Abstract This is an essay about a certain family of elements in the general linear group \( GL(d, q) \) called primitive prime divisor elements, or ppd-elements. A classification of the subgroups of \( GL(d, q) \) which contain such elements is discussed, and the proportions of ppd-elements in \( GL(d, q) \) and the various classical groups are given. This study of ppd-elements was motivated by their importance for the design and analysis of algorithms for computing with matrix groups over finite fields. An algorithm for recognising classical matrix groups, in which ppd-elements play a central role is described.

1 Introduction.

The central theme of this essay is the study of a special kind of element of the general linear group \( GL(d, q) \) of nonsingular \( d \times d \) matrices over a finite field \( GF(q) \) of order \( q \). We define these elements, which we call \textit{primitive prime divisor elements} or \textit{ppd-elements}, and give good estimates of the frequencies with which they occur in \( GL(d, q) \) and the various classical matrix groups. Further we describe a classification of the subgroups of \( GL(d, q) \) which contain ppd-elements, and explore their role in the design and analysis of a randomised algorithm for recognising the classical matrix groups computationally.

Perhaps the best way to introduce these ideas, and to explain the reasons for investigating this particular set of research questions, may be to give a preliminary discussion of a generic recognition algorithm for matrix groups. We wish to determine whether a given subgroup \( G \) of \( GL(d, q) \) contains a certain subgroup \( \Omega \). We design the algorithm to study properties of randomly selected elements from \( G \) in such a way that, if \( G \) contains \( \Omega \) then with high probability we will gain sufficient information from these elements to conclude with certainty that \( G \) does contain \( \Omega \). A skeleton outline of the algorithm could be written as follows.

\textbf{Algorithm 1.1} To recognise whether a given subgroup of \( GL(d, q) \) contains a certain subgroup \( \Omega \).

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Input: $G = \langle X \rangle \leq \text{GL}(d, q)$ and possibly some extra information about $G$.

Output: Either

(a) $G$ contains the subgroup $\Omega$, or
(b) $G$ does not contain $\Omega$.

If Algorithm\[1\] returns option (a) then $G$ definitely contains $\Omega$. However if option (b) is returned there is a possibility that this response is incorrect. In other words Algorithm\[1\] is a Monte Carlo algorithm. It proceeds by making a sequence of random selections of elements from the group $G$, seeking a certain kind of subset $E$ of $G$, which if found will greatly assist in deciding whether or not $G$ contains $\Omega$. The essential requirements for $E$ are two-fold:

1. If $G$ contains a subset $E$ with the required properties, then either $G$ contains $\Omega$, or $G$ belongs to a short list of other possible subgroups of $\text{GL}(d, q)$ (and the algorithm must then distinguish subgroups in this list from subgroups containing $\Omega$).

2. If $G$ contains $\Omega$, then the event of not finding a suitable subset $E$ in $G$ after a reasonable number $N(\varepsilon)$ of independent random selections of elements from $G$ has probability less than some small pre-assigned number $\varepsilon$.

In order to make the first requirement explicit, we need a classification of the subgroups of $\text{GL}(d, q)$ which contain a suitable subset $E$. Similarly in order to make the second requirement explicit, we need good estimates for the proportions of “$E$-type elements” in groups containing $\Omega$. Moreover, if these two requirements are to lead to an efficient algorithm for recognising whether $G$ contains $\Omega$, the proportions of $E$-type elements in groups containing $\Omega$ must be fairly large to guarantee that we have a good chance of finding a suitable subset $E$ after a reasonable number of random selections; and in practice we need good heuristics for producing approximately random elements from a group. Also, among other things, we need efficient procedures to identify $E$-type elements, and to distinguish between the subgroups on the short list and the subgroups which contain $\Omega$. The aim of this paper is to present and discuss results of these types, and the corresponding recognition algorithms, in the cases where $\Omega$ is one of the classical matrix groups. In these cases the subset $E$ consists of certain ppd-elements.

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2 Classical groups.

We consider certain subgroups of $GL(d, q)$ where $d$ is a positive integer and $q = p^a$, a power of a prime $p$, and we let $V$ denote the underlying vector space of $d$-dimensional row vectors over $GF(q)$ on which $GL(d, q)$ acts naturally.

