A CHARACTERISATION OF WEAKLY LOCALLY PROJECTIVE AMALGAMS RELATED TO A_{16} AND THE SPORADIC SIMPLE GROUPS M_{24} AND He

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Abstract. A simple undirected graph is weakly G-locally projective, for a group of automorphisms G, if for each vertex x, the stabiliser $G(x)$ induces on the set of vertices adjacent to x a doubly transitive action with socle the projective group $L_{n_x}(q_x)$ for an integer n_x and a prime power q_x. It is G-locally projective if in addition G is vertex transitive. A theorem of Trofimov reduces the classification of the G-locally projective graphs to the case where the distance factors are as in one of the known examples. Although an analogue of Trofimov's result is not yet available for weakly locally projective graphs, we would like to begin a program of characterising some of the remarkable examples. We show that if a graph is weakly locally projective with each $q_x = 2$ and $n_x = 2$ or 3, and if the distance factors are as in the examples arising from the rank 3 tilde geometries of the groups M_{24} and He, then up to isomorphism there are exactly two possible amalgams. Moreover, we consider an infinite family of amalgams of type $\mathcal{U}_n$ (where each $q_x = 2$ and $n = n_x + 1 \geq 4$) and prove that if $n \geq 5$ there is a unique amalgam of type $\mathcal{U}_n$ and it is unfaithful, whereas if $n = 4$ then there are exactly four amalgams of type $\mathcal{U}_4$, precisely two of which are faithful, namely the ones related to M_{24} and He, and one other which has faithful completion A_{16}.

1. Introduction

Let Δ be an undirected, connected locally finite graph and G be an automorphism group of Δ. For a vertex x of Δ let $G(x)$ be the stabilizer of x in G and $\Delta(x)$ be the set of neighbours of x in Δ. Let $G(x)^{\Delta(x)}$ denote the permutation group induced by $G(x)$ on $\Delta(x)$. We say that the action of G on Δ is *weakly locally projective* if the following holds: for every vertex $x \in \Delta$ there is a positive integer n_x and a prime power q_x such that

$$|\Delta(x)| = (q_x^{n_x} - 1)/(q_x - 1);$$

$$L_{n_x}(q_x) \leqslant G(x)^{\Delta(x)} \leqslant \mathrm{P\Gamma L}(n_x, q_x).$$

Here $L_{n_x}(q_x)$ is considered as a doubly transitive permutation group of degree $|\Delta(x)|$ and $\mathrm{P\Gamma L}(n_x, q_x)$ is the normalizer of $L_{n_x}(q_x)$ in the symmetric group on $\Delta(x)$. If a weakly locally projective action is also vertex-transitive it is said to be *locally projective*. Since every weakly locally projective action is edge-transitive it is easy to see that either

2010 *Mathematics Subject Classification.* 20B25, 05C25, 20D08.

The first and third authors are supported by the Australian Research Council Discovery Project Number DP120100446 while the fourth author is supported by DP130100106.

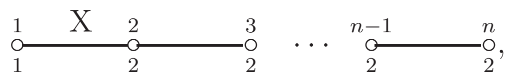
(a) the action is locally projective and there is a pair (n, q) such that $n_x = n$ and $q_x = q$ for every $x \in \Delta$, or
(b) there is bipartition $\Delta = \Delta_1 \cup \Delta_2$ of Δ and a quadruple of parameters $(n_1, q_1; n_2, q_2)$ such that $n_x = n_i$, $q_x = q_i$ whenever $x \in \Delta_i$.

The sequence (n, q) or $(n_1, q_1; n_2, q_2)$ is said to be the *type* of a locally or weakly locally projective action, respectively.

Suppose that G acts (weakly) locally projectively on Δ and $\{x, w\}$ is an edge of Δ. Then the amalgam $\mathcal{A} = \{G(x), G(w)\}$ formed by the stabilizers of the vertices incident to the edge is called a *(weakly) locally projective amalgam.* The *shape* of $\mathcal{A}$ is the type of the action of G on Δ. Since G is an automorphism group of Δ, it acts faithfully on the set of edges. Thus $G(x) \cap G(w) = G(x, w)$ does not contain a nontrivial subgroup which is normal in both $G(x)$ and $G(w)$. Amalgams with this property are called faithful.

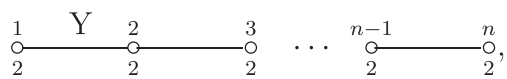
The present paper makes a modest contribution to the classification of the weakly locally projective amalgams. We were motivated by the following geometrical constructions.

Construction 1.1. Let $\mathcal{G}$ be a geometry with a diagram



for some X (see for example [8, p 2] for notation), let G be a flag-transitive automorphism group of $\mathcal{G}$ such that the stabilizer of an element of type 1 induces the full automorphism group $L_n(2)$ of the corresponding residue. Let Δ be a graph whose vertices are the elements of type 1 in $\mathcal{G}$ and two such vertices are adjacent in Δ if in $\mathcal{G}$ they are incident to a common element of type 2. Then the action of G on Δ is locally projective of type $(n, 2)$.

Construction 1.2. Let $\mathcal{G}$ be a geometry with a diagram



for some Y, let G be a flag-transitive automorphism group of $\mathcal{G}$ such that the stabilizer of an element of type 1 induces the full automorphism group $L_n(2)$ of the corresponding residue and the stabilizer of an element of type 2 induces the group $S_3 \cong L_2(2)$ on the set of elements of type 1 incident to that element. Let Δ be a graph whose vertices are the elements of type 1 and 2 in $\mathcal{G}$ and two vertices are adjacent if they are incident as elements of $\mathcal{G}$. Then Δ is bipartite and the action of G on Δ is weakly locally projective of type $(n, 2; 2, 2)$.

Remarkable locally projective amalgams can be obtained by Construction 1.1 with X being the geometry of edges and vertices of the Petersen graph. The examples include geometries of the Mathieu groups M_{22}, M_{23}, the Conway group Co_2, the Janko group J_4 and the Baby Monster group, (see [7, 8]).

Similarly remarkable weakly locally projective amalgams can be obtained by Construction 1.2 with Y being the rank 2 tilde geometry (the triple cover of the generalized quadrangle of order $(2, 2)$ associated with $3 \cdot S_6$). The examples include geometries of the Mathieu group M_{24}, the Conway group Co_1 and the Monster group M.

On the other hand, the classification of certain locally and weakly locally projective amalgams restricts the possibilities for the rank 2 residues X and Y in geometries as in Constructions 1.1 and 1.2.

Building on work of Tutte [14], in 1980 Djoković and Miller [3] classified the locally projective amalgams of type $(2,2)$. In the same year the weakly locally projective amalgams of type $(2,2;2,2)$ were classified by Goldschmidt [5].

In 1991 Trofimov [12] proved a fundamental result on locally projective amalgams. The main consequence of Trofimov's Theorem is a bound on the order of the vertex stabilizer $G(x)$ in a locally projective action of type (n,q) by a function of n and q. Furthermore Trofimov determined the possibilities for the so-called *distance factors* $G_i(x)/G_{i+1}(x)$ (here $G_i(x)$ is the stabilizer in $G(x)$ of all the vertices whose distance from x is at most i). All the possibilities for the distance factors are realized in known examples (see [13]).

Thus Trofimov's theorem reduces the classification problem of locally projective amalgams to its 'restricted' version when the distance factors $G_i(x)/G_{i+1}(x)$ are assumed to be as in one of the known examples.

The restricted problem was solved in 2004 by Ivanov and Shpectorov in [10] for the amalgams obtained by Construction 1.1. It is worth mentioning that a few new amalgams were found within this project whose distance factors are exactly as in the examples known before. As a consequence of the classification it was shown that the residue X possesses a covering onto the geometry of edges and vertices of one of the following three graphs: the complete graph K_4 on 4 vertices; the complete bipartite graph $K_{3,3}$ on 6 vertices; the Petersen graph.

The analogue of Trofimov's theorem for weakly locally projective actions is not yet available. Nevertheless we would like to solve the restricted problem for such amalgams coming from Construction 1.2. In this paper we assume that Δ is a graph, G is an automorphism group of Δ whose action on Δ is weakly locally projective of type $(3,2;2,2)$ and if x is a vertex of valency 7 in Δ then the distance factors are as follows:

$$\begin{array}{rcl} G(x)/G_1(x) & \cong & L_3(2); \\ G_1(x)/G_2(x) & \cong & C_2^3; \\ G_2(x)/G_3(x) & \cong & C_2^3; \\ G_3(x)/G_4(x) & \cong & 1; \\ G_4(x)/G_5(x) & \cong & C_2; \\ G_5(x) & \cong & 1. \end{array} \tag{1.1}$$

This pattern appears in graphs obtained via Construction 1.2 from the rank 3 tilde geometries of the groups M_{24} and He. Our main result is the following theorem.

Theorem 1.3. *Let Δ be a connected graph with automorphism group G whose action is weakly locally projective of type $(3,2;2,2)$. Let x and w be adjacent vertices such that x has valency 7 and w has valency 3. Moreover, suppose that the distance factors are as in (1.1). Then the following hold:*

(1) $G(x) \cong 2_+^{1+6} \rtimes L_3(2)$, *the centraliser of a 2A-involution in* $L_5(2)$*;*

(2) *$G(w) = N \rtimes K$ where $N \cong C_2^4$ and $K \cong C_2^4 \rtimes (S_3 \times S_3)$, the stabiliser in $L_4(2)$ of a 2-dimensional subspace of N;*
(3) *$G(x) \cap G(w) \cong 2_+^{1+6} \rtimes S_4$;*
(4) *Up to isomorphism, there are two faithful amalgams $\mathcal{A} = \{G(x), G(w)\}$.*

We see in Theorem 4.7 that given $G_1 \cong 2_+^{1+6} \rtimes L_3(2)$ and $G_2 \cong N \rtimes K$ such that $G_1 \cap G_2 \cong 2_+^{1+6} \rtimes S_4$, there are four amalgams $\{G_1, G_2\}$. However, only two arise from weakly locally projective actions as in the remaining two, $G_1 \cap G_2$ contains a nontrivial subgroup which is normal in both G_1 and G_2. For one of the two faithful amalgams the sporadic groups M_{24} and He are completions, for the other A_{16} is a completion, see Section 5 for details. In fact, Theorem 4.7 applies to the infinite sequence of amalgams $(\mathcal{U}_n)_{n\geqslant 4}$ and we see that each member of this sequence is a unique (unfaithful) amalgam, unless $n = 4$. This highlights how remarkable the amalgams related to the Held and Mathieu groups are.

2. Distance two graphs and triangles

Since we are studying the structure of the vertex stabilisers, as opposed to the structure of Δ, we may assume that Δ is a tree [11, Chapter 1,§4]. Moreover, we do our analysis in the *distance two graph* of Δ, that is, the graph with the same vertex set as Δ but where two vertices are adjacent if and only if they are at distance two in Δ. Since Δ is connected and bipartite, the distance two graph of Δ has two connected components, one containing the vertices of Δ of valency 7 and the other containing the vertices of Δ of valency 3. For a graph Σ with vertex v, we denote the set of vertices at distance i from v by $\Sigma_i(v)$ and write $\Sigma(v)$ for $\Sigma_1(v)$. For vertices u and v we will write $u \sim v$ to indicate that $u \in \Sigma(v)$. If a group R acts on Σ and L is a subgroup of R which stabilises a set Π of vertices of Σ, we write L^{Π} for the permutation group induced on Π by L. Typically we use this notation where L fixes a vertex u and $\Pi = \Sigma(u)$.

