$ $(i = 1, 2, 3, 4)$ of S_3-subgroups, where each element from the same conjugacy class shares a common C_3-subgroup. For each i, $\Gamma(G, X, \mathcal{S}_i) \cong 3K_3$, while if $\mathcal{S} = \bigcup_i \mathcal{S}_i$ then $\Gamma(G, X, \mathcal{S}) \cong K_9$.

2.1. Automorphisms Given an S_3-involution graph $\Gamma(G, X, \mathcal{S})$, the group G acts on the set of vertices by conjugation and preserves adjacency. The following lemma collects some information about the action of G.

LEMMA 2.8. *Let G be a group with a set X of involutions closed under conjugation and a set $\mathcal{S}$ of S_3-subgroups closed under conjugation. Let $\Gamma = \Gamma(G, X, \mathcal{S})$.*

(1) *The orbits of G on the set of vertices are the conjugacy classes in X.*
(2) *The kernel of the action of G on the set of vertices is $C_G(\langle X\rangle)$.*
(3) *The orbits of G on the set of arcs are $\{(x, y) \mid \langle x, y\rangle \in \mathcal{S}_i\}$ for each conjugacy class $\mathcal{S}_i \subseteq \mathcal{S}$.*

PROOF. Part (1) is trivial. Part (2) follows as an element lies in the kernel if and only if it centralizes each element of X and hence of $\langle X\rangle$. Part (3) follows from the fact that two arcs lie in the same G-orbit if and only if the S_3-subgroups generated by their vertices are conjugate. □

The next lemma gives a natural way to find extra automorphisms of S_3-involution graphs.

LEMMA 2.9. *Let G be a group with conjugacy classes X_1, X_2 of involutions and $\mathcal{S}_1$, $\mathcal{S}_2$ of S_3-subgroups. If there exists $g \in \mathrm{Aut}(G)$ such that $X_1^g = X_2$ and $\mathcal{S}_1^g = \mathcal{S}_2$, then $\Gamma(G, X_1, \mathcal{S}_1) \cong \Gamma(G, X_2, \mathcal{S}_2)$.*

PROOF. The element g provides the isomorphism. □

COROLLARY 2.10. *If $g \in \mathrm{Aut}(G)$ fixes X_1 and $\mathcal{S}_1$ setwise, then g induces an automorphism of $\Gamma(G, X_1, \mathcal{S}_1)$.*

By [8, Theorem 9.1.2], if $n \geq 5$ then $\mathrm{Aut}(J(n, 2)) = S_n$ and so the graphs in Example 2.1 give examples where the full automorphism group of $\Gamma(G, X, \mathcal{S})$ is G. The line graph of the Petersen graph is the S_3-involution graph of A_5 and has full automorphism group S_5. This provides examples of automorphism provided by Corollary 2.10. We saw in Example 2.3 that M_{11} has an S_3-involution graph isomorphic to $J(11, 3)$. Thus its full automorphism group is S_{11}, which is much bigger than G. Similarly, M_{12} has an S_3-involution graph isomorphic to $J(12, 4)$ whose full automorphism group is S_{12}.

2.2. Connectivity We begin with the following lemma.

LEMMA 2.11. *Let $\Gamma = (G, X, \mathcal{S})$ be an S_3-involution graph. If $X = \dot{\bigcup} X_i$ with each X_i a G-conjugacy class, then Γ is the vertex disjoint union of the graphs $\Gamma(G, X_i, \mathcal{S})$.*

PROOF. Suppose that $x, y \in X$ lie in the same connected component of Γ. Then there exists a path $x = x_0, x_1, \ldots, x_d = y$ in Γ. For each $i = 0, \ldots, d-1$, we have $\langle x_i, x_{i+1}\rangle \cong S_3$ and so x_i is conjugate to x_{i+1}. Hence x is conjugate to y and the result follows. □

Note that if X is a single conjugacy class then $\Gamma(G, X, \mathcal{S})$ is not necessarily connected. For example, let $G = S_3 \mathrm{\ wr\ } S_2$ and let

$$X = \{(x, 1), (1, y) \mid x, y \in S_3, o(x) = o(y) = 2\},$$

a conjugacy class of involutions. If $\mathcal{S} = \{S_3 \times 1, 1 \times S_3\}$ then $\Gamma(G, X, \mathcal{S})$ consists of two disjoint triangles. (Note that in this example $\langle X\rangle \neq G$ and X is not an $\langle X\rangle$-conjugacy class.)

We have the following lemma.

LEMMA 2.12. *Let $N \lhd G$ and let $X \subset N$. Then $\Gamma(G, X, \mathcal{S}) = \Gamma(N, X, \mathcal{S})$. In particular, $\Gamma(G, X, \mathcal{S}) = \Gamma(\langle X\rangle, X, \mathcal{S})$.*

PROOF. Since the involutions of elements of $\mathcal{S}$ lie in X and hence N, it follows that for $S \in \mathcal{S}$ we have $S \leqslant N$. □

Lemma 2.12 leads to the following necessary condition for connectedness.

LEMMA 2.13. *If $\Gamma(G, X, \mathcal{S})$ is connected, then X is both a G-conjugacy class and an $\langle X\rangle$-conjugacy class.*

PROOF. By Lemma 2.12, $\Gamma(G, X, \mathcal{S}) = \Gamma(\langle X\rangle, X, \mathcal{S})$. By Lemma 2.11, if $\Gamma(G, X, \mathcal{S})$ is connected, then X is a G-conjugacy class; while if $\Gamma(\langle X\rangle, X, \mathcal{S})$ is connected, then X is an $\langle X\rangle$-conjugacy class. □

Example 2.7 is an example where X is both a G-conjugacy class and an $\langle X\rangle$-conjugacy class but $\Gamma(G, X, \mathcal{S}_i)$ is still disconnected.

To obtain a necessary and sufficient condition for an S_3-involution graph $\Gamma(G, X, \mathcal{S})$ to be connected with X a single conjugacy class, we need to introduce a general method for constructing vertex-transitive graphs (see for example [20]).