The classical groups preserve certain bilinear, sesquilinear or quadratic forms on $V$. To describe them we adapt some notation from the book of Kleidman and Liebeck [26].

A subgroup $G$ of $GL(d, q)$ is said to preserve a form $\kappa$ modulo scalars if there exists a homomorphism $\mu : G \to GF(q)^\#$ such that, in the case of a bilinear or sesquilinear form, $\kappa(ug, vg) = \mu(g) \cdot \kappa(u, v)$, or, in the case of a quadratic form, $\kappa(vg) = \mu(g) \cdot \kappa(v)$, for all $u, v \in V$ and $g \in G$. A matrix $g$ in such a group is said to preserve $\kappa$ modulo scalars, and if $\mu(g) = 1$ then $g$ is said to preserve $\kappa$. We denote by $\Delta$ or $\Delta(V, \kappa)$ the group of all matrices in $GL(d, q)$ which preserve $\kappa$ modulo scalars, and by $S$ the subgroup of $\Delta$ consisting of those matrices which preserve $\kappa$ and which have determinant 1.

The subgroup $\Omega$ which we shall seek to recognise is equal to $S$ unless $\kappa$ is a non-degenerate quadratic form, and in this latter case $\Omega$ has index 2 in $S$ and is the unique such subgroup of $S$. There are four families of subgroups which we shall consider, and by a classical group in $GL(d, q)$ we shall mean a subgroup $G$ which satisfies $\Omega \leq G \leq \Delta$, for $\Omega$, $\Delta$ in one of these families. The four families are as follows.

(i) Linear groups: $\kappa = 0$, $\Delta = GL(d, q)$ and $\Omega = SL(d, q)$;

(ii) Symplectic groups: $d$ is even, $\kappa$ is a non-degenerate alternating bilinear form on $V$, $\Delta = GSp(d, q)$ and $\Omega = Sp(d, q)$;

(iii) Orthogonal groups: $\kappa$ is a non-degenerate quadratic form on $V$, $\Delta = GO^\varepsilon(d, q)$, and $\Omega = \Omega^\varepsilon(d, q)$, where $\varepsilon = \pm$ if $d$ is even, and $\varepsilon = \circ$ if $d$ is odd. If $d$ is odd then also $q$ is odd since $\kappa$ is non-degenerate;

(iv) Unitary groups: $q$ is a square, $\kappa$ is a non-degenerate unitary form on $V$, that is a non-degenerate sesquilinear form with respect to the automorphism of $GF(q)$ of order 2, $\Delta = GU(d, q)$ and $\Omega = SU(d, q)$.

The books [26, 40] are good references for information about the finite classical groups.

3 Primitive prime divisors and $ppd$-elements.

Let $b, e$ be positive integers with $b > 1$. A prime $r$ dividing $b^e - 1$ is said to be a primitive prime divisor of $b^e - 1$ if $r$ does not divide $b^i - 1$ for any $i$ such that $1 \leq i < e$. It was proved
by Zsigmondy [11] in 1892 that \( b^e - 1 \) has a primitive prime divisor unless either the pair \((b, e)\) is \((2, 6)\), or \( e = 2 \) and \( b + 1 \) is a power of 2. Observe that

\[
|\text{GL}(d, q)| = q^{\binom{d}{2}} \prod_{1 \leq i \leq d} (q^i - 1).
\]

This means that primitive prime divisors of \( q^e - 1 \) for various values of \( e \leq d \) divide \( |\text{GL}(d, q)| \), and indeed divide \( |\Omega| \) for various of the classical groups \( \Omega \) in \( \text{GL}(d, q) \). We define primitive prime divisor elements, sometimes called ppd-elements, in \( \text{GL}(d, q) \) to be those elements with order a multiple of some such primitive prime divisor. Thus we define an element \( g \in \text{GL}(d, q) \) to be a ppd \((d, q; e)\)-element if its order \( o(g) \) is divisible by some primitive prime divisor of \( q^e - 1 \).

