Hydraulics and Mixing in Controlled Exchange Flows

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Abstract

Internal hydraulic theory is often used to describe idealised bi-directional exchange flow through a constricted channel. This approach is formally applicable to layered flows in which velocity and density are represented by discontinuous functions that are constant within discrete layers. The layered structure of the hydraulic solution complicates its application to geophysical flows in which strong velocity gradients between layers may produce turbulent mixing which alters the two-layer structure. In this study we investigate mixing in exchange flows through a lateral contraction. The results fall into three distinct parts: a set of numerical simulations used to examine the effect of mixing upon exchange flux, a theoretical analysis of the mechanism of hydraulic control in stratified environments, and a laboratory model which is used to examine the physics of mixing processes in the flow.

First, numerical simulations of bi-directional density-driven exchange flows are used to study the effect of turbulent mixing upon these flows. The numerical experiments are designed so that it is possible to specify the intensity of mixing. The simulated flows are compared to two analytical solutions, first, the two-layer hydraulic solution which has no mixing, and second, a solution in which turbulent mixing dominates the flow. The simulations demonstrate that the two analytical solutions form the limits of a wide class of problems and that the flow regime in between the limiting solutions can be described by a single nondimensional parameter $G_{T-A^2}$, where $G_T$ is a turbulent Grashof number, and $A$ aspect ratio.

Hydraulic theory relies on the determination of flow conditions at points of hydraulic control, where long interracial waves have zero phase speed. In the second stage we focus upon the propagation characteristics of the gravest vertical mode internal waves in exchange flows where mixing has produced a continuous stratification. Two approaches are used to determine the behaviour of the waves. In the first, waves are mechanically excited at discrete locations within a numerically simulated bi-directional exchange flow and allowed to evolve under linear dynamics. A second approach, based on the stability theory of parallel viscous shear flows is used to interpret the direct excitation experiments. Two types of gravest mode solutions are identified: vorticity modes and density modes. These modes are used to generalise the notion of hydraulic control in continuously stratified flows.

Thirdly, laboratory models of exchange flow through a contracting channel are used to investigate the processes leading to mixing between two flowing layers. Shear-driven mixing is sensitive to changes in parameters such as the height of the water column and the density difference of the layers. Two regimes are found: the first where limited mixing is due to Holmboe instabilities, and the second where vigorous mixing is produced by Kelvin–Helmholtz billows.
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Preface

The main body of this thesis is comprised of three chapters (2–4), each of which is a paper written for journal publication. Chapter 2 and Appendix A were published in the *Journal of Geophysical Research* under the title "Hydraulics and mixing in controlled exchange flows" by A. M. Hogg, G. N. Ivey & K. B. Winters, volume 103, number C1, pages 30695–30711, 2001. Chapter 3 was accepted for publication in the *Journal of Fluid Mechanics* on 14 June 2001 under the title "Linear internal waves and the control of stratified exchange flows" by A. McC. Hogg, K. B. Winters & G. N. Ivey, and is currently in press. This chapter refers to the derivation of a sixth-order viscous stability equation which is shown in detail in Appendix B. Chapter 4 was submitted to the *Journal of Fluid Mechanics* on 8 August 2001, entitled "The Kelvin-Helmholtz to Holmboe instability transition in stratified exchange flows" by A. McC. Hogg & G. N. Ivey.

Except where referenced, the material presented in this thesis is a synthesis of my own ideas and work undertaken by myself under the supervision of G. N. Ivey and K. B. Winters.
CHAPTER 1

Introduction

1.1 Motivation

This thesis examines the role played by mixing in two-layer exchange flows. The study of exchange flows is relevant principally to oceanographic or limnological applications, where two water bodies with different density are connected by a narrow channel or constriction. Flow within the channel (which is of primary interest here) is driven by the density difference between the end reservoirs. Under certain conditions the channel may support transport of dense water in one direction, overlain by a counterflowing layer of fresh water. Thus, flow in the channel or constriction acts to continuously exchange fluid between the reservoirs.

Prominent examples of density-driven exchange flow includes the Bosphorus Strait between the Black Sea and the Mediterranean Sea in Turkey (Oğuz et al., 1990; Gregg et al., 1999; Gregg & Özsoy, 2001), the Strait of Gibraltar at the mouth of the Mediterranean Sea (Bray et al., 1995) and Bab al Mandab at the mouth of the Red Sea (Pratt et al., 1999, 2000). At smaller scales, exchange flow occurs at the mouths of estuaries and inlets and between basins in fresh water bodies (Greco, 1998). The processes investigated here may also apply to flow of bottom water between ocean basins. Other applications for this type of flow includes flow of air through doorways and windows in air-conditioned buildings, where the heat flux between rooms must be calculated in the design of the air-conditioning system.

Exchange flow is routinely observed in the Bosphorus Strait, as shown in figures 1.1 and 1.2. These images, reproduced from Gregg & Özsoy (2001), show a map of the bathymetry of the Bosphorus Strait (figure 1.1), with the Black Sea to the north, and the Mediterranean Sea to the south. The heavy black line in the channel shows the thalweg, the position of the deepest part of the strait. Using a number of transects along the thalweg, two-dimensional time-averaged fields of density and velocity can be calculated as shown in figure 1.2. Here the two-layer nature of the flow is demonstrated in both the velocity and density fields. The open circles show that the fresher upper layer is flowing south out of the Black Sea. The saline lower layer transports water to the north into the Black Sea.

The transport in the channel will redistribute both active and passive tracers and thereby influence the hydrodynamics and the water quality of the two water bodies at the ends of the channel. For example, the climate and ecology of the Black Sea is dependent upon the balance between water and salt, and therefore slight changes to the continuous exchange flow through the Bosphorus will have a
significant impact upon the Sea in the long term (Ünlüata et al., 1993). In other cases, the active tracer may be temperature, such as in the Burlington Ship Canal which divides Hamilton Harbour from the main body of Lake Ontario (Greco, 1998). In this case, while the temperature difference drives exchange flow in the Ship Canal during summer months, the water quality in Hamilton Harbour is dependent upon the flushing of nutrients (passive tracers) out of the Harbour.

Previous theoretical studies of these systems have frequently made the assumption that mixing plays a minor role in governing the development of the exchange flow. If one makes this assumption (as well as several additional assumptions as discussed in detail in chapter 2) it is possible to analytically solve for the velocity and layer thickness of each layer. Thus, one can calculate the flux through the channel based only on external parameters. This calculation, often called the two-layer hydraulic solution, is particularly relevant to the modelling community. If one wishes to model a system which includes inflow via a narrow channel, the model can be simplified considerably by parameterising flux of the inflow as a function of known variables. In addition such a parameterisation may

Figure 1.1: Map of the Bosphorus region. Reproduced with permission from Gregg & Özsöy (2001).
1.1. Motivation

Figure 1.2: Quasi-steady exchange flow through the Bosphorus Strait. Average potential density contours are depicted by solid lines, with the heavy contours showing the 13 and 26 kg m$^{-3}$ contours. The velocity field is shown by northward (solid circles) and southward (open circles) fluid velocity, with the zero-isotach represented by the dash-dot line. Reproduced with permission from Gregg & Özsoy (2001).

prove more accurate than direct calculation using a model in which the channel may be under-resolved. Thus the ability to directly calculate flux in exchange flow may assist in modelling of inlets and estuaries, as well as modelling the effect of large embayments on Ocean Global Circulation Models.

However, available data demonstrates the difficulty involved in applying the two-layer hydraulic solution to geophysical examples of this type of flow. For example, in the Bosphorus (figure 1.2) mixing between the two layers produces a finite-thickness interfacial region in which density and velocity varies continuously. The thickness of this region is dependent upon turbulent mixing between the two layers which is inconsistent with the basic assumptions underlying the two-layer solution. It is shown in the following chapters that such mixing alters the flux of active and passive tracers through the channel, and affects the physics of the exchange flow.
1.2 Overview

This thesis is a compilation of numerical, analytical and laboratory studies into the effects of mixing in exchange flows. In each case, every attempt is made to isolate the effects of mixing, and thus the conditions under which the flow occurs is simplified as much as possible. As a result the studies described here cannot be directly applied to field scenarios where topography, wind stress and tides complicate flow. Rather than applying the results of this thesis to a specific field site, the intent is to provide generalised observations which can be adapted to a range of situations.

Three different approaches are used to investigate the exchange flows. In each case, simple domains are used: flat-bottomed channels with gradually varying sidewalls which produce a constriction in the centre of the domain. The flow is initialised with homogeneous reservoirs at either end of the channel, and exchange flow allowed to develop to a steady state. It is the steady-state flow which is of primary interest.

Each of the three approaches has been written as a separate journal paper, and is reproduced in full as a chapter of this thesis. In the first approach, a numerical technique is used to simulate flow through a contracting channel. Second, a combination of numerical and analytical techniques applied to the results of the first study are used to produce a description of the physics governing the exchange. In the third paper, a laboratory investigation into exchange flow is used to study the physics of interfacial mixing processes.

The numerical study concentrates on the effect of mixing upon the flux of active and passive tracers along the channel. The results of this investigation are outlined in chapter 2. The philosophy governing the design of the numerical experiments is to simplify the processes governing the mixing in the flow: therefore instead of calculating the amount of mixing due to different turbulent processes, turbulent mixing is considered as a free parameter. By specifying the level of turbulent mixing in a range of different exchange flows it is possible to measure the flux as a function of mixing. Comparisons with reported studies using field measurements of exchange flows demonstrates the variability of behaviour in geophysical exchange flows.

The physics of exchange flows is frequently described in terms of the two-layer hydraulic solution which is described in detail in chapters 2 and 3. This solution allows the calculation of a maximal flux, an upper bound of the exchange flux, using only a handful of external parameters. The solution breaks down when mixing occurs, but it is shown in chapter 2 that the flux prediction of this solution
is a good approximation to measurements of flux when mixing is small. The calculation of the maximal flux in the hydraulic solution is physically dependent upon the speed of long linear internal waves, and chapter 3 describes the study into the role of long linear internal waves in flows altered by the onset of mixing. Two approaches are used here: a numerical approach and an analytical approach. The aim is to determine the effectiveness of these techniques in gauging whether an exchange flow is close to the maximal flux limit.

The laboratory investigation outlined in chapter 4 gives some insight into the processes which lead to mixing in exchange flows. Experiments demonstrate the occurrence of mixing in a simple exchange flow due to interfacial instability. The physics of shear instability is investigated in some detail, yielding insight into the variability and sensitivity of the instabilities as a function of external parameters.
CHAPTER 2

Hydraulics and mixing in controlled exchange flows

Abstract

Numerical simulations of bi-directional density-driven exchange flows are used to study the effects of turbulent mixing in these flows. The numerical experiments are designed so that it is possible to specify the intensity of mixing, which allows the investigation of a wide range of flows that are difficult to model in the laboratory. The simulated flows are compared to two analytical solutions, first, the two-layer hydraulic solution which has no mixing, and second, a solution in which turbulent mixing dominates the flow. We are able to model exchange flows similar to either of these limits by modifying the turbulent mixing, as well as simulating behaviour between these two extremes. The simulations demonstrate that the two analytical solutions form the limits of a wide class of problems and that the flow regime in between the limiting solutions can be described by a single nondimensional parameter $GrTA^2$.

2.1 Introduction

Differences in density between two connected reservoirs or seas may drive a bi-directional exchange flow in which the flux of exchange between two such water bodies is limited by some topographic constriction (either a sill or a contraction). In this chapter we present the results of a numerical study of bi-directional exchange flow through a lateral contraction (illustrated in plan view in figure 2.1a).

To predict flux in bi-directional exchange flows, two very different analytical solutions have been proposed. First, in the absence of mixing, viscosity, and friction, internal hydraulic theory (Wood, 1970; Armi, 1986; Armi & Farmer, 1986; Lawrence, 1990; Dalziel, 1991) can be used to predict the velocities and depths of two distinct layers. Alternatively, if the exchange flow is dominated by turbulent mixing, we can find a solution where velocity is limited by turbulent eddy viscosity, and transport of mass is due to a combination of advection and turbulent diffusion (Cormack et al., 1974; Officer, 1976). We refer to the second solution as the viscous advective diffusive (VAD) solution. The two analytical solutions describe the behaviour of exchange flows at extreme limits that comprise the endpoints of a range of flows. Between these two limits there is a range of flows which cannot be represented accurately by either limit. Our interest in this work is to present a unifying description of the entire range of flows, not just the endpoints.
Chapter 2. Hydraulics and mixing in controlled exchange flows

Figure 2.1: (a) Plan view of channel with contraction; (b) elevation view of flow.

Internal hydraulic theory is based on a simple balance between inertial and buoyancy forces. Flow is assumed to consist of two layers, each having constant velocity and density. The layer thicknesses vary monotonically through the region of the contraction and can be predicted from the hydraulic equations (Lawrence, 1990). The power of hydraulic theory is illustrated by Dalziel (1991) who showed that there are regions of the flow where long waves can travel only in one direction, or in other words, information can only propagate in one direction. Conditions in one part of the domain may therefore vary without altering the rest of the flow. Hydraulic theory provides an upper bound on the flux through the contraction which is both useful and simple to calculate. There are factors, however, which are not considered by the hydraulic solution. Some of these factors, such as nonhydrostatic effects (Zhu & Lawrence, 1998) and friction (Assaf & Hecht, 1974) can be incorporated as a correction into the hydraulic equations, but there is no such correction for the effect of mixing on exchange flows.

Exchange flow experiments designed to verify the hydraulic solution demonstrate that, in some cases, mixing between the two layers will play a significant role in determining the flow. Wood (1970) remarked upon the difficulties faced in designing exchange flow experiments which did not result in interfacial instabilities leading to mixing. The generation of these instabilities have been studied in
detail (Pawlak & Armi, 1998); however, the effect of the resulting mixing has not been quantified. Investigations into the effect of tidal oscillations on exchange in a salt-stratified fluid showed that, predominantly due to interfacial mixing, the flux of salt was 20% less than the hydraulic prediction (Helfrich, 1995). The effect of mixing upon the flux of volume through a contraction was investigated in a series of numerical simulations which showed that mixing leads to entrainment between the two flowing layers and thereby reduces the total flux (Winters & Seim, 2000).

Two well-known examples of geophysical exchange flows are the Bosphorus (Oğuz et al., 1990; Gregg et al., 1999) and the Straits of Gibraltar (Farmer & Armi, 1986; Bray et al., 1995). The Bosphorus is a thin channel which links the Mediterranean and Black Sea, and Gregg et al. (1999) show that a dense Mediterranean underflow is divided from the return flow by an interfacial region which is 10 m deep in a 35 m deep channel. While the observed density field is similar in some respects to the hydraulic solution, the finite thickness of the interface dividing the two layers indicates vertical mixing. Similarly, field data of exchange flow near the Camarinal Sill at the exit of the Straits of Gibraltar (Wesson & Gregg, 1994) shows strong turbulent mixing which creates observable volumes of intermediate density water. Flow in both Gibraltar and the Bosphorus exhibit some patterns consistent with hydraulic theory; however, the occurrence of mixing and finite-thickness interfacial layers implies that the hydraulic solution is not directly applicable.

If mixing is very strong we can find an analytical solution (the VAD solution) for flow in the limit where buoyancy forces are balanced by turbulent viscosity and diffusion. Turbulent mixing can be parameterised by a constant turbulent eddy viscosity $K_v$ and eddy diffusivity $K_\rho$, respectively, and we can then use the asymptotic analysis of Cormack et al. (1974) to predict the flow. The flux of volume and density will depend upon three dimensionless parameters: the aspect ratio $A = H/L$, the turbulent Grashof number $Gr_T = g'H^3/K_\rho^2$, and the turbulent Prandtl number $Pr_T = K_v/K_\rho$, where $H$ is the height of the channel, $L$ is the length of the mixed region and $g' = g\Delta \rho/\rho_0$ is the reduced gravity which depends upon gravitational acceleration $g$, the difference in density from one end of the channel to the other $\Delta \rho$ and the reference density $\rho_0$. The turbulent Grashof number is a dimensionless quantity which compares the effect of buoyancy forces to the turbulent viscous forces, and in combination with the turbulent Prandtl number is thus a natural parameter to use to describe behaviour in the VAD limit. This study is confined to flows where the turbulent Prandtl number is unity, so the turbulent Grashof number may equally well represent the effect of
turbulent diffusion on buoyancy forces. Burling et al. (1999) review the analytical predictions for the VAD solution and show field data from Herald Loop Channel, Shark Bay, Western Australia. This channel is so shallow and the surface wind stress so high that the mixing dominates inertial effects, and the flow in this channel resembles the VAD limit.

It is reasonable to expect that most geophysical exchange flows will lie in the range between the hydraulic and the VAD limits, and we describe this range of behaviours (including a review of previous work) in section 2.2 in terms of two dimensionless parameters $Gr_T$ and $A$. The numerical code and modelling methods are described in section 2.3. Our main objectives are to vary the amount of mixing in the channel and thereby determine conditions where either of the limiting solutions are valid and to investigate conditions where the three-way balance is required. Section 2.4 describes the results of the effect of mixing. In section 2.5 we discuss these results and use field data to show where a number of geophysical flows lie in this parameter space.

2.2 Analytical solutions for the limiting cases

2.2.1 Review of two-layer hydraulic theory

In this section we pick out the salient points of previous work on internal hydraulics to illustrate the nature of this theory. Two-layer internal hydraulic theory has been considered by a number of authors including Wood (1970), Armi (1986), Armi & Farmer (1986), Lawrence (1990), Dalziel (1991), and Baines (1995). Figure 2.1 shows the general configuration we consider: two reservoirs of homogeneous density fluid separated by a contraction. We assume that flow in the contraction has two distinct layers, that each layer has a constant velocity and density, that horizontal variations in the channel width are small enough to be able to ignore vertical velocities and that friction and mixing are negligible. As shown in Appendix A we can use Bernoulli’s equations and mass conservation to write a condition on the flow,

\[
\frac{\partial h_1}{\partial x} = \frac{1}{b} \frac{\partial b}{\partial x} \left( \frac{-u_1^2/2 + u_2^2/2 + (1 - r)u_1^2u_2^2/4h_2}{G^2 - 1} \right),
\]

\[
\frac{\partial h_2}{\partial x} = \frac{1}{b} \frac{\partial b}{\partial x} \left( \frac{ru_1^2/2 - u_2^2/2 + (1 - r)u_1^2u_2^2/4h_1}{G^2 - 1} \right)
\]

where $h_i$ are layer heights (nondimensionalised by half the depth of the left reservoir $H/2$), $b$ is channel width (nondimensionalised by the minimum width $B$), $u_i$ are layer velocities (nondimensionalised by $1/2(g'H)^{1/2}$), $r \equiv \rho_1/\rho_2$ is the ratio of
2.2. Analytical solutions for the limiting cases

layer densities and,

\[ G^2 \equiv \frac{u_1^2}{2h_1} + \frac{u_2^2}{2h_2} - \frac{(1 - r)u_1^2u_2^2}{4h_1h_2}, \]  

(2.3)
is the composite Froude number (Armi, 1986).

The power of internal hydraulics is illustrated by (2.1) and (2.2). At the minimum width of the contraction (where \( \partial h/\partial x = 0 \)), either both layers are flat (\( \partial h_i/\partial x = 0 \)) or the composite Froude number is fixed at

\[ G^2 = 1. \]  

(2.4)

In the latter case the flow is said to be controlled (or critical), as flow variables are constrained by (2.4) there. We refer to this control point as the topographic control. The flow may also be controlled at another point when \( G^2 = 1 \), and this second control is called a virtual control (Wood, 1970). Further constraints at the virtual control can be derived from (2.1) and (2.2), as the numerator in the brackets on the right-hand side of these equations must be zero,

\[ -u_1^2 + u_2^2 + \frac{(1 - r)u_1^2u_2^2}{2h_2} = 0, \]  

(2.5)

\[ ru_1^2 - u_2^2 + \frac{(1 - r)u_1^2u_2^2}{2h_1} = 0. \]  

(2.6)

It can be shown that only two of (2.4), (2.5), and (2.6) are independent (Armi, 1986). A flow is said to be maximal if it has both a virtual control and a topographic control (Armi & Farmer, 1987).

The nature of control points is illustrated by Dalziel (1991), who writes

\[ G^2 = 1 + \frac{C_1C_2}{h_1h_2}, \]  

(2.7)

where \( C_i \) are the long-wave speeds of internal waves (nondimensionalised by \( 1/2(g' H)^{1/2} \)). In a stationary two-layer fluid, \( C_1 \) and \( C_2 \) are of opposite sign, representing a wave travelling in each direction. However, one of \( C_1 \) or \( C_2 \) must be zero at the control points (\( G^2 = 1 \)), and for supercritical flow (\( G^2 > 1 \)), \( C_1 \) and \( C_2 \) are of the same sign. Therefore at a control point the composite Froude number indicates that information can only propagate in one direction but does not discern which direction the waves can propagate (Dalziel, 1991), and so we refer back to the full solution of the hydraulic equations.

To find the hydraulic solution using the method outlined by Lawrence (1990), we require one further parameter, the volume flux ratio \( q_r = q_1/q_2 \), where \( q_i \) is the volume flux in each layer. If we assume \( q_r \) is specified, we can determine layer heights and velocities everywhere in the channel. Lawrence (1990) used the
Bernoulli equations (A.2) and (A.3) and made the Boussinesq assumption to show that at the virtual control,

\[ h_1 \approx h_2, \]  

(2.8)

and hence

\[ u_1 \approx -u_2. \]  

(2.9)

The position of the virtual control is determined by the magnitude of the volume flux ratio \( q_r \) and is upstream of the contraction relative to the fastest flowing layer. The solution for a case with \( q_r = 2 \) is shown in figure 2.2, where we have not considered the occurrence of hydraulic jumps at the ends of the contracting region to return fluid velocities to reservoir conditions. In figure 2.2 we show the channel in plan view, the shape of the interface dividing the two layers in elevation view, as well as the variation of the composite Froude number \( G^2 \) and wave speeds \( C_1 \) and \( C_2 \). When \( q_r = 1 \) the topographic and virtual control coalesce, and there is no subcritical region (Armi, 1986). In all other cases there is a subcritical region \( (G^2 < 1) \) between the two controls as in figure 2.2(c). Figure 2.2(d) shows that in the subcritical region, information can be communicated (that is, long waves can propagate) in either direction from the subcritical region through the control points and into the reservoirs. On the other hand, no waves can propagate from the reservoirs through the supercritical region to reach the control points. This solution is for the maximal exchange state (as given by Armi & Farmer, 1987) where the control points are isolated from changes to the interfacial level in the reservoirs.