We have the following lemma.

Lemma 2.1. *Let Δ be a connected tree with automorphism group G whose action is weakly locally projective of type $(3, 2; 2, 2)$ and suppose that the distance factors are as in (1.1). Let Γ be the connected component of the distance two graph of Δ which contains all the vertices of valency 7. Let $x \in V\Gamma$ and let $Q_i(x)$ be the stabiliser in $G(x)$ of all vertices of Γ of distance at most i from x. The following all hold.*

(A1) *Γ is a connected graph of valency 14.*
(A2) *Every edge of Γ is in a unique triangle.*
(A3) *G is an arc-transitive automorphism group of Γ,*
(A4) *$G(x)$ induces $L_3(2)$ on the set of seven triangles in Γ containing x.*
(A5) *For a triangle T of Γ, the setwise stabiliser in G of T induces S_3 on T.*
(A6) *The kernel of $G(x)^{\Gamma(x)}$ acting on the set of triangles containing x is C_2^3.*

(A7) *The distance factors for* Γ *are as follows:*

$$\begin{aligned} G(x)/Q_1(x) &\cong C_2^3.L_3(2);\\ Q_1(x)/Q_2(x) &\cong C_2^3;\\ Q_2(x)/Q_3(x) &\cong C_2;\\ Q_3(x) &\cong 1. \end{aligned}$$

Proof. Since Δ is a tree, (A1) holds and given two vertices x, y of valency 7 in Δ that are at distance two there is a unique vertex z of Δ at distance two from both x and y. Thus (A2) holds. Since for each vertex v of Δ we have that $G(v)$ is 2-transitive on $\Delta(v)$, it follows that $G(x)$ acts transitively on the set of paths of length two starting at x. Moreover, G is a vertex-transitive automorphism group of Γ and since arcs in Γ correspond to paths of length two in Δ with initial vertex having valency 7 it follows that G is an arc-transitive group of automorphisms of Γ. Thus (A3) holds.

For a given vertex $x \in V\Gamma$, the action of $G(x)$ on the set of triangles containing x is equivalent to $G(x)^{\Delta(x)} \cong G(x)/G_1(x)$ and so by (1.1), (A4) holds. Moreover, since $G(v)^{\Delta(v)} \cong L_2(2) \cong S_3$ for each vertex v of Δ of valency three (A5) holds. For a positive integer i we have $\Gamma_i(x) = \Delta_{2i}(x)$ and $Q_i(x) = G_{2i}(x)$. Hence the kernel of $G(x)^{\Gamma(x)}$ acting on the set of triangles containing x is $G_1(x)/G_2(x)$ and so (A6) holds. We obtain (A7) from (1.1). □

Note that each triangle in Γ corresponds to a vertex of valency 3 in Δ. Thus the problem of identifying the isomorphism type of $\{G(x), G(w)\}$ for a given pair x, w of adjacent vertices in Δ is equivalent to identifying the isomorphism type of $\mathcal{A} = \{G(x), G\{T\}\}$ where $G\{T\}$ is the setwise stabiliser in G of the triangle $T = \{x, y, z\}$ in Γ. Note in particular that $\mathcal{A}$ is faithful since $\langle G(x), G\{T\}\rangle = G$ by connectivity.

3. Determining the structure of $G(x)$ and $G\{T\}$

Throughout this section we assume the following hypothesis:

Γ is a connected graph on which G acts faithfully such that (A1)–(A7) of Lemma 2.1 hold.

We first establish some notation for the action of G on Γ which will hold for the rest of the paper. We let $Q(x)$ denote the kernel of the action of $G(x)$ on the set of triangles containing x. Then $Q(x) = G_1(x) = O_2(G(x))$ and $G(x)/Q(x) \cong L_3(2)$. As in Lemma 2.1 we write $Q_i(x)$ for the stabiliser in $G(x)$ of all vertices of distance at most i from x in Γ. We fix a triangle $T = \{x, y, z\}$ containing x and write $G\{T\}$ for the setwise stabiliser in G of T. We write $G(T)$ for the pointwise stabiliser of T in G, so that

$$G(T) = G(x) \cap G(y) \cap G(z)$$

and $G(T)$ is a normal subgroup of $G\{T\}$. Moreover $G\{T\}/G(T) \cong S_3$. Some more normal subgroups of $G\{T\}$ which we will need are

$$\begin{aligned} F &= O_2(G\{T\}),\\ N &= Q(x) \cap Q(y) \cap Q(z). \end{aligned}$$

Since G is both vertex and triangle transitive, statements proved about $G(x)$ and $G\{T\}$ apply to arbitrary vertex and triangle stabilisers, and we will use appropriate notation for subgroups conjugate to named subgroups of $G(x)$ and $G\{T\}$.

We now collate information about the actions of various subgroups of $G(x)$. We have the following lemma since $\{G(x), G\{T\}\}$ is a faithful amalgam.

Lemma 3.1. *Let R be a subgroup of $G(x) \cap G\{T\}$ and suppose that R is normal in both $G(x)$ and $G\{T\}$. Then $R = 1$.*

Lemma 3.2. *We have $G(x) \cap G\{T\} = Q(x)G(T)$.*

Proof. Suppose that $Q(x) \leqslant G(T)$. The normality of $Q(x)$ in $G(x)$ now implies that $Q(x)$ fixes $\Gamma(x)$ pointwise which gives $Q(x) = Q_1(x)$, a contradiction to (A7). Hence $Q(x)G(T) > G(T)$. By (A5), $G\{T\}$ is 2-transitive on T and therefore contains an element fixing x and interchanging y and z. Hence

$$|G(x) \cap G\{T\} : G(T)| = 2$$

and the result follows. □

By (A7) we have $|Q_2(x)| = 2$. For each $u \in V\Gamma$ let $e_u \in G(u)$ be such that $Q_2(u) = \langle e_u \rangle$. Put

$$\begin{aligned} E_x &= \langle e_u \mid u \in \Gamma(x) \rangle, \\ E_T &= \langle e_x, e_y, e_z \rangle. \end{aligned}$$

Observe that E_x is a normal subgroup of $G(x)$ contained in $Q_1(x)$ and that E_T is a normal subgroup of $G\{T\}$ contained in $G(T)$. Above we mentioned that we will use similar notation for subgroups conjugate to E_x, E_T, etc. As an example of this, we write

$$E_y = \langle e_v \mid v \in \Gamma(y) \rangle.$$

Lemma 3.3. *The involutions e_x, e_y and e_z are all distinct.*

Proof. Since $G\{T\}$ acts on T primitively, if the assertion fails, $e_x = e_y = e_z$. Then $Q_2(x) = Q_2(y) = Q_2(z)$ is normalised by $G(x)$ and by $G\{T\}$, a contradiction to Lemma 3.1. □

Lemma 3.4. *The following hold.*

1. *$[e_u, e_v] = 1$ for all $u, v \in \Gamma$ with $d(u, v) \leqslant 2$.*
2. *We have $e_x e_y e_z = 1$.*
3. *The involutions e_u for $u \in \Gamma(x)$ are pairwise distinct.*
4. *$E_x = Q_1(x) \cong C_2^4$ and $E_x = \langle e_x \rangle \cup \{e_u \mid u \in \Gamma(x)\}$.*
5. *The action of $G(x)$ on the nontrivial elements of $E_x/\langle e_x \rangle$ is equivalent to the action of $G(x)$ on the set of triangles containing x.*

Proof. Let u and v be as in (1). By definition, we have $\langle e_u \rangle = Q_2(u)$ so that $e_u \in G(v)$. Since $Q_2(v)$ is a normal subgroup of $G(v)$ of order two, we have $e_v \in \mathrm{Z}(G(v))$ whence $[e_u, e_v] = 1$. Thus (1) holds.

We now define an equivalence relation on the set $\Gamma(x)$ which will aid us in proving (2)–(5). For $u, v \in \Gamma(x)$ we say $u \approx v$ if and only if $\langle e_x, e_u \rangle = \langle e_x, e_v \rangle$. It is immediate that $\approx$ is an equivalence relation. Since $e_x \in \mathrm{Z}(G(x))$ and $G(x)$ preserves the set $\Gamma(x)$ we see that $\approx$ is a $G(x)$-invariant relation. If $\approx$ is the universal relation, we have that

$$\langle E_x, e_x \rangle = \langle e_x, e_y \rangle = \langle e_x, e_y, e_z \rangle = E_T$$

and so E_T is a normal subgroup of $G(x)$ and of $G\{T\}$, a contradiction to Lemma 3.1.

Suppose now that $\approx$ is the trivial relation. Then for all $u, v \in \Gamma(x)$ we have $e_u \neq e_v$, so that

$$|E_x| \geqslant |\{e_u \mid u \in \Gamma(x)\}| = 14.$$

By (1) E_x is an elementary abelian 2-group and by definition, $E_x \leqslant Q_1(x)$. Since $|Q_1(x)| = 2^4$ by (A7) we have that $E_x = Q_1(x)$ and since $e_x \in Q_1(x)$

$$E_x = \{1, e_x\} \cup \{e_u \mid u \in \Gamma(x)\}.$$

On the other hand, $|\langle e_x, e_y \rangle| = 2^2$ and $d = e_x e_y$ is distinct from 1, e_x and e_y. Now $d \in E_x$ and therefore $d = e_f$ for some $f \in \Gamma(x)$. This implies $y \approx f$, a contradiction to our assumption that $\approx$ is trivial.

Suppose now that the blocks of $\approx$ have size seven. Since $G(x)$ is transitive on the triangles which contain x, it must be that the two blocks divide each triangle into two and for $u \in \Gamma(x)$ one of $u \approx y$ or $u \approx z$ holds. This means that $\langle e_x, e_u \mid u \in \Gamma(x) \rangle = \langle e_x, e_y, e_z \rangle = E_T$, a contradiction to Lemma 3.1. Hence the blocks for $\approx$ have size two. Let B be a block containing y. Since $|G(x) : G(x)_B| = 7$ we see that $Q(x) \leqslant G(x)_B$ and since $G(T)$ fixes y we have $G(T) \leqslant G(x)_B$. Now $G(x)_B \geqslant Q(x)G(T)$ and so Lemma 3.2 shows that $G(x)_B = G(x) \cap G\{T\}$. Hence the relation $\approx$ is the same as the relation "in a triangle". Thus $y \approx z$ and by Lemma 3.3 we have $e_x e_y = e_z$, which is (2).

Now if $u, v \in \Gamma(x)$ are such that $e_u = e_v$ then $u \approx v$ which means that u and v are in some triangle, S say. Since there is $g \in G(x)$ with $S^g = T$ this means $e_y = e_z$, a contradiction to Lemma 3.3. Thus (3) holds.