Our interest is mainly in ppd \((d, q; e)\)-elements with \( e > d/2 \) and we shall describe in Section 5 a classification by Guralnick, Penttila, Saxl and the author in [17] of all subgroups of \( \text{GL}(d, q) \) containing such an element. We shall henceforth reserve the term ppd-elements to refer to elements of \( \text{GL}(d, q) \) which are ppd \((d, q; e)\)-elements for some \( e > d/2 \). Note that, if \( g \in \text{GL}(d, q) \) is a ppd \((d, q; e)\)-element with \( e > d/2 \), then there is a unique \( g \)-invariant \( e \)-dimensional subspace of the underlying vector space \( V \) on which \( g \) acts irreducibly, and also the characteristic polynomial for \( g \) has an irreducible factor over \( \text{GF}(q) \) of degree \( e \). While neither of these two conditions is sufficient to guarantee that an element is a ppd \((d, q; e)\)-element, it turns out that most elements satisfying either of them are in fact ppd \((d, q; e)\)-elements. In addition, a large proportion of elements in any of the classical groups are ppd-elements, and this fact has proved to be very important for the development of recognition algorithms for classical groups.

In 1974 Hering [19] investigated subgroups of \( \text{GL}(d, q) \) containing ppd \((d, q; d)\)-elements. Such subgroups act irreducibly on \( V \). Hering was interested in applications of these results to geometry, in particular for constructing finite translation planes. He was also interested in the link between such groups and finite affine 2-transitive permutation groups. If \( G \) is a finite affine 2-transitive permutation group acting on a set \( X \), then \( X \) may be taken as the set of vectors of a finite vector space, say \( V = V(d, q) \) of dimension \( d \) over \( \text{GF}(q) \), and \( G = NG_o \) where \( N \) is the group of translations of \( V \) and \( G_o \) is a subgroup of \( \text{GL}(d, q) \) acting transitively on \( V^\# \), that is \( G_o \) is a transitive linear group. Conversely if \( G_o \) is a transitive linear group on \( V \), and \( N \) is the group of translations of \( V \), then \( NG_o \) is a 2-transitive permutation group of affine type acting on \( V \). Thus the problems of classifying finite affine 2-transitive groups, and classifying finite transitive linear groups are equivalent. Moreover if \( G_o \) is transitive on \( V^\# \) then \( q^d - 1 \) divides \( |G_o| \) so that \( G_o \) contains a ppd \((d, q; d)\)-element. Hering’s work led to a classification of finite affine 2-transitive permutation groups, see [20] and also [27, Appendix]. In common with most of the classifications we shall mention related to ppd-elements, this classification depends on the classification of the finite simple groups. Merkt [29] extended Hering’s work obtaining a better description of certain of the subgroups.
of $\text{GL}(d, q)$ containing a ppd $(d, q; d)$-element.

Dempwolff [12] in 1987 began an investigation of subgroups of $\text{GL}(d, q)$ containing a ppd $(d, q; e)$-element for some $e \geq d/2$. His analysis is independent of the work of Aschbacher which we shall describe in the next section, and he made significant progress on describing what we shall call (and shall define in the next section) the “geometric subgroups” containing such ppd-elements. He also did some work on the nearly simple examples. The classification in [17] of all subgroups of $\text{GL}(d, q)$ containing a ppd $(d, q; e)$-element for some $e > d/2$ uses the work of Aschbacher to guide both the analysis and the presentation of the examples. Similar results may be obtained if the condition $e > d/2$ is relaxed, but their proofs become more technical.

4 Aschbacher’s classification of finite linear groups.

Aschbacher’s description [2] of subgroups of $\text{GL}(d, q)$, where $q = p^a$ with $p$ prime, has been very influential both on the way problems concerning linear groups are analysed and on the way results about such groups are presented. Aschbacher defined eight families of subgroups $\mathcal{C}_1, \ldots, \mathcal{C}_8$ of $\text{GL}(d, q)$ as follows. These families are usually defined in terms of some geometrical property associated with the action on the underlying vector space $V$, and in all cases maximal subgroups of $\text{GL}(d, q)$ in the family can be identified. Subgroups of $\text{GL}(d, q)$ in these families are therefore called geometric subgroups. We indicate in parentheses the rough structure of a typical maximal subgroup in the family. Note that $Z$ denotes the subgroup of scalar matrices in $\text{GL}(d, q)$. Also, as in [26], we denote by $b$ a cyclic group of order $b$, and for a prime $r$ we denote by $r^{1+2c}$ an extraspecial group of that order.