Let G be a group with subgroup H and D a union of double cosets of H in G which is closed under inverses. We can define a graph $\mathcal{G}(G, H, D)$ with vertices the right cosets of H in G such that Hg_1 is adjacent to Hg_2 if and only if $g_1g_2^{-1} \in D$. Then G acts vertex-transitively on $\mathcal{G}(G, H, D)$ by right multiplication preserving adjacency. Moreover, $\mathcal{G}(G, H, D)$ is connected if and only if D generates G (see [20, Theorem 7]). Conversely, given a G-vertex-transitive graph Γ with arbitrary vertex v, $\Gamma \cong \mathcal{G}(G, H, D)$, where $H = G_v$ and D is the set of all elements of G that map v to a vertex adjacent to v (see [20, Theorem 1]).

LEMMA 2.14. *Let X be a conjugacy class of involutions in G and let $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_t$ be a union of t conjugacy classes of S_3-subgroups of G such that involutions in elements of $\mathcal{S}$ are contained in X. Let $x \in X$ and, for each i, let $y_i \in X$ such that $\langle x, y_i\rangle \in \mathcal{S}_i$. Then $\Gamma(G, X, \mathcal{S})$ is connected if and only if $\langle C_G(x), y_1, \ldots, y_t\rangle = G$.*

PROOF. Let $\Gamma = \Gamma(G, X, \mathcal{S})$ and let $H = C_G(x)$. For each i, $\{x, y_i, y_ixy_i\}$ forms a triangle in Γ and y_i maps x to the adjacent vertex y_ixy_i. Moreover, the double coset Hy_iH is the set of all elements of G mapping x to a neighbour u of x such that $\langle x, u\rangle \in \mathcal{S}_i$. Thus letting $D = Hy_1H \cup \cdots \cup Hy_tH$, [20, Theorem 1] implies that $\Gamma \cong \mathcal{G}(G, H, D)$. Hence by [20, Theorem 7], Γ is connected if and only if D generates G. Since $\langle D\rangle = \langle C_G(x), y_1, \ldots, y_t\rangle$ the result follows. □

2.3. Triangles

Let $S \in \mathcal{S}$ and $T(S) = \{x, y, z\}$ be the set of three involutions in S. Then $T(S)$ is a triangle in $\Gamma(G, X, \mathcal{S})$. In particular, note that all S_3-involution graphs have girth three.

Given a graph Γ and partition $\mathcal{P}$ of the edge-set of Γ, we say that $(\Gamma, \mathcal{P})$ is a *G-arc-symmetrical decomposition* if G preserves $\mathcal{P}$, $G^{\mathcal{P}}$ is transitive, G acts transitively on the set of arcs of Γ and, for $P \in \mathcal{P}$, G_P is transitive on the set of arcs of P. Arc-symmetrical decompositions were introduced in [15].

LEMMA 2.15. *Let G be a group with conjugacy class X of involutions and conjugacy class $\mathcal{S}$ of S_3-subgroups containing elements of X. Let $\Gamma = \Gamma(G, X, \mathcal{S})$ and $\mathcal{P} = \{T(S) \mid S \in \mathcal{S}\}$. Then $(\Gamma, \mathcal{P})$ is a G-arc-symmetrical decomposition.*

PROOF. Each edge of Γ lies in a unique triangle $T(S)$, $S \in \mathcal{S}$, and so $\mathcal{P}$ is a partition of $E\Gamma$ preserved by G. Since $\mathcal{S}$ is a G-conjugacy class, $G^{\mathcal{P}}$ is transitive. Moreover, given $P \in \mathcal{P}$, $G_P^P \cong S_3$, which is transitive on the six arcs of P. Hence $(\Gamma, \mathcal{P})$ is a G-arc-symmetrical decomposition. □

An S_3-involution graph may or may not contain triangles other than those of the form $T(S)$ for $S \in \mathcal{S}$. The following theorem determines what subgroups arise in G if $\Gamma(G, X, \mathcal{S})$ has extra triangles.

THEOREM 2.16. *Let G be a finite group with conjugacy class X of involutions and union of conjugacy classes $\mathcal{S}$ of S_3-subgroups. If $\{x, y, z\}$ is a triangle in $\Gamma(G, X, \mathcal{S})$, then $\langle x, y, z\rangle \cong C_n^2 \rtimes S_3$ or $(C_{3n} \times C_n) \rtimes S_3$ for some $n \geq 1$.*

PROOF. Let $x, y \in S$ be involutions and hence $\{x, y\}$ is an edge of $\Gamma = \Gamma(G, X, \mathcal{S})$. Suppose that $z \in X$ such that $\{x, y, z\}$ is a triangle of Γ not obtained by a subgroup in $\mathcal{S}$ and let $R = \langle x, y, z\rangle$. Note that $\langle x, y\rangle < R$ and R satisfies the relations $x^2 = y^2 = z^2 = (xy)^3 = (xz)^3 = (yz)^3 = 1$, and hence is a finite quotient of the affine Coxeter group $\tilde{A}_2$, which is isomorphic to $\mathbb{Z}^2 \rtimes S_3$. We model this Coxeter group by the group $H = L \rtimes W$ where $L = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1 + x_2 + x_3 = 0\}$ and $W \cong S_3$ acts on L by naturally permuting the coordinates. Note that L is generated by $(1, -1, 0)$ and $(0, -1, 1)$. The normal subgroups of H are determined in [21, Theorem 0.2] and are either subgroups of L or kernels of homomorphisms from H to a Coxeter group induced by a homomorphism from the Coxeter graph of H. As the Coxeter graph for H is a triangle, the only possible images of such homomorphisms have Coxeter graph a single edge or a single vertex and so the quotients obtained are C_2 or S_3. Since R contains the proper subgroup $\langle x, y\rangle \cong S_3$, it follows that R is the quotient of H by a proper subgroup of L. By [21, Proposition 7.2], the normal subgroups contained in L are integral multiples of one of the lattices $\Lambda_1 = L + \mathbb{Z}\omega$ and $\Lambda_3 = 3L + \mathbb{Z}\omega$, where $\omega = (1, 1, -2) \in L$. Since $\omega \in L$ we have $\Lambda_1 = L$ and so nontrivial integral multiples of Λ_1 give the quotients $C_n^2 \rtimes S_3$ for some integer $n \geq 2$. Now $L/\Lambda_3 = \langle (1, -1, 0) + \Lambda_3\rangle \cong C_3$. Let $W = \langle \sigma, \tau\rangle$ with $\sigma^3 = \tau^2 = 1$ such that $(1, -1, 0)^\sigma = (0, 1, -1)$ and $(1, -1, 0)^\tau = (-1, 1, 0)$. Then