The flux of both volume and mass through the contraction for the maximal case can be calculated from the knowledge of only a few parameters \( (B, g', q_r, H) \) and is independent of channel shape. For example, if \( q_r = 1 \), we can write nondimensional volume flux at the topographic control as the product of layer depth \( (h_1 \approx h_2 \approx 1) \) and velocity \( (u_1 \approx -u_2 \approx 1) \) at the contraction \( (b = 1) \) to give the nondimensional flux for the hydraulic solution,

\[ q_h = u_1 h_1 b = 1, \]  

(2.10)

which is nondimensionalised by the hydraulic scaling \( 1/4 B g'^{1/2} H^{3/2} \). For all fluxes we use lowercase letters to indicate the dimensionless quantity. Net mass flux, nondimensionalised by \( 1/4 \Delta \rho B g'^{1/2} H^{3/2} \) is then simply

\[ m_h = 1, \]  

(2.11)

for the maximal hydraulic prediction.
2.2. Analytical solutions for the limiting cases

Figure 2.2: The hydraulic solution for bi-directional exchange flow. (a) Channel shape; (b) velocity vectors and the interface dividing layers; (c) variation of the composite Froude number with $x/L$; (d) variation of long-wave speeds $C_1$ and $C_2$ with $x/L$. The vertical dashed lines delineate the supercritical regions to the right and left, with arrows showing the direction of propagation of waves. The central region is subcritical as $C_1$ propagates to the right and $C_2$ to the left.
2.2.2 Review of the VAD solution

If exchange flow occurs in a channel with strong mixing, horizontal velocity may be controlled by the turbulent eddy viscosity and diffusivity rather than the (inviscid) speed of internal waves. Since wave speeds do not limit the flow, the role of the contraction is altered in this type of exchange flow. We can make a good estimate of the flux using a straight-edged channel which is as wide as the minimum width of the contraction $B$, such as the channel shown in figure 2.3 where the shaded region is strongly mixed.

The solution for flow in a strongly mixed channel is identical to the problem considered by Cormack et al. (1974). They derived the asymptotic solution of natural convection in a cavity with small aspect ratio and with vertical endwalls maintaining the density contrast. The leading order formal asymptotic solution for the core flow gives the same result as the simple theory of Officer (1976), and so for simplicity we follow the informal method of Officer (1976) to determine flux through the contraction in the VAD limit. We assume the aspect ratio is small (that is, the channel is long compared to its depth), so that velocities can be considered horizontal and constant in $x$, and the pressure everywhere hydrostatic.

Figure 2.3: (a) Plan view of channel with the region of mixing shaded; (b) elevation view of the expected VAD solution.
The equations of motion for steady flow are then

\[ \frac{\partial p}{\partial x} = \rho K_v \frac{\partial^2 u}{\partial z^2}, \]  

(2.12)

\[ p = \rho g (\zeta + z), \]  

(2.13)

where \( p \) is pressure and \( \zeta \) is the surface elevation. We show the solution of this problem over a two-dimensional domain stretching from \( x = -L/2 \) to \( x = L/2 \), and \( z = 0 \) to \( z = H \) as shown in figure 2.3 and with all solid boundaries being free-slip and zero flux, and we constrain the barotropic flow rate \( \int_0^H u \, dz = 0 \). Using the result from the leading order asymptotic solution that \( \rho \) and \( \zeta \) are linear in \( x \) (see Cormack et al., 1974), we can eliminate pressure from (2.12) and (2.13) and integrate with free-slip top and bottom boundaries to give

\[ u = \frac{Gr_T K_v}{24L} (-4z'^3 + 6z'^2 - 1), \]  

(2.14)

where \( z' = z/H \) is the dimensionless height. The dimensional volume flux in either direction is then

\[ Q_v = \left| BH \int_0^{1/2} u \, dz' \right|, \]

\[ Q_v = \frac{5}{384} Gr_T AB K_v. \]  

(2.15)

To find the net mass flux, we need an equation for density which simply balances vertical turbulent diffusion and horizontal advection (Officer, 1976),

\[ \frac{\partial^2 \rho}{\partial z^2} = -\frac{u \partial \rho}{K_v \partial x}. \]  

(2.16)

Using zero-flux boundary conditions at top and bottom and (2.14), we integrate to get

\[ \rho' = x' \Delta \rho + \frac{Gr_T Pr_T A^2 \Delta \rho}{240} (-2z'^6 + 5z'^4 - 5z'^2), \]  

(2.17)

where \( x' = x/L \) is the dimensionless length coordinate and \( \rho' = \rho - \rho_0 \) is the perturbation density. The second term on the right-hand side of (2.17) describes a constant vertical density gradient profile for all \( x' \), while the first term represents the linear variation of density with \( x' \). The net flux of mass through the contraction is therefore due to both horizontal diffusion and the advection of density by the velocity field (2.14),

\[ M_v = \left| BH \int_0^1 \frac{\partial \rho'}{\partial x} K_v + \rho' u \, dz' \right| \]

\[ M_v = \frac{A}{Pr_T} BK_v \Delta \rho + \frac{31}{362880} Gr_T^2 A^3 Pr_T BK_v \Delta \rho. \]  

(2.18)
This is the solution that we call the VAD solution and has been verified in the laboratory (Imberger, 1974) and using field data (Burling et al., 1999). In this chapter, we are not concerned with verifying behaviour in this limit but rather to compare the VAD solution with the hydraulic solution and investigate the parameter range over which each solution can be considered valid.

### 2.2.3 Comparison of the hydraulic and VAD limit

The two limits we have examined above are inherently difficult to compare, as they apply in different circumstances and scale with different parameters. The hydraulic solution requires a channel which varies slowly in width, has a simple contraction, and negligible mixing. The velocity scale is the long-wave speed and so the Froude number arises naturally in hydraulic analysis. The VAD solution requires constant, strong mixing with low aspect ratio, so that a natural scaling of variables is based on the turbulent eddy viscosity, and hence it is usual to use a turbulent Reynolds number. The two solutions can be compared, however, by their prediction of volume and mass flux through the channel, provided we scale the fluxes in each limit in the same way. We choose to nondimensionalise using the hydraulic scaling, so that flux for the maximal hydraulic solution with \( q_r = 1 \) is as given in (2.10) and (2.11). The VAD flux predictions (2.15) and (2.18) can be nondimensionalised by the same factor as the hydraulic fluxes to give

\[
q_v = \frac{5(GrTA^2)^{1/2}}{96},
\]

\[
m_v = \frac{4A^2}{(GrTA^2)^{1/2}Pr_T} + \frac{31(GrTA^2)^{3/2}Pr_T}{90720}.
\]

The flux due to advection near the VAD limit is therefore a function of the naturally arising parameter \( GrTA^2 \); however, the mass flux is also modulated by the effect of horizontal diffusion when \( GrTA^2 \) becomes very small. Using (2.10)–(2.11) and (2.19)–(2.20), we can plot both hydraulic and VAD fluxes on the one graph (figure 2.4) against the factor \( GrTA^2 \), provided we assume \( Pr_T = 1 \). Note that we show curves for two different values of \( A \) (0.25 and 0.0625) to plot (2.20).

Using figure 2.4, we can make some predictions about the behaviour of this flow. For large \( GrTA^2 \) the flux should approach the hydraulic solution, and figure 2.4 implies if \( GrTA^2 \) is small we expect the flux to obey the VAD prediction. However, in the region of transition between the two limits we have insufficient information to be able to predict the dependence of flux upon \( GrTA^2 \). We expect the dynamics to be controlled by a three-way force balance with inertia, buoyancy and viscous/diffusive effects all playing significant roles. For example, while
2.3 Numerical experiments

Exchange flow is simulated numerically using a code called S-FIT (Winters et al., 2000). The equations of motion

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0} \nabla p - \frac{\gamma g \rho'}{\rho_0} + \nabla \cdot K_p \nabla \mathbf{u},
\]

(2.21)

\[
\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho' = \nabla \cdot K_p \nabla \rho',
\]

(2.22)

\[
\nabla \cdot \mathbf{u} = 0,
\]

(2.23)
where \( \mathbf{u} = (u, v, w) \) is the velocity vector, are solved over a curvilinear domain which can be distorted in one dimension. The facility enabling equations to be solved in curvilinear coordinates allows us to realistically model flow through a contraction (Winters & Seim, 2000), with a domain as shown in plan view in figure 2.5.

The standard implementation allows either direct numerical simulations or large eddy simulations using a Smagorinsky closure scheme to relate mixing parameters to resolved properties such as shear and stratification. How well such schemes work depends on both grid resolution and flow conditions, though large eddy simulations are run under the assumption that resolution is high enough that details do not matter. In principle, one must defend the ability of some scheme to relate outer processes to mixing. Here we do not attempt to do this. Rather, we use the turbulent mixing as an independent, albeit artificial parameter which is varied in each experiment. The philosophy here is that one power of numerical investigation is to be able to model flows which are not physically realistic (e.g., Jiménez & Pinelli, 1999). Using mixing as a free parameter we are able to simulate the effect of turbulence caused by an unspecified source, for example, wind stress, whereas such an experiment is more difficult to control in the laboratory or the field. Therefore we use the closure scheme to determine the dynamic eddy viscosity and diffusivity (note that \( PrT = 1 \) in all cases, so that \( K_p = K_\nu \)), and in regions of the domain where we want to amplify mixing, we can add an extra constant value \( K' \), so that the total viscosity or diffusivity is given by

\[
K = K_p + K'.
\] (2.24)

![Figure 2.5: Graph of channel shape and curvilinear coordinates in plan view.](image)
We choose a channel of arbitrary length $l$ and height $H$ with the channel shape as shown in figure 2.5. There is a contracting region of length $L = l/3$, which has a minimum in width at $x = 0$ and the channel width exponentially decays toward the maximum width outside the contracting region. The boundary conditions are free-slip on the top, bottom, and side walls with a rigid lid on the top surface. We specify the pressure at the top of the channel at each end and adjust the pressure difference between the ends to force an exchange flow with $q_r \approx 1$ (zero barotropic flow). While we calculate full hydrodynamic pressure throughout the domain, pressure is close to hydrostatic near the open boundaries. We impose hydrostatic pressure at these boundaries, and the viscosity and diffusivity is increased near the end regions to inhibit reflection of internal waves.

In each experiment we vary only the aspect ratio $A = H/L$ and the added eddy viscosity/diffusivity $K'$ in the central contracting region and thereby simulate flows with a range of values of $Gr_r A^2$. The extra diffusivity/viscosity is nonzero only in the contraction region, so that VAD flows and hydraulic flows will have the same horizontal length scale $L$. Using this technique, we can vary $Gr_r A^2$ from 30 through to $5 \times 10^5$. Each experiment is run to a steady state, at which time we compare the volume flux, mass flux, and velocity and density fields with the two limiting solutions.

<table>
<thead>
<tr>
<th>Run</th>
<th>$A$</th>
<th>$K$ (m$^2$/s)</th>
<th>$Gr_r A^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.25</td>
<td>$7.4 \times 10^{-4}$</td>
<td>$5.4 \times 10^5$</td>
</tr>
<tr>
<td>B</td>
<td>0.25</td>
<td>$2.3 \times 10^{-3}$</td>
<td>$5.6 \times 10^4$</td>
</tr>
<tr>
<td>C</td>
<td>0.25</td>
<td>$5.0 \times 10^{-3}$</td>
<td>$1.2 \times 10^4$</td>
</tr>
<tr>
<td>D</td>
<td>0.25</td>
<td>$9.7 \times 10^{-3}$</td>
<td>$3.2 \times 10^3$</td>
</tr>
<tr>
<td>E</td>
<td>0.25</td>
<td>$1.9 \times 10^{-2}$</td>
<td>$8.2 \times 10^2$</td>
</tr>
<tr>
<td>F</td>
<td>0.25</td>
<td>$4.8 \times 10^{-2}$</td>
<td>$1.3 \times 10^2$</td>
</tr>
<tr>
<td>G</td>
<td>0.25</td>
<td>$9.8 \times 10^{-2}$</td>
<td>$3.1 \times 10^1$</td>
</tr>
<tr>
<td>H</td>
<td>0.0625</td>
<td>$3.3 \times 10^{-4}$</td>
<td>$2.6 \times 10^3$</td>
</tr>
<tr>
<td>I</td>
<td>0.0625</td>
<td>$6.6 \times 10^{-4}$</td>
<td>$6.7 \times 10^2$</td>
</tr>
<tr>
<td>J</td>
<td>0.0625</td>
<td>$1.0 \times 10^{-3}$</td>
<td>$2.6 \times 10^2$</td>
</tr>
<tr>
<td>K</td>
<td>0.0625</td>
<td>$2.0 \times 10^{-3}$</td>
<td>$7.4 \times 10^1$</td>
</tr>
</tbody>
</table>

Table 2.1: Summary of numerical experiments.
Chapter 2. Hydraulics and mixing in controlled exchange flows

2.4 Results

Results from 11 numerical experiments are summarised in table 2.1. Two different aspect ratios (0.25 and 0.0625) and a range of eddy viscosities were used covering a wide spectrum of flows from close to the hydraulic limit to the VAD limit. The flow structure in four representative experiments which span the range which was modelled can be seen in figure 2.6 which shows isopycnals and velocity vectors in the contraction region, with figure 2.6(a) close to the hydraulic solution (case A), and figure 2.6(d) close to the VAD limit (case G). The notable features

Figure 2.6: Comparison of flow dynamics with varying mixing, $A = 0.25$. (a) $Gr_T A^2 = 5.4 \times 10^5$; (b) $Gr_T A^2 = 1.2 \times 10^4$; (c) $Gr_T A^2 = 8.2 \times 10^2$; (d) $Gr_T A^2 = 3.1 \times 10^1$. 
in figure 2.6(a) are that the density interface is very thin and overlies the velocity interface. The thin interface separates two definite layers, in which velocity and density are constant as predicted by hydraulic theory. The one difference from the hydraulic prediction is that the interfacial region has a finite thickness; however, in this case the interface is so thin that it is unlikely to alter the flux significantly. The opposite end of the spectrum is shown in figure 2.6(d), with a near-horizontal velocity interface, and density gradient profiles which are constant in $x'$ as in the solution for the VAD limit. While cases A and G illustrate the limits of our problem, it is the intermediate cases which we are primarily concerned with in this study.

Two intermediate flows are shown in the central panels in figure 2.6 (cases C and E) and illustrate features of exchange flows which differ from both the hydraulic and the VAD solution. In both cases we can identify a general layer structure from the velocity field and the density field, with the lower layer flowing to the left and the upper layer flowing rightwards. Both layers accelerate as they pass through the contraction, a feature which is consistent with a transition to supercritical conditions at the throat. However, as demonstrated by Winters & Seim (2000), the layer structure differs from the hydraulically controlled solution in two ways. First, there is a region of finite thickness dividing the two layers where velocities are low, and this region thickens as mixing increases. Second, the position of zero velocity no longer matches the middensity isopycnal: streamlines are crossing isopycnals due to the mixing.

We can consider the effect of finite-thickness interfacial layers by taking the hydraulic solution and inserting a layer of thickness $h_t$ (nondimensionalised by $H$) in which velocity and density gradients are linear. We assume that the thickness $h_t$ will increase with mixing, and hence will scale with diffusivity and viscosity, and the timescale $T$ which the fluid spends in the contraction, 

$$h_t \sim \frac{(TK)^{1/2}}{H}. \quad (2.25)$$

The timescale $T$ can be evaluated from the length of the contracting region and the velocity of the fluid, giving 

$$T \sim \frac{L}{(g'H)^{1/2}}, \quad (2.26)$$

so that 

$$h_t \sim (GrTA^2)^{-1/4}. \quad (2.27)$$

Here we see that, even for cases close to the hydraulic limit, the natural parameter which describes the effect of mixing is again $GrTA^2$. From (2.27) we make
the assumption that interface thickness depends only upon this parameter and is given by

\[ h_t = \alpha (GrT A^2)^{-1/4}. \]  

(2.28)

Since velocity and density gradients in the interfacial layer are assumed linear, we can integrate idealised finite-thickness interface layer profiles to estimate the volume and mass flux of exchange as a function of \( GrT A^2 \) in the regime where neither the VAD nor the hydraulic limit apply,

\[ q = (1 - \alpha (GrT A^2)^{-1/4}/2), \]  

(2.29)

\[ m = (1 - 2\alpha (GrT A^2)^{-1/4}/3). \]  

(2.30)

If we constrain the three-layer prediction of volume flux \( (q) \) to match the VAD prediction \( (q_V \) in (2.19)) and also require that the first derivatives of each of these functions are equal at the same point, then we find that the solutions intersect when \( GrT A^2 \approx 40 \) and that the coefficient in (2.28) is \( \alpha \approx 3.4 \).

We measure the nondimensional interface thickness \( h_t \) from the density profile at the contraction and plot this parameter against \( (GrT A^2)^{-1/4} \) for all 11 runs in figure 2.7. The interface thickness is defined to be the vertical distance between points where \( \rho = \rho_0 - 0.4\Delta \rho \) and \( \rho = \rho_0 + 0.4\Delta \rho \) at the throat of the contraction (the vertical distance between the highest and lowest isopycnals on figure 2.6), with the condition that \( 0 \leq h_t \leq 1 \). Figure 2.7 strongly supports the scaling argument of (2.28) and demonstrates that a value of \( \alpha \approx 3.4 \) leads to an accurate prediction of interface thickness.

Equations (2.29) and (2.30) predict that flux will be decreased by mixing, a result which is consistent with previous work on this problem. For example, Winters & Seim (2000) show how entrainment between the two layers causes recirculation within the channel, thereby reducing the total volume flux, and Helfrich (1995) estimates the reduction in mass flux due to mixing in one case. However, (2.29) and (2.30) give an indication that the reduction in flux may be able to be quantified in terms of a dimensionless parameter. In figure 2.8 we plot the limiting predictions for flux ((2.10),(2.11),(2.19) and (2.20); dash-dot lines), the finite-thickness interface model prediction ((2.29) and (2.30); solid line) and the measured fluxes from all numerical experiments against the parameter \( GrT A^2 \).

The open circles show the measured fluxes from experiments with \( A = 0.25 \), and the crosses are fluxes from experiments with \( A = 0.0625 \). The relationship between flux and \( GrT A^2 \) is of primary importance in figure 2.8. First, we note that the reduction in mass flux from the hydraulic limit for a given value of \( GrT A^2 \) is greater than the variation in volume flux, as reflected
2.5. Discussion and conclusions

2.5.1 Transition between hydraulic and VAD limits

The flux of volume and mass in exchange flows is dependent on turbulent mixing. While the behaviour of this type of flow is complicated, it is encouraging to note that we have been able to demonstrate that for flows with a turbulent Prandtl number of unity, the flux scales with a single nondimensional parameter $Gr_T A^2$. However, several qualifying remarks must be made about this scaling. The relationship derived in (2.29) and (2.30) was tested for exchange flows where all parameters were constant except for $K$ and $H$. We have not investigated the sensitivity of the flow to other parameters. Some parameters which do not

in (2.29) and (2.30). However, for $Gr_T A^2 \gtrsim 10^5$, both mass and volume flux are within approximately 10% of the hydraulic prediction. Second, if $Gr_T A^2 \lesssim 40$, the VAD solution is a good predictor of flux. Third, for $40 \lesssim Gr_T A^2 \lesssim 10^5$ flux in the intermediate regime follows a smooth monotonic curve which is well predicted by the scaling argument which leads to (2.29) and (2.30).

Figure 2.7: Variation of interface thickness $h_i$ with mixing.
affect either of the limiting solutions, but may change behaviour in the intermediate regime, include the shape of the channel and the density difference between the reservoirs $\Delta \rho$. Given the lack of physical understanding of the intermediate regime, we cannot predict how channel shape or density difference may alter the problem in this regime.

We have formulated a conceptual framework suitable for a wide class of problems in between the limiting cases of the hydraulic solution and the VAD solution. The two limits have analytical solutions based on two-term force balances. It follows that the regime between these limits is characterised by a three-way balance between viscosity, inertia, and buoyancy. The mathematical analysis of flows in this regime is complicated by the feedback between wave speeds and viscosity, diffusivity, and buoyancy. Wave speeds are decreased by viscosity and diffusivity and are also affected by gradients in velocity and density which result from mix-

![Diagram](image)

Figure 2.8: Comparison of hydraulic and VAD solution predictions with numerical calculation. (a) Volume flux; (b) net mass flux.
2.5. Discussion and conclusions

Thus we propose that the relationship between the three controlling forces is highly nonlinear and that currently we can only use empirical methods to predict flux as a function of $Gr_T A^2$. The scaling (2.27) can be used to show that interface thickness, and therefore flux, depends on $Gr_T A^2$. The numerical results demonstrate that a linear relationship between $(Gr_T A^2)^{-1/4}$ and interface thickness $h_t$ is a reasonable approximation, at least for flows characterised by constant eddy viscosity and diffusivity. Using the finite-thickness interface model, and requiring that predictions of volume flux smoothly intersect with the VAD solution, we are able to predict the flux through the contraction as a function of $Gr_T A^2$.

