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The m -covers and m -ovoids of generalised quadrangles and related structures

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For Emily –

I saw so much of myself in you. I wish life had been better to you during your time on this pale blue dot, and that we could remember you in the way that you deserve.

Abstract

This dissertation explores methods of generalising hemisystems and relative hemisystems on the Hermitian surface $H(3, q^2)$.

Segre determined in 1965 that the only nontrivial m -covers of $H(3, q^2)$, q odd, have $m = \frac{q+1}{2}$. He named these covers *hemisystems* because they constitute half of the lines on every point. Despite a forty year period of only one example being known, hemisystems have piqued the interest of many finite geometers and algebraic graph theorists because of their links to strongly regular graphs, partial quadrangles and association schemes. In 2011, Penttila and Williford defined the concept of a *relative hemisystem* on $H(3, q^2)$, for q a power of 2. In contrast to hemisystems, relative hemisystems are only known to give rise to Q -polynomial association schemes that cannot be constructed from distance regular graphs. These particular sorts of association schemes were considered rare before the introduction of relative hemisystems, and the first infinite family of relative hemisystems gave rise to the first infinite family of such association schemes.

There have been a variety of generalisations of hemisystems since the first infinite family of them was discovered by Cossidente and Penttila in 2005. Many of these generalisations rely on the fact that $H(3, q^2)$ falls into a broader class of objects called *generalised quadrangles*. In 2010, Bamberg, Giudici and Royle showed that every *flock generalised quadrangle* has a hemisystem. In Chapter 6, we search for relative hemisystems on the smallest nonclassical flock generalised quadrangle and the dual of the generalised quadrangle $T_3(\mathcal{O})$ and give the first set of examples of relative hemisystems there.

We also prove an analogous version of Segre's result which initiated the study of hemisystems. In particular, we prove that every nontrivial relative m -cover is a relative hemisystem. We also briefly explore other generalisations of relative hemisystems to higher dimensions, and search for relative hemisystems on $H(3, 16^2)$.

In 2011, Vanhove also generalised the concept of a dual hemisystem, that is, a $\frac{q+1}{2}$ -*ovoid* of the dual Hermitian space $DH(3, q^2)$. He showed that a $\frac{q+1}{2}$ -ovoid of an odd-dimensional dual Hermitian space $DH(2d-1, q^2)$ gives rise to a distance regular graph, and that for $d \geq 3$, such graphs would be new. We give some computational results on the nonexistence of $\frac{q+1}{2}$ -ovoids of $DH(5, q^2)$ for $q = 3$ and 5, using a simplification of the problem to $\frac{q+1}{2}$ -ovoids of $DW(5, q)$.

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Contents

Abstract	v
Acknowledgements	vii
1 Introduction	1
2 Geometries	5
2.1 Projective geometries	5
2.2 Polar spaces	8
2.2.1 Classical polar spaces	8
2.3 Polarities and dualities	12
2.3.1 Dualities	12
2.3.2 Polarities	13
2.4 The Klein correspondence	14
3 Groups and actions	17
3.1 Group actions	17
3.2 Group products	20
3.2.1 Semidirect products	20
3.2.2 Wreath products	21
3.3 Classical groups	22
3.3.1 Linear and projective linear groups	22
3.3.2 Unitary groups	24
3.3.3 Symplectic groups	25
3.3.4 Orthogonal groups	26
4 Generalised quadrangles	27
4.1 Definitions and examples	27
4.2 Generalised quadrangles from ovals and ovoids	28
4.2.1 Automorphism groups of $T_3(\mathcal{O})$ and $T_2(\mathcal{O}')$	30
4.3 Flock generalised quadrangles and q -clan geometries	30
4.3.1 q -clans	30
4.3.2 Flocks of quadratic cones	31
4.3.3 Herds of ovals	32
4.3.4 Flock generalised quadrangles	32
4.3.5 The known q -clans, q even	35
4.3.6 Automorphism groups of flock generalised quadrangles	36
4.4 Subquadrangles	38
4.4.1 The classical case	38
4.4.2 Flock generalised quadrangles	39
4.4.3 $T_3(\mathcal{O})$	41
4.4.4 Constructing subquadrangles	42

4.4.5	Doubly subtended subquadrangles	43
4.5	m -covers and their generalisations	46
5	Association schemes	49
5.1	Graphs	49
5.2	Association schemes	51
5.3	Association schemes of dual polar spaces	56
5.4	Association schemes of relative hemisystems	57
6	New generalisations of relative hemisystems	61
6.1	Relative m -covers	61
6.2	Techniques	69
6.3	Relative hemisystems of nonclassical GQ($8^2, 8$)	72
6.3.1	Relative hemisystems of flock GQs	72
6.3.2	Relative hemisystems of the dual of $T_3(\mathcal{O})$	76
6.4	Classical relative hemisystems for $q = 16$	77
6.5	Higher dimensions	77
6.6	$\frac{q+1}{2}$ -ovoids of DH($5, q^2$)	79
7	Concluding remarks	83
	Bibliography	85
A	Association scheme parameters of some dual polar spaces	91
B	Constructing nonclassical generalised quadrangles in GAP	93
B.1	Flock generalised quadrangles	93
B.2	$T_3(\mathcal{O})$ and $T_2(\mathcal{O})$	96
C	An alternate proof of Theorem 6.1.6	101

1. Introduction

A Hermitian space $H(3, q^2)$ is an incidence structure consisting of points and lines, and can be constructed from taking the totally isotropic subspaces of $PG(3, q^2)$ with respect to a Hermitian form. In his treatise on Hermitian spaces in 1965, Beniamino Segre [66] pondered upon the existence of m -covers of $H(3, q^2)$. An m -cover of $H(3, q^2)$ is a set of lines S such that every point is incident with m lines of S . Segre proved that the only m -cover of a Hermitian space for q odd that is not simply all or none of the lines has $m = \frac{q+1}{2}$. Since such an m -cover would consist of half of the lines on every point of $H(3, q^2)$, he called a $(\frac{q+1}{2})$ -cover a *hemisystem*. He gave an example of a hemisystem on $H(3, 3^2)$, and showed that it was the only example up to equivalence on this Hermitian space.

In 1978, Bruen and Hirschfeld proved that there are no nontrivial m -covers of $H(3, q^2)$, q even [14]. Around the same time, Cameron et al. proved that every hemisystem gives rise to a strongly regular graph [20]. However, they did not only show this for $H(3, q^2)$. They recalled that $H(3, q^2)$ also belongs to a class of geometric objects called *generalised quadrangles*.

A generalised quadrangle of order (s, t) is an incidence structure with points and lines satisfying the following axioms:

- (i) Every two points lie on at most one line together.
- (ii) Every line is incident with exactly $s + 1$ points.
- (iii) Every point is incident with exactly $t + 1$ lines.
- (iv) (*The GQ axiom*) For every point P , and every line ℓ not incident with P , there is a unique point K incident with ℓ and collinear with P .

A Hermitian space $H(3, q^2)$ is a generalised quadrangle of order (q^2, q) . There are also examples of generalised quadrangles (indeed, even ones of the same order) other than $H(3, q^2)$. There are the flock generalised quadrangles of order (q^2, q) that arise from q -clans, and when q is an odd power of 2, there are also the generalised quadrangles $T_3(\mathcal{O})$, for a Suzuki–Tits ovoid \mathcal{O} of $PG(3, q)$. Cameron et al. proved their result for any generalised quadrangle of order (q^2, q) , independent of its construction. That is, a hemisystem of any generalised quadrangle of order (q^2, q) gives rise to a strongly regular graph. In 1989, Thas generalised Segre’s result about m -covers of $H(3, q^2)$ to show that it holds for all generalised quadrangles of order (q^2, q) [69].

Despite these developments, no new examples of hemisystems had been found on any Hermitian space $H(3, q^2)$ for $q \geq 5$. Thas conjectured in 1995 that Segre’s example of a hemisystem is the only example on any Hermitian space [70]. This conjecture was disproved ten years later, when Cossidente and Penttila

gave an example of an infinite family of hemisystems, one on each Hermitian space $H(3, q^2)$ for $q \geq 5$ [29]. This was followed by the construction of several infinite families of hemisystems on both classical and nonclassical generalised quadrangles, discovered by a variety of different authors, and the search is still ongoing. Perhaps one of the most remarkable results in this vein is the result of Bamberg, Giudici and Royle that every flock generalised quadrangle has a hemisystem [5].

In 2011, Penttila and Williford considered an analogous version of hemisystems on $H(3, q^2)$, in the case where q is even [64]. They were motivated by the desire to construct new examples of primitive Q -polynomial association schemes that do not arise from distance regular graphs. Previously, only sporadic examples of these sorts of association schemes were known. To this end, they defined the notion of a relative hemisystem.

Let R be a generalised quadrangle of order (q^2, q) containing a generalised subquadrangle R' of order (q, q) , q even. We call a subset \mathcal{H} of the lines disjoint from R' a *relative hemisystem* of R with respect to R' if for each point $x \in R \setminus R'$, exactly half of the lines through x disjoint from R' lie in \mathcal{H} . Penttila and Williford proved that every relative hemisystem of $H(3, q^2)$ gives rise to a primitive Q -polynomial association scheme not arising from a distance regular graph. They also constructed the first infinite family of relative hemisystems of $H(3, q^2)$, and in doing so, constructed the first infinite family of the aforementioned association schemes. Since then, two infinite families of relative hemisystems have been constructed by Cossidente [27, 26], and one further example conjectured to be sporadic has been constructed by Cossidente and Pavese [28].

However, partly because relative hemisystems were defined quite recently, there are no references in the literature to relative hemisystems of nonclassical generalised quadrangles of order (q^2, q) , nor is there any consideration of the existence of relative m -covers that are not relative hemisystems.

In Chapter 6, we address this by proving a series of results about relative m -covers that culminate in an analogous version of Segre's theorem:

Theorem 1.0.1. Let R be a generalised quadrangle of order (q^2, q) containing a doubly subtended subquadrangle R' of order (q, q) . Then a nontrivial relative m -cover of the external lines is a relative hemisystem.

By [71], the only example of a generalised quadrangle of order (q^2, q) , q even, with a doubly subtended subquadrangle is $H(3, q^2)$, so we indeed have an analogue of Segre's result. However, when q is odd, there exists at least one additional family of generalised quadrangles of order (q^2, q) containing doubly subtended subquadrangles. This other known family was discovered by Kantor [47].

Hermitian spaces also fall into a category of geometric objects called *polar*

spaces. A non-degenerate (*finite*) *polar space* is an incidence geometry whose points and lines satisfy the following axioms:

- (i) Any two points are incident with at most one common line.
- (ii) Every line is incident with at least three points.
- (iii) Any two lines meet in at most one point.
- (iv) There is no point that is collinear to every other point.
- (v) Given a point P and a line ℓ that are not incident, there is either a unique point on ℓ collinear with P , or every point on ℓ is collinear with P .

A polar space can contain higher-dimensional subspaces such as planes, solids and so on, depending on its *rank*. A Hermitian space $H(3, q^2)$ is a rank 2 polar space because it contains two types of subspaces, namely points and lines. We obtain the *dual polar space* of a polar space by applying a bijection that makes the set of points of the dual polar space the set of maximal subspaces of the polar space, makes the set of lines of the dual polar space the next-to-maximal subspaces of the polar space and so on, while reversing inclusion.

An *m-ovoid* S of a dual polar space is a set of points such that every line of the dual polar space is incident with m points of S . A problem that generalises hemisystems in a slightly different manner is the search for $\frac{q+1}{2}$ -ovoids of the dual Hermitian space $DH(2d-1, q^2)$. Vanhove showed that every $\frac{q+1}{2}$ -ovoid of $DH(2d-1, q^2)$ gives rise to a distance regular graph [78]. In particular, for $d \geq 3$, the existence of a $\frac{q+1}{2}$ -ovoid would mean the construction of a previously unknown distance regular graph. Notice that when $d = 2$, finding $\frac{q+1}{2}$ -ovoids of $DH(3, q^2)$ is equivalent to finding hemisystems of $H(3, q^2)$. However, once we consider $d \geq 3$, we move outside the realm of hemisystems and into uncharted territory.

In this dissertation, we aim to extend the concepts of hemisystems and relative hemisystems in a variety of different ways. In many cases, our focus is on exploring generalisations that have been considered for hemisystems, but not for relative hemisystems.

In Chapter 2, we recall the theory of projective spaces, which often provide the ambient geometries of the incidence geometries that we are studying. In Chapter 3, we study the symbiotic connection between projective geometries and the groups that describe their symmetries, as well as stating some group-theoretic results that will help us search for, and justify the existence of, geometric objects that are of interest to us in later chapters. In Chapter 4, we cover the theory of the main focus of our research: generalised quadrangles. We provide a census of the generalised quadrangles of order (q^2, q) , their subquadrangles of order (q, q) and how to construct them, as well as some discussion of the automorphism groups of flock generalised quadrangles. In Chapter 5, we delve into the theory of association schemes. These highly symmetric combinatorial objects allow

us to better understand the relationships between points in dual polar spaces and m -covers and m -ovals. They also provide the motivation for studying relative hemisystems and $\frac{q+1}{2}$ -ovals of the dual Hermitian space $\text{DH}(n, q^2)$, n odd. Chapter 6 presents the main results of this dissertation. We describe the computational techniques, based on group-theoretic results, employed to search for relative hemisystems of nonclassical generalised quadrangles of order (q^2, q) , and $\frac{q+1}{2}$ -ovals of $\text{DH}(5, q^2)$.

We also explore relative m -covers, and apart from proving Theorem 1.0.1, we prove computationally that there are no nontrivial relative m -covers of the unique nonclassical flock generalised quadrangle of order $(8^2, 8)$ for $m \neq \frac{q}{2}$.

We then provide the following partial classifications of relative hemisystems of nonclassical generalised quadrangles of order $(8^2, 8)$.

Theorem 1.0.2. Let R be the unique nonclassical flock generalised quadrangle of order $(8^2, 8)$. Then a relative hemisystem of R with respect to the subquadrangle $R' = S_{(1,0)}$ is equivalent to one of 17 examples, or it has a stabiliser that is either an elementary abelian 2-group or the trivial group.

Theorem 1.0.3. A relative hemisystem of the dual of $T_3(\mathcal{O})$, \mathcal{O} a Suzuki–Tits ovoid, with respect to the dual of a generalised subquadrangle $T_2(\mathcal{O}')$ of order $(8, 8)$, is one of 11 examples, or has a stabiliser that is either trivial or an elementary abelian 2-group.

In the former case, we give a construction of a family of groups that contains the largest known stabiliser of a relative hemisystem on the unique flock generalised quadrangle of order $(8^2, 8)$. We also search for relative hemisystems of $\text{H}(3, 16^2)$, and the results of this may be found in Section 6.4.

Furthermore, we state some nonexistence results for $\frac{q+1}{2}$ -ovals of $\text{DH}(5, q^2)$ for some small values of q . In particular, we have the following theorem.

Theorem 1.0.4. $\text{DH}(5, q^2)$ has no $\frac{q+1}{2}$ -ovals when $q = 3$ or 5 .

Finally, our concluding remarks in Chapter 7 summarise the results of this dissertation and highlight avenues for future work.

2. Geometries

In this chapter, we define many of the fundamental geometric objects that will form the basis of our studies in later chapters.

2.1 Projective geometries

An *incidence geometry* of rank n is a 4-tuple $(\mathcal{S}, I, \Delta, \sigma)$ comprising a nonempty set of *varieties* \mathcal{S} , a symmetric binary incidence relation $I \subseteq \mathcal{S} \times \mathcal{S}$, a set Δ of size n of *types* and an onto *type map* $\sigma : \mathcal{S} \rightarrow \Delta$ such that I does not contain any pairs of varieties of the same type. In this dissertation, we assume that all of our incidence geometries are finite.

A *(point–line) incidence structure* is a rank 2 incidence geometry where the varieties are a set \mathcal{P} called *points* and a set \mathcal{B} called *lines*. We say that a point P and a line ℓ are *incident* if $(P, \ell) \in I$. We also sometimes say that P is *on* ℓ and vice versa. Two points P, Q are said to be *collinear* if there exists a line ℓ such that $(P, \ell) \in I$ and $(Q, \ell) \in I$. We sometimes denote the line incident with both P and Q by PQ . Similarly, we say two lines ℓ and m are *concurrent* or that they *meet* or *intersect* if there exists a point P such that $(P, \ell) \in I$ and $(P, m) \in I$. A point–line incidence structure is a *partial linear space* if any two distinct points are incident with at most one common line and every line is incident with at least two points (this can also be seen to be a consequence of the first part of the definition). If a partial linear space is such that any pair of distinct points are incident with exactly one common line, then we say it is a *linear space*.

Definition 2.1.1. A *projective plane* is a linear space satisfying the following axioms:

- (i) Any two distinct lines meet in a unique point.
- (ii) There exist four points such that no three of them are collinear.

Suppose that P is a point not incident with a line ℓ in a projective plane. Then for every point K on ℓ , there must exist a line incident with both K and P , since a projective plane is a linear space. Every line through P must also meet ℓ in a point. Therefore, there is a one-to-one correspondence between the lines on a point and points on a line, and therefore there are the same number of each. If this number is $q + 1$, we say that the projective plane has *order* q .

We may construct a projective plane $\text{PG}(2, q)$ by taking the set of nonzero vectors in the three-dimensional vector space $V(3, q)$ over a finite field $\text{GF}(q)$. We take the 2-spaces and 1-spaces of $V(3, q)$ through the origin to be the lines and points of $\text{PG}(2, q)$ respectively. We define incidence by taking the “inclusion of subspaces” relation and making it symmetric. For example, if a line ℓ is

contained in a plane π in $V(3, q)$, then the projective point corresponding to ℓ will be incident with the projective line corresponding to π , and vice versa.

Two triangles in a projective plane are *in perspective* from a point P if the lines spanned by their corresponding vertices are incident with P . Similarly, two triangles are said to be in perspective from a line ℓ if the points of intersection between corresponding edges of the triangle are incident with ℓ . We say that a projective plane is *Desarguesian* if and only if whenever two triangles are in perspective from a line, they are also in perspective from a point. For example, $\text{PG}(2, q)$ is Desarguesian [75, Theorem 2.4].

Hilbert proved that every Desarguesian projective plane arises from a weaker version of the construction of $\text{PG}(2, q)$ given above, namely from a vector space over a division ring [42]. Wedderburn [49] showed that every finite division ring is a field. Wedderburn's original proof had a mistake that was corrected by Dickson [34].

Theorem 2.1.2. A finite projective plane is isomorphic to some $\text{PG}(2, q)$ if and only if it is Desarguesian.

Hilbert also showed that there exist non-Desarguesian projective planes, the smallest being three examples of order nine [83]. We can generalise the definition of a projective plane to a projective space as follows.

Definition 2.1.3. A *projective space of rank n* is an incidence geometry of rank n containing two types called “points” and “lines” which satisfy the following axioms:

- (i) Any two points P and Q are incident with exactly one common line.
- (ii) Each line is incident with at least three points.
- (iii) If A, B, C and D are distinct points such that AB is concurrent with CD , then AC is concurrent with BD .

The final axiom is referred to as the *Veblen–Young* axiom, and was first described in 1908 [80]. It ensures that every two lines on a plane meet in a point. Note that if we know that every two lines in an incidence structure are concurrent, we do not have to prove that the incidence structure satisfies the Veblen–Young axiom to prove that it is a projective space, because satisfying the other axioms implies that the geometry is a projective plane.

Example 2.1.4. A projective plane \mathcal{P} with at least three points on every line is a projective space, since \mathcal{P} is a linear space.

A projective space is said to be *nondegenerate* if it contains at least two lines. We always assume that our projective spaces are nondegenerate.

Example 2.1.5. The smallest example of a nondegenerate projective space

is the Fano plane, which contains seven points and seven lines, and each line is incident with three points. We may assign a nonzero vector in the three-dimensional vector space $V(3, 2)$ over $\text{GF}(2)$ to each of the points, in such a way that the lines of the Fano plane can be described by taking the span of two points.

More generally we construct the incidence geometry $\text{PG}(n, q)$ by defining the k -dimensional subspaces of $\text{PG}(n, q)$ to be the $(k + 1)$ -dimensional subspaces of $V(n + 1, q)$ through the origin, and take the incidence relation I on subspaces of $\text{PG}(n, q)$ to be the symmetric version of inclusion of subspaces. For example, the points of $\text{PG}(n, q)$ are the lines of $V(n + 1, q)$ through the origin, and the lines of $\text{PG}(n, q)$ are the planes of $V(n + 1, q)$ through the origin. We sometimes say that $\text{PG}(n, q)$ has *rank* n . To avoid confusion, we also call the dimension of a subspace in $\text{PG}(n, q)$ the *projective dimension*. We call a subspace of $\text{PG}(n, q)$ of projective dimension $n - 1$ a *hyperplane*.

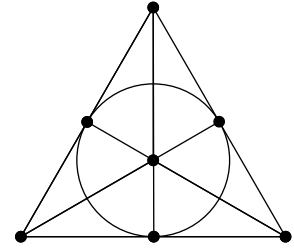


Figure 2.1: The Fano plane.

Theorem 2.1.6. The incidence geometry $\text{PG}(n, q)$ satisfies the axioms of a projective space.

We will not prove this result here; see [75, Theorem 2.2]. Now $\text{PG}(n, q)$ is Desarguesian [75, Theorem 2.4], and Vahlen [76] and Hessenberg [41] showed that every projective space of rank at least three is Desarguesian, although this result is usually attributed to Veblen and Young [80]. We obtain the following result as an immediate corollary.

Theorem 2.1.7. Any finite projective space with rank $n \geq 3$ is isomorphic to the projective space $\text{PG}(n, q)$ with the same order.

Recall that in Example 2.1.5, we assigned a nonzero vector of $V(3, 2)$ to each of the seven points of the Fano plane. We can generalise this concept to higher dimensions by representing the points of $\text{PG}(n, q)$ using *homogeneous coordinates*, that is, by associating each point with a nonzero vector of $V(n + 1, q)$. Two projective points are regarded to be the same if their corresponding homogeneous coordinates are scalar multiples of each other. This makes sense if we recall that the definition of the set of points of $\text{PG}(n, q)$ is the set of lines of $V(n + 1, q)$ through the origin. Each line of $V(n + 1, q)$ through the origin can be expressed as the span of a nonzero vector, and furthermore, two vectors that are scalar multiples of each other span the same line, which makes our definition of homogeneous coordinates consistent with the definition of projective points.

The projective spaces that we study in this dissertation are all isomorphic to $\text{PG}(n, q)$. We will therefore refer to the varieties of the associated incidence geometry as *subspaces*, since they arise from subspaces of vector spaces, and

define the *span* of a set of points in a projective space to be the smallest subspace containing every point in the set. We will also do the same when discussing substructures of projective spaces in the rest of this dissertation.

The remainder of this chapter is devoted to discussion about subsets of projective spaces that form nice incidence geometries, beginning with polar spaces.

2.2 Polar spaces

The axiomatic definition of a polar space was first stated by Veldkamp [81] in 1959, but it was shown to be equivalent to the following definition due to Buekenhout and Shult [15].

Definition 2.2.1. A *polar space of rank n* ($n \geq 2$) is an incidence geometry of rank n , whose points and lines form a partial linear space $(\mathcal{P}, \mathcal{B}, I)$ satisfying the following axioms.

- (i) Every line is incident with at least three points.
- (ii) Any two lines meet in at most one point.
- (iii) There is no point that is collinear to every other point.
- (iv) Given a point P and a line ℓ that are not incident, there is either a unique point on ℓ collinear with P , or every point on ℓ is collinear with P .

If an incidence geometry satisfies all of the conditions in Definition 2.2.1 except (iii), we say that it is a *degenerate polar space*. As we will see in the next section, we may talk about polar spaces of rank at least 2 as subspaces of a projective space. In this case, a subspace of the projective space lies in the polar space if every point in the subspace is also in the polar space. As indicated in the definition, a polar space of rank n has subspaces of projective dimension at most $n - 1$. We call subspaces of projective dimension $n - 1$ *maximals*, and we call projective $(n - 2)$ -dimensional subspaces *next-to-maximals*.

2.2.1 Classical polar spaces

Let σ be an automorphism of $\text{GF}(q)$ (in the sense defined in Section 3.1) and V a finite dimensional vector space over $\text{GF}(q)$. A σ -*sesquilinear form* over $\text{GF}(q)$ is a map $\beta : V \times V \rightarrow \text{GF}(q)$ satisfying the identities

- $\beta(u + v, w) = \beta(u, w) + \beta(v, w)$,
- $\beta(u, v + w) = \beta(u, v) + \beta(u, w)$, and
- $\beta(au, bv) = ab^\sigma \beta(u, v)$

for all $u, v, w \in V$ and $a, b \in \text{GF}(q)$. If σ is the identity map, then β is linear in both coordinates, and is called a *bilinear* form.

We say that two vectors $u, v \in V$ are *orthogonal* with respect to β if $\beta(u, v) = 0$.

A form β is *nondegenerate* if the radical

$$\text{rad}(\beta) = \{u \in V \mid \beta(u, v) = 0 \text{ for all } v \in V\}$$

of the form consists only of the zero vector. A subspace U of V is said to be *nondegenerate with respect to β* if the form restricted to U is nondegenerate. Furthermore, U is said to be a *totally isotropic* subspace if $\beta(u, v) = 0$ for all $u, v \in U$.

Let V be a vector space equipped with a sesquilinear form β . A *hyperbolic pair* is a set of two vectors $u, v \in V$ such that $\beta(u, v) = 1$ and $\beta(u, u) = \beta(v, v) = 0$. A hyperbolic pair generates a two-dimensional space. We will call the subspace of V generated by u and v a *hyperbolic line*, since it can be thought of as a projective line.

A bilinear form is said to be *alternating* if $\beta(u, u) = 0$ for all $u \in V$. A *Hermitian form* is a σ -sesquilinear form such that σ is an involutory automorphism (in other words, it has order 2), and $\beta(u, v) = \beta(v, u)^\sigma$ for all $u, v \in V$.

Example 2.2.2. The symplectic space $W(3, q)$ is the rank 2 polar space whose points are the one-dimensional subspaces of a four-dimensional vector space V over $\text{GF}(q)$. The lines of $W(3, q)$ are the totally isotropic two-dimensional subspaces of the alternating form $\beta(u, v) = u_1v_4 - v_1u_4 + u_2v_3 - v_2u_3$.

A *quadratic form* $Q : V \rightarrow \text{GF}(q)$ is a map satisfying $Q(\lambda v) = \lambda^2Q(v)$ for all $v \in V$, and such that the form defined by $\beta(u, v) = Q(u + v) - Q(u) - Q(v)$ is bilinear. In this case, we call β the *bilinear form associated to Q* . Notice that when q is even (that is, the characteristic of the field is two), we find that $\beta(u, u) = 4Q(u) - Q(u) - Q(u) = 2Q(u) = 0$, so the bilinear form associated to Q is also alternating. A subspace U of V is said to be *totally singular* if $Q(u) = 0$ for all points $u \in U$. A quadratic form is said to be nondegenerate if either the associated bilinear form is nondegenerate, or the only totally singular element of the radical of the associated bilinear form is the zero vector. We say that a subspace U is *anisotropic* if no vector contained in U is totally singular.

Example 2.2.3. The map $Q(v) = v_1^2 + v_2^2 + v_1v_2 + v_3v_4$ defines a quadratic form over $\text{GF}(q)$, with associated bilinear form

$$\beta(u, v) = 2v_1u_1 + u_1v_2 + v_1u_2 + 2v_2u_2 + u_4v_3 + u_3v_4.$$

Lemma 2.2.4 ([45, Proposition 103.31]). Let V be a vector space, and let Q be a nondegenerate quadratic form on V . Then we may write

$$V = H_1 \oplus \cdots \oplus H_k \oplus U,$$

where the H_i are mutually orthogonal hyperbolic lines and U is an anisotropic subspace with respect to Q which is orthogonal to all of the H_i , and such that $\dim(U) \leq 2$.

Suppose V_1 and V_2 are vector spaces over $\text{GF}(q)$ equipped with sesquilinear or quadratic forms β_1 and β_2 respectively. When β_1, β_2 are sesquilinear forms, a *semisimilarity* is an invertible linear map $f : V_1 \rightarrow V_2$ such that there exist $\lambda \in \text{GF}(q)$ and $\sigma \in \text{Aut}(\text{GF}(q))$ so that $\beta_2(f(u), f(v)) = \lambda\beta_1(u, v)^\sigma$ for all vectors $u, v \in V_1$. When β_1 and β_2 are quadratic forms, we then require that $\beta_2(f(v)) = \lambda\beta_1(v)$. If σ is the identity map, then the semisimilarity is called a *similarity*. When we also have $\lambda = 1$, the map is called an *isometry*. The collections of all semisimilarities, similarities and isometries of a vector space equipped with a form β onto itself are groups called the semisimilarity, similarity and isometry groups of (V, β) respectively.

For example, consider the vector space $V = V(4, q)$ equipped with the Hermitian form $\beta(u, v) = u_1v_2^q + u_2v_1^q + u_3v_4^q + u_4v_3^q$ for $v \in V$. The map given by $(v_1, v_2, v_3, v_4) \mapsto (v_1^q, v_2^q, v_3^q, v_4^q)$ defines a semisimilarity from V to itself.

There are two types of quadratic forms up to isometry [16, §3.5]. When the dimension of V is odd, the dimension of U is 1 and the two types of quadratic forms are similar (that is, there is a similarity of the underlying vector space mapping one to the other) and so there is one type of quadratic form up to equivalence. When the dimension of V is even, there is no such similarity and there are two types of quadratic forms, according to whether the dimension of U is 0 or 2. We therefore have a total of three types of quadratic forms:

- *Elliptic*: In this case, V is even dimensional and can be written as a subspace decomposition $V = H_1 \oplus \cdots \oplus H_k \oplus \langle u, v \rangle$, with $\langle u, v \rangle$ anisotropic.
- *Hyperbolic*: In this case, V is even dimensional and can be written as a subspace decomposition $V = H_1 \oplus \cdots \oplus H_k$.
- *Parabolic*: In this case, V is odd dimensional and can be written as a subspace decomposition $V = H_1 \oplus \cdots \oplus H_k \oplus \langle u \rangle$, where $\langle u \rangle$ is anisotropic.

Theorem 2.2.5 (Witt's Theorem [84]). Suppose that f is a nondegenerate σ -sesquilinear or quadratic form on a vector space V . Then any isometry between subspaces U_1 and U_2 of (V, f) can be extended to an isometry of (V, f) .

One implication of this theorem is that any two maximal totally isotropic subspaces of (V, f) have the same dimension, and we call this the *Witt index* of (V, f) .

We may construct polar spaces by taking the set of totally isotropic or totally singular subspaces in $V(n+1, q)$ of a σ -sesquilinear or quadratic form, respectively. Polar spaces constructed in this manner are called *classical*. Note that in this case, the rank of the polar space is the same as its Witt index, but one more than the projective dimension of the corresponding maximal subspaces. We talk about polar spaces being embedded into projective spaces by considering the totally isotropic or totally singular subspaces as being projective subspaces of $\text{PG}(n, q)$. The complete list of examples of classical polar spaces is as follows:

- $H(2n, q^2)$, called a *Hermitian space*, is the set of totally isotropic subspaces of a Hermitian form on $\text{PG}(2n, q^2)$. This polar space has rank n .
- $H(2n + 1, q^2)$, the Hermitian space on $\text{PG}(2n + 1, q^2)$. This polar space has rank $n + 1$.
- $W(2n + 1, q)$, called a *symplectic space*, is the set of totally isotropic subspaces of an alternating form on $\text{PG}(2n + 1, q)$. This polar space has rank $n + 1$.
- $Q(2n, q)$, called a *parabolic quadric*, is the set of totally singular subspaces of a parabolic quadratic form on $\text{PG}(2n, q)$, and has rank n .
- $Q^+(2n + 1, q)$, called a *hyperbolic quadric*, is the set of totally singular subspaces of a hyperbolic quadratic form on $\text{PG}(2n + 1, q)$, and has rank $n + 1$.
- $Q^-(2n + 1, q)$, called an *elliptic quadric*, is the set of totally singular subspaces of an elliptic quadratic form on $\text{PG}(2n + 1, q)$, and has rank n .

In fact, for large enough rank, these are the only polar spaces we must consider:

Theorem 2.2.6 ([73]). Every finite polar space with rank at least 3 is classical.

Polar space	Rank	Number of Points	Number of Maximals
$H(2n + 1, q^2)$	$n + 1$	$\frac{(q^{2n+1}+1)(q^{2n+2}-1)}{q^2-1}$	$(q + 1)(q^3 + 1) \dots (q^{2n+1} + 1)$
$H(2n, q^2)$	n	$\frac{(q^{2n+1}+1)(q^{2n}-1)}{q^2-1}$	$(q^3 + 1)(q^5 + 1) \dots (q^{2n+1} + 1)$
$W(2n + 1, q)$	$n + 1$	$\frac{(q^{n+1}+1)(q^{n+1}-1)}{q-1}$	$(q + 1)(q^2 + 1) \dots (q^{n+1} + 1)$
$Q^+(2n + 1, q)$	$n + 1$	$\frac{(q^n+1)(q^{n+1}-1)}{q-1}$	$2(q + 1)(q^2 + 1) \dots (q^n + 1)$
$Q^-(2n + 1, q)$	n	$\frac{(q^{n+1}+1)(q^n-1)}{q-1}$	$(q^2 + 1)(q^3 + 1) \dots (q^{n+1} + 1)$
$Q(2n, q)$	n	$\frac{(q^n+1)(q^n-1)}{q-1}$	$(q + 1)(q^2 + 1) \dots (q^n + 1)$

Table 2.1: Information about classical polar spaces [31, p. 35].

We summarise some key information about classical polar spaces in Table 2.1.

We say that two polar spaces are *isomorphic* if there is a bijection from the set of points, lines, planes etc. to the set of points, lines, planes etc. of the other that preserves incidence. In even characteristic we find the following isomorphism between symplectic and orthogonal polar spaces.

Theorem 2.2.7 ([19, Theorem 8.5]). There is an isomorphism between the two rank n polar spaces $W(2n - 1, q)$ and $Q(2n, q)$ over $\text{GF}(q)$, q even.

2.3 Polarities and dualities

In this section we present the theory of constructions of polar spaces and dual spaces using maps between subspaces of projective spaces.

2.3.1 Dualities

The *dual* of a projective space $\text{PG}(n, q)$ is the incidence geometry where the subspaces of projective dimension k are exchanged with those of projective dimension $n - k - 1$, with the incidence relation between subspaces preserved. Note that this new incidence geometry is a projective space. A bijection that takes a projective space to its dual space and preserves incidence while reversing inclusion is called a *duality*. If a space is isomorphic to its dual, we say it is *self dual*.

Theorem 2.3.1 ([43, p. 31]). The projective space $\text{PG}(n, q)$ is self dual when $n \geq 2$.

A *polarity* is an involutory duality acting on a self-dual projective space. One of the simplest and most commonly used polarities is the *standard duality* τ defined by

$$W^\tau = \{v \in V \mid vw^T = 0 \text{ for all } w \in W\}.$$

This duality is of order 2, so it is also a polarity. We can also define a duality as simply a bijection from the set of points of a projective space to the set of hyperplanes of the space such that if two points are collinear, their images intersect in a space of dimension $n - 2$. We can then “lift” this duality of the points to a duality of the whole projective space using the following theorem.

Theorem 2.3.2. Let Π be a projective space, and γ a duality of Π . If U is a subspace of Π , then

$$U^\gamma = \bigcap_{u \in \mathcal{P}(U)} u^\gamma,$$

where $\mathcal{P}(U)$ is the set of points of Π contained in U .

Proof. Since γ is a duality, the set $S_U := \{u^\gamma \mid u \in U\}$ is a subspace of the dual of Π . Therefore,

$$U^\gamma = S_U = \langle u^\gamma \mid u \in U \rangle = \bigcap_{u \in U} u^\gamma$$

as required. \square

We can also define the *dual polar space* of a polar space, which is defined analogously to a dual projective space. It is the incidence geometry obtained by exchanging the subspaces of the polar space of projective dimension k with those of projective dimension $n - k - 1$, where n is the rank of the polar space.