$$((1, -1, 0) + \Lambda_3)^\sigma = (1, -1, 0) + \Lambda_3 \quad \text{and}$$
$$((1, -1, 0) + \Lambda_3)^\tau = (2, -2, 0) + \Lambda_3.$$

Hence $H/\Lambda_3 \cong C_3^2 \rtimes C_2$. For $n \geq 2$, we have

$$L/(n\Lambda_3) = \langle (1, -1, 0) + n\Lambda_3, (1, 1, -2) + n\Lambda_3\rangle \cong C_{3n} \times C_n$$

and hence $H/(n\Lambda_3) \cong (C_{3n} \times C_n) \rtimes S_3$. □

We can now prove Theorem 1.3.

PROOF OF THEOREM 1.3. By Theorem 2.16, if G does not contain any subgroups of the form $C_n^2 \rtimes S_3$ for $n \geq 2$, or $(C_{3n} \times C_n) \rtimes S_3$ for some $n \geq 1$, then the only triangles in $\Gamma(G, X, \mathcal{S})$ are those arising from subgroups in $\mathcal{S}$. Note that if p is a prime dividing n then $C_n^2 \rtimes S_3$ contains $C_p^2 \rtimes S_3$. Also $R = (C_{3n} \times C_n) \rtimes S_3$ contains a subgroup $C_3^2 \rtimes C_2$ seen as follows. Using the notation from the proof of Theorem 1.3, if $R = H/(n\Lambda_3)$ then R contains the subgroup $\langle (n, -n, 0) + n\Lambda_3\rangle \cong C_3$, which is normalized by W and centralized by σ. Hence $H/(n\Lambda_3)$ contains a subgroup isomorphic to $C_3^2 \rtimes C_2$. □

TABLE 1. PSL(2, q) information.

q mod 12	$\|X\|$	$C_G(x)$, $x \in X$	# of classes of S_3	$\|\mathcal{S}\|$
4, 8	$q^2 - 1$	C_2^r	1	$\|G\|/6$
1	$q(q+1)/2$	D_{q-1}	2	$\|G\|/12$
3	$q(q-1)/2$	D_{q+1}	0	
5	$q(q+1)/2$	D_{q-1}	1	$\|G\|/6$
7	$q(q-1)/2$	D_{q+1}	1	$\|G\|/6$
11	$q(q-1)/2$	D_{q+1}	2	$\|G\|/12$
9	$q(q+1)/2$	D_{q-1}	2	$\|G\|/6$

The converse of Theorem 1.3 is not true. For example, we saw in Example 2.7 that if $G = C_3^2 \rtimes C_2$ then G has four conjugacy classes of S_3-subgroups and, for each of the classes $\mathcal{S}_i$, $\Gamma(G, X, \mathcal{S}_i) \cong 3K_3$ with the three triangles arising from the three subgroups in $\mathcal{S}_i$. The problem is that if, in the proof of Theorem 1.3, $R = C_3^2 \rtimes C_2$ then not all S_3-subgroups of R are R-conjugate and so are not necessarily contained in $\mathcal{S}$. The situation is similar, if $R = C_n^2 \rtimes S_3$ with 3 dividing n. On the other hand, if $(3, n) = 1$ then all S_3-subgroups of $R = C_n^2 \rtimes S_3$ are conjugate in R and hence contained in $\mathcal{S}$, so in this case we definitely obtain extra triangles.

3. PSL(2, q) graphs

In this section we investigate the S_3-involution graphs for PSL(2, q). Note that PSL(2, 2) $\cong S_3$ while PSL(2, 3) $\cong A_4$, which does not contain any S_3-subgroups.

Table 1 collates information about the involutions and S_3-subgroups of PSL(2, q). This mostly follows from a theorem of Dickson [12, pp. 285–286] (and see [17, Theorem 2.1]). When there are two conjugacy classes of S_3-subgroups, the two classes are fused in PGL(2, q). Lemma 3.1 determines the conjugacy classes of S_3-subgroups when q is a power of 3.

LEMMA 3.1. *Let G = PSL(2, 3^r). Then the number and sizes of conjugacy classes of S_3-subgroups is given by the last two columns of Table 1. In particular, G contains S_3-subgroups if and only if r is even.*

PROOF. Let $q = 3^r$ and consider an S_3-subgroup S, which contains a C_3-subgroup C. Since the projective line contains $q + 1$ points and a C_3 cannot fix four points, it follows that C must be contained in a unique point stabilizer $P \cong C_3^r \rtimes C_{(q-1)/2}$, where P is the index-two subgroup of AGL(1, 3^r) given in Example 2.6. Since S normalizes C, it fixes the unique fixed point of C and hence $S \leqslant P$. Thus $(q - 1)/2$ is even and so r is even. We saw in Example 2.6 that P has two conjugacy classes of S_3-subgroups and hence so does G. □

First we show that the S_3-involution graphs for PSL(2, q) are always connected.