As argued above, it follows immediately from (3) that (4) holds. For (5) we let $\mathcal{T}$ be the set of triangles containing x. By (4) for each $u \in \Gamma(x)$ we let u' be the unique vertex in $\Gamma(x)$ distinct from u such that $u \approx u'$. Then we may define $\phi : E_x/\langle e_x \rangle^{\#} \to \mathcal{T}$ by

$$\phi : \langle e_u, e_x \rangle / \langle e_x \rangle \mapsto \{x, u, u'\}.$$

Since $G(x)$ preserves the set of triangles and $\langle e_u, e_x \rangle = \langle e_{u'}, e_x \rangle$ for all $u \in \Gamma(x)$ it follows that ϕ is a well defined $G(x)$-invariant map. □

In the next lemma, the natural homomorphism $\alpha : G(x) \to \mathrm{Aut}(E_x)$ is the homomorphism induced by the conjugation action of $G(x)$ on E_x.

Lemma 3.5. *Let $\alpha : G(x) \to \mathrm{Aut}(E_x)$ be the natural homomorphism. Then*

(1) $\ker(\alpha) = E_x$;

(2) $\mathrm{Im}(\alpha)$ *is the stabiliser in $GL_4(2)$ of the 1-space $\langle e_x \rangle$ and $\alpha(Q(x))$ is the group of transvections of E_x with axis e_x;*

(3) *$E_x/\langle e_x\rangle = Q_1(x)/Q_2(x)$ and $Q(x)/Q_1(x) \cong Im(\alpha)$ are dual as modules for $G(x)/Q(x) \cong L_3(2)$;*
(4) *$Q(x)$ is extraspecial of plus type and with centre $Q_2(x) = \langle e_x\rangle$.*

Proof. Since E_x is abelian we have $E_x \leqslant \ker(\alpha)$. As the action of $G(x)$ on $E_x\backslash\{1\}$ is equivalent to its action on $\Gamma(x) \cup \{x\}$, it follows that $\ker(\alpha) = Q_1(x) = E_x$. Hence (1) holds. Moreover, $\langle e_x\rangle = Q_2(x) \lhd G(x)$ and so $\mathrm{Im}(\alpha)$ is contained in the stabiliser S in $\mathrm{GL}_4(2)$ of $\langle e_x\rangle$. Now $S = C_2^3 \rtimes L_3(2)$ and using the First Isomorphism Theorem, $\mathrm{Im}(\alpha) = S$. Since $Q(x)$ acts trivially on the set of triangles containing x, $Q(x)$ centralises $E_x/\langle e_x\rangle$ by Lemma 3.4(5). The kernel of S on $E_x/\langle e_x\rangle$ is C_2^3, which gives (2). For (3) a straightforward computation shows that the actions of $G(x)/Q(x)$ on $E_x/\langle e_x\rangle$ and $Q(x)/E_x \cong \alpha(Q(x))$ are dual.

Now $Q(x)$ is nonabelian as $Q(x)$ acts nontrivially on $\Gamma(x)$ and hence on E_x. Moreover, $E_x/\langle e_x\rangle$ is a normal subgroup of order 2^3 of $Q(x)/\langle e_x\rangle$ and $L_3(2)$ acts as a group of automorphisms of $Q(x)/\langle e_x\rangle$ so that it acts dually on $Q(x)/E_x$ and $E_x/\langle e_x\rangle$. Thus by [9, Lemma 3.4], $Q(x)/\langle e_x\rangle$ is elementary abelian. Now $Z(Q(x))$ is contained in E_x and the action of $L_3(2)$ tells us that $Z(Q(x)) = \langle e_x\rangle$ or E_x. Since $Q(x)$ does not centralise E_x we have that $Z(Q(x)) = \langle e_x\rangle$. Hence $Q(x)$ is extraspecial and since $Q(x)$ contains the elementary abelian subgroup $E_x \cong C_2^4$, it follows that $Q(x)$ is of plus type. □

At this stage we can say that $G(x)$ is an extension of 2_+^{1+6} by $L_3(2)$. To determine the isomorphism type of $G(x)$ we need to determine the extension involved.

Lemma 3.6. (1) *$G(T)$ induces an irreducible action of S_3 on $E_x/E_T \cong C_2^2$.*
(2) *$E_y \cap Q(x) = E_T$.*

Proof. Part (1) follows from Lemma 3.5(3). By definition we have $E_T \leqslant E_y$ and so $E_T \leqslant E_y \cap Q(x)$. Suppose equality does not hold and recall that $E_y \cong C_2^4$ and $E_T \cong C_2^2$. Since $G(T)$ normalises $E_y \cap Q(x)$ and $E_y \neq E_T$, part (1) implies that $E_y = E_y \cap Q(x)$. By symmetry we have $E_x \leqslant Q(y)$. Thus $E_xE_y \leqslant Q(x) \cap Q(y)$. Since $Q(x)$ and $Q(y)$ are extraspecial with derived subgroups $\langle e_x\rangle$ and $\langle e_y\rangle$ respectively, we have $[E_xE_y, E_xE_y] \leqslant \langle e_x\rangle \cap \langle e_y\rangle = 1$. Thus E_xE_y is an abelian subgroup of $Q(x)$. However, E_x is a maximal elementary abelian subgroup of $Q(x)$ and so $E_xE_y = E_x$ and hence $E_y = E_x$. This contradicts Lemma 3.1 for $Q_1(x)$. Thus $E_y \cap Q(x) = E_T$. □

Lemma 3.7. *The following hold:*
(1) *$N = Q(x) \cap Q(y) = Q(x) \cap Q(z) = Q(y) \cap Q(z)$ and $|Q(x) \cap G(T)| = 2^6$;*
(2) *$|N| = 2^4$ and $E_T < N$;*
(3) *N is a maximal elementary abelian subgroup of $Q(x)$.*

Proof. Since $Q(x)$ acts transitively on $T - \{x\}$ there is $t \in Q(x)$ interchanging y and z. Since $Q(x)/\langle e_x\rangle$ is abelian, every subgroup of $Q(x)$ that contains $\langle e_x\rangle$ is normal in $Q(x)$. We have $e_x \in Q(x) \cap Q(y)$, hence

$$Q(x) \cap Q(y) = (Q(x) \cap Q(y))^t = Q(x) \cap Q(z).$$

The same argument shows that $Q(y) \cap Q(z) = Q(x) \cap Q(z)$.

Since $x^{Q(y)} = \{x, z\}$ is of size two, it follows that

$$|Q(y) \cap G(T)| = |Q(y) \cap G(x)| = 2^6.$$

and so (1) holds.

Since $G(x) \cap G\{T\} = Q(x)G(T)$ we see that both $E_yQ(x)$ and $(Q(y) \cap G(x))Q(x)$ are normal 2-subgroups of $G(x) \cap G\{T\}$. Since $(G(x) \cap G\{T\})/Q(x) \cong S_4$ we have $E_yQ(x) = (Q(y) \cap G(x))Q(x)$ and

$$|E_yQ(x)/Q(x)| = 2^2 = |Q(y) \cap G(x) : Q(y) \cap G(x) \cap Q(x)|.$$

Plainly $Q(y) \cap G(x) \cap Q(x) = Q(y) \cap Q(x) = N$ and therefore $|N| = 2^4$. The second assertion of (2) is immediate.

We have that $Q(x) \cong 2^{1+6}_+$ and by (2) $|N| = 2^4$, so for (3) we just need to see that N is elementary abelian. Observe that

$$\Phi(Q(x))\Phi(Q(y)) = \langle e_x, e_y \rangle \leqslant Q(x) \cap Q(y),$$

and therefore $\Phi(Q(x) \cap Q(y)) \leqslant \Phi(Q(x)) \cap \Phi(Q(y)) = 1$ and so the result follows from (1). □

The next three lemmas expose detailed structure of $G\{T\}$. Surprisingly the outcome of these results is not the identification of $G\{T\}$ but rather of $G(x)$, which we complete in Lemma 3.11 and Proposition 3.12.

Lemma 3.8. *The following hold.*

(1) $F = \mathrm{O}_2(G(T)) = (Q(x) \cap G(T))E_y$.
(2) $\mathrm{C}_{G\{T\}}(F) = E_T$ *is an irreducible* $G\{T\}$*-module.*
(3) $E_T = \Phi(F) = [F, F] = \mathrm{Z}(F)$, *in particular,* F *is a special 2-group.*
(4) $G\{T\}/\mathrm{F} \cong S_3 \times S_3$.
(5) $\mathrm{C}_{G\{T\}}(\mathrm{F}/E_T) = \mathrm{F}$.
(6) $\mathrm{C}_{G\{T\}}(N/E_T)/F \cong S_3$ *and* $\mathrm{C}_{G\{T\}}(N/E_T) \cap G(T) = F$. *In particular,* N/E_T *is irreducible as a* $G(T)$*-module.*
(7) *Write* $\overline{F} = F/E_T$. *Then* $\overline{F} = \overline{E_x} \oplus \overline{E_y} \oplus \overline{N}$ *as a* $G(T)$*-module.*
(8) $\mathrm{C}_{G\{T\}}(N) = \mathrm{C}_F(N)$.
(9) $\mathrm{C}_{G\{T\}}(F/N) = F$.

Proof. (1) We have that $G\{T\}/G(T)$ is isomorphic to S_3, so $\mathrm{O}_2(G\{T\}) \leqslant G(T)$ and the first equality of (1) holds since $G(T)$ is normal in $G\{T\}$. For the second equality, we just calculate the orders of the subgroups involved. First note that since $G(x) \cap G\{T\} = G(T)Q(x)$ we have that

$$G(T)/(G(T) \cap Q(x)) \cong S_4$$

and so $|F| = 2^2|Q(x) \cap G(T)|$. By Lemma 3.7 (1) we have that $|Q(x) \cap G(T)| = 2^6$, hence $|F| = 2^8$. Now since $E_y \leqslant G(T)$ we have

$$E_y \cap (Q(x) \cap G(T)) = E_y \cap Q(x)$$

and so $|E_y(Q(x) \cap G(T))| = 2^8$ by Lemma 3.6 (2). This completes the proof of (1).

(2) Since $E_T = \langle e_x, e_y\rangle$ we see that $G\{T\}$ acts irreducibly on E_T. Let $C = \mathrm{C}_{G\{T\}}(F)$. Then C centralises E_T, so $C \leqslant G(T)$. Since

$$E_x \leqslant Q(x) \cap G(T) \leqslant F$$

we have $C \leqslant \mathrm{C}_{G(x)}(E_x) = E_x$, and in particular, $C = \mathrm{Z}(F)$. Similarly, $C \leqslant E_y$, so $C \leqslant E_x \cap E_y = E_T$. Since C is non-trivial and E_T is an irreducible $G\{T\}$-module, we see that $C = E_T = \mathrm{Z}(F)$.

(3) If P is a p-group with normal subgroups A and B such that $P = AB$, then it can be shown that $\Phi(P) = \Phi(A)\Phi(B)[A, B]$. We use this identity and (1) to see that

$$\Phi(F) = \Phi(E_y)\Phi(Q(x) \cap G(T))[E_y, Q(x) \cap G(T)] \leqslant \langle e_x\rangle E_T = E_T.$$

This implies $[F, F] \leqslant \Phi(F) \leqslant E_T$. Since $\Phi(F)$ and $[F, F]$ are normal in $G\{T\}$, and E_T is an irreducible $G\{T\}$-module (3) holds unless F is abelian. However if F is abelian then, using $E_xE_y \leqslant F$, we see that $E_y \leqslant \mathrm{C}_{G(x)}(E_x) = E_x$, a contradiction.

