Classifying uniformly generated groups

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Abstract. A finite group $G$ is called uniformly generated, if whenever there is a (strictly ascending) chain of subgroups $1 < \langle x_1 \rangle < \langle x_1, x_2 \rangle < \cdots < \langle x_1, x_2, \ldots, x_d \rangle = G$, then $d$ is the minimal number of generators of $G$. Our main result classifies the uniformly generated groups without using the simple group classification. These groups are related to finite projective geometries by a result of Iwasawa on subgroup lattices.

1. Introduction

Let $G$ be a finite group. A chain $1 = G_0 < G_1 < \cdots < G_d = G$ of subgroups of a $G$ is called unrefinable if $G_i$ is maximal in $G_{i+1}$ for each $i$. The length of $G$, denoted $\ell(G)$, is the maximum length of an unrefinable chain, and the depth of $G$, denoted $\lambda(G)$, is the minimum length of an unrefinable chain. By [4], a nonabelian simple group $G$ satisfies

$$\lambda(G) \leq (1 + o(1)) \frac{\ell(G)}{\log_2(\ell(G))}.$$

It was shown in [5] that $\ell(\text{Alt}_n) = \left\lfloor \frac{3(n-1)}{2} \right\rfloor - s_2(n)$ where $\text{Alt}_n$ is the alternating group of degree $n$, and $s_p(n) = \sum_{i \geq 0} n_i$ is the sum of the digits of the base-$p$ expansion of $n = \sum_{i \geq 0} n_i p^i$. In [3] and [4] the length and depth of finite groups, and algebraic groups, are studied. These references review some of the earlier work in this area.

Iwasawa [8] proved a striking result, namely $\ell(G) = \lambda(G)$ if and only if $G$ is supersolvable. Inspired by this result, [3] classifies the finite groups $G$ for which $\ell(G) - \lambda(G)$ is ‘small’. An elementary proof of Iwasawa’s result is given in [7, Theorem 19.3.1].

We say that $G$ is $d$-uniformly generated if for all $(x_1, x_2, \ldots, x_d) \in G^d$ with

$$1 < \langle x_1 \rangle < \langle x_1, x_2 \rangle < \cdots < \langle x_1, x_2, \ldots, x_d \rangle$$
we have $G = \langle x_1, x_2, \ldots, x_d \rangle$. In Lemma 2.1, we will prove that $G$ is $d$-uniformly generated if and only if $d = \ell(G)$. In particular, this implies that $G$ can be $d$-uniformly generated for at most one choice of $d$. The minimal number of generators of $G$ is denoted $d(G)$. Clearly $G = \langle x_1, x_2, \ldots, x_d \rangle$ implies $d \geq d(G)$. Recall that a generating set $S$ for a group $G$ is called independent (sometimes called irredundant) if $\langle S \setminus \{s\} \rangle < G$ for all $s \in S$. Let $m(G)$ denote the maximal size of an independent generating set for $G$. For example, $d(\text{Sym}_n) = 2$ for $n \geq 3$, and $m(\text{Sym}_n) = n - 1$ for $n \geq 1$ by [9]. The finite groups with $m(G) = d(G)$ are classified by Apisa and Klopsch in [1, Theorem 1.6].

We say that $G$ is uniformly generated if $G$ is $d(G)$-uniformly generated. By Lemma 2.1, $G$ is uniformly generated if and only if $d(G) = \ell(G)$. We classify such groups in Theorem 1.1. Our first proof of this result (see [6, p. 4]) relied on the Classification of Finite Simple Groups (CFSG). This dependence seemed undesirable as the conclusion did not involve any nonabelian simple groups. The proof we give appeals to Iwasawa’s result, and is completely elementary.

**Theorem 1.1.** Let $G$ be a finite group, and let $C_n$ denote a cyclic group of order $n$. Then $G$ is uniformly generated if and only if either $G \cong (C_p)^d$ is elementary or $G \cong (C_p)^{d-1} \rtimes C_q$ where $p, q$ are primes and $C_q$ acts as a nontrivial scalar on $(C_p)^{d-1}$.

**Remark 1.2.** There are two key ideas for the proof of Theorem 1.1. First, for any group $G$, we have $d(G) \leq m(G) \leq \ell(G)$ and $d(G) \leq \lambda(G) \leq \ell(G)$, and second

(1) if $G$ is uniformly generated, then $d(G) = \ell(G)$ and hence $\ell(G) = \lambda(G) = m(G)$.

Since $\lambda(G) = \ell(G)$, a uniformly generated group $G$ must be supersolvable by [8]. Further, since $d(G) = m(G)$ it is amongst the (solvable) groups classified by Apisa and Klopsch in [1, Theorem 1.6]. Their groups are structurally similar to ours, but with a more general module action. Our proof does not refer to [1], even though it would be natural to do so, because we want our proof to be independent of the CFSG.

**Remark 1.3.** The groups we classify in Theorem 1.1 arise in connection with other very natural characterizations. For example, Iwasawa [8] classified the groups $G$ whose subgroup lattice forms a finite projective geometry with at least three points on a line, and found the same groups. Further, Baer [2, Theorem 11.2(b)] determined the same groups when considering “subgroup-isomorphisms” and “ideal-cyclic” groups [2, p. 2, p. 8].
2. Proof

The characterization of $d$-uniformly generated groups in Lemma 2.1 below helps to prove Theorem 1.1.