TABLE 2. Valencies of $\Gamma(\mathrm{PSL}(2, q), X, \mathcal{S})$.

q mod 12	Valency
4, 8	q
1	$(q-1)/2$
5	$q-1$
7	$q+1$
11	$(q+1)/2$
9	$q-1$

LEMMA 3.2. *Let $G = \mathrm{PSL}(2, q)$, X the set of involutions in G and $\mathcal{S}$ a conjugacy class of S_3-subgroups. Then $\Gamma(G, X, \mathcal{S})$ is connected.*

PROOF. By Lemma 2.14, we need to prove that in each case $\langle C_G(x), y\rangle = G$ for some $x, y \in X$ such that $\langle x, y\rangle \in \mathcal{S}$.

Suppose first that $q \equiv 1 \bmod 4$. Then $C_G(x) \cong D_{q-1}$, which is maximal in G for $q \geq 13$. Since $y \notin C_G(x)$, we are finished if $q \geq 13$. If $G = \mathrm{PSL}(2, 5)$, then $C_G(x) \cong C_2^2$ is contained only in maximal subgroups isomorphic to A_4. However, C_2^2 contains all involutions of the A_4-subgroup, and so $y \notin A_4$. Hence $\langle C_G(x), y\rangle = G$. If $G = \mathrm{PSL}(2, 9)$, then $C_G(x) \cong D_8$ is contained only in maximal subgroups isomorphic to S_4. Looking at the permutation representation on four points, we see that a central involution in a D_8 cannot be in an S_3-subgroup of S_4. Therefore $y \notin S_4$ and so $\langle C_G(x), y\rangle = G$.

Next suppose that $q \equiv 3 \bmod 4$. Then $C_G(x) \cong D_{q+1}$, which is maximal in G for $q \geq 11$, and so $y \notin C_G(x)$ and we are finished. If $G = \mathrm{PSL}(2, 7)$, then $C_G(x) \cong D_8$ is contained only in maximal subgroups isomorphic to S_4 and we can conclude as above.

Finally, suppose $q = 2^r$. Then $C_G(x) \cong C_2^r$, which lies in a unique maximal subgroup $H \cong C_2^r \rtimes C_{q-1}$. If $y \in H$, then y lies in the unique Sylow 2-subgroup of H, and so commutes with x. Therefore $y \notin H$ and hence $\langle C_G(x), y\rangle = G$. □

THEOREM 3.3. *Let $G = \mathrm{PSL}(2, q)$, X the set of involutions in G and $\mathcal{S}$ a conjugacy class of S_3-subgroups. Then the valency of $\Gamma(G, X, \mathcal{S})$ is given by Table 2 according to the value of q mod 12. Moreover if there are two conjugacy classes of S_3-subgroups in G, the corresponding graphs are isomorphic.*

PROOF. Knowing the number $|X|$ of involutions, and hence of vertices, and the number of S_3-subgroups in a conjugacy class $\mathcal{S}$, that is, a third of the number of edges, it is immediate to deduce the valency of $\Gamma(G, X, \mathcal{S})$. If there are two conjugacy classes of S_3-subgroups, the two classes are fused in $\mathrm{PGL}(2, q)$ and hence by Lemma 2.9 the corresponding graphs are isomorphic. □

3.1. Cliques First we analyse triangles.

LEMMA 3.4. *Let* $G = \mathrm{PSL}(2, q)$ *with* q *even or* $q \equiv \pm 3 \bmod 8$. *Then the only triangles in* $\Gamma(G, X, \mathcal{S})$ *are those arising from the elements of* $\mathcal{S}$.

PROOF. If q is a power of 3 and $q \equiv \pm 3 \bmod 8$ then q is an odd power of 3 and so G contains no S_3-subgroups. For the values of q given in the statement, we can see from [12, pp. 285–286] that G does not contain any subgroups of the form $C_3^2 \rtimes C_2$ or $C_p^2 \rtimes S_3$. Hence by Theorem 1.3, the only triangles in $\Gamma(G, X, \mathcal{S})$ are those arising from the elements of $\mathcal{S}$. □

COROLLARY 3.5. *Let* $G = \mathrm{PSL}(2, q)$ *with* q *even or* $q \equiv \pm 3 \bmod 8$. *Then each edge of* $\Gamma(G, X, \mathcal{S})$ *lies in a unique triangle and the size of the largest clique is three.*

For the values of q where Γ contains triangles other than the natural ones, it is obvious to ask what is the size of the largest clique.

THEOREM 3.6. *Let* $G = \mathrm{PSL}(2, 9^e)$, X *the unique conjugacy class of involutions in* G *and* $\mathcal{S}$ *a conjugacy class of* S_3*-subgroups of* G. *Then the size of the largest clique in* $\Gamma(G, X, \mathcal{S})$ *is* 3^e.

PROOF. Let $\{x, y\}$ be an edge of $\Gamma = \Gamma(G, X, \mathcal{S})$. Let $\{x, y, z\}$ be a triangle such that $\langle x, y, z\rangle \cong S_3$ and suppose that $\{x, y, u\}$ is another triangle. By Theorem 2.16 and the subgroup structure of G, $\langle x, y, u\rangle \cong C_2^2 \rtimes S_3 \cong S_4$ or $\langle x, y, u\rangle \cong C_3^2 \rtimes C_2$. Indeed, either $\langle x, y, u\rangle$ is isomorphic to $C_n^2 \rtimes S_3$, and the only such possibility in G is for $n = 2$, or it contains an abelian subgroup $C_{3n} \times C_n$, and the only such possibility in G is for $n = 1$.

Since G has two conjugacy classes of S_4-subgroups [17] and two conjugacy classes of S_3-subgroups, it follows that all S_4-subgroups of G containing $\langle x, y\rangle$ are conjugate. There are $|G|/6$ conjugates of $\langle x, y\rangle$ and $|G|/24$ conjugates in each class of S_4-subgroups. Thus $\langle x, y\rangle$ lies in a unique S_4-subgroup. The edge $\{x, y\}$ lies in the two triangles $\{x, y, z\}$ and $\{x, y, u_1\}$ where $\langle x, y, u_1\rangle \cong S_4$. Under this isomorphism we can make the identifications $x = (1, 2)$, $y = (1, 3)$, $z = (2, 3)$ and $u_1 = (1, 4)$. Hence z is not adjacent to u_1 and so we do not obtain a clique of size four in this way.