(4) In the proof of (1) we saw that $G(T)/F \cong S_3$. Working in $G\{T\}/F$ we see that the centraliser of $G(T)/F$ meets $G(T)/F$ trivially, and therefore complements $G(T)/F$ in $G\{T\}/F$. Since $(G\{T\}/F)/(G(T)/F) \cong S_3$ we obtain (4).

(5) By (3) we have that $[F, F] = E_T$, so $F \leqslant \mathrm{C}_{G\{T\}}(F/E_T)$. Hence if (5) doesn't hold, then by (4) there is $D \leqslant G\{T\}$ of order three that centralises F/E_T. Now (3) says that D centralises $F/\Phi(F)$ and by [6, 5.3.5, pg.180] D must centralise F which is a contradiction to (2). Thus (5) holds.

(6) By (3) we have $[F, N] \leqslant [F, F] = E_T$ so $F \leqslant \mathrm{C}_{G\{T\}}(N/E_T)$. Since $G(T)/F$ is isomorphic to S_3, if $F < \mathrm{C}_{G(T)}(N/E_T)$ then a Sylow 3-subgroup X of $G(T)$ centralises N/E_T. In particular, $N/E_T \cong C_2^2$ is contained in $\mathrm{C}_{Q(x)/E_T}(X)$. As an X-module $Q(x)/E_T$ is semisimple with two 2-dimensional summands and one trivial summand, hence $|\mathrm{C}_{Q(x)/E_T}(X)| = 2$, a contradiction. Since $\mathrm{Aut}(N/E_T) \cong S_3$ we obtain (6) after using (4).

(7) Notice that $E_yE_x \cap N \leqslant E_yE_x \cap Q(x) = E_x(E_y \cap Q(x)) = E_x$. Similarly, we obtain $E_yE_x \cap N \leqslant E_y$, and so $E_yE_x \cap N \leqslant E_x \cap E_y = E_T$. Thus $\overline{E_xE_y} \cap \overline{N} = 1$. Now $\overline{E_xE_y} = \overline{E_x} \oplus \overline{E_y}$. All of these spaces are $G(T)$ invariant, which yields (7).

(8) Using $\mathrm{C}_{G\{T\}}(E_T) \leqslant G(T)$, part (6) and the fact that

$$\mathrm{C}_{G\{T\}}(N) \leqslant \mathrm{C}_{G\{T\}}(E_T) \cap \mathrm{C}_{G\{T\}}(N/E_T)$$

we obtain $N \leqslant \mathrm{C}_{G\{T\}}(N) \leqslant F$. This is (8).

(9) By part (6) we can choose $d \in G\{T\}$ of order three with $d \notin G(T)$ so that $[N, d] \leqslant E_T$. Suppose that $d \in \mathrm{C}_{G\{T\}}(F/N)$. Then $[F, d, d] \leqslant [N, d] \leqslant E_T \leqslant E_x$. Since $E_x \leqslant F$ this implies that d normalises E_x, a contradiction to Lemma 3.1 (since $\langle G(x), d\rangle = \langle G(x), G\{T\}\rangle = G$). Hence we have that $\mathrm{C}_{G\{T\}}(F/N) \leqslant G(T)$. If there is $d \in G(T)$ of order three so that $[F, d] \leqslant N$ then $[E_x, d] \leqslant E_x \cap N = E_T$. Since $G(T)$ centralises E_T we see that $[E_x, d, d] = [E_x, d] = 1$, a contradiction to $\mathrm{C}_{G(x)}(E_x) = E_x$. Hence $\mathrm{C}_{G\{T\}}(F/N) \leqslant F$ and since F/N is abelian we see that equality holds. □

Lemma 3.9. *The following hold:*

1. *F/N is an irreducible $G\{T\}/F$-module;*
2. *N is self-centralising in $G\{T\}$.*

Proof. Set $\overline{F} = F/N$. By Lemma 3.8 we see that $\overline{F} = \overline{E_x} \oplus \overline{E_y}$ as a $G(T)$-module. Moreover there is a third submodule $\overline{E_z}$ which (by the 2-transitive action of $G\{T\}$ on $\{x, y, z\}$) is distinct from $\overline{E_x}$ and $\overline{E_y}$. Part (4) of Lemma 3.8 shows that these three submodules are permuted transitively by $G\{T\}$, so we conclude that $\overline{F}$ is an irreducible $G\{T\}$-module and part (9) of the same lemma shows that $\overline{F}$ is an irreducible $G\{T\}/F$-module.

By (1) we have $\mathrm{C}_F(N) = N$ or F, but the latter is ruled out by part (3) of Lemma 3.8. Now Lemma 3.8 (8) shows that $\mathrm{C}_F(N) = \mathrm{C}_{G\{T\}}(N)$ which is (2). □

Lemma 3.10. *Let X be a Sylow 3-subgroup of $G\{T\}$. Then $\mathrm{N}_{G\{T\}}(X) \cong S_3 \times S_3$. In particular, both $G\{T\}$ and $G(T)$ split over F.*

Proof. Let X be a Sylow 3-subgroup of $G\{T\}$ and let $\overline{G\{T\}} = G\{T\}/F$. Since $(|F|, |X|) = 1$ we have

$$\mathrm{N}_{\overline{G\{T\}}}(\overline{X}) = \overline{\mathrm{N}_{G\{T\}}(X)}.$$

Now Lemma 3.8 (4) shows that $\overline{X}$ is normal in $\overline{G\{T\}}$ whence $G\{T\} = \mathrm{N}_{G\{T\}}(X)F$. Note that $\mathrm{N}_{G\{T\}}(X) \cap F = \mathrm{C}_F(X)$. We claim that $\mathrm{C}_F(X) = 1$, from this it follows that $\mathrm{N}_{G\{T\}}(X) \cong S_3 \times S_3$ and we obtain the lemma.

Indeed, since the action of X on F is coprime we have $\mathrm{C}_{F/N}(X) = \mathrm{C}_F(X)N/N$ and Lemma 3.9(1) shows that $\mathrm{C}_{F/N}(X) = 1$, thus $\mathrm{C}_F(X) = \mathrm{C}_N(X)$. By Lemma 3.8 (4) and (6) we have $\mathrm{C}_{N/E_T}(X) = 1$ so that $\mathrm{C}_N(X) = \mathrm{C}_{E_T}(X)$. Finally we see that X is transitive on $\{x, y, z\}$ and therefore on $\{e_x, e_y, e_z\}$, hence $\mathrm{C}_{E_T}(X) = 1$. This proves our claim. □

By [10, Lemma 3.1(iii)] there is a unique indecomposable $\mathrm{GF}(2)L_3(2)$-module which is an extension of V by V^*, for V the natural module. We give some brief details of a construction of this module. Let $\epsilon : V^* \times V^* \to V$ be defined as follows for $\alpha, \beta \in V^*$:

$$\epsilon : (\alpha, \beta) \mapsto \begin{cases} 0 & \text{if } \alpha = \beta \text{ or if one of } \alpha, \beta \text{ is } 0, \\ (\ker \alpha \cap \ker \beta)^\# & \text{otherwise.} \end{cases}$$

Now we define

$$W = \{(v, \alpha) \mid v \in V, \alpha \in V^*\} \tag{3.1}$$

and for $v, w \in V$ and $\alpha, \beta \in V^*$ we set

$$(v, \alpha) + (w, \beta) = (v + w + \epsilon(\alpha, \beta), \alpha + \beta).$$

It is easy to check that W is an elementary abelian 2-group and the actions of $L_3(2)$ on V and V^* induce an action on W. Moreover,

$$V_0 := \{(v, 0) \mid v \in V\} \cong V$$

is the only nontrivial proper submodule of W whilst $W/V_0 \cong V^*$. Finally $L_3(2)$ respects a unique quadratic form q_W defined on W by

$$q_W(v, \alpha) = \begin{cases} 0 & \text{if } \alpha = 0 \text{ or if } \alpha \neq 0 \text{ and } v \notin \ker \alpha, \\ 1 & \text{if } \alpha \neq 0 \text{ and } v \in \ker \alpha. \end{cases} \tag{3.2}$$

Lemma 3.11. *As a module for $G(x)/Q(x) \cong L_3(2)$ we have*

$$Q(x)/\langle e_x \rangle \cong V \oplus V^*.$$

Proof. We set $\overline{Q(x)} = Q(x)/\langle e_x \rangle$ and use the bar notation. By Lemma 3.5 and the uniqueness of W proved in [10, Lemma 3.1(iii)] the only possibility other than $\overline{Q(x)} \cong V \oplus V^*$ is that $\overline{Q(x)}$ is the module W defined in (3.1) and $\overline{E_x}$ is the unique submodule of $\overline{Q(x)}$ of dimension three. Moreover the quadratic form q_E defined on $\overline{Q(x)}$ that $G(x)/Q(x)$ respects is given by

$$q_E : \ \overline{a} \mapsto a^2, \quad a \in Q(x). \tag{3.3}$$

Therefore, since E_x is elementary abelian, $q_E(\overline{E_x}) = 0$. Let ϕ be the $G(x)/Q(x)$-isomorphism $\phi : \overline{Q(x)} \to W$. Then by the uniqueness of q_W, for all $a \in Q(x)$ we have

$$q_E(\overline{a}) = q_W(\phi(\overline{a}))$$

with q_W as in (3.2).

Let $S \cong S_3$ be a subgroup of $G(T)$ provided by Lemma 3.10. Since $S \cap Q(x) \leqslant S \cap \mathrm{O}_2(G(T)) = 1$ we see that $SQ(x)/Q(x) \cong S_3$ and so S acts on $\overline{E_x}$ as a stabiliser some non-zero vector $v \in \overline{E_x}$ and some 2-space $U \leqslant \overline{E_x}$ such that $v \notin U$. Since $S \leqslant G(T)$ we have that S normalises N which is elementary abelian and has order 2^4 by Lemma 3.7. Moreover S centralises $E_T = \langle e_x, e_y, e_z \rangle$ which is contained in N. By Lemma 3.9 we have that $[N, S] \neq 1$ and so we conclude $N = E_T \times [N, S]$.

Set $M := [N, S]$ and observe that $M = [N, S] \leqslant [Q(x), S]$. Note that $\overline{M} \cong M$ since $M \cap \langle e_x \rangle = 1$. Hence $\overline{M}$ is a 2-dimensional subspace of $\overline{Q(x)}$ which we claim satisfies the following three properties:

(1) $\overline{M}$ is a faithful S-submodule of $\overline{Q(x)}$;
(2) $\overline{M}$ is totally isotropic, that is, $q_E(\overline{M}) = 0$;
(3) $\overline{M}$ is not contained in $[\overline{E_x}, S]$.