2.5.2 Relevance to laboratory testing

One result from the numerical simulations deserves further consideration. Laboratory experiments designed to determine the accuracy of the hydraulic prediction (Wood, 1970; Armi & Farmer, 1986; Lawrence, 1993) measured the variation in layer heights with $x$. In these experiments the interface is usually located using dye visualisation, so it is reasonable to assume that such a method would locate the mid-density isopycncal. However, we can show that while the interface thickens, the mid-density isopycncal retains the shape of the interface described by the hydraulic solution. The mid-density isopycncal for seven different runs is illustrated in figure 2.9, showing that there is no discernible change to the layer heights in

Figure 2.9: Comparison of contour of mid-density for a range of runs with $A = 0.25$, with values of $Gr_T A^2$ as shown.
the contraction region until the flow is altered so much that the middensity con­
tour intersects that floor and roof of the tank. In fact, we can reduce the mass flux to 50% of the hydraulic prediction before we see a change in the middensity isopycnal. This result confirms the necessity of measuring fluxes and velocities to
determine whether a flow is close to the hydraulic prediction or not.

2.5.3 Application to geophysical examples

One of the objectives of this work is to determine the suitability of the hy­
draulic and VAD limits to analysing field data. To compare field observations with the numerical results presented, we select a number of geophysical exchange flows and determine an approximate position for each flow on the $Gr_T A^2$ range. We do this in two ways, first, with published field measurements of $L$, $H$, $\Delta \rho$, and $K_p$ (with the assumption that $Pr_T = 1$) to directly compute $Gr_T A^2$. Second, we use density and velocity transects to estimate the average interface thickness $h_t$, which is then correlated with numerical data to estimate $Gr_T A^2$.

The first of these two options will be more accurate if data are available; how­ever, the selection of scale parameters to determine $Gr_T$ and $A$ is not straight­forward. In the numerical experiments, $K$ was roughly constant; however, in any field situation, diffusivity/viscosity is likely to be highly variable both spatially and temporally. In addition, our experiments were designed so that the same length scale was used for the contraction and the mixed region, making it easy to define height and length scales. In field situations, it is likely that both $H$ and $L$ will be difficult to quantify precisely. The second method is less accurate, but it is easier to measure the interface thickness than parameters such as eddy viscosity. We will use both methods to determine the class of behaviour of some geophysical flows.

<table>
<thead>
<tr>
<th>Location</th>
<th>$A$</th>
<th>$\Delta \rho$ (kg/m$^3$)</th>
<th>$K_p$ (m$^2$/s)</th>
<th>$Gr_T A^2$</th>
<th>$h_t$</th>
<th>$Gr_T A^2$</th>
</tr>
</thead>
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<tr>
<td>Shark Bay</td>
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<td>10</td>
<td>$10^{-3}$</td>
<td>$10^{-1}$</td>
<td>1.0</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>Gibraltar</td>
<td>$8 \times 10^{-3}$</td>
<td>3.5</td>
<td>$10^{-2}$</td>
<td>$10^6$</td>
<td>0.15</td>
<td>$10^5$</td>
</tr>
<tr>
<td>Bosphorus</td>
<td>$1 \times 10^{-3}$</td>
<td>16</td>
<td></td>
<td>0.35</td>
<td>$5 \times 10^3$</td>
<td></td>
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<tr>
<td>Burlington</td>
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<td>2</td>
<td></td>
<td>0.25</td>
<td>$3 \times 10^4$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2: Estimates of $Gr_T A^2$ from field measurements. The quantity $Gr_T A^2$ is calculated by two methods here. The first method uses field estimates of parameters to directly calculate $Gr_T A^2$. The second method uses an estimate of the average interface thickness and direct comparison with the numerical simulations to estimate $Gr_T A^2$. 
Table 2.2 shows field data from three exchange flow sites. Some of these (Gibraltar and Shark Bay) involve flow over a sill combined with flow through a contraction. While the sill may produce a slightly different flow from those modelled, it is not unreasonable to expect mixing will alter the exchange in a similar way and in fact the properties of the interfacial layer in Gibraltar (Bray et al., 1995) are remarkably similar to those described by Winters & Seim (2000) for flow through a contraction. In addition to complex topography, many natural flows are influenced by time-dependent features, such as tidal oscillations, which are not considered in this study.

Table 2.2 illustrates that there is a large range of natural behaviours of channel flow, spanning from the VAD to the hydraulic limit. A clear example of geophysical flow in the VAD limit is the Herald Loop Channel in Shark Bay, Western Australia. Burling et al. (1999) present field data from this site where there is a channel with very small aspect ratio which is well mixed by strong winds. Using the same solution as we have used for the VAD solution, Burling et al. (1999) conclude that this flow lies inside the regime where turbulent viscosity and diffusion controls the flux. From the data presented by Burling et al. (1999) (with an approximate estimate of $K_p$), we estimate $Gr_TA^2 \approx 10^{-1}$ in this case. Conversely, the Strait of Gibraltar gives rise to tidally modified exchange flow where three distinct layers can be defined (Bray et al., 1995). Using measurements of $K_p$ at the Carmarinal Sill (Wesson & Gregg, 1994), we estimate that $Gr_TA^2 \approx 10^5$. This estimate is supported by observations of an 80 m thick interface (Bray et al., 1995), giving $h_t \approx 0.15$. From direct comparison with numerical simulations (2.28), we can estimate $Gr_TA^2 \approx 10^8$ using the second method of calculating the mixing parameter. The two methods of calculating $Gr_TA^2$ are consistent to within an order of magnitude and demonstrate that the flow through the Straits of Gibraltar may be well approximated by the hydraulic solution.

The Bosphorus Strait is the same length as Gibraltar but is roughly one tenth of the depth and is therefore much more likely to lie away from the hydraulic limit. Whereas we have no field estimates of $K_p$ from this site, we do have density and velocity transects (Gregg et al., 1999). Using an estimate of $h_t \approx 0.35$, we find from (2.28) $Gr_TA^2 \approx 5 \times 10^3$. Simulations of this section of the $Gr_TA^2$ range showed flows which had a density interface slope similar to the hydraulic solution but with fluxes only 60-70% of the hydraulic prediction.

Using the same method, we are able to estimate $Gr_TA^2$ for the Burlington Ship Canal, a channel between Hamilton Harbour and the main body of Lake Ontario. Available field data (Greco, 1998) allows us to estimate the strength of
exchange through the canal which is forced in the summer by cooler conditions in the main body of Lake Ontario. The average interface thickness is estimated at 0.25, leading to a value of $GrTA^2 \approx 3 \times 10^4$, which indicates flux though the canal may be approximately 80% of the hydraulic prediction.

Figure 2.10 shows estimates of the position of the Bosphorus, Burlington Ship Canal, and Gibraltar on the $GrTA^2$ scale, with the net mass flux plotted on the vertical axis. The field data shown here illustrates the range of behaviour of geophysical exchange flows and suggests that geophysical flows may lie over a wide range along the $GrTA^2$ parameter space. While the numerical simulations outlined here go some way toward empirically predicting flux in exchange flows, we do not yet have a comprehensive physical understanding of behaviour in this regime. There is a good case to support the possibility that wave speeds may play a controlling role in part of the intermediate regime, in a similar way to the long waves which control flux in the hydraulic limit. It is possible that an analysis of the waves which can propagate in the flows investigated here may be used to develop a theory which can predict exchange flows from external parameters.
Linear internal waves and the control of stratified exchange flows

Abstract

Internal hydraulic theory is often used to describe idealised bi-directional exchange flow through a constricted channel. This approach is formally applicable to layered flows in which velocity and density are represented by discontinuous functions that are constant within discrete layers. The theory relies on the determination of flow conditions at points of hydraulic control, where long interfacial waves have zero phase speed. In this chapter, we consider hydraulic control in continuously stratified exchange flows. Such flows occur, for example, in channels connecting stratified reservoirs and between homogeneous basins when interfacial mixing is significant. Our focus here is on the propagation characteristics of the gravest vertical mode internal waves within a laterally contracting channel.

Two approaches are used to determine the behaviour of waves propagating through a steady, continuously sheared and stratified exchange flow. In the first, waves are mechanically excited at discrete locations within a numerically simulated bi-directional exchange flow and allowed to evolve under linear dynamics. These waves are then tracked in space and time to determine propagation speeds. A second approach, based on the stability theory of parallel shear flows and examination of solutions to a sixth-order eigenvalue problem, is used to interpret the direct excitation experiments. Two types of gravest mode eigensolutions are identified: vorticity modes, with eigenfunction maxima centred above and below the region of maximum density gradient, and density modes with maxima centred on the strongly stratified layer. Density modes have phase speeds that change sign within the channel and are analogous to the interfacial waves in hydraulic theory. Vorticity modes have finite propagation speed throughout the channel but undergo a transition in form: upwind of the transition point the vorticity mode is trapped in one layer. It is argued that modes trapped in one layer are not capable of communicating interfacial information, and therefore that the transition points are analogous to control points. The location of transition points are identified and used to generalise the notion of hydraulic control in continuously stratified flows.

3.1 Introduction

Internal hydraulic theory (see, for example, Wood, 1968, 1970; Armi, 1986) can be used to describe density-driven flows in which fluid motion is determined
by a buoyancy and inertia force balance. In particular, inviscid flow through constricted channels is the major application of hydraulic theory. If mixing and friction are negligible, the hydraulic solution can be used to calculate fluid velocity in distinct layers which interact in only a limited way. The velocities scale with the speed of interfacial waves and depend upon the depth of the interface.

The key to understanding hydraulic flows lies in the existence of control points. At a control point, the flow is constrained in that velocities are such that the phase speed of one of two interfacial waves is zero. The existence of control points has two major ramifications. First, given knowledge of the flow at the control points, it is possible to calculate global information about the flow field within a simple domain, provided the fundamental assumptions are correct. Second, since velocities are limited by internal wave speeds at the control points, hydraulic theory places an upper bound on the total flux through the channel, the quantity of paramount importance to estimating transport in geophysical systems.

It is often difficult to determine whether an exchange flow between two basins is hydraulically controlled. One reason for this is that hydraulic control is formally defined for layered flow, that is, horizontal flow with uniform velocity and density within discrete layers and discontinuities in these quantities between layers. In contrast, observed flows are continuously sheared and stratified. Though large gradients in velocity and density are often present, these finite-thickness interfaces may be displaced vertically from one another, making an approximate decomposition into layers difficult and subjective. Unfortunately, using the results of hydraulic theory to determine whether such flows are controlled depends sensitively on how one defines the layers (see, for example, Gregg & Örsoy, 2001; Pratt et al., 1999). Furthermore, flux estimates based on layered hydraulic solutions are often significant overestimates of the true flux in continuously stratified flows (chapter 2; Winters & Seim, 2000).

Our objective here is to extend the notion of hydraulic control to continuously stratified flows in such a way that determining whether a flow is controlled and locating the position of control points can be done in an objective manner. To simplify the presentation we consider in detail the limited class of flows resulting from a classical lock-exchange problem for a fluid with non-negligible viscosity and diffusivity. In particular, we consider exchange flow through a laterally contracting channel separating two infinite basins filled initially with homogeneous fluid of different densities. For channels with slowly varying width, the solution in the inviscid, non-diffusive limit can be obtained using two-layer hydraulic theory as described in Armi (1986) and Lawrence (1990). When mixing and dissipation are
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significant, a finite-thickness interfacial layer is produced with the zero-isotach displaced from the position of the maximum density gradient.

Before considering the continuously stratified case, we briefly review the concept of hydraulic control in the idealised two-layer theory. Consider a flat-bottomed channel of length $L$ and depth $H$, with a simple minimum width $W$ as shown in figure 3.1(a), where spatial variables have already been non-dimensionalised by the size of the domain. At the left-hand end of the channel is an infinite reservoir of density $\rho_1$, and at the right-hand end of the channel is a similar reservoir with a higher density $\rho_2$. Assuming that viscosity and mixing are negligible, and that the width of the channel varies slowly with horizontal distance, fluid velocities in two discrete layers will be constant within the layer, and approximately horizontal. Under these assumptions, a simple model based on conservation of mass within each of the two layers in combination with conservation of energy (Bernoulli’s equation) captures the nonlinear dynamics of exchange flow in the system (see, for example Wood, 1970; Armi, 1986; Armi & Farmer, 1986; Lawrence, 1990; Baines, 1995). The resulting flow is shown in figure 3.1(b).

Points of control occur when the composite Froude number,

$$G^2 = u_1^2/h_1 + u_2^2/h_2 - (1 - r)u_1^2u_2^2/h_1h_2 = 1,$$  \hspace{1cm} (3.1)

where $h_i$ is the dimensionless depth of each layer (non-dimensionalised by $H$), and $u_i$ the dimensionless velocity (normalised by the inviscid wave speed $(g'H)^{1/2}$). The ratio of layer densities is $r \equiv \rho_1/\rho_2$, and $g' = g(1 - r)$ is the reduced gravity. As shown by the variation of $G^2$ with $x$ in figure 3.1(c), in the case where the channel has a simple width minimum and a controlled flow has a net barotropic component (defined by the ratio of layer fluxes $q_r = q_1/q_2$), there are two points of control. A topographic control is located at the width minimum ($x = 0$ in figure 3.1) and a virtual control is located upstream of the width minimum with respect to the barotropic flow direction ($x \approx -0.13$). Between these two points, the flow is said to be subcritical while the flow between the reservoirs and the control points is said to be supercritical. We are ignoring here the possibility of hydraulic jumps as they do not formally appear in the theory.

It can be shown (Dalziel, 1991; Baines, 1995) that for a two-layer flow the hydraulic control condition can be written

$$G^2 = 1 + \frac{c_1c_2}{h_1h_2} = 1,$$ \hspace{1cm} (3.2)

where $c_1$ and $c_2$ are the phase speeds of the two interfacial waves non-dimensionalised by $(g'H)^{1/2}$. In a quiescent fluid, we expect two interfacial waves to exist:
Figure 3.1: Characteristics of the hydraulic solution with \( q_r = 2 \). (a) Channel in plan view; (b) layer height and velocity vectors in elevation view; (c) variation of the composite Froude number \( G^2 \); (d) plot of wave speeds.

one travelling to the right, and one to the left. When the layers flow in opposite directions, (3.2) implies that at least one wave is arrested at a point of hydraulic
control. This is demonstrated in figure 3.1(d), where normalised waves speeds are plotted as a function of $x$. In subcritical flow, $G^2 < 1$ and the two waves travel in opposite directions. In supercritical flow, $G^2 > 1$ and both waves propagate in the same direction, away from the contraction and toward one of the reservoirs. A central region of subcritical flow is said to be insulated from interface disturbances in either basin in the sense that such changes cannot be communicated to the subcritical region via long interfacial internal waves. In this way, changes can occur in the reservoirs which do not alter the flux through the channel.

The two long-wave speeds coalesce downstream of the contraction (relative to the fastest flowing layer). Mathematically, this means that the wave speeds are complex, implying that the modes are unstable there. This aspect of the hydraulic solution is described by Lawrence (1990) who introduces a stability Froude number. The implication is that hydraulic theory is not self-consistent: instabilities will be generated in part of the domain leading to mixing of density and momentum between the two layers which violates the fundamental assumptions. Nonetheless, numerical simulations have demonstrated that weak damping suppresses long-wave instabilities (Winters & Seim, 2000), and that the hydraulic solution is a good approximation to modelled flows in the limit of weak damping (chapter 2).

Hydraulic control of a multi-layered exchange flow is investigated by Engqvist (1996). By defining two groups of layers, each group flowing in the opposite direction, with a stagnant layer in between, it is possible to calculate the position of a topographic control and a number of virtual controls. An extra virtual control is formed for each extra layer and the virtual controls refer to points at which the higher vertical modes become critical. In this case the flow is still controlled by the lowest mode. The multi-layered technique is useful for conditions where end reservoirs are stratified, but the condition that groups of layers are decoupled prevents the application of this technique to flows where mixing is significant.

In this chapter we consider the propagation of internal waves in a flow similar to the two-layer solution discussed above but where mixing and dissipation have acted to smooth the discontinuities between layers producing a continuously sheared and stratified exchange flow. A steady-state exchange flow is generated through direct simulation of a lock-exchange problem with finite mixing and dissipation. The simulation is run to a quasi-steady state, at which point it is averaged with respect to the cross-channel coordinate to produce a background flow field $\bar{u}(x, z)$, $\bar{w}(x, z)$ and $\bar{p}(x, z)$. Two complementary techniques are then used to quantify the propagation characteristics of linear internal waves through
Chapter 3. Linear internal waves in exchange flows

3.2 Numerical experiments

3.2.1 Numerical simulation of exchange flows

The numerical experiments in this study are based on a three-dimensional model called S-FIT (Winters et al., 2000) which solves the fluid equations of motion in a curvilinear coordinate system. The coordinates are orthogonal, however the boundary can be distorted in one dimension as shown in figure 3.2 to simulate flow down a channel of varying width. The domain shown in figure 3.2 is used for all numerical experiments described here.

S-FIT solves the following equations:

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0} \nabla p - \frac{\hat{z} \rho g}{\rho_0} + \nabla \cdot K \nabla \mathbf{u}, \]  

\[ \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \nabla \cdot K \nabla \rho, \]  

\[ \nabla \cdot \mathbf{u} = 0, \]
where $\mathbf{u} = (u, v, w)$ is the velocity vector, $\rho$ is the density and $p$ the pressure. In addition, we define reference density $\rho_0$, a turbulent eddy viscosity $K$, and assume a turbulent Prandtl number of one, so that $K$ also represents turbulent eddy diffusivity. Note that $K$ is dimensionless (normalised by $(g' H^3)^{1/2}$). Usually, turbulent parameters such as eddy viscosity and eddy diffusivity are calculated explicitly by a closure scheme (see Winters & Seim, 2000). We represent the effect of such mixing only crudely here and prescribe a constant eddy viscosity/diffusivity throughout the contraction region, as discussed in chapter 2. The purpose here is to generate a continuously sheared and stratified flow similar to that expected if strong mixing were to occur.

The flows simulated are identical to the flow detailed in chapter 2. The surface is a fixed lid, with free-slip conditions on upper, lower and sidewall boundaries. The streamwise end conditions are that inflowing density is specified (representing infinite homogeneous reservoirs), and a surface pressure difference across the length of the channel is imposed. The channel is $120 \times 10 \times 10$ m, and has $129 \times 17 \times 65$ gridpoints. The resulting base flow, for $K \approx 10^{-3}$ and $q_r \approx 1$ can be seen in figure 3.3 with velocity vectors and selected isopycnals shown.

![Figure 3.3: Numerical simulation of exchange flow showing isopycnals and velocity vectors in elevation view.](image-url)
3.2.2 Direct excitation experiments

We use a slightly adapted version of S-FIT to look at the evolution of linear perturbations in a steady-state flow. Starting with (3.3)–(3.5), assume that the steady-state solution satisfies these equations in our curvilinear domain. Each variable is then separated into two parts: a perturbation quantity (prime) and a background steady state (overbar), by writing

\[
 u(x, y, z, t) = \bar{u}(x, z) + u'(x, y, z, t), \tag{3.6}
\]

\[
 \rho(x, y, z, t) = \bar{\rho}(x, z) + \rho'(x, y, z, t), \tag{3.7}
\]

\[
 p(x, y, z, t) = \bar{p}(x, z) + p'(x, y, z, t). \tag{3.8}
\]

This is substituted into (3.3)–(3.5), to give

\[
 \frac{\partial u'}{\partial t} + u' \cdot \nabla u' + u' \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u' = -\frac{1}{\rho_0} \nabla p' - \frac{\dot{z} g \rho'}{\rho_0} + \nabla \cdot K \nabla u', \tag{3.9}
\]

\[
 \frac{\partial \rho'}{\partial t} + u' \cdot \nabla \bar{\rho} + \bar{u} \cdot \nabla \rho' + u' \cdot \nabla \rho' = \nabla \cdot K \nabla \rho', \tag{3.10}
\]

\[
 \nabla \cdot u' = 0, \tag{3.11}
\]

where we have removed background variables which independently satisfy (3.3)–(3.5). Since we are interested in the linear evolution of these perturbations, we remove the possibility of nonlinear interactions by eliminating double perturbation quantities, giving,

\[
 \frac{\partial u'}{\partial t} + u' \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u' = -\frac{1}{\rho_0} \nabla p' - \frac{\dot{z} g \rho'}{\rho_0} + \nabla \cdot K \nabla u', \tag{3.12}
\]

\[
 \frac{\partial \rho'}{\partial t} + u' \cdot \nabla \bar{\rho} + \bar{u} \cdot \nabla \rho' = \nabla \cdot K \nabla \rho', \tag{3.13}
\]

\[
 \nabla \cdot u' = 0. \tag{3.14}
\]

It is now apparent that (3.12)–(3.14) are almost identical to (3.3)–(3.5), with the exception of the cross-terms between the background state and the perturbation variables. Therefore, the existing code which solves (3.3)–(3.5) can be easily adapted to solve for the perturbation quantities in (3.12)–(3.14) using the background (steady) state as a set of input variables. The response to small perturbations introduced into the flow is then calculated using the adapted numerical model. The initial perturbations are typically confined in area, but slowly evolving in time, so as to simulate long waves in the domain.
3.2.3 Calculation of eigenmodes

Simulations of the evolution of perturbations provide some information on the structure of hydraulic control in stratified flows. However, perturbations will inevitably project onto a number of superposed travelling modes, and thus it is not possible to extract information about individual modes. Here we outline an analytical technique to separate the modes available in a turbulent stratified shear flow.