Lemma 2.3.3 ([11, §9.4]). The number of points incident with each line in a dual classical polar space is $q^e + 1$, where e is equal to 0, 1, 1, 1, 3 or 2 for the dual of $Q^+(2d - 1, q)$, $H(2d - 1, q^2)$, $Q(2d, q)$, $W(2d - 1, q)$, $H(2d, q^2)$ or $Q^-(2d + 1, q)$ respectively.

2.3.2 Polarities

We now focus on polarities. As we will see, polarities allow us to construct polar spaces.

Proposition 2.3.4 ([75, Theorem 4.5]). Suppose Π is a projective space, and let γ be a bijection (not equal to the identity map if $\Pi = \text{PG}(1, q)$) from the set of points of Π to the set of hyperplanes of Π . Then the following statements are equivalent:

- (i) The map γ is a polarity.
- (ii) If P, Q are points of Π , then $P \in Q^\gamma$ implies $Q \in P^\gamma$.

We say that a point P in a projective space Π is *absolute* with respect to a polarity γ if $P \in P^\gamma$. Moreover, a subspace V of Π is said to be *absolute* with respect to γ if $V \subseteq V^\gamma$.

Proposition 2.3.5 ([75, Theorem 4.6]). Suppose Π is a projective space and γ is a polarity of Π .

- (i) If a subspace U of Π is absolute with respect to γ , then every point contained in U is absolute with respect to γ .
- (ii) If P and Q are two points of Π that are absolute with respect to γ , then the line PQ is absolute with respect to γ if and only if $Q \in P^\gamma$.

Proof. For the proof of (i), by the definition of a polarity, for any $P \in U$ we have $P \in U \subseteq U^\gamma \subseteq P^\gamma$.

As for (ii), we begin with the ‘if’ implication. Suppose $\ell = PQ$ is an absolute line with respect to γ . Again, by definition we have $Q \in \ell \subseteq \ell^\gamma \subseteq P^\gamma$. Conversely, suppose P and Q are two absolute points of Π such that $Q \in P^\gamma$. Since γ is a polarity, we also have $P \in Q^\gamma$. Furthermore, since P and Q are absolute, $P \in P^\gamma$ and $Q \in Q^\gamma$. So $\ell = PQ \subseteq P^\gamma \cap Q^\gamma$. Suppose that K is a point incident with ℓ but distinct from P and Q . Since γ is a polarity, $K \in PQ$ implies that $(PQ)^\gamma \subseteq K^\gamma$, and in particular from Theorem 2.3.2, $P^\gamma \cap Q^\gamma \subseteq K^\gamma$ and so $\ell \subseteq K^\gamma$. Now $\ell^\gamma = \bigcap_{R \in \ell} R^\gamma \supseteq \ell$. Therefore, ℓ is absolute. \square

We can produce polarities of finite polar spaces using the associated form. Recall that a classical polar space of rank n is constructed by taking the totally isotropic or totally singular subspaces of an $(n + 1)$ -dimensional vector space, where $n \geq 2$, over a finite field $\text{GF}(q)$ with respect to a σ -sesquilinear or

quadratic form, together with the natural incidence relation. We now have the following theorem.

Theorem 2.3.6. Suppose Π is a projective space, and let γ be a polarity of Π such that there is at least one absolute line with respect to γ . Then the absolute lines and absolute points with respect to γ define a (possibly degenerate) polar space.

Proof. We simply check the axioms of a polar space given in Definition 2.2.1. Every line is incident with at least three points because any line of Π is incident with at least three points, and all of the points on an absolute line are absolute by Proposition 2.3.5. The property that any two lines meet in at most one point is also inherited from the underlying projective space Π . Suppose ℓ is an absolute line of Π , and let P be an absolute point not on ℓ , and Q an absolute point on ℓ . Then the line PQ is absolute if and only if $Q \in P^\gamma$, by Proposition 2.3.5. Now, P^γ is a hyperplane of Π and so ℓ is either contained in P^γ or meets it in a point. The former case implies that every point on ℓ is contained within P^γ , and so every line incident with P that is also incident with a point of ℓ is absolute. Alternatively, if ℓ meets P^γ in a single point, then that is the only point of ℓ incident with an absolute line on P . As for axiom (iii), if there no absolute point not incident with the absolute line ℓ , then the polar space is degenerate. Otherwise, the axiom is satisfied. \square

For a quadratic form Q , let f be the associated symmetric bilinear form, and for a σ -sesquilinear form β , let $f = \beta$. Then, if W is a subspace of a vector space $V(n+1, q)$, we define

$$W^\perp = \{v \in V \mid f(v, w) = 0 \text{ for all } w \in W\}.$$

Viewing W as a set of subspaces of $\text{PG}(n, q)$, the map \perp that takes W to W^\perp is a polarity. The subspaces of a polar space are exactly those that satisfy $W \subseteq W^\perp$.

2.4 The Klein correspondence

The Klein correspondence is a way of connecting three-dimensional projective space with a hyperbolic quadric $Q^+(5, q)$ known as the *Klein quadric* in $\text{PG}(5, q)$. This correspondence gives a geometric understanding of the isomorphism between certain polar spaces, and therefore their automorphism groups. The latter will be explored further in Chapter 3.

Recall that we can think of $\text{PG}(3, q)$ as the space formed by taking the 1-, 2- and 3-dimensional subspaces through the origin of a 4-dimensional vector space to be the points, lines and planes of $\text{PG}(3, q)$ respectively. As mentioned in Section 2.1, we can represent each of the projective points using *homogeneous coordinates*. Now, if we think of a plane of $\text{PG}(3, q)$ as being the orthogonal

complement of a point, then it seems fitting to adopt the notation $[A, B, C, D]$ to denote the plane $Ax_1 + Bx_2 + Cx_3 + Dx_4 = 0$. We can immediately define a duality τ between the points and planes of $\text{PG}(3, q)$ via $(w, x, y, z) \mapsto [w, x, y, z]$, and that this is in fact the standard duality.

It remains to find a representation of lines. We can express a line ℓ as the span of two points, and in particular, as the span of their homogeneous coordinates $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$. We consider the matrix formed by taking these vectors as the rows:

$$M_{u,v} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix}.$$

By the definition of homogeneous coordinates, $M_{u,v}$ has full rank, and in particular, any 2×2 submatrix has nonzero determinant. There are six 2×2 submatrices (up to column ordering), and if we write

$$c_{ij} = \det \begin{pmatrix} u_i & u_j \\ v_i & v_j \end{pmatrix},$$

then we can define the *Plücker coordinates* for ℓ to be $(c_{12}, c_{13}, c_{14}, c_{23}, c_{42}, c_{34})$. If we express ℓ as the span of two different points, u' and v' , then we can write the corresponding matrix $M_{u',v'}$ as $XM_{u,v}$, for some 2×2 matrix X . The Plücker coordinates for this representation of ℓ are therefore $\det(X)(c_{12}, c_{13}, c_{14}, c_{23}, c_{42}, c_{34})$ since the determinant function is multiplicative. Therefore, if we think of the Plücker coordinates as the homogeneous coordinates of a point of $\text{PG}(5, q)$, the correspondence is independent of our representation of ℓ .

Now consider the 4×4 matrix formed by taking two copies of M_{uv} :

$$\bar{M}_{u,v} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix}.$$

This matrix has rank 2, and therefore determinant 0. By appropriately rearranging the expression for the determinant, we obtain $c_{12}c_{34} + c_{13}c_{42} + c_{14}c_{23} = 0$. Therefore the Plücker coordinates of a line of $\text{PG}(3, q)$ are a zero of the following quadratic form of $\text{PG}(5, q)$:

$$Q(x_1, x_2, x_3, x_4, x_5, x_6) = x_1x_6 + x_2x_5 + x_3x_4.$$

The totally singular subspaces of this quadratic form define a hyperbolic quadric $Q^+(5, q)$, called the *Klein quadric*. Incredibly, we find that the converse holds as well, and there is a bijection between lines of $\text{PG}(3, q)$ and points of the Klein quadric. Moreover, concurrent lines in $\text{PG}(3, q)$ map to collinear points of the Klein quadric. The Klein quadric also contains planes of $\text{PG}(5, q)$. For example, consider $\pi : x_1 = x_3 = x_5 = 0$. Suppose P is a point of $\text{PG}(3, q)$. The

image of the set of lines incident with P under the Klein correspondence is a set of pairwise collinear points of the Klein quadric that together form a plane. Similarly, if we take the image of a set of lines lying on a plane of $\text{PG}(3, q)$ under the Klein correspondence, we also construct a plane of the Klein quadric.

Consider the symplectic space $W(3, q)$ defined by the bilinear alternating form

$$\beta(u, v) = u_1v_2 - u_2v_1 + u_3v_4 - v_3u_4.$$

If $\ell = M_{u,v}$ is a totally isotropic line of this symplectic space, then we must have $\beta(u, v) = c_{12} + c_{34} = 0$. Thus the lines of $W(3, q)$ are mapped to points on the hyperplane $x_1 + x_6 = 0$ under the Klein correspondence. This hyperplane is nondegenerate and so we have a bijection from $W(3, q)$ to $Q(4, q)$, since the number of lines of $W(3, q)$ is equal to the number of points of $Q(4, q)$ (see Table 2.1). Therefore, $W(3, q)$ and $Q(4, q)$ are dual to each other, and furthermore, they are generalised quadrangles of order (q, q) .

In a similar way, the Klein correspondence can be used to show that $H(3, q^2)$ and $Q^-(5, q)$ are duals of each other. The construction is quite involved, so we do not give it here. See [19, p. 111].

3. Groups and actions

In this section we assume that the reader has an undergraduate knowledge of group theory. For further background reading, and the proofs of any basic group theory results outlined here, see [38].

3.1 Group actions

In our context, we are interested in the relationship between groups and geometries. This relationship relies heavily on group actions, and we briefly review the theory of group actions here.

Definition 3.1.1. An action of a group G on a set Ω is a map $G \times \Omega \rightarrow \Omega$, denoted by $(g, \omega) \mapsto \omega^g$, such that

- (i) $\omega^1 = \omega$, and
- (ii) $\omega^{gh} = (\omega^g)^h$.

We say that G acts on Ω .

We say that G acts *faithfully* on Ω if $\omega^g = \omega$ for all $\omega \in \Omega$ implies that g is the identity element of G . Note that if G acts faithfully on Ω then distinct elements of G give rise to distinct permutations of Ω .

Example 3.1.2. The finite field $\text{GF}(q)$ acts on itself by right multiplication, since $x^1 = x1 = x$ for all $x \in \text{GF}(q)$, and $x^{ab} = xab = (xa)b = (x^a)^b$ for all $x, a, b \in \text{GF}(q)$. This action is faithful, since for all $x \in G$, if $x^a = x^b$, then $xa = xb$, and so $a = b$.

The action of G defines an equivalence relation \sim_G on Ω , with $\omega_1 \sim_G \omega_2$ if and only if there exists a group element $g \in G$ such that $\omega_1^g = \omega_2$. The equivalence classes are called the *orbits* of G on Ω . We denote the orbit of $\omega \in \Omega$ by $\omega^G = \{\omega^g \mid g \in G\}$.

Example 3.1.3. Let G be a group. Then G acts on itself by conjugation by $\varphi_g : h \mapsto g^{-1}hg$ for all $g \in G$. Two elements $h, k \in G$ are said to be *conjugate* if they lie in the same orbit under this action. The orbits are called *conjugacy classes*. Notice that every element of a conjugacy class has the same order.

If G has one orbit on Ω (that is, $\omega^G = \Omega$ for all $\omega \in \Omega$), we say that G acts *transitively* on Ω . We define the image of a subset W of Ω under $g \in G$ to be $\{w^g \mid w \in W\}$, and denote this set by W^g . The orbit of W under G is defined analogously as $W^G = \{W^g \mid g \in G\}$. The *stabiliser* of $\omega \in \Omega$ is the set $G_\omega = \{g \in G \mid \omega^g = \omega\}$. We say that a group acts *regularly* on a set Ω if it

acts transitively and there is no nonidentity element of G that fixes an element of Ω .

We can also define the stabiliser of a subset of Ω in two different ways. The *setwise stabiliser* of $W \subseteq \Omega$ is the set $\{g \in G \mid \omega^g \in W \text{ for all } \omega \in W\}$. On the other hand, the *pointwise stabiliser* of $W \subseteq \Omega$ is the set of elements of G that fix every element of W individually, or alternatively, it is the intersection of the stabilisers of each element of W . We will denote both the setwise stabiliser and pointwise stabiliser of a set W by G_W , and it will be clear from the context which definition is being adopted.

Example 3.1.4. Suppose that H is a subgroup of G . Then G acts on the set of right cosets of H by right multiplication. In fact, G acts transitively on the set of cosets of H , because given two cosets Hg_1 and Hg_2 , we can map the first to the second via the element $g_1^{-1}g_2 \in G$. Moreover, if $g \in G$ is in the stabiliser of H , then $Hg = H$. This occurs if and only if $g \in H$, by the definition of a coset.

Theorem 3.1.5 (Orbit–Stabiliser Theorem). Suppose G is a finite group acting on a set Ω . Then

$$|G| = |G_\omega| |\omega^G|$$

for all $\omega \in \Omega$.

The set of permutations of a set Ω forms the *symmetric group* of Ω under composition, and is denoted $\text{Sym}(\Omega)$. If a group G acts on a set Ω , then, since every element of G induces a permutation $\omega \mapsto \omega^g$ of Ω , G is homomorphic to a subgroup of $\text{Sym}(\Omega)$. If G acts faithfully on Ω , then it is isomorphic to a subgroup of $\text{Sym}(\Omega)$. We call a subgroup of $\text{Sym}(\Omega)$ a *permutation group* on Ω .

We are often interested in group actions that preserve some additional structure of the underlying set, such as multiplication in a finite field, or collinearity in a finite incidence geometry. If Ω is a set with ‘structure’, then we say that a group G acts on Ω if it acts as a group of permutations on Ω and preserves this structure. The largest subgroup of $\text{Sym}(\Omega)$ that preserves the structure of Ω is called the *automorphism group* of Ω , denoted $\text{Aut}(\Omega)$, and the elements of $\text{Aut}(\Omega)$ are called *automorphisms*. Any group G that preserves the structure of Ω is isomorphic to a subgroup of $\text{Aut}(\Omega)$.

Example 3.1.6. Let $\text{GF}(q)$ be the finite field of order $q = p^k$ for some prime p , and $k \in \mathbb{N}$. The automorphism group of $\text{GF}(q)$, when considered as a group under composition, is cyclic of order k , and generated by the *Frobenius automorphism* $\sigma_p : x \mapsto x^p$. Note that the ‘structure’ that we are preserving here is multiplication, and in particular, every automorphism must be a homomorphism. This explains why the action by right multiplication described in Example 3.1.2 is not an automorphism of $\text{GF}(q)$, since a map $\phi_y : x \mapsto xy$ is not a homomorphism.

Consider two projective spaces $\text{PG}(n_1, q_1)$ and $\text{PG}(n_2, q_2)$ over $\text{GF}(q_1)$ and $\text{GF}(q_2)$ respectively. Then a bijection f from the set of points of $\text{PG}(n_1, q_1)$ to the set of points of $\text{PG}(n_2, q_2)$ is called a *collineation* if for every three points P, P', P'' in $\text{PG}(n_1, q_1)$, their images under f are collinear in $\text{PG}(n_2, q_2)$ if and only if $P, P',$ and P'' are collinear in $\text{PG}(n_1, q_1)$. A group of collineations from a projective space to itself is called a *collineation group*.

Proposition 3.1.7 ([75, Theorems 4.1 & 4.2]). Consider two finite-dimensional projective spaces $\text{PG}(n, q_1)$ and $\text{PG}(n, q_2)$ over $\text{GF}(q_1)$ and $\text{GF}(q_2)$ respectively, and let $f : \text{PG}(n, q_1) \rightarrow \text{PG}(n, q_2)$ be a collineation. Then the image under f of a subspace $U \subseteq \text{PG}(n, q_1)$ of projective dimension k is a subspace of projective dimension k in $\text{PG}(n, q_2)$. Furthermore, f is a bijection between the sets of subspaces of $\text{PG}(n, q_1)$ and $\text{PG}(n, q_2)$ of projective dimension k .

Definition 3.1.8. A *Frobenius group* $F \leq \text{Sym}(\Omega)$ is a transitive permutation group on a finite set Ω such that some nontrivial element of F fixes an element of Ω , and every nontrivial element of F fixes at most one element of Ω .

For example, we can generate a Frobenius group as a subgroup of the collineation group of the Fano plane defined in Example 2.1.5. Let τ_1 be the collineation of the Fano plane corresponding to a rotation by $\frac{2\pi}{3}$, and let τ_2 be a cyclic permutation of order 7 of the points of the Fano plane, such that $\tau_1\tau_2 = \tau_2^2\tau_1$. Then the group generated by τ_1 and τ_2 is a Frobenius group of order 21, and we denote it by F_{21} .

Lemma 3.1.9. Suppose G is a group that acts on a set Ω and let $H \leq G$. Then H also acts on Ω , and the orbits of G on Ω are unions of the orbits of H on Ω .

We now present some group theoretic results that will assist us in chapters to come.

Theorem 3.1.10 (Cauchy's Theorem). Let G be a group. If p is a prime such that p divides the order of G , then there exists an element of order p in G .

Theorem 3.1.11 (Sylow's First Theorem). Let G be a group. Suppose p is a prime such that $|G| = p^k n$ with $p \nmid n$. Then there exists a subgroup of G of order p^k .

Let p be a prime. A group G is called a *p-group* if every element of G has order a power of p .

Lemma 3.1.12. A finite group G is a *p-group* if and only if $|G| = p^k$, for some $k \in \mathbb{N}$.

Proof. Suppose G is a finite *p-group*. Then every element of G has order a power of p . By Cauchy's Theorem, if there was another prime p' dividing the

order of G , then there would exist an element of order p' , contradicting that G is a p -group. Therefore, $|G| = p^k$ for some k . Suppose on the other hand that $|G| = p^k$, for some $k \in \mathbb{N}$. Then by Lagrange's Theorem, every element of G must have order dividing p^k , that is, every element must have order a power of p . \square

We say that a group is *elementary abelian* if it is a p -group for some prime p , and every element of the group has order p .

3.2 Group products

Apart from investigating and describing the actions of groups on finite geometries and generalised quadrangles, we also need some machinery to describe the structure of groups. We do this using a variety of group products.

3.2.1 Semidirect products

Definition 3.2.1. Suppose G is a group with subgroups N and K such that

- N is a normal subgroup of G ,
- $N \cap K = \{1\}$, and
- $NK = G$.

Then G is the *semidirect product* of N and K , denoted $G = N \rtimes K$, or sometimes $G = N : K$.

If G is the semidirect product of N and K , then the third condition of the definition tells us that we can write each element of G as nk for some $n \in N$ and $k \in K$. If $n_1k_1 = n_2k_2$ for some $n_1, n_2 \in N$ and $k_1, k_2 \in K$, then we must have $1 = n_1^{-1}n_2k_2k_1^{-1}$. This implies that $(n_1^{-1}n_2)^{-1} = k_2k_1^{-1}$, which only holds if both expressions are equal to 1, since $N \cap K = \{1\}$. Therefore, $n_1 = n_2$ and $k_1 = k_2$ by uniqueness of inverses in a group, and every element $g \in G$ can be written uniquely as $g = nk$ for some $n \in N$ and $k \in K$.

The following proposition gives insight into how multiplication in a semidirect product maintains the group structure.

Proposition 3.2.2. Multiplication of elements in $G = N \rtimes K$ can be written as

$$(n_1k_1)(n_2k_2) = n_1\phi_{k_1}(n_2)k_1k_2,$$

where $\phi_k : n \mapsto knk^{-1}$, and the inverse of any element $nk \in G$ can be written as $(nk)^{-1} = \phi_{k^{-1}}(n^{-1})k^{-1}$.

Proof. First, $(n_1k_1)(n_2k_2) = n_1k_1n_2(k_1^{-1}k_1)k_2 = n_1\phi_{k_1}(n_2)k_1k_2$. The inverse of nk is $\phi_{k^{-1}}(n^{-1})k^{-1}$ since $nk\phi_{k^{-1}}(n^{-1})k^{-1} = nk(k^{-1}n^{-1}k)k^{-1} = 1$. \square

Therefore, in order to define the group structure of the semidirect product of N and K , we only need N , K and the function $\phi : K \rightarrow \text{Aut}(N)$ which maps an element of $k \in K$ to the corresponding conjugation map ϕ_k .

Example 3.2.3. Let F be a Frobenius group, acting on a set Ω . The *Frobenius complement* of F is a subgroup H of F such that $H \cap g^{-1}Hg = \{1\}$ whenever we have $g \notin H$. The *Frobenius kernel* K is defined as the identity element of F , together with all of the elements of F that are not conjugate to elements of H . Not only does the Frobenius kernel form a group, but a theorem of Frobenius [37] shows that it is normal. By definition, $H \cap K = \{1\}$. In addition, $|F| = \frac{|F|}{|H|}(|H| - 1) + |K|$, which implies that $|F| = |H||K|$, so $F = HK$ by the trivial intersection of H and K , and hence $F = K \rtimes H$.

Conjugation is not the only automorphism of N that gives us the group structure of a semidirect product. In fact, if we consider the converse and start with groups N and K , as well as a homomorphism $\phi : K \rightarrow \text{Aut}(N)$, we can construct a group that is the semidirect product of N and K .

Construction 3.2.4. Let N and K be groups, and let $\phi : K \rightarrow \text{Aut}(N)$ be a homomorphism. Then we can construct a group G that is the semidirect product of N and K as follows. The elements of G are the tuples (n, k) with $n \in N$ and $k \in K$. We define multiplication of elements of G by

$$(n_1, k_1)(n_2, k_2) = (n_1\phi_{k_1}(n_2), k_1k_2).$$

By this definition, the identity of G is $(1, 1)$. Then the inverse of each element $(n, k) \in G$ is given by

$$(n, k)^{-1} = (\phi_{k^{-1}}(n^{-1}), k^{-1}).$$

Since we just need any homomorphism ϕ between K and $\text{Aut}(N)$ to define the semidirect product of N and K , sometimes the notation $G = N \rtimes K$ is insufficient. To be more explicit, we write $G = N \rtimes_{\phi} K$ when it is not clear which homomorphism is being used.

3.2.2 Wreath products

Another type of group product that is useful in describing groups in future chapters is the wreath product.

Let Δ and $\Omega = \{1, 2, \dots, n\}$ be finite sets, and $H \leq \text{Sym}(\Delta)$ and $K \leq \text{Sym}(\Omega)$ be groups. Set $G = H^n$ (the n -fold direct product of H). Then the *wreath product* W of H and K , denoted $H \wr K$, is $G \rtimes K$. The multiplication of elements $(h_1, h_2, \dots, h_n, k)$ and $(h'_1, h'_2, \dots, h'_n, k')$ in W is as follows:

$$(h_1, h_2, \dots, h_n, k)(h'_1, h'_2, \dots, h'_n, k') = (h_1h'_{1k^{-1}}, h_2h'_{2k^{-1}}, \dots, h_nh'_{nk^{-1}}, kk').$$

The wreath product W acts on $\Sigma = \Delta \times \Omega$ by the following operation:

$$(h_1, h_2, \dots, h_n, k) : (\delta, i) \mapsto (\delta^{h_i}, i^k).$$

Example 3.2.5. The smallest example of a wreath product is $C_2 \wr C_2$, which is isomorphic to the group of symmetries of a square, namely the dihedral group D_8 .

3.3 Classical groups

Classical groups are fundamental to our understanding of groups acting on classical polar spaces, since they both originate from the theory of sesquilinear and quadratic forms on finite-dimensional vector spaces. We always assume that our groups are finite.

3.3.1 Linear and projective linear groups

Let V be an n -dimensional vector space over a field $\text{GF}(q)$, and let \mathcal{B} be a basis for V . The set of all invertible linear transformations $f : V \rightarrow V$ forms a group under composition, called the *general linear group*, denoted $\text{GL}(n, q)$. We can also think of $\text{GL}(n, q)$ as a matrix group, that is, the set of $n \times n$ invertible matrices over $\text{GF}(q)$ under matrix multiplication. The centre of $\text{GL}(n, q)$ is comprised of maps $x \mapsto \lambda x$ for $\lambda \in \text{GF}(q)^\times$, the multiplicative group of $\text{GF}(q)$. In particular, the centre is the kernel of $\text{GL}(n, q)$ in its action on one-dimensional subspaces of V through the origin. When we take the quotient of $\text{GL}(n, q)$ by its centre, we obtain the *projective general linear group*, denoted by $\text{PGL}(n, q)$. Let us consider the projective space $\text{PG}(n-1, q)$ constructed from V . The projective general linear group acts faithfully on one-dimensional subspaces of V , and therefore on the points of $\text{PG}(n-1, q)$. Hence, $\text{PGL}(n, q)$ is isomorphic to a permutation group on the points of $\text{PG}(n-1, q)$ preserving the structure (which in this case is incidence), since every element of $\text{PGL}(n, q)$ is a linear transformation.

The *special linear group* $\text{SL}(n, q)$ is the subgroup of $\text{GL}(n, q)$ comprised of isometric linear transformations. If we view $\text{GL}(n, q)$ as a matrix group, we can think of $\text{SL}(n, q)$ as the subgroup of matrices of determinant 1. The *projective special linear group* $\text{PSL}(n, q)$ is constructed by taking the quotient of $\text{SL}(n, q)$ by its centre (which is a subgroup of the centre of $\text{GL}(n, q)$).

A *semilinear transformation* on V is a map $f : V \rightarrow V$ such that for all $u, v \in V$ and $\lambda \in \text{GF}(q)$ the following conditions hold:

- (i) $f(u + v) = f(u) + f(v)$,
- (ii) $f(\lambda v) = \lambda^\sigma f(v)$,

where σ is a field automorphism of $\text{GF}(q)$.

The set of all semilinear transformations on V forms a group called the *general semilinear group* of V , denoted $\Gamma\mathrm{L}(n, q)$. Notice that $\mathrm{GL}(n, q) \leq \Gamma\mathrm{L}(n, q)$, since we can take the field automorphism σ to be the identity automorphism.

Proposition 3.3.1. The general semilinear group $\Gamma\mathrm{L}(n, q)$ over a vector space $V = V(n, q)$ is generated by the elements of $\mathrm{GL}(n, q)$, together with the Frobenius automorphism $\sigma_p \in \mathrm{Aut}(\mathrm{GF}(q))$. The Frobenius automorphism and its powers act on subspaces of V by acting coordinatewise on points of the subspace.

Analogously, we define the *projective semilinear group* $\mathrm{P}\Gamma\mathrm{L}(n, q)$, which is the extension of $\mathrm{PGL}(n, q)$ by field automorphisms.

The projective semilinear group of V acts on the set of points of the projective space $\mathrm{PG}(n-1, q)$ constructed from V . It also preserves the collinearity property, by the following argument. Suppose $P = \langle u \rangle$, $Q = \langle v \rangle$ are two points of $\mathrm{PG}(n-1, q)$, or equivalently, two one-dimensional subspaces of V . Then any projective point (resp. one-dimensional subspace) lying in the span of P and Q can be written as $\langle \lambda u + \mu v \rangle$ for $\lambda, \mu \in \mathrm{GF}(q)$. If $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$, then $\langle \lambda u + \mu v \rangle^\sigma = \langle \lambda^\sigma u^\sigma + \mu^\sigma v^\sigma \rangle$, where u^σ and v^σ are the componentwise application of σ to u and v respectively. Thus, $\langle \lambda u + \mu v \rangle^\sigma$ is contained in the span of $P^\sigma = \langle u^\sigma \rangle$ and $Q^\sigma = \langle v^\sigma \rangle$. Therefore, incidence in $\mathrm{PG}(n-1, q)$ is preserved by field automorphisms and $\mathrm{P}\Gamma\mathrm{L}(n, q)$ does indeed act on the set of points of $\mathrm{PG}(n-1, q)$. In fact, the following results show us that these are the only automorphisms of $\mathrm{PG}(n-1, q)$ that we need to consider.

Theorem 3.3.2 (The Fundamental Theorem of Projective Geometry [79]). Suppose $V_1 = V(n_1, q_1)$ and $V_2 = V(n_2, q_2)$ are vector spaces over $\mathrm{GF}(q_1)$ and $\mathrm{GF}(q_2)$ respectively, with $n_1 \geq 3$. Let $f : \mathrm{PG}(V_1) \rightarrow \mathrm{PG}(V_2)$ be a collineation between the projective spaces constructed from V_1 and V_2 respectively. Then $n_1 = n_2$, $q_1 = q_2$ and there exists a semilinear map ϕ that has the same action as f .

Corollary 3.3.3. The collineation group of $\mathrm{PG}(n-1, q)$, $n \geq 2$, is isomorphic to $\mathrm{P}\Gamma\mathrm{L}(n, q)$.

Proof. Recall that we construct $\mathrm{PG}(n-1, q)$ from a vector space $V = V(n, q)$. By Theorem 3.3.2, every collineation from $\mathrm{PG}(n-1, q)$ to itself is induced by a semilinear map. Conversely, we have already shown that every projective semilinear map acts on the points of $\mathrm{PG}(n-1, q)$. The result immediately follows. \square

We now discuss subgroups of linear groups that we will later find are the automorphism groups of the classical polar spaces discussed in Section 2.2. As might be suspected, the definition of these groups relies on the theory of sesquilinear and quadratic forms, and the fact that they are automorphism groups of polar

spaces relies on the following result.

Theorem 3.3.4 ([73]). An isomorphism between classical polar spaces of rank at least 2, which are not of symplectic or orthogonal type in even characteristic, is induced by a semilinear transformation of the underlying vector spaces.

Symplectic and orthogonal polar spaces in even characteristic are excluded from Theorem 3.3.4 because as in Theorem 2.2.7, there are isomorphisms between these two polar spaces that are not semisimilarities of the underlying vector spaces.

3.3.2 Unitary groups

Let $\beta : V \times V \rightarrow \text{GF}(q^2)$ be a nondegenerate Hermitian form on a vector space $V = V(n, q^2)$. Recall that an *isometry* of β is a map $f : V \rightarrow V$ such that $\beta(u, v) = \beta(f(u), f(v))$ for all $u, v \in V$, and that the set of all isometries on a vector space equipped with a form β is a group, called the *isometry group* of (V, β) . The isometry group of a vector space equipped with a Hermitian form is called the *general unitary group*, denoted $\text{GU}(n, q)$. As in the linear case, we can also think of the general unitary group as a matrix group $\text{GU}(n, q) = \{M \in \text{GL}(n, q^2) \mid A(A^q)^T = I\}$, where A^q is the matrix obtained by raising each entry of A to the power of q . The *special unitary group* $\text{SU}(n, q)$ is the intersection of $\text{GU}(n, q)$ with the special linear group $\text{SL}(n, q^2)$. Equivalently, in the matrix model of $\text{GU}(n, q)$, the special unitary group is the subgroup of all matrices with determinant 1.

We denote the group of all similarities of (V, β) by $\Delta\text{U}(n, q)$, and the group of all semisimilarities by $\Gamma\text{U}(n, q)$. Let Z be the set of matrices $\{\lambda I_n \mid \lambda \in \text{GF}(q^2)^\times\}$. Notice that we can construct a normal subgroup of any of $\text{GU}(n, q)$, $\text{SU}(n, q)$, $\Delta\text{U}(n, q)$ or $\Gamma\text{U}(n, q)$ by taking its intersection with Z . We define the projective versions of these groups as their quotients by their intersections with Z . The projective versions of $\text{GU}(n, q)$, $\text{SU}(n, q)$, $\Delta\text{U}(n, q)$ and $\Gamma\text{U}(n, q)$ are denoted by $\text{PGU}(n, q)$, $\text{PSU}(n, q)$, $\text{P}\Delta\text{U}(n, q)$ and $\text{P}\Gamma\text{U}(n, q)$, respectively. Notice that these groups are all subgroups of $\text{P}\Gamma\text{L}(n, q^2)$.

Recall that a Hermitian space $\text{H}(n, q^2)$ is the set of totally isotropic subspaces with respect to a Hermitian form β on $\text{PG}(n, q^2)$, the projective space constructed from $V(n+1, q^2)$. Therefore, the automorphism group of $\text{H}(n, q^2)$ must both preserve the form β on $\text{PG}(n, q^2)$, and by Theorem 3.3.4, all of the automorphisms must arise from semisimilarities of the underlying vector space of $\text{PG}(n, q^2)$. Notice that the totally isotropic subspaces of $\text{H}(n, q^2)$ are preserved by $\text{P}\Gamma\text{U}(n+1, q)$. Therefore, the full automorphism group of $\text{H}(n, q^2)$ is $\text{P}\Gamma\text{U}(n+1, q^2)$.

3.3.3 Symplectic groups

We begin this section by looking at vector spaces equipped with alternating forms more closely. Recall from Section 2.2.1 the definition of a hyperbolic pair.

Lemma 3.3.5. Suppose V is a vector space equipped with a nondegenerate alternating form β . Then V has a hyperbolic pair.

Proof. Let u be a nonzero vector in V . Then since β is nondegenerate, there exists a vector $v \in V$ such that $\beta(u, v) \neq 0$. Define a new vector $w = \beta(u, v)^{-1}v$. Then since β is alternating, $\beta(u, w) = \beta(u, v)^{-1}\beta(u, v) = 1$, as required. \square

Lemma 3.3.6. Suppose β is a nondegenerate alternating form on a vector space $V = V(n, q)$. Then n is even.

Proof. We will prove this result using induction on n . Let $\{u, v\}$ be a hyperbolic pair of (V, β) , and define $U = \langle u, v \rangle$. We claim that $V = U \oplus U^\perp$, where \perp is the duality induced by β . Let $w \in U$. The map $\varphi_w : V \rightarrow \text{GF}(q)$ defined by $x \mapsto \beta(x, w)$ is linear, by the definition of an alternating form. Since β is nondegenerate, φ_w is surjective, and so the dimension of the kernel of φ_w is $n-1$. Now, we may write $U^\perp = \ker(\varphi_u) \cap \ker(\varphi_v)$. Since $u \in \ker(\varphi_u)$, but $u \notin \ker(\varphi_v)$, we have that the dimension of U^\perp is $n-2$. Furthermore, $U \cap U^\perp = \{0\}$ because U is nondegenerate. Therefore, $V = U \oplus U^\perp$ and our claim is proved. We now show that U^\perp is nondegenerate. If x is an element of the radical of β restricted to U^\perp , then x must be in the radical of β , by the last argument. Now, since β is nondegenerate, x is the zero vector and U^\perp is nondegenerate. We can repeat this argument on U^\perp and so on, and therefore, by induction, the dimension of U^\perp is even and hence so is the dimension of V . \square

Suppose $V = V(2n, q)$ is a vector space equipped with a nondegenerate alternating form β . The isometry group of (V, β) is called a *symplectic group*, denoted $\text{Sp}(2n, q)$. The similarity and semisimilarity groups of (V, β) are denoted $\text{GSp}(2n, q)$ and $\Gamma\text{Sp}(2n, q)$, respectively. As in the last section, let Z be the set of scalars $\{\lambda I_n \mid \lambda \in \text{GF}(q)^\times\}$. Again, we construct the projective versions of $\text{Sp}(2n, q)$, $\text{GSp}(2n, q)$ and $\Gamma\text{Sp}(2n, q)$ by taking the quotient of each of them by their intersections with Z . Note that $\text{Sp}(2n, q) \cap Z = \{\pm I_n\}$ if q is odd, and $\text{Sp}(2n, q) \cap Z = \{I_n\}$ if q is even. Moreover, if q is even, then $\text{PSp}(2n, q) = \text{Sp}(2n, q)$. As in the linear case, $\text{P}\Gamma\text{Sp}(2n, q) = \langle \text{GSp}(2n, q), \sigma_p \rangle$, where σ_p is the Frobenius automorphism of $\text{GF}(q)$, and this group not only preserves the symplectic space defined by β , but comprises its full set of automorphisms by Theorem 3.3.4.