We have (1) by definition of M and observe that (2) holds since M is elementary abelian. If (3) is false then $\overline{M} \leqslant \overline{E_x}$ and this implies $N = M\langle e_x, e_y \rangle \leqslant E_x$ which gives $E_x = N$, a contradiction to Lemma 3.1. Hence indeed $\overline{M}$ satisfies (1), (2) and (3).

Now $J := [\overline{Q(x)}, S] = \overline{M} \oplus [\overline{E_x}, S]$ is 4-dimensional and every faithful S-submodule of $\overline{Q(x)}$ must be contained in J. Moreover, as S-modules, $[\overline{E_x}, S] \cong \overline{M}$, so there are exactly three S-submodules of J. Since S preserves the decomposition $\langle v \rangle \oplus U$ of $\overline{E_x}$ we have that $[\overline{E_x}, S] = U$. Write $U = \{u_1, u_2, u_3, 0\}$ and then let $U_1 = \phi(U)$. We label the elements of U_1 as follows (recall (3.1))

$$U_1 = \{(0, 0), (u_1, 0), (u_2, 0), (u_3, 0)\}.$$

For each u_i we have a 2-space $V_i := \langle v, u_i \rangle \leqslant \overline{E_x}$ and S permutes the subspaces V_1, V_2, V_3 transitively. Define $\alpha_i : V \to V/V_i$ (that is, $\alpha_i \in V^*$) and observe that S has equivalent actions on $\{u_1, u_2, u_3\}$ and $\{\alpha_1, \alpha_2, \alpha_3\}$. So we may define

$$U_2 := \{(0, 0), (v, \alpha_1), (v, \alpha_2), (v, \alpha_3)\}$$

and note that U_2 is an S-invariant subset of W. Since $\alpha_i+\alpha_j=\alpha_k$ and $\epsilon(\alpha_i,\alpha_j)=v$ for $\{i,j,k\}=\{1,2,3\}$, a quick calculation shows that U_2 is in fact a subspace of W. Finally, we set

$$U_3 := \{(0,0),(u_1+v,\alpha_1),(u_2+v,\alpha_2),(u_3+v,\alpha_3)\}$$

and again observe that U_3 is an S-invariant submodule of W. Since S acts non-trivially on U_2 and U_3, we have $U_2=[U_2,S]$ and $U_3=[U_3,S]$. Thus $U_2,U_3\leqslant[W,S]$ and U_1, U_2 and U_3 are the three non-trivial S-submodules of $[W,S]$.

Now $[\overline{E_x},S]$ and $\overline{M}$ are both totally isotropic subspaces of J. On the other hand, we have $q_W(U_2)\neq 0$ since, by (3.2), $q_W(v,\alpha_1)=1$ and $q_W(U_3)\neq 0$ since $q_W(u_1+v,\alpha_1)=1$. Since $[\overline{Q(x)},S]$ and $[W,S]$ are isomorphic as S-modules, we see that $[\overline{Q(x)},S]$ has a unique totally isotropic subspace that is S-invariant. Then properties (1) and (2) imply that $[\overline{E_x},S]=\overline{M}$, a contradiction to (3) which completes the proof. □

We will denote by E^x the unique normal subgroup of $G(x)$ of order 2^4 which is contained in $Q(x)$ and is not equal to E_x, thus we have

$$Q(x)/\langle e_x\rangle = E^x/\langle e_x\rangle \oplus E_x/\langle e_x\rangle$$

as a $G(x)/Q(x)$-module.

For the next result we need information about the 1-cohomology of $L:=L_3(2)$ on the natural module V (we refer the reader to [1, §17] for any unexplained notation). By [9, Lemma 3.1(i)] we have $\mathrm{H}^1(L,V)\cong C_2$. That is, there exists a unique indecomposable L-module W which has a submodule isomorphic to V and such that $[W/V,L]=1$ (see [1, (17.11)]). Choosing $M=\mathrm{N}_L(R)$ for some Sylow 7-subgroup R of L we observe that W is a semisimple M-module. Picking $w\in W$ such that $W=\langle w\rangle\oplus V$ (as an M-module) we define the 1-cocycle $\mu: L\to V$ by

$$\mu:\ell\mapsto[w,\ell]\in V.$$

Since M is a maximal subgroup of L we have $M=\mathrm{C}_L(w)$, in particular, $\mu(\ell)\neq 0$ for $\ell\notin M$. This gives us a concrete description of $\mathrm{H}^1(L,V)=\langle\mu\rangle$. Similarly we define $\gamma: L\to V^*$ so that $\gamma(m)=0$ for $m\in M$ and $\gamma(\ell)\neq 0$ for $\ell\notin M$. Thus we have

$$\mathrm{H}^1(L,V\oplus V^*)=\langle\mu,\gamma\rangle$$

(although we have identified μ and γ with their images in the 1-cohomology group). As in [1, (17.1)] for a 1-cocycle $\alpha: L\to V\oplus V^*$ we set

$$S(\alpha)=\{(\alpha(\ell),\ell)\mid \ell\in L\}\leqslant(V\oplus V^*)\rtimes L.$$

Then $S(0)$, $S(\mu)$, $S(\gamma)$ and $S(\mu+\gamma)$ are the *standard complements* and form a transversal of the conjugacy classes of complements to $V\oplus V^*$ in the semidirect product. Note that the intersection of each pair of these groups is precisely M.

Proposition 3.12. *In $G(x)$ there exist three conjugacy classes of subgroups isomorphic to $L_3(2)$ and one conjugacy class of subgroups isomorphic to $SL_2(7)$. Moreover, there is a unique class of complements to $Q(x)$ in $G(x)$ for which both E_x and E^x are semisimple. In particular, $G(x)$ is isomorphic to the centraliser of a $2A$-involution in $L_5(2)$.*

Proof. We set $\overline{G(x)} = G(x)/\langle e_x\rangle$ and use the bar notation. Since

$$\mathrm{C}_{G(x)}(Q(x)) = \langle e_x\rangle$$

we identify $\overline{G(x)}$ with a subgroup of $\mathrm{Aut}(Q(x)) \cong 2^6.O_6^+(2)$ which contains $\overline{Q(x)}$, the normal subgroup of order 2^6. Since $G(x)$ is perfect, $\overline{G(x)}$ is contained in the derived subgroup of $\mathrm{Aut}(Q(x))$ which is isomorphic to $2^6 \rtimes A_8$. In particular, we see that $\overline{G(x)}$ splits over $\overline{Q(x)}$, which means that we may identify $\overline{G(x)}$ with $(V \oplus V^*) \rtimes L_3(2)$. Clearly if $L \leqslant G(x)$ and $L \cong L_3(2)$ then $\overline{L} \cong L_3(2)$, whereas if $L \cong SL_2(7)$ then we have $L \cap Q(x) = \mathrm{Z}(L) = \langle e_x\rangle$ so that $\overline{L} \cong L_3(2)$ also.

From the remarks above we know that there are four conjugacy classes of subgroups of $\overline{G(x)}$ isomorphic to $L_3(2)$. Moreover, our standard complements are representatives of each class, let them be $\overline{L_1}$, $\overline{L_2}$, $\overline{L_3}$ and $\overline{L_4}$. There is a subgroup $\overline{M}$ of $\overline{G(x)}$ with $\overline{M} \cong C_7 \rtimes C_3$ such that $\overline{L_i} \cap \overline{L_j} = \overline{M}$ for $i \neq j$. Since the $\overline{L_i}$ belong to distinct conjugacy classes of $\overline{G(x)}$ we see that the preimages L_i of the $\overline{L_i}$ are also non-conjugate. Moreover, a preimage is isomorphic to one of $SL_2(7)$ or $C_2 \times L_3(2)$. We will show that exactly one of the L_i is isomorphic to $SL_2(7)$.

Let $S = \langle X, e\rangle$ be the subgroup of $G(T)$ isomorphic to S_3 with Sylow 3-subgroup X delivered by Lemma 3.10. Then $\overline{S} \leqslant \mathrm{N}_{\overline{G(x)}}(\overline{X})$. Moreover, since $(|\overline{X}|, |\overline{Q(x)}|) = 1$ and the normaliser of a Sylow 3-subgroup of $L_3(2)$ is isomorphic to S_3, we have that

$$\mathrm{N}_{\overline{G(x)}}(\overline{X}) = \mathrm{N}_{\overline{Q(x)}}(\overline{X})\overline{S} = \mathrm{C}_{\overline{Q(x)}}(\overline{X})\overline{S}.$$

It follows from Lemma 3.11 that

$$\mathrm{C}_{\overline{Q(x)}}(\overline{X}) = \mathrm{C}_{\overline{Q(x)}}(\overline{S}) = \langle \overline{e_y}, \overline{r}\rangle$$

where $\overline{r}$ is some element of $\overline{E^x}$ which, when we identify $\overline{E_x}$ and $\overline{E^x}$ with V and V^* respectively, is a line on which $\overline{e_y}$ does not lie. In particular, $\overline{r}$ is contained in a totally isotropic subspace and $[e_y, r] \neq 1$. Now $\mathrm{N}_{\overline{G(x)}}(\overline{X})$ contains exactly four subgroups isomorphic to S_3, namely

$$\begin{aligned} \overline{N_1} &= \langle \overline{X}, \overline{e}\rangle, \\ \overline{N_2} &= \langle \overline{X}, \overline{ee_y}\rangle, \\ \overline{N_3} &= \langle \overline{X}, \overline{er}\rangle, \\ \overline{N_4} &= \langle \overline{X}, \overline{ee_yr}\rangle. \end{aligned}$$

By Sylow's Theorem, Sylow 3-subgroups of $\overline{M}$ and $\overline{X}$ are conjugate, so we may assume that $\overline{X} \leqslant \overline{M}$. Now we see that $\mathrm{N}_{\overline{L_i}}(\overline{X}) = \overline{N_j}$ for some j. Since $\overline{L_i} \cap \overline{L_k} = \overline{M}$ for $i \neq k$, we may relabel the $\overline{N_j}$ so that $\mathrm{N}_{\overline{L_i}}(\overline{X}) = \overline{N_i}$ for $i = 1, 2, 3, 4$. Since $e \in G(T)$ we have that e centralises e_y, moreover, since e is an involution, both e and e_ye are involutions, so the preimages of $\overline{N_1}$ and $\overline{N_2}$ are isomorphic to $C_2 \times S_3$. Now e normalises $\langle r, e_x\rangle$, so either $r^e = r$ or $r^e = e_xr$. In the latter case we see that $r^{ee_y} = r$. Without loss of generality therefore, we can assume that $r^e = r$. Hence the preimage of N_3 is isomorphic to $C_2 \times S_3$ also. Now $(ee_yr)^2 = e_x$ so the preimage of N_4 is isomorphic to $C_3 \rtimes C_4$. Considering the number of involutions in the L_i, we see that L_1, L_2 and L_3 are isomorphic to $C_2 \times L_3(2)$ and $L_4 \cong SL_2(7)$.