$\mathcal{C}_1$ These subgroups act reducibly on $V$, and maximal subgroups in the family are the stabilisers of proper subspaces (maximal parabolic subgroups).

$\mathcal{C}_2$ These subgroups act irreducibly but imprimitively on $V$, and maximal subgroups in the family are the stabilisers of direct sum decompositions $V = \oplus_{i=1}^t V_i$ with $\dim V_i = d/t$ (wreath products $\text{GL}(d/t, q) \wr S_t$).

$\mathcal{C}_3$ These subgroups preserve on $V$ the structure of a vector space over an extension field of $\text{GF}(q)$, and maximal subgroups in the family are the stabilisers of extension fields of $\text{GF}(q)$ of degree $b$, where $b$ is a prime dividing $d$ (the groups $\text{GL}(d/b, q^b), b$).

$\mathcal{C}_4$ These subgroups preserve on $V$ the structure of a tensor product of subspaces, and maximal subgroups in the family are the stabilisers of decompositions $V = V_1 \otimes V_2$ (central products $\text{GL}(b, q) \circ \text{GL}(c, q)$ where $d = bc$).
These subgroups preserve on $V$ the structure of a vector space over a proper subfield of $\text{GF}(q)$; such a subgroup is said to be realisable over a proper subfield. The maximal subgroups in the family are the stabilisers modulo scalars of subfields of $\text{GF}(q)$ of prime index $b$ dividing $a$ (central products $\text{GL}(d, q^{1/b}) \circ Z$).

These subgroups have as a normal subgroup an $r$-group $R$ of symplectic type ($r$ prime) which acts absolutely irreducibly on $V$, and maximal subgroups in the family are the normalisers of these subgroups, $(Z_{q-1} \circ R).\text{Sp}(2c, r)$, where $d = r^c$ and $R$ is an extraspecial group $r^{1+2c}$, or if $r = 2$ then $R$ may alternatively be a central product $4 \circ 2^{1+2c}$.

These subgroups preserve on $V$ a tensor decomposition $V = \bigotimes_{i=1}^{t} V_i$ with $\dim V_i = c$, and maximal subgroups in the family are the stabilisers of such decompositions ($(\text{GL}(c, q) \circ \ldots \circ \text{GL}(c, q)).S_t$, where $d = c^t$).

These subgroups preserve modulo scalars a non-degenerate alternating, or sesquilinear, or quadratic form on $V$, and maximal subgroups in the family are the classical groups.

The main result of Aschbacher’s paper [2] (or see [26, Theorem 1.2.1]) states that, for a subgroup $G$ of $\text{GL}(d, q)$ which does not contain $\text{SL}(d, q)$, either $G$ is a geometric subgroup, or the socle $S$ of $G/(G \cap Z)$ is a nonabelian simple group, and the preimage of $S$ in $G$ is absolutely irreducible on $V$, is not realisable over a proper subfield, and is not a classical subgroup (as defined in Section 2). The family of such subgroups is denoted $S$, and subgroups in this family will often be referred to as nearly simple subgroups. Aschbacher [2] also defined families of subgroups of each of the classical subgroups $\Delta$ in $\text{GL}(d, q)$, analogous to $C_1, \ldots, C_8, S$, and proved that each subgroup of a classical group $\Delta$ which does not contain $\Omega$ belongs to one of these families.

5 Linear groups containing ppd-elements.

The analysis in [17] to determine the subgroups of $\text{GL}(d, q)$ which contain a ppd $(d, q; e)$-element for some $e > d/2$, was patterned on a similar analysis carried out in [30] to classify subgroups of $\text{GL}(d, q)$ which contain both a ppd $(d, q; d)$-element and a ppd $(d, q; d-1)$-element. Moreover the results in [17] seek to give information about the smallest subfield over which such a subgroup $G$ is realisable modulo scalars. We say that $G$ is realisable modulo scalars over a subfield $\text{GF}(q_0)$ of $\text{GF}(q)$ if $G$ is conjugate to a subgroup of $\text{GL}(d, q_0) \circ Z$.