3.3.4 Orthogonal groups

Recall from Section 2.2 that there are three types of quadratic forms: parabolic type on odd-dimensional vector spaces, and hyperbolic and elliptic types on even-dimensional vector spaces.

The *orthogonal group* is the isometry group of a vector space V equipped with a quadratic form Q . In odd dimension $2k + 1$, we denote the orthogonal group by $O(2k + 1, q)$. In even dimension, we have two types of orthogonal groups, the so-called *plus* and *minus* types $O^+(2k, q)$ and $O^-(2k, q)$, which are the isometry groups corresponding to hyperbolic and elliptic quadratic forms respectively. The *special orthogonal group* $SO(n, q)$ is the subgroup of the orthogonal group consisting of matrices in the kernel of the determinant map. We also define $\Omega(n, q)$ to be the derived subgroup of the special orthogonal group, and set

$$P\Omega(n, q) = \Omega(n, q) / (\Omega(n, q) \cap Z),$$

where Z is defined as before. Again, the precise definition of these groups will vary depending on what type of quadratic form we are working with, and the notation is analogous to that used in the definition of the orthogonal group.

We further denote the similarity and semisimilarity groups of (V, Q) by $GO(n, q)$ and $\Gamma O(n, q)$. The projective versions of all of the groups just defined are constructed in the same way as in the symplectic and unitary cases.

4. Generalised quadrangles

Generalised polygons were introduced by Tits in 1959 to describe the rank 2 residues of spherical buildings [72]. Feit and Higman [35] proved that nontrivial finite generalised n -gons exist only when $n \in \{2, 3, 4, 6, 8\}$. We are exclusively interested in generalised 4-gons, also known as *generalised quadrangles*.

4.1 Definitions and examples

A *generalised quadrangle* of order (s, t) is a partial linear space $(\mathcal{P}, \mathcal{B}, I)$ that satisfies the following conditions:

- (i) Every line is incident with exactly $s + 1$ points.
- (ii) Every point is incident with exactly $t + 1$ lines.
- (iii) (*The GQ axiom.*) For every point P and every line ℓ not incident with P , there is a unique point K on ℓ collinear with P .

For conciseness, we sometimes denote a generalised quadrangle of order (s, t) by $\text{GQ}(s, t)$. Notice that the GQ axiom implies the nonexistence of triangles in generalised quadrangles.

Example 4.1.1. The smallest example of a generalised quadrangle $\text{GQ}(s, t)$ with $s, t > 1$ is the unique generalised quadrangle of order $(2, 2)$, pictured in Figure 4.1. It has 15 points and lines, and is isomorphic to the symplectic space $W(3, 2)$.

Suppose $R = (\mathcal{P}, \mathcal{B}, I)$ is a generalised quadrangle of order (s, t) . If we switch the set we call ‘points’ and the set we call ‘lines’, the resulting incidence structure $\bar{R} = (\mathcal{B}, \mathcal{P}, I)$ is called the *dual* of R , and is a generalised quadrangle of order (t, s) .

Lemma 4.1.2. Suppose R is a generalised quadrangle of order (s, t) , and let ℓ and n be two nonconcurrent lines of R . Then the number of lines concurrent with both ℓ and n is $s + 1$.

Proof. By the GQ axiom, every point on ℓ must be collinear with a unique point of n . In particular, two points P and Q incident with ℓ must be collinear with different points on n , because otherwise we form a triangle. Thus the line incident with P and concurrent with n must be different to the line incident with Q and concurrent with n . Therefore, there are $s + 1$ lines concurrent with both ℓ and n . □

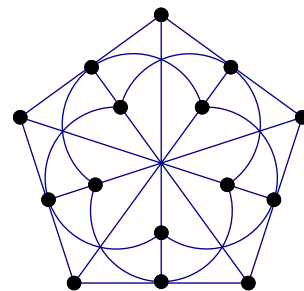


Figure 4.1: The unique generalised quadrangle of order $(2, 2)$.

We are mostly interested in generalised quadrangles of order (q^2, q) , for q a power of 2, since the geometric objects we examine in future chapters are defined on generalised quadrangles with these parameters.

A generalised quadrangle may be considered *classical* or *nonclassical*, depending on how it arises. Classical generalised quadrangles arise from totally isotropic subspaces of sesquilinear and quadratic forms. As an example, a classical generalised quadrangle of order (q^2, q) , q even, is a Hermitian space $H(3, q^2)$. The points of the generalised quadrangle are the totally isotropic 1-spaces with respect to the Hermitian form, the lines are the totally isotropic 2-spaces, and the incidence relation is defined by symmetrised inclusion. Sometimes, we want to consider the dual of a Hermitian generalised quadrangle. This is a generalised quadrangle of order (q, q^2) , with points and lines the totally singular points and lines of an elliptic quadric $Q^-(5, q)$.

All generalised quadrangles not arising from forms in this way are considered to be *nonclassical*. We are interested in nonclassical generalised quadrangles arising from two different constructions: flock generalised quadrangles arising from so-called *q-clans*, and the generalised quadrangles $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$ arising from *ovals* of $PG(2, q)$ and *ovoids* of $PG(3, q)$ respectively.

4.2 Generalised quadrangles from ovals and ovoids

An *oval* of a projective plane $PG(2, q)$ is a set of $q + 1$ points such that no three are collinear. If these $q + 1$ points are the zeros of some nondegenerate quadratic form, then we call the oval a *conic*. Segre [65] proved that when q is odd, the only ovals of $PG(2, q)$ are conics. When q is even, there are many examples of ovals that are not conics. A line that meets the oval in one point is called a *tangent*, while a line meeting the oval in two points is called a *secant*. The construction of some ovals that are not conics stems from the following result.

Lemma 4.2.1. Let \mathcal{O} be an oval of $PG(2, q)$, q even. Then the tangents to the $q + 1$ points of \mathcal{O} intersect in a single point, called the *nucleus* of \mathcal{O} .

Proof. Suppose P is a point not contained in \mathcal{O} . Then from the axioms of a projective space, the lines on P partition the points of \mathcal{O} , and in particular, every line of P is incident with either 0, 1 or 2 points of \mathcal{O} . Since $q + 1$ is odd, P must be incident with at least one tangent line to \mathcal{O} . Suppose that ℓ is a line that is secant to \mathcal{O} , with points of intersection Q and K . Then the tangent lines to the points in $\mathcal{O} \setminus \{Q, K\}$ are concurrent to ℓ , and in particular, they meet ℓ in distinct points (otherwise some points on ℓ would not be incident with a tangent to \mathcal{O}). Hence every point that lies on a secant line to \mathcal{O} must be incident with exactly one tangent. Now consider the intersection of two tangent lines to \mathcal{O} . The point of intersection cannot lie on any secants, since every point on a secant line must be a point of \mathcal{O} or incident with exactly one

tangent. Therefore, the point of intersection of two tangents must lie on all of the tangent lines to \mathcal{O} . \square

We call the set of $q+2$ points comprising an oval and its nucleus a *hyperoval*. We can construct a further $q+1$ ovals from any given oval by creating the associated hyperoval, and then choosing a subset of $q+1$ points. Unlike when q is odd, there are ovals that are not conics for q even, and there are also hyperovals that do not arise from adding the nucleus to a conic. We will see two examples of families of ovals that are not conics in Section 4.3.3, namely the *Subiaco* and *Adelaide* ovals.

An *ovoid* of $\text{PG}(3, q)$ is a set of $q^2 + 1$ points, no three collinear. Here the analogous concept to a conic is an elliptic quadric. Barlotti and Panella [8, 55] proved that every ovoid of $\text{PG}(3, q)$, q odd, is an elliptic quadric. When q is an odd power of 2, apart from elliptic quadrics there are Suzuki–Tits ovoids (see [74]), and it is conjectured that these two examples are the only ovoids of $\text{PG}(3, q)$, q even.

Tits proved that we can construct a generalised quadrangle $T_2(\mathcal{O})$ of order (q, q) from an oval \mathcal{O} of $\text{PG}(2, q)$, and a generalised quadrangle $T_3(\mathcal{O}')$ of order (q, q^2) from an ovoid \mathcal{O}' of $\text{PG}(3, q)$ as follows.

Let \mathcal{O} be an oval of $H = \text{PG}(2, q)$, and embed H as a hyperplane of $A = \text{PG}(3, q)$. We define a point–line incidence structure as follows. The points are

- (i) the points of A not on H ,
- (ii) the hyperplanes of A that intersect \mathcal{O} in a single point, and
- (iii) an extra point labelled (∞) .

The lines are

- (i) the lines of A not contained in H and concurrent with exactly one point of \mathcal{O} , and
- (ii) the points of \mathcal{O} .

We define incidence as follows:

- A point of type (i) is incident with lines of type (i), with incidence inherited from $\text{PG}(3, q)$.
- A point of type (ii) is incident with the lines of type (i) contained within it and the unique point of \mathcal{O} that it meets (again with incidence inherited from $\text{PG}(3, q)$).
- The point (∞) is incident with all lines of type (ii).

This incidence structure forms a generalised quadrangle of order (q, q) , denoted $T_2(\mathcal{O})$.

Now $T_2(\mathcal{O}) \cong Q(4, q)$ if and only if \mathcal{O} is a conic [63, 3.3.2]. Note that when q is even, $T_2(\mathcal{O}) \cong Q(4, q) \cong W(3, q)$. See Section 2.4 for details of the second isomorphism. When q is even and \mathcal{O} is not a conic, we have constructed a new generalised quadrangle of order (q, q) .

On the other hand, let \mathcal{O} be an ovoid of $H = PG(3, q)$, and embed H into $A = PG(4, q)$. Using the same procedure as for $T_2(\mathcal{O})$, we construct $T_3(\mathcal{O})$, a generalised quadrangle of order (q, q^2) . Analogously to $T_2(\mathcal{O})$, we have that $T_3(\mathcal{O}) \cong Q^-(5, q)$ if and only if \mathcal{O} is an elliptic quadric. Therefore the only known extra examples of generalised quadrangles of the form $T_3(\mathcal{O})$ arise when \mathcal{O} is a Suzuki–Tits ovoid.

4.2.1 Automorphism groups of $T_3(\mathcal{O})$ and $T_2(\mathcal{O}')$

The automorphism groups of $T_3(\mathcal{O})$ and $T_2(\mathcal{O}')$ are slightly more complicated to define than in the classical case. However, due to the nature of the constructions of these generalised quadrangles, their automorphism groups may still be described in a geometric manner. The following characterisations were first given by O’Keefe and Penttila.

Theorem 4.2.2 ([52]). Let \mathcal{O} be an ovoid of $PG(3, q)$ which is not an elliptic quadric. Then the automorphism group of $T_3(\mathcal{O})$ is the stabiliser of \mathcal{O} in $PTL(5, q)$. Moreover, if \mathcal{O}' is an oval of $PG(2, q)$ which is not a conic, then the automorphism group of $T_2(\mathcal{O}')$ is the stabiliser of \mathcal{O}' in $PTL(4, q)$.

Knowing these automorphism groups allows us to more comprehensively describe relative hemisystems of $T_3(\mathcal{O})$, for \mathcal{O} a Suzuki–Tits ovoid, in Chapter 6.

4.3 Flock generalised quadrangles and q -clan geometries

4.3.1 q -clans

A matrix M is said to be *anisotropic* if the only vector x that satisfies $xMx^T = 0$ is the zero vector. We say that two 2×2 matrices M and N are *equivalent* (and write $M \equiv N$) if they have the same main diagonal entries and the sums of the entries on their antidiagonals are the same. Note that $M \equiv N$ if and only if $xMx^T = xNx^T$, for all vectors x .

Definition 4.3.1. A family $\mathbf{C} = \{M_t \mid t \in GF(q)\}$ of pairwise inequivalent 2×2 matrices is said to be a q -clan if the difference between any two of them is anisotropic.

Since

$$M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \equiv \begin{bmatrix} m_1 & m_2 + m_3 \\ 0 & m_4 \end{bmatrix},$$

we assume that the q -clans we are working with are comprised of upper triangular matrices. Notice that if $\mathbf{C} = \{A_t \mid t \in \text{GF}(q)\}$ is a q -clan, then so is $\mathbf{C}' = \{A_t - A_0 \mid t \in \text{GF}(q)\}$, since the difference between any two anisotropic matrices is anisotropic. Therefore, we always assume that A_0 is the zero matrix. Given two matrices

$$M_t = \begin{bmatrix} a_t & b_t \\ 0 & c_t \end{bmatrix} \text{ and } M_{t'} = \begin{bmatrix} a_{t'} & b_{t'} \\ 0 & c_{t'} \end{bmatrix}$$

contained in a q -clan, we must have $b_t \neq b_{t'}$, otherwise their difference is a diagonal matrix and therefore not anisotropic. Hence the map $t \mapsto b_t$ is a permutation of $\text{GF}(q)$, and we may relabel the matrices in the q -clan so that $b_t = t^\tau$ for some τ . A q -clan written in this way is termed a τ -normalised q -clan. In the majority of our work, we assume that our q -clans are $\frac{1}{2}$ -normalised, that is, $b_t = t^{1/2} = t^{q/2}$.

Suppose $q = p^k$ for some prime p and $k \in \mathbb{N}$. We define the *trace function* $\text{Tr} : \text{GF}(q) \rightarrow \text{GF}(p)$ by $\text{Tr}(x) = x + x^q + x^{q^2} + \dots + x^{q^{n-1}}$. Note that the trace function satisfies $\text{Tr}(x + y) = \text{Tr}(x) + \text{Tr}(y)$ for all $x, y \in \text{GF}(q)$.

Example 4.3.2. The *classical* q -clan is the set

$$\mathbf{C} = \left\{ \begin{bmatrix} t^{q/2} & t^{q/2} \\ 0 & \kappa t^{q/2} \end{bmatrix} \mid t \in \text{GF}(q) \right\}$$

of $\frac{1}{2}$ -normalised anisotropic matrices over $\text{GF}(q)$, with $\kappa \in \text{GF}(q)$ fixed and $\text{Tr}(\kappa) = 1$. We will see in Section 4.3.4 that this q -clan gives rise to a generalised quadrangle isomorphic to the Hermitian space $\text{H}(3, q^2)$.

4.3.2 Flocks of quadratic cones

Consider embedding $\text{PG}(2, q)$ as a hyperplane in $\text{PG}(3, q)$, and take a point P not on this hyperplane. By the axioms of a projective space, each point of a conic of $\text{PG}(2, q)$ is incident with a unique line that is also incident with P . Let \mathcal{C} be the set comprising all of the $q^2 + q + 1$ points on these $q + 1$ lines (there are $q + 1$ points on every line, and we double count the point P a total of q times). We say that \mathcal{C} is a *quadratic cone* with vertex P .

A *flock* of \mathcal{C} is a partition of $\mathcal{C} \setminus P$ into q conics. Each of these conics is the intersection of \mathcal{C} with a plane. In 1987, Thas [68] showed that every q -clan gives rise to a flock of a quadratic cone, and vice versa.

Theorem 4.3.3 ([21, Theorem 1.3.2]). Let $f_1 : t \mapsto a_t$, $f_2 : t \mapsto b_t$ and $f_3 : t \mapsto c_t$ be three functions on $\text{GF}(q)$, and let \mathcal{C} be a quadratic cone of $\text{PG}(3, q)$. For $t \in \text{GF}(q)$, define planes $\pi_t = [a_t, b_t, c_t, 1]$ and conics $C_t = \pi_t \cap \mathcal{C}$, and let $\mathcal{F} = \{C_t \mid t \in \text{GF}(q)\}$. Set

$$M_t = \begin{bmatrix} a_t & b_t \\ 0 & c_t \end{bmatrix},$$

and $\mathcal{C} = \{M_t \mid t \in \text{GF}(q)\}$. Then \mathcal{C} is a q -clan if and only if \mathcal{F} is a flock.

Thas and Fisher [36], and Walker [82] independently showed that we can also construct *translation planes* from flocks of quadratic cones and therefore from q -clans. We do not discuss this here, but see [61].

4.3.3 Herds of ovals

Recall that an *oval* in $\text{PG}(2, q)$ is a set of $q + 1$ points with no three collinear. When q is even, there is a unique tangent line through each point of the oval, and these tangent lines meet in a point called the *nucleus*. This point, when added to the $q + 1$ points of the oval gives a set of $q + 2$ points, no three collinear, called a *hyperoval*.

Let $f_0, g : \text{GF}(q) \rightarrow \text{GF}(q)$ be functions such that $f_0(0) = g(0) = 0$, and $f_0(1) = g(1) = 1$. Also, choose $a \in \text{GF}(q)$ such that $\text{Tr}(a) = 1$. A *herd* of ovals in $\text{PG}(2, q)$, q even, is a family $\{\mathcal{O}_s \mid s \in \text{GF}(q) \cup \{\infty\}\}$ of $q + 1$ ovals, with each containing the points $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 1)$, and the nucleus $(0, 0, 1)$, where

$$\mathcal{O}_s = \{(1, t, f_s(t)) \mid t \in \text{GF}(q)\} \cup \{(0, 1, 0)\}$$

and

$$\mathcal{O}_\infty = \{(1, t, g(t)) \mid t \in \text{GF}(q)\} \cup \{(0, 1, 0)\},$$

with

$$f_s(t) = \frac{f_0(t) + asg(t) + s^{1/2}t^{1/2}}{1 + as + s^{1/2}}.$$

Cherowitzo et al. [22] proved the following result linking q -clans and herds of ovals.

Theorem 4.3.4. The set of matrices

$$\mathcal{C} = \left\{ \begin{bmatrix} f_0(t) & t^{1/2} \\ 0 & ag(t) \end{bmatrix} \right\}$$

is a q -clan if and only if $\{\mathcal{O}_s \mid s \in \text{GF}(q) \cup \{\infty\}\}$ is a herd.

4.3.4 Flock generalised quadrangles

Suppose R is a generalised quadrangle of order (s, t) , and let P be a point of R . An *elation* about P is an automorphism of R that fixes P , and all of the lines incident with P , but does not fix any point not collinear with P . A group G comprised entirely of elations together with the identity automorphism, and which acts regularly on the set of points of R not collinear with P , is called an *elation group*, and P is called the *base point*. Note that G has order s^2t by the Orbit–Stabiliser Theorem. In this case, R is called an *elation generalised quadrangle*.

In 1980, Kantor [46] introduced a new way of constructing generalised quadrangles using group cosets. The following is the dual statement of Kantor's construction, but it is the standard version used in the literature.

Let G be a group of order s^2t , for some $s, t > 1$. Let $\{A_i\}_{i=0}^t$ be a family of subgroups of G of order s , and $\{A_i^*\}_{i=0}^t$ a family of subgroups of order st , such that A_i is a subgroup of A_i^* for each i . We then define an incidence structure with point set \mathcal{P} comprised of

- (i) the elements of G ,
- (ii) the cosets A_i^*g for all $g \in G$,
- (iii) another point labelled (∞) ;

and set of lines \mathcal{B} comprised of

- (i) the right cosets $A_i g$, for all $g \in G$,
- (ii) the symbols $[A_i]$.

We define the incidence relation I individually for each of the types of points as follows:

- A point $g \in G$ is incident with each of the lines of the form $A_i g$ for all $i \in \{0, \dots, t\}$.
- A point A_i^*g is incident with the line $[A_i]$ and each line $A_i h$ satisfying $A_i h < A_i^*g$.
- The point (∞) is incident with every line $[A_i]$.

Theorem 4.3.5. The incidence structure $(\mathcal{P}, \mathcal{B}, I)$ is a generalised quadrangle R of order (s, t) if and only if

- (i) $A_i A_j \cap A_k = \{1\}$ for distinct i, j, k , and
- (ii) $A_i^* \cap A_j = \{1\}$ for $i \neq j$.

If $(\mathcal{P}, \mathcal{B}, I)$ is a generalised quadrangle, then the triple $(G, \{A_i^*\}_{i=0}^t, \{A_i\}_{i=0}^t)$ is called a *Kantor family* and we also say that $\{A_i\}_{i=0}^t$ is a Kantor family for G . Kantor used this construction to create a new family of generalised quadrangles of order (q^2, q) , using the exceptional group of Lie type $G_2(q)$ [46].

It turns out that the group G can be considered as a group of automorphisms of R that fixes the point (∞) and the lines $[A_i]$, and acts on the remaining points and lines by right multiplication. In particular, every nontrivial element of G is an elation about (∞) , and, because $|G| = s^2t$, it is an elation group of R , and so R is an elation generalised quadrangle with base point (∞) . In addition, every elation generalised quadrangle can be constructed from a Kantor family.

Suppose that $\mathbf{C} = \{M_t \mid t \in \text{GF}(q)\}$ is a q -clan. Let

$$\mathcal{G} = \{(\mathbf{a}, c, \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \text{GF}(q)^2, c \in \text{GF}(q)\},$$

with multiplication defined by

$$(\mathbf{a}, c, \mathbf{b})(\mathbf{a}', c', \mathbf{b}') = (\mathbf{a} + \mathbf{a}', c + c' + \mathbf{b} \circ \mathbf{a}', \mathbf{b} + \mathbf{b}'),$$

where

$$\mathbf{b} \circ \mathbf{a} = \mathbf{b} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{a}^T.$$

We construct two families of subgroups of \mathcal{G} as follows. First, we create a family of $q + 1$ subgroups $\{A_t \mid t \in \text{GF}(q) \cup \{\infty\}\}$ by defining

$$A_\infty = \{(0, 0, \mathbf{b}) \mid \mathbf{b} \in \text{GF}(q)^2\} \quad \text{and} \quad A_t = \{(\mathbf{a}, g_t(\mathbf{a}), b_t \mathbf{a}) \mid \mathbf{a} \in \text{GF}(q)^2\},$$

where $t \in \text{GF}(q)$ and $g_t(\mathbf{a}) = \mathbf{a} M_t \mathbf{a}^T$. Notice that each of these subgroups has order q^2 . The centre of \mathcal{G} is $Z(\mathcal{G}) = \{(0, c, 0) \mid c \in \text{GF}(q)\}$, which has order q . Now we define for each $t \in \text{GF}(q) \cup \{\infty\}$ the subgroup $A_t^* = A_t Z(\mathcal{G})$. These subgroups have order q^3 .

These definitions follow the style of Payne [61]. The standard definitions given in other parts of the literature have

$$\mathbf{b} \circ \mathbf{a} = \sqrt{\mathbf{b} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{a}^T}$$

and

$$A_t = \{(\mathbf{a}, \sqrt{\mathbf{a} M_t \mathbf{a}^T}, b_t \mathbf{a}) \mid \mathbf{a} \in \text{GF}(q)^2\}, t \in \text{GF}(q).$$

The following result was proved by Payne [56] in 1980.

Theorem 4.3.6. The group \mathcal{G} , along with the families of subgroups

$$\mathcal{J}(\mathbf{C}) = \{A_i \mid i \in \text{GF}(q) \cup \{\infty\}\} \quad \text{and} \quad \mathcal{J}^*(\mathbf{C}) = \{A_i^* \mid i \in \text{GF}(q) \cup \{\infty\}\},$$

form a Kantor family.

Therefore, every q -clan \mathbf{C} gives rise to an elation generalised quadrangle $\text{GQ}(\mathbf{C})$. Every generalised quadrangle $\text{GQ}(\mathbf{C})$ that arises from a q -clan is called a *flock generalised quadrangle*. The name is derived from the connection between q -clans and flocks of quadratic cones.

4.3.5 The known q -clans, q even

There are five known families of q -clans over $\text{GF}(q)$, q even. There are no sporadic q -clans for q even, but this is not the case for q odd (and indeed there are infinite families of q -clans that only exist for q odd; see [60]).

Recall the *classical* q -clan from Example 4.3.2. Each such q -clan gives rise to a generalised quadrangle isomorphic to the Hermitian space $\text{H}(3, q^2)$. As mentioned earlier, the normalised q -clan is

$$\mathcal{C} = \left\{ \left[\begin{array}{cc} t^{q/2} & t^{q/2} \\ 0 & \kappa t^{q/2} \end{array} \right] \mid t \in \text{GF}(q) \right\},$$

with $\kappa \in \text{GF}(q)$ fixed and $\text{Tr}(\kappa) = 1$. Each of these q -clans gives rise to one type of oval in its herd, and one flock [60].

The Fisher–Thas–Walker–Kantor–Betten (*FTWKB*) q -clan arises for every q an odd power of 2. The normalised form of this q -clan is

$$\left\{ \left[\begin{array}{cc} t^{q/4} & t^{q/2} \\ 0 & \kappa t^{3q/4} \end{array} \right] \mid t \in \text{GF}(q) \right\},$$

with κ fixed as before. The associated flocks were discovered first by Fisher and Thas [36], with the q -clan constructed later. This q -clan gives rise to one flock and one oval, up to isomorphism, for each value of q . The associated generalised quadrangle was constructed by Kantor [46] and is nonclassical for $q > 2$.

The *Payne* q -clans occur for q an odd power of 2. The associated normalised q -clan is

$$\left\{ \left[\begin{array}{cc} t^{q/6} & t^{q/2} \\ 0 & t^{5q/6} \end{array} \right] \mid t \in \text{GF}(q) \right\}.$$

This q -clan is classical if $q = 2$, and FTWKB for $q = 8$. When $q \geq 32$, it gives rise to two flocks and two ovals up to isomorphism.

The other two examples of q -clans are the Subiaco and Adelaide examples. Their constructions are quite involved, so we state their forms via a remarkable theorem proved by Cherowitzo, et al. unifying four of the five known families of q -clans over $\text{GF}(q)$, q even.

Theorem 4.3.7 ([23]). Let $\text{GF}(q^2)$ be a quadratic extension of $\text{GF}(q)$, where $q = 2^e$. Choose $\beta \in \text{GF}(q^2) \setminus \{1\}$ such that $\beta^{q+1} = 1$, and let $T(x) = x + x^q$ for all $x \in \text{GF}(q^2)$. Let $a \in \text{GF}(q)$, define the functions $f, g : \text{GF}(q) \rightarrow \text{GF}(q)$ by

$$\begin{aligned} a &= \frac{T(\beta^m)}{T(\beta)} + \frac{1}{T(\beta^m)} + 1, \\ f(t) = f_{m,\beta}(t) &= \frac{T(\beta^m)(t+1)}{T(\beta)} + \frac{T((\beta t + \beta^q)^m)}{T(\beta)(t + T(\beta)t^{1/2} + 1)^{m-1}} + t^{1/2}, \\ ag(t) = ag_{m,\beta}(t) &= \frac{T(\beta^m)}{T(\beta)}t + \frac{T((\beta^2 t + 1)^m)}{T(\beta)T(\beta^m)(t + T(\beta)t^{1/2} + 1)^{m-1}} + \frac{1}{T(\beta^m)}t^{1/2}, \end{aligned}$$

and let

$$\mathbf{C} = \mathbf{C}_{m,\beta} = \left\{ \left[\begin{array}{cc} f(t) & t^{1/2} \\ 0 & ag(t) \end{array} \right] \mid t \in \text{GF}(q) \right\}.$$

Then the following hold:

- (i) If $m \equiv \pm 1 \pmod{q+1}$, then \mathbf{C} is the classical q -clan for all $q = 2^e$ and for all β .
- (ii) If $q = 2^e$, with e odd and $m \equiv \pm \frac{q}{2} \pmod{q+1}$, then \mathbf{C} is the FTWKB q -clan for all β .
- (iii) If $q = 2^e$, with $e > 2$ and $m \equiv 5 \pmod{q+1}$, then \mathbf{C} is the *Subiaco* q -clan for all β such that if λ is a primitive element of $\text{GF}(q^2)$ with $\beta = \lambda^{k(q-1)}$, then $q+1 \nmid km$.
- (iv) If $q = 2^e$ with $e > 2$ even and $m \equiv \frac{q-1}{3} \pmod{q+1}$, then \mathbf{C} is the *Adelaide* q -clan for all β .

The Subiaco geometries were first discovered by Cherowitzo et al. [22]. For a fixed value of $q = 2^e$, the Subiaco q -clan is unique up to equivalence, and there is only one associated flock. When $e \equiv 2 \pmod{4}$, the Subiaco herd contains two types of ovals: $\frac{q+1}{5}$ of one type, and $\frac{4(q+1)}{5}$ of another. These arise because the subgroup of $\text{GQ}(\mathbf{C})$ stabilising $((0, 0), 0, (0, 0))$ and (∞) is not transitive on the ovals. Apart from this case, all other Subiaco herds contain only one type of oval.

The Adelaide geometries were discovered by Cherowitzo et al. [23]. They showed that up to isomorphism, there is one associated flock. Payne and Cherowitzo [24] showed that apart from $q = 16$, when the Adelaide flock generalised quadrangle is isomorphic to the Subiaco flock generalised quadrangle, each Adelaide q -clan gives rise to a new flock generalised quadrangle and one new oval up to equivalence.

4.3.6 Automorphism groups of flock generalised quadrangles

The automorphism groups of nonclassical flock generalised quadrangles are more difficult to describe than in the classical case. There are a number of different descriptions of the groups and actions, most of which are quite technical (for example, see [54, 59]). We mostly adopt the notation and descriptions used in Payne's Subiaco Notebook [61]. The collineation group of a flock generalised quadrangle R is the semidirect product of \mathcal{G} with the subgroup of $\text{Aut}(R)$ that stabilises $(\mathbf{0}, 0, \mathbf{0})$ [54]. The action of \mathcal{G} on R is by right multiplication on the points and lines of R , fixing the lines of type (ii) and the point (∞) . Notice that any automorphism of \mathcal{G} has an induced action on the subgroups $\{A_t \mid t \in \text{GF}(q) \cup \{\infty\}\}$ (and $\{A_t^* \mid t \in \text{GF}(q) \cup \{\infty\}\}$), for $t \in \text{GF}(q) \cup \{\infty\}$, and therefore also induces a permutation on the elements of $\tilde{F} = \text{GF}(q) \cup \{\infty\}$. We

describe the automorphisms of \mathcal{G} based on whether the induced permutation on \tilde{F} is trivial or not. The q -clan kernel is the set of automorphisms of \mathcal{G} that act trivially in their induced action on \tilde{F} . We now take advantage of our use of $\frac{1}{2}$ -normalised q -clans to construct our flock generalised quadrangles and state some useful results.

Theorem 4.3.8 ([61, Theorem 1.10.3]). If C is a nonclassical $\frac{1}{2}$ -normalised q -clan, then the q -clan kernel is the group of collineations of the associated generalised quadrangle $GQ(C)$ that fixes all of the lines on the points (∞) and $(\mathbf{0}, 0, \mathbf{0})$.

In particular, the lines incident with the point $(\mathbf{0}, 0, \mathbf{0})$ are exactly the subgroups $\{A_t \mid t \in \text{GF}(q) \cup \{\infty\}\}$ and so the q -clan kernel must induce the identity permutation on these subgroups.

Theorem 4.3.9 ([61, Theorem 10.10.1]). An automorphism of \mathcal{G} that induces the identity permutation on $\{A_t \mid t \in \text{GF}(q) \cup \{\infty\}\}$ must have the form $\varphi_\lambda : (\mathbf{a}, c, \mathbf{b}) \mapsto (\lambda\mathbf{a}, \lambda^2c, \lambda\mathbf{b})$ for $\lambda \in \text{GF}(q) \setminus \{0\}$.

We show that these are indeed automorphisms of \mathcal{G} , and hence that they induce automorphisms of the flock generalised quadrangle R . As before, we write $g_t(\alpha) = \alpha M_t \alpha^T$ for $t \in \text{GF}(q)$ and $\alpha \in \text{GF}(q)^2$. Notice that $g_t(\lambda\alpha) = \lambda^2 g_t(\alpha)$, so $\varphi_\lambda : A_t \rightarrow A_t$ with $(\alpha, \alpha M_t \alpha^T, t^{1/2}\alpha) \mapsto (\lambda\alpha, (\lambda\alpha) M_t (\lambda\alpha)^T, t^{1/2}(\lambda\alpha))$. Due to the nature of these automorphisms, we sometimes refer to them as ‘‘scaling automorphisms’’. Now that we have completely described the automorphisms of \mathcal{G} that act trivially on \tilde{F} , it remains to describe those that act nontrivially. We do that using the following theorem.

Theorem 4.3.10 ([61, Theorem 1.9.4]). Let C, C' be τ^{-1} -normalised q -clans. Then, modulo the q -clan kernel \mathcal{N} , each isomorphism $\theta : GQ(C) \rightarrow GQ(C')$ mapping $(\infty), A_\infty$ and $(\mathbf{0}, 0, \mathbf{0})$ to $(\infty), A'_\infty$ and $(\mathbf{0}, 0, \mathbf{0})$, respectively, is given by an automorphism mapping the family of subgroups $\mathcal{J}(C)$ to $\mathcal{J}(C')$ and the subgroup A_∞ to A'_∞ . There is a unique such isomorphism θ for each 4-tuple $(\lambda, B, \sigma, \pi) \in \text{GF}(q)^\times \times \text{SL}(2, q) \times \text{Aut}(\text{GF}(q)) \times \text{Sym}(\text{GF}(q))$ for which

- (i) $\theta : A_t \rightarrow A'_t$, where $\pi : t \mapsto \bar{t}$ satisfies $\bar{t} = (\lambda t^{\sigma/\tau} + \bar{0}^{1/\tau})^\tau$,
- (ii) $A'_t \equiv \lambda B^{-1} A_t^\sigma B^{-T} + A'_0$ for all $t \in F$.

The associated automorphism $\theta : \mathcal{G} \rightarrow \mathcal{G}$ is given by

- (iii) $\theta(\sigma, \begin{pmatrix} 1 & \bar{0}^{1/\tau} \\ 0 & \lambda \end{pmatrix} \otimes B) : (\mathbf{a}, \mathbf{b}, c) \mapsto ((\mathbf{a}^\sigma, \mathbf{b}^\sigma) \left(\begin{pmatrix} 1 & \bar{0}^{1/\tau} \\ 0 & \lambda \end{pmatrix} \otimes B \right), \lambda c^\sigma + g'_0(\mathbf{a}^\sigma B))$,
- where $g'_0(\mathbf{a}^\sigma B) = (\mathbf{a}^\sigma B), A'_0(\mathbf{a}^\sigma B)^T$.

Therefore, there are three ‘flavours’ of automorphisms of a flock generalised quadrangle: those arising from the elements of \mathcal{G} acting by right multiplication, the scaling automorphisms, and those acting nontrivially on the lines on (∞) .

4.4 Subquadrangles

A (*generalised*) *subquadrangle* of a generalised quadrangle $R = (\mathcal{P}, \mathcal{B}, I)$ is a generalised quadrangle $R' = (\mathcal{P}', \mathcal{B}', I')$ with $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{B}' \subseteq \mathcal{B}$, both nonempty, and I' the incidence relation induced by the restriction of I to \mathcal{P}' and \mathcal{B}' .

Notice that if $|\mathcal{P}| = |\mathcal{P}'|$, then $|\mathcal{B}| = |\mathcal{B}'|$ and $R = R'$, otherwise the GQ axiom fails for at least one disjoint point–line pair. We will therefore assume that all of our generalised quadrangles are proper subquadrangles (that is, the point set of the subquadrangle is a proper subset of the point set of the parent generalised quadrangle).

Lemma 4.4.1. Suppose $R = (\mathcal{P}, \mathcal{B}, I)$ is a generalised quadrangle with subquadrangle $R' = (\mathcal{P}', \mathcal{B}', I')$. Every line $\ell \in \mathcal{B}$ is either contained in \mathcal{B}' , incident with exactly one point of \mathcal{P}' , or disjoint from R' .

Proof. Suppose $\ell \in \mathcal{B}$ meets R' in at least two points, P and Q . Suppose $m \in \mathcal{B}'$ is a line on Q not equal to ℓ . Then P is not incident with m and there must exist a unique line contained in R' that is incident with both P and a point of m . Any such choice of line that is not ℓ would create a triangle in R . Therefore, ℓ must be completely contained in \mathcal{B}' . \square

We are mainly interested in subquadrangles of order (q, q) of generalised quadrangles of order (q^2, q) .