We choose a preimage M of $\overline{M}$ so that $X \leqslant M$ and $M \leqslant L_i \cap L_j$ for $i, j \in \{1, 2, 3, 4\}$. As M-modules we have

$$E_x = \langle e_x \rangle \oplus [E_x, M] \text{ and } E^x = \langle e_x \rangle \oplus [E^x, M].$$

As N_i-modules, we have

$$E_x = \langle e_x, e_y \rangle \oplus [E_x, X] \text{ and } E^x = \langle e_x, r \rangle \oplus [E^x, X]$$

with $[E_x, X] \cong [E^x, X] \cong C_2^2$ and of course $[E_x, X] \leqslant [E_x, M]$ and $[E^x, X] \leqslant [E^x, M]$. Since $L_i = \langle M, z_i \rangle$, where $z_1 = e$, $z_2 = ee_y$, $z_3 = er$ and $z_4 = ere_y$, to determine $[E_x, L_i]$ we just need to evaluate $[z_i, e_y]$ and $[z_i, r]$ for each i. Since $[e_y, r] = e_x$ we have that

$$\begin{aligned}
&[E_x, L_1] = [E_x, L_2] = [E_x, M],\\
&[E^x, L_1] = [E^x, L_3] = [E^x, M],\\
&[E_x, L_3] = [E_x, L_4] = E_x \text{ and}\\
&[E^x, L_2] = [E^x, L_4] = E^x.
\end{aligned}$$

Hence both E_x and E^x are semisimple modules for L_1 only. Clearly the decompositions are invariant under conjugation by $G(x)$, so we obtain the second part of the proposition.

We have now seen that $G(x)$ is a split extension of the extraspecial group $Q(x) \cong 2_+^{1+6}$ by $L_3(2)$ and have completely determined the action of a complement on $Q(x)$. This uniquely determines the isomorphism type of $G(x)$. After inspecting the centraliser of a $2A$-involution in $L_5(2)$, we see that this group is isomorphic to $G(x)$. □

We now introduce some notation for subgroups of $G(x)$.

Notation 3.13. Recall that we can identify $G(x)$ with the centraliser of an involution in $L_5(2)$. Hence $G(x) = \langle a_1, a_2, \ldots, a_{12} \rangle$ where a_i is the matrix with 1's on the diagonal and 0's everywhere else except for the coordinate given by a_i in the matrix below.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0\\
a_1 & 1 & a_{11} & 0 & 0\\
a_2 & a_3 & 1 & a_{12} & 0\\
a_4 & a_5 & a_6 & 1 & 0\\
a_7 & a_8 & a_9 & a_{10} & 1
\end{pmatrix}$$

With this identification we have $e_x = a_7$, $e_y = a_4$, $E_T = \langle e_x, e_y \rangle$, $E_x = \langle a_1, a_2, a_4, a_7 \rangle$, $E^x = \langle a_7, a_8, a_9, a_{10} \rangle$, $Q(x) = \langle a_1, a_2, a_4, a_7, a_8, a_9, a_{10} \rangle$ and $L = \langle a_{11}, a_{12}, a_3, a_5, a_6 \rangle$. Now $G(x) \cap G\{T\}$ is the stabiliser in $G(x)$ of E_T and in particular $(G(x) \cap G\{T\})/Q(x) \cong S_4$ is the stabiliser in $G(x)/Q(x) \cong L_3(2)$ of $E_T/\langle a_7 \rangle$. Since $G(x)$ acts dually on $E_x/\langle e_x \rangle$ and $E^x/\langle e_x \rangle$, it follows that $G(x) \cap G\{T\}$ fixes a 3-subspace U of E^x containing $\langle a_7 \rangle$. Moreover, we must have that $U = \langle a_7, a_8, a_9 \rangle$. We note that $E_T/\langle a_7 \rangle$ and $U/\langle e_x \rangle$ are respectively the unique 1-dimensional and 2-dimensional subspaces of $Q(x)/\langle a_7 \rangle$ fixed by $G(x) \cap G\{T\}$. There are exactly four elementary abelian normal subgroups of $G(x) \cap G\{T\}$ of order 2^4, these are $E_x, E^x, W_1 = \langle a_7, a_4, a_8, a_9 \rangle$ and $W_2 = \langle a_7, a_4, a_1a_9, a_2a_8 \rangle$. (The number of such subgroups can either be directly calculated using a computer algebra package, or see Lemma 4.2.)

Now N is an elementary abelian subgroup of order 2^4 normalised by $G(x) \cap G\{T\}$ with $N \cap E_x = E_T$. Then $N/\langle a_7 \rangle$ is totally singular and so contained in $\langle a_4 \rangle^\perp = \langle a_4, a_2, a_1, a_8, a_9 \rangle$. Note that W_2 is self-centralising in $G(x) \cap G\{T\}$ while W_1 is not, thus Lemma 3.9(2) implies $N = W_2$.

Lemma 3.14. *$G\{T\}$ splits over N and $G\{T\} = N \rtimes K$, where $K \cong C_2^4 \rtimes (S_3 \times S_3)$ is the stabiliser in $L_4(2)$ of the subgroup $E_T \cong C_2^2$ of N.*

Proof. We first show that $G\{T\}$ splits over N. Since N is a normal abelian subgroup of $G\{T\}$, by Gaschütz's Lemma [1, (10.4)] it suffices to show that a Sylow 2-subgroup of $G\{T\}$ splits over N. Such a subgroup is contained in $G(x) \cap G\{T\}$. We will use Notation 3.13. Then a complement to N is given by $\langle a_8, a_9, a_{10} \rangle \rtimes \mathrm{Dih}(8)$ where the $\mathrm{Dih}(8)$ subgroup is a Sylow 2-subgroup of a standard complement which preserves both E_x and E^x as semisimple spaces.

Let K be a complement to N in $G\{T\}$. Then $K \cong KN/N$ can be identified with a subgroup of a 2-space stabiliser in $GL_4(2) = L_4(2)$, since E_T is a normal subgroup of $G\{T\}$. By comparing orders, we see that K is the full stabiliser in $L_4(2)$ of a 2-dimensional subspace and so $K \cong C_2^4 \rtimes (S_3 \times S_3)$. □

Hence we have proved the following.

Proposition 3.15. *Parts (1)-(3) of Theorem 1.3 hold.*

4. Determination of the amalgams

In Proposition 3.15 we determined the isomorphism type of the amalgam $\{G(x), G\{T\}\}$ appearing in Theorem 1.3. We now wish to determine the number of isomorphism classes of amalgams of this type. Two amalgams $\mathcal{A} = \{A_1, A_2\}$, $\mathcal{B} = \{B_1, B_2\}$ are *isomorphic* if there exists a bijection $\varphi : A_1 \cup A_2 \to B_1 \cup B_2$ that maps A_i onto B_i such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A_i$ and for $i = 1, 2$. Let $B = A_1 \cap A_2$, $D = \mathrm{Aut}(B)$ and D_i be the image in D of $\mathrm{N}_{\mathrm{Aut}(A_i)}(B)$. Then Goldschmidt's Theorem [5, (2.7)] states that the number of nonisomorphic amalgams of the same type as $\mathcal{A}$ is equal to the number of double cosets of D_1 and D_2 in D.

We determine the number of isomorphism classes of amalgams of type $\{G(x), G\{T\}\}$ whilst considering an infinite family of amalgams. For $n \geqslant 4$ we define the amalgam $\mathcal{U}_n$ as follows. Let $H = \mathrm{AGL}_n(2) = R \rtimes L$ with $R \cong 2^n$ and $L \cong \mathrm{L}_n(2)$. Picking $r, s \in R^\#$ with $r \neq s$ we let $B_n = \mathrm{N}_H(\langle r \rangle)$ and $C_n = N_H(\langle r, s \rangle)$. Then set

$$\mathcal{U}_n = \{B_n, C_n\}. \tag{4.1}$$

Each amalgam in the sequence $(\mathcal{U}_n)_{n \geq 4}$ is unfaithful since B_n and C_n both normalise the elementary abelian subgroup R of order 2^n. Note that $\mathcal{U}_4$ and $\{G(x), G\{T\}\}$ have the same type but $\mathcal{U}_4$ is unfaithful while the amalgam appearing in Theorem 1.3 is faithful since it is a weakly locally projective amalgam.

We now assemble results on the automorphism groups of B_n, C_n and $B_n \cap C_n$. We let $Q_n = O_2(B_n)$ and observe that $Q_n \cong 2_+^{1+2(n-1)}$. In particular, $Q_n = E_1E_2$ where $s \in E_1$, $E_1 \cong E_2 \cong 2^n$ and $E_1 \cap E_2 = Z(Q_n) = \langle r \rangle$. Moreover, B_n is the centraliser of an appropriate involution in $L_{n+1}(2)$. We choose a subgroup L of B_n such that

$$B_n = Q_n \rtimes L$$

and $L \cong L_{n-1}(2)$ decomposes both E_1 and E_2 into semisimple modules. Note that $E_1/\langle r \rangle$ and $E_2/\langle r \rangle$ are dual and the module $Q_n/\langle r \rangle = E_1/\langle r \rangle \oplus E_2/\langle r \rangle$ admits an alternating bilinear form defined by commutators in Q_n.

Lemma 4.1. *Q_n is characteristic in $B_n \cap C_n$.*

Proof. Suppose for a contradiction there is $\alpha \in \mathrm{Aut}(B_n \cap C_n)$ such that $(Q_n)^\alpha \neq Q_n$ and let $P = (Q_n)^\alpha$. Then $Q_nP > Q_n$ and is normal in $B_n \cap C_n$. Since $B_n \cap C_n/Q_n \cong 2^{n-2}.L_{n-2}(2)$, we see that $\mathrm{O}_2(B_n \cap C_n/Q_n)$ is a minimal normal subgroup of order 2^{n-2}. This implies $|Q_nP| = 2^{n+(n-1)+(n-2)}$ and $|Q_n \cap P| = 2^{n+1}$. Since $(Z(B_n \cap C_n))^\alpha = \mathrm{Z}(B_n \cap C_n) = Z(Q_n)$ we have $Z(Q_n) \leqslant Q_n \cap P$.

Let $\overline{Q_n} = Q_n/Z(Q_n)$. Then $\overline{Q_n \cap P}$ has order 2^n. Note that

$$[Q_n \cap P, P] \leqslant [P, P] = Z(Q_n),$$

so PQ_n/Q_n centralises $\overline{Q_n \cap P}$. Now we have

$$PQ_n/Q_n = \mathrm{O}_2(B_n \cap C_n/Q_n) \cong 2^{n-2}.$$

With V the natural module for $\mathrm{L}_{n-1}(2)$ we see that $\overline{Q_n} = E_1/\langle r \rangle \oplus E_2/\langle r \rangle \cong V \oplus V^*$. An easy calculation shows that subspace of fixed points of PQ_n/Q_n acting on $\overline{Q_n}$ has order 2^{n-1}, a contradiction to the fact that PQ_n/Q_n centralises $\overline{Q_n \cap P}$ which has order 2^n. □

In the next lemma we see that the amalgam $\mathcal{U}_4$ has properties different from $\mathcal{U}_n$ for $n \geqslant 5$. Recall that for $n = 4$ we use Notation 3.13 for elements of B_4.

Lemma 4.2. *Let $\Gamma_n = \mathrm{Aut}(B_n \cap C_n)$. Then*

$$|\Gamma_n : \mathrm{C}_{\Gamma_n}(Q_n)(\mathrm{Inn}(B_n \cap C_n))| = \begin{cases} 2 & \text{if } n = 4, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, if $n = 4$ and

$$\tau \in \Gamma_n - \mathrm{C}_{\Gamma_n}(Q_n)(\mathrm{Inn}(B_n \cap C_n))$$

then τ interchanges N and E_x.