The propagation of waves in inviscid continuously stratified fluids is described by the Taylor-Goldstein equation (see, for example, Banks et al., 1976)

\[(\bar{u} - c)(\partial^2 - k^2)\hat{\psi} - \bar{u}_{zz}\hat{\psi} + \frac{N^2}{(\bar{u} - c)}\hat{\psi} = 0, \tag{3.15}\]

where \(\partial\) and \(z\) subscripts are used to represent the derivative with respect to \(z\). \(N^2 = -g/\rho_0(\partial\bar{\rho}/\partial z)\) is the buoyancy frequency which is non-dimensionalised by a time scale using the channel height and hydraulic velocity scale. The perturbation streamfunction \(\psi\) is based on the perturbation velocities, so that

\[u' = \frac{\partial \psi}{\partial z}, \tag{3.16}\]
\[w' = -\frac{\partial \psi}{\partial x}, \tag{3.17}\]

and waves are assumed to be of the form,

\[\psi(x, z, t) = \text{Re}[\hat{\psi}(z)e^{i(k(z-c)t)}]. \tag{3.18}\]

Therefore \(k\) is the horizontal wavenumber and \(c\) the horizontal phase speed. The Taylor-Goldstein equation describes the behaviour of waves in fluids with shear and stratification and the eigenvalue formulation produces the vertical modal structure (\(\hat{\psi}(z)\)) as well as the phase speed of waves (\(c\)).

The simplest approach to estimating the hydraulic state of a stratified sheared exchange flow would be to extract the vertical profiles of \(\bar{u}(z)\) and \(\bar{\rho}(z)\) from the background flow. The phase speed of the two lowest modes might then be determined by solving the Taylor-Goldstein equation many times for different profiles at different points in the channel. We would then interpret the flow to be critical at points where one of the two lowest modes has zero phase speed. The difficulty with this approach is that the Taylor–Goldstein equation becomes singular at levels where \(\bar{u} - c = 0\) (see Pratt et al., 2000, for a more complete description of this problem). Such levels are called critical layers (see, for example, Drazin & Reid, 1981), not to be confused with critical points in the context of
two-layer hydraulic theory. Near critical layers, the viscosity and diffusivity, which are neglected by the Taylor–Goldstein equation, play a greater role (Drazin & Reid, 1981). In the exchange flow considered here, viscosity and diffusivity play a significant role in determining the flow. This requires the use of a more generalised wave equation originally derived by Koppel (1964). This is the sixth-order viscous stability equation, which can be derived in the same way as the Taylor–Goldstein equation (see Appendix B for details), using (3.3)–(3.5). The stability equation is written as follows:

\[-c^2[\partial^2 - k^2]\psi + c[2i K_c(\partial^2 - k^2)^2 + 2\bar{u}(\partial^2 - k^2) - \bar{u}_{zz}]\psi + [K_c^2(\partial^2 - k^2)^3
\]

\[2i\bar{u}K_c(\partial^2 - k^2)^2 - 2i\bar{u}_z K_c \partial (\partial^2 - k^2) - \bar{u}^2(\partial^2 - k^2)
\]

\[+ 2i\bar{u}_{zzz} K_c \partial + \bar{u}_{zz} + i\bar{u}_{zzzz} K_c - N^2] \psi = 0, \quad (3.19)\]

where $K_c$ is defined as

\[K_c = \frac{K}{k}, \quad (3.20)\]

and is non-dimensionalised by the factor $(g'H^2)^{1/2}$. $K_c$ can be considered to be a ratio of the time taken for a wave to travel one wavelength, $1/(k(g'H)^{1/2})$, to the time for momentum to diffuse a distance $H$ vertically. As for the underlying simulations, we have assumed a turbulent Prandtl number of unity in all of these calculations. The sixth-order stability equation reduces to the Taylor–Goldstein equation if one removes the effect of viscosity and diffusion by setting $K_c = 0$. We can make an immediate simplification to (3.19) by assuming that we are interested in long waves (small wavenumber), so that $k^2\psi$ will be small compared to $\partial^2\psi$. Therefore we eliminate those terms, simplifying (3.19) to

\[-c^2[\partial^2 - k^2]\psi + c[2i K_c \partial^4 + 2\bar{u}\partial^2 - \bar{u}_{zz}]\psi + [K_c^2 \partial^6 - 2i\bar{u} K_c \partial^4 - 2i\bar{u}_z K_c \partial^3
\]

\[-\bar{u}^2\partial^2 + 2i\bar{u}_{zzz} K_c \partial + \bar{u}_{zz} + i\bar{u}_{zzzz} K_c - N^2] \psi = 0. \quad (3.21)\]

This problem is reduced to a matrix eigenvalue equation, following the procedure of Winters & Riley (1992) and solved numerically. Start by writing (3.21) as

\[[A_0 + cA_1 - c^2A_2]\psi = 0, \quad (3.22)\]

where

\[A_0 = [K_c^2 \partial^6 - 2i\bar{u} K_c \partial^4 - 2i\bar{u}_z K_c \partial^3 - \bar{u}^2\partial^2
\]

\[+ 2i\bar{u}_{zzz} K_c \partial + \bar{u}_{zz} + i\bar{u}_{zzzz} K_c - N^2], \quad (3.23)\]

\[A_1 = [2i K_c \partial^4 + 2\bar{u}\partial^2 - \bar{u}_{zz}], \quad (3.24)\]
3.2. Numerical experiments

\[ A_2 = [\partial^2]. \] (3.25)

Linearisation of (3.22) (following Winters & Riley, 1992) gives

\[
\begin{bmatrix}
A_1 & A_0 \\
A_2 & 0
\end{bmatrix}
\begin{bmatrix}
c\psi \\
\psi
\end{bmatrix}
- c
\begin{bmatrix}
A_2 & 0 \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
c\psi \\
\psi
\end{bmatrix} = 0.
\] (3.26)

This equation can be solved numerically using the complex analogue of the QZ algorithm (Golub & van Loan, 1983; Moler & Stewart, 1973).

The equation is solved using the same boundary conditions as the model, with free-slip rigid surfaces at top and bottom. This translates to

\[ \psi = 0, \quad z = 0, H, \] (3.27)

and

\[ \psi_{zz} = 0, \quad z = 0, H. \] (3.28)

In addition, there is an adiabatic condition on density which imposes the restriction

\[ 2\psi_x \bar{u}_{zzz} = 2(\bar{u} - c)\psi_{zzzz} + iK_c \psi_{zzzzzz}, \quad z = 0, H. \] (3.29)

Numerical solutions to (3.26) require vertical profiles of \( \bar{u} \) and \( N^2 \), and a constant value for \( K_c \). The vertical profiles are taken from the background steady state, with the implicit assumption being that wavelengths are small enough so that the background steady state does not change significantly over one wavelength. A value of \( K_c \) requires an estimate of both eddy viscosity and wavenumber. The eddy viscosity is already set in the numerical simulation, however the selection of wavenumber is not so straightforward. The derivation of (3.21) from (3.19) is dependent on long wavelengths, however there will be upper limits on the length of the wave for two reasons. First, if we choose infinitely long waves \( (\kappa \to 0) \), the parameter \( K_c \) will become very large indicating that momentum and density will diffuse so as to significantly alter the waveform over one wavelength. Secondly, the background flow we are investigating is changing with horizontal distance, and therefore we need to select a wavelength over which the flow does not change dramatically. The wavenumber is chosen so that wavelengths are 1/8th of the channel length, resulting in a value of \( K_c = 0.003 \). We have run additional cases examining the sensitivity to the viscosity parameter and found that the results are not altered by variations in \( K_c \) of an order of magnitude.

The solution to (3.26) will then give information about the modes which can exist, and the phase speed (the real part of \( c \)) and rates of growth or decay (the imaginary part of \( c \)) of those modes. The eigenvectors have a real and imaginary
part, and therefore we take the absolute value \(((\hat{\psi}^2)\)^{1/2}\) to obtain a simple eigenvector whose structure can be used to identify the lowest modes. The structure and phase speed of vertical mode-1 waves as a function of position in the channel is of primary concern, and will be used to characterise the behaviour of waves in the flow.

3.3 Results

3.3.1 Direct excitation experiments

Perturbations to the steady exchange shown in figure 3.3 are applied to a small area \((0.07L \times 0.08H)\) on the region of maximum density gradient. The density is perturbed within this region over a dimensionless time of 0.8. This disturbance is designed to stimulate a long mode-1 wave travelling on the density interface, however it is found that results are not particularly sensitive to the length of time, or area over which the perturbation is applied, provided that it is centred on the maximum density gradient.

Figure 3.4 demonstrates the evolution of a perturbation generated at \(x = -0.06\), just to the left of the contraction. The evolution is described by the density variation of the perturbation field on the mid-density isopycnal of the background flow. At any one time this produces the perturbation amplitude as a function of \(x\). Therefore, we combine a number of perturbation amplitude snapshots taken at different times to construct an \(x,t\) plot as shown in figure 3.4. In this plot, one can clearly see the evolution of three different wave extrema. The initial placement of the perturbation produces a positive peak which travels to the left. When the perturbation ends (at dimensionless time 0.8), a trough and a peak remain. The trough propagates to the left at a similar speed to the first peak, while the large peak slowly dissipates, and breaks up into several wave packets travelling to the left and right.

The hydraulic solution for flow with the same reservoir conditions and barotropic flow rate predicts that a wave generated to the left of the contraction cannot propagate information to the right-hand end of the domain. As figure 3.4 shows, when diffusion is introduced, information propagates to the left, but some adjustment occurs at the right-hand end of the channel, implying that energy is able to leak through. This simulation can be used to infer the speed of propagation of waves as a function of position. The technique used here is simply to identify the position of a peak as a function of time, and find the derivative to give the wave velocity at each point. By exciting a number of perturbations at different positions, the speed of waves at different positions can be calculated. The procedure
here is to track wave troughs and peaks as they propagate through the domain, and determine their velocity at each gridpoint. Each dot in figure 3.5 represents a measured wave velocity at the position shown. Note that there is significant scatter in this data, as the initial perturbation may break into a number of different modes. These modes are compared to the hydraulic prediction of wave speeds, shown as the solid curves. The vertical line is the position of the topographic and

Figure 3.4: Evolution of a linear perturbation in time. Contours show the elevation of the interface from the equilibrium condition in dimensionless units. The perturbation inserted at $x = -0.06$ for time $t = 0.8$ spawns a number of disturbances which mostly travel leftwards (away from the contraction), however some of the disturbance is communicated to the right-hand side of the domain.
Figure 3.5: Measurements of speed of propagation of disturbances against position for $q_r \approx 1.0$. Each dot represents the speed of one wave, although it is not possible to distinguish the different modes which are created. The two-layer hydraulic prediction of wave speeds is shown as solid curves, with the position of the virtual and topographic control (coincident in this case) shown as a vertical solid line.

virtual controls in the hydraulic solution, which are coincident.

Figure 3.5 demonstrates that the two-layer hydraulic solution generally overestimates the phase speeds which are observed. However, from the point of view of critical flow, it is the direction of wave propagation rather than the phase speed which is of interest. Notably there are no right-going waves ($c > 0$) for $x \approx -0.08$, and no left-going modes ($c < 0$) for $x \approx 0$. This implies supercritical flow in the sense that information about interfacial disturbances is not communicated from one end of the channel to the other.

For cases when the barotropic flow rate is finite, two-layer hydraulics predicts that there is some finite subcritical region in the centre of the contraction due to the displacement of the virtual control point. The hydraulic phase speed for a case with $q_r = 4.2$ is shown in figure 3.6, along with linear perturbation phase speed data using a background flow with the same mixing as in figure 3.3, but
with $q_r \approx 4.2$. Again there are regions of supercritical flow for $x \gtrsim -0.06$ and $x \lesssim -0.28$, implying that there must be some restriction on the propagation of internal waves in these regions. While the position of these control points differs between the hydraulic and the linear perturbation analysis, the end result is that both of these techniques predict that mode-1 interfacial waves are unable to propagate through the channel domain between reservoirs, so that the two reservoirs are isolated from each other.

The propagation of linear disturbances on a background exchange flow demonstrates that while there are differences, the concept of hydraulic control may have relevance to stratified flows where mixing occurs. Qualitative features of hydraulic control appear to be preserved: a subcritical region in the centre of the domain and two supercritical regions on either side. The control points do not occur at the position predicted in the hydraulic solution.

![Figure 3.6: Measurements of speed of propagation of disturbances against position for $q_r \approx 4.2$ using the same methodology as figure 3.5. Virtual and topographic control points predicted by hydraulic theory are shown by vertical lines.](image-url)
3.3.2 Calculation of eigenmodes

Solutions of (3.26) are sought, with input conditions being vertical profiles of velocity and density from the background states used in the above direct simulation of waves. The numerical solution of this equation produces a large number of modes, and requires significant interpretation to be able to use these results. We assume that the lowest modes (those with a simple eigenvector structure) will be the most important from the point of view of communicating information. In addition, we eliminate modes which are 'trapped' in one layer, so that we predominantly investigate vertical mode-1 waves which are centred on the interfacial region dividing the two layers. While the instantaneous gradient Richardson number is less than 1/4 at the interface, the modes we show are stabilised by the effect of viscosity, and thus in all cases the waves are decaying. However the growth or decay is not related to formal definitions of hydraulic control, and therefore we do not specify the imaginary part of the wave speeds.

Figure 3.7 shows density (a) and velocity profiles (b) at $x = 0$, with eigenvectors of the lowest mode waves (defined to be those modes with only one turning point). The modes shown here are typical of the modes observed at any point in the channel. In panels (c), (e) and (f) we see modes which are trapped in either one of the two layers, and are propagating in the same direction as the layer velocity. These modes are not of interest to the concept of hydraulic control, as the information which they communicate is by and large related to the portion of the domain which is homogeneous in density and thus irrelevant to the baroclinic flow. Modes shown in (d), (g) and (h) are of greater interest as they are centred on the interfacial region, and thus have the potential to carry information about any possible variation in the main baroclinic state of either reservoir.

The effect of strong shear on the low modes can be seen in figure 3.7 (g,h). Without shear or viscosity, one would find two mode-1 waves travelling in opposite directions, and the eigenfunction would have a maximum value on the density interface. The addition of viscosity does not significantly alter the velocity or shape of these two waves. However, by incrementally increasing the magnitude of the velocity field, it can be demonstrated that shear distorts these modes so that the mode travelling to the left (figure 3.7g) is skewed upwards, and conversely the mode travelling to the right (figure 3.7h) is skewed vertically downwards. For the purposes of this chapter we will call these two modes the vorticity modes. The peak of each of the two vorticity modes is coincident with one edge of the interfacial region (the region of maximum curvature in the velocity profile), rather than with the region of maximum density gradient. Mathematically, this result is due to the
relative importance of the $u_{zz}$ terms (representing the largest vorticity gradients) in (3.26), and demonstrates that the inclusion of shear is crucial in determining the effect of internal modes as carriers of information about the density structure.

There is also a single mode which propagates on the density interface (figure 3.7d) and we therefore refer to this mode as the density mode. The density mode is only present when viscosity is finite, and while it is a persistent feature at all points of the channel, it appears to be unaffected by the shear. Instead, the speed of propagation of this mode is best approximated by the velocity of fluid at the mid-isopycnal (see the centre contour in figure 3.3).

It is possible to track the density mode, as well as the two vorticity modes along the whole channel. On the assumption that these three are the most important mode-1 waves from the point of view of communication of information about stratification, we show the variation of phase speed and modal structure with $x$ in figure 3.8 ($q_r \approx 1$). Here the dots show the phase speed, while the curved lines

![Figure 3.7: Modal structure of selected eigenvectors at $x = 0$ for $q_r \approx 1.0$ as determined by the numerical solution of (3.26). (a) Input density profile; (b) input velocity profile; (c) eigenvector for mode with phase speed $c = -0.61$; (d) $c = 0.0013$ (density mode); (e) $c = -0.27$; (f) $c = 0.28$; (g) $c = -0.51$ (vorticity mode); (h) $c = 0.51$ (vorticity mode).]
show the hydraulic prediction of wave speeds for the case $q_r \approx 1$. At every second gridpoint a panel shows the modal structure of each of the three modes. For the vorticity modes (upper and lower rows of panels) we plot two lines showing the two curvature extrema, and for the density mode (central row of panels) one horizontal line is shown at the maximum of $N^2$.

Before trying to relate the modes in figure 3.8 to hydraulic control, it is instructive to use this information to explain the behaviour observed in the numerical simulation of linear waves (section 3.3.1). Figure 3.8 shows very different phase speed information than was obtained in the direct simulation of waves. For example, direct simulation demonstrated that waves generally do not travel to the right from the left-hand side of the channel. Yet the eigenvalue solutions show that modes exist which propagate rightwards at these points. Upon examination of the top row of panels in figure 3.8 one sees that the rightward travelling vorticity
mode undergoes a change on the left-hand side of the domain. As we travel from the centre to the left, a second peak in the eigenvector materialises at $x \approx -0.08$, and at $x \approx -0.12$ the second peak dominates the eigenvector profile. The transition from a vorticity mode to a mode which is trapped in the upper layer is complete at $x \approx -0.20$. This transition is crucial to understanding the results of the direct simulation of linear waves. In the direct simulation, modes are excited by a perturbation centred on the maximum density gradient. At $x \approx -0.24$ (for example), that disturbance at the density interface projects predominantly onto the density mode (which is travelling to the left), and partly onto the leftward propagating vorticity mode. The disturbance will not project significantly onto the rightward travelling vorticity mode. Therefore, while this mode exists, it is not being excited in the linear perturbation experiments. The same argument can be applied to the leftward travelling vorticity mode on the right-hand side of the domain. The wave speeds measured in the perturbation experiments are in general bounded by the density mode and the leftward propagating vorticity mode on the left-hand side of the contraction, and bounded by the density mode and the rightward propagating vorticity mode on the right hand side of the contraction.

We now consider the implications of the modal structure for hydraulic control. The data of interest in figure 3.8 pertain to the physical communication of information in the channel which would usually be carried by the baroclinic mode-1 wave. This information is likely to be carried by the density mode, since the maximum of the eigenfunction of this mode is coincident with the interface. The phase speed of the density mode varies along the length of the channel, and is positive on the right-hand side, negative on the left-hand side, and intersects with $c = 0$ at the centre of the contraction. Therefore, this mode is carrying information away from the centre of the contraction. For the density mode, we can specify the critical point to be close to the minimum of the contraction.

The two vorticity modes are also capable of carrying interfacial information, however as the wave propagates there is a progressive transformation in the eigenfunction. A transition occurs at $x \approx -0.1$ for the right travelling mode, and $x \approx 0.1$ for the left travelling mode. The end result is that these modes are capable of carrying interfacial information away from the contraction, but are not effective carriers of interfacial information from the reservoirs towards the contraction. Unlike the density mode which has a definable critical point, their inability to carry information the length of the channel relies upon the gradual evolution of eigenvector shape. However, the vorticity modes do illustrate that interfacial variations in either reservoir will not be able to propagate into the channel.
A feature of hydraulic control is that the position of the virtual control depends upon the barotropic flow rate. This was seen to hold for the direct excitation of waves as shown in figure 3.6, and we can test if it applies to the eigenvalue analysis using the case with \( q_r \approx 4.2 \). Figure 3.9 shows the density mode, and two vorticity modes for the case with a finite barotropic flow rate, where the field of view has been shifted to the left. Note that the two solid vertical lines show the hydraulic prediction of the virtual and topographic control points from hydraulic theory. The density mode behaves similarly to the zero barotropic flow case, and the phase speed intersects with zero at \( x \approx -0.25 \). This is consistent with the observation that waves cannot travel to the right for \( x \lesssim -0.28 \) in the direct simulation of waves (see figure 3.6). It is notable that hydraulic theory predicts the position of the virtual control at \( x \approx -0.16 \) in this case. This may imply that the zero crossing of the density mode may represent an analogy to the hydraulic virtual control point, however the analogy is not complete, because the density mode only appears when viscosity is finite, and thus cannot be part of the two-layer inviscid hydraulic solution.

The behaviour of the vorticity modes presents a different picture in this case. A transition occurs at \( x \approx 0.2 \) (not shown) for the leftward vorticity mode, and at
3.4. Discussion and conclusions

$x \approx 0$ for the rightward vorticity mode. As was the case with no barotropic flow rate, this indicates that reservoir-to-reservoir communication will not be possible via the vorticity modes. One might claim that the rightward vorticity mode transition at $x \approx 0$ represents a topographic control. However the case for the leftward vorticity mode transition as a virtual control region is weak, as it is on the opposite side of the contraction to the predicted hydraulic virtual control.

3.4 Discussion and conclusions

Two different methods of evaluating the behaviour of linear internal waves in exchange flows have been demonstrated. The first method relates to the direct simulation of linear perturbations by solving (3.12)–(3.14) using an adaptation of the numerical model S-FIT. The second method uses the solution of the sixth-order stability equation to find both the phase speed and modal structure of waves. The direct simulation method considers all available modes which are excited by an initial disturbance, whereas the second method produces a large number of modes which are subjectively assessed, to determine which are the important modes. The techniques reveal different but complementary information about the propagation of low-mode internal waves through exchange flows. Both show that, as in two-layer flows, control may be thought of in terms of information propagation. More precisely, it is the propagation of information regarding the vertical location of the maximum density gradient that is important.

The eigenvalue method is perhaps the more useful of the two approaches as it gives the modal structure at any point, and requires only profiles of velocity and density, and estimates of the turbulent eddy viscosity and wavenumber. A single mode, centred on the maximum density gradient, appears to conform to the behaviour of the interfacial mode in two-layer theory. It is the intersection of the phase speed with zero which indicates the position of a control in the flow. The control point shifts upstream relative to the barotropic flow as barotropic flow rate increases, indicating that this control point may be analogous to the virtual control point identified in hydraulic theory.