4.4.1 The classical case

For the classical case, we work with the dual of the Hermitian space $H(3, q^2)$, namely the elliptic quadric $Q^-(5, q)$.

There are two classical generalised quadrangles of order (q, q) that we consider in this section. The symplectic space $W(3, q)$ is created by taking the totally isotropic subspaces of an alternating form. Its dual is the parabolic quadric $Q(4, q)$, which is created by taking the totally isotropic subspaces of a quadratic form. When q is even, we have $W(3, q) \cong Q(4, q)$, that is, $W(3, q)$ is self dual [63].

Recall that $Q^-(5, q)$ lies in the projective space $PG(5, q)$. Take a nondegenerate hyperplane $\pi \cong PG(4, q)$ of $PG(5, q)$ that meets $Q^-(5, q)$. The points and lines of the intersection $\pi \cap Q^-(5, q)$ form a subquadrangle $Q(4, q)$ of $Q^-(5, q)$ [63]. Taking the dual, we have $W(3, q)$ as a subquadrangle of order (q, q) of $H(3, q^2)$. The following nice result classifies the subquadrangles of $H(3, q^2)$.

Theorem 4.4.2 ([53, Theorem 6]). Every subquadrangle of $H(3, q^2)$ of order (q, q) is isomorphic to $W(3, q)$, and arises as in the construction above.

We may also construct a subquadrangle of $H(3, q^2)$ isomorphic to $W(3, q)$ in the following way. Let $H(3, q^2)$ be defined by the form $v_1^q u_2 + v_2^q u_1 + v_3^q u_4 + v_4^q u_3$ over $GF(q^2)$. We can define an embedded symplectic space by restricting the form to $GF(q)$. This causes it to become alternating, and the sets of totally isotropic points and lines form a symplectic space $W(3, q)$.

4.4.2 Flock generalised quadrangles

All of the flock generalised quadrangles constructed from the Kantor family construction are elation generalised quadrangles with base point (∞) . This makes the following result of O’Keefe and Penttila particularly useful.

Theorem 4.4.3 ([53]). *If R is a flock generalised quadrangle of order (q^2, q) , q even, with base point (∞) , then R contains exactly $q^3 + q^2$ subquadrangles of order q containing the point (∞) .*

Suppose R is a generalised quadrangle constructed from a Kantor family of \mathcal{G} . Payne and Maneri [62] show that subquadrangles of R can be constructed using a Kantor family of a certain type of subgroup of \mathcal{G} . For $\alpha \in GF(q)^2$, let $\mathcal{G}_\alpha = \{(x\alpha, z, y\alpha) \mid x, y, z \in GF(q)\}$. This is a subgroup of \mathcal{G} of order q^3 , and contains the centre $Z(\mathcal{G})$. Define $A_{t,\alpha} = \mathcal{G}_\alpha \cap A_t$ for all $t \in \tilde{F}$, and notice that $|A_{t,\alpha}| = q$. Then $\{A_{t,\alpha} \mid t \in \tilde{F}\}$ is a Kantor family for \mathcal{G}_α . Hence we can construct a generalised quadrangle S_α of order (q, q) containing the points (∞) and $(\mathbf{0}, \mathbf{0}, \mathbf{0})$ from this Kantor family. The generalised quadrangle obtained through this construction can be seen to be a subquadrangle of R by the following injective homomorphism from S_α to R .

Points:

$$\begin{aligned} (\infty) &\mapsto (\infty), \\ (x\alpha, z, y\alpha) &\mapsto (x\alpha, z, y\alpha), \\ A_{t,\alpha}^*(x\alpha, z, y\alpha) &\mapsto A_t^*(x\alpha, z, y\alpha). \end{aligned}$$

Lines:

$$\begin{aligned} A_{t,\alpha}(x\alpha, z, y\alpha) &\mapsto A_t(x\alpha, z, y\alpha), \\ [A_{t,\alpha}] &\mapsto [A_t]. \end{aligned}$$

It follows that S_α is isomorphic to its image under this homomorphism. Notice also that the homomorphism is well-defined, since we have, for example, $A_t^*(x\alpha, z, y\alpha) = A_t^*(x'\alpha, z', y'\alpha)$ if and only if $A_{t,\alpha}^*(x\alpha, z, y\alpha) = A_{t,\alpha}^*(x'\alpha, z', y'\alpha)$ by definition. Now, $\mathcal{G}_\alpha = \mathcal{G}_\beta$ if and only if $\alpha = \gamma\beta$ for some $\gamma \in GF(q) \setminus \{0\}$. Therefore, there are $q + 1$ distinct subgroups \mathcal{G}_α , and thus a total of $q + 1$ distinct subquadrangles S_α . For conciseness, we sometimes write $(x\alpha, z, y\alpha) \in S_\alpha$ as $((x, y) \otimes \alpha, z)$.

O'Keefe and Penttila [53] proved that the $q^3 + q^2$ subquadrangles described in Theorem 4.4.3 are actually just those described by Payne and Maneri, and images of these subquadrangles under automorphisms of the flock generalised quadrangle R .

The following result of Brown and Thas shows that these are the only subquadrangles we must consider.

Theorem 4.4.4 ([13, Theorem 4.3]). Let R be a nonclassical flock generalised quadrangle of order (q^2, q) , q even, with base point (∞) . Then any subquadrangle of R of order q contains (∞) .

In other words, if we can find a subquadrangle of order (q, q) of a flock generalised quadrangle of order (q^2, q) , q even, that does not contain the point (∞) , then the flock generalised quadrangle must be the classical generalised quadrangle $H(3, q^2)$, and the subquadrangle must be the symplectic space $W(3, q)$.

We now examine the subquadrangles S_α further.

Proposition 4.4.5. The map $\phi : (x\alpha, z, y\alpha) \mapsto (x^2, y^2, z)$ is an isomorphism between \mathcal{G}_α and $V(3, q)$.

Proof. Since q is even, the map $x \mapsto x^2$ is an automorphism of $\text{GF}(q)$, and so is its inverse. A preimage of $(x, y, z) \in V(3, q)$ is $(\sqrt{x}\alpha, z, \sqrt{y}\alpha)$, and this is the unique preimage. It remains to check that ϕ is a homomorphism. We have

$$\begin{aligned} \phi((x_1\alpha, z_1, y_1\alpha)(x_2\alpha, z_2, y_2\alpha)) &= \phi(((x_1 + x_2)\alpha, z_1 + z_2 + 0, (y_1 + y_2)\alpha)) \\ &= ((x_1 + x_2)^2, (y_1 + y_2)^2, z_1 + z_2) \\ &= (x_1^2, y_1^2, z_1) + (x_2^2, y_2^2, z_2) \\ &= \phi((x_1\alpha, z_1, y_1\alpha)) + \phi((x_2\alpha, z_2, y_2\alpha)). \end{aligned}$$

Therefore, ϕ is an isomorphism. □

We can also consider ϕ as a map onto the homogeneous coordinates of points of $\text{PG}(2, q)$. Under this isomorphism, the image of the subgroup $A_{\infty, \alpha}$ is the point $(0, 1, 0) \in \text{PG}(2, q)$ and the image of $A_{t, \alpha}$ is the point $(1, t, \alpha M_t \alpha^T) \in \text{PG}(2, q)$. Ranging over $t \in \text{GF}(q)$, we define, for each $\alpha \in \text{GF}(q)^2 \setminus \{0\}$,

$$\mathcal{O}_\alpha = \{(1, t, \alpha M_t \alpha) \mid t \in \text{GF}(q)\} \cup \{(0, 1, 0)\}.$$

Payne [57] proved that the set \mathcal{O}_α is an oval of $\text{PG}(2, q)$ for each $\alpha \in \text{GF}(q)^2 \setminus \{0\}$, and that the nucleus of each of these ovals is $(0, 0, 1)$. Notice that this is precisely the image of $Z(\mathcal{G}) \subset \mathcal{G}_\alpha$ in $\text{PG}(2, q)$ under ϕ . Moreover, the isomorphism exhibited in Proposition 4.4.5 gives a correspondence between the Kantor family of \mathcal{G}_α and the Kantor family described in [56, Section IV, Example 1], which shows that the generalised subquadrangle S_α is isomorphic to the generalised quadrangle $T_2(\mathcal{O}_\alpha)$. Since there are $q+1$ distinct subquadrangles S_α , Payne [57]

has constructed $q + 1$ ovals associated to the generalised quadrangle R . These $q + 1$ ovals form a herd of ovals, as defined in Section 4.3.3.

As in [21], we embed this $\text{PG}(2, q)$ as the hyperplane $x_1 = 0$ into $\text{PG}(3, q)$. Below, we give the explicit isomorphism between a subquadrangle S_α and $\text{T}_2(\mathcal{O}_\alpha)$ described above.

Points:

$$\begin{aligned} (\infty) &\mapsto (\infty), \\ (\gamma \otimes \alpha, c) &\mapsto (1, \gamma^2, c), \\ A_{t,\alpha}^*(\gamma \otimes \alpha, c) &\mapsto \langle (0, 1, t, g(\alpha, t)), (0, 0, 0, 1), (1, \gamma^2, c) \rangle, \\ A_{\infty,\alpha}^*(\gamma \otimes \alpha, c) &\mapsto \langle (0, 0, 1, 0), (0, 0, 0, 1), (1, \gamma^2, c) \rangle. \end{aligned}$$

Lines:

$$\begin{aligned} A_{t,\alpha}(\gamma \otimes \alpha, c) &\mapsto \langle (0, 1, t, g(\alpha, t)), (1, \gamma^2, c) \rangle, \\ A_{\infty,\alpha}(\gamma \otimes \alpha, c) &\mapsto \langle (0, 0, 1, 0), (1, \gamma^2, c) \rangle, \\ [A_{t,\alpha}] &\mapsto (0, 1, t, g(\alpha, t)), \\ [A_{\infty,\alpha}] &\mapsto (0, 0, 1, 0). \end{aligned}$$

We make use of this isomorphism extensively later.

4.4.3 $\text{T}_3(\mathcal{O})$

We now investigate the subquadrangles of $\text{T}_3(\mathcal{O})$, for \mathcal{O} an ovoid of $\text{PG}(3, q)$. As mentioned in Section 4.2, we are mostly interested in the case where \mathcal{O} is not an elliptic quadric, since otherwise the associated generalised quadrangle is classical.

Recall from Section 4.2 that we can construct $\text{T}_3(\mathcal{O})$ by first taking an embedding $H = \text{PG}(3, q)$, the projective space containing \mathcal{O} , into $A = \text{PG}(4, q)$. Take a plane π of H such that $\mathcal{O} \cap \pi = \mathcal{O}'$ is an oval. Now take a hyperplane Σ of A such that $\Sigma \cap H = \pi$. Then we can construct a subquadrangle $S(\Sigma) = \text{T}_2(\mathcal{O}')$ with the following sets of points and lines and incidence inherited from $\text{T}_3(\mathcal{O})$ [53]. The points of $\text{T}_2(\mathcal{O}')$ are

- (i) the points of Σ not on π ,
- (ii) the hyperplanes of A that intersect \mathcal{O}' in a single point, and
- (iii) the point labelled (∞) .

The lines of $\text{T}_2(\mathcal{O}')$ are

- (i) the lines of Σ not contained in π and concurrent with exactly one point of \mathcal{O}' , and

(ii) the points of \mathcal{O}' .

The following result shows that subquadrangles constructed in this manner are the only subquadrangles of $T_3(\mathcal{O})$ that we need to consider.

Theorem 4.4.6 ([53, Theorem 7]). Let \mathcal{O} be an ovoid of $PG(3, q)$, q even, that is not an elliptic quadric. Then a subquadrangle R' of order (q, q) of the associated generalised quadrangle $R = T_3(\mathcal{O})$ arises from the construction above. In particular, $R' = S(\Sigma)$ for some three-dimensional projective space Σ intersecting \mathcal{O} in an oval.

4.4.4 Constructing subquadrangles

In practice, it is useful to know how to construct subquadrangles of order (q, q) within generalised quadrangles of order (q^2, q) .

Suppose \mathcal{S} is a set of points (lines) of a generalised quadrangle. We define \mathcal{S}^\perp to be the set of points (lines) that are collinear (concurrent) with every member of \mathcal{S} .

A triple (K, P, Q) of points in a generalised quadrangle is said to be a *triad* if the points are pairwise noncollinear. Bose and Shrikande [10] proved that every triad (K, P, Q) in a generalised quadrangle of order (q^2, q) has the property that $|\{K, P, Q\}^\perp| = q + 1$. This implies that $|\{K, P, Q\}^{\perp\perp}| \leq q + 1$. We say that a triad (K, P, Q) contained in a $GQ(s, t)$ is a *3-regular triad* if it satisfies $|\{K, P, Q\}^{\perp\perp}| = s + 1$. Furthermore, we say that a point K is *3-regular* if every triad of points containing K is 3-regular.

Theorem 4.4.7 ([63, §2.6.2]). Suppose (K, P, Q) is a 3-regular triad of the generalised quadrangle $R = (\mathcal{P}, \mathcal{B}, I)$ of order (q, q^2) , q even. Let \mathcal{P}' be the set of all points incident with lines of the form $\ell = \text{span}\{u, v\}$, with $u \in \{K, P, Q\}^\perp$ and $v \in \{K, P, Q\}^{\perp\perp}$, let \mathcal{B}' be the set of all lines in \mathcal{B} that are incident with at least two points of \mathcal{P}' , and let I' be the restriction of I to \mathcal{P}' and \mathcal{B}' . Then $R' = (\mathcal{P}', \mathcal{B}', I')$ is a subquadrangle of R of order (q, q) .

This result becomes very useful when we consider that every point of $Q^-(5, q)$ is 3-regular [44, Theorem 5.8.1.(i)], and the point (∞) is 3-regular when the generalised quadrangle is isomorphic to a flock generalised quadrangle [58] or $T_3(\mathcal{O})$ [63, 3.3.2.(i)].

We used the dual statement of Theorem 4.4.7 initially to generate subquadrangles of flock generalised quadrangles in the computer algebra system GAP [3, 39]. However, it would be more convenient for us to construct and describe a subquadrangle of a flock generalised quadrangle in terms of the cosets of the elation group that form the points and lines, and so we use Payne's construction described in Section 4.4.2.

4.4.5 Doubly subtended subquadrangles

Let R be a generalised quadrangle of order (q^2, q) and R' a subquadrangle of R of order (q, q) , q even.

Lemma 4.4.8. Every line of R meets R' in exactly 0 or $q + 1$ points.

Proof. From Lemma 4.4.1, every line of R must meet R' in exactly 0, 1 or $q + 1$ points. Suppose ℓ is a line that meets R' in one point, and call this point P . Since P is a point of R' , by definition, it must have $q + 1$ lines on it, each incident with $q + 1$ points lying in R' , and none of these lines are equal to ℓ . Therefore, P must have at least $q + 2$ lines incident with it in R , which is a contradiction because R has order (q^2, q) . \square

In a slight abuse of notation, we will denote the set of objects that lie in R but are disjoint from R' by $R \setminus R'$. We define *external points* to be points that lie in $R \setminus R'$ and denote the set of external points be \mathcal{P}_E . Similarly, we define *external lines* to be lines that lie in R but do not meet R' and denote the set of external lines by \mathcal{L}_E .

The number of points of a generalised quadrangle of order (s, t) is $(s + 1)(st + 1)$, and the number of lines is $(t + 1)(st + 1)$. From this, the following are obtained.

- The number of points and the number of lines in R are $(q^2 + 1)(q^3 + 1)$ and $(q + 1)(q^3 + 1)$, respectively.
- The number of points and the number of lines in R' are both equal to $(q + 1)(q^2 + 1)$.
- The number of external points is $(q^2 + 1)(q^3 + 1) - (q + 1)(q^2 + 1)$, which is equal to $(q^2 + 1)(q^3 - q)$.
- The number of external lines is $(q + 1)(q^3 + 1) - (q + 1)(q^2 + 1) = q^2(q^2 - 1)$.

Lemma 4.4.9. Each external point of R has exactly one line on it meeting R' . This line meets R' in $q + 1$ points.

Proof. By the definition of a generalised quadrangle, every external point P is incident with $q + 1$ lines. Suppose for a contradiction that at least two of these lines meet R' . Denote one of these lines by k . Then, since R' is itself a generalised quadrangle, by the GQ axiom, a point of k that lies in R' must be collinear with some point in R' on each of the other lines on P that meet R' , as by Lemma 4.4.8, such lines meet R' in $q + 1$ points (that is, in a line of R'). This is a contradiction, because in every case, we will have made a triangle. So every external point P has at most one line on it meeting R' . Let ℓ be an external line. The number of points in R' is $(q^2 + 1)(q + 1)$, and by the GQ axiom, each of these must be incident with a point of ℓ . Recall that ℓ has $q^2 + 1$ points on it. Since every point of ℓ can have at most one line on it meeting R' in $q + 1$

points, the only way for all of the points of R' to satisfy the GQ axiom with ℓ is if each point on ℓ has a line on it meeting R' in $q + 1$ points. \square

A *spread* of a generalised quadrangle is a set of lines S such that every point is incident with exactly one line of S .

Corollary 4.4.10. Let R be a generalised quadrangle of order (q^2, q) with a generalised subquadrangle R' of order (q, q) . Then every external line ℓ in R induces a spread of R' .

Proof. Suppose ℓ is an external line. Each of the $q^2 + 1$ points on ℓ has exactly one line on it meeting R' . None of these lines can be concurrent because otherwise they would form a triangle with ℓ . Therefore, these lines meet $(q^2 + 1)(q + 1)$ points of R' , which is the whole of the point set. Therefore, these lines form a spread of R' . \square

Using the notation developed by Brown [12], we say that the spread given by Corollary 4.4.10 is *subtended* by ℓ , and denote it \mathcal{S}_ℓ . We say that a spread is doubly subtended if it is subtended by exactly two lines of $R \setminus R'$. The subquadrangle R' is said to be doubly subtended if every subtended spread of R' is doubly subtended. We denote the other line subtending a spread by $\bar{\ell}$, and call ℓ and $\bar{\ell}$ *antipodes*.

Lemma 4.4.11 ([12, Corollary 2.2]). The size of the intersection of two subtended spreads \mathcal{S}_ℓ and \mathcal{S}_n is one of the following:

- (i) $q^2 + 1$, if ℓ and n subtend the same spread;
- (ii) 1, if ℓ and n subtend different spreads and n is concurrent with either ℓ or $\bar{\ell}$;
- (iii) $q + 1$, if ℓ and n subtend different spreads and n is not concurrent with ℓ nor $\bar{\ell}$.

Knowing the size of the intersection of the spreads subtended by two external lines ℓ and n underpins the definition of an association scheme on the external lines of $H(3, q^2)$, which will be described later in Section 5.4. It also implies the following result.

Lemma 4.4.12. Let ℓ and n be two nonconcurrent external lines in a generalised quadrangle R of order (q^2, q) containing a doubly subtended subquadrangle R' of order (q, q) . Then the number of external lines concurrent with both ℓ and n is as follows:

- (i) 0, if $n = \bar{\ell}$,
- (ii) $q - 2$, if n is concurrent with ℓ ,
- (iii) $q^2 - q$ if n is not concurrent with either ℓ or $\bar{\ell}$,

(iv) q^2 , if n is concurrent with $\bar{\ell}$.

Proof. Suppose n is concurrent with ℓ and that P is their point of intersection. To prevent the existence of a triangle, the only lines that can be concurrent with both ℓ and n (but not equal to either) are the $q - 2$ remaining external lines that are incident with P . Now suppose n is not concurrent with or equal to ℓ . By Lemma 4.1.2, ℓ and n are concurrent with $q^2 + 1$ lines together in R . We must determine how many of these are external lines. The lines concurrent with both ℓ and n and meeting R are exactly the lines in the intersection of the spreads subtended by ℓ and n . Therefore, by Lemma 4.4.11, if ℓ and n subtend the same spread (that is, $n = \bar{\ell}$), then there must be zero external lines concurrent with both. If the intersection of the spreads subtended by ℓ and n has size $q + 1$ (that is, n is concurrent with neither ℓ or $\bar{\ell}$), then ℓ and n are concurrent with $q^2 - q$ external lines together. Finally, if n is concurrent with $\bar{\ell}$, which means that size of the intersection of their subtended spreads is equal to one, then ℓ and n must be concurrent with q^2 external lines together. \square

The following result gives a characterisation of generalised quadrangles with doubly subtended subquadrangles.

Lemma 4.4.13 ([12, Lemma 3.1.5]). A subquadrangle R' is doubly subtended in R if and only if R has an involutory automorphism that fixes R' pointwise.

Essentially, the automorphism described in Lemma 4.4.13 fixes the points of R' , while swapping antipodes.

Thas showed that the only generalised quadrangle of order (q^2, q) that has a doubly subtended subquadrangle of order (q, q) is $H(3, q^2)$ [71].

Recall that in the classical case, if our Hermitian space $H(3, q^2)$ is defined by the form $v_1^q u_2 + v_2^q u_1 + v_3^q u_4 + v_4^q u_3$ over $\text{GF}(q^2)$, we can define an embedded symplectic space by restricting the form to $\text{GF}(q)$. This causes the form to become alternating and the sets of totally isotropic points and lines to make a symplectic space. Notice that any projective point with all coordinates in $\text{GF}(q)$ will be included in this symplectic space. The associated involutory automorphism is the map taking the subspace U associated to a point or line in $H(3, q^2)$ to U^q . We sometimes call this automorphism the q -map. Notice that a projective point will be fixed under the q -map if and only if all of its coordinates lie in $\text{GF}(q)$. Therefore, an external line ℓ cannot be concurrent with its image $\bar{\ell}$ under the q -map because the point of intersection will be fixed under the q -map, and so it will lie in R' , contradicting our assumption that ℓ was an external line.

Lemma 4.4.14. Let ℓ be an external line. Then a line meeting ℓ is concurrent with $\bar{\ell}$ if and only if it also meets R' .

Proof. We begin with the ‘only if’ implication. By definition, ℓ and $\bar{\ell}$ subtend the same spread of R' , which means that every line concurrent with ℓ that meets R' must also be concurrent with $\bar{\ell}$. As for the ‘if’ implication, suppose k is a line concurrent with both ℓ and $\bar{\ell}$. Then by Lemma 4.4.9, the point that is the intersection of ℓ and k must have one line on it meeting R' , and, by the previous argument, that line must be concurrent with $\bar{\ell}$. To prevent the existence of a triangle, k must meet R' . \square

4.5 m -covers and their generalisations

An m -cover (sometimes also called a *regular m -system* [66]) of a generalised quadrangle is a set of lines such that every point is incident with m elements of the set. Notice that a 1-cover is just a spread. The dual notion of an m -cover is an m -ovoid, and a 1-ovoid is simply called an *ovoid*. Note that this is a different concept of ovoid to the one introduced in Section 4.2. We will say that an m -cover or m -ovoid is *nontrivial* if it is not simply all or none of the lines or points respectively.

Segre [66] showed that the only nontrivial m -cover of a Hermitian space $H(3, q^2)$, q odd, arises when $m = \frac{q+1}{2}$. Since such an m -cover consists of half of the lines on every point, Segre called this m -cover a *hemisystem*. He gave an example of a hemisystem on $H(3, 3^2)$, admitting $\text{PSL}(3, 4)$ as a setwise stabiliser, and proved that this is the only hemisystem on this Hermitian space.

In 1978, Bruen and Hirschfeld [14] proved that there are no nontrivial m -covers on $H(3, q^2)$, q even. In the same year, Cameron et al. [20] investigated dual hemisystems in generalised quadrangles of order (q, q^2) , and gave an alternative construction of Segre’s hemisystem in this setting. They also proved that the *point graph* of the dual of a hemisystem is strongly regular. We will discuss point graphs further in Section 5.1. This generalised work by Thas, who proved this in the context of $H(3, q^2)$, and reproved Segre’s original result using this construction [67]. For example, Segre’s hemisystem on $H(3, 3^2)$ gives rise to the Gewirtz graph on 56 vertices [20]. In 1989, Thas generalised Segre’s result by proving that the only nontrivial m -cover of a generalised quadrangle of order (q^2, q) is a hemisystem [69].

For 40 years, the only known example of a hemisystem (up to equivalence) was Segre’s example on $H(3, q^2)$. Thas conjectured in 1995 that there are no hemisystems on $H(3, q^2)$ that are inequivalent to Segre’s hemisystem [70]. In 2005, 40 years after Segre’s result and ten years after Thas conjectured the nonexistence of other hemisystems, Penttila and Cossidente [29] discovered an infinite family of hemisystems on $H(3, q^2)$, q odd, with each hemisystem admitting $\text{P}\Omega^-(4, q)$. They also gave a sporadic example of a hemisystem on $H(3, 5^2)$, admitting $(3.A_7).2$. Since then, five more infinite families on both classical and nonclassical generalised quadrangles of order (q^2, q) have been discovered. Bamberg, Giudici and Royle [5] also proved that every flock generalised quadrangle,

q odd, has a hemisystem.

Hemisystems are of interest because they give rise to strongly regular graphs, 4-class imprimitive cometric Q -antipodal association schemes [77] and partial quadrangles [17].

The definition of *relative m -covers* arises as a generalisation of the concept of a relative hemisystem, which was in turn defined by Penttila and Williford in 2011 as an analogous concept to hemisystems for q even [64].

Definition 4.5.1 (Relative hemisystem). Let R be a generalised quadrangle of order (q^2, q) containing a generalised subquadrangle R' of order (q, q) , q even. Then every line of R either meets R' in $q + 1$ points, or is disjoint from R' . We call a subset \mathcal{H} of the lines disjoint from R' a *relative hemisystem* of R with respect to R' if for each point $x \in R \setminus R'$, exactly half of the lines through x disjoint from R' lie in \mathcal{H} .

Penttila and Williford [64] constructed an infinite family of relative hemisystems admitting $P\Omega^-(4, q)$, for every even $q > 2$. Cossidente proved the existence of two more infinite families admitting $\text{PSL}(2, q)$ [26] and a group of order $q^2(q+1)$ [27] respectively, for every even $q > 2$. There is also a relative hemisystem arising from a Suzuki–Tits ovoid, discovered by Cossidente and Pavese, which is conjectured to be sporadic [28]. Bamberg, Lee and Swartz unified the three infinite families of relative hemisystems by providing a set of sufficient criteria to determine a relative hemisystem that all of them satisfy [6]. Unlike hemisystems, relative hemisystems do not give rise to partial quadrangles or strongly regular graphs. However, each relative hemisystem does give rise to a type of association scheme that is cometric and Q -bipartite, but is not Q -antipodal and does not arise from a P -polynomial association scheme (see [64]). Prior to the definition of relative hemisystems, the only known examples of such schemes were sporadic.

Despite the discovery of these examples of relative hemisystems, there is no reference in the literature to the more general idea of relative m -covers, nor is there any exploration of relative hemisystems of flock generalised quadrangles. We will discuss this further in Chapter 6.

5. Association schemes

5.1 Graphs

The theory of graphs provides a taste for the immense symmetry and regularity present in association schemes. We will have a particular focus on distance regular graphs, which were first introduced by Biggs [9].

A *graph* $\Gamma = (V(\Gamma), E(\Gamma))$ is an ordered pair consisting of a set $V = V(\Gamma)$ of *vertices* and a set of *edges* $E = E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$ with the property that $(v_1, v_2) \in E(\Gamma)$ if and only if $(v_2, v_1) \in E(\Gamma)$. We say that two vertices v_1 and v_2 are *adjacent* or *neighbours* if $(v_1, v_2) \in E(\Gamma)$, and we call them *nonadjacent* otherwise. We can define a $|V| \times |V|$ binary matrix $A = A(\Gamma) = (A_{ij})$, termed the *adjacency matrix* of Γ , by indexing the vertices of Γ with natural numbers and setting

$$A_{ij} = \begin{cases} 1 & (v_i, v_j) \in E(\Gamma), \\ 0 & (v_i, v_j) \notin E(\Gamma). \end{cases}$$

We usually illustrate graphs by drawing the vertices as dots, and drawing a line between a pair of vertices if there is an edge between them. A *path* is a sequence of vertices $\{v_1, v_2, \dots, v_k\}$ where every vertex appears at most once and $(v_i, v_{i+1}) \in E(\Gamma)$. A graph is said to be *connected* if there exists a path between any two vertices v and w ; otherwise we say that the graph is *disconnected*. We define the *distance* $d(v, w)$ between two vertices v and w in a connected graph to be the length of the shortest path between them. The maximum distance d between vertices in the graph is called the *diameter*. The set of vertices at distance i from a given vertex v is denoted by $\Gamma_i(v)$.

Example 5.1.1. Suppose \mathcal{S} is a polar space. We define the *point graph* $\Gamma(\mathcal{S})$ of \mathcal{S} by taking the vertex set to be the set of points of \mathcal{S} , with $(v, v') \in E(\Gamma(\mathcal{S}))$ if and only if the points v and v' are not equal and are collinear in \mathcal{S} .

Suppose Γ is a connected graph. Let $v, w \in V(\Gamma)$ be vertices at distance i from each other, where $0 \leq i \leq d$. We define the following quantities:

$$\begin{aligned} k_i(v) &= |\Gamma_i(v)|, \\ a_i(v, w) &= |\Gamma_1(v) \cap \Gamma_i(w)|, \\ b_i(v, w) &= |\Gamma_1(v) \cap \Gamma_{i+1}(w)|, \\ c_i(v, w) &= |\Gamma_1(v) \cap \Gamma_{i-1}(w)|. \end{aligned}$$

Note that we can also write $a_i(v, w) = k_1(v) - b_i(v, w) - c_i(v, w)$. Note that $b_d(v, w) = c_0(v, w) = 0$, $c_1(v, w) = 1$ and $k_0(v) = 1$ for all $v, w \in V(\Gamma)$. We say that Γ is *distance regular* if b_i and c_i do not depend on our choice of v and w at distance i . If Γ is a connected distance regular graph with diameter 2, then we

say Γ is *strongly regular*. The sequence $\iota(\Gamma) = \{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$ is called the *intersection array* associated to a distance regular graph.

Every vertex in a distance regular graph has the same number $k = k_1 = b_0$ of neighbours, and we call this the *valency* of the graph. We can define the k_i recursively by noting that $k_{j+1} = \frac{k_j b_j}{c_{j+1}}$ [11, §4.1].

Example 5.1.2. Suppose \mathcal{S} is a polar space with dual polar space \mathcal{S}' . Then the *dual polar graph* of \mathcal{S} is the point graph of \mathcal{S}' . In other words, it is the graph formed by taking the maximals of \mathcal{S} as the vertex set, with an edge between two vertices if their corresponding maximals meet in a next-to-maximal. The dual polar graph of a classical polar space is a distance regular graph [18]. Moreover, the point graph of \mathcal{S} is strongly regular [11, p. 440].

Let b be an integer. We define the following quantities, called the *Gaussian binomial coefficients with basis b* :

$$\begin{bmatrix} i \\ \ell \end{bmatrix}_b = \begin{cases} \prod_{n=0}^{\ell-1} \frac{b^i - b^n}{b^{\ell-n} - b^n} & b \neq 1, \\ \prod_{n=0}^{\ell-1} \frac{i-n}{\ell-n} & b = 1. \end{cases} \quad (5.1)$$

Many distance regular graphs that arise from polar spaces and related structures have intersection arrays that can be expressed using the diameter d of the graph and three other parameters, α, β and b , in the following way:

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b \right), \quad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b \right),$$

for $0 \leq i \leq d$. We call the entries of the tuple (d, b, α, β) the *classical parameters* of the graph. From these, we may also obtain expressions for k_1 and therefore a_i :

$$k_1 = \frac{k_0 b_0}{c_1} = b_0 = \begin{bmatrix} d \\ 1 \end{bmatrix}_b \beta,$$

$$a_i = k_1 - b_i - c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(\beta - 1 + \alpha \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b \right) \right).$$

Theorem 5.1.3 ([11, Theorem 9.4.3.]). Suppose Γ is a dual polar graph of a classical polar space of rank d over $\text{GF}(q)$. Then Γ has classical parameters $(d, q, 0, q^e)$ and therefore has intersection array given by

$$b_j = q^{j+e} \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \quad \text{and} \quad c_j = \begin{bmatrix} j \\ 1 \end{bmatrix}_q,$$

where $0 \leq j \leq d$, e as in Theorem 2.3.3.

We are now ready to define association schemes, which add a further level of abstraction to the neighbourhood regularity that occurs in distance regular graphs. We use the association schemes of distance regular graphs as a recurring example.

5.2 Association schemes

A d -class symmetric association scheme $(X; R_0, R_1, \dots, R_d)$ is a finite set X equipped with a set of symmetric binary relations R_0, R_1, \dots, R_d that satisfy the following axioms:

- (i) $R_0 = \{(x, x) \mid x \in X\}$.
- (ii) R_0, R_1, \dots, R_d partition $X \times X$.
- (iii) If $(x, y) \in R_i$, then $(y, x) \in R_i$.
- (iv) For all $i, j, k \in \{0, 1, \dots, d\}$, there exist integers p_{ij}^k such that for every $x, y \in X$ with $(x, y) \in R_k$ we have

$$p_{ij}^k = |\{z \in X \mid (x, z) \in R_i, (y, z) \in R_j\}|.$$

The p_{ij}^k are called *intersection numbers*.

Proposition 5.2.1 ([11, Chapter 4]). The vertex set of a distance regular graph along with the relations $R_i = \{(v, w) \mid d(v, w) = i\}$, $0 \leq i \leq d$, forms a d -class association scheme. We say that this is the association scheme *associated to* Γ .

This result is particularly useful to us because we may take advantage of association scheme results to find intersection numbers, which may otherwise be difficult to compute geometrically.

Lemma 5.2.2 ([11, Lemma 2.1.1.]). Suppose $(X; R_0, R_1, \dots, R_d)$ is a d -class association scheme. Then the intersection numbers p_{ij}^k satisfy the following properties:

- (i) $p_{0j}^k = \delta_{jk}$,
- (ii) $p_{ij}^0 = \delta_{ij}p_{ii}^0$,
- (iii) $p_{ij}^k = p_{ji}^k$,
- (iv) $p_{ij}^k p_{kk}^0 = p_{ik}^j p_{jj}^0$,
- (v) $\sum_j p_{ij}^k = p_{ii}^0$,
- (vi) $\sum_l p_{ij}^l p_{lk}^m = \sum_r p_{ir}^m p_{jk}^r$.

An association scheme $(X; R_0, R_1, \dots, R_d)$ is called *metric* if

- $p_{ij}^{i+j} \neq 0$, and
- $p_{ij}^k \neq 0$ implies that $k \leq i + j$

for all choices of i, j, k with $0 \leq i, j, k \leq d$.

Theorem 5.2.3 ([11, §2.7]). An association scheme $(X; R_0, R_1, \dots, R_d)$ is metric if and only if (X, R_1) defines a distance regular graph Γ and $(x, y) \in R_i$ if and only if x, y are distance i in Γ .

An association scheme is called *imprimitive* if a union of a subset of the relations that is not just the union of all the relations or just equality is an equivalence relation. Otherwise, the association scheme is called *primitive*.

The following lemma enables us to use the machinery developed for calculating the intersection arrays of graphs with classical parameters in Section 5.1 to calculate intersection numbers.

Lemma 5.2.4 ([11, Lemma 4.1.7]). Suppose Γ is a distance regular graph. Then the intersection numbers of the association scheme associated to Γ can be calculated recursively using the following equations:

$$p_{0h}^i = \delta_{ih}, \quad p_{j0}^i = \delta_{ij}, \quad p_{jd+1}^i = 0,$$

where δ_{ij} is the Kronecker delta function,

$$p_{1h} = \begin{cases} c_i & \text{if } h = i - 1, \\ a_i & \text{if } h = i, \\ b_i & \text{if } h = i + 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$p_{j+1h}^i = \frac{1}{c_{j+1}} (p_{jh-1}^i b_{h-1} + p_{jh}^i (a_h - a_j) + p_{jh+1}^i c_{h+1} - p_{j-1h}^i b_{j-1}),$$

where $0 \leq h, i, j \leq d$.