Now $\langle x, y\rangle$ is contained in a unique parabolic subgroup $P \cong C_3^{2e} \rtimes C_{(9^e-1)/2}$ and if $\langle x, y, u\rangle \cong C_3^2 \rtimes C_2$ then $\langle x, y, u\rangle \leqslant P$. Now P is isomorphic to the index-two subgroup of $\mathrm{AGL}(1, 3^{2e})$ as in Example 2.6. Moreover, the two conjugacy classes of S_3-subgroups of P remain separate conjugacy classes in G. Thus the restriction of Γ to the involutions of P is the Paley graph of $\mathrm{GF}(9^e)$. By [7] the largest clique in the Paley graph of $\mathrm{GF}(9^e)$ has size 3^e. Moreover, [5] proved that such cliques are affine images of subfields $\mathrm{GF}(3^e)$. Thus a clique in P of size 3^e containing $x = t_{-1,a}$ and $y = t_{-1,b}$ corresponds to $(b - a)\mathrm{GF}(3^e) + a$ containing a, b and $-b - a$ (with notation for elements as in Example 2.6). Therefore $z = x^y = t_{-1,-a-b}$ also lies in this clique. Since z is not adjacent to u_1 we cannot make the clique larger by adding u_1. Thus the largest clique size of Γ is 3^e. □

THEOREM 3.7. *Let $G = \mathrm{PSL}(2, q)$ with $q \equiv \pm 1 \bmod 8$ not a power of 3. Then the size of the largest clique in $\Gamma(G, X, \mathcal{S})$ is four if $q = 25^e$ and three otherwise.*

PROOF. Assume that $\Gamma(G, X, \mathcal{S})$ contains a clique $\{x, y, z, t\}$ of size four. Let $S = \langle x, y\rangle \cong S_3$. Then S contains only one more involution, so we may assume without loss of generality that $\{x, y, t\}$ is a triangle not generating an S_3. By Theorem 2.16 and the subgroup structure of G it follows that $\langle x, y, t\rangle \cong S_4$. Note that G has two classes of S_4-subgroups each of length $|G|/24$ (see [12]) and has $|G|/6$ S_3-subgroups (in one or two conjugacy classes). An easy counting argument shows that S is contained in two S_4-subgroups B_1 and B_2. Obviously, all S_3-subgroups in a given S_4 are conjugate. Without loss of generality, we may assume that $\langle x, y, t\rangle = B_1$ and that the only other involution of B_1 adjacent to x and y is x^y. Moreover, t is not adjacent to x^y and so we must have that $\langle x, y, z\rangle = B_2$. Since z, t and x^y are the only elements of X adjacent to both x and y, we conclude that $\{x, y, z, t\}$ is a maximal clique. By symmetry, any three involutions in this clique generate an S_4-subgroup.

Let us look at the relations in $\langle x, y, z, t\rangle$. Because of the clique structure,

$$1 = x^2 = y^2 = z^2 = t^2 = (xy)^3 = (xz)^3 = (yz)^3 = (xt)^3 = (yt)^3 = (zt)^3.$$

Since x, y, z are three involutions generating an S_4 with pairs generating various S_3, we also have $1 = (xyxz)^2$ and the relations obtained from this one by permuting the three letters. Of course we have similar relations for any 3-subset of $\{x, y, z, t\}$. Now let $x' = x$, $y' = x^y$, $z' = y^z$ and $t' = z^t$. It is easily seen that $\langle x, y, z, t\rangle = \langle x', y', z', t'\rangle$. It is also easily proved from the relations described above that

$$1 = x'^2 = y'^2 = z'^2 = t'^2 = (x'y')^3 = (y'z')^3 = (z't')^3 = (z'x')^2 = (t'x')^2 = (y't')^2.$$

These relations yield a Coxeter group of type A_4, and so we have $\langle x, y, z, t\rangle = \langle x', y', z', t'\rangle \cong S_5$. Now $\mathrm{PSL}(2, q)$ contains a subgroup isomorphic to S_5 if and only if $q = 25^e$ (note that $\mathrm{PGL}(2, 5) \cong S_5$) and if G contains an S_5 then it does indeed have a clique of size four. Thus the size of the largest clique in $\Gamma(G, X, \mathcal{S})$ is four if $q = 25^e$ and three otherwise. □

Theorem 1.4 follows from Corollary 3.5 and Theorems 3.6 and 3.7.

3.2. Duality The *dual graph* of an S_3-involution graph $\Gamma(G, X, \mathcal{S})$ is the graph whose vertices are the S_3-triangles of $\Gamma(G, X, \mathcal{S})$ (that is, which correspond to elements of $\mathcal{S}$), with two triangles being adjacent if they share a vertex. It was seen in [11] that the graph $\Gamma(\mathrm{PSL}(2, 11), X, \mathcal{S})$, with X a conjugacy class of involutions and $\mathcal{S}$ a conjugacy class of S_3-subgroups, is isomorphic to its dual graph, with the duality between X and $\mathcal{S}$ induced by elements of $\mathrm{PGL}(2, 11) \setminus \mathrm{PSL}(2, 11)$. We now show that the only other value of q for which this happens is $q = 13$. Note that by Corollary 3.5 in both cases the only triangles in $\Gamma(G, X, \mathcal{S})$ are S_3-triangles.

THEOREM 3.8. *Let $G = \mathrm{PSL}(2, q)$, X the unique conjugacy class of involutions and $\mathcal{S}$ a conjugacy class of S_3-subgroups. Then $\Gamma(G, X, \mathcal{S})$ is isomorphic to its dual graph if and only if $q = 11$ or 13.*

PROOF. For $\Gamma(G, X, \mathcal{S})$ to have a duality between X and $\mathcal{S}$, the number of vertices must equal the number of S_3-subgroups in $\mathcal{S}$. It follows from Table 1 that $q = 11$ or 13. It remains to prove that in these two cases we do indeed have a duality.