There are two other important modes, which have a maximum near the interface, but are skewed from the interface so that the maximum in the eigenvector coincides with the maximum vorticity gradients. While neither of these modes has zero phase speed, the vorticity modes can nonetheless be used to infer details about control. In this case, the mode undergoes a transition so that upstream of the transition (relative to the direction of travel of the wave) the eigenfunction peak is not in the interfacial region. This implies that the mode is an ineffective
Chapter 3. Linear internal waves in exchange flows

carrier of interfacial information upstream of the transition point. Conversely, downstream of the transition, the vorticity mode takes a form which indicates it could be an effective carrier of interfacial information. In one sense this can be considered to result in control of the flow of information, however this is a different concept from hydraulic control, as the transition occurs gradually and thus instead of a critical point at which wave speeds are zero we find a critical region over which control gradually takes effect. In any case, this results in the same conclusion: that the system is in a state where end conditions can change without altering the flow.

There are implications in these findings for the extension of hydraulic control to cases with continuous stratification and mixing. Two primary features of hydraulic control outlined in section 3.1 are first that local information (at a control point) can be extrapolated to give global information, and secondly that the hydraulic solution gives an upper bound on flux through the channel. Extrapolation of global information from local variables is difficult in cases where mixing is significant, since energy is continually being lost to mixing. However, the above analysis demonstrates that in stratified exchange flows, reservoir conditions can be altered without affecting the flow. Thus it is possible that a solution may exist which can be used to place an upper bound on the flux for a given rate of mixing.

Another application of this work is in the analysis of geophysical field data to determine whether flows are supercritical. This may have relevance to both uni-directional flows (for example, abyssal flow over mid-ocean ridges) and bi-directional flows (such as the Bosphorus). In the case of uni-directional flow, control points indicate that downstream conditions may vary without altering the flux. In such cases it may be possible to calculate flux with only limited knowledge of the downstream conditions. Alternatively, if it can be determined that bi-directional exchange flow is controlled, then the implication is that changes to one reservoir may not affect the other reservoir. For example, profiles of velocity and density at discrete locations taken from field measurements such as Gregg & Özsöy (2001) may be used to calculate the eigenvalue solution to the viscous stability equation. By selecting the lowest modes from the solution, it may be possible to determine whether the flow is controlled, and if so, which parts of the channel are super- and subcritical. This can be used to predict flux variations due to changes in external conditions.
CHAPTER 4

The Kelvin–Helmholtz to Holmboe instability transition in stratified exchange flows

Abstract

A laboratory investigation of exchange flows near the two-layer hydraulic limit is used to examine the generation of shear instability at the interface dividing the two layers. Regimes characterised by either Kelvin–Helmholtz or Holmboe's instability are found to be separated by a well-defined transition. Observations of the transition from Kelvin–Helmholtz to Holmboe's instability are compared to predictions from scaling arguments that draw on elements of both two-layer hydraulic theory and linear stability theory. The characteristics of unstable modes near the transition, and the structure of both classes of instability are examined in detail.

4.1 Introduction

An understanding of the role played by shear instability in the transition from laminar flow to turbulence is crucial to characterising the evolution of density stratified flows. Shear instability is the process by which kinetic energy is drawn from the sheared velocity field, and converted to potential energy via mixing of a stable background density gradient. The conditions required for shear instability in a stratified fluid can be described to first order by defining a Richardson number to be the ratio of destabilising parameters (velocity shear) to stabilising effects (density stratification). When the Richardson number drops below a certain critical value then the flow may become unstable (Turner, 1973).

The most commonly studied class of instability in stratified flows is the Kelvin–Helmholtz (KH) instability (see Thorpe, 1973; Koop & Browand, 1979; Klaassen & Peltier, 1985, for example). KH instability produces a billow which grows and rotates until secondary and convective instabilities complete the transition to turbulence. Another instability was predicted theoretically by Holmboe (1962), but has rarely been observed in experiments, and has been called Holmboe's instability (Browand & Wang, 1972). This instability is composed of a pair of oppositely propagating modes which interact to produce limited mixing of the background density gradient.

In this chapter we present results of laboratory experiments which enable us to examine the transition from KH to Holmboe's instability. These instabilities are observed in a bi-directional exchange flow which resembles the two-layer hydraulic solution (Armi, 1986) for exchange flow through a contraction. Before describing the experiments we give a basic description of hydraulic exchange flows, as
well as providing a brief overview of shear flow instability, and the KH–Holmboe transition.

4.1.1 Two-layer hydraulic exchange flows

The steady-state solution for density-driven exchange flow through a contracting channel is originally due to Wood (1970), with further major developments from Armi (1986), Lawrence (1990) and Dalziel (1991). This solution is derived simply from the conservation of energy and mass in a two-layer, inviscid, non-diffusive fluid, and is frequently referred to as the two-layer hydraulic solution. This hydraulic solution cannot be applied to the initiation or time dependence of exchange flow, however experiments have shown that it is a good approximation to steady flows (Wood, 1970; Armi, 1986; Lawrence, 1993; Helfrich, 1995; Zhu & Lawrence, 1996). The hydraulic solution, derived by following Lawrence (1990) is shown in figure 4.1. Interested readers can refer to the above references for a detailed understanding of the physics of this flow. It is sufficient for the purposes of this chapter to note that, if the volume flux in each direction is equal, the difference in velocity between the two layers is constant in the streamwise \( x \) direction, and is given by

\[
\Delta U = u_1 - u_2 = (g' H)^{1/2}, \tag{4.1}
\]

where \( u(x) \) is the layer velocity which is constant in \( z \), \( H \) is fluid height and \( g' = g \Delta \rho/\rho_0 \) is the reduced gravity based in the density difference \( \Delta \rho \) between the layers, the gravitational constant \( g \) and reference density \( \rho_0 \). Furthermore, the arithmetic mean velocity at any point,

\[
\bar{U}(x) = \frac{u_1 + u_2}{2}, \tag{4.2}
\]

is always directed away from the centre of the contracting region.

The derivation of the hydraulic solution requires that there is no communication between the two layers via either viscous effects or diffusion. This assumption is clearly unrealistic, and Lawrence (1990) demonstrates that the solution shown in figure 4.1 is only marginally stable to long waves along the entire channel. Turbulent mixing has been observed in laboratory experiments (Wood, 1970) and field investigations (Gregg & Özsöy, 2001). Numerical simulations (chapter 2; Winters & Seim, 2000) show that mixing and communication between the layers is crucial to predicting the flux which decreases as intensity of vertical mixing increases. However, for small rates of mixing the hydraulic solution is still a good approximation to the flow (chapter 2).
4.1. Introduction

4.1.2 Shear instability

We begin our discussion of shear instability by considering a background flow with simple idealised profiles of fluid velocity and density while neglecting viscosity and diffusion. One can estimate the growth rate of linear disturbances in such a flow. The fastest growing linear wave will amplify by extracting kinetic energy from the velocity field and grow into a nonlinear instability. Therefore, the dominant wavelength and phase speed of instabilities can be predicted with some confidence simply by finding the fastest growing mode (see Lawrence et al., 1991).

The background flow used here is the piecewise velocity and density profiles seen in figure 4.2. Based on Hazel (1972), we define the bulk Richardson number to be

\[
J \equiv \frac{g' \delta}{(\Delta U)^2},
\]

(4.3)

where \( \delta \) is the thickness of the shear layer. We also define a thickness of the density interface \( \eta \) as shown in figure 4.2, and the ratio of layer thicknesses \( R \equiv \delta/\eta \). In

Figure 4.1: Hydraulic solution for exchange flow through a contracting channel. (a) Channel in plan view; (b) velocity vectors and interface position in elevation view.
the absence of viscosity and diffusion the linear stability characteristics of the flow are dependent only upon the Richardson number and $R$. The linear stability diagram for the case when $R \to \infty (\eta = 0)$ is shown in figure 4.3 (reproduced from Haigh & Lawrence, 1999). This diagram shows positive growth rates as a function of $J$ and wavenumber $\alpha$ (which has been normalised by multiplication with the velocity interface thickness $\delta$).

The key feature of figure 4.3 is that there are two distinct unstable regions in the diagram, which are divided by the heavy dashed contour in the lower left-hand corner. If the most unstable mode lies beneath this dashed contour, an instability will form which has zero phase speed and relatively high growth rate. This is a Kelvin–Helmholtz instability (Haigh & Lawrence, 1999). It occurs when the destabilising effect of shear overcomes buoyancy, and can therefore form even when $\eta \approx \delta$ as demonstrated by Scotti & Corcos (1972) and Thorpe (1973). A KH billow is roughly symmetrical and grows to large amplitude which is limited by the formation of secondary and convective instabilities (Thorpe, 1973; Smyth & Peltier, 1991). The formation and dynamics of KH billows is detailed in the numerical experiments of Klaassen & Peltier (1985).

At higher values of bulk Richardson number, the fastest growing instability is Holmboe's instability which differs from the KH instability in several ways. First,
it has a finite phase speed relative to the mean shear (Hazel, 1972; Smyth et al., 1988). Second, growth rates are generally smaller and the amount of mixing due to the instability is less than for KH instability (Koop & Browand, 1979). Third, in the absence of viscosity there is no theoretical upper limit to the Richardson number at which Holmboe's instability can occur (Holmboe, 1962). It can be demonstrated (Smyth et al., 1988) that the instability is made up of two modes, with equal but opposite phase speed. When $R < 2.4$, the Holmboe mode is suppressed (Smyth & Peltier, 1989).

Baines & Mitsudera (1994) argued that all shear instability can be derived
from the phase-locking of two freely propagating modes with opposite celerity. This phase locking is due to the shear, while amplification is caused by the two modes combining to kinematically increase the size of the total perturbation. For arbitrary shear profiles with a stable density gradient, while this technique recovers the same stability criteria generated by linear stability analysis, it has the advantage that it lends itself to a physical interpretation of the mechanism by which shear instability occurs. In the background flow shown in figure 4.2, there are four distinct freely propagating modes which can form. Two of these modes travel on the density interface (one in each direction). The other two travel on the vorticity interface, with the mode travelling on the upper interface going to the left, and the mode on the lower interface travelling right. Baines & Mitsudera (1994) show that when the two vorticity modes are able to lock phase, KH instability forms. However, when $J$ is increased, the two vorticity modes are unable to communicate, either because $g'$ is large or the distance $S$ between the two vorticity interfaces is large. In such cases, two unstable waves are formed by the interaction of one density mode with either one of the two vorticity modes. These two unstable modes are Holmboe’s instability.

According to figure 4.3, when $J < 0.071$ both KH and Holmboe’s instability are possible, with smaller values of Richardson number favouring KH instability and high Richardson number favouring Holmboe’s instability. For $J < 0.046$, KH modes have higher growth rates, while for $0.046 < J < 0.071$, Holmboe modes grow faster (Haigh & Lawrence, 1999). One therefore expects a transition between the two types of instability at $J \approx 0.046$. At the transition point, the wavelength changes discontinuously, as does the phase speed which is zero for the KH instability. This discontinuity is shown in figure 4.3 by the open circles on the ends of the dash-dot lines.

Several studies have investigated factors not considered by linear stability theory. The KH–Holmboe transition was investigated numerically by Smyth & Peltier (1991) with the inclusion of nonlinear and viscous effects, and finite $R$. This study demonstrated that once the instability grows to a finite size, nonlinear effects play a role in determining its evolution. The two oppositely propagating Holmboe modes interact as they approach each other and result in increased mixing in the form of a wisp of fluid being ejected through the cusp of the Holmboe modes.

While Smyth & Peltier (1991) were able to demonstrate the existence of a transition between the KH and Holmboe regimes, characteristics of KH instability are observed on the Holmboe side of the transition in the form of complete overturning of the density interface. Likewise, on the KH side of the transition,
the two counter-propagating modes are visible in the early part of the experiment, but these two modes lock phase and roll up. It is possible (Collins & Maslowe, 1988; Smyth & Peltier, 1991) that nonlinearity acts to increase the likelihood of interactions between modes. Such interactions effectively reduce the stability of the flow, thereby increasing the values of the bulk Richardson number at the transition point. If this type of interaction were to occur, then one would expect the component modes of Holmboe's instability would attenuate each other as they pass, and this attenuation would increase near the KH-Holmboe transition. This consequence is tested in the present experiments.

The effect of finite viscosity is to damp the high wavenumber instabilities. Smyth et al. (1988) included viscosity in the stability analysis and demonstrated that while the general features of the stability diagram are unchanged, viscosity may act to preferentially dampen the Holmboe modes (see figure 2 of that paper). Accordingly, the value of Richardson number at which the KH-Holmboe transition occurs may be greater than that predicted by inviscid theory. The extent of viscous damping will depend upon the Reynolds number,

\[ Re \equiv \frac{\Delta U \delta}{\nu} \]  

(4.4)

It is claimed (Browand & Wang, 1972) that for \( Re \approx 100 \), growth rates are an order of magnitude lower than the value predicted by inviscid linear stability theory.

Other corrections to linear stability theory are considered by Haigh & Lawrence (1999). Both a vertical shift in the position of the density interface and the presence of boundaries alter the stability of the flow. It is expected that boundaries have an effect when the total fluid height \( H < 10 \delta \), and that the vertical asymmetry produces a difference in growth rate and phase speed between the left-going and right-going Holmboe modes (Haigh & Lawrence, 1999). Nonetheless, even though these modes have different growth rates, symmetric Holmboe instabilities may still be possible, because the amplitude of the component Holmboe modes is limited by nonlinear damping, and not growth rate (Smyth et al., 1988).

In order to verify the predictions of linear stability theory, a number of mixing layer experiments have been conducted. For example, Scotti & Corcos (1972) demonstrated that the onset of instability could be estimated based on the Richardson number. Lawrence et al. (1991) used mixing layer experiments to predict the formation of asymmetric KH and Holmboe instabilities, and showed a strong correlation between observed and predicted wavelengths. Observations of the transition from KH to Holmboe modes are reported by both Koop & Browand.
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Figure 4.4: Experimental configuration. (a) Tank in elevation view; (b) plan view.

(1979) and Lawrence et al. (1991), however asymmetry of the velocity and density interface positions may have affected the stability characteristics of these experiments.

The current experiments are designed to test a number of facets of linear stability theory as they apply to both KH and Holmboe instability. In addition we investigate the applicability of recent findings of numerical investigations. In particular, we focus on three main issues which remain unclear from previous studies:

1. Showing the density and velocity structure of the instabilities, in particular Holmboe’s instability;

2. The extent of nonlinear interactions between component modes of Holmboe’s instability;

3. Using \( J \) to predict the type of instability which will form, and the wavelength and wave speed of that instability.

4.2 Experiments

The experiments were conducted in a \( 2.58 \times 0.53 \times 0.60 \) m tank, with a 0.50 m long perspex insert forming the contracting region as shown schematically in figure
4.4. The insert included a false bottom 0.15 m above the floor of the tank, and the contracting channel width varied smoothly from 0.12 m (at the centre) to 0.3 m. The insert was placed to the left of centre of the tank so that the reservoir on the left-hand side of the insert was 0.93 m long, and on the right-hand side was 1.15 m.

A sluice gate at the left-hand end of the insert separated fresh water in the left-hand reservoir from saline water in the right-hand reservoir. Once the sluice gate was opened, dense water flowed through the contracting channel, spilling into the left-hand reservoir and settling beneath the level of the false bottom in the test section, and as such, did not effect flow in the channel. Likewise, fresh water flowed rightwards and resided in the upper part of the right-hand reservoir. The fresh water level in the right-hand reservoir limited duration of a given experiment. When this level became too large, it flooded the hydraulic control and altered the quasi-steady exchange flow in the contraction. The false bottom in the insert, in combination with the asymmetrical positioning of the insert was designed to maximise the length of experiments.

The tank was filled to a level $H$ above the floor of the channel. The sluice gate was closed and salt added to the right-hand reservoir to increase the density on that side. Experiments were initiated by lifting the sluice gate, allowing exchange between the two basins. After 20–50 seconds (depending on the hydraulic velocity scale given by (4.1)), the exchange reached a quasi-steady state (maximal exchange as defined by Armi & Farmer, 1987). The exchange was roughly constant for between 3 and 9 minutes. Superimposed on this quasi-steady state, instabilities formed on the interface dividing the two layers.

Measurements of the density and velocity fields were taken during the experiments. Several attempts were made to measure the density field. First, conductivity probes were used, however these measurements were unsuccessful as these probes could not resolve the density interface. The conductivity probes have a physical cross-section of 1 mm, which, as shown in section 4.4.2, was in some cases smaller than the size of the density interface thickness. Second, the evolution of the interface was observed by seeding fluid in the left-hand reservoir with Sodium Fluorescein, a fluorescent dye. The flow was illuminated from below with a rack of 50 W halogen bulbs with a double slit arrangement producing a vertical light sheet of width between 5 and 15 mm. Images were taken with a CCD video camera placed 1.6 m from the tank perpendicular to the flow. The video images were written directly to disk using an ITEX PC-COMP frame-grabber and were simultaneously stored on S-VHS tape. This technique was useful for visualising
the instabilities, however the resolution was too low to accurately measure the density interface thickness. Therefore a third technique was employed. Still photographs were taken with a Nikon F90 SLR camera with a 500mm zoom lens, a 42 mm extension tube and using 3200 ASA black and white film. Negatives were scanned to a resolution of 1200 horizontal lines (limited by the resolution of the film), allowing interface thickness $\eta$ to be measured by fitting a tanh function to the vertical profiles of intensity $I$ using the definition

$$\eta = \frac{\Delta I}{(\partial I/\partial z)_{\text{max}}},$$

(4.5)

where $\Delta I$ is the difference in intensity between the upper and lower layers. The external parameters for experiments which used this technique are shown in table 4.1, and can be identified by the prefix S. In this table, experiments with the prefix I are experiments used for visualising the evolution of instabilities recorded from video.

The velocity interface thickness was measured by dropping crystals of Potassium Permanganate of diameter 0.5–1 mm into the fluid. This technique allowed higher resolution measurements of the velocity interface than the PIV measurements outlined below. For these experiments (labelled by K in table 4.1) the test section was back-lit, and images were taken using colour video recorded onto S-VHS tape, and converted to a digital image using the PC-COMP frame-grabber at a later date. The crystal left a streak of dye in the fluid, allowing measurement of the fluid displacement as a function of $z$. By tracking the fall of the crystal it was possible to convert the displacement into a horizontal velocity profile by scaling the $x$ coordinates of the video image. The interface thickness was measured directly from the velocity profile using

$$\delta = \frac{\Delta U}{(\partial U/\partial z)_{\text{max}}}. $$

(4.6)

To measure two-dimensional velocity fields, Particle Image Velocimetry (PIV) was used (see Stevens & Coates, 1994, for a description of the algorithm). The flow was seeded with pliolite particles of diameter 45–75 $\mu$m, and illuminated by the light sheet. Images were taken from a distance of 2.0 m using a Hamamatsu C4742 digital camera with $1017 \times 996$ pixels resolution and 10 bit pixel depth. The images were downloaded to a PC using an ITEX IC-DIG-16-D frame-grabber. Two images were taken at a separation of 140–150 ms which was limited by the speed of the digital camera. The first image was stored in PC RAM, and written to hard disk after both images were collected. Image pairs were taken at a separation of 1 second.
The processing of the PIV images was complicated by the high velocities in the channel relative to the frame rate of the camera. Refinements to the technique were required to determine velocity fields, although quality degenerated as fluid velocity increased. The refinements involved three phases. Firstly, using a relatively large box size (40 pixels, or 4.8 mm), estimates of the velocity were obtained. Poor quality vectors were eliminated by applying an adaptive Gaussian filter (identical to that used by Cowen & Monismith, 1997). The remaining vectors were averaged in time, and used as input velocities for a second pass of the PIV routine, allowing a reduction in the size of the search radius, and hence decreasing

<table>
<thead>
<tr>
<th>Experiment name</th>
<th>$H$ (mm)</th>
<th>$g'$ (m/s²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>90</td>
<td>0.0189</td>
</tr>
<tr>
<td>S2</td>
<td>70</td>
<td>0.0152</td>
</tr>
<tr>
<td>S3</td>
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<tr>
<td>I1</td>
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<td>0.0221</td>
</tr>
<tr>
<td>I2</td>
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<td>0.0191</td>
</tr>
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</tr>
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<tr>
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<tr>
<td>V4</td>
<td>60</td>
<td>0.0106</td>
</tr>
<tr>
<td>V5</td>
<td>50</td>
<td>0.0209</td>
</tr>
<tr>
<td>V6</td>
<td>50</td>
<td>0.0202</td>
</tr>
<tr>
<td>V7</td>
<td>50</td>
<td>0.0353</td>
</tr>
<tr>
<td>V8</td>
<td>40</td>
<td>0.0137</td>
</tr>
</tbody>
</table>

Table 4.1: List of experiments undertaken. Experiments S are those used for taking still photographs, I refers to experiments using analogue video, K for experiments in which Potassium Permanganate crystals were used to estimate velocity profiles and V for PIV velocity experiments.
errors. The results of the second pass were averaged over a 20 second time interval, and these averages were used as input velocities for a third pass of PIV. The third pass allowed use of a smaller box size (16-20 pixels or 1.9–2.4 mm), and a further reduction in the search radius, resulting in higher quality measurements of regions with high velocity shear. Eight experiments were conducted using these techniques and are identified by the prefix V in table 4.1. Although we were unable to look at both velocity and density fields simultaneously (as was done with simultaneous PIV and Laser Induced Fluorescence in Pawlak & Armi, 1998a; Zhu & Lawrence, 2001), these techniques allowed visualisation of the velocity and density evolution of the flow at very high spatial resolution.

4.3 Results

When the sluice gate is removed, some mixing occurs at the boundary between the two fluids. Therefore, when the quasi-steady exchange flow forms, the interface contains some mixed fluid. The diffuse interface subsequently undergoes sharpening as a result of advection of reservoir fluid into the channel, and the divergence of the fluid as it accelerates through the contraction. In some experiments a sharp stable interface forms, which is ultimately disturbed by instabilities. In such cases the timing of the onset of instability appears to be dependent upon background noise, and instability can be initiated by creating a small disturbance to the interface. In other cases the onset of instability occurs shortly after the exchange begins.