We now define adjacency matrices A_i for each of the relations R_i , with rows and columns indexed by the elements of X . For elements $x, y \in X$, we set $(A_i)_{xy} = 1$ if and only if $(x, y) \in R_i$, and $(A_i)_{xy} = 0$ otherwise. We can then restate the axioms of an association scheme in terms of these adjacency matrices. Here, J denotes the all-ones matrix.

- (i)' $A_0 = I$,
- (ii)' $\sum_{i=0}^d A_i = J$,
- (iii)' $A_i^\top = A_i$,
- (iv)' $A_i A_j = \sum_{k=0}^d p_{ij}^k A_k$.

Note that when we are talking about the association scheme of a distance regular graph Γ , the matrix A_1 is the adjacency matrix of Γ , as defined in Section 5.1. From (ii)', we see that the $d + 1$ adjacency matrices are linearly independent, in particular over \mathbb{C} . Furthermore, (iii)' shows that multiplication of adjacency matrices is commutative since $A_i A_j = (A_i A_j)^\top = A_j^\top A_i^\top = A_j A_i$. Moreover, (iv)' tells us that the span of the set of adjacency matrices is closed under multiplication. Therefore, the adjacency matrices generate a $(d+1)$ -dimensional

commutative algebra. If we set the base field of this algebra to be \mathbb{C} , then the algebra is called the *Bose–Mesner algebra* \mathcal{A} associated to the association scheme. Any matrix M contained in \mathcal{A} must be symmetric, since all of the adjacency matrices that form the basis of the algebra are symmetric. Therefore, M is diagonalisable over \mathbb{R} (which is a subfield of the base field \mathbb{C} of \mathcal{A}). Thus M has real distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_t$. In particular, the corresponding eigenspaces $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_t$ are orthogonal and give a decomposition of \mathbb{R}^X , that is, the vector space over \mathbb{R} with the elements of X as a basis. Since the scalars of \mathcal{A} are contained in the complex numbers, applying these scalars to the eigenspaces of M gives an orthogonal decomposition of \mathbb{C}^X .

Lemma 5.2.5 ([1, Corollary 2.5]). If M and N are commuting symmetric matrices in $\mathbb{R}^{X \times X}$, then \mathbb{R}^X is the direct sum of the nonzero intersections of the eigenspaces of M and N . Moreover, these spaces are mutually orthogonal and their orthogonal projectors (that is, the matrices that project onto the intersections of the eigenspaces of M and N) are polynomials in M and N .

Therefore, by Lemma 5.2.5, we may decompose \mathbb{C}^X into a set of orthogonal subspaces, $\mathbb{C}^X = V_0 \perp V_1 \perp \dots \perp V_d$, where each V_i is a subset of an eigenspace of every matrix in \mathcal{A} [51]. One of these orthogonal subspaces is the one-dimensional subspace spanned by the all-ones vector j , and we set this subspace as V_0 . Let f_i be the dimension of the subspace V_i . We call the f_i the *multiplicities* of the association scheme. Define E_i to be the matrix representing the orthogonal projection of vectors in V onto V_i , where $0 \leq i \leq d$. These matrices are idempotents (that is, they satisfy $E_i^2 = E_i$), and they are minimal (that is, they cannot be written as the sum of two other idempotents). Together they form a basis $\{E_i\}_{i=0}^d$ of the Bose–Mesner algebra \mathcal{A} , which is the unique basis comprised of minimal idempotents [51]. The matrices E_0, E_1, \dots, E_d satisfy $E_i E_j = \delta_{ij} E_i$, where δ_{ij} is the Kronecker delta function, and their sum is equal to the identity matrix. We set $E_0 = \frac{1}{|X|} J$, where J is the all-ones matrix, which is consistent with the definition of V_0 . We also note that the row vectors of E_i form a basis for the subspace V_i , and therefore $\text{rank}(E_i) = \text{trace}(E_i) = f_i$, since the rank of an idempotent matrix is the same as its trace.

Not only is the Bose–Mesner algebra of an association scheme closed under matrix multiplication, it is also closed under entrywise multiplication of matrices (also known as *Hadamard multiplication*). We denote this multiplication by \circ . The adjacency matrices of the association scheme are minimal idempotents with respect to this entrywise multiplication. Since they also form a basis of \mathcal{A} , we can write the following relation between the E_i matrices:

$$E_i \circ E_j = \frac{1}{n} \sum_{k=0}^d q_{ij}^k E_k$$

for some real numbers q_{ij}^k . We call the q_{ij}^k *dual intersection numbers*.

If two association schemes are such that the intersection numbers of the first are the dual intersection numbers of the second, then the converse also holds and the two association schemes are said to be (*formally*) *dual*. Note that the formal dual of an association scheme need not be unique, nor does it need to exist [11, §2.3]. An association scheme is said to be *cometric* if $q_{ij}^{i+j} \neq 0$, and $q_{ij}^k \neq 0$ implies that $k \leq i + j$ for all choices of i, j, k with $0 \leq i, j, k \leq d$. Notice that this is very similar to our definition of a metric association scheme, except here we are using the q_{ij}^k rather than the p_{ij}^k . In particular, the formal dual of a cometric association scheme is metric and vice versa.

We define the first eigenmatrix $P = (P_{ik})$ and the second eigenmatrix $Q = (Q_{ki})$ of the association scheme to be the $(d + 1) \times (d + 1)$ matrices satisfying

$$A_k = \sum_{i=0}^d P_{ik} E_i \quad \text{and} \quad E_i = \frac{1}{n} \sum_{k=0}^d Q_{ki} A_k,$$

where $n = |X|$. Now $PQ = QP = nI$, and since the minimal idempotents are orthogonal, $A_j E_i = P_{ij} E_i$, so P_{ij} is an eigenvalue of A_j , and in fact the P_{ij} values are precisely the eigenvalues of A_j . Since the rank of E_i is f_i , the multiplicity of the eigenvalue P_{ij} is also f_i .

Lemma 5.2.6 ([11, Lemma 2.2.1]). The first and second eigenmatrices P and Q of an association scheme satisfy the following equations:

- (i) $P_{i0} = Q_{i0} = 1$,
- (ii) $P_{0j} = n_j$ and $Q_{0j} = f_j$,
- (iii) $P_{ij} P_{ik} = \sum_l p_{jk}^l P_{il}$,
- (iv) $|P_{ij}| \leq n_j$ and $|Q_{ij}| \leq f_j$,
- (v) $\sum_j P_{jk} = \sum_i p_{ik}^i$.

Here, $n_j = p_{jj}^0$, the number of elements of the underlying set that are R_j related to a particular element.

Define B_i to be the $(d+1) \times (d+1)$ matrix whose (kj) th entry is p_{ij}^k . Then Property (iii) of Lemma 5.2.6 can be rewritten as $P B_j P^{-1} = \text{diag}(P_{0j}, P_{1j}, \dots, P_{dj})$, the diagonal matrix with entries $P_{0j}, P_{1j}, \dots, P_{dj}$ on the main diagonal and zero elsewhere. Hence the values P_{ij} are the eigenvalues of B_j . We call the B_i the *intersection matrices* of the association scheme. The algebra generated by the intersection matrices is isomorphic to the Bose–Mesner algebra of the association scheme [11, p. 45].

Proposition 5.2.7 ([11, Proposition 2.2.2]). Suppose that $\{v_i \mid 0 \leq i \leq d\}$ are the common left eigenvectors of the intersection matrices B_i , scaled so that the first entry of each v_i is equal to 1. Then the rows of P are the vectors v_i .

It is now clear why P is called the *first eigenmatrix* of the association scheme. Therefore, from Proposition 5.2.7, if one of our intersection matrices has $d + 1$ distinct eigenvalues, we may use its eigenvectors to compute the matrix P because all of the eigenspaces are one dimensional and cannot be further decomposed by taking intersections with eigenspaces of other intersection matrices. When our association scheme arises from a distance regular graph, we find the following relationship between the adjacency matrices [11, §4.1]:

$$A_1 A_i = c_{i+1} A_{i+1} + a_i A_i + b_{i-1} A_{i-1},$$

for $0 \leq i \leq d$, and where $A_{-1} = A_{d+1} = 0$. This relation implies that we may write each of the adjacency matrices $A_i = \nu_i(A_1)$ as a degree i polynomial in A_1 , where the polynomials ν_i are defined recursively by

$$\nu_{-1}(x) = 0, \quad \nu_0(x) = 1, \quad \nu_1(x) = x, \quad (5.2)$$

$$c_{j+1} \nu_{j+1}(x) = (x - a_j) \nu_j(x) - b_{j-1} \nu_{j-1}(x), \quad (5.3)$$

for $j \in \{0, 1, \dots, d\}$. Now since the A_i matrices are linearly independent and can be written as functions of A_1 , it follows that A_1 has at least $d + 1$ distinct eigenvalues. Indeed, A_1 has exactly $d + 1$ eigenvalues since $A_{d+1} = \nu_{d+1}(A_1) = 0$. The eigenvalues of A_1 are precisely the roots of ν_{d+1} . Therefore, we can use the corresponding eigenvectors of A_1 to compute the first eigenmatrix P by using the following result.

Proposition 5.2.8 ([11, Proposition 4.1.1.]). Suppose Γ is a distance regular graph with intersection array $\iota(\Gamma) = \{b_0, b_1, \dots, b_d; c_1, c_2, \dots, c_d\}$, and with the polynomials ν_i as defined in (5.2) and (5.3). Then the eigenvalues of Γ are the roots of $\nu_{d+1}(x)$. Furthermore, these are the eigenvalues of the intersection matrix B_1 , and for any eigenvalue λ_i , a corresponding left eigenvector of B_1 is given by $v_{\lambda_i} = (\nu_0(\lambda_i), \nu_1(\lambda_i), \dots, \nu_d(\lambda_i))^T$.

Suppose $(X; R_0, R_1, \dots, R_d)$ is an association scheme with first and second eigenmatrices P and Q . This scheme is said to be *P-polynomial* (respectively *Q-polynomial*) if there exist $z_0, z_1, \dots, z_d \in \mathbb{R}$ and polynomials p_0, p_1, \dots, p_d (resp. q_0, q_1, \dots, q_d) with real coefficients such that p_j (resp. q_j) has degree j and $P_{ij} = p_j(z_i)$ (resp. $Q_{ij} = q_j(z_i)$) for all i, j with $0 \leq i, j \leq d$. Note that the property of being *P-polynomial* (or *Q-polynomial*) is dependent on the ordering of the relations R_0, R_1, \dots, R_d , and a reordering of relations of a *P-polynomial* (or *Q-polynomial*) scheme may not be *P-polynomial* (*Q-polynomial*) and vice versa. The following powerful results by Delsarte link together the metric and cometric association schemes discussed earlier with *P-* and *Q-polynomial* association schemes. Since the two results are so closely related, we include them in one theorem.

Theorem 5.2.9 ([33, Theorems 5.6 and 5.16]). The following are equivalent:

- (i) $(X; R_0, R_1, \dots, R_d)$ is a metric association scheme.

(ii) $(X; R_0, R_1, \dots, R_d)$ is a P -polynomial association scheme.

Dually, the following are equivalent:

(i) $(X; R_0, R_1, \dots, R_d)$ is a cometric association scheme.

(ii) $(X; R_0, R_1, \dots, R_d)$ is a Q -polynomial association scheme.

A Q -polynomial scheme is said to be Q -bipartite if the dual intersection number q_{ij}^k is equal to 0 whenever $i + j + k$ is odd. A Q -polynomial scheme is called Q -antipodal if $q_{1j+1}^j = q_{1d-j-1}^{d-j}$ for all j with $0 \leq j \leq d$, except maybe when $j = \lfloor \frac{d}{2} \rfloor$ (see [50]).

In the next section, we use the results of this section to calculate the association scheme parameters of some dual polar graphs.

5.3 Association schemes of dual polar spaces

In Chapter 6, we discuss $\frac{q+1}{2}$ -ovoids of the dual polar spaces $\text{DH}(5, q^2)$ and $\text{DW}(5, q)$. We begin this section with a rather powerful result that motivates the study of association schemes on these dual polar spaces.

Theorem 5.3.1 ([18]). The point graph of a dual polar space is distance regular.

We are particularly interested in the association schemes of the point graphs of $\text{DH}(5, q^2)$, q odd, due to the following result.

Theorem 5.3.2 ([78, Theorem 6.7.8]). Suppose S is a $\frac{q+1}{2}$ -ovoid of the dual Hermitian space $\text{DH}(2d-1, q^2)$, q odd and $d \geq 2$. The subgraph induced by S on the point graph of $\text{DH}(2d-1, q^2)$ is distance regular with classical parameters

$$(d, b, \alpha, \beta) = \left(d, -q, -\left(\frac{q+1}{2}\right), -\frac{(-q)^d + 1}{2} \right).$$

Moreover, the distance between any two vertices in the induced subgraph is the same as in the original point graph.

Note that if $d = 2$, a $\frac{q+1}{2}$ -ovoid of $\text{DH}(3, q^2)$ is a hemisystem of $\text{H}(3, q^2)$, and the graphs obtained are the same as the strongly regular graphs in the result by Cameron et al. [20]. This is the only value of d for which significant work has been done, and in fact no distance regular graphs with these classical parameters are known for $d \geq 3$ [78, p. 210].

We investigate the existence of $\frac{q+1}{2}$ -ovoids of $\text{DH}(5, q^2)$ as the smallest open case. In this section, we calculate some intersection arrays and first and second eigenmatrices, and in Chapter 6, we will present some computational results.

There are some further results specific to association schemes of dual polar graphs that provide the final pieces of information necessary to compute the intersection arrays, valencies, a_i values and first and second eigenmatrices of the dual polar graph of $H(5, q^2)$.

Theorem 5.3.3 ([11, Theorem 9.4.3]). Suppose Γ is the dual polar graph of a classical polar space \mathcal{S} of rank d over $\text{GF}(q)$. The eigenvalues and multiplicities of the adjacency matrix A_1 of the related association scheme are as follows:

$$\lambda_i = q^e \begin{bmatrix} d-i \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix},$$

$$f'_i = f_i = q^i \begin{bmatrix} d \\ i \end{bmatrix} \frac{1 + q^{d+e-2i}}{1 + q^{d+e-i}} \prod_{j=1}^i \frac{1 + q^{d+e-j}}{q + q^{j-e}},$$

where e is as in Theorem 2.3.3.

Dual polar spaces also fall into a class of geometric objects called *regular near polygons* [11, §6.4]. In particular, the dual of a polar space of rank d is a regular near $2d - gon$. A regular near polygon is said to be of order (s, t) if there are $s + 1$ points incident with every line, and $t + 1$ lines incident with every point. Notice that this is the same definition of order that we have for generalised quadrangles. We may now make use of the following result.

Theorem 5.3.4 ([78, Theorem 6.4.8.]). Suppose S is an m -ovoid of a regular near $2d$ -gon $(\mathcal{P}, \mathcal{B}, I)$ of order (s, t) , $d \geq 2$. Then for any point $P \in \mathcal{P}$, the number of points in S at distance i from P is

$$\begin{cases} k_i \left(\frac{m}{s+1} + \left(1 - \frac{m}{s+1}\right) \left(-\frac{1}{s}\right)^i \right) & \text{if } P \in S, \\ k_i \frac{m}{s+1} \left(1 - \left(-\frac{1}{s}\right)^i\right) & \text{if } P \notin S. \end{cases}$$

Corollary 5.3.5. The subgraph induced by the set of points of an m -ovoid of a regular near $2d$ -gon on its point graph is distance regular. Therefore, it gives rise to an association scheme.

We have calculated all of the valencies and intersection arrays of $DH(5, q^2)$, $DW(5, q)$ and a $\frac{q+1}{2}$ -ovoid of $DH(5, q^2)$, as well as the first and second eigenmatrices of $DH(5, q^2)$ and $DW(5, q)$. They may be found in Appendix A. We explain in Section 6.6 why we are also interested in the association scheme of the point graph of $DW(5, q)$, and we also present some computational results for small values of q .

5.4 Association schemes of relative hemisystems

In Section 4.5 we briefly mentioned the association schemes that arise from relative hemisystems. We now have the machinery to explore these and explicitly state the relations that define them.

Recall from Section 4.4.5 that given a generalised quadrangle R of order (q^2, q) with a subquadrangle R' of order (q, q) , every external line ℓ subtends a spread \mathcal{S}_ℓ of R' . When $R = \text{H}(3, q^2)$ and $R' = \text{W}(3, q)$, the size of the intersection of the subtended spreads of two external lines ℓ and n is either 1, $q + 1$ or $q^2 + 1$, depending on the relationship between ℓ and n . In the dual of R , which is a generalised quadrangle of order (q, q^2) , the subtended spread \mathcal{S}_ℓ of an external line ℓ becomes a subtended ovoid \mathcal{O}_x of the external point x which is the image of ℓ in the dual. Note that here, we mean an ovoid in the sense of Section 4.5.

Theorem 5.4.1 ([64, Theorem 1]). Let R be a generalised quadrangle of order (q, q^2) , containing a doubly subtended generalised quadrangle R' of order (q, q) . Let X be the set of points of $R \setminus R'$. Define the following relations on X , together with the identity relation R_0 :

- R_1 : $(x, y) \in R_1$ if and only if x and y are not collinear and $|\mathcal{O}_x \cap \mathcal{O}_y| = 1$.
- R_2 : $(x, y) \in R_2$ if and only if x and y are not collinear and $|\mathcal{O}_x \cap \mathcal{O}_y| = q + 1$.
- R_3 : $(x, y) \in R_3$ if and only if x and y are collinear in R .
- R_4 : $(x, y) \in R_4$ if and only if $\mathcal{O}_x = \mathcal{O}_y$.

When all of the above relations are non-empty, they form a cometric association scheme on X .

If we use Lemma 4.4.11 to reinterpret the relations in the dual statement of Theorem 5.4.1, then we obtain the following result.

Theorem 5.4.2. Let R be a generalised quadrangle of order (q^2, q) , containing a doubly subtended generalised quadrangle R' of order (q, q) . Let \mathcal{L}_E be the set of external lines of R relative to R' . Then the following set of relations on \mathcal{L}_E , together with the identity relation R_0 , form a cometric association scheme on \mathcal{L}_E .

- R_1 : $(\ell, n) \in R_1$ if and only if ℓ and n are not concurrent in R and n is concurrent with $\bar{\ell}$.
- R_2 : $(\ell, n) \in R_2$ if and only if ℓ and n are not concurrent in R and n is not concurrent with $\bar{\ell}$.
- R_3 : $(\ell, n) \in R_3$ if and only if ℓ and n are concurrent in R .
- R_4 : $(\ell, n) \in R_4$ if and only if $n = \bar{\ell}$.

The P and Q matrices of this association scheme are the same for the original

scheme and the dual and are as follows

$$P = \begin{pmatrix} 1 & (q-1)(q^2+1) & (q^2-2q)(q^2+1) & (q-1)(q^2+1) & 1 \\ 1 & q^2+1 & 0 & -(q^2+1) & -1 \\ 1 & q-1 & -2q & q-1 & 1 \\ 1 & -q+1 & 0 & q-1 & -1 \\ 1 & -(q-1)^2 & 2q(q-2) & -(q-1)^2 & 1 \end{pmatrix}, \quad (5.4)$$

$$Q = \begin{pmatrix} 1 & \frac{q(q-1)^2}{2} & \frac{(q-2)(q+1)(q^2+1)}{2} & \frac{q(q-1)(q^2+1)}{2} & \frac{q(q^2+1)}{2} \\ 1 & \frac{q(q-1)}{2} & \frac{(q-2)(q+1)}{2} & \frac{-q(q-1)}{2} & \frac{-q(q-1)}{2} \\ 1 & 0 & -(q+1) & 0 & q \\ 1 & \frac{-q(q-1)}{2} & \frac{(q-2)(q+1)}{2} & \frac{q(q-1)}{2} & \frac{-q(q-1)}{2} \\ 1 & \frac{-q(q-1)^2}{2} & \frac{(q-2)(q+1)(q^2+1)}{2} & \frac{-q(q-1)(q^2+1)}{2} & \frac{q(q^2+1)}{2} \end{pmatrix}. \quad (5.5)$$

Recall from Section 4.4.5 that the only example of a generalised quadrangle R of order (q^2, q) , q even, containing a doubly subtended subquadrangle R' of order q is in the classical case when $R = H(3, q^2)$ and $R' = W(3, q)$. Therefore, the association scheme mentioned above cannot be related to flock generalised quadrangles, and cannot be used to directly prove results about them (as far as we are aware). However, we can use it to construct an association scheme arising from a relative hemisystem of $H(3, q^2)$ arising from $W(3, q)$.

Theorem 5.4.3 ([64, Theorem 4]). Suppose S is a relative hemisystem of $H(3, q^2)$, $q > 2$, with respect to $W(3, q)$. Then a primitive Q -polynomial association scheme can be constructed on the lines of the relative hemisystem, with the following relations:

- R_0 : $(\ell, m) \in R_0$ if and only if $\ell = m$ and $|\mathcal{S}_\ell \cap \mathcal{S}_m| = q^2 + 1$.
- R_1 : $(\ell, m) \in R_1$ if and only if ℓ, m are not concurrent and $|\mathcal{S}_\ell \cap \mathcal{S}_m| = 1$.
- R_2 : $(\ell, m) \in R_2$ if and only if ℓ, m are not concurrent and $|\mathcal{S}_\ell \cap \mathcal{S}_m| = q + 1$.
- R_3 : $(\ell, m) \in R_3$ if and only if ℓ, m are concurrent and $|\mathcal{S}_\ell \cap \mathcal{S}_m| = 1$.

The association scheme described in Theorem 5.4.3 is neither a P -polynomial association scheme, nor the dual of one. Only sporadic examples of such primitive Q -polynomial association schemes were known before relative hemisystems were introduced, and so with the first infinite family of relative hemisystems came the first infinite family of primitive Q -polynomial association schemes not arising from distance regular graphs [64].

6. New generalisations of relative hemisystems

Unlike with hemisystems, there has not been an exploration of generalisations of relative hemisystems, not even to the concept of relative m -covers. In this chapter, we will aim to address this by providing a variety of generalisations of relative hemisystems and results about them. We begin with the natural generalisation of relative hemisystems to relative m -covers.

6.1 Relative m -covers

Let R be a generalised quadrangle of order (q^2, q) , containing a subquadrangle R' of order (q, q) . Recall that we denote incidence of a point P with a line ℓ by $P I \ell$, and that we denote the set of external points and external lines by \mathcal{P}_E and \mathcal{L}_E respectively. Suppose \mathcal{S} is a set of points (lines) of R . Define $\mathcal{S}^{\perp E}$ to be the set of external points (external lines) that are collinear (concurrent) with every member of \mathcal{S} . We call a subset S of external lines a *relative m -cover* of R with respect to R' if every external point is incident with exactly m lines of S . There is always a relative 0-cover and a relative q -cover of R with respect to R' , obtained by taking none or all of the external lines. We will call these two cases *trivial*.

Lemma 6.1.1. Suppose that S is a relative m -cover of R . Then $|S| = m(q^3 - q)$.

Proof. We double count pairs (X, ℓ) , where X is an external point and ℓ is a line of the relative m -cover incident with X . Recalling the number of external points from Section 4.4.5, we have $(q^2 + 1)(q^3 - q) \cdot m = |S|(q^2 + 1)$, and so $|S| = m(q^3 - q)$. \square

We will now focus on relative m -covers of generalised quadrangles of order (q^2, q) with doubly subtended subquadrangles of order (q, q) .

Define a *characteristic vector* χ to be a vector indexed by an ordering of the external lines of R . If \mathcal{S} is a set of lines, then $\chi_{\mathcal{S}}$ has a 1 in the i th position if and only if the i th external line lies in \mathcal{S} , and a 0 otherwise. If \mathcal{S} is a set of points, then $\chi_{\mathcal{S}}$ has a 1 in the i th position if and only if the i th external line is incident with a point in \mathcal{S} , and a 0 otherwise.

Let $\chi_{[P]}$ denote the characteristic vector of the set of lines incident with a particular point P , and let χ_S denote the characteristic vector of an m -cover S .

Note that $\chi_{[P]} \cdot \chi_S = m$, by the definition of an m -cover, and moreover that $\chi_S \cdot j = |S| = m(q^3 - q)$, where j is the all-ones vector and \cdot is the dot product.

Now, we have the following:

$$((q^3 - q)\chi_{[P]} - j) \cdot \chi_S = (q^3 - q)\chi_{[P]} \cdot \chi_S - j \cdot \chi_S = m(q^3 - q) - m(q^3 - q) = 0.$$

Also, recalling the association scheme on external lines given in Theorem 5.4.2, and the associated adjacency and minimal idempotent matrices, we have

$$\begin{aligned} ((q^3 - q)\chi_{[P]} - j) E_0 &= \frac{1}{q^2(q^2 - 1)} ((q^3 - q)\chi_{[P]} - j) J \\ &= \frac{(q^3 - q)}{q^2(q^2 - 1)} \chi_{[P]} J - \frac{1}{q^2(q^2 - 1)} j J \\ &= \frac{(q^3 - q)}{q^2(q^2 - 1)} (qj) - \frac{q^2(q^2 - 1)}{q^2(q^2 - 1)} j \\ &= 0. \end{aligned}$$

Now, using the matrix Q in equation (5.5), we can express each of the other E_i matrices in terms of adjacency matrices. Recall that the size of the set \mathcal{L}_E of external lines is $q^2(q^2 - 1)$. We have

$$E_1 = \frac{1}{|\mathcal{L}_E|} \left(\frac{q(q-1)^2}{2} \mathbf{I} + \frac{q(q-1)}{2} A_1 - \frac{q(q-1)}{2} A_3 - \frac{q(q-1)^2}{2} A_4 \right), \quad (6.1)$$

$$\begin{aligned} E_2 &= \frac{1}{|\mathcal{L}_E|} \left(\frac{(q-2)(q+1)(q^2+1)}{2} \mathbf{I} + \frac{(q-2)(q+1)}{2} A_1 - (q+1) A_2 \right. \\ &\quad \left. + \frac{(q-2)(q+1)}{2} A_3 + \frac{(q-2)(q+1)(q^2+1)}{2} A_4 \right), \quad (6.2) \end{aligned}$$

$$E_3 = \frac{1}{|\mathcal{L}_E|} \left(\frac{q(q-1)(q^2+1)}{2} \mathbf{I} - \frac{q(q-1)}{2} A_1 + \frac{q(q-1)}{2} A_3 - \frac{q(q-1)(q^2+1)}{2} A_4 \right), \quad (6.3)$$

$$E_4 = \frac{1}{|\mathcal{L}_E|} \left(\frac{q(q^2+1)}{2} \mathbf{I} - \frac{q(q-1)}{2} A_1 + q A_2 - \frac{q(q-1)}{2} A_3 + \frac{q(q^2+1)}{2} A_4 \right). \quad (6.4)$$

Let σ be the involutory automorphism of $H(3, q^2)$ introduced in Lemma 4.4.13 fixing $W(3, q)$ pointwise.

Proposition 6.1.2. Let $\chi_{[P]}$ be the characteristic vector of an external point P . Then $((q^3 - q)\chi_{[P]} - j) E_1 = 0$ and $((q^3 - q)\chi_{[P]} - j) E_i \neq 0$ for $i \in \{2, 3, 4\}$.

Proof. Define $\chi_{[\bar{P}]}$ to be the characteristic vector of $\bar{P} = P^\sigma$. Also, define χ_W to be the characteristic vector of the set of external lines concurrent with the line on P meeting the doubly subtended quadrangle R' , but not incident with P or \bar{P} . As mentioned earlier, incidence is preserved under σ because it is an automorphism, so $P \ell$ if and only if $\bar{P} \bar{\ell}$. Now, $\chi_{[P]} \mathbf{I} = \chi_{[P]}$ trivially, and $\chi_{[P]} A_4 = \chi_{[\bar{P}]}$ because the i th position in the resulting vector will be 1 if and only if the image of the corresponding external line under σ is incident with P . We now calculate $v = \chi_{[P]} A_1$. This is equivalent to counting how many

external lines on P are concurrent with the image of each of the other external lines under σ . First, if an external line ℓ is incident with P , then Lemma 4.4.14 implies that $\bar{\ell}$ must be concurrent with the line on P meeting the subquadrangle R' . It cannot be concurrent with any other lines on P (otherwise we have a triangle). Therefore, v has the value 0 in positions corresponding to external lines on P . On the other hand, if we have an external line ℓ such that $\bar{\ell} \cap P$, then every other external line incident with P is concurrent with $\bar{\ell}$, so v will have $q - 1$ in every position where the corresponding line has its image under σ incident with P . Each of the remaining external lines must have a 1 in their corresponding positions in v by the GQ axiom, unless it is concurrent with the line on P meeting R' , in which case the corresponding entry in v is 0 because it is concurrent with no external line incident with P . Summarising, we have

$$\chi_{[P]}A_1 = (q - 1)\chi_{[\bar{P}]} + (j - \chi_{[P]} - \chi_W - \chi_{[\bar{P}]}) = (q - 2)\chi_{[\bar{P}]} + j - \chi_W - \chi_{[P]}.$$

Let us now calculate $u = \chi_{[P]}A_3$. This is equivalent to counting how many external lines on P are concurrent with each of the other external lines. If ℓ is an external line incident with P , then the other $q - 1$ external lines on P are concurrent with it and so the corresponding positions in u will have value $q - 1$. If ℓ is an external line not incident with P , then, by the GQ axiom, either ℓ is concurrent with an external line on P , in which case the corresponding value in u will be 1, or ℓ is concurrent with the line on P meeting the subquadrangle R' , and so contributes 0 to u . Summarising, we have

$$\chi_{[P]}A_3 = (q - 1)\chi_{[P]} + (j - \chi_{[P]} - \chi_W - \chi_{[\bar{P}]}) = j + (q - 2)\chi_{[P]} - \chi_W - \chi_{[\bar{P}]}.$$

We will now calculate $\chi_{[P]}A_2$, using the fact that the sum of the adjacency matrices is the all-ones matrix J :

$$\begin{aligned} \chi_{[P]}A_2 &= \chi_{[P]}J - (\chi_{[P]}I + \chi_{[P]}A_1 + \chi_{[P]}A_3 + \chi_{[P]}A_4) \\ &= qj - \left(\chi_{[P]} + (q - 2)\chi_{[\bar{P}]} + j - \chi_W - \chi_{[P]} + j \right. \\ &\quad \left. + (q - 2)\chi_{[P]} - \chi_W - \chi_{[\bar{P}]} + \chi_{[\bar{P}]} \right) \\ &= (q - 2)j - (q - 2)(\chi_{[\bar{P}]} + \chi_{[P]}) + 2\chi_W. \end{aligned}$$

Now, considering the product of the all-ones vector j with the adjacency matrices, we have $j\mathbf{I} = jA_4 = j$, since each line is only equal to itself, and there is only one line equal to the image of a line under σ . We also have $jA_1 = (q^2 + 1)(q - 1)j = jA_3$, since every line has $(q^2 + 1)(q - 1)$ external lines concurrent with it. Finally, we calculate jA_2 by again using the fact that the sum of the adjacency matrices is the all-ones matrix J :

$$\begin{aligned} jA_2 &= jJ - (jI + jA_1 + jA_3 + jA_4) \\ &= q^2(q^2 - 1)j - (j + (q^2 + 1)(q - 1)j + (q^2 + 1)(q - 1)j + j) \\ &= q(q^3 - 2q^2 + q - 2)j. \end{aligned}$$

Now,

$$\begin{aligned}
q^2(q^2 - 1)\chi_{[P]}E_1 &= \chi_{[P]} \left(\frac{q(q-1)^2}{2}I + \frac{q(q-1)}{2}A_1 - \frac{q(q-1)}{2}A_3 - \frac{q(q-1)^2}{2}A_4 \right) \\
&= \frac{q(q-1)^2}{2}\chi_{[P]} - \frac{q(q-1)^2}{2}\chi_{[\bar{P}]} \\
&\quad + \frac{q(q-1)}{2}((q-2)\chi_{[\bar{P}]} + j - \chi_W - \chi_{[P]}) \\
&\quad - \frac{q(q-1)}{2}(j + (q-2)\chi_{[P]} - \chi_W - \chi_{[\bar{P}]}) \\
&= 0.
\end{aligned}$$

Therefore, $\chi_{[P]}E_1 = 0$. Similarly, we can calculate $\chi_{[P]}E_i$ for $i \in \{2, 3, 4\}$:

$$\begin{aligned}
\chi_{[P]}E_2 &= \frac{(q-2)(q+1)}{2(q-1)}(\chi_{[P]} + \chi_{[\bar{P}]}) - \frac{1}{q(q-1)}\chi_W, \\
\chi_{[P]}E_3 &= \frac{1}{2}(\chi_{[P]} - \chi_{[\bar{P}]}) \\
\chi_{[P]}E_4 &= \frac{1}{q(q-1)}(\chi_{[P]} + \chi_{[\bar{P}]} + \chi_W) - \frac{1}{q(q^2-1)}j.
\end{aligned}$$

None of these are ever equal to the zero vector. Furthermore, $jE_i = 0$ for every i with $1 \leq i \leq 4$ because j is the basis vector for V_0 , and E_i is the projection onto V_i , which is orthogonal to V_k when $i \neq k$. Therefore,

$$\begin{aligned}
((q^3 - q)\chi_{[P]} - j)E_1 &= 0, \\
((q^3 - q)\chi_{[P]} - j)E_2 &= \frac{q(q-2)(q+1)^2}{2}(\chi_{[P]} + \chi_{[\bar{P}]}) - (q+1)\chi_W, \\
((q^3 - q)\chi_{[P]} - j)E_3 &= \frac{q(q^2 - 1)}{2}(\chi_{[P]} - \chi_{[\bar{P}]}) \\
((q^3 - q)\chi_{[P]} - j)E_4 &= (q+1)(\chi_{[P]} + \chi_{[\bar{P}]} + \chi_W) - j. \quad \square
\end{aligned}$$

Theorem 6.1.3. Let $\chi_{[P]}$ be the characteristic vector of the set of external lines incident with an external point P . Then the set of vectors $\{\chi_{[P]} \mid P \in \mathcal{P}_E\}$ is a spanning set of $V_0 \perp V_2 \perp V_3 \perp V_4$.