Proof. Let $\gamma \in \Gamma_n$. By Lemma 4.1, Q_n is characteristic in $B_n \cap C_n$ and so γ normalises Q_n and $\langle r \rangle = Z(Q_n)$, and therefore acts on $\overline{B_n \cap C_n} = (B_n \cap C_n)/\langle r \rangle$. Since $\overline{\langle r, s \rangle}$ and $\overline{\langle C_{E_2}(s) \rangle}$ are the only totally singular 1- and $(n-2)$-spaces of $\overline{Q_n}$ fixed by $B_n \cap C_n$ each of them must be stabilised by γ. Set

$$\mathcal{I} = \{\overline{U} \subset \overline{Q_n} \mid \overline{U} \text{ is totally isotropic, } B_n \cap C_n\text{-invariant and } \dim \overline{U} = n-1\}.$$

Observe that γ permutes the elements of $\mathcal{I}$ and $\overline{E_1}$, $\overline{E_2}$ are elements of $\mathcal{I}$. Let $E_3 = \langle r, s, C_{E_2}(s) \rangle$, then $\overline{E_3} \in \mathcal{I}$. Suppose that $\overline{E_4}$ is a fourth element of $\mathcal{I}$. Consider the quotient $\widetilde{Q_n} := \overline{Q_n}/\overline{\langle r,s\rangle}$ where $\overline{E_4}$ projects to an $n-2$, respectively, $n-1$ dimensional subspace if $\overline{E_4}$ contains $\overline{\langle r,s\rangle}$, respectively, doesn't contain $\langle r,s\rangle$. We have $\widetilde{Q_n} = \widetilde{E_1} \oplus \widetilde{E_2}$ and $\widetilde{E_3}$ is the unique $(B_n \cap C_n)$-invariant proper subspace of $\widetilde{E_2}$. If $n \neq 4$ then $\widetilde{E_3}$ and $\widetilde{E_1}$ are dual non-isomorphic $(B_n \cap C_n)$-modules. Thus one of $\widetilde{E_4} = \widetilde{E_1}$, $\widetilde{E_4} = \widetilde{E_3}$ or $\widetilde{E_4} = \widetilde{E_2}$. In the respective cases we obtain $\overline{E_4} = \overline{E_1}$, $\overline{E_4} = \overline{E_3}$ or $\overline{E_4} = \overline{E_2}$, and so for $n \neq 4$ we have $|\mathcal{I}| = 3$. For $n = 4$ we see that there is exactly one more option for $\widetilde{E_4}$, the unique diagonal submodule of $\widetilde{E_1} \oplus \widetilde{E_3}$. This is the image of N defined in Notation 3.13, and therefore $|\mathcal{I}| = 4$ for $n = 4$.

Since $\langle r,s\rangle$ is fixed by γ and $\overline{E_2}$ is the only element of $\mathcal{I}$ not containing $\overline{\langle r,s\rangle}$, E_2 is fixed by γ. Since E_2 is fixed and $E_3 = \langle r, s, C_{E_2}(s)\rangle$ we see that E_3 is fixed by γ also. Hence E_1 is fixed by γ unless $n = 4$ and possibly γ interchanges $E_1 = E_x$ and N as above.

Now $\Gamma_n/\mathrm{C}_{\Gamma_n}(Q_n)$ is isomorphic to a subgroup of $\mathrm{Aut}(Q_n) = 2^{2(n-1)}.O^+_{2(n-1)}(2)$ containing $\mathrm{Inn}(Q_n)$. The stabiliser in $O^+_{2(2n-1)}(2)$ of two complementary totally isotropic $(n-1)$-spaces is $L_{n-1}(2)$ and the stabiliser in this of a totally singular 1-space contained in one of the $(n-1)$-spaces is $2^{n-2}.L_{n-2}(2)$. Since each element of Γ_n fixes $\overline{E_2}$ and $\overline{\langle r,s\rangle}$, it follows that $\mathrm{C}_{\Gamma_n}(Q_n)\,\mathrm{Inn}(B_n\cap C_n)$ is the stabiliser in Γ_n of E_1. In the previous paragraph we saw that $|E_1^{\Gamma_n}| = 1$ if $n \neq 4$ and $|E_1^{\Gamma_n}| \leqslant 2$ if $n = 4$. In particular, $\Gamma_n = \mathrm{C}_{\Gamma_n}(Q_n)\,\mathrm{Inn}(B_n \cap C_n)$ if $n \neq 4$ and for $n = 4$ we have

$$|\Gamma_n : \mathrm{C}_{\Gamma_n}(Q_n)\,\mathrm{Inn}(B_n \cap C_n)| \leqslant 2.$$

Recall Notation 3.13 for elements of B_4. We define an automorphism β of $B_4 \cap C_4$ as follows

$$\beta : \begin{array}{ccc} a_1 & \mapsto & a_1a_7a_9 \\ a_2 & \mapsto & a_2a_7a_8 \end{array} \tag{4.2}$$

and β fixes $a_3, \dots, a_{11}$. Clearly $\beta \notin \mathrm{C}_{\Gamma_n}(Q_n)\,\mathrm{Inn}(B_n \cap C_n)$ and thus we obtain the equality stated in the lemma. □

Proposition 4.3. *Let* $\Gamma_n = \mathrm{Aut}(B_n \cap C_n)$*. If* $n \neq 4$ *then* $\Gamma_n = \mathrm{Inn}(B_n \cap C_n)$*. If* $n = 4$ *then* $|\mathrm{C}_{\Gamma_n}(Q_n)| = 2$ *and* $|\,\mathrm{Out}(B_n \cap C_n)| = 4$*.*

Proof. Let $C = \mathrm{C}_{\Gamma_n}(Q_n)$. Suppose there is $g \in B_n \cap C_n$ such that $c_g \in C \cap \mathrm{Inn}(B_n \cap C_n)$ (where c_g denotes the automorphism induced by conjugation by g). Then for all $w \in Q_n$ we have $w = wc_g = w^g$, so $g \in \mathrm{C}_{B_n}(Q_n) = \mathrm{Z}(B_n)$, whence $c_g = 1$. Hence

$$[C, \mathrm{Inn}(B_n \cap C_n)] = 1.$$

Now suppose that $\alpha \in C$ and let $g \in B_n \cap C_n$ be arbitrary. Since α centralises c_g we have $c_g = c_{g\alpha}$. It follows that $g\alpha = g$ or $g\alpha = gr$ (where $\langle r\rangle = \mathrm{Z}(B_n \cap C_n) \cong C_2$). Since $r\alpha = r$ we have that $g\alpha^2 = g$ and $\alpha^2 = 1$ by the arbitrary choice of g. For $g_1, g_2 \notin C_{B_n\cap C_n}(\alpha)$ we have $(g_1g_2^{-1})\alpha = g_1rg_2^{-1}r = g_1g_2^{-1}$. Hence $g_1C_{B_n\cap C_n}(\alpha) = g_2C_{B_n\cap C_n}(\alpha)$, that is, $C_{B_n\cap C_n}(\alpha)$ is a subgroup of index at most two containing Q_n. If $n \neq 4$ then $B_n \cap C_n = Q_n \rtimes 2^{n-2}.L_{n-2}(2)$ has no such proper subgroup, and we have $C = 1$. If $n = 4$ then $B_n \cap C_n$ has a unique such subgroup and it follows that $|C| \leqslant 2$. We now show that in the case of $n = 4$ we have equality. Let D be the unique index two subgroup of $B_n \cap C_n$ that contains Q_n. We define $\alpha : B_n \cap C_n \to B_n \cap C_n$ by the following,

$$\alpha : g \mapsto \begin{cases} g & \text{if } g \in D \\ gr & \text{otherwise.} \end{cases} \tag{4.3}$$

Since r is an element of order two in the centre of B_n, it is easy to check that α is a homomorphism. Moreover, by definition $\ker \alpha = 1$, so α is a non-trivial automorphism of $B_n \cap C_n$ which lies in $\mathrm{C}_{\Gamma_n}(Q_n)$.

The result now follows from Lemma 4.2. □

For the next three results we restrict our attention to $n = 4$. Therefore we set

$$B = B_4,$$
$$C = C_4,$$
$$Q = Q_4,$$

and we use Notation 3.13 and the results of Section 3.

Proposition 4.4. $\mathrm{Aut}(B) = (Q/Z(Q)) \rtimes \mathrm{Aut}(L)$ *where* $(Q/Z(Q)) \cong 2^6$ *decomposes into dual* L*-modules which are interchanged by the inverse-transpose automorphism of* L*.*

Proof. Since the centraliser of an appropriate involution in $\mathrm{Aut}(L_4(2))$ is isomorphic to $Q \rtimes \mathrm{Aut}(L_3(2))$ and $Z(Q) = \mathrm{Z}(B)$, we have that $Q/Z(Q) \rtimes \mathrm{Aut}(L_3(2)) \leqslant \mathrm{Aut}(B)$. Let g be an automorphism of B. Then L^g is a complement of Q and L^g must decompose Q in the same manner. Since all complements of Q with this property are conjugate to L by Proposition 3.12, we can adjust g by an inner automorphism so that g normalises L and then by an inverse-transpose automorphism if necessary so that g fixes E_1 and E_2 setwise. However, L is the full stabiliser in $\mathrm{Aut}(Q)$ of E_1 and E_2. Hence $g \in Q/Z(Q) \rtimes \mathrm{Aut}(L)$. □

Lemma 4.5. *The stabiliser of* $B \cap C$ *in* $\mathrm{Aut}(B)$ *induces the inner automorphism group of* $B \cap C$*.*

Proof. Since the stabiliser of a 1-space is self-normalising in $\mathrm{Aut}(L)$ it follows that $B \cap C$ is selfnormalising in $Q \rtimes \mathrm{Aut}(L)$. The result follows. □

Lemma 4.6. *We have* $\mathrm{Aut}(C) = \mathrm{Inn}(C) \cong C$ *and the stabiliser of* $B \cap C$ *in* $\mathrm{Aut}(C)$ *induces the inner automorphism group of* $B \cap C$*.*

Proof. Write $C = NRS$ where S is the normaliser of a Sylow 3-subgroup X of C and RS is a complement to N in C (so $R \cong C_2^4$). Observe that $F = O_2(C)$ and $E_T = Z(F)$ are characteristic subgroups of C. We claim that N is the unique normal subgroup of C of order 2^4, and is therefore characteristic. Suppose that M is another such subgroup. Then $M \leqslant F$ and so $M \cap Z(F) = M \cap E_T$ is a nontrivial normal subgroup of C. Since E_T is irreducible as a C-module, we have $E_T \leqslant M$. If $M \neq N$ then $M \cap N = E_T$ by Lemma 3.8 (6) and therefore MN/N has order 2^2 and is a normal subgroup of C/N contained in F/N. This contradicts Lemma 3.9 (1).