4.3.1 Observations of instabilities

Both KH and Holmboe's instability are observed in experiments S1–S4. In figure 4.5 we show a representative image from each of the four experiments. Estimates of bulk Richardson number and Reynolds number for each experiment are shown in the caption, and are discussed more fully in section 4.4. In experiment S1, KH instability is observed, as shown in figure 4.5(a). Large billows are formed which do not travel with respect to the mean fluid flow: the instabilities are dragged away from the centre of the contraction by the mean shear. The billows grow to large amplitude and then collapse, after which the shear flow advects fluid away from the mixed region and acts to reduce the interface thickness. Sharpening of the interface ensues, thereby decreasing stability until another perturbation forms.

Some interface perturbations in experiment S2 exhibit features of both KH and Holmboe's instability. For instance, figure 4.5(b) captures one such instability which overturns the central density interface, yet is travelling with respect to the
4.3. Results

Figure 4.5: Still photographs from different parameter regimes. (a) Experiment S1, \( J \approx 0.058, \Re \approx 220 \); (b) experiment S2, \( J \approx 0.084, \Re \approx 190 \); (c) experiment S3, \( J \approx 0.121, \Re \approx 190 \); (d) experiment S4, \( J \approx 0.103, \Re \approx 220 \).

mean flow. This is consistent with flows simulated by Smyth & Peltier (1991) which are close to the KH–Holmboe transition point.
By decreasing the fluid height again (experiment S3, figure 4.5c) we reach a state which clearly shows the characteristics of Holmboe's instability. The two component Holmboe modes are shown: the downward cusping wave is travelling to the left and the upward cusping mode is travelling right. The two modes are confined in their region of influence (as shown in the numerical simulations of Smyth et al., 1988). The large amplitude of the modes demonstrates that the disturbances are nonlinear and it will be shown below that they interact with each other during their evolution.

Experiment S4 shown in figure 4.5(d) has the same $H$ as experiment S3, but double the density difference so that, according to two-layer hydraulic solution, the fluid velocities are the same as in experiment S1. The observed flow shows the character of Holmboe's instability with two oppositely propagating modes, demonstrating that the velocity difference across the interface is not the only parameter which governs the mode of instability.

The evolution of instabilities with time is best visualised with space-time diagrams. To construct these diagrams we use a number of sequential images from experiment series I, which show the density field as indicated by the fluorescein dye. At any point in time ($t$) and space ($x$), we can define a density profile. A tanh function fitted to the density profile is used to determine the vertical position of the interface. By finding interface position at all points in $x$ and $t$ we are able to plot the perturbation of the interface height from the running mean. This is done in figure 4.6 for experiments 11–14.

The transition from KH to Holmboe's instability is clearly illustrated by figure 4.6. A signature of KH instability is that disturbances propagate away from the centre of the contraction. The instabilities do not travel with constant velocity as shown in figure 4.6(a). In contrast, the velocity of the Holmboe modes is very well defined. Theoretically these modes should have constant velocity relative to the mean shear, and to some extent this is shown in figure 4.6(c,d), with waves travelling slowly as they propagate towards the centre of the contraction, and speeding up as they exit the contraction. In figure 4.6(b), the $x,t$ diagram for experiment I2, both KH and Holmboe's instability are present, particularly for early time.

### 4.3.2 Velocity structure of instabilities

Using PIV we obtained two-dimensional measurements of fluid velocity in the contracting channel. Figure 4.7(a) shows velocity vectors for experiment V2 which is in the KH regime. Despite some patchiness in the coverage of the vectors, one can clearly resolve several elliptical shaped KH eddies at different stages of their
evolution.

The structure of the instabilities alters at the KH–Holmboe transition. Figure 4.7(b) shows the case close to the transition (experiment V3) with the left-travelling component mode shown. The two differences between this mode and the KH instability in figure 4.7(a) is that the streamlines in the high vorticity region are not closed, and the mode now propagates with respect to the mean shear.

Figure 4.6: The evolution of instabilities shown in $x,t$ diagrams. (a) Experiment I1, $J \approx 0.065$, $Re \approx 220$; (b) experiment I2, $J \approx 0.079$, $Re \approx 200$; (c) experiment I3, $J \approx 0.096$, $Re \approx 200$; (d) experiment I4, $J \approx 0.119$, $Re \approx 190$;
Figure 4.7: Velocity vectors for (a) experiment V2, $J \approx 0.073$, $Re \approx 190$; (b) experiment V3, $J \approx 0.083$, $Re \approx 190$; (c) experiment V5, $J \approx 0.118$, $Re \approx 190$;

Features of Holmboe modes are more clearly shown in figure 4.7(c) (experiment V5). Here two component Holmboe modes are propagating to the right. As is the
case with density visualisation of the waves, the region of influence of the wave is very small. Within the confined region of perturbation, the vertical velocity is high.

The general structure of the component Holmboe modes compares favourably to that predicted by numerical simulations of nonlinear Holmboe instabilities (Smyth et al., 1988). The structural features which include a region of high vorticity on the face of the modes are observed. The modes travel slower than the fluid in the adjacent layer, so that fluid is continually passing over the top of cusp, and the region of high vorticity sits on the lee side of the wave relative to the fast flowing fluid.

4.3.3 Interactions between two component Holmboe modes

The interaction between component modes of Holmboe’s instability is of interest for several reasons. Firstly, Smyth & Peltier (1989) demonstrate that it is during the interaction of two modes that the instability is amplified, and some mixing occurs. Secondly, if nonlinear phase locking alters the point of KH–Holmboe transition, then it is possible that the component modes may be capable of attenuating each other as they pass one another. It was noted by Zhu & Lawrence (2001) that whilst linear theory predicts that the waves should speed up as they approach one another, observations of Holmboe modes indicated that the converse in fact occurs. Figure 4.6 shows that it is difficult to see any sign of interaction between the component modes. This indicates that if phase locking of component modes is the mechanism responsible for KH instability, then this mechanism is not due to the gradual increased interaction of component modes. Instead, our observations support the prediction of Baines & Mitsudera (1994) that the transition to KH instability occurs when vorticity modes can communicate due to reduced $J$, and that the likelihood of communication is not enhanced by nonlinear interaction between the component Holmboe modes.

While the mixing done by Holmboe’s instability is weak, Smyth & Peltier (1991) indicate that accelerated mixing occurs in the form of a wisp of fluid being ejected through the cusp of a component mode as two modes pass one another. This wisp is observed in the present experiments, as shown in figure 4.5(b). As expected, the frequency of observations of wisping is greater near the KH–Holmboe transition. However, the occurrence of the wisp is not correlated with the interaction between component modes. This may be an indication that, as postulated by Smyth & Peltier (1991), the wisping depends upon three-dimensional effects, while the measurement techniques used here are only capable of visualising a two-dimensional field.
The amplification of component modes just after they pass one another as described by Smyth et al. (1988) was tested in these experiments. In general, interacting waves amplify each other, although there is some variability in this process. The amplification of interacting modes is easiest to observe in the velocity field. An example is shown in figure 4.8 where we have 4 panels separated by 1 second. Panel (a) shows a strong right-propagating mode, with a weak left-propagating mode, both having vorticity of the same sign, just before interaction. One second later, the modes are positioned so that the two vortices produce a large single vortex. The vortex acts to kinematically amplify both component modes, as described by Baines & Mitsudera (1994). However, the modes travel past one another so that they do not wrap up into a billow. Instead, the next frame shows that the waves continue with negligible attenuation. In this frame they appear to be linearly superposed, leaving a residual upwards vertical velocity (since the left-going disturbance is smaller), but a smaller vortex than before interaction. In panel (d) the two modes break free from one another. The vertical velocity field demonstrates that the right-travelling mode has increased in amplitude by 20%, and that the left-travelling mode has increased by 50%. This sort of interaction occurs frequently in the experiments, and while there is significant variability, as a general rule the two modes are both amplified during the interaction. In some cases, interacting waves destroy one of the modes, although it's once again possible that this is caused by three-dimensional effects or secondary instability.

4.4 Predicting modes of instability

4.4.1 Scaling of stability parameters

In this section we consider some simple scaling arguments which allow us to include the first-order effects of molecular viscosity and diffusion in determining the finite thicknesses $\eta$ and $\delta$ in a hydraulic exchange flow. This, combined with knowledge of $\Delta U$ and $\Delta \rho$, can be used to determine the bulk Richardson number, the Reynolds number and the ratio of thicknesses $R$.

The finite thickness of the velocity interface, $\delta$ is due to the viscous diffusion of momentum over some timescale $\tau$, or

$$\delta \sim (\nu \tau)^{1/2}. \quad (4.7)$$

The most appropriate timescale will be related to the fluid velocity and a horizontal length scale $L$ over which the two fluid layers are in contact, that is

$$\tau \sim \frac{L}{\Delta U}. \quad (4.8)$$
This can be combined with (4.1) and (4.7) to give
\[ \delta \sim \frac{(\nu L)^{1/2}}{(g'H)^{1/4}}, \] (4.9)
which shows the dependence of the velocity interface thickness upon external parameters.

The density interface thickness can be estimated using the same technique, using the molecular diffusion of salt \( \kappa \) instead of \( \nu \),
\[ \eta \sim (\kappa T)^{1/2}. \] (4.10)

The application of this argument is complicated by the density interface being thinner than the velocity interface. Therefore, the fluid velocity at the edge of the

Figure 4.8: Velocity vectors for experiment V5, \( J \approx 0.118, \ Re \approx 190 \) showing the interaction between two component modes. Each frame is separated by 1 second.
density interface will be reduced in proportion to the ratio $R$ of the two thicknesses. For this reason, the timescale used for estimating the density interface is

$$\tau \sim \frac{L}{\Delta U(\eta/\delta)},$$

(4.11)
giving

$$\eta \sim \frac{\kappa^{1/3} \nu^{1/6} L^{1/2}}{(g'H)^{1/4}}.$$  

(4.12)

The interface thickness is dependent on the choice of horizontal lengthscale $L$. The dominant horizontal lengthscale in the problem is the length of the channel so that the lengthscale is the same value for both velocity and density interfaces, and the value of the ratio of thicknesses is

$$R \sim \left(\frac{\nu}{\kappa}\right)^{1/3} = Sc^{1/3}.$$  

(4.13)

This estimate matches the scaling of Bejan (1995) for the growth of boundary layers in a large Prandtl number fluid, but differs from that of Smyth et al. (1988), because the periodic boundary conditions in that paper mean that the differential velocity plays no role in determining the thicknesses. Instead, Smyth et al. (1988) estimate that $R = Pr^{1/2}$ provided that $R$ starts at that value. Note also that Smyth et al. (1988) considered heat as the stratifying species, and therefore used the Prandtl number as the ratio of diffusivities, whereas in a salt-stratified fluid the Schmidt number is the relevant parameter.

We are now able to estimate the bulk Richardson number in this flow. From (4.1) and (4.3) we write

$$J = \frac{\delta}{H},$$

(4.14)

and using (4.9) we obtain

$$J \sim \frac{(\nu L)^{1/2}}{g^{1/4} H^{5/4}}.$$  

(4.15)

We are also able to estimate the Reynolds number using (4.1), (4.4) and (4.9) which gives

$$Re \sim \frac{(g'H)^{1/4} L^{1/2}}{\nu^{1/2}}.$$  

(4.16)

The scaling arguments shown here demonstrate the dependence of stability parameters upon relevant external controllable parameters for the exchange flows considered here. By varying $g'$ and $H$, both $Re$ and $J$ change, but $R$ is constant. Note, however, that $\delta$ and $Re$ depend weakly upon both external parameters, while $J$ varies strongly with $H$. Therefore we expect variations of $H$ to be the primary cause of variations in stability. At large values of $H$, $J$ will decrease, and
4.4. Predicting modes of instability

KH billows are likely; as $H$ is decreased we may pass through the transition to Holmboe's instability.

We now evaluate the consistency of the scaling results by directly measuring $\eta$ and $\delta$ in the experiments. We then determine $R$, $J$ and $Re$ using (4.13), (4.14) and (4.16) respectively. This allows comparison of linear inviscid stability theory with the observed behaviour.

4.4.2 Comparison of experiments with scaling arguments

An image of the velocity profile measured by falling Potassium Permanganate crystals for experiment K3 is shown in figure 4.9. This image shows the symmetrical form of the interface with the exception of the influence of the bottom

![Figure 4.9: Snapshot of the velocity profile for experiment K3, before the onset of instability. The solid line shows the idealised interface giving $\delta \approx 6.2$ mm.](image-url)
Figure 4.10: Measurements of $\delta$ for experiments K2 (□), K3 (○), K4 (△). The solid line is the relationship described by (4.17).

boundary layer. The interface thickness measured by (4.6) gives $\delta = 6.2 \pm 0.3$ mm, and the solid line in figure 4.9 shows the maximum slope of the interface. The experimental uncertainty here is generated from uncertainty in estimating the maximum gradient for use in the denominator in (4.6). By measuring $\delta$ from several profiles in each of experiments K2–K4, we are able to plot $\delta$ against the right-hand side of (4.9) in figure 4.10. In this case, the solid line shows (4.9) using a scaling coefficient of 1.55. Therefore, we represent the interface thickness by writing

$$\delta \approx 1.55 \frac{(\nu L)^{1/2}}{\left(g' H^2\right)^{1/4}}. \quad (4.17)$$

Measurements of density interface thickness defined according to (4.5) were taken from experiments S2-S4 by fitting a tanh function to vertical profiles of light intensity. The results from these three measurements including experimental uncertainties are plotted in figure 4.11 against the predicted thickness from (4.12). The large experimental error is due to the effect of refraction of light in the interfacial region, and makes it difficult to validate the weak dependence of $\eta$ upon these two parameters $g'$ and $H$, however the evidence shown here is consistent with a scaling coefficient of 1.1 in (4.12), giving the value of $\eta$ (in mm) to be

$$\eta \approx 1.1 \frac{\kappa^{1/3} \nu^{1/6} L^{1/2}}{\left(g' H\right)^{1/4}}. \quad (4.18)$$
4.4. Predicting modes of instability

The ratio of thicknesses can be easily calculated from (4.17) and (4.18) to give

\[
R \approx 1.4Sc^{1/3} \approx 12. \tag{4.19}
\]

This is much larger than the threshold value for the formation of Holmboe's instability, validating the use of \( R \to \infty \) in the stability calculation leading to figure 4.3.

The results in (4.17) and (4.18) are not consistent with measurements of \( \delta \) and \( \eta \) in experiments K1 and S1 respectively, when fluid height is \( H = 90 \text{mm} \). In these cases, interface thicknesses before the onset of instability are \( \delta \approx 13 \pm 1 \text{mm} \), and \( \eta \approx 1.5 \pm 0.3 \text{ mm} \), which are both a factor of 4 greater than expected. The onset of instability occurs early in this experiment; as soon as the steady exchange begins. We hypothesise that because of the decreased stability of this experiment, the flow is required to pass through a number of unstable states as the interface sharpens. Therefore it becomes unstable before reaching the quasi-steady background state predicted by (4.17) and (4.18).

By combining (4.17) with (4.14) we can write an expression for the bulk Richardson number \( J \) in the exchange flow,

\[
J \approx 1.5 \left( \frac{\nu L}{g' H^5} \right)^{1/4}. \tag{4.20}
\]

A summary of the observed instabilities is shown in figure 4.12, where this functional dependence of \( J \) on \( g' \) and \( H \) is shown by contours of constant \( J \). The

---

**Figure 4.11:** Measurements of \( \eta \) for experiments S2 (□), S3 (○), S4 (△). The solid line is the relationship described by (4.18).
right-hand solid contour \(J = 0.046\) is the point at which the KH–Holmboe transition occurs in the linear theory, although nonlinearity and viscosity may both act to increase the value of \(J\) at the transition as discussed in section 4.1.2. The left-hand solid contour \(J = 0.071\) is the point above which KH instability cannot occur in the linear theory. Figure 4.12 also contains information about the type of instability observed in each experiment showing Holmboe's instability occurring at high \(J\) and KH instability at low \(J\). From figure 4.12, the transition point occurs at \(J \approx 0.08\), larger than predicted by linear theory.

Finally we use (4.16) to predict the variation of Reynolds number in the flow.

![Graph showing the relationship between \(g'\) and \(H\).](image)

Figure 4.12: The value of \(g'\) versus \(H\) for each experiment. Dashed contours are levels of constant \(J\), with the two solid contours at \(J = 0.071\) and \(J = 0.046\). Experiments I are shown by a \(\Box\), with \(V\) (\(\bigcirc\)), \(S\) (\(\bigtriangleup\)) and \(K\) (\(\backslash\)). The open symbols indicate Holmboe's instability was observed, and symbols filled with black indicate KH instability. For experiments in which both forms of instability were observed the symbol is filled grey.
Again we substitute (4.17) to give
\[ Re \approx 1.5 \frac{L^{1/2}(g'H)^{1/4}}{\nu^{1/2}}. \] (4.21)

Note that there is quite a weak dependence of Reynolds number on experimental parameters, so that the range of Reynolds numbers is quite small. Nonetheless, the values of Reynolds numbers in the flow range between 160 and 220, so that we can expect a significant reduction in amplification rates compared to the inviscid prediction (Browand & Wang, 1972; Smyth et al., 1988).

### 4.4.3 Measuring wave speeds and wavelengths

Measurements of the speed of Holmboe modes can be compared to the predictions of linear stability, using values of \( J \) from (4.20) in figure 4.3 which shows dimensionless phase speeds as dashed contours. This theory predicts that wave speeds should be of the order 4-6 mm/s (relative to the mean velocity) for experiments 12-14. Observed wave speeds indicate a velocity differential between the left-going and right-going modes at a given position of 16-18 mm/s in all experiments in which Holmboe's instability occurs. This implies that each component mode is travelling at 8-9 mm/s relative to the mean velocity, which is approximately double the predicted wave speed.

The linear stability prediction (figure 4.3) can be used to predict wavelengths of the order of 50 mm for experiments 12-14. In general it is not possible to measure a coherent wavelength of Holmboe modes, because of the high variability in the distance between wave crests. For example, in figure 4.6(d) we are able to measure wavelengths varying from 50 to 120 mm. Predictions of KH instability wavelengths are of order 100 mm in experiment II. This prediction cannot be verified because the images we use only show 100 mm either side of the contraction.

### 4.5 Discussion

It has been proposed (Baines & Mitsudera, 1994) that the two component modes which make up Holmboe's instability are each produced by the interaction of two modes, one travelling on the density interface, and one on the vorticity interface. According to this theory, KH instability forms when the two vorticity modes (one from each component Holmboe mode) lock phase. While Smyth & Peltier (1991) observe the phase locking of two nonlinear component Holmboe modes which are fired towards one another, the present experiments show no evidence component modes attenuate each other. Phase locking of two vorticity modes should not be confused with phase locking of component Holmboe modes.
When $J$ is sufficiently small that phase locking of vorticity modes can occur, the growth of KH instability outstrips the growth of Holmboe’s instability, so that Holmboe’s instability is not observed.

The experiments presented in this chapter demonstrate several features of Holmboe’s instability and the KH–Holmboe transition which have not been investigated in the laboratory before. Observations of the KH–Holmboe transition showed a strong dependence on the height of the exchange flow. This feature is at first startling, but can be reconciled by scaling arguments which predict the bulk Richardson number (which governs linear stability) is strongly dependent on $H$, the fluid height, as shown in (4.20), and depends weakly upon reduced gravity. In contrast, the Reynolds number and interface thicknesses vary only weakly with either height or reduced gravity. This scaling was used to determine a critical value for the KH–Holmboe transition, which was found to be significantly larger than the linear stability prediction of the transition point. The disparity between the predicted and observed behaviour may be due to three factors: the finite value of $R$, nonlinear amplification and viscosity. The variation of the stability diagram with $R$ was investigated by Smyth & Peltier (1989), who showed that for values of $R = 8$, the stability diagram had similar characteristics to the diagram for infinite $R$. Given that these experiments were conducted with $R \approx 12$, it is unlikely that a finite value of $R$ can account for the difference between experiments and theory. Likewise, we have described above results that show nonlinearity does not lead to attenuation of counter-propagating component modes, and thus should not alter the dynamics of the KH–Holmboe transition. Therefore it is most likely that viscosity alters the transition point. The physical mechanism for this process, as described in section 4.1.2, is that higher wavenumbers are likely to feel the effects of viscosity to a greater degree, and therefore that Holmboe modes (which occur at higher wavenumbers as shown in figure 4.3) are preferentially damped, allowing KH instability to dominate.

The linear stability prediction of wave speeds was also poor compared to observed velocities. While there may be several reasons for this, the primary candidate is nonlinearity. When the waves grow to finite size, they are asymmetric so that the deviation of the interface is greater in one direction than the other (see figure 4.7). In the case of the left-travelling mode, the wave extends into the region of fluid travelling to the left. In other words the wave is more likely to feel additional drag from the fluid travelling in the same direction so that wave speeds will be greater than predicted.

In general it is difficult to reconcile predictions from stability theory with ob-
servations, as stability theory requires the assumption of a steady background state upon which it predicts the growth of perturbations. In a flow where perturbations do grow, they may grow to a finite size and thereby alter the background state. It is the altered state which is generally observed in measurements. Thus it may not be possible to estimate the stability of a flow based on flow variables. Instead, one needs to analytically determine a background state in the absence of disturbances, and calculate the stability of the fluid from these variables. This was shown in the results described here: the scaling arguments for bulk Richardson number were found to hold in cases where perturbations were slow to develop (in particular cases where Holmboe's instability occurred) but the predicted background state was never achieved in flows prone to KH instability. Nonetheless, the scaling based on the theoretical background state was found to be well correlated with the observed mode of instability.
Chapter 4. KH-Holmboe transition in exchange flows
5.1 Summary

Density-driven exchange flow occurs in channels which connect the ocean to marginal seas, estuaries or inlets. The exchange of density, as well as passive tracers, plays a role in determining the hydrodynamics and water quality of these regions. The exchange flow can be described by the two-layer hydraulic solution, an analytical solution which has two key features: firstly, it allows the calculation of flux through a channel based only on external parameters, and secondly the flux is considered to be maximal, or controlled, so that changes may occur at the ends of the channel without altering the flux. The hydraulic solution requires the assumption that mixing in the channel is negligible. In the preceding pages we have examined how mixing alters the calculation of flux through the channel, and have investigated whether flows with mixing can be considered to be controlled.