Proof. By the proof of Proposition 6.1.2, $\chi_{[P]} \in V_0 \perp V_2 \perp V_3 \perp V_4$. Let A be the matrix whose rows are the $\chi_{[P]}$ vectors. To prove that $\{\chi_{[P]} \mid P \in \mathcal{P}_E\}$ spans $V_0 \perp V_2 \perp V_3 \perp V_4$, it is sufficient to show that the rank of A is equal to $\dim(V_0 \perp V_2 \perp V_3 \perp V_4) = |\mathcal{L}_E| - \dim(V_1)$. Consider the matrix $M = A^T A$. The rows of M correspond to counting how many points incident with a particular external line are incident with each of the other external lines. The diagonal entries of M are equal to $q^2 + 1$, the entries whose row and column correspond to concurrent lines are equal to 1, and the entries are zero otherwise. In particular, the rows of M are of the form $(q^2 + 1)\chi_\ell + \chi_{\{\ell\}^\perp_E}$ for some external line ℓ . Denote

the row of M corresponding to the external line ℓ by M_ℓ , and recall that the size of $\{\ell\}^{\perp E}$ is $(q-1)(q^2+1)$, since each of the points of ℓ are incident with $q-1$ external lines not equal to ℓ . Note that $\text{rank}(M) \leq \text{rank}(A)$, since M is constructed by taking linear combinations of the rows of A . Also notice that M has constant column sum equal to $q(q^2+1)$, and so the all-ones vector j lies in the row space of M (implying that $V_0 = \langle j \rangle$ does too). We first show that $\chi_\ell + \chi_{\bar{\ell}}$ is in the row space of M . Define the set

$$U_\ell := \{M_\ell\} \cup \{M_{\ell'} \mid \ell' \text{ is concurrent with } \ell\}.$$

Then the sum of the vectors in U_ℓ is

$$\begin{aligned} & (q^2+1)\chi_\ell + \chi_{\{\ell\}^{\perp E}} + (q^2+1)\chi_{\{\ell\}^{\perp E}} + (q-1)(q^2+1)\chi_\ell \\ & + (q-2)\chi_{\{\ell\}^{\perp E}} + q^2\chi_{\{\bar{\ell}\}^{\perp E}} + (q^2-q)\chi_{\{\ell, \bar{\ell}\}^{\perp 2}}, \end{aligned}$$

where $\chi_{\{\ell, \bar{\ell}\}^{\perp 2}}$ is the set of lines that are neither concurrent with nor equal to ℓ or $\bar{\ell}$. The first two terms originate from M_ℓ . The third term arises because $M_{\ell'}$ contributes $(q^2+1)\chi_{\ell'}$ for every $\ell' \in \{\ell\}^{\perp E}$. Every line of $\{\ell\}^{\perp E}$ is concurrent with ℓ by definition, and this is how the fourth term arises. The fifth term comes about because every element of $\{\ell\}^{\perp E}$ is also concurrent with the $q-2$ elements of $\{\ell\}^{\perp E}$ that lie on the same point of intersection with ℓ (and no other elements of $\{\ell\}^{\perp E}$). The sixth and seventh terms arise because for every external line n not equal to ℓ or $\bar{\ell}$ and not concurrent with ℓ , Lemma 4.4.12 implies that the size of $\{\ell, n\}^{\perp E}$ is q^2 if n is concurrent with $\bar{\ell}$, and q^2-q otherwise. Simplifying slightly, we write the sum as

$$q(q^2+1)\chi_\ell + (q^2+q)\chi_{\ell^{\perp E}} + q^2\chi_{\bar{\ell}^{\perp E}} + (q^2-q)\chi_{\{\ell, \bar{\ell}\}^{\perp 2}}.$$

Consider the sum of all of the vectors in $U_\ell \cup U_{\bar{\ell}}$. After some simplification, we arrive at the expression

$$q(q^2+1)(\chi_\ell + \chi_{\bar{\ell}}) + (2q^2+q)(\chi_{\ell^{\perp E}} + \chi_{\bar{\ell}^{\perp E}}) + (2q^2-2q)\chi_{\{\ell, \bar{\ell}\}^{\perp 2}}. \quad (6.5)$$

Recall that j is in the row space of M . We calculate the difference between (6.5) and $(2q^2-2q)j$ and arrive at the following expression:

$$q(q^2-2q+3)(\chi_\ell + \chi_{\bar{\ell}}) + 3q(\chi_{\ell^{\perp E}} + \chi_{\bar{\ell}^{\perp E}}).$$

We now subtract $3q(M_\ell + M_{\bar{\ell}})$:

$$\begin{aligned} & q(q^2-2q+3)(\chi_\ell + \chi_{\bar{\ell}}) + 3q(\chi_{\ell^{\perp E}} + \chi_{\bar{\ell}^{\perp E}}) - 3q(M_\ell + M_{\bar{\ell}}) \\ & = q(q^2-2q+3)(\chi_\ell + \chi_{\bar{\ell}}) - 3q(q^2+1)(\chi_\ell + \chi_{\bar{\ell}}) \\ & = -2q^2(q+1)(\chi_\ell + \chi_{\bar{\ell}}). \end{aligned}$$

Therefore, $\chi_\ell + \chi_{\bar{\ell}}$ is in the row space of M . Based on this, we can now show that V_2, V_3 and V_4 lie in the row space of M . The χ_ℓ vectors form a basis for the vector space over $\mathbb{C}^{\mathcal{L}^E}$ by definition. Therefore, the set $\{\chi_\ell E_i \mid \ell \in \mathcal{L}^E\}$ forms

a basis for V_i for $0 \leq i \leq 4$. Hence, to show that V_i lies in the row space of M , it is sufficient to show that $\chi_\ell E_i$ does, for any choice of external line ℓ . We first calculate the $\chi_\ell E_i$, based on their expressions as the linear combinations of adjacency matrices above (see Equations (6.1),(6.2), (6.3), (6.4)):

$$\begin{aligned}\chi_\ell E_0 &= \frac{1}{q^2(q^2-1)}j, \\ \chi_\ell E_1 &= \frac{1}{2q(q+1)}((q-1)(\chi_\ell - \chi_{\bar{\ell}}) + \chi_{\bar{\ell}^\perp E} - \chi_{\ell^\perp E}), \\ \chi_\ell E_2 &= \frac{1}{2q^2(q-1)}(-2j + q((q-1)^2\chi_\ell + \chi_{\ell^\perp E} + (q-1)^2\chi_{\bar{\ell}} + \chi_{\bar{\ell}^\perp E})), \\ \chi_\ell E_3 &= \frac{1}{2q(q+1)}((q^2+1)\chi_\ell - (q^2+1)\chi_{\bar{\ell}} - \chi_{\bar{\ell}^\perp E} + \chi_{\ell^\perp E}), \\ \chi_\ell E_4 &= \frac{1}{2q(q^2-1)}(2j + (q+1)((q-1)(\chi_\ell + \chi_{\bar{\ell}}) - \chi_{\bar{\ell}^\perp E} - \chi_{\ell^\perp E})).\end{aligned}$$

Recall that the rows M_ℓ of M are of the form $(q^2+1)\chi_\ell + \chi_{\{\ell\}^\perp E}$ for each external line ℓ . Then $\chi_\ell E_3$ is in the row space of M , since it is a linear combination of the $M_\ell - M_{\bar{\ell}}$ vectors. Now consider $\chi_\ell E_2$. Since j is in the row space of M , we only need to show that $(q-1)^2\chi_\ell + \chi_{\ell^\perp E} + (q-1)^2\chi_{\bar{\ell}} + \chi_{\bar{\ell}^\perp E}$ is also in the row space of M . We can rewrite this vector as

$$((q^2+1)\chi_\ell + \chi_{\ell^\perp E}) + ((q^2+1)\chi_{\bar{\ell}} + \chi_{\bar{\ell}^\perp E}) + ((q-1)^2 - (q^2+1))(\chi_\ell + \chi_{\bar{\ell}}).$$

This lies in the row space of M since the first two terms are M_ℓ and $M_{\bar{\ell}}$, and $\chi_\ell + \chi_{\bar{\ell}}$ also lies in the row space of M . Finally, consider $\chi_\ell E_4$. Using the same reasoning as before, it is sufficient to show that $(q-1)(\chi_\ell + \chi_{\bar{\ell}}) - \chi_{\bar{\ell}^\perp E} - \chi_{\ell^\perp E}$ is in the row space of M . We can rewrite this expression as

$$-((q^2+1)\chi_\ell + \chi_{\ell^\perp E}) - ((q^2+1)\chi_{\bar{\ell}} + \chi_{\bar{\ell}^\perp E}) + ((q-1) + (q^2+1))(\chi_\ell + \chi_{\bar{\ell}}),$$

which is in the row space of M . Therefore, $V_0 \perp V_2 \perp V_3 \perp V_4$ is a subspace of the row space of M . This implies that M , and therefore A , has rank at least the dimension of $V_0 \perp V_2 \perp V_3 \perp V_4$. Hence A has rank equal to the dimension of $V_0 \perp V_2 \perp V_3 \perp V_4$ and its rows (that is, the set $\{\chi_{[P]} \mid P \in \mathcal{P}_E\}$) form a spanning set of $V_0 \perp V_2 \perp V_3 \perp V_4$. \square

Corollary 6.1.4. The set of vectors $\{(q^3 - q)\chi_{[P]} - j \mid P \in \mathcal{P}_E\}$ forms a spanning set of $V_2 \perp V_3 \perp V_4$.

Proof. By Theorem 6.1.3, the set $\{\chi_{[P]} \mid P \in \mathcal{P}_E\}$ forms a spanning set of $V_0 \perp V_2 \perp V_3 \perp V_4$, and therefore so does $\{(q^3 - q)\chi_{[P]} \mid P \in \mathcal{P}_E\}$. Since j lies in V_0 , any set $\{\alpha\chi_{[P]} + \beta j \mid P \in \mathcal{P}_E\}$ will span at least $V_2 \perp V_3 \perp V_4$ for any choice of $\alpha, \beta \in \mathbb{C}$. Now, $((q^3 - q)\chi_{[P]} - j)E_0 = \frac{1}{q^2(q^2-1)}((q^3 - q)q - q^2(q^2 - 1)) = 0$, so $(q^3 - q)\chi_{[P]} - j \in V_2 \perp V_3 \perp V_4$ for every external point P . Therefore, the set $\{(q^3 - q)\chi_{[P]} - j \mid P \in \mathcal{P}_E\}$ forms a spanning set of $V_2 \perp V_3 \perp V_4$. \square

Corollary 6.1.5. Let S be a relative m -cover of a generalised quadrangle R of order (q^2, q) relative to a doubly subtended subquadrangle R' of order (q, q) . The characteristic vector χ_S of the external lines contained in S lies in $V_0 \perp V_1$.

Proof. Since $\{(q^3 - q)\chi_{[P]} - j \mid P \in \mathcal{P}_E\}$ forms a spanning set of $V_2 \perp V_3 \perp V_4$, it is sufficient to show that $\chi_S \cdot ((q^3 - q)\chi_{[P]} - j) = 0$ for all $P \in \mathcal{P}_E$. By Lemma 6.1.1, we have

$$\chi_S \cdot ((q^3 - q)\chi_{[P]} - j) = m(q^3 - q) - m(q^3 - q) = 0.$$

The result immediately follows. \square

Theorem 6.1.6. Let R be a generalised quadrangle of order (q^2, q) containing a doubly subtended subquadrangle R' of order (q, q) . If S is a nontrivial relative m -cover of the external lines, then

- (i) S is a relative hemisystem, that is, $m = \frac{q}{2}$;
- (ii) the image of S under the involutory automorphism σ fixing R' pointwise is its complement.

Proof. Let χ_S be the characteristic vector of the external lines in S , let S^σ denote the image of S under σ , and let χ_{S^σ} denote the characteristic vector of the external lines in S^σ . We first calculate the product of χ_S with each of the adjacency matrices. First, $\chi_S I = \chi_S$ trivially, and $\chi_S A_4 = \chi_{S^\sigma}$ since there will be a 1 in the j th position of $\chi_S A_4$ if and only if the image of the corresponding line under σ is contained in S , and 0 otherwise. Let us now calculate $\chi_S A_3$. This is equivalent to counting how many elements of S are concurrent with each external line. If ℓ is an external line in S , then it will be concurrent with $(m - 1)(q^2 + 1)$ lines of S (that is, $m - 1$ for each point on the line). If ℓ is not in S , then there will be $m(q^2 + 1)$ lines of S concurrent with ℓ . Summarising,

$$\chi_S A_3 = (q^2 + 1)(m - 1)\chi_S + m(q^2 + 1)(j - \chi_S) = m(q^2 + 1)j - (q^2 + 1)\chi_S.$$

We now calculate $\chi_S A_1$. This is equivalent to counting the number of elements of S that are concurrent with the image $\bar{\ell}$ of each external line ℓ under σ . Since σ preserves incidence, if s is a line in S , then s is concurrent with $\bar{\ell}$ if and only if \bar{s} is concurrent with ℓ . Therefore, we may equivalently count how many lines in S^σ are concurrent with each external line ℓ , which is the same as calculating $\chi_{S^\sigma} A_3$. Using the calculation we just completed, we have

$$\chi_S A_1 = \chi_{S^\sigma} A_3 = m(q^2 + 1)j - (q^2 + 1)\chi_{S^\sigma}.$$

Now, it remains to calculate $\chi_S A_2$. We will make use of the fact that the sum

of all of the adjacency matrices is equal to J . Therefore,

$$\begin{aligned}\chi_S A_2 &= \chi_S J - (\chi_S I + \chi_S A_1 + \chi_S A_3 + \chi_S A_4) \\ &= m(q^3 - q)j - (\chi_S + m(q^2 + 1)j - (q^2 + 1)\chi_{S^\sigma}) \\ &\quad + m(q^2 + 1)j - (q^2 + 1)\chi_S + \chi_{S^\sigma} \\ &= m(q^3 - 2q^2 - q - 2)j + q^2(\chi_S + \chi_{S^\sigma}).\end{aligned}$$

We now make use of Equations (6.2), (6.3) and (6.4) to compute the projections of χ_S into V_2 , V_3 and V_4 . They are as follows:

$$\begin{aligned}\chi_S E_2 &= \frac{1}{q^2(q^2 - 1)} (2mq(q + 1)j - q^2(q + 1)(\chi_S + \chi_{S^\sigma})), \\ \chi_S E_3 &= 0, \\ \chi_S E_4 &= \frac{1}{q^2(q^2 - 1)} \left(\frac{q^2(q + 1)^2}{2} (\chi_S + \chi_{S^\sigma}) - mq(q + 1)^2 j \right).\end{aligned}$$

By Corollary 6.1.5, $\chi_S \in V_0 \perp V_1$. In particular, the projection of S into V_2 , V_3 and V_4 must be equal to zero. Therefore, it is sufficient to show that $q^2(q^2 - 1)\chi_S E_i = 0$ for $i \in \{2, 3, 4\}$. First note that the entries in $\chi_S + \chi_{S^\sigma}$ must be equal to 0, 1 or 2, since both χ_S and χ_{S^σ} are vectors composed of zeros and ones. In order for $q^2(q^2 - 1)\chi_S E_2 = 2mq(q + 1)j - q^2(q + 1)(\chi_S + \chi_{S^\sigma})$ to be equal to the zero vector, $\chi_S + \chi_{S^\sigma}$ must be a constant vector, since j is a constant vector. In other words, $\chi_S + \chi_{S^\sigma}$ must be the zero vector, j or $2j$. The first and last cases correspond to when S consists of none or all of the external lines respectively. Since S is a nontrivial relative m -cover, it follows that $\chi_S + \chi_{S^\sigma} = j$. Since $|S| = |S^\sigma|$, it follows that S consists of half of the external lines and so $m = \frac{q}{2}$. That is, S is a relative hemisystem. Moreover, we must have that S^σ is the complement of S . Now, taking $m = \frac{q}{2}$ and $\chi_S + \chi_{S^\sigma} = j$, we find that

$$\begin{aligned}q^2(q^2 - 1)\chi_S E_2 &= 2mq(q + 1)j - q^2(q + 1)(\chi_S + \chi_{S^\sigma}) \\ &= q^2(q + 1)j - q^2(q + 1)j = 0, \\ q^2(q^2 - 1)\chi_S E_3 &= 0, \\ q^2(q^2 - 1)\chi_S E_4 &= \frac{q^2(q + 1)^2}{2} (\chi_S + \chi_{S^\sigma}) - mq(q + 1)^2 j \\ &= \frac{q^2(q + 1)^2}{2} j - \frac{q^2(q + 1)^2}{2} j = 0,\end{aligned}$$

as required. \square

Note that we have now proved Theorem 1.0.1. An alternate proof of part (i) of Theorem 6.1.6, using techniques similar to those used by Bamberg et. al to prove to show the nonexistence of m -covers that are not hemisystems [4, Section 8] is given in Appendix C.

From Section 4.4.5, the only generalised quadrangle of order (q^2, q) that has a doubly subtended subquadrangle of order (q, q) is $H(3, q^2)$, so we have the following immediate corollary.

Corollary 6.1.7. Let $R = H(3, q^2)$, a generalised quadrangle of order (q^2, q) with $R' = W(3, q)$ a subquadrangle of order (q, q) . Suppose that S is a nontrivial relative m -cover of the external lines. Then:

- (i) S is a relative hemisystem. That is, $m = \frac{q}{2}$.
- (ii) The image of S under the involutory automorphism σ fixing R' pointwise is its complement.

We have also proved the following result computationally.

Lemma 6.1.8. There are no nontrivial relative m -covers that are not relative hemisystems on the unique nonclassical flock generalised quadrangle of order $(8^2, 8)$.

6.2 Techniques

The techniques employed to achieve the computational results included in this chapter are extremely similar for all of the problems we address, and we now outline them. In order to be as specific as possible, we will describe our techniques in the context of finding relative hemisystems of nonclassical generalised quadrangles. Recall that we define relative hemisystems of a generalised quadrangle R of order (q^2, q) relative to a particular generalised subquadrangle R' of order (q, q) . Therefore, when considering how to construct the stabiliser of a relative hemisystem S , our first step is to stabilise R' . We say that two relative hemisystems on R (relative to R') are *equivalent* if one may be mapped to the other by an automorphism contained in $\text{Aut}(R)_{R'}$. Therefore, the set of relative hemisystems equivalent to a given relative hemisystem S is precisely the set of images of S under the group $\text{Aut}(R)_{R'}$ of automorphisms of R stabilising R' .

Lemma 6.2.1. Let R be a generalised quadrangle of order (q^2, q) , let R' be a generalised subquadrangle of R of order (q, q) , and let S_1 and S_2 be relative hemisystems of R relative to R' . Then S_1 is equivalent to either S_2 or a relative hemisystem stabilised by the stabiliser of S_2 if and only if the setwise stabilisers of S_1 and S_2 in $\text{Aut}(R)_{R'}$ are conjugate.

Proof. First suppose that S_1 and S_2 are equivalent. Let G_1 and G_2 denote their respective setwise stabilisers, and choose an automorphism $h \in \text{Aut}(R)_{R'}$ mapping S_1 to S_2 . Let $g_1 \in G_1$, and $\ell \in S_2$. Then $\ell^{h^{-1}g_1h} \in S_2$. Therefore, $h^{-1}G_1h \subseteq G_2$. By an analogous argument, we have $hG_2h^{-1} \subseteq G_1$, which combined with the previous inclusion implies $G_2 = h^{-1}G_1h$. Therefore, G_1 and G_2 are conjugate. An analogous argument holds if S_1 is instead equivalent to another relative hemisystem stabilised by G_2 .

Conversely, suppose S_1 and S_2 are relative hemisystems, with stabilisers G_1 and G_2 respectively, such that $h^{-1}G_1h = G_2$ for some $h \in \text{Aut}(R)_{R'}$. Then for $\ell \in S_2$, $\ell^{h^{-1}g_1h} \in S_2$ for all $g_1 \in G_1$. This implies that $\ell^{h^{-1}g_1} \in S_2^{h^{-1}}$. Since $\ell^{h^{-1}} \in S_2^{h^{-1}}$, G_1 must stabilise the set $S_2^{h^{-1}}$, which is a relative hemisystem equivalent to S_2 . \square

Our method for classifying relative hemisystems of the unique nonclassical flock generalised quadrangle of order $(8^2, 8)$, and the dual of $T_3(\mathcal{O})$, \mathcal{O} a Suzuki–Tits ovoid, is therefore dependent on classifying their stabilisers up to conjugacy.

For both of these choices of generalised quadrangle R , the generalised subquadrangle R' is isomorphic to $T_2(\mathcal{O})$, with \mathcal{O} not a conic. The stabiliser $\text{Aut}(R)_{R'}$ of R' in the automorphism group of R is isomorphic to the full automorphism group of R' , and for $q = 8$, this group is of order $602112 = 2^{12} \cdot 3 \cdot 7^2$. If S is a relative hemisystem of R relative to R' and $\text{Stab}(S) \leq \text{Aut}(R)_{R'}$ is its stabiliser, then Lagrange’s Theorem tells us that the order of $\text{Stab}(S)$ divides the order of $\text{Aut}(R)_{R'}$. Moreover, if p is a prime dividing the order of $\text{Stab}(S)$ (and hence $\text{Aut}(R)_{R'}$), then by Cauchy’s Theorem, there must exist an element of order p in $\text{Stab}(S)$.

Therefore, we take a set of conjugacy class representatives in $\text{Aut}(R)_{R'}$, and consider the cyclic groups generated by each of them. If we can classify all of the relative hemisystems stabilised by all of the cyclic groups of a particular order k , then we will have classified up to equivalence all of the relative hemisystems with a stabiliser containing an element of order k . In particular, if k is prime, then we will have classified the relative hemisystems with stabilisers of order divisible by k .

We use the mixed integer linear programming software Gurobi [40] to exhaustively search for relative hemisystems.

Suppose G is a subgroup of the automorphism group of an incidence geometry containing points and lines. Let P be a point and ℓ be a line of the incidence geometry. We define the *line orbit incidence number* as follows:

$$n_{P,\ell}^G = |\{m \in \ell^G \mid PIm\}|.$$

Lemma 6.2.2. Let G be a subgroup of the automorphism group of an incidence geometry as defined above. Then for every pair of points P and Q that lie in the same orbit under G , we have $n_{P,\ell}^G = n_{Q,\ell}^G$, for all $\ell \in \mathcal{B}$.

Proof. Let $P^g = Q$, for some $g \in G$. Then since G preserves incidence, if P is incident with a line $\ell \in \mathcal{B}$, then $P^g = Q$ is incident with ℓ^g . The result immediately follows. \square

A relative hemisystem is a union of external line orbits of its stabiliser. We can therefore check whether a set of lines S forms a relative hemisystem by taking

representatives of the point orbits of the setwise stabiliser of S and making sure that each of the representatives is incident with exactly $\frac{q}{2}$ lines of S by calculating the associated line orbit incidence numbers.

We write a mixed integer linear program (MILP) to search for all relative hemisystems stabilised by a group $G \leq \text{Aut}(R)_{R'}$ as follows. We begin by computing the orbits $\ell_1^G, \ell_2^G, \dots, \ell_a^G$ of G on external lines, and indexing them by binary variables r_1, r_2, \dots, r_a . Second, we compute the orbits of G on external points $P_1^G, P_2^G, \dots, P_b^G$, and take representatives P_1, P_2, \dots, P_b of each of these orbits. Next, for each point orbit representative P_i , we add a constraint to our MILP by taking the coefficients of r_j to be the corresponding line orbit incidence numbers n_{P_i, ℓ_j}^G , and set the sum of these terms equal to $\frac{q}{2}$ to satisfy the definition of a relative hemisystem.

From here, we run Gurobi using one of two methods. If the number of external line orbits (and therefore the number of variables in the MILP) of G is small, we simply run the program in Gurobi through the Gurobify package in GAP [7, 39]. If the number of external line orbits is large, we further exploit the action of the group on the geometry to make the constraints stronger.

Consider the images of each of the ℓ_i^G under $\text{Aut}(R)_{R'}$. If ℓ_i^G is the image of some ℓ_j^G under an element of $\text{Aut}(R)_{R'}$, then any relative hemisystem containing one of them must be equivalent to one containing the other, by Lemma 6.2.1. We can define an equivalence relation $\sim_{R'}$ by making $\ell_i^G \sim_{R'} \ell_j^G$ if and only if the elements of one are the images of the elements of the other under an element of $\text{Aut}(R)_{R'}$. Therefore, it is sufficient to consider those relative hemisystems containing certain sets of lines that constitute equivalence class representatives of $\sim_{R'}$ on $\{\ell_1^G, \ell_2^G, \dots, \ell_a^G\}$. We can apply the same logic to generate all of the sets of a given size n of external line orbits of G up to equivalence. For convenience, we usually do this using a top-down approach: we take the equivalence class representatives of the action of $\text{Aut}(R)_{R'}$ on the elements of external line orbits of G , fix them, find equivalence class representatives of the action on elements of the remaining external line orbits, fix them, and so on. This enables us to construct a computation tree, namely a tab-delimited list of the tuples of indices of size at most n , up to equivalence. We call the tuples that are not subsets of any other tuples in the computation tree *leaves*. Once we have checked all of the tuples of size n , we have found all of the relative hemisystems stabilised by G (up to equivalence).

As an example, suppose that we want to construct a computation tree of point orbit representatives from the automorphism group of $W(3, 2)$. This symplectic space is isomorphic to the unique generalised quadrangle of order $(2, 2)$ discussed in Example 4.1.1. Here, we will take our group G to be the trivial group. From Section 3.3.3, the automorphism group of $W(3, 2)$ is $H = \text{P}\Gamma\text{Sp}(4, 2)$. Since $q = 2$, H is isomorphic to $\text{P}\text{Sp}(4, 2)$, because the automorphism group of $\text{GF}(2)$ is the trivial group. We index the points of $\text{GQ}(2, 2)$ with numbers $\{1, 2, \dots, 15\}$.

Suppose we want to calculate all of the 3-tuples of points up to equivalence under the action of G . Now, H is transitive on the points of $\text{GQ}(2, 2)$, so we fix a point. Without loss of generality, we fix the point indexed with 1. Then $H_{\{1\}}$ has two orbits on the remaining set, so we fix one representative of the first orbit. The group then has three orbits on the remaining elements, so each of these form the leaves of our computation tree. We then go back to the second orbit of $H_{\{1\}}$, and fix an orbit representative there. The new group has two orbits on the remaining points, and so each of these forms another leaf of our computation tree. The resulting tab delimited computation tree is shown in Figure 6.1.

We can take advantage of this further reduction of the problem as follows. We compute the images of each of the leaves of the computation tree, and add constraints to the original MILP. One constraint forces the elements of a leaf to be included in the relative hemisystem we are trying to construct, while the other added constraints force the relative hemisystem to contain at most $n - 1$ of the elements of tuples that are images of the leaf under G , to ensure that we are not finding relative hemisystems that are equivalent. Here we have significantly tighter constraints on the mixed integer linear program, which causes it to compute solutions more quickly, especially with larger values of n . This causes the overall process of classifying relative hemisystems stabilised by a group G to be much quicker than the straightforward approach, albeit with a larger number of MILPs to run.

1
1, 2
1, 2, 3
1, 2, 4
1, 2, 12
1, 8
1, 8, 9
1, 8, 11

Figure 6.1: A sample computational tree for $\text{PSp}(4, q)$.

6.3 Relative hemisystems of nonclassical $\text{GQ}(8^2, 8)$

As noted in Section 4.5, there is no mention in the literature of relative hemisystems of nonclassical generalised quadrangles. In this section, we achieve a partial classification of relative hemisystems on two different nonclassical generalised quadrangles of order $(8^2, 8)$, namely the unique nonclassical flock generalised quadrangle of this order, and the dual of $T_3(\mathcal{O})$ for a Suzuki–Tits ovoid \mathcal{O} in $\text{PG}(3, 8)$.

6.3.1 Relative hemisystems of flock GQs

The smallest nonclassical flock generalised quadrangle R of order (q^2, q) , q even, is of order $(8^2, 8)$, and it belongs to both the Fisher–Thas–Walker–Kantor–Betten (FTWKB) family and the Payne family of generalised quadrangles. Using the FinInG [3] package and Theorem 4.4.7, we were able to construct this flock generalised quadrangle and its subquadrangles, each isomorphic to $T_2(\mathcal{O}')$,

for \mathcal{O}' an oval in PG(2, 8) that is not a conic, in GAP [39]. Some code written to construct flock generalised quadrangles in even characteristic is given in Appendix B.

We focused on finding relative hemisystems of R relative to the subquadrangle $R' = S_{(1,0)}$ (see Section 4.4.2). There are a further 8 inequivalent subquadrangles of R of order (8, 8), and the relative hemisystems that we find in this section may not necessarily exist in these subquadrangles as well.

We were then able to classify all of the relative hemisystems with stabilisers containing an element of order greater than 2, using the method described in Section 6.2. Our findings are summarised in the following result, first stated in Theorem 1.0.2.

Theorem 6.3.1. Let R be the unique nonclassical flock generalised quadrangle of order (8², 8). Then a relative hemisystem of R with respect to the subquadrangle $R' = S_{(1,0)}$ is equivalent to one of the 11 examples in Table 6.1 or its complement, or it has a stabiliser that is either an elementary abelian 2-group or the trivial group.

Group	Size	Gap ID	Orbit Lengths	SC
$C_2^3 : F_{21}$	168	[168,43]	$56^6, 168^{22}$	False
$C_2^3 : F_{21}$	168	[168,43]	$56^6, 168^{22}$	False
A_4	12	[12,3]	$4^{12}, 6^{32}, 12^{316}$	True
$C_2^3 \times C_2^3 : F_{21}$	1344	[1344,11690]	$224^2, 448^2, 1344^2$	True
$((C_2 \times ((C_4 \times C_2) : C_2)) : C_2) : C_3$	192	[192,191]	$16^4, 48^4, 64^2, 96^2, 192^{18}$	True
$C_7 : A_4$	84	[84,11]	$28^{12}, 84^{44}$	True
$C_2^3 : F_{21}$	168	[168,43]	$28^4, 56^4, 84^4, 168^{20}$	False
$(C_2^2 \times C_2^2 \times C_3) : D_8$	384	[384,5790]	$32^2, 64^2, 96^4, 192^{18}$	False
$(C_2^3 \wr C_2) : F_{21}$	2688	N/A	$224^2, 448^2, 1344^2$	False
$C_2 \times (C_2^3 : C_7) : A_4$	1344	[1344,11689]	$224^2, 448^2, 1344^2$	False
$C_2^2 \times (C_2^3 : C_7)$	224	[224,195]	$112^4, 224^{16}$	False

Table 6.1: Information about relative hemisystems of the nonclassical flock generalised quadrangle of order (8², 8).

Note that in Table 6.1, F_{21} denotes the Frobenius group of order 21, and the column labelled ‘‘SC’’ denotes whether the relative hemisystem is *self complementary*, that is, equivalent to its complement.

We will look in detail at the relative hemisystem with stabiliser of order 2688 on the unique nonclassical flock generalised quadrangle of order (8², 8) and describe how to construct it using a ‘top-down’ approach. In particular, we will construct a family of groups, one for each $q = 2^e$, e odd, containing this group of order 2688. Recall that we have chosen the subquadrangle $S_{(1,0)}$, and this contains all of the points $((a_1, a_2), c, (b_1, b_2))$ and cosets $A_t((a_1, a_2), c, (b_1, b_2))$

with $a_2 = b_2 = 0$. Moreover, when we are considering external points and lines, we must always assume that at least one of a_2 and b_2 are nonzero.

Let R be a FTWKB generalised quadrangle of order (q^2, q) , $q = 2^e$, e odd, with associated q -clan $\mathbf{C} = \left\{ \begin{pmatrix} t^{1/4} & t^{1/2} \\ 0 & t^{3/4} \end{pmatrix} \mid t \in \text{GF}(q) \right\}$. Recall from Section 4.4.2 that $R' = S_{(1,0)}$ is a subquadrangle of R of order (q, q) . According to the isomorphism described by Payne (see Section 4.4.2) between a subquadrangle S_α and $V(3, q)$ (or indeed the homogeneous coordinates of $\text{PG}(2, q)$), the symbols $[A_i]$, $i \in \text{GF}(q) \cup \{\infty\}$ form an oval $\mathcal{O} = \{(1, t, t^{1/4}) \mid t \in \text{GF}(q)\} \cup \{(0, 1, 0)\}$ of $\text{PG}(2, q)$. Embed this $\text{PG}(2, q)$ as the hyperplane $\pi : x_1 = 0$ in $\text{PG}(3, q)$. Consider the hyperplane $\rho : x_2 + x_3 + x_4 = 0$ of $\text{PG}(3, q)$. This is disjoint from \mathcal{O} , since $1 + t + t^{1/4} = 0$ implies $t^4 + t + 1 = 0$. This has no solution in $\text{GF}(q)$, otherwise $\text{GF}(q)$ would contain a subfield isomorphic to $\text{GF}(2^4)$, but by the subfield criterion [48, p. 46], this can occur if and only if $4 \mid e$, which is false because e is odd. The affine points of ρ (that is, the points of ρ not in π) are of the form $(1, x^2, y^2, x^2 + y^2)$. The set of preimages of these points in $S_{(1,0)}$ under Payne's isomorphism is $\mathcal{O}' = \{((x, 0), x^2 + y^2, (y, 0)) \mid x, y \in \text{GF}(q)\}$. Together with the point (∞) , \mathcal{O}' comprises an ovoid of $S_{(1,0)}$ [63, 3.4.2]. We will now stabilise this ovoid, as well as stabilising R' . Note that without loss of generality, we may determine the stabiliser of the ovoid by finding the stabiliser of \mathcal{O}' , since every automorphism of R fixes (∞) . Recall from Section 4.3.6 that the automorphism group of a flock generalised quadrangle is generated by three types of automorphisms, namely elations, the so-called 'scaling' automorphisms, and automorphisms which induce a nontrivial permutation of the underlying q -clan.

Proposition 6.3.2. Let R be the FTWKB generalised quadrangle of order (q^2, q) , for $q = 2^e$ with e odd, and let R' be the subquadrangle $S_{(1,0)}$ of R' of order (q, q) . The group $G_{\mathcal{O}'} \leq \text{Aut}(R)_{R'}$ stabilising the ovoid $\mathcal{O}' \cup \{(\infty)\}$ in $\text{GF}(q)$ constructed above has order $q^2(q-1)2e$. It is generated by elations of the form $((x, 0), x^2 + y^2, (y, 0))$, the scaling automorphisms φ_λ with $\lambda \in \text{GF}(q) \setminus \{0\}$, and automorphisms of the form $\theta(\sigma, \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix})$, where $\sigma \in \text{Aut}(\text{GF}(q))$ and $s \in \{0, 1\}$.

Proof. Suppose that $(\mathbf{a}, c, \mathbf{b})$ is an elation fixing R' and \mathcal{O}' . The elation must be of the form $((a_1, 0), c, (b_1, 0))$, otherwise it would not stabilise R' . Furthermore, applying this elation to an element of \mathcal{O}' gives

$$((x, 0), x^2 + y^2, (y, 0)) ((a_1, 0), c, (b_1, 0)) = ((x + a_1, 0), x^2 + y^2 + c, (y + b_1, 0)).$$

For this point to be in the ovoid, we require $(x + a_1)^2 + (y + b_1)^2 = x^2 + y^2 + c$, which implies that we must have $c = a_1^2 + b_1^2$. Therefore, the elations fixing the ovoid have the form $((a_1, 0), a_1^2 + b_1^2, (b_1, 0))$, for $a_1, b_1 \in \text{GF}(q)$.

Now consider the scaling automorphisms φ_λ . Acting on an element of \mathcal{O}' , we

have

$$\begin{aligned}\varphi_\lambda : ((x, 0), x^2 + y^2, (y, 0)) &\mapsto ((\lambda x, 0), \lambda^2(x^2 + y^2), (\lambda y, 0)) \\ &= ((\lambda x, 0), (\lambda x)^2 + (\lambda y)^2, (\lambda y, 0)),\end{aligned}$$

with $\lambda \in \text{GF}(q)^\times$. Note that every choice of λ preserves \mathcal{O} , and these automorphisms also preserve R' .