Suppose that $G \leq \text{GL}(d, q)$ and that $G$ contains a ppd $(d, q; e)$-element for some $e > d/2$, and let $r$ be a primitive prime divisor of $q^e - 1$ which divides $|G|$. Suppose moreover that $\text{GF}(q_0)$ is the smallest subfield of $\text{GF}(q)$ such that $G$ is realisable modulo scalars over $\text{GF}(q_0)$.  


There is a recursive aspect to the description in [17] of such subgroups $G$ which are geometric subgroups. For example, the reducible subgroups $G$ leave invariant some subspace or quotient space $U$ of $V$ of dimension $m \geq e$, and the subgroup $G^U$ of $\text{GL}(m, q)$ induced by $G$ in its action on $U$ contains a ppd $(m, q; e)$-element. In [17] no further description is given of these examples, though extra information may be obtained about the group $G^U$ by applying the results recursively.

Although the classification of the geometric examples is not difficult, care needs to be taken in order not to miss some of them. For example, while at first sight it might appear that a maximal imprimitive subgroup $\text{GL}(d/t, q) \wr S_t$ (where $t > 1$) cannot contain a ppd $(d, q; e)$-element since $r$ does not divide $|\text{GL}(d/t, q)|$, it is possible sometimes for $r$ to divide $|S_t| = t!$, so that we do have some examples in the family $C_2$.

To understand how this can happen, observe that the defining condition for $r$ to be a primitive prime divisor of $q^e - 1$, namely that $e$ is the least positive integer $i$ such that $r$ divides $q^i - 1$, is equivalent to the condition that $q$ has order $e$ modulo the prime $r$. Thus $r = ke + 1 \geq e + 1$ for some $k \geq 1$. Sometimes we can have $r = e + 1$ (which satisfies $d/2 < r \leq d$) and hence in these cases an imprimitive subgroup $\text{GL}(1, q) \wr S_d$ will contain ppd $(d, q; e)$-elements.

Both of the above observations come into play in describing the examples in the family $C_3$. Here either the prime $r = e + 1 = d$ and the group $G$ is conjugate to a subgroup of $\text{GL}(1, q^d).d$, or $e$ is a multiple of a prime $b$ where $b$ is a proper divisor of $d$ and, replacing $G$ by a conjugate if necessary, $G \leq \text{GL}(d/b, q^b).b$ such that $G \cap \text{GL}(d/b, q^b)$ contains a ppd $(d/b, q^b; e/b)$-element.

After determination of the geometric examples there remains the problem of finding the nearly simple examples. So suppose that $G$ is nearly simple and $S \leq G/(Z \cap G) \leq \text{Aut} S$ for some nonabelian simple group $S$. What we need is a list of all possible groups $G$ together with the values of $d, e$ and $q_0$. Although there is no classification of all the nearly simple subgroups of $\text{GL}(d, q)$ in general, it is possible to classify those which contain a ppd $(d, q; e)$-element. The reason we can do this is that, for each simple group $S$, the presence of a ppd $(d, q; e)$-element in $G$ leads to both upper and lower bounds for $d$ in terms of the parameters of $S$ strong enough to lead to a complete classification.

On the one hand $d$ is at least the minimum degree of a faithful projective representation of $S$ over a field of characteristic $p$, and lower bounds are available for this in terms of the parameters of $S$. On the other hand we have seen that $r = ke + 1 \geq e + 1 \geq (d + 3)/2$, and in all cases we may deduce that $r$ divides $|S|$. Moreover we have an upper bound on the size of prime divisors of $S$ in terms of the parameters of $S$. For some simple groups $S$ the upper and lower bounds for $d$ obtained in this way conflict and we have a proof that there are no examples involving $S$. In many cases however this line of argument simply narrows down the range of possible values for $d$, $e$ and $r$. Often there are examples involving $S$, but, in order to complete the classification, we need to have more information about small dimensional
representations of $S$ in characteristic $p$ than simply the lower bound for the dimension of such representations.