Evidence for mixing in exchange flows comes from a variety of sources. The laboratory experiments detailed in chapter 4 shows exchange flow in which mixing between the two layers occurs. The mixing is due to shear instability, and the type of instability which occurs is highly sensitive to external parameters. This chapter concentrated on a regime in which the type of shear instability changed from Kelvin–Helmholtz to Holmboe’s instability. The transition was found to depend upon several external parameters which determined both the bulk Richardson number and the Reynolds number. Importantly, this work demonstrated that interfacial instability does lead to mixing between the two layers. This finding is in keeping with a variety of field results (Wesson & Gregg, 1994; Bray et al., 1995; Greco, 1998; Pratt et al., 1999, 2000; Gregg & Özsoy, 2001) which demonstrate that mixing plays a role in determining the exchange flux. In some cases, mixing may be due to interfacial instability, but in field scenarios, mixing is also likely to be caused by interactions between flow and topography, wind stress and tidal currents.

The first body of work in this thesis concentrated on determining how mixing altered the flux through a simple contraction. Mixing was parameterised via a constant, specified turbulent eddy viscosity and diffusivity. It was shown that the parameter $GrTA^2$ governed the effect of mixing in the exchange flow, and that flux varied monotonically as a function of $GrTA^2$ between two analytical limits: the hydraulic limit where mixing was small and the viscous-advective-diffusive limit where turbulent mixing dominated the nonlinear terms representing momentum. Comparison with field data demonstrated that exchange flows in different parts
of the world spanned a wide range in $Gr_T A^2$ space.

Mixing also alters the controlled nature of the flow. The mechanism of control in a two-layer hydraulic exchange flow physically translates to long internal waves being unable to propagate information from one end of the channel to the other. It was demonstrated in chapter 3 that the same physical mechanism of control may apply to flows where some mixing occurs. While the exact nature of control is altered, the effect is the same: the propagation of information via long internal waves from reservoir to reservoir is not possible. The implication is that alterations in one reservoir may occur without altering the flux through the channel.

5.2 Future work

There are several avenues of research which would complement the results described here. There is a significant amount of work required to reconcile the numerical and laboratory results described here with field data, and modelling objectives. The most pressing task is to determine whether the technique described in chapter 3 allows the determination of the presence of control points from field measurements. To undertake such work, one will need measurements of sufficient resolution to allow the calculation of all terms in (3.26). The terms which require calculation of third or fourth derivatives of horizontal velocity may be problematic. In addition, the identification of a transition in the form of the wavefunctions in field data, where flow may be complicated by three-dimensional or tidal effects, will be non-trivial. The verification of these fundamental concepts applied to field data is required for the application of this theory.

There are a number of features in field examples of exchange flows which complicate the application of results described here. One important example is the problem of dealing with mixing which varies spatially and temporally when calculating the reduction in flux due to mixing. The ultimate goal in this regard is to parameterise the effect of interfacial mixing from a knowledge of external parameters. Laboratory experiments in chapter 4 demonstrated the difficulty of predicting the rate of mixing even in the simplest configuration. This implies that there is a significant amount of work to be undertaken to resolve the question of how mixing can be predicted a priori, and thus how the exchange flux can be parameterised based only on external variables. Other complications which arise in geophysical systems include unsteadiness and rotation. Unsteadiness may be due to tides in small estuaries and inlets, or variations in air pressure in larger systems. In such cases the average flux will depend upon the length of the channel, and the timescale and magnitude of barotropic variations. Rotation will alter
5.2. Future work

flow in larger channels, and has been the focus of much research, especially for uni-directional flows. The concepts outlined in this thesis may be adapted for application in flows where rotation and unsteadiness are important.

Another body of work which has not been investigated fully is the effect of stratified end reservoirs on the exchange. Exchange between stratified reservoirs has more relevance to oceanographic and limnological applications than homogeneous reservoirs. Such experiments would be difficult in the laboratory, but are possible with the numerical model S-FIT which was used in chapters 2 and 3. Preliminary investigations into this phenomena revealed a rich variation in dynamic behaviour. A complete study of this behaviour would provide a significant contribution to this field.
Details of the hydraulic solution

Here we write the details of the derivation of the hydraulic solution. We assume two layers of thickness $d_i$, with different densities ($\rho_1 < \rho_2$) and velocities $v_i$ parallel to the $x$ axis. Reservoir conditions are defined at a point where width is infinite, hence the fluid height is $H$ on the left and $\eta$ on the right, and the density constant with a value of $\rho_i$. If we assume that the fluid is inviscid, nonrotating, and immiscible and that pressure is hydrostatic, we can use volume flux and Bernoulli's equation to solve the problem. The volume flux of each layer is

$$Q_i = w v_i d_i.$$  \hspace{1cm} (A.1)

Bernoulli's equation can be applied along the surface streamline as well as the stagnation streamline between the layers. On the surface we use the condition at the left-hand reservoir to give

$$\rho_1 g H = \frac{\rho_1 v_1^2}{2} + \rho_1 g (d_1 + d_2).$$  \hspace{1cm} (A.2)

On the interface we use the conditions at the right-hand reservoir, where we assume that the upper layer is infinitely thin at the reservoir and contributes pressure $p = \rho_1 g d_1$ in the channel:

$$\rho_2 g \eta = \frac{\rho_2 v_2^2}{2} + \rho_1 g d_1 + \rho_2 g d_2.$$  \hspace{1cm} (A.3)

We can put (A.1)-(A.3) in more convenient forms where variables are nondimensionalised as follows:

$$h_i = \frac{d_i}{H/2},$$

$$u_i = \frac{v_i}{(g' H)^{1/2}},$$

$$q_i = \frac{Q_i}{B (g' H^3)^{1/4}},$$

$$b = \frac{w}{B},$$

where $g' \equiv (1 - r) g$, and $r \equiv \frac{\rho_1}{\rho_2}$. Using the convention of Lawrence (1990), (A.1) is now written

$$q_i = b u_i h_i,$$  \hspace{1cm} (A.4)

and by dividing (A.2) and (A.3) by $\rho_1 g H$, Bernoulli's equations become

$$1 = \frac{u_i^2 (1 - r)}{8} + \frac{h_1}{2} + \frac{h_2}{2},$$  \hspace{1cm} (A.5)
If we differentiate (A.4)-(A.6), we can eliminate velocity gradients to give the two equations in layer height gradient shown in (2.1) and (2.2).
APPENDIX B

Derivation of linear waves propagating in a continuously stratified, sheared viscous fluid

B.1 Equations of motion

Here we describe the derivation of (3.21) which is used in chapter 3 to solve for eigenvalues and eigenfunctions of modes which can propagate in a continuously sheared background flow. We assume an arbitrary background state, and derive a dispersion relation for linear waves travelling in this flow. The background state has horizontal velocity $u$, vertical velocity $w$, pressure $p$, density $\bar{\rho}$. These variables are assumed to be functions of horizontal $(x)$ and vertical $(z)$ space, but not of time $(t)$, and satisfy the steady-state two-dimensional equations of motion,

$$
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + K \frac{\partial^2 u}{\partial x^2} + K \frac{\partial^2 u}{\partial z^2},
$$

(B.1)

$$
\frac{\partial \bar{\rho}}{\partial t} + u \frac{\partial \bar{\rho}}{\partial x} + w \frac{\partial \bar{\rho}}{\partial z} = -\frac{g \bar{\rho}}{\rho_0} + K \frac{\partial^2 \bar{\rho}}{\partial x^2} + K \frac{\partial^2 \bar{\rho}}{\partial z^2},
$$

(B.2)

$$
\frac{\partial \bar{p}}{\partial t} + u \frac{\partial \bar{p}}{\partial x} + w \frac{\partial \bar{p}}{\partial z} = K \frac{\partial^2 \bar{p}}{\partial x^2} + K \frac{\partial^2 \bar{p}}{\partial z^2},
$$

(B.3)

$$
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0.
$$

(B.4)

The background state is determined by the numerical model S-FIT (see chapter 2), and is considered to be a known input parameter. Note that the turbulent eddy viscosity $K$ is used to model turbulent processes, and that the turbulent Prandtl number is assumed to be unity, allowing $K$ to be used as turbulent eddy diffusivity. We have also made the Boussinesq approximation, thereby introducing $\rho_0$ as the reference density.

We now write the time-dependent two-dimensional equations of motion to be

$$
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + K \frac{\partial^2 u}{\partial x^2} + K \frac{\partial^2 u}{\partial z^2},
$$

(B.5)

$$
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} - g \rho + K \frac{\partial^2 w}{\partial x^2} + K \frac{\partial^2 w}{\partial z^2},
$$

(B.6)

$$
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} = K \frac{\partial^2 \rho}{\partial x^2} + K \frac{\partial^2 \rho}{\partial z^2},
$$

(B.7)

$$
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,
$$

(B.8)

where $u$, $w$, $p$ and $\rho$ are dependent on time $t$ as well as $x$ and $z$. 
B.2 Scaling of the equations

In order to scale these equations in a way which is relevant for exchange flow we define the following scales:

- Horizontal length scale $L$ of the channel;
- Vertical length scale $H$ which is the height of the water column;
- Velocity scale $U = (g'H)^{1/2}$;
- Aspect ratio $A = H/L$;
- Turbulent Reynolds number $Re_T = UH/K$.

From these scales we define the following dimensionless variables:

$$
x' = x/H, \\
z' = z/H, \\
t' = tUA/H, \\
u' = u/U, \\
w' = w/U, \\
p' = p/pQU^2, \\
\rho' = \rho/\rho_0,
$$

and substitute these variables into (B.5)-(B.8), giving

$$
\frac{\partial U'u'}{\partial t'UA} + U'u'\frac{\partial U'u'}{\partial Hx'} + U'w'\frac{\partial U'u'}{\partial Hz'} = \frac{-1}{\rho_0} \frac{\partial \rho_0 U'^2 p'}{\partial Hx'} + K \frac{\partial^2 U'u'}{\partial Hx'^2} + K \frac{\partial^2 U'u'}{\partial Hz'^2}, \quad (B.9)
$$

$$
\frac{\partial U'w'}{\partial t'UA} + U'u'\frac{\partial U'w'}{\partial Hx'} + U'w'\frac{\partial U'w'}{\partial Hz'} = \frac{-1}{\rho_0} \frac{\partial \rho_0 U'^2 p'}{\partial Hx'} + K \frac{\partial^2 U'w'}{\partial Hx'^2} + K \frac{\partial^2 U'w'}{\partial Hz'^2}, \quad (B.10)
$$

$$
\frac{\partial \rho_0 \rho'}{\partial t'UA} + U'u'\frac{\partial \rho_0 \rho'}{\partial Hx'} + U'w'\frac{\partial \rho_0 \rho'}{\partial Hz'} = K \frac{\partial^2 \rho_0 \rho'}{\partial Hx'^2} + K \frac{\partial^2 \rho_0 \rho'}{\partial Hz'^2}, \quad (B.11)
$$

$$
\frac{\partial U'u'}{\partial Hx'} + \frac{\partial U'w'}{\partial Hz'} = 0. \quad (B.12)
$$

These equations can be rearranged to give

$$
A \frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + w' \frac{\partial u'}{\partial z'} = -\frac{\partial p'}{\partial x'} + \frac{1}{Re_T} \frac{\partial^2 u'}{\partial x'^2} + \frac{1}{Re_T} \frac{\partial^2 u'}{\partial z'^2}, \quad (B.13)
$$
B.3 Linear stability analysis

We will apply a linear stability analysis to (B.13)-(B.16) to determine the waves which can propagate in the background state. Each variable is written in terms of a time-averaged quantity denoted by the overbar and a perturbation variable denoted by a tilde:

\[
\begin{align*}
    u' &= \bar{u}(x,z) + \tilde{u}(x,z,t), \\
    w' &= \bar{w}(x,z) + \tilde{w}(x,z,t), \\
    p' &= \bar{p}(x,z) + \tilde{p}(x,z,t), \\
    \rho' &= \bar{\rho}(x,z) + \tilde{\rho}(x,z,t).
\end{align*}
\]

This gives a new set of equations

\[
A \frac{\partial (\bar{u} + \tilde{u})}{\partial t'} + (\bar{u} + \tilde{u}) \frac{\partial (\overline{\bar{u}} + \tilde{u})}{\partial x'} + (\bar{w} + \tilde{w}) \frac{\partial (\bar{u} + \tilde{u})}{\partial z'} = \\
- \frac{\partial (\bar{p} + \tilde{p})}{\partial x'} + \frac{1}{Re_T} \frac{\partial^2 (\bar{u} + \tilde{u})}{\partial x'^2} + \frac{1}{Re_T} \frac{\partial^2 (\bar{u} + \tilde{u})}{\partial z'^2}, 
\]

\[
\frac{\partial (\bar{w} + \tilde{w})}{\partial t'} + (\bar{u} + \tilde{u}) \frac{\partial (\bar{w} + \tilde{w})}{\partial x'} + (\bar{w} + \tilde{w}) \frac{\partial (\bar{w} + \tilde{w})}{\partial z'} = \\
- \frac{\partial (\bar{p} + \tilde{p})}{\partial z'} - g(\bar{\rho} + \tilde{\rho}) + \frac{1}{Re_T} \frac{\partial^2 (\bar{w} + \tilde{w})}{\partial x'^2} + \frac{1}{Re_T} \frac{\partial^2 (\bar{w} + \tilde{w})}{\partial z'^2},
\]

\[
A \frac{\partial (\bar{p} + \tilde{p})}{\partial t'} + (\bar{u} + \tilde{u}) \frac{\partial (\bar{p} + \tilde{p})}{\partial x'} + (\bar{w} + \tilde{w}) \frac{\partial (\bar{p} + \tilde{p})}{\partial z'} = \\
\frac{1}{Re_T} \frac{\partial^2 (\bar{p} + \tilde{p})}{\partial x'^2} + \frac{1}{Re_T} \frac{\partial^2 (\bar{p} + \tilde{p})}{\partial z'^2},
\]

which are expanded as follows:

\[
A \frac{\partial \bar{u}}{\partial t} + A \frac{\partial \tilde{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \bar{w} \frac{\partial \bar{u}}{\partial z} + \tilde{w} \frac{\partial \tilde{u}}{\partial z} + \bar{w} \frac{\partial \bar{u}}{\partial z} + \tilde{w} \frac{\partial \tilde{u}}{\partial z} = \\
- \frac{\partial \bar{p}}{\partial x} + \frac{1}{Re_T} \frac{\partial^2 \bar{u}}{\partial x'^2} + \frac{1}{Re_T} \frac{\partial^2 \bar{u}}{\partial x'^2} + \frac{1}{Re_T} \frac{\partial^2 \tilde{u}}{\partial x'^2} + \frac{1}{Re_T} \frac{\partial^2 \tilde{u}}{\partial x'^2},
\]
Appendix B. Derivation of linear wave propagation

\[ A \frac{\partial \bar{w}}{\partial t} + A \frac{\partial \bar{w}}{\partial t} + \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{w} \frac{\partial \bar{w}}{\partial z} + \bar{w} \frac{\partial \bar{w}}{\partial z} + \bar{w} \frac{\partial \bar{w}}{\partial z} = - \frac{\partial \bar{p}}{\partial z} - \frac{\partial \bar{p}}{\partial z} - g \bar{p} + \frac{1}{Re} \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{1}{Re} \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{1}{Re} \frac{\partial^2 \bar{w}}{\partial z^2} + \frac{1}{Re} \frac{\partial^2 \bar{w}}{\partial z^2}, \quad (B.26) \]

\[ A \frac{\partial \bar{p}}{\partial t} + A \frac{\partial \bar{p}}{\partial t} + \bar{u} \frac{\partial \bar{p}}{\partial x} + \bar{u} \frac{\partial \bar{p}}{\partial x} + \bar{u} \frac{\partial \bar{p}}{\partial x} + \bar{w} \frac{\partial \bar{p}}{\partial z} + \bar{w} \frac{\partial \bar{p}}{\partial z} + \bar{w} \frac{\partial \bar{p}}{\partial z} = \frac{1}{Re} \frac{\partial^2 \bar{p}}{\partial x^2} + \frac{1}{Re} \frac{\partial^2 \bar{p}}{\partial x^2} + \frac{1}{Re} \frac{\partial^2 \bar{p}}{\partial z^2} + \frac{1}{Re} \frac{\partial^2 \bar{p}}{\partial z^2}, \quad (B.27) \]

\[ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{w}}{\partial z} = 0. \quad (B.28) \]

Note that we have now removed the primes from the coordinate variables, but continue to use the nondimensional variables.

We now begin to eliminate terms. The background case is steady, so we eliminate the time derivatives of averaged quantities. Background variables which independently satisfy (B.1)-(B.4) can also be eliminated. In addition, we disregard double perturbation quantities, leaving only linear terms,

\[ A \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{w} \frac{\partial \bar{u}}{\partial z} + \bar{w} \frac{\partial \bar{u}}{\partial z} = \frac{\partial \bar{p}}{\partial z} + \frac{1}{Re} \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{1}{Re} \frac{\partial^2 \bar{u}}{\partial z^2}, \quad (B.29) \]

\[ A \frac{\partial \bar{w}}{\partial t} + \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{w} \frac{\partial \bar{w}}{\partial z} + \bar{w} \frac{\partial \bar{w}}{\partial z} = - \frac{\partial \bar{p}}{\partial z} + \frac{1}{Re} \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{1}{Re} \frac{\partial^2 \bar{w}}{\partial z^2}, \quad (B.30) \]

\[ A \frac{\partial \bar{p}}{\partial t} + \bar{u} \frac{\partial \bar{p}}{\partial x} + \bar{u} \frac{\partial \bar{p}}{\partial x} + \bar{w} \frac{\partial \bar{p}}{\partial z} + \bar{w} \frac{\partial \bar{p}}{\partial z} = \frac{1}{Re} \frac{\partial^2 \bar{p}}{\partial x^2} + \frac{1}{Re} \frac{\partial^2 \bar{p}}{\partial z^2}, \quad (B.31) \]

\[ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0. \quad (B.32) \]

To further reduce this set of equations, make the assumptions that \( \bar{u} \) and \( \bar{p} \) are independent of \( x \), and that \( \bar{w} \approx 0 \), leaving

\[ A \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{w} \frac{\partial \bar{u}}{\partial z} = \frac{\partial \bar{p}}{\partial z} + \frac{1}{Re} \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{1}{Re} \frac{\partial^2 \bar{u}}{\partial z^2}, \quad (B.33) \]

\[ A \frac{\partial \bar{w}}{\partial t} + \bar{u} \frac{\partial \bar{w}}{\partial x} = - \frac{\partial \bar{p}}{\partial z} - g \bar{p} + \frac{1}{Re} \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{1}{Re} \frac{\partial^2 \bar{w}}{\partial z^2}, \quad (B.34) \]

\[ A \frac{\partial \bar{p}}{\partial t} + \bar{u} \frac{\partial \bar{p}}{\partial x} + \bar{w} \frac{\partial \bar{p}}{\partial z} = \frac{1}{Re} \frac{\partial^2 \bar{p}}{\partial x^2} + \frac{1}{Re} \frac{\partial^2 \bar{p}}{\partial z^2}, \quad (B.35) \]

\[ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0. \quad (B.36) \]
B.4 Reduction of system to a single equation

It is possible to reduce this set of four equations to only one equation. Begin by defining a perturbation streamfunction $\psi$ so that

$$\tilde{u} = -\frac{\partial \psi}{\partial z},$$  
(B.37)

$$\tilde{w} = \frac{\partial \psi}{\partial x},$$  
(B.38)

which automatically satisfies (B.36) and leaves the other equations as

$$-A\frac{\partial^2 \psi}{\partial t \partial z} - \tilde{u} \frac{\partial^2 \psi}{\partial x \partial z} + \frac{\partial \psi}{\partial z} \frac{\partial \tilde{u}}{\partial z} = -\frac{\partial \tilde{p}}{\partial z} - \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial x^2 \partial z^2} - \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial z^4},$$  
(B.39)

$$A\frac{\partial^2 \psi}{\partial t \partial x} + \tilde{u} \frac{\partial^2 \psi}{\partial x^2} = -\frac{\partial \tilde{p}}{\partial x} - g\tilde{p} + \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial x^4} + \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial z^2 \partial x^2},$$  
(B.40)

Cross-differentiate (B.39) and (B.40) to eliminate pressure,

$$-A\frac{\partial^2 \psi}{\partial x^2 \partial t} - \frac{\partial}{\partial z} \left( \tilde{u} \frac{\partial^2 \psi}{\partial x \partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \tilde{u}}{\partial z} \right) = -\frac{\partial \tilde{p}}{\partial x} - \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial x^2 \partial z^2} - \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial z^4},$$  
(B.42)

$$A\frac{\partial^2 \psi}{\partial x^2 \partial t} + \tilde{u} \frac{\partial^2 \psi}{\partial x^4} = -\frac{\partial \tilde{p}}{\partial x} - g\tilde{p} + \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial x^4} + \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial z^2 \partial x^2}. $$  
(B.43)

From (B.43) subtract (B.42) to give

$$A\frac{\partial^3 \psi}{\partial x^2 \partial t} + \tilde{u} \frac{\partial^3 \psi}{\partial x^4} + A\frac{\partial^3 \psi}{\partial t \partial z} + \tilde{u} \frac{\partial^2 \psi}{\partial z \partial x} + \tilde{u} \frac{\partial^2 \psi}{\partial x^2 \partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \tilde{u}}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \tilde{u}}{\partial z} =$$

$$-g\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial x^4} + \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial z^2 \partial x^2} + \frac{1}{Re_T} \frac{\partial^4 \psi}{\partial z^4}, $$  
(B.44)

and rearrange so that the problem is reduced to just two equations, (B.41) and

$$A\frac{\partial}{\partial t} \nabla^2 \psi + \tilde{u} \frac{\partial}{\partial x} \nabla^2 \psi - \frac{\partial \psi}{\partial x} \frac{\partial \tilde{u}}{\partial z} = -g\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re_T} \nabla^2 \nabla^2 \psi.$$  
(B.45)

To combine these two we write them in the form

$$\left( A\frac{\partial}{\partial t} + \tilde{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right) \nabla^2 \psi - \frac{\partial \psi}{\partial x} \frac{\partial \tilde{u}}{\partial z} = -g\frac{\partial \tilde{p}}{\partial x}, $$  
(B.46)

$$\left( A\frac{\partial}{\partial t} + \tilde{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right) \tilde{p} + \frac{\partial \psi}{\partial x} \frac{\partial \tilde{p}}{\partial z} = 0.$$  
(B.47)

We multiply (B.46) by the differential operator in the brackets,

$$\left( A\frac{\partial}{\partial t} + \tilde{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right)^2 \nabla^2 \psi - \left( A\frac{\partial}{\partial t} + \tilde{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right) \frac{\partial \psi}{\partial x} \frac{\partial \tilde{u}}{\partial z}$$

$$+ g \left( A\frac{\partial}{\partial t} + \tilde{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right)\frac{\partial \tilde{p}}{\partial x} = 0.$$  
(B.48)
Equation (B.47) is differentiated with respect to \(x\) and multiplied by \(g\), and thus becomes

\[
g \left( A \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right) \frac{\partial \Phi}{\partial x} = -g \frac{\partial^2 \psi}{\partial x^2} \frac{\partial \rho}{\partial x}. \tag{B.49}
\]

We can then reduce our system of equations to one partial differential equation,

\[
\left( A \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right)^2 \nabla^2 \psi - \left( A \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right) \frac{\partial \psi}{\partial x} \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} N^2 = 0, \tag{B.50}
\]

where we have defined \(N = (-g \partial \rho/\partial z)^{1/2}\) to be the buoyancy frequency.