When $q = 11$ or 13, the group G has two conjugacy classes $\mathcal{S}$ and $\mathcal{S}'$ of S_3-subgroups and $\Gamma(G, X, \mathcal{S}) \cong \Gamma(G, X, \mathcal{S}')$. By Theorem 3.3, $\Gamma(G, X, \mathcal{S})$ has valency six, which means that each involution is contained in three subgroups of $\mathcal{S}$ and in three subgroups of $\mathcal{S}'$. The centralizer in G of an involution x is isomorphic to D_{12} and each D_{12}-subgroup contains a unique S_3-subgroup in each conjugacy class. We will denote the unique subgroup of $\mathcal{S}$ (respectively $\mathcal{S}'$) in $C_G(x)$ by $s(x)$ (respectively $s'(x)$). Moreover, each subgroup $S \in \mathcal{S} \cup \mathcal{S}'$ is contained in a unique subgroup isomorphic to D_{12}, whose central involution will be denoted by $i(S)$. Notice that s and i are inverse bijections between involutions and elements of $\mathcal{S}$ and s' and i are inverse bijections between involutions and elements of $\mathcal{S}'$.

We claim that for $x \in X$, $T \in \mathcal{S}$, we have $x \in T$ if and only if x and $i(T)$ commute and $T \cap s'(x) = 1$.

First suppose that $x \in T$. Then $i(T)$ commutes with all elements in the D_{12} containing T and hence with x. We have $C_T(x) = \langle x \rangle$ and $x \notin s'(x) \subset C_G(x)$, and therefore $T \cap s'(x) = 1$.

Conversely, suppose that x and $i(T)$ commute, $T \cap s'(x) = 1$, and that $x \notin T$. If $x = i(T)$, then $T = s(x)$, and $s(x) \cap s'(x) \cong C_3$, a contradiction. Hence $x \neq i(T)$ and x is one of the six noncentral involutions in $C_G(i(T))$. Since $x \notin T = s(i(T))$, we have $x \in s'(i(T))$ and there exists a unique involution $t \in T$ commuting with x. Since $T \cap s'(x) = 1$ it follows that $t \in s(x)$. Now $s(x)$ and T are both in $\mathcal{S}$ and the normalizer of an element in $\mathcal{S}$ acts transitively on its three involutions. Thus there exists $g \in G$ mapping $s(x)$ to T and fixing t. Hence $g \in C_G(t)$ and g must map x onto $i(T)$. Moreover, x and $i(T)$ are two commuting involutions of $C_G(t)$ other than t, which implies that one is in $s(t)$ and the other in $s'(t)$. In other words, they are not conjugate in $C_G(t)$, and we get a contradiction. Hence the claim is proved.

Of course, by symmetry, we also have that a vertex $x \in T' \in \mathcal{S}'$ if and only if x and $i(T')$ commute and $T' \cap s(x) = 1$.

Consider the map $s' : X \to \mathcal{S}'$. By the claim, the vertices x and y are adjacent in $\Gamma(G, X, \mathcal{S})$ if and only if there exists $T \in \mathcal{S}$ such that $i(T)$ commutes with both x and y, and $T \cap s'(x) = T \cap s'(y) = 1$. Since $x = i(s'(x))$, $y = i(s'(y))$ and $s(i(T)) = T$, using the claim again implies that x and y are adjacent if and only if the involution $i(T)$ is contained in both $s'(x)$ and $s'(y)$. Therefore s' yields an isomorphism from $\Gamma(G, X, \mathcal{S})$ onto the dual graph of $\Gamma(G, X, \mathcal{S}')$. Since $\Gamma(G, X, \mathcal{S}) \cong \Gamma(G, X, \mathcal{S}')$, it follows that $\Gamma(G, X, \mathcal{S})$ is isomorphic to its dual graph. □

We note that the proof given in [11] for the duality in the case where $q = 11$ is different from the one here. It relies on a geometrical description of the graph.

3.3. Automorphism groups We now need more details on the different projective groups. The group PGL(2, q) is the group of all fractional linear transformations

$$t_{a,b,c,d} : z \mapsto \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

of the projective line $L = \{\infty\} \cup \mathrm{GF}(q)$ with the conventions that $1/0 = \infty$ and $(a\infty + b)/(c\infty + d) = a/c$. Note that $t_{a,b,c,d} = t_{a',b',c',d'}$ if and only if

$$(a, b, c, d) = \lambda(a', b', c', d') \quad \text{for some } \lambda \neq 0.$$

The group PSL(2, q) is then the set of all $t_{a,b,c,d}$ such that $ad - bc$ is a square in GF(q). The Frobenius map $\phi : z \mapsto z^p$ also acts on L and $\phi^{-1} t_{a,b,c,d} \phi = t_{a^p,b^p,c^p,d^p}$. Then $\mathrm{P\Gamma L}(2, q) = \langle \mathrm{PGL}(2, q), \phi \rangle$ and $\mathrm{P\Sigma L}(2, q) = \langle \mathrm{PSL}(2, q), \phi \rangle$.

We split the determination of the full automorphism group into the cases where q is even and odd.

THEOREM 3.9. *Let $G = \mathrm{PSL}(2, q)$ with q odd, X the set of involutions in G, $\mathcal{S}$ a conjugacy class of S_3-subgroups, and let $\Gamma = \Gamma(G, X, \mathcal{S})$. If $q \equiv 1, 9, 11 \bmod 12$, then* $\mathrm{Aut}(\Gamma) = \mathrm{P\Sigma L}(2, q)$. *If $q \equiv 5, 7 \bmod 12$, then* $\mathrm{Aut}(\Gamma) = \mathrm{P\Gamma L}(2, q)$.