Finally, we consider those automorphisms of the generalised quadrangle R that have a nontrivial action on the q -clan. For this, we will rely on the description of these automorphisms given in Theorem 4.3.10. We begin with part (iii) of this theorem. Let $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \text{SL}(2, q)$. We now act on an element of \mathcal{O} as follows:

$$\begin{aligned}\theta = \theta \left(\sigma, \begin{pmatrix} 1 & \bar{0}^{1/\tau} \\ 0 & \lambda \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \right) : ((x, 0), x^2 + y^2, (y, 0)) &\mapsto \\ ((x^\sigma, \bar{0}^{1/2}x^\sigma + \lambda y^\sigma) \otimes (b_1, b_2), \lambda(x^2 + y^2)^\sigma + (x^\sigma)^2(b_1\bar{0}^{1/4} + \bar{0}^{1/2}b_1b_2 + \bar{0}^{3/4}b_2^2)) &.\end{aligned}$$

Now, $b_2 = 0$, otherwise the image of θ would not lie in \mathcal{O} . Hence the image becomes $((x^\sigma, \bar{0}^{1/2}x^\sigma + \lambda y^\sigma) \otimes (b_1, 0), \lambda(x^\sigma)^2 + \lambda(y^\sigma)^2 + (x^\sigma)^2b_1\bar{0}^{1/4})$. In order for this to lie in \mathcal{O} , we require

$$(b_1x^\sigma)^2 + (b_1(\bar{0}^{1/2}x^\sigma + \lambda y^\sigma))^2 = \lambda(x^\sigma)^2 + \lambda(y^\sigma)^2 + (x^\sigma)^2b_1\bar{0}^{1/4},$$

which implies that $(b_1^2 + b_1^2\bar{0})(x^\sigma)^2 + b_1^2\lambda^2(y^\sigma)^2 = (\lambda + b_1^2\bar{0}^{1/4})(x^\sigma)^2 + \lambda(y^\sigma)^2$. Since this must hold for all choices of x and y (and in particular, when one of them is 0), we have the following equations:

$$b_1^2 + b_1^2\bar{0} = \lambda + b_1^2\bar{0}^{1/4}, \quad (6.6)$$

$$b_1^2\lambda^2 = \lambda. \quad (6.7)$$

Solving the second equation, we have $b_1^2 = \lambda^{-1}$, and substituting into the first equation, we have

$$1 + \bar{0} + \lambda^2 + \bar{0}^{1/4} = 0. \quad (6.8)$$

We will come back to this. We now consider part (i) of Theorem 4.3.10. Since we are mapping our FTWKB q -clan to itself, we have $A_t = A'_t$. So the equation we are considering is $A_{\bar{t}} \equiv \lambda B^{-1}A_t^\sigma B^{-T} + A_{\bar{0}}$ for all $t \in \text{GF}(q)$. Let $s = \bar{0}$ and recall that $b_2 = 0$. Then substituting the appropriate matrices into the equation (and recalling the definition of equivalence of anisotropic matrices from Section 4.3.1), we have

$$A_{\bar{t}} \equiv \begin{pmatrix} s^{1/4} + b_4^2\lambda(t^\sigma)^{1/4} & s^{1/2} + \lambda(b_3b_4(t^\sigma)^{1/4} + b_1b_4(t^\sigma)^{1/2}) \\ b_3b_4\lambda(t^\sigma)^{1/4} & s^{3/4} + \lambda(b_1(b_3(t^\sigma)^{1/2} + b_1(t^\sigma)^{3/4}) + b_3^2(t^\sigma)^{1/4}) \end{pmatrix}.$$

Now, since $A_{\bar{t}} = \begin{pmatrix} \bar{t}^{1/4} & \bar{t}^{1/2} \\ 0 & \bar{t}^{3/4} \end{pmatrix}$, we must have $w^2 = x + y$ for any matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ equivalent to $A_{\bar{t}}$. Therefore, we require

$$(s^{1/4} + b_4^2\lambda(t^\sigma)^{1/4})^2 = s^{1/2} + \lambda(b_3b_4(t^\sigma)^{1/4} + b_1b_4(t^\sigma)^{1/2}) + b_3b_4\lambda(t^\sigma)^{1/4},$$

which simplifies to

$$b_4^2 \lambda^2 (t^\sigma)^{1/2} = \lambda b_1 b_4 (t^\sigma)^{1/2}. \quad (6.9)$$

This expression must hold for all $t \in \text{GF}(q)$ (and in particular $t = 1$), so we have $b_4^2 \lambda^2 = \lambda b_1 b_4$. Since $B = \begin{pmatrix} b_1 & 0 \\ b_3 & b_4 \end{pmatrix} \in \text{SL}(2, q)$, we must have $b_4 = b_1^{-1}$. Also recall from (6.7) that $b_1^2 = \lambda^{-1}$, namely $\lambda = b_1^{-2} = b_4^2$. Substituting, we have $\lambda^3 = \lambda$, and since $\lambda \neq 0$ and q is even, we must have $\lambda = b_1 = b_4 = 1$. We now calculate b_3 by recalling that equivalent matrices have identical main diagonal entries and therefore using the fact that in $A_{\bar{t}}$, the bottom-right entry is the cube of the top-left entry. We have $(s^{1/4} + (t^\sigma)^{1/4})^3 = s^{3/4} + b_3 (t^\sigma)^{1/2} + (t^\sigma)^{3/4} + b_3^2 (t^\sigma)^{1/4}$, which, after some simplification, implies that $b_3 = s^{1/2}$. Recall that we defined $s = \bar{0}$, so going back to (6.8) and substituting in the values we have solved for, we have $\bar{0} = \bar{0}^{1/4}$, so $\bar{0} = s$ is equal to either 0 or 1 and so is b_3 . \square

We have shown computationally that the group of order 2688 stabilising the ninth relative hemisystem listed in Table 6.1 is a member of the family of groups described in Proposition 6.3.2. We do not know whether other groups in this family give rise to relative hemisystems. We leave this as an open problem.

6.3.2 Relative hemisystems of the dual of $\text{T}_3(\mathcal{O})$

We were also able to achieve a partial classification of relative hemisystems of the dual of $\text{T}_3(\mathcal{O})$, \mathcal{O} a Suzuki–Tits ovoid for $q = 8$ using the techniques described in Section 6.2. The construction of $\text{T}_3(\mathcal{O})$ and a subquadrangle $\text{T}_2(\mathcal{O}')$, \mathcal{O}' an oval in GAP [39] is given in Appendix B. The following is a more explicit version of Theorem 1.0.3.

Theorem 6.3.3. Let $\text{T}_3(\mathcal{O})$ be the generalised quadrangle of order $(8^2, 8)$ constructed from a Suzuki–Tits ovoid \mathcal{O} in $\text{PG}(3, 8)$. Let S be a relative hemisystem of the dual of $\text{T}_3(\mathcal{O})$ with respect to the dual of the generalised subquadrangle $\text{T}_2(\mathcal{O}')$ of order $(8, 8)$, \mathcal{O}' an oval, constructed in Appendix B. Then either S appears in Table 6.2, its complement does, or it has a stabiliser that is either trivial or an elementary abelian 2-group.

Stabiliser	Size	GAP ID	Orbit Lengths	SC?
$(C_2^3 : C_7) : C_3$	168	[168,43]	$28^4, 56^4, 84^4, 168^{20}$	False
$(C_2^3 : C_7) : C_3$	168	[168,43]	$28^4, 56^4, 84^4, 168^{20}$	False
$C_2 \times A_4$	24	[24, 13]	$2^2, 4^4, 6^{10}, 8^2, 12^{20}, 24^{154}$	False
$(C_2^3 : C_7) : C_3$	168	[168,43]	$28^4, 56^4, 84^4, 168^{20}$	False
$C_2^2 \times (C_2^4 : C_3)$	192	[192,1540]	$32^6, 96^{40}$	True
$(C_2^3 : C_7) : C_3$	168	[168,43]	$28^4, 56^4, 84^4, 168^{20}$	False
$(C_2^6 : C_7) : C_3$	1344	[1344, 11690]	$224^6, 672^4$	True

Table 6.2: A partial classification of relative hemisystems on the dual of $\text{T}_3(\mathcal{O})$.

As before, the final column in Table 6.2 states whether each relative hemisystem is *self-complementary*; that is, equivalent to its complement.

6.4 Classical relative hemisystems for $q = 16$

Bamberg, Lee and Swartz classified all of the relative hemisystems on $H(3, 8^2)$, up to those with a trivial stabiliser [6]. We use the methods described in Section 6.2 to begin to classify the relative hemisystems on $H(3, 16^2)$. The stabiliser of $W(3, 16)$ in $PGU(4, 16)$ is isomorphic to $PSp(4, 16)$ extended by the group generated by the Frobenius automorphism σ of $GF(16^2)$, which of order 8. This stabiliser has order $8761599590400 = 2^{19} \cdot 3^2 \cdot 5^2 \cdot 17^2 \cdot 257$. We know that the stabiliser H of a relative hemisystem must be a subgroup of $PSp(4, 16) \rtimes \langle \sigma \rangle$. If H stabilises a relative hemisystem S , then S must be a union of the orbits of H on external lines. Our results thus far are summarised in Table 6.3.

Group	Size	Comments
$Syl_{257}(PSp(4, 16) \rtimes \langle \sigma \rangle)$	257	Two relative hemisystems, both with stabiliser of order $2^{11} \cdot 3 \cdot 5 \cdot 17 \cdot 257$.
$Syl_{17}(PSp(4, 16) \rtimes \langle \sigma \rangle)$	17^2	No relative hemisystems.
$Syl_2(PSp(4, 16) \rtimes \langle \sigma \rangle)$	2^{19}	No relative hemisystems (group is transitive on all of the lines incident with each external point P).
$H_1 \leq Syl_2(PSp(4, 16) \rtimes \langle \sigma \rangle)$	2^{18}	No relative hemisystems (group is transitive on all of the lines incident with each external point P).
$H_2 \leq Syl_2(PSp(4, 16) \rtimes \langle \sigma \rangle)$	2^{17}	No relative hemisystems (group is transitive on all of the lines incident with each external point P).

Table 6.3: Some classification results for relative hemisystems of $H(3, 16^2)$.

Lemma 6.4.1. Let $G = PSp(4, q)$, and let G_P denote the point stabiliser of an external point P . Then $G_{Pg} = g^{-1}(G_P)g = (G_P)^g$, for all external points P and $g \in G$.

Proof. Suppose $x \in G_{Pg}$. Then $(P^g)^x = P^g$, which is if and only if $P^{g x g^{-1}} = P$. Thus $g x g^{-1} \in G_P$, and $x \in g^{-1}(G_P)g = (G_P)^g$. \square

Corollary 6.4.2. All of the point stabilisers G_P are conjugate.

Proof. $PSp(4, q)$ is transitive on external points [27, §2.1]. \square

6.5 Higher dimensions

In the classical case at least, we define analogous versions of m -covers and relative m -covers for higher dimensions.

In [30], Penttinen and Cossidente defined a *regular system of order m of $H(n, q^2)$* to be a set \mathcal{H} of maximals in $H(n, q^2)$ with the property that every point lies on exactly m maximals of \mathcal{H} .

When $n = 3$, these maximals are lines, which is consistent with our definition of m -covers of $H(3, q^2)$. When $n = 5$, we have a rank 3 polar space, and we now consider m -covers comprised of planes.

We now redefine a hemisystem of $H(n, q^2)$ to be a set of *maximals*, instead of a set of lines, as it is in $H(3, q^2)$. Penttinen and Cossidente [30] proved the existence of three infinite families of hemisystems of $H(5, q^2)$, q odd. These infinite families admit $P\Omega^-(6, q)$, $P\Omega^+(6, q)$, and $P\Omega(5, q)$ respectively. They also showed from computational results that there exist m -covers of $H(5, q^2)$ for $q = 2, 3$ that are not hemisystems. This implies that the result of Segre [66] that a nontrivial m -cover of $H(3, q^2)$ is a hemisystem does not hold for higher dimensions.

Our main focus is relative m -covers, which have not been generalised to higher dimensions in the literature. We begin by exploring possible analogous versions of relative hemisystems on $H(5, q^2)$. As is the case when $n = 3$, we may embed a symplectic space $W(5, q)$ into $H(5, q^2)$ by restricting the associated form to $\text{GF}(q)$. Now, there are $(q^5 + 1)(q^4 + q^2 + 1)$ points of $H(5, q^2)$ and $(q^4 + 1)(q^2 + q + 1)$ points of $W(5, q)$. Hence using the same definition as before, the number of external points with respect to $W(5, q)$ is $q(q^8 + q^6 - q^5 - 1)$.

Planes of $H(5, q^2)$ meet $W(5, q)$ in a single point or a projective plane that is a plane of $W(5, q)$. We only consider the planes that meet $W(5, q)$ in a point, and in a slight abuse of terminology, we will call the set of such planes *external planes*. There are $(q + 1)(q^3 + 1)(q^5 + 1)$ planes of $H(3, q^2)$, and moreover, there are $(q + 1)(q^2 + 1)(q^3 + 1)$ planes of $W(5, q)$, which means that the number of external planes is $q^2(q + 1)(q^3 + 1)(q^3 - 1)$.

Each external point of $H(5, q^2)$ is incident with exactly $(q + 1)(q^3 + 1)$ planes, and of these, $q^3(q + 1)$ meet $W(3, q)$ in a single point. When q is even, the number of external planes on each point is even, and so we may define a *tangent relative hemisystem* to be a set of external planes \mathcal{H} such that every external point is incident with $\frac{q^3(q+1)}{2}$ planes of \mathcal{H} .

Index	Comments
1	This may be the 12-regular system of $H(3, 4)$ described by Penttinen and Cossidente in [30].
2	This tangent relative hemisystem also meets every point of $W(5, 2)$ in six planes.

Table 6.4: Information about two tangent relative hemisystems of $H(5, 2^2)$.

With the aid of GAP [39], the FinInG package in GAP [3] and Gurobi [40], we were able to construct a mixed integer linear program to search for these

tangent relative hemisystems on $\text{H}(5, 2^2)$. We have found two tangent relative hemisystems to date, with properties summarised in Table 6.4.

6.6 $\frac{q+1}{2}$ -ovoids of $\text{DH}(5, q^2)$

Recall from Section 5.3 that every $\frac{q+1}{2}$ -ovoid of a dual Hermitian polar space $\text{DH}(2d-1, q^2)$, of rank $d \geq 2$ with q odd, gives rise to a distance regular graph. As mentioned, the only value of d that has been investigated in the literature is $d = 2$, in which case the problem is equivalent to finding hemisystems of $\text{H}(3, q^2)$. We investigate the next smallest open case: $\frac{q+1}{2}$ -ovoids of $\text{DH}(5, q^2)$, q odd, and more broadly, m -ovoids. The number of points of $\text{DH}(2d-1, q^2)$ (which by definition is the same as the number of maximals of $\text{H}(2d-1, q^2)$) is $(q+1)(q^3+1)\dots(q^{2d-1}+1)$, which is exponential in d . Therefore, even for small values of q , this problem is much harder than when $d = 2$. We can simplify the problem by noting that we can embed $\text{DW}(2d-1, q)$ into $\text{DH}(2d-1, q^2)$ [32, Theorem 1.5]. We then make use of the following lemma.

Lemma 6.6.1. Suppose S is an m -ovoid of $\text{DH}(5, q^2)$, and let $\text{DW}(5, q)$ be a subgeometry of $\text{DH}(5, q^2)$. Then the intersection of S with the point set of $\text{DW}(5, q)$ forms an m -ovoid of $\text{DW}(5, q)$.

Proof. Suppose S is an m -ovoid of $\text{DH}(5, q^2)$. Consider a dual symplectic space $\text{DW}(5, q)$ embedded in $\text{DH}(5, q^2)$. By Lemma 2.3.3, every line in $\text{DH}(5, q^2)$ has $q+1$ points on it, and the same is true for every line of $\text{DW}(5, q)$. This implies that each line ℓ of $\text{DW}(5, q)$ is incident with all of the points that it is incident with when considered as a line of $\text{DH}(5, q^2)$. In particular, the m points on ℓ contained in S are also part of the point set of $\text{DW}(5, q)$. Therefore, the intersection of S with the point set of $\text{DW}(5, q)$ forms an m -ovoid of $\text{DW}(5, q)$. \square

Corollary 6.6.2. If $\text{DW}(5, q)$ has no m -ovoids, then neither does $\text{DH}(5, q^2)$.

Therefore, we can examine whether there exist m -ovoids of $\text{DH}(5, q^2)$ by first checking whether m -ovoids of $\text{DW}(5, q)$ exist. The size of an m -ovoid of $\text{DW}(5, q)$ is given by the following lemma.

Lemma 6.6.3. The size of an m -ovoid S of $\text{DW}(5, q)$ is $m(q^2+1)(q^3+1)$.

Proof. We first calculate the number of lines of $\text{DW}(5, q)$ by double counting incident point–line pairs (P', ℓ') . The number of lines on a point of $\text{DW}(5, q)$ is the same as the number of lines of $\text{PG}(2, q)$, which is q^2+q+1 [19, pp. 20, 89], and we recall from Lemma 2.3.3 that each line of $\text{DW}(5, q)$ is incident with

$q + 1$ points . We therefore have

$$\begin{aligned} \sum_{P' \in \text{DW}(5, q)} q^2 + q + 1 &= \sum_{\ell' \in \text{DW}(5, q)} q + 1, \\ (q + 1)(q^2 + 1)(q^3 + 1)(q^2 + q + 1) &= (\# \text{ lines of } \text{DW}(5, q))(q + 1), \\ \# \text{ lines of } \text{DW}(5, q) &= (q^2 + 1)(q^3 + 1)(q^2 + q + 1). \end{aligned}$$

Note that we have made use of Table 2.1 for the number of points of $\text{DW}(5, q)$. Thus the number of lines of $\text{DW}(5, q)$ is $(q^2 + 1)(q^3 + 1)(q^2 + q + 1)$. We now double count incident point–line pairs (P, ℓ) such that $P \in S$:

$$\begin{aligned} \sum_{P \in S} (q^2 + q + 1) &= \sum_{\ell \in \text{DW}(5, q^2)} m, \\ |S|(q^2 + q + 1) &= (q^3 + 1)(q^2 + 1)(q^2 + q + 1)m, \\ |S| &= m(q^2 + 1)(q^3 + 1). \quad \square \end{aligned}$$

Let $\text{Aut}(\text{DW}(5, q))$ denote the automorphism group of $\text{DW}(5, q)$. As per our definition for relative hemisystems, we say that two m -ovoids of $\text{DW}(5, q)$ are *equivalent* if there is an automorphism of $\text{DW}(5, q)$ mapping one to the other. The following result shows that we can use the same symmetry breaking techniques to find $\frac{q+1}{2}$ -ovoids of $\text{DW}(5, q)$ as we did to find relative hemisystems of nonclassical generalised quadrangles.

Lemma 6.6.4. Suppose S_1 and S_2 are two m -ovoids of $\text{DW}(5, q)$. Then S_1 is equivalent to S_2 or an m -ovoid stabilised by the stabiliser of S_2 if and only if their stabilisers in $\text{Aut}(\text{DW}(5, q))$ are conjugate.

Proof. Let S_1 and S_2 be equivalent m -ovoids of $\text{DW}(5, q)$, with setwise stabilisers G_1 and G_2 . Let $h \in \text{Aut}(\text{DW}(5, q))$ be a automorphism that maps S_1 to S_2 . Let $P \in S_2$ and $g_1 \in G_1$. Then $P^{h^{-1}g_1h} \in S_2$, so $h^{-1}G_1h \subseteq G_2$. Similarly, $hG_2h^{-1} \subseteq G_1$, so $h^{-1}G_1h = G_2$ and the stabilisers are conjugate. An analogous argument holds when S_1 is equivalent to an m -ovoid stabilised by G_2 that is not S_2 . Conversely, suppose S_1 and S_2 are two m -ovoids of $\text{DW}(5, q)$ with conjugate stabilisers G_1 and G_2 such that $h^{-1}G_1h = G_2$ for some automorphism h . Let $P \in S_2$. Then $P^{h^{-1}g_1h} \in S_2$ for all $g_1 \in G_1$. This implies that $P^{h^{-1}g_1} \in S_2^{h^{-1}}$. Now, since $P^{h^{-1}} \in S_2^{h^{-1}}$, G_1 must stabilise $S_2^{h^{-1}}$, which is an m -ovoid equivalent to S_2 . \square

Therefore, it is sufficient to classify m -ovoids of $\text{DW}(5, q)$ up to the conjugacy of stabilisers, and we may use the computation tree construction techniques described in Section 6.2. Using these methods, we obtain the following computational results.

Theorem 6.6.5. $\text{DW}(5, 3)$ and $\text{DW}(5, 5)$ have no m -ovoids.

Corollary 6.6.6. $\text{DH}(5, 3^2)$ and $\text{DH}(5, 5^2)$ have no m -ovoids.

We obtain Theorem 1.0.4 as an immediate corollary of Corollary 6.6.6. This leads us to the following conjecture.

Conjecture 6.6.7. There are no m -ovoids in $\text{DW}(5, q)$, q odd.

We remark that Cooperstein and Pasini have proved this conjecture for $m = 1$ (that is, when the m -ovoid is an ovoid) [25].

7. Concluding remarks

In this dissertation, we have set out to explore and extend problems related to line covers of generalised quadrangles and polar spaces.

We have provided the first set of examples of relative hemisystems on two nonclassical generalised quadrangles of order $(8^2, 8)$, namely the smallest nonclassical example of a flock generalised quadrangle and the dual of $T_3(\mathcal{O})$, for a Suzuki–Tits ovoid \mathcal{O} . In the case of the nonclassical flock generalised quadrangle of order $(8^2, 8)$, we have described in general for $q = 2^e$, e odd, how to construct the automorphism group $G_{\mathcal{O}}$ of the relative hemisystem with the largest stabiliser. We do not know whether this family of subgroups gives rise to relative hemisystems for $q > 8$ and we leave this as an open question.

Other possible directions for future work include investigating whether any of the other examples of relative hemisystems on nonclassical generalised quadrangles extend to infinite families, and completing the classification of relative hemisystems on nonclassical generalised quadrangles of order $(8^2, 8)$. In particular, in the flock generalised quadrangle case, this means classifying relative hemisystems relative to all of the $q + 1$ inequivalent subquadrangles S_{α} , since in this dissertation we have only looked for relative hemisystems relative to $S_{(1,0)}$. After classifying relative hemisystems on these generalised quadrangles, the next step would be classifying relative hemisystems on generalised quadrangles of order $(16^2, 16)$. Apart from $H(3, 16^2)$, there is one other known generalised quadrangle of this order up to equivalence; namely the generalised quadrangle arising from a Subiaco q -clan (which is isomorphic to that arising from an Adelaide q -clan). A classification of relative hemisystems for $q = 16$ has not been completed for the classical case, so that would seem to be a good place to start.

The motivation for studying relative hemisystems in the first place, apart from studying them as an analogous version of a hemisystem for q even, is that each relative hemisystem of $H(3, q^2)$ relative to a $W(3, q)$ subquadrangle gives rise to a primitive Q -polynomial association scheme that does not arise from a distance regular graph (or the dual of one). However, the construction of this association scheme is only valid for relative hemisystems in the classical case, due to the nonexistence of doubly subtended generalised subquadrangles of nonclassical generalised quadrangles of order (q^2, q) . This is in particular due to the absence of an involutory automorphism that fixes a subquadrangle of order (q, q) pointwise (see Lemma 4.4.13). Let S be a relative hemisystem of a generalised quadrangle R of order (q^2, q) relative to a doubly subtended subquadrangle R' of order (q, q) , q even. We proved in Theorem 6.1.6 that the involutory automorphism σ fixing R' pointwise (see Lemma 4.4.13) maps S to its complement, and therefore, every relative hemisystem of $H(3, q^2)$ is equivalent to its complement. As we have seen in Sections 6.3.1 and 6.3.2, there exist relative hemisystems of nonclassical generalised quadrangles that are not

equivalent to their complements, and there also exist some that are. Another possible avenue for further research would be to investigate whether we can construct association schemes on self complementary relative hemisystems of nonclassical generalised quadrangles using the automorphisms that map them to their complements, or indeed if there exist other combinatorial structures that arise from relative hemisystems of nonclassical generalised quadrangles.

We have also proved that relative hemisystems are the only relative m -covers on $H(3, q^2)$, q even, relative to $W(3, q)$. This is analogous to Segre's original result for $H(3, q^2)$, q odd, which initiated the study of hemisystems. Although this conclusion is remarkable in and of itself, it would be nice to be able to either extend the proof to include all generalised quadrangles of order (q^2, q) , relative to their subquadrangles of order (q, q) , or find a nontrivial relative m -cover that is not a relative hemisystem. This would require a different approach to the one used in this dissertation, which relies heavily on the association scheme on the external lines of $H(3, q^2)$ relative to $W(3, q)$.

We also investigated m -ovals of the dual Hermitian polar space $DH(5, q^2)$, q odd. Our motivation for this was a result by Vanhove [78, Theorem 6.7.8] showing that every $\frac{q+1}{2}$ -oval of $DH(2d-1, q^2)$ gives rise to a distance regular graph with classical parameters. In particular, for $d \geq 3$, an example of a $\frac{q+1}{2}$ -oval of $DH(2d-1, q^2)$ would give a previously unknown distance regular graph. We worked on the smallest open case, $d = 3$. Lemma 6.6.1 allowed us to first consider searching for $\frac{q+1}{2}$ -ovals of the dual symplectic space $DW(5, q)$ to test for the existence of $\frac{q+1}{2}$ -ovals of $DH(5, q^2)$. We were able to prove computationally that both $DW(5, 3)$ and $DW(5, 5)$ do not have m -ovals (and in particular, $\frac{q+1}{2}$ -ovals) which implies that the same is true on $DH(5, 3^3)$ and $DH(5, 5^2)$. We conjecture that no m -ovals exist on $DW(5, q)$ and thus $DH(5, q^2)$ for any odd prime power q .

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A. Association scheme parameters of some dual polar spaces

	DW(5, q)	DH(5, q^2)	$\frac{q+1}{2}$ -ovoid of DH(5, q^2)
b_0	$q(q^2 + q + 1)$	$q(q^4 + q^2 + 1)$	$\frac{1}{2}(q - 1)(q^4 + q^2 + 1)$
b_1	$q^2(q + 1)$	$q^3(q^2 + 1)$	$\frac{1}{2}q^2(q^2 + 1)(q - 1)$
b_2	q^3	q^5	$\frac{1}{2}q^4(q - 1)$
b_3	0	0	0
c_0	0	0	0
c_1	1	1	1
c_2	$q + 1$	$q^2 + 1$	$\frac{1}{2}(q - 1)^2$
c_3	$q^2 + q + 1$	$q^4 + q^2 + 1$	$\frac{1}{2}(q^2 + 1)(q^2 - q + 1)$
a_0	0	0	0
a_1	$q - 1$	$q - 1$	$\frac{q-3}{2}$
a_2	$q^2 - 1$	$(q^2 + 1)(q - 1)$	$\frac{1}{2}(q^2 - q + 2)(q - 1)$
a_3	$(q - 1)(q^2 + q + 1)$	$(q - 1)(q^4 + q^2 + 1)$	$\frac{1}{2}(q^3 - q^2 - 2)(q^2 - q + 1)$
k_0	1	1	1
k_1	$q(q^2 + q + 1)$	$q(q^4 + q^2 + 1)$	$\frac{1}{2}(q - 1)(q^4 + q^2 + 1)$
k_2	$q^3(q^2 + q + 1)$	$q^4(q^4 + q^2 + 1)$	$\frac{1}{2}q^2(q^2 + 1)(q^4 + q^2 + 1)$
k_3	q^6	q^9	$\frac{1}{2}q^6(q - 1)(q^2 + q + 1)$
P	$P_{\text{DW}(5,q)}$	$P_{\text{DH}(5,q^2)}$	N/A
Q	$Q_{\text{DW}(5,q)}$	$Q_{\text{DH}(5,q^2)}$	N/A

where

$$P_{\text{DW}(5,q)} = \begin{pmatrix} 1 & q(q^2 + q + 1) & q^3(q^2 + q + 1) & q^6 \\ 1 & -q^2 - q - 1 & q(q^2 + q + 1) & -q^3 \\ 1 & q^2 + q - 1 & q(q^2 - q - 1) & -q^3 \\ 1 & -1 & -q^2 & q^2 \end{pmatrix},$$

$$P_{\text{DH}(5,q^2)} = \begin{pmatrix} 1 & q(q^4 + q^2 + 1) & q^4(q^4 + q^2 + 1) & q^9 \\ 1 & -q^4 - q^2 - 1 & q^2(q^4 + q^2 + 1) & -q^6 \\ 1 & q^3 + q - 1 & q(q^3 - q^2 - 1) & -q^4 \\ 1 & -q^2 + q - 1 & -q(q^2 - q + 1) & q^3 \end{pmatrix},$$

$$Q_{\text{DW}(5,q)} = \begin{pmatrix} 1 & \frac{q(q^2+1)(q^2-q+1)}{2} & \frac{q(q^2+1)(q^2+q+1)}{2} & q^6 + q^4 + q^2 \\ 1 & -\frac{(q^2+1)(q^2-q+1)}{2} & \frac{(q^2+1)(q^2+q-1)}{2} & -q(q^2 - q + 1) \\ 1 & \frac{(q^2+1)(q^2-q+1)}{2q} & -\frac{-q^4+q^3+q+1}{2q} & -q(q^2 - q + 1) \\ 1 & -\frac{(q^2+1)(q^2-q+1)}{2q^2} & -\frac{(q^2+1)(q^2+q+1)}{2q^2} & q^2 + 1 + \frac{1}{q^2} \end{pmatrix},$$

$$Q_{\text{DH}(5,q^2)} = \begin{pmatrix} 1 & q(q^4 - q^3 + q^2 - q + 1) & q^2(q^2 - q + 1)^2(q^2 + q + 1) & q^3(q^2 + q + 1)(q^4 - q^3 + q^2 - q + 1) \\ 1 & -q^4 + q^3 - q^2 + q - 1 & q(q^2 - q + 1)(q^3 + q - 1) & -q^2(q^4 - q^3 + q^2 - q + 1) \\ 1 & \frac{q^4 - q^3 + q^2 - q + 1}{q} & \frac{(q^2 - q + 1)(q^3 - q^2 - 1)}{q} & -q^4 + q^3 - q^2 + q - 1 \\ 1 & -\frac{q^4 - q^3 + q^2 - q + 1}{q^2} & -\frac{(q^2 - q + 1)^2(q^2 + q + 1)}{q^3} & \frac{(q^2 + q + 1)(q^4 - q^3 + q^2 - q + 1)}{q^3} \end{pmatrix}.$$

B. Constructing nonclassical generalised quadrangles in GAP

B.1 Flock generalised quadrangles

The vast majority of the computational work done to construct generalised quadrangles and their automorphism groups was done using GAP [39] and the Finite Incidence Geometry (FinInG) package [3]. However, the functions in this package that relate to constructing flock generalised quadrangles require the associated fields to have odd characteristic, and in particular, there were no functions that construct q -clans that only exist over even order fields. As a result, we developed patches for functions in FinInG where needed, and wrote new functions that construct even characteristic q -clans so that we could create flock generalised quadrangles over fields with even order. We include the code here so that any other researchers interested in constructing these flock generalised quadrangles in even characteristic can make use of this resource.

```

KantorFamilybyqClanEven := function( clan )
  local g, q, f, i, omega, mat, at, ainf, ainfstar, clanmats,
        ainfgens, centregens, as, astars, basis, remgens, k;

  #Getting field information and q-clan
  f := clan!.basefield;
  clanmats := Elements(clan);
  q := Size(f);
  i := One(f);
  #Converting into matrices so binary operation of G is
  equivalent to matrix multiplication
  mat := function(a1, a2, c, b1, b2)
    return i * [[1, a1, a2, c], [0, 1, 0, b1],
                [0, 0, 1, b2], [0, 0, 0, 1]];
  end;
  basis := AsList(Basis(f));

  centregens := [];
  ainfgens := [];
  remgens := [];
  for omega in basis do
    Add(remgens, mat(0,0,0,omega,0));
    Add(remgens, mat(0,0,0,0,omega));
    Add(centregens, mat(0,0,omega,0,0));
    Add(ainfgens, mat(0,omega,0,0,0));
    Add(ainfgens, mat(omega,0,0,0,0));
  od;

  #Making A∞ and A∞*.
  ainf := Group(ainfgens);
  ainfstar := Group(Union(ainfgens, centregens));

```

```

at := function( m )
  # returns generators for Kantor family element defined by
  #q-Clan element m
  local a1, a2, k, gens, bas, zero;
  gens := [];
  bas := AsList(Basis(f));
  zero := Zero(f);

  for a1 in bas do
    k := [a1, zero] * (m+TransposedMat(m));
    Add(gens, mat(k[1], k[2], ([a1, zero]*m*[a1, zero]), a1,
                  zero) );
  od;

  for a2 in bas do
    k := [zero, a2] * (m+TransposedMat(m));
    Add(gens, mat(k[1], k[2], ([zero, a2]*m*[zero, a2]), zero, a2)
        );
  od;
  return gens;
end;

#Making  $\mathcal{G}$ 
g := Group(Union( ainfgens, centregens, remggens));

#Making the 4-gonal family
as := List(clanmats, m -> Group( at(m) ));
Add(as, ainf);
astars := List(clanmats, m -> Group(Union(at(m), centregens)));
Add(astars, ainfst);

#Returns Kantor family
return [g, as, astars];
end;

#This function makes the FTWKB q-clan in q even as well as odd.
FTWKBqClan := function(q)
  local f, g, h, clan;

  if not IsPrimePowerInt(q) then
    Error("Argument must be a prime power");
  fi;
  if q mod 3 <> 2 then
    Error("q must be congruent to 2 mod 3");
  fi;

  f := t -> t^(q/4);
  g := t -> t^(q/2);
  h := t -> t^(3*q/4);
  clan := List(GF(q), t -> [[f(t), g(t)], [0*Z(q), h(t)]]);
  return qClan(clan, GF(q));
end;

#This function makes the Payne q-clan for any odd power of 2.

```



```

PayneqClan := function(q)
  local f,g,h, clan;
  if not IsPrimePowerInt(q) then
    Error("Argument must be a prime power");
  fi;
  if q mod 2 <> 0 then
    Error("q must be an odd power of 2.");
  fi;
  f := t -> t^(q/6);
  g := t -> t^(q/2);
  h := t^(5*q/6);
  clan := List(GF(q), t -> [[f(t), g(t)], [0*Z(q), h(t)]]);
  return qClan(clan, GF(q));
end;

```

#This function makes the Subiaco q-clan for any even power of 2.

```

SubiacoqClan := function(q)
  local a,d, ds, invds, f,g,h, clan,inv;
  if not IsPrimePowerInt(q) then
    Error("Argument must be a prime power");
  fi;
  if q mod 2 <> 0 then
    Error("q must be a power of 2.");
  fi;

  invds := Filtered(GF(q), t -> Trace(GF(q), t) = Z(q)^0);
  ds := List(invds, s -> Inverse(s));
  ds := Filtered(ds, t -> t^2+t+1 <> 0*Z(q));
  d:= Random(ds);

  a := (d^2 + d^5 + d^(q/2))/(d*(1+d+d^2));
  f := t -> (d^2*(t^4 + t)+d^2*(1+d+d^2)*(t^3+t^2))/((t^2+d*t
    +1)^2) + t^(q/2);
  h := t -> (d^4*t^4 + d^3*(1+d^2+d^4)*t^3+ d^3*(1+d^2)*t)/((
    d^2+d^5+d^(q/2))*(t^2+d*t+1)^2) + ((d^(q/2))/(d^2+d^5+d
    ^2)))*t^(q/2);
  g := t -> t^(q/2);

  clan := List(GF(q), t -> [[f(t), g(t)], [0*Z(q), a*
    h(t)]]);
  qClan(clan, GF(q));
  return qClan(clan, GF(q));
end;

```

#This function makes the Adelaide q-clan for any even power of 2.

```

AdelaideqClan := function(q)
  local a,d, ds, invds, f,g,h, clan,inv;
  if not IsPrimePowerInt(q) then
    Error("Argument must be a prime power");
  fi;
  if q mod 2 <> 0 then
    Error("q must be a power of 2.");
  fi;
  fi;

```

```

bs := Filtered(GF(q^2), t-> t <> Z(q^2)^0 and t^(q+1) = Z(
  q^2)^0);;
b := Random(bs);
m:= First([1..100], i -> (i + (q-1)/3) mod (q+1) = 0);
T := x -> Trace(GF(q^2), GF(q), x);
a := T(b^m)/(T(b)) + (Z(q)^0)/(T(b^m)) + (Z(q)^0);
f := t -> (T(b^m)*(t+1))/(T(b)) + T((b*t+b^q)^m)/(T(b)*(t+
  (b)*t^(q/2) + 1)^(m-1)) + t^(q/2);
g := t -> t^(q/2);
h := t -> T(b^m)/(T(b))*t + T((b^2*t+1)^m)/(T(b)*T(b^m)*(t+
  T(b)*t^(q/2) + 1)^(m-1)) + Z(q)^0/(T(b^m)) * t^(q/2);

clan := List(GF(q), t -> [[f(t), g(t)], [0*Z(q), h(t)]]);
qClan(clan, GF(q));
return qClan(clan, GF(q));
end;

```

B.2 $T_3(O)$ and $T_2(O)$

The FinInG package also does not have functions that construct the generalised quadrangles $T_3(O)$ and $T_2(O)$, although we can make use of the existing functions in order to construct it. We use the methods described in Sections 4.2 and 4.4.4. Again, we include the code here for use in future work. The construction of $T_3(O)$ given here was written in collaboration with John Bamberg [2].