The square of the differential equation operator is expanded as follows:

\[
\left( A \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right)^2 = \left( A \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right) \left( A \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right) = A^2 \frac{\partial^2}{\partial t^2} + \bar{u}^2 \frac{\partial^2}{\partial x^2} + \frac{1}{Re_T} \nabla^4 + 2A \bar{u} \frac{\partial^2}{\partial t \partial x} - 2 \frac{A}{Re_T} \nabla^2 \frac{\partial}{\partial t} - \frac{\bar{u}}{Re_T} \nabla^2 \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \left( \bar{u} \frac{\partial}{\partial x} \right) = A^2 \frac{\partial^2}{\partial t^2} + \bar{u}^2 \frac{\partial^2}{\partial x^2} + \frac{1}{Re_T} \nabla^4 + 2A \bar{u} \frac{\partial^2}{\partial t \partial x} - 2 \frac{A}{Re_T} \nabla^2 \frac{\partial}{\partial t} - 2 \frac{\bar{u}}{Re_T} \nabla^2 \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \frac{\partial}{\partial x} = \frac{1}{Re_T} \frac{\partial^2 \bar{u}}{\partial x^2} \frac{\partial^2}{\partial t^2} + \frac{1}{Re_T} \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2}{\partial t^2} + \frac{1}{Re_T} \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2}{\partial x^2} \tag{B.51}
\]

where the last two terms are inserted because of the Laplacian operator applied to the average velocity. This can now be inserted into (B.50),

\[
\left( A^2 \frac{\partial^2}{\partial t^2} + \bar{u}^2 \frac{\partial^2}{\partial x^2} + \frac{1}{Re_T} \nabla^4 + 2A \bar{u} \frac{\partial^2}{\partial t \partial x} - 2 \frac{A}{Re_T} \nabla^2 \frac{\partial}{\partial t} \right) \nabla^2 \psi - \left( A \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} - \frac{1}{Re_T} \nabla^2 \right) \frac{\partial \psi}{\partial x} \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} N^2 = 0. \tag{B.52}
\]

For the next part of this problem it is easier to adjust our notation to use subscripts.
to represent partial derivatives, giving

\[
A^2(\psi_{xtt} + \psi_{ztt}) + \bar{u}^2(\psi_{xxxx} + \psi_{zzzz}) + \Re_T^{-2}(\psi_{xxxxxxx} + 3\psi_{xxxxxx} + 3\psi_{xxxx} + \psi_{zzzzzz}) \\
+ 2A\bar{u}(\psi_{xxxt} + \psi_{zzzt}) - 2A\Re_T^{-1}(\psi_{xxxx} + 2\psi_{xxxxxx} + \psi_{zzzzzz}) \\
- 2\bar{u}\Re_T^{-1}(\psi_{xxxx} + 2\psi_{xxxxxx} + \psi_{zzzzzz}) - \bar{u}_{zzz}\Re_T^{-1}(\psi_{zzzzzz} + \psi_{zzzzzzzz}) \\
- 2\bar{u}_z\Re_T^{-1}(\psi_{xxxx} + \psi_{zzzzzz}) - A\bar{u}_{zzz}\psi_{zx} - \bar{u}_{zzz}\psi_{zz} + \bar{u}_z\Re_T^{-1}(\psi_{zzzz} + \psi_{zzzzzz}) \\
+ 2\bar{u}_{zzz}\Re_T^{-1}\psi_{zx} + \bar{u}_{zzzzz}\Re_T^{-1}\psi_z + \psi_{zzzzzzzz}N^2 = 0. \tag{B.53}
\]

\section*{B.5 Looking for eigenfunctions}

Equation (B.53) cannot be directly solved. To make the solution of this equation possible, we assume that we are looking at perturbations which are sinusoidal in form, and propagate in the $x$ direction,

\[
\psi = \hat{\psi}(z)e^{(kz-\omega t)}. \tag{B.54}
\]

Here we use dimensionless wavenumber $k$ and dimensionless frequency. Substituting (B.54) into (B.53) gives us a nonlinear ordinary differential equation in $z$,

\[
A^2(k^2\omega^2\hat{\psi} - \omega^2\hat{\psi}_{zz}) + \bar{u}^2(k^4\hat{\psi} - k^2\hat{\psi}_{zz}) + \Re_T^{-2}(-k^6\hat{\psi} + 3k^4\hat{\psi}_{zz} - 3k^2\hat{\psi}_{zzzz} + \hat{\psi}_{zzzzzz}) \\
+ 2A\bar{u}(-k^3\omega\hat{\psi} + k\omega\hat{\psi}_{zz}) - 2A\Re_T^{-1}(-ik^4\omega\hat{\psi} + 2ik^2\omega\hat{\psi}_{zz} - i\omega\hat{\psi}_{zzzz}) \\
- 2\bar{u}\Re_T^{-1}(ik^3\hat{\psi} - 2ik^3\hat{\psi}_{zz} + ik\hat{\psi}_{zzzz}) - \bar{u}_{zzz}\Re_T^{-1}(-ik^3\hat{\psi} + ik\hat{\psi}_{zz}) \\
- 2\bar{u}_z\Re_T^{-1}(-ik^3\hat{\psi} + ik\hat{\psi}_{zzzz}) - A\bar{u}_{zzz}k\omega\hat{\psi} + \bar{u}_{zzz}k^2\hat{\psi} + \bar{u}_z\Re_T^{-1}(-ik^3\hat{\psi} + ik\hat{\psi}_{zz}) \\
+ 2\bar{u}_{zzzzz}\Re_T^{-1}ik\hat{\psi}_z + \bar{u}_{zzzzzzz}\Re_T^{-1}ik\hat{\psi} - k^2\hat{\psi}N^2 = 0. \tag{B.55}
\]

We want to write this equation in terms of phase speed $c$ which we define by

\[
c = \frac{A\omega}{k}, \tag{B.56}
\]

so we divide (B.55) by $k^2$ to give

\[
(k^2c^2\hat{\psi} - c^2\hat{\psi}_{zz}) + \bar{u}^2(k^2\hat{\psi} - \hat{\psi}_{zz}) + \Re_T^{-2}(-k^4\hat{\psi} + 3k^2\hat{\psi}_{zz} - 3\hat{\psi}_{zzzz} + \hat{\psi}_{zzzzzzzz}/k^2) \\
+ 2\bar{u}(-k^3c\hat{\psi} + c\hat{\psi}_{zz}) - 2\Re_T^{-1}(-ik^3c\hat{\psi} + 2ikc\hat{\psi}_{zz} - ic\hat{\psi}_{zzzz}/k) \\
- 2\bar{u}\Re_T^{-1}(ik^3\hat{\psi} - 2ik\hat{\psi}_{zz} + ic\hat{\psi}_{zzzz}/k) - \bar{u}_{zzz}\Re_T^{-1}(-ik\hat{\psi} + ic\hat{\psi}_{zz}/k) \\
- 2\bar{u}_z\Re_T^{-1}(-ik\hat{\psi}_z + ic\hat{\psi}_{zzzz}/k) - \bar{u}_{zzz}c\hat{\psi} + \bar{u}_{zzz}\hat{\psi} + \bar{u}_z\Re_T^{-1}(-ik\hat{\psi} + ic\hat{\psi}_{zz}/k) \\
+ 2\bar{u}_{zzzz}\Re_T^{-1}ic\hat{\psi}_z/k + \bar{u}_{zzzzzzz}\Re_T^{-1}ic\hat{\psi}/k - \hat{\psi}N^2 = 0. \tag{B.57}
\]
Appendix B. Derivation of linear wave propagation

This equation will ultimately be written as an eigenvalue equation where $c$ is the eigenvalue, so we collect like terms of $c$,

\[
c^2[(k^2 - \partial^2)]\hat{\psi} - c[21Re_T^{-1}(-k^3\hat{\psi} + 2k\hat{\psi}_{zz} - \hat{\psi}_{zzzz}/k) - \hat{u}_{zz}\hat{\psi} + 2\hat{u}(-k^2\hat{\psi} + \hat{\psi}_{zz})] + \hat{u}^2(k^2 - \partial^2) + Re_T^2(-k^4 + 3k^2\hat{\psi}_{zz} - 3\hat{\psi}_{zzzz} + \hat{\psi}_{zzzzzz}/k^2) - 2i\hat{u}Re_T^{-1}(k^3\hat{\psi} - 2k\hat{\psi}_{zz} + \hat{\psi}_{zzzz}/k) - i\hat{u}_{zz}Re_T^{-1}(-k\hat{\psi} + \hat{\psi}_{zz}/k)
- 2i\hat{u}_{zz}Re_T^{-1}(-k\hat{\psi}_{zz} + \hat{\psi}_{zzzz}/k) + \hat{u}_{zz}\hat{\psi} + i\hat{u}_{zz}Re_T^{-1}(-k\hat{\psi} + \hat{\psi}_{zz}/k)
+ 2i\hat{u}_{zzzz}Re_T^{-1}\hat{\psi}/k + i\hat{u}_{zzzz}Re_T^{-1}\hat{\psi}/k - \hat{\psi}N^2 = 0. \tag{B.58}
\]

To make it easier to write our final eigenvalue equation, we again change the notation for derivatives, $\partial\hat{\psi} = \hat{\psi}_z$, giving us a more compact form,

\[
c^2[(k^2 - \partial^2)]\hat{\psi} - c[21(Re_Tk)^{-1}(-k^4 + 2k^2\partial^2 - \partial^4) - \hat{u}_{zz} + 2\hat{u}(-k^2 + \partial^2)]\hat{\psi} + [\hat{u}^2(k^2 - \partial^2) + (Re_Tk)^{-2}(-k^6 + 3k^4\partial^2 - 3k^2\partial^4 + \partial^6) - 2i\hat{u}(Re_Tk)^{-1}(k^4 - 2k^2\partial^2 + \partial^4) - 2i\hat{u}_{zz}(Re_Tk)^{-1}(-k^2 + \partial^2) - 2i\hat{u}_{zzzz}Re_T^{-1}(-k^2 + \partial^2) + 2i\hat{u}_{zzzz}Re_T^{-1}\partial + i\hat{u}_{zzzz}Re_T^{-1} - N^2]\hat{\psi} = 0. \tag{B.59}
\]

A new parameter $K_c$ is introduced, where we define

\[
K_c \equiv \frac{1}{Re_Tk}, \tag{B.60}
\]

so that the new parameter represents the ratio of wavelength to the effect of viscosity upon the momentum balance. When $K_c$ is zero, we have inviscid flow, or are representing modes which have zero wavelength and (B.59) reduces to the Taylor–Goldstein equation. When $K_c$ is infinite, we are modelling modes which are long, but are dominated by viscosity. The modes we are interested in will be described by a particular range of values of $K_c$. The equation can now be written

\[
c^2[(k^2 - \partial^2)]\hat{\psi} - c[21K_c(-k^4 + 2k^2\partial^2 - \partial^4) - \hat{u}_{zz} + 2\hat{u}(-k^2 + \partial^2)]\hat{\psi} + [\hat{u}^2(k^2 - \partial^2) + K_c^2(-k^6 + 3k^4\partial^2 - 3k^2\partial^4 + \partial^6) - 2i\hat{u}K_c(k^4 - 2k^2\partial^2 + \partial^4) - 2i\hat{u}_{zz}K_c(-k^2 + \partial^2) + 2i\hat{u}_{zzzz}K_c\partial + i\hat{u}_{zzzz}K_c - N^2]\hat{\psi} = 0. \tag{B.61}
\]

Within each group of terms, we will arrange terms by the power of the derivative,

\[
c^2[k^2 - \partial^2]\hat{\psi} + c[21K_c\partial^4 + (-4iK_ck^2 + 2\hat{u})\partial^2 + 2iK_ck^4 - 2\hat{u}k^2 - \hat{u}_{zz}]\hat{\psi} + [K_c^2\partial^6 - (3K_c^2k^2 + 2i\hat{u}K_c)\partial^4 - 2i\hat{u}_{zz}K_c\partial^3 + (-\hat{u}^2 + 4i\hat{u}K_ck^2 + 3K_c^2k^4)\partial^2 + (-2i\hat{u}_{zz}K_c^2 + 2i\hat{u}_{zzzz}K_c)\partial - k^2\hat{u}^2 - K_c^2k^6 - 2i\hat{u}K_ck^4 + \hat{u}_{zz} + i\hat{u}_{zzzz}K_c - N^2]\hat{\psi} = 0. \tag{B.62}
\]
B.6. Boundary conditions

It can be shown that (B.62) is the same sixth-order linear stability equation from Winters & Riley (1992) which was originally derived by Koppel (1964).

We now make the pivotal assumption in this derivation: that we are investigating long waves (i.e. $k$ is small), and that our parameter $K_c$ is non-negligible resulting in a much simplified equation,

$$-c^2[\partial^2]\hat{\psi} + c[2iK_c\partial^4 + 2u\partial^2 - u_{zzz}]\hat{\psi} + [K_c^2\partial^6 - 2iuK_c\partial^4 - 2i\hat{u}_zK_c\partial^3 - u^2\partial^2 + 2iu_{zzz}K_c\partial + \hat{u}_zzz + i\hat{u}_{zzzz}K_c - N^2]\hat{\psi} = 0. \quad (B.63)$$

This is identical to (3.21) which is solved in chapter 3.

B.6 Boundary conditions

The boundary conditions for the above equations are set by the boundary conditions of the background flow: in our case this is a fixed lid with free-slip conditions on the upper and lower boundaries. In other words, we can write this in terms of our original variables to be

$$w = 0, \quad z = 0, H, \quad (B.64)$$

$$\frac{\partial u}{\partial z} = 0, \quad z = 0, H, \quad (B.65)$$

$$\frac{\partial \rho}{\partial z} = 0, \quad z = 0, H. \quad (B.66)$$

Since $\hat{u}$, $\hat{w}$ and $\hat{\rho}$ already satisfy these conditions, then (B.17), (B.18) and (B.20) demonstrate that after non-dimensionalising $\hat{u}$, $\hat{w}$ and $\hat{\rho}$ must also satisfy these conditions. Using (B.37)-(B.38), the velocity boundary conditions translate to

$$\frac{\partial \hat{\psi}}{\partial z} = 0, \quad z = 0, H, \quad (B.67)$$

$$\frac{\partial^2 \hat{\psi}}{\partial z^2} = 0, \quad z = 0, H. \quad (B.68)$$

By including (B.54) we can write

$$ik\hat{\psi}e^{i(kz-\omega t)} = 0, \quad z = 0, H, \quad (B.69)$$

$$\frac{\partial^2 \hat{\psi}}{\partial z^2}e^{i(kz-\omega t)} = 0, \quad z = 0, H, \quad (B.70)$$

so that the final boundary conditions are written as

$$\hat{\psi} = 0, \quad z = 0, H, \quad (B.71)$$

$$\hat{\psi}_{zz} = 0, \quad z = 0, H. \quad (B.72)$$
Boundary conditions on the density field may result in further restrictions to boundary values of the streamfunction. To determine if this is the case, we use equations (B.41) and (B.45). First, assume that \( \tilde{\rho} \) behaves sinusoidally like \( \psi \),

\[
\tilde{\rho} = \tilde{\rho}(z)e^{i(kz-\omega t)}. \tag{B.73}
\]

By substituting (B.54) and (B.73) into (B.41) and (B.45), we obtain

\[
-i\omega A(-k^2+\partial^2)\psi + ik\tilde{u}(-k^2+\partial^2)\psi - ik\tilde{\psi}u_{zz} = -ikg\tilde{\rho} + Re^{-1}(-k^2-\partial^2)^2\psi, \tag{B.74}
\]

\[
A(-i\omega)\hat{\rho} + \hat{u}(ik)\hat{\rho} + (ik)\hat{\psi}\hat{\rho}_z = Re^{-1}(-k^2+\partial^2)\hat{\rho}. \tag{B.75}
\]

We divide by \( ik \) and use (B.56) and (B.60) to write

\[
-c(-k^2+\partial^2)\psi + \tilde{u}(-k^2+\partial^2)\psi - \psi u_{zz} = -g\tilde{\rho} - iKc(-k^2+\partial^2)(-k^2+\partial^2)\psi, \tag{B.76}
\]

\[
-c\tilde{\rho} + \tilde{u}\tilde{\rho} + \psi\tilde{\rho}_z = -iKc(-k^2+\partial^2)\tilde{\rho}. \tag{B.77}
\]

At this stage we use the long-wave approximation to eliminate explicit use of \( k \), simplifying the equations to

\[
(\bar{u} - c)\psi_{zz} - \psi u_{zz} = -g\tilde{\rho} - iKc\psi_{zzzz}, \tag{B.78}
\]

\[
(\bar{u} - c)\tilde{\rho} + \psi\tilde{\rho}_z = -iKc\psi_{zzzz}. \tag{B.79}
\]

A simple condition can be found by applying (B.71) and (B.72) to (B.78), giving

\[
\psi_{zzzz} = -\frac{g}{iKc}\tilde{\rho}, \quad z = 0, H. \tag{B.80}
\]

In addition we use (B.71) and (B.79) to write

\[
\tilde{\rho}_{zz} = -\frac{(\bar{u} - c)}{iKc}\tilde{\rho}, \tag{B.81}
\]

or,

\[
\tilde{\rho}_{zz} = \frac{(\bar{u} - c)}{g}\psi_{zzzz}, \quad z = 0, H. \tag{B.82}
\]

Of course (B.81) also means that

\[
\tilde{\rho}_{zz} = -\frac{(\bar{u} - c)}{iKc}\tilde{\rho}_z - \frac{\bar{u}_z}{iKc}\tilde{\rho}. \tag{B.83}
\]

Since

\[
\bar{u}_z = 0, \quad z = 0, H, \tag{B.84}
\]

\[
\tilde{\rho}_z = 0, \quad z = 0, H, \tag{B.85}
\]
we can write

\[ \dot{\rho}_{zz} = 0, \quad z = 0, H. \] (B.86)

We go back to (B.78) and differentiate twice, giving

\[
(\ddot{u} - c)\dot{\psi}_{zzzz} + 2\dot{u}_z \dot{\psi}_{zz} + \ddot{u}_{zz} \dot{\psi}_{zz} - \dddot{\psi}_{zz} \ddot{u}_{zz} - 2\dddot{\psi}_z \ddot{u}_{zz} - \dddot{\psi} \dddot{u}_{zzz} \]
\[
\quad = -g\dot{\rho}_{zz} - iK_c \dot{\psi}_{zzzzzz}. \quad (B.87)
\]

At the boundaries, we find many of the terms in (B.87) are zero, because of (B.71), (B.72) and (B.84)-(B.86), leaving

\[
(\ddot{u} - c)\dot{\psi}_{zzzz} - 2\dddot{\psi}_z \ddot{u}_{zzz} = -g\dot{\rho}_{zz} - iK_c \dot{\psi}_{zzzzzz}, \quad z = 0, H. \quad (B.88)
\]

We now substitute (B.82) to give

\[ 2\dddot{\psi}_z \ddot{u}_{zzz} = 2(\ddot{u} - c)\dot{\psi}_{zzzz} + iK_c \dot{\psi}_{zzzzzz}, \quad z = 0, H. \quad (B.89)\]

If (B.71) and (B.72) are satisfied, then (B.63) automatically satisfies (B.89), demonstrating that no additional boundary conditions are required.
Appendix B. Derivation of linear wave propagation
Bibliography