PROOF. Let $A = \mathrm{Aut}(\Gamma)$ and note that $G \leqslant A \leqslant \mathrm{Sym}(V\Gamma)$. Using MAGMA [6] the result can be verified for $q = 5, 7, 9$ and 11. Thus we may assume that $q \geq 13$. For $q \equiv \pm 1 \bmod 4$, we have that the stabilizer in G of a vertex is $D_{q\mp 1}$ and $|V\Gamma| = q(q \pm 1)/2$. Moreover, by Theorem 3.3, Γ has valency $q \mp 1$ or $(q \mp 1)/2$ and so $\mathrm{Alt}(V\Gamma)$ is not contained in A. Since $q \geq 13$, the subgroups D_{q-1} and D_{q+1} are maximal in G (see [12]) and hence both G and A act primitively on $V\Gamma$. Moreover, G and A share a common nontrivial orbital given by the edges of Γ. Using [19, Theorem 1], we conclude that either $\mathrm{soc}(A) = \mathrm{PSL}(2, q)$ or $q \equiv 1 \bmod 4$ and $\mathrm{soc}(A) = A_{q+1}$. In this last case, A acts on $V\Gamma$ as on 2-sets of a $(q+1)$-set. Therefore, the stabilizer in A of a vertex has orbit sizes 1, $2(q-1)$ and $\binom{q-1}{2}$ on vertices. Since Γ has valency $q - 1$ or $(q-1)/2$, we conclude that $\mathrm{soc}(A) \neq A_{q+1}$. Hence $\mathrm{soc}(A) = \mathrm{PSL}(2, q)$ and so $G \leqslant A \leqslant \mathrm{P\Gamma L}(2, q)$.

When $q \equiv 5, 7 \bmod 12$, Table 1 states that G has a unique conjugacy class of S_3-subgroups. Thus $\mathrm{P\Gamma L}(2, q)$ fixes X and $\mathcal{S}$ and so, by Corollary 2.10, $A = \mathrm{P\Gamma L}(2, q)$.

For $q \equiv 1, 9, 11 \bmod 12$, Table 1 states that G has two conjugacy classes $\mathcal{S}$ and $\mathcal{S}'$ of S_3-subgroups. These two classes are fused by any $g \in \mathrm{PGL}(2, q) \setminus \mathrm{PSL}(2, q)$ and so such elements do not induce automorphisms of Γ. Thus if $q = p$ then we have $A = G = \mathrm{P\Sigma L}(2, q)$. When $q = p^f$ with $f \geq 2$ and $p \neq 3$, by Table 1, the PSL(2, p)-subgroup centralized by ϕ contains an S_3 and so ϕ fixes $\mathcal{S}$ and $\mathcal{S}'$. Thus again $A = \mathrm{P\Sigma L}(2, q)$. Finally when $p = 3$, by Lemma 3.1, f is even and so ϕ centralizes a PGL(2, 3)-subgroup that contains an S_3. Hence ϕ fixes $\mathcal{S}$ and $\mathcal{S}'$ and so $A = \mathrm{P\Sigma L}(2, q)$. □

To deal with the q even case, we need the following lemma about the structure of the graph.

LEMMA 3.10. *Let $G = \mathrm{PSL}(2, q)$ for $q = 2^e \geq 4$, X be the unique conjugacy class of involutions of G and $\mathcal{S}$ the unique class of S_3-subgroups. Let $\Gamma = \Gamma(G, X, \mathcal{S})$. Then X can be partitioned into $q + 1$ blocks $X_\alpha = X \cap G_\alpha$ of size $q - 1$ ($\alpha \in \mathrm{GF}(q) \cup \{\infty\}$), such that the subgraph induced by Γ on any two blocks is a matching between those blocks.*

PROOF. Let $\alpha, \beta \in \mathrm{GF}(q) \cup \{\infty\}$ with $\alpha \neq \beta$. Then $G_\alpha \cong \mathrm{AGL}(1, q) \cong C_2^e \rtimes C_{q-1}$ and $G_{\alpha,\beta} \cong C_{q-1}$. Thus each element of X lies in a unique point stabilizer of G. Moreover, since each G_α has a unique Sylow 2-subgroup and this subgroup is abelian, all involutions in X_α commute. Thus Γ is a multipartite graph with block set $\mathcal{P} = \{X_\alpha \mid \alpha \in \mathrm{GF}(q) \cup \{\infty\}\}$. Now $G_{X_\alpha} = G_\alpha$ and $G_{\alpha,\beta} = G_{X_\alpha, X_\beta} \cong C_{q-1}$ acts regularly on both X_α and X_β. Since G acts arc-transitively on Γ (Lemma 2.8), it follows that each vertex in X_α is adjacent to at most one vertex in any other block. Using the fact from Theorem 3.3 that Γ has valency q, we can conclude that the subgraph induced on $X_\alpha \cup X_\beta$ is a matching. □

THEOREM 3.11. *Let $G = \mathrm{PSL}(2, q)$ for $q \geq 4$ even, X the unique conjugacy class of involutions of G and $\mathcal{S}$ the unique class of S_3-subgroups. Let $\Gamma = \Gamma(G, X, \mathcal{S})$. Then* $\mathrm{Aut}(\Gamma) = \mathrm{P\Gamma L}(2, q)$.

PROOF. Let $A = \mathrm{Aut}(\Gamma)$. Since G has a unique conjugacy class of involutions and of S_3-subgroups it follows from Corollary 2.10 that $\mathrm{P\Gamma L}(2, q) \leqslant A$. By Lemma 3.10, Γ is a multipartite graph with block set $\mathcal{P} = \{X_\alpha \mid \alpha \in \mathrm{GF}(q) \cup \{\infty\}\}$, where $X_\alpha = X \cap G_\alpha$. Let $x \in X_\infty$. By Lemma 3.10, for each $\alpha \neq \infty$, $|\Gamma(x) \cap X_\alpha| = 1$. By Corollary 3.5, each edge of Γ lies in a unique triangle. Hence, A_x preserves a partition of $\mathcal{P}$ into blocks of size two given by the triangles containing x.