For example if $S = A_n$ with $n \geq 9$ then $d \geq n - 2$ if $p$ divides $n$ and $d \geq n - 1$ otherwise, by [41, 42, 43]. Moreover $r \leq n$, so $(d + 3)/2 \leq n$, and we obtain $n - 2 \leq d \leq 2n - 3$ and $r = e + 1$. The upper bound for $d$ cannot be improved since we may have $r = n = e + 1$ infinitely often. Thus we need more information about small dimensional representations of $A_n$ in characteristic $p$. For $n \geq 15$ this is available from a combination of results of James [22] and Wagner [43]. We see that the representations of $A_n$ and $S_n$ of dimension $n - 1$ or $n - 2$ are those coming from the deleted permutation module in the natural representation. These give an infinite family of examples with $q_0 = p$. All other faithful projective representations of $A_n$ have dimension greater than the upper bound on $d$. For the remaining cases, where $n < 15$, special arguments are required, making full use of information in [11, 23]. The result of this analysis is an explicit list of examples for alternating groups $S$.

The list of examples of linear groups containing ppd-elements can be found in [17, Section 2] and is not reproduced here. Note that completing the classification of the nearly simple examples for classical groups $S$ over fields of characteristic different from $p$ involved proving new results about small dimensional representations of such groups over fields of characteristic $p$.

6 Various applications of the “ppd classification”.

The classification of subgroups of $\mathrm{GL}(d, q)$ containing ppd-elements has already been used in a variety of applications concerning finite classical groups. In particular the papers [16, 18] make use of it to answer questions concerning the generation of finite classical groups, while in [28] it is used to show that the finite classical groups are characterised by their orbit lengths on vectors in their natural modules. Information about the invariant generation of classical simple groups (see [32, 38]) can be deduced from the classification (in [32], or see Section 7) of subgroups of classical groups containing two different ppd-elements. (Elements $x_1, \ldots, x_s$ of a group $G$ are said to generate $G$ invariably if $\langle x_1^{g_1}, \ldots, x_s^{g_s} \rangle$ is equal to $G$ for all $g_1, \ldots, g_s \in G$.)

Similarly in [4] the ppd classification, or more accurately the more specialised classification based on it (and described in Section 7), can be used to deal with the finite classical groups in an analysis of finite groups with the permutizer property. A group $G$ is said to have the permutizer property if, for every proper subgroup $H$ of $G$, there is an element $g \in G \setminus H$ such that $H$ permutes with $\langle g \rangle$, that is $\langle g \rangle H = H \langle g \rangle$. The main result of [4] is that all finite groups with the permutizer property are soluble. The proof consists of an examination of a minimal counterexample to this assertion, and the ppd classification can be used to show that the minimal counterexample cannot be an almost simple classical group.
Two different ppd-elements in linear groups

The principal application up to now of the classification of linear groups containing ppd-elements has been the development by Niemeyer and the author in [32] of a recognition algorithm for finite classical groups in their natural representation. The basic idea of this algorithm is as described in Section 1. Given a subgroup \( G \) of a classical group \( \Delta \) in \( \text{GL}(d, q) \) (as described in Section 2), we wish to determine if \( G \) contains the corresponding classical group \( \Omega \). We do this by examining randomly selected elements from \( G \). The elements of \( G \) which we seek by random selection are ppd \((d, q; e)\)-elements for various values of \( e > d/2 \), and an appropriate set of such elements will form the subset \( E \) mentioned in Section 1.

It turns out that the proportion of ppd \((d, q; e)\)-elements in any of the classical groups is very high (as shown in Section 8), so we are very likely to find such an element after a few independent random selections from any subgroup of \( \Delta \) which contains \( \Omega \). Suppose then that we have indeed found a ppd \((d, q; e)\)-element in our group \( G \), for some \( e > d/2 \). The ppd-classification just described then provides a restricted list of possibilities for the group \( G \). The task is to distinguish subgroups containing \( \Omega \) from the other possibilities, and this task is a nontrivial one.