#Requires the FinInG package

```
LoadPackage("fining");
```

```
q := 8;
H := PG(3,q);
A := PG(4,q);
f := GF(q);
```

#Constructing the Suzuki-Tits ovoid

```
vecs := Union(List(f, x->List(f, y->[One(f), x*y+x^6+y^4, x, y])));;
Add(vecs, [0, 1, 0, 0]*Z(q)^0);
ovoid := List(vecs, x->VectorSpaceToElement(H, x));;
Ahyps := AsList(Hyperplanes(A));
H2 := Random(Ahyps);
```

#Embedding ovoid on a hyperplane in PG(4, q)

```
em := NaturalEmbeddingBySubspace(H, A, H2);;
H2pts := ImagesSet(em, Points(H));;
H2lines := ImagesSet(em, Lines(H));;
Alines := AsList(Lines(A));;
```

#Suzuki-Tits Ovoid

```
stov := ImagesSet(em, ovoid);;
```

#Points

```
pts1 := AsSet(Difference(AsSet(Points(A)), AsSet(H2pts)));;
pts2 := Union(List(stov, t -> Filtered(Solids(t), i -> Number(stov,
  u -> u * i)=1)));;
```

```

#This should be the point ( $\infty$ ), but we define it to be H2 to make
  the definition of the incidence relation simpler.
pts3 := [H2];

#Lines
ls1 := Union(List(stov, p -> Filtered(Lines(p), u -> not u * H2)))
;;
ls2 := AsSet(stov);;

points := AsSet(Concatenation(pts1, pts2, pts3));;
lines := AsSet(Concatenation(ls1, ls2));;
inc := \*;

#Stabilising the hyperplane H2 in A.
affineG := FiningStabiliser(CollineationGroup(A), H2);

#Creates the collineation group of the generalised quadrangle
StabSet := function(g, s, act, pts)
  local hom, u, omega, imgs, stab, gens;
  hom := ActionHomomorphism(g, pts, act);
  u := UnderlyingExternalSet(hom);;
  omega := HomeEnumerator(u);;
  imgs := Filtered([1..Size(omega)], x -> omega[x] in s);;
  stab := Stabilizer(Image(hom), imgs, OnSets);
  if IsTrivial(stab) then
    return Kernel(hom);
  else
    gens := GeneratorsOfGroup(stab);
    gens := List(gens, x -> PreImagesRepresentative(hom, x));
    return GroupWithGenerators(gens);
  fi;
end;

#Make the collineation group of our GQ
group := StabSet(affineG, stov, OnProjSubspaces, H2pts);
GQ := GeneralisedPolygonByElements(points, lines, inc, group,
  OnProjSubspaces);;
pts := AsSet(Points(GQ));;
ls := AsSet(Lines(GQ));;

#Here we start constructing the subGQ
#The triad needs to be of points. Only 3-regular point is ( $\infty$ )
x := First(pts, t -> t!.obj = pts3[1]);
y := First(pts, t -> not Span(x, t) in ls);
z := First(pts, t -> not Span(x, t) in ls and not Span(y, t) in ls);
triad := [x, y, z];
slines := Filtered(pts, p1 -> ForAll(triad, p2 -> Span(p1, p2) in
  ls));
pplines := Filtered(pts, p1 -> ForAll(slines, p2 -> Span(p1, p2) in
  ls));

spanlines := [];
for p1 in slines do
for p2 in pplines do

```

```

Add(spanlines , Span(p1,p2));
od;
od;
subgqpts := Union(List(spanlines , t -> AsList(Points(t))));;
subgqlines := [];
comb := Combinations(subgqpts , 2);;
for c in comb do
if Span(c[1] , c[2]) in ls then
Add(subgqlines , Span(c[1] , c[2]));
fi ;
od;
subgqpts := AsSet(subgqpts);;
subgqlines := AsSet(subgqlines);;
subgqinc := \*;
#In many cases, we don't need to construct the subGQ object, we
  only have to know its points and lines.

#Using the incidence graph to create a permutation group of the
  points as the collineation group.
inc := IncidenceGraph(GQ);;
d2g := DistanceGraph(inc , 2);;
conn := ConnectedComponents(d2g);;
ptgr := InducedSubgraph(d2g , conn[1]);;
verts := VertexNames(ptgr);;

#making the permutation collineation grp
imsubgqpts := Filtered([1..Size(verts)], i -> pts[i] in subgqpts);;
Size(last);
#Reading in the generators of the permutation group constructed
  #previously
Read("newestT30gens");
#Stabilising the subGQ
stab := Stabiliser(G, imsubgqpts , OnSets);;

extpts := AsSet(Difference(Set(pts) , Set(subgqpts)));;
extlines := AsSet(Difference(Set(ls) , Set(subgqlines)));;
#Getting the indices of the external points in the original point
  set
extptin := [];
for p in extpts do
Add(extptin , Position(pts , p));
od;
extptin := AsSet(extptin);;

#Creating a set expressing lines as tuples of points.
blocks := [];
for l in extlines do
bl := List(AsList(Points(l)) , t -> Position(pts , t));;
Add(blocks , bl);
od;
Collected(List(blocks , Size));
blockset := List(blocks , AsSet);;

```

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C. An alternate proof of Theorem 6.1.6

Here we give an alternate proof of Theorem 6.1.6 using a more combinatorial approach. We will denote incidence of a point P with a line ℓ by $PI\ell$ and concurrency of two lines ℓ and n by $\ell \sim n$.

Lemma C.0.1. Suppose ℓ and n are two nonconcurrent external lines of a generalised quadrangle R of order (q^2, q) , with respect to a generalised subquadrangle R' of order (q, q) . Then in the association scheme on the external lines of R defined in Theorem 5.4.1, the vector $w = q\chi_{\{\ell, n\}} + \chi_{\{\ell, n\}^\perp E}$ is orthogonal to E_1 .

Proof. Let ℓ be an external line, and let σ be the involutory automorphism fixing R' pointwise (see Lemma 4.4.13). We first evaluate $(q\chi_{\{\ell, n\}})E_1$, since it is independent of our choice of the external line n nonconcurrent with ℓ . By equation (6.1), we can express E_1 as a linear combination of adjacency matrices:

$$\begin{aligned} q\chi_{\{\ell, n\}}E_1 &= \frac{q\chi_{\{\ell, n\}}}{q^2(q^2-1)} \left(\frac{q(q-1)^2}{2} I + \frac{q(q-1)}{2} A_1 - \frac{q(q-1)}{2} A_3 - \frac{q(q-1)^2}{2} A_4 \right) \\ &= \frac{q(q-1)}{2q^2(q^2-1)} (q(q-1)(\chi_{\{\ell, n\}} - \chi_{\{\bar{\ell}, \bar{n}\}}) \\ &\quad - q(\chi_{\{\ell\}^\perp E} + \chi_{\{n\}^\perp E} - \chi_{\{\bar{\ell}\}^\perp E} - \chi_{\{\bar{n}\}^\perp E})). \end{aligned}$$

We will split the rest of the proof into three cases.

First case: $n = \bar{\ell}$.

Recall from Lemma 4.4.12 that a line meeting both ℓ and $\bar{\ell}$ also meets R' . So $\{\ell, n\}^\perp E = \emptyset$, and $\chi_{\{\ell, n\}^\perp E} = \mathbf{0}$. We are left with the following calculation:

$$\begin{aligned} q\chi_{\{\ell, \bar{\ell}\}}E_1 &= \frac{q(q-1)}{2q^2(q^2-1)} (q(q-1)(\chi_{\{\ell, \bar{\ell}\}} - \chi_{\{\bar{\ell}, \ell\}}) \\ &\quad - q(\chi_{\{\ell\}^\perp E} + \chi_{\{\bar{\ell}\}^\perp E} - \chi_{\{\bar{\ell}\}^\perp E} - \chi_{\{\ell\}^\perp E})) \\ &= 0. \end{aligned}$$

Second case: n is concurrent with but not equal to $\bar{\ell}$.

By Lemma 4.4.11, since n is concurrent with $\bar{\ell}$, the size of the intersection of their subtended spreads of R' is 1. In other words, $|\{\ell, n\}^\perp E| = q^2$. Now, by Lemma 4.4.14, since n is concurrent with $\bar{\ell}$, the line k through their point of intersection meeting ℓ must also meet R' . Since incidence is preserved under σ , ℓ must be concurrent with \bar{n} , and so, similarly, the line through their point of intersection meeting n must also meet R' . Since there is only one line meeting

R' concurrent with both ℓ and n , this line must be k . That is, the line k is incident with the point of intersection of ℓ and \bar{n} , and the point of intersection of n and $\bar{\ell}$.

We now evaluate $v = \chi_{\{\ell, n\}^{\perp E}} A_3$, which is equivalent to counting the number of lines in $\{\ell, n\}^{\perp E}$ concurrent with (but not equal to) each of the external lines. Clearly ℓ and n are concurrent with all of the lines in $\{\ell, n\}^{\perp E}$, so their corresponding entries in the vector v will be q^2 . Each of the q^2 lines in $\{\ell, n\}^{\perp E}$ cannot be concurrent with another line in that set because otherwise a triangle will be formed, so their corresponding entries in v will be 0.

Suppose r is an external line concurrent with ℓ but not n . Then r is concurrent with a single line meeting both ℓ and n , namely the line incident with the point of intersection of r and ℓ , because otherwise a triangle is formed. Therefore, a line on ℓ not concurrent with n has an entry of 1 in its corresponding position in v , unless it is concurrent with the line on ℓ and n meeting R' , in which case its corresponding entry in v is 0. An analogous argument holds for external lines that are concurrent with n but not ℓ .

The remainder of the external lines do not meet ℓ or n . Each of these lines s then forms a triad with ℓ and n . Recall from Section 4.4.4 that the number of lines concurrent with all elements of a triad is $q + 1$. Therefore, s is concurrent with $q + 1$ lines of $\{\ell, n\}^{\perp E}$, unless it is concurrent with k , in which case it is concurrent with q lines of $\{\ell, n\}^{\perp E}$. Summarising, we have

$$\begin{aligned} \chi_{\{\ell, n\}^{\perp E}} A_3 &= (q + 1)j + (q^2 - q - 1)\chi_{\{\ell, n\}} \\ &\quad - (q + 1) \left(\chi_{\{\ell, n\}^{\perp E}} + \chi_{\{\bar{\ell}, \bar{n}\}} + \chi_{\{\ell, \bar{n}\}^{\perp E}} + \chi_{\{\bar{\ell}, n\}^{\perp E}} \right) \\ &\quad - q \left(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} - 2\chi_{\{\ell, n\}^{\perp E}} - \chi_{\{\bar{\ell}, \bar{n}\}} - \chi_{\{\ell, \bar{n}\}^{\perp E}} - \chi_{\{\bar{\ell}, n\}^{\perp E}} \right) \\ &\quad - \left(\chi_{\{k\}^{\perp E}} - \chi_{\{\ell, \bar{n}\}^{\perp E}} - \chi_{\{\bar{\ell}, n\}^{\perp E}} - \chi_{\{\ell, n\}} - \chi_{\{\bar{\ell}, \bar{n}\}} \right) \\ &= (q + 1)j + (q^2 - q)\chi_{\{\ell, n\}} + (q - 1)\chi_{\{\ell, n\}^{\perp E}} \\ &\quad - q(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}}) - \chi_{\{k\}^{\perp E}}. \end{aligned}$$

We now evaluate $\chi_{\{\ell, n\}^{\perp E}} A_1$. This is equivalent to taking each external line and counting how many lines concurrent with it are equal to the images under σ of a line in $\{\ell, n\}^{\perp E}$. Since incidence is preserved under σ , the set of images of $\{\ell, n\}^{\perp E}$ under σ is equal to $\{\bar{\ell}, \bar{n}\}^{\perp E}$. Thus $\chi_{\{\ell, n\}^{\perp E}} A_1 = \chi_{\{\bar{\ell}, \bar{n}\}^{\perp E}} A_3$, and so we can make use of our previous argument:

$$\begin{aligned} \chi_{\{\ell, n\}^{\perp E}} A_1 &= \chi_{\{\bar{\ell}, \bar{n}\}^{\perp E}} A_3 \\ &= (q + 1)j + (q^2 - q)\chi_{\{\bar{\ell}, \bar{n}\}} + (q - 1)\chi_{\{\bar{\ell}, \bar{n}\}^{\perp E}} \\ &\quad - q(\chi_{\{\bar{\ell}\}^{\perp E}} + \chi_{\{\bar{n}\}^{\perp E}}) - \chi_{\{k\}^{\perp E}}. \end{aligned}$$

Notice that the $\chi_{\{k\}^{\perp E}}$ term is fixed, because k meets R' and so is fixed under

σ by Lemma 4.4.13. We can now calculate $\chi_{\{\ell, n\}^{\perp E}} E_1$:

$$\begin{aligned}
\chi_{\{\ell, n\}^{\perp E}} E_1 &= \frac{\chi_{\{\ell, n\}^{\perp E}}}{q^2(q^2-1)} \left(\frac{q(q-1)^2}{2} I + \frac{q(q-1)}{2} A_1 - \frac{q(q-1)}{2} A_3 - \frac{q(q-1)^2}{2} A_4 \right) \\
&= \frac{q(q-1)}{2q^2(q^2-1)} \left((q-1)(\chi_{\{\ell, n\}^{\perp E}} - \chi_{\{\bar{\ell}, \bar{n}\}^{\perp E}}) \right. \\
&\quad + \left[(q+1)j + (q^2-q)\chi_{\{\bar{\ell}, \bar{n}\}} + (q-1)\chi_{\{\bar{\ell}, \bar{n}\}^{\perp E}} \right. \\
&\quad \quad \left. - q \left(\chi_{\{\bar{\ell}\}^{\perp E}} + \chi_{\{\bar{n}\}^{\perp E}} \right) - \chi_{\{k\}^{\perp E}} \right] \\
&\quad \left. - \left[(q+1)j + (q^2-q)\chi_{\{\ell, n\}} + (q-1)\chi_{\{\ell, n\}^{\perp E}} \right. \right. \\
&\quad \quad \left. \left. - q \left(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} \right) - \chi_{\{k\}^{\perp E}} \right] \right) \\
&= \frac{q(q-1)}{2q^2(q^2-1)} \left((q^2-q) \left[\chi_{\{\bar{\ell}, \bar{n}\}} - \chi_{\{\ell, n\}} \right] \right. \\
&\quad \left. + q \left[\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} - \chi_{\{\bar{\ell}\}^{\perp E}} - \chi_{\{\bar{n}\}^{\perp E}} \right] \right).
\end{aligned}$$

Therefore, $(q\chi_{\{\ell, n\}} + \chi_{\{\ell, n\}^{\perp E}})E_1$ is equal to

$$\begin{aligned}
&\left(\frac{q(q-1)}{2q^2(q^2-1)} \left(q(q-1)(\chi_{\{\ell, n\}} - \chi_{\{\bar{\ell}, \bar{n}\}}) - q \left[\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} - \chi_{\{\bar{\ell}\}^{\perp E}} - \chi_{\{\bar{n}\}^{\perp E}} \right] \right) \right. \\
&\quad \left. + \left(\frac{q(q-1)}{2q^2(q^2-1)} \left((q^2-q) \left[\chi_{\{\bar{\ell}, \bar{n}\}} - \chi_{\{\ell, n\}} \right] + q \left[\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} - \chi_{\{\bar{\ell}\}^{\perp E}} - \chi_{\{\bar{n}\}^{\perp E}} \right] \right) \right) \right),
\end{aligned}$$

which is equal to 0.

Third case: n is not concurrent with $\bar{\ell}$.

By Lemma 4.4.11, the size of the intersection of the spreads of R' subtended by ℓ and n is $q+1$. That is, $|\{\ell, n\}^{\perp E}| = q^2 - q$. The lines meeting R' and concurrent with ℓ and n we denote by γ_i , for $i \in \{1, \dots, q+1\}$. Like in the previous case, we evaluate $v = \chi_{\{\ell, n\}^{\perp E}} A_3$, which is equivalent to counting the number of lines in $\{\ell, n\}^{\perp E}$ concurrent (but not equal to) to each of the external lines.

Again, ℓ and n are concurrent with all lines in $\{\ell, n\}^{\perp E}$, so their entries in v will be equal to $q^2 - q$. Also, as in the previous case, a line in $\{\ell, n\}^{\perp E}$ cannot be concurrent with any other line in the set because otherwise we have a triangle. If r is an external line concurrent with ℓ but not n , then r is concurrent with a single line meeting both ℓ and n , namely the line incident with the point of intersection of r and ℓ . Therefore, a line on ℓ not concurrent with n has an entry of 1 in its corresponding position in v , unless it is concurrent with one of the $q+1$ lines on ℓ and n meeting R' , in which case its corresponding entry in v is 0. An analogous argument holds for external lines that are concurrent with n but not ℓ . In addition, $\bar{\ell}$ and \bar{n} are concurrent with zero lines of $\{\ell, n\}^{\perp E}$ by Lemma 4.4.14. By the same lemma, $\bar{\ell}$ and \bar{n} are concurrent with every line γ_i . Each of the lines in $\{\bar{\ell}\}^{\perp E}$ is therefore concurrent with q lines of $\{\ell, n\}^{\perp E}$ if they are incident with a point of intersection of $\bar{\ell}$ and some γ_i , and $q+1$ points of $\{\ell, n\}^{\perp E}$ otherwise. The same argument holds for the lines in $\{\bar{n}\}^{\perp E}$.

The remaining lines are not concurrent with ℓ , $\bar{\ell}$, n or \bar{n} . Recall that every external line s not concurrent with ℓ and n forms a triad with ℓ and n , and that the number of lines concurrent with all three of them is $q + 1$. Therefore, to determine how many lines of $\{\ell, n\}^{\perp E}$ are concurrent with s , we simply need to know how many lines in the set $\{\gamma_i \mid 1 \leq i \leq q + 1\}$ are concurrent with s . To investigate this, we will switch to the duals of R and R' . The dual \hat{R} of R is an elliptic quadric $Q^-(5, q)$, with the dual \hat{R}' of R' being embedded as a parabolic quadric $Q(4, q)$. The duals of ℓ and n are two noncollinear points, P and Q , such that there are $q^2 + 1$ points collinear with both of them. A total of $q + 1$ of these points (namely the points that are duals of γ_i) have $q + 1$ lines incident with each of them meeting \hat{R}' . These $q + 1$ points form a conic \mathcal{C} of \hat{R} . A point of \hat{R} can be collinear with 0, 1, 2 or $q + 1$ points of \mathcal{C} . Back in R , this is equivalent to each external line being concurrent with 0, 1, 2 or $q + 1$ of the lines γ_i and therefore $q + 1$, q , $q - 1$ or 0 lines of $\{\ell, n\}^{\perp E}$ respectively. Let χ_0, χ_1, χ_2 and χ_{q+1} be the characteristic vectors of lines not concurrent with $\ell, \bar{\ell}, n$ and \bar{n} that meet $\{\gamma_i \mid 1 \leq i \leq q + 1\}$ in 0, 1, 2 and $q + 1$ lines respectively. Notice that an external line s meets γ_i if and only if \bar{s} does, because σ is incidence preserving, so each of these characteristic vectors is made up of conjugate pairs. Then we have

$$\begin{aligned}
\chi_{\{\ell, n\}^{\perp E}} A_3 &= (q + 1)j + (q^2 - 2q - 1)\chi_{\{\ell, n\}} - \chi_1 - 2\chi_2 \\
&\quad - (q + 1) \left(\chi_{q+1} + \chi_{\{\ell, n\}^{\perp E}} + \chi_{\{\bar{\ell}, \bar{n}\}} + \sum_{i=1}^{q+1} \chi_{\{\ell, \gamma_i\}^{\perp E}} + \sum_{i=1}^{q+1} \chi_{\{n, \gamma_i\}^{\perp E}} \right) \\
&\quad - q \left(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} - 2\chi_{\{\ell, n\}^{\perp E}} - \sum_{i=1}^{q+1} \chi_{\{\ell, \gamma_i\}^{\perp E}} - \sum_{i=1}^{q+1} \chi_{\{n, \gamma_i\}^{\perp E}} \right) \\
&\quad - \left(\sum_{i=1}^{q+1} \chi_{\{\bar{\ell}, \gamma_i\}^{\perp E}} + \sum_{i=1}^{q+1} \chi_{\{\bar{n}, \gamma_i\}^{\perp E}} \right) \\
&= (q + 1)j + (q^2 - 2q - 1)\chi_{\{\ell, n\}} - \chi_1 - 2\chi_2 - (q + 1)\chi_{q+1} \\
&\quad - \left(\sum_{i=1}^{q+1} \chi_{\{\ell, \gamma_i\}^{\perp E}} + \sum_{i=1}^{q+1} \chi_{\{n, \gamma_i\}^{\perp E}} \right) - \left(\sum_{i=1}^{q+1} \chi_{\{\bar{\ell}, \gamma_i\}^{\perp E}} + \sum_{i=1}^{q+1} \chi_{\{\bar{n}, \gamma_i\}^{\perp E}} \right) \\
&\quad + (q - 1)\chi_{\{\ell, n\}^{\perp E}} - q \left(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} \right) - (q + 1)\chi_{\{\bar{\ell}, \bar{n}\}}.
\end{aligned}$$

We now evaluate $\chi_{\{\ell, n\}^{\perp E}} A_1$, which by the argument in the previous case is equal to $\chi_{\{\bar{\ell}, \bar{n}\}^{\perp E}} A_3$. Since incidence is preserved and lines meeting R' are fixed under σ , \bar{n} and $\bar{\ell}$ are both concurrent with all of the lines γ_i . Therefore, we have

$$\begin{aligned}
\chi_{\{\ell, n\}^{\perp E}} A_1 &= (q + 1)j + (q^2 - 2q - 1)\chi_{\{\bar{\ell}, \bar{n}\}} - \chi_1 - 2\chi_2 - (q + 1)\chi_{q+1} \\
&\quad - \left(\sum_{i=1}^{q+1} \chi_{\{\bar{\ell}, \gamma_i\}^{\perp E}} + \sum_{i=1}^{q+1} \chi_{\{\bar{n}, \gamma_i\}^{\perp E}} \right) - q \left(\chi_{\{\bar{\ell}\}^{\perp E}} + \chi_{\{\bar{n}\}^{\perp E}} \right) \\
&\quad + (q - 1)\chi_{\{\bar{\ell}, \bar{n}\}^{\perp E}} - \sum_{i=1}^{q+1} \chi_{\{\ell, \gamma_i\}^{\perp E}} - \sum_{i=1}^{q+1} \chi_{\{n, \gamma_i\}^{\perp E}} - (q + 1)\chi_{\{\ell, n\}}.
\end{aligned}$$

We are now ready for the following calculation.

$$\begin{aligned}
\chi_{\{\ell,n\}^{\perp E}}(A_1 - A_3) &= \left((q+1)j + (q^2 - 2q - 1)\chi_{\{\bar{\ell},\bar{n}\}} - \chi_1 - 2\chi_2 - (q+1)\chi_{q+1} \right. \\
&\quad + (q-1)\chi_{\{\bar{\ell},\bar{n}\}^{\perp E}} - \left(\sum_{i=1}^{q+1} \chi_{\{\bar{\ell},\gamma_i\}^{\perp E}} + \sum_{i=1}^{q+1} \chi_{\{\bar{n},\gamma_i\}^{\perp E}} \right) \\
&\quad - q(\chi_{\{\bar{\ell}\}^{\perp E}} + \chi_{\{\bar{n}\}^{\perp E}}) - \sum_{i=1}^{q+1} \chi_{\{\ell,\gamma_i\}^{\perp E}} \\
&\quad \left. - \sum_{i=1}^{q+1} \chi_{\{n,\gamma_i\}^{\perp E}} - (q+1)\chi_{\{\ell,n\}} \right) \\
&\quad - \left((q+1)j + (q^2 - 2q - 1)\chi_{\{\ell,n\}} - \chi_1 - 2\chi_2 - (q+1)\chi_{q+1} \right. \\
&\quad + (q-1)\chi_{\{\ell,n\}^{\perp E}} - \sum_{i=1}^{q+1} \chi_{\{\ell,\gamma_i\}^{\perp E}} - \sum_{i=1}^{q+1} \chi_{\{n,\gamma_i\}^{\perp E}} \\
&\quad - \left(\sum_{i=1}^{q+1} \chi_{\{\bar{\ell},\gamma_i\}^{\perp E}} + \sum_{i=1}^{q+1} \chi_{\{\bar{n},\gamma_i\}^{\perp E}} \right) \\
&\quad \left. - (q+1)\chi_{\{\bar{\ell},\bar{n}\}} - q(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}}) \right) \\
&= (q^2 - q)(\chi_{\{\bar{\ell},\bar{n}\}} - \chi_{\{\ell,n\}}) + (q-1)(-\chi_{\{\ell,n\}^{\perp E}} + \chi_{\{\bar{\ell},\bar{n}\}^{\perp E}}) \\
&\quad + q(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} - \chi_{\{\bar{\ell}\}^{\perp E}} - \chi_{\{\bar{n}\}^{\perp E}}).
\end{aligned}$$

It follows that

$$\begin{aligned}
\chi_{\{\ell,n\}^{\perp E}} E_1 &= \frac{\chi_{\{\ell,n\}^{\perp E}}}{q^2(q^2 - 1)} \left(\frac{q(q-1)^2}{2} I + \frac{q(q-1)}{2} A_1 - \frac{q(q-1)}{2} A_3 - \frac{q(q-1)^2}{2} A_4 \right) \\
&= \frac{q(q-1)}{2q^2(q^2 - 1)} \left((q-1)(\chi_{\{\ell,n\}^{\perp E}} - \chi_{\{\bar{\ell},\bar{n}\}^{\perp E}}) + (q^2 - q)(\chi_{\{\bar{\ell},\bar{n}\}} - \chi_{\{\ell,n\}}) \right. \\
&\quad \left. + (q-1)(-\chi_{\{\ell,n\}^{\perp E}} + \chi_{\{\bar{\ell},\bar{n}\}^{\perp E}}) \right. \\
&\quad \left. + q(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} - \chi_{\{\bar{\ell}\}^{\perp E}} - \chi_{\{\bar{n}\}^{\perp E}}) \right) \\
&= \frac{q(q-1)}{2q^2(q^2 - 1)} \left((q^2 - q)(\chi_{\{\bar{\ell},\bar{n}\}} - \chi_{\{\ell,n\}}) \right. \\
&\quad \left. + q(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} - \chi_{\{\bar{\ell}\}^{\perp E}} - \chi_{\{\bar{n}\}^{\perp E}}) \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(q\chi_{\{\ell,n\}} + \chi_{\{\ell,n\}^{\perp E}})E_1 &= \left(\frac{q(q-1)}{2q^2(q^2-1)} \left(q(q-1)(\chi_{\{\ell,n\}} - \chi_{\{\bar{\ell},\bar{n}\}}) \right. \right. \\
&\quad \left. \left. - q(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} - \chi_{\{\bar{\ell}\}^{\perp E}} - \chi_{\{\bar{n}\}^{\perp E}}) \right) \right) \\
&\quad + \left(\frac{q(q-1)}{2q^2(q^2-1)} \left((q^2-q)(\chi_{\{\bar{\ell},\bar{n}\}} - \chi_{\{\ell,n\}}) \right. \right. \\
&\quad \left. \left. + q(\chi_{\{\ell\}^{\perp E}} + \chi_{\{n\}^{\perp E}} - \chi_{\{\bar{\ell}\}^{\perp E}} - \chi_{\{\bar{n}\}^{\perp E}}) \right) \right) \\
&= 0. \quad \square
\end{aligned}$$

We now prove the following result, making use of the association scheme and a technique employed by Bamberg et al. to show the nonexistence of m -covers that are not hemisystems [4, Section 8].

Theorem C.0.2. Let R be a generalised quadrangle of order (q^2, q) containing a doubly subtended subquadrangle R' of order (q, q) . Suppose that S is a nontrivial relative m -cover of the external lines. Then $m = \frac{q}{2}$.

Proof. First, by Lemma 6.1.1, $|S| = m(q^3 - q)$. Suppose ℓ and n are two nonconcurrent external lines of $H(3, q^2)$ such that ℓ does not lie in S . By Lemma 4.4.12, the size of $\{\ell, n\}^{\perp E}$ is q^2 if n is concurrent with $\bar{\ell}$, and $q^2 - q$ if it is not. Recall from Section 5.4 that we can find a vector space V with the set of external lines as a basis, and that we can decompose V into orthogonal components as $V = V_0 \perp V_1 \perp \cdots \perp V_4$ such that each V_i is the row space of the corresponding minimal idempotent E_i . Consider the vector $w = q\chi_{\{\ell,n\}} + \chi_{\{\ell,n\}^{\perp E}}$. Now, w is orthogonal to E_1 by Lemma C.0.1, so it must lie in $V_0 \perp V_2 \perp V_3 \perp V_4$. Notice that $w \cdot j = (q\chi_{\{\ell,n\}} + \chi_{\{\ell,n\}^{\perp E}}) \cdot j = 2q + d$, where $d \in \{q^2 - q, q^2\}$. We also have

$$\left(w - \frac{w \cdot j}{j \cdot j} j \right) \cdot j = 0.$$

Recalling that V_0 is spanned by j , we conclude that $w - \frac{w \cdot j}{j \cdot j} j \in V_2 \perp V_3 \perp V_4$. Now $\chi_S \cdot ((q^3 - q)\chi_{[P]} - j) = m(q^3 - q) - m(q^3 - q) = 0$ for all external points P , and so $\chi_S \in V_0 \perp V_1$, since the set of vectors $\{(q^3 - q)\chi_{[P]} - j \mid P \in \mathcal{P}_E\}$ forms a spanning set of $V_2 \perp V_3 \perp V_4$ by Corollary 6.1.4. It follows that $(w - \frac{w \cdot j}{j \cdot j} j) \cdot \chi_S = 0$. We can rewrite this as

$$w \cdot \chi_S = \frac{w \cdot j}{j \cdot j} m(q^3 - q) = \frac{2q + d}{q^2(q^2 - 1)} m(q^3 - q) = \frac{m(2q + d)}{q}.$$

Therefore,

$$|\{\ell, n\}^{\perp E} \cap S| = \chi_{\{\ell,n\}^{\perp E}} \cdot \chi_S = (w - q\chi_{\{\ell,n\}}) \cdot \chi_S = \frac{m(2q + d)}{q} - q\chi_n \cdot \chi_S, \quad (\text{C.1})$$

where d is equal to q^2 or $q^2 - q$, depending on our choice of n .

We now double count pairs (n, k) where $n, k \in S$, n is not concurrent with ℓ , and $k \in \{\ell, n\}^{\perp E}$. First notice that the number of lines of S concurrent with ℓ is $m(q^2 + 1)$, since there are m lines of S concurrent with each point of ℓ . Therefore, $|S \setminus \ell^{\perp E}| = m(q^3 - q) - m(q^2 + 1) = m(q^3 - q^2 - q - 1)$.

First count: $\sum_{n \in S \setminus \ell^{\perp E}} |S \cap \{\ell, n\}^{\perp E}|$.

The value of $|S \cap \{\ell, n\}^{\perp E}|$ depends on whether the line in $S \setminus \ell^{\perp E}$ is concurrent with $\bar{\ell}$ or not. By equation (C.1), each of the lines in $S \setminus \ell^{\perp E}$ concurrent with $\bar{\ell}$ has $|\{\ell, n\}^{\perp E} \cap S| = m(q + 2) - q$, and the rest of the lines of $S \setminus \ell^{\perp E}$ (that is, those not concurrent with $\bar{\ell}$) have $|\{\ell, n\}^{\perp E} \cap S| = m(q + 1) - q$.

The number of these lines depends on whether $\bar{\ell}$ is in S . If $\bar{\ell}$ is not in S , then the number of lines of $S \setminus \ell^{\perp E}$ concurrent with $\bar{\ell}$ is $m(q^2 + 1)$, and so there are $m(q^3 - 2q^2 - q - 2)$ lines of $S \setminus \ell^{\perp E}$ not concurrent with $\bar{\ell}$. Therefore, we have

$$\begin{aligned} \sum_{n \in S \setminus \ell^{\perp E}} |S \cap \{\ell, n\}^{\perp E}| &= m(q^2 + 1)(m(q + 2) - q) \\ &\quad + m(q^3 - 2q^2 - q - 2)(m(q + 1) - q) \\ &= mq(1 + q + q^2 - q^3 + m(-2 - q + q^3)). \end{aligned}$$

Suppose, on the other hand, that $\bar{\ell} \in S$. Then $\bar{\ell} \in S \setminus \ell^{\perp E}$, but $|S \cap \{\ell, \bar{\ell}\}^{\perp E}| = 0$ because $\{\ell, \bar{\ell}\}^{\perp E}$ is empty (if we only count external lines). There are also $(m - 1)(q^2 + 1)$ lines belonging to $S \setminus \ell^{\perp E}$ concurrent with $\bar{\ell}$, and we have

$$\begin{aligned} \sum_{n \in S \setminus \ell^{\perp E}} |S \cap \{\ell, n\}^{\perp E}| &= (m - 1)(q^2 + 1)(m(q + 2) - q) \\ &\quad + (m(q^3 - q^2 - q - 1) - (m - 1)(q^2 + 1) - 1)(m(q + 1) - q) \\ &= q + m^2q(-2 - q + q^3) + m(-2 + q^3 - q^4). \end{aligned}$$

Second count: $\sum_{k \in S \cap \ell^{\perp E}} |(k^{\perp E} \setminus P_{\ell k}) \cap S|$.

Here, $P_{\ell k}$ is the point of intersection of ℓ and k . Since $k \in S$, we have that $|(k^{\perp E} \setminus P_{\ell k}^{\perp E}) \cap S| = q^2(m - 1)$, because each of the q^2 points on k (excluding $P_{\ell k}$) have $m - 1$ lines of S incident with them not equal to k . The size of $S \cap \ell^{\perp E}$ is $m(q^2 + 1)$. We have

$$\sum_{k \in S \cap \ell^{\perp E}} |(k^{\perp E} \setminus P_{\ell k}) \cap S| = m(q^2 + 1)q^2(m - 1).$$

Now, combining the double counts, if $\bar{\ell} \notin S$, we have

$$\begin{aligned}mq(1 + q + q^2 - q^3 + m(-2 - q + q^3)) &= m(q^2 + 1)q^2(m - 1), \\mq(1 + q + q^2 - q^3 + m(-2 - q + q^3)) - m(q^2 + 1)q^2(m - 1) &= 0, \\m(q(1 + q + q^2 - q^3 + m(-2 - q + q^3)) - (q^2 + 1)q^2(m - 1)) &= 0.\end{aligned}$$

Observe that $m = 0$ is a solution. In addition,

$$q(1 + q + q^2 - q^3 + m(-2 - q + q^3)) - (q^2 + 1)q^2(m - 1) = 0.$$

which when expanded and simplified gives $m = \frac{q+1}{2}$. However, since q is even, $\frac{q+1}{2}$ is not an integer, so we disregard this solution. If, on the other hand, we have $\bar{\ell} \in S$, then

$$q + m^2q(-2 - q + q^3) + m(-2 + q^3 - q^4) = m(q^2 + 1)q^2(m - 1).$$

Rearranging and simplifying gives

$$m = \frac{q}{2} \quad \text{or} \quad m = -\frac{1}{q(1+q)}.$$

Since q is even, $\frac{q}{2}$ is an integer, and we have a relative hemisystem. The other solution is negative, and never an integer for any positive prime power, so we disregard it. \square

Notice that we did not need to assume Theorem 6.1.6(ii) in order to prove this result.