Let $y \in X_0$ be the unique element of X_0 adjacent to x and let $\{x, y, z\}$ be a triangle. Then $z \in X_\alpha$ for some $\alpha \in \mathrm{GF}(q) \backslash \{0\}$. Note that $G_{X_\infty, X_0, X_\alpha} = 1$. Now let $x' \in X_\infty$ with $x' \neq x$ and let $y' \neq y$ be the unique element of X_0 adjacent to x'. Since G is arc-transitive, there exists $g \in G$ such that $(x, y)^g = (x', y')$. Moreover, $\{x', y', z^g\}$ is the unique triangle of Γ containing $\{x', y'\}$. Since $G_{X_\infty, X_0, X_\alpha} = 1$ and g fixes X_∞ and X_0, it follows that $z^g \notin X_\infty \cup X_0 \cup X_\alpha$. Hence if $h \in A_{X_\infty, X_0, X_\alpha}$ then $h \in A_x$. Since x was arbitrary, if K is the kernel of the action of A on $\mathcal{P}$, it follows that K fixes X_∞ pointwise. Since $K \lhd A$ and A acts transitively on $\mathcal{P}$ we conclude that $K = 1$, that is, A acts faithfully on $\mathcal{P}$. Thus $\mathrm{P\Gamma L}(2, q) \leqslant A \leqslant S_{q+1}$. By [18] it follows that either $\mathrm{soc}(A) = \mathrm{PSL}(2, q)$ or $A_{q+1} \leqslant A$. If $q = 4$ then $\mathrm{PSL}(2, 4) = A_5$ and $\mathrm{P\Gamma L}(2, 4) = S_5$. Hence $\mathrm{Aut}(\Gamma) = \mathrm{P\Gamma L}(2, 4)$ in this case. Suppose now that $q \geq 8$ and $A_{q+1} \leqslant A$. Then there exists $h \in A$ that induces a 3-cycle on $\mathcal{P}$ and fixes X_∞, X_0 and X_α. As we have seen, this implies that $h \in A_x$ and so preserves a partition of $\mathcal{P}$ into blocks of size two, contradicting h inducing a 3-cycle on $\mathcal{P}$. Hence A_{q+1} is not contained in A and so $\mathrm{soc}(A) = \mathrm{PSL}(2, q)$. As $\mathrm{P\Gamma L}(2, q) \leqslant A$ it follows that $A = \mathrm{P\Gamma L}(2, q)$. □

Lemma 3.10 also enables us to determine the diameter of the S_3-involution graph of $\mathrm{PSL}(2, q)$ for q even.

THEOREM 3.12. *Let $G = \mathrm{PSL}(2, q)$ for $q \geq 4$ even, X the unique conjugacy class of involutions of G and $\mathcal{S}$ the unique class of S_3-subgroups. Let $\Gamma = \Gamma(G, X, \mathcal{S})$. Then Γ has diameter three.*

PROOF. For each $\alpha \in \mathrm{GF}(q) \cup \{\infty\}$ let $X_\alpha = X \cap G_\alpha$. By Lemma 3.10, Γ is multipartite with blocks $\{X_\alpha \mid \alpha \in \mathrm{GF}(q) \cup \{\infty\}\}$ such that the graph induced between any two blocks is a complete matching. Let $x \in X$. Without loss of generality we may assume $x \in X_\infty$. By Theorem 3.3, x has q neighbours and these are each in a different X_α, $\alpha \in \mathrm{GF}(q)$. By Corollary 3.5, each of these q vertices is adjacent to exactly one other neighbour of x. Hence each is adjacent to $q - 2$ vertices at distance 2 from x, and, by Lemma 3.10, none of these are in X_∞. Moreover, we claim that Γ contains no 4-cycle, and so there are exactly $q(q - 2)$ vertices at distance 2 from x. Therefore, the vertices at distance at most 2 from x cover exactly the involutions not in X_∞. Since the subgraph induced on two blocks of the partition is a matching, all vertices distinct from x in X_∞ are at distance 3 from x. Therefore any vertex is at distance at most 3 from x.

It remains to prove that there is no 4-cycle in Γ. Suppose that (x, y, z, t) is a 4-cycle. By Lemma 3.10, the four vertices are in four distinct blocks of the partition. Since G has one orbit on $V\Gamma$ (Lemma 2.8), we may assume that $x = t_{1,1,0,1} \in X_\infty$. A calculation shows that the unique involution in X_i ($i \in \mathrm{GF}(q)$) adjacent to x is $t_{i+1,i^2,1,i+1}$. Moreover, for $j \in \mathrm{GF}(q) \setminus \{i\}$, the image of the edge $\{t_{1,1,0,1}, t_{0,1,1,0}\}$ under

$$t_{i^2+ij,i^2+ij+j,i+j,i+j+1} \in G$$

is

$$\{t_{i+1,i^2,1,i+1}, t_{i^2+j^2+j,j^2,1,i^2+j^2+j}\}.$$

Thus the unique involution in X_j ($j \in \mathrm{GF}(q) \setminus \{i\}$) adjacent to $t_{i+1,i^2,1,i+1}$ is $t_{i^2+j^2+j,j^2,1,i^2+j^2+j}$. Hence

$$y = t_{i+1,i^2,1,i+1} \quad \text{and} \quad t = t_{i'+1,i'^2,1,i'+1}$$

for distinct $i, i' \in \mathrm{GF}(q)$. We know that $z \in G_j$ for some $j \in \mathrm{GF}(q) \setminus \{i, i'\}$. Since z is adjacent to both y and t, we have

$$z = t_{i^2+j^2+j,j^2,1,i^2+j^2+j} = t_{i'^2+j^2+j,j^2,1,i'^2+j^2+j}.$$

Hence $i^2 + j^2 + j = i'^2 + j^2 + j$, and so $i = i'$, which contradicts the fact that y and t are distinct. □

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