**Lemma 2.1.** A finite group $G$ is $d$-uniformly generated if and only if $d = \ell(G)$.

**Proof.** The inequality $d \leq \ell(G)$ is clear. Suppose now that $G$ is $d$-uniformly generated and $d < \ell(G)$. Then there exists an unrefinable chain

$$1 = G_0 < G_1 < \cdots < G_{\ell(G)} = G.$$ 

Since $G_i$ is maximal in $G_{i+1}$ we have $G_{i+1} = \langle G_i, x_i \rangle$ for all $x_i \in G_i \setminus G_{i-1}$. It follows that $G_i = \langle x_1, \ldots, x_i \rangle$ and $1 = G_0 < G_1 < \cdots < G_d < G_{\ell(G)} = G$. Consequently, $G$ is not $d$-uniformly generated. This contradiction proves the result. \qed

Recall the following definitions. The Frattini subgroup, $\Phi(G)$, is the intersection of the maximal subgroups of $G$; so the elements of $\Phi(G)$ are precisely the elements of $G$ contained in no independent generating sets of $G$. The Fitting subgroup, $F(G)$, is the largest normal nilpotent subgroup of $G$.

**Lemma 2.2.** Let $G$ be a finite uniformly generated group.

(a) If $1 \trianglelefteq N \trianglelefteq G$, then $N$ and $G/N$ are both uniformly generated.
(b) The Frattini subgroup $\Phi(G)$ is trivial.

**Proof.** (a) Suppose $1 \trianglelefteq N \trianglelefteq G$. For any group $G$ we have $d(G) \leq d(G/N) + d(N)$ and $\ell(G) = \ell(G/N) + \ell(N)$, see [5, Lemma 2.1]. Since $G$ is uniformly generated,

$$d(G) = \ell(G) = \ell(G/N) + \ell(N) \geq d(G/N) + d(N) \geq d(G).$$

Therefore, $\ell(G/N) = d(G/N)$ and $\ell(N) = d(N)$, implying that $G/N$ and $N$ are uniformly generated by Lemma 2.1.

(b) Assume that $\Phi(G) \neq 1$, and choose $1 \neq x_1 \in \Phi(G)$. Suppose $Y = \{y_1, \ldots, y_d\}$ generates $G$, where $d = d(G)$. The minimality of $d$ implies $\langle y_2, \ldots, y_d \rangle < G$, and hence $\langle x_1, y_2, \ldots, y_d \rangle < G$ as $x_1 \in \Phi(G)$. If for some $i < d$, the subgroup $\langle x_1, y_2, \ldots, y_i \rangle$ equals $\langle x_1, y_2, \ldots, y_{i+1} \rangle$, then $y_{i+1} \in \langle x_1, y_2, \ldots, y_i \rangle$. In this case, we therefore have

$$G = \langle y_1, \ldots, y_i, y_{i+1}, \ldots, y_d \rangle = \langle y_1, \ldots, y_i, x_1, y_{i+2}, \ldots, y_d \rangle = \langle y_1, \ldots, y_{i-1}, y_{i+2}, \ldots, y_d \rangle < G.$$
This contradiction shows that there is a strictly ascending chain

\[ 1 < \langle x_1 \rangle < \langle x_1, y_2 \rangle < \cdots < \langle x_1, y_2, \ldots, y_d \rangle < G \]

with too many subgroups, contradicting the fact that \( G \) is uniformly generated. \( \square \)

**Proof of Theorem 1.1.** Assume that \( G \) is uniformly generated and \( d = d(G) \). Then \( \ell(G) = \lambda(G) \) by (1), and \( G \) is supersolvable by [8]. Assume \( G \neq 1 \) and \( N := F(G) \). Then \( N \neq 1 \) since \( G \) is solvable. Lemma 2.2(a,b) imply that \( \Phi(N) = 1 \). If \( |N| \) is divisible by two primes, then we have a smaller generating set. Hence \( N \) must be elementary abelian. The first possibility is \( G = N \cong (C_p)^d \). Suppose now that \( N \) is a proper subgroup of \( G \). Since \( G \) is supersolvable, the derived subgroup \( G' \) is nilpotent, so \( G' \leq F(G) \) and \( G/F(G) \) is abelian. The above argument shows that \( G/N \) is an elementary \( q \)-group. Clearly \( q \neq p \). Let \( g \in G \) have order \( q \). By Lemma 2.2(a), \( N \langle g \rangle \) is uniformly generated, and by Maschke’s theorem \( N \) is a direct sum of simple \( \langle g \rangle \)-submodules which must have dimension 1 and be isomorphic. Therefore \( g \) acts as a scalar matrix on \( N \). The scalar has order \( q \), and not 1 because \( N = F(G) \). Also, if \( |G/N| = q^k \), then we must have \( k = 1 \), otherwise we could find an element of order \( q \) centralizing \( N \) and hence \( N < F(G) \), a contradiction. In summary, either \( G \cong (C_p)^d \) or \( G \cong (C_p)^{d-1} \rtimes C_q \) where \( C_q \) acts as a nontrivial scalar on \( (C_p)^{d-1} \). Conversely, such groups are easily shown to be uniformly generated and to have \( d = d(G) \). \( \square \)

We conclude with two open problems.

**Problem 2.3.** Classify the finite groups \( G \) with \( m(G) - d(G) \leq 1 \).

**Problem 2.4.** Bound the difference \( m(G) - d(G) \), for a connected algebraic group \( G \).

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References


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