Let $\Gamma = \mathrm{Aut}(C)$. By the Frattini argument we have $\Gamma = \mathrm{N}_\Gamma(X)I$, for $I := \mathrm{Inn}(C)$. Let $h \in \mathrm{N}_\Gamma(X)$ and note that h normalises $\mathrm{N}_I(X) = S$, which is isomorphic to $S_3 \times S_3$ by

Lemma 3.10. Let T_1, T_2 be subgroups of S so that $S = T_1 \times T_2$ and $T_1 \cong T_2 \cong S_3$ and label so that $T_1 = \mathrm{C}_S(E_T)$ and T_2 acts faithfully on E_T. Since E_T is characteristic in C we see that h must normalise both T_1 and T_2. Thus, after adjusting h by an inner automorphism if necessary, we may assume $h \in \mathrm{C}_\Gamma(S)$.

Now h normalises each of E_T, N and T_2, so h normalises the complement $E := \mathrm{C}_N(T_2)$ to E_T in N. Since E_T and E are irreducible modules for T_1 and T_2, we see that h centralises both E_T and E, whence h centralises N. We now claim that h normalises R. Since R is an absolutely irreducible module for S and h centralises S, it will follow that h centralises R and therefore h centralises $NRS = C$, from which we conclude $h \in I$ as desired. We now prove the claim. Note that T_2 centralises N/E_T and that T_2 preserves a decomposition of R into two irreducible modules. Since h acts on F/E_T and normalises $[T_2, F/E_T] = RE_T/E_T$ we see that h normalises RE_T. Now T_1 centralises E_T and also preserves a decomposition of R into two irreducible modules, thus $[T_1, E_T R] = R$. Since h normalises $E_T R$ and T_1 we see that h normalises R. This proves the claim and we obtain the lemma. □

We are now in a position to determine the number of isomorphism classes of amalgams of type $\mathcal{U}_n$. Recall that an amalgam $\{B, C\}$ is faithful if there is no normal subgroup contained in $B \cap C$.

Theorem 4.7. *There are four amalgams of type $\mathcal{U}_4$, and precisely two of these are faithful. For $n \geqslant 5$ there is a unique amalgam of type $\mathcal{U}_n$.*

Proof. We use Goldschmidt's Theorem [5, (2.7)]. By Lemmas 4.4 and 4.6 this says that the number of isomorphism classes of amalgams of the same type as $\mathcal{U}_n$ is the number of double cosets of $I := \mathrm{Inn}(B_n \cap C_n)$ in $T := \mathrm{Aut}(B_n \cap C_n)$. Hence for $n \geqslant 5$ there is a unique isomorphism class of amalgams of type $\mathcal{U}_n$ by Proposition 4.3. Now consider the case $n = 4$ and let $B = B_4$, $C = C_4$ and $Q = Q_4$. Let α and β be the automorphisms of $B \cap C$ defined in (4.3) and (4.2) respectively. Proposition 4.3 shows that there are four cosets of I in T. It is easy to check that the cosets I, $I\alpha$, $I\beta$ and $I\alpha\beta$ are distinct. To see then that there are exactly four amalgams of this type, we just need to show that all of these cosets are in distinct I-orbits. If this is not the case, then we must have $I\alpha\beta I = I\beta I$. This implies there are $g, h \in I$ such that $\beta = h\alpha\beta g$. That is $h^{-1}\beta g^{-1}\beta^{-1} = \alpha$, which gives $\alpha \in I$, a contradiction.

The automorphism α of $B \cap C$ preserves faithfulness since every normal subgroup of B contained in $B \cap C$ is contained in Q. There exist faithful and unfaithful amalgams of type $\mathcal{U}_4$ inside M_{24} and $\mathrm{AGL}_4(2)$ respectively. Thus we see that exactly two of the four isomorphism classes of amalgams of type $\mathcal{U}_4$ are faithful. □

Theorem 1.3 now follows from Proposition 3.15 and Theorem 4.7.

5. Completions

In this final section we find presentations for the universal completions of the two faithful amalgams appearing in Theorem 1.3 and we give finite completions for both. To derive these presentations, it is convenient to begin with an unfaithful amalgam of the same type and use Theorem 4.7 to obtain the faithful amalgams. Recall the definition of $\mathcal{U}_4$ from the beginning of Section 4. We let $\mathcal{U}_4 = \{G_1, G_2\}$ and view $\mathrm{AGL}_4(2)$ as a subgroup of $\mathrm{L}_5(2)$. We then have that

$$\begin{aligned} G_1 &= \langle a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12} \rangle, \\ G_2 &= \langle a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{13} \rangle, \\ G_1 \cap G_2 &= \langle a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11} \rangle \end{aligned}$$

where a_i is the element of $\mathrm{L}_5(2)$ with 1's on the diagonal and 0's everywhere except for the position of a_i given below.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a_1 & 1 & a_{11} & 0 & 0 \\ a_2 & a_3 & 1 & a_{12} & 0 \\ a_4 & a_5 & a_6 & 1 & a_{13} \\ a_7 & a_8 & a_9 & a_{10} & 1 \end{pmatrix}$$

To obtain the amalgams of the same type as $\mathcal{U}_4$ in the different isomorphism classes we need to use the automorphisms α (see (4.3)) and β (see (4.2)) of $G_1 \cap G_2$ given below (where α and β act trivially on the generators not listed).

$$\alpha \; : \; \begin{cases} a_3 & \mapsto \; a_7 a_3 \\ a_{11} & \mapsto \; a_7 a_{11} \end{cases}$$

$$\beta \; : \; \begin{cases} a_1 & \mapsto \; a_1 a_7 a_9 \\ a_2 & \mapsto \; a_2 a_7 a_8 \end{cases}$$

Let us write $\mathcal{U}_4^\sigma$ for the amalgam obtained from $\mathcal{U}_4$ using the map $\sigma \in \{\alpha, \beta, \alpha\beta\}$. Note that $E_1 := \langle a_7 \rangle$, $E_2 := \langle a_1, a_2, a_4, a_7 \rangle$ and $E_3 = \langle a_7, a_8, a_9, a_{10} \rangle$ are the only normal subgroups of G_1 contained in $G_1 \cap G_2$. The amalgams $\mathcal{U}_4$ and $\mathcal{U}_4^\alpha$ are unfaithful precisely because E_2 is normalised by G_1, G_2 and by α. On the other hand E_1 and E_3 are normalised by β, but not by a_{13}, and $[E_2^\beta, a_{13}] \not\leqslant E_2^\beta$. Thus the amalgams $\mathcal{U}_4^{\alpha\beta}$ and $\mathcal{U}_4^\beta$ are faithful.

For $\sigma \in \{\beta, \alpha\beta\}$ we denote the universal completion of the amalgam $\mathcal{U}_4^\sigma$ by $\mathcal{G}^\sigma$. For $1 \leqslant i \leqslant j \leqslant 13$ we let $R(i,j)$ be a relation between a_i and a_j that holds in $\mathrm{L}_5(2)$ and for $1 \leqslant i \leqslant 11$ we write $R^\sigma(i,j)$ for a relation between a_i^σ and a_j. Then we obtain

$$\mathcal{G}^\sigma = \left\langle \begin{array}{c} a_1, a_2, a_3, a_4, a_5, a_6, a_7, \\ a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13} \end{array} \middle| \begin{array}{c} R(i,i) \text{ for } 1 \leqslant i \leqslant 13, \; R(i,j) \text{ for } 1 \leqslant i < j \leqslant 12, \\ R^\sigma(i,13) \text{ for } 1 \leqslant i \leqslant 11 \end{array} \right\rangle.$$

For the relations we have $R(i,i) = a_i^2$ for $1 \leqslant i \leqslant 13$, $R(3,11) = (a_3 a_{11})^3$, $R(10,11) = (a_{10} a_{11})^3$, $R(6,12) = (a_6 a_{12})^3$, $R(11,12) = (a_{11} a_{12})^4$, and the remaining relations are of the form $R(i,j) = [a_i, a_j] w(i,j)$ for some $w(i,j) \in G_1 \cap G_2$ which can be calculated by directly multiplying the matrices above. We note explicitly that $R^\beta(1,13) = [a_1, a_{13}] a_4 a_6$, $R^\beta(2,13) = [a_2, a_{13}] a_4 a_5$, $R^{\alpha\beta}(3,13) = [a_3, a_{13}] a_4$ and $R^{\alpha\beta}(11,13) = [a_{11}, a_{13}] a_4$.

We observe that the subgroup $L = \langle a_3, a_5, a_6, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13} \rangle$ of $\mathcal{G}^\beta$ is complemented by the elementary abelian subgroup $\langle a_1, a_2, a_4, a_7 \rangle$ of order 2^4. This gives a representation π of $\mathcal{G}^\beta$ of degree 16. Moreover, since each normal subgroup of G_1 contains a_7 and each normal subgroup of G_2 contains the subgroup $\langle a_4, a_7 \rangle$, π restricted to G_1 or

G_2 is faithful. We claim that A_{16} is a faithful completion of $\mathcal{G}^\beta$, that is, $\mathcal{G}^\beta\pi = A_{16}$. Since G_1 is perfect, we have $G_1\pi \leqslant A_{16}$ and it is easy to check that $a_{13}\pi$ is of cycle type $(a,b)(c,d)(e,f)(g,h)$. Thus $\mathcal{G}^\beta\pi$ is an insoluble transitive subgroup of A_{16} so if the claim is false it must be that $\mathcal{G}^\beta\pi$ is contained in a transitive insoluble maximal subgroup, conjugate to one of

$$\mathrm{AGL}_4(2),\ S_2 \wr S_8,\ S_8 \wr S_2.$$

We note that $G_2\pi$ preserves only blocks of size four by examining the action on the regular normal subgroups of order 2^4. If $G_i\pi \leqslant \mathrm{AGL}_4(2)$ for $i \in \{1,2\}$ then order considerations show that $G_i\pi$ contains a Sylow 2-subgroup of $\mathrm{AGL}_4(2)$ and therefore contains the regular normal subgroup of order 2^4. Since $\mathcal{U}_4^\beta$ is faithful therefore, we have $\mathcal{G}^\beta\pi \not\leqslant \mathrm{AGL}_4(2)$ and this gives the claim.

In the introduction we noted that M_{24} and He are completions of faithful amalgams of type $\mathcal{U}_4$. Indeed, using Magma [2] we see that adding the following set of relations to $\mathcal{G}_4^{\alpha\beta}$ gives M_{24} as a quotient:

$$\{(a_6a_{12}a_{13})^5, (a_{11}a_{12}a_{13})^{11}, (a_{10}a_{12}a_{13})^5\}$$

and adding the following set of relations gives He as a quotient:

$$\{(a_{12}a_2a_8a_{13})^5,\ (a_6a_{12}a_2a_7a_8a_{13})^5,\ (a_{10}a_8a_{13}a_{12}a_7)^5\}.$$

In the same way we find $\mathcal{G}_4^\beta$ has no index 24 subgroup and that $\mathcal{G}_4^{\alpha\beta}$ has no index 16 subgroup. Thus A_{16} is a completion of $\mathcal{U}_4^\beta$ alone and the sporadic groups M_{24} and He are completions of $\mathcal{U}_4^{\alpha\beta}$ only.

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