For the purposes of presenting the basic strategy, we assume that \( G \) is irreducible on \( V \) and that we have complete information about any \( G \)-invariant bilinear, sesquilinear or quadratic forms on \( V \). There are standard tests in practice which may be used to determine whether \( G \) is irreducible on \( V \) and to find all \( G \)-invariant forms (see [21, 35]). Note that in an implementation of the algorithm in [32] a different protocol may be followed for deciding the stage at which to obtain this precise information about \( G \). Nevertheless, we may and shall assume that \( G \) does not lie in the Aschbacher classes \( C_1 \) or \( C_8 \). Then, having found a ppd \((d, q; e)\)-element in \( G \) for some \( e > d/2 \), the ppd-classification would still allow the possibility that \( G \) lies in one of \( C_2, C_3, C_5, C_6 \), or that \( G \) is nearly simple, as well as the desired conclusion that \( G \) contains \( \Omega \). In the nearly simple case, the classification in [17] shows that there are approximately 30 infinite families and 60 individual examples of nearly simple groups in explicitly known representations.

Guided by the original SL-recognition algorithm developed in [30], we decided to seek, in the first instance, two different ppd-elements in \( G \) by which we mean a ppd \((d, q; e)\)-element and a ppd \((d, q; e')\)-element, where \( d/2 < e < e' \leq d \). We also decided to strengthen the ppd-property required of these elements in two different ways, by requiring at least one of the ppd-elements to be large and at least one of them to be basic.

Let \( q = p^a \), and let \( r \) be a primitive prime divisor of \( q^e - 1 \). Recall that \( r = ke + 1 \) for some integer \( k \). We say that \( r \) is a basic primitive prime divisor if \( r \) is a primitive prime divisor of \( p^{(ae)} - 1 \), and that \( r \) is a large primitive prime divisor if either \( r \geq 2e + 1 \), or \( r = e + 1 \) and \( (e + 1)^2 \) divides \( q^e - 1 \). Correspondingly we say that a ppd \((d, q; e)\)-element \( g \) is basic if \( o(g) \) is divisible by a basic primitive prime divisor of \( q^e - 1 \), and that \( g \) is large if \( o(g) \)
is divisible by a large primitive prime divisor $r$ of $q^e - 1$ and either $r \geq 2e + 1$ or $r = e + 1$ and $(e + 1)^2$ divides $o(g)$. Note that, for $e \geq 2$, if $q^e - 1$ has a primitive prime divisor, then $q^e - 1$ has a basic primitive prime divisor unless $(q, e) = (4, 3)$ or $(8, 2)$. Similarly an explicit list can be given for pairs $(q, e)$ for which $q^e - 1$ has a primitive prime divisor but does not have a large primitive prime divisor (see [15, 19] or [32, Theorem 2.2]). Thus in most cases $q^e - 1$ has both a large primitive prime divisor and a basic primitive prime divisor; and many ppd-elements will be both large and basic.

We shall see in Section 8 that requiring the additional condition of being large or basic does not alter significantly the very good upper and lower bounds we can give for the proportion of ppd-elements in subgroups of $\Delta$ containing $\Omega$.

Suppose that we now have $G \subseteq \Delta$ for some classical group $\Delta$ in $GL(d, q)$, with $G$ irreducible on the underlying vector space $V$, and suppose also that we have complete information about $G$-invariant forms so that we can guarantee that $G$ is not contained in the class $C_8$ of subgroups of $\Delta$. Further we suppose that $G$ contains two different ppd-elements, say a ppd $(d, q; e)$-element $g$ and a ppd $(d, q; e')$-element $h$, where $d/2 < e < e' \leq d$.

In [32, Theorem 4.7], Niemeyer and the author refined the classification in [17] to find all possibilities for the group $G$. These possibilities comprise groups containing $\Omega$, members of the Aschbacher families $C_2, C_3$ and $C_5$, and some nearly simple examples. The presence of two different ppd-elements certainly restricts the possibilities within these families, but it is still difficult to distinguish some of them from groups containing $\Omega$.

If we require that at least one of $g, h$ is large and at least one is basic then, as was shown in [32, Theorem 4.8], the possibilities for irreducible subgroups $G$ which do not contain $\Omega$ are certain subgroups in $C_3$ and nearly simple groups in a very short list comprising explicit representations of one infinite family and five individual nearly simple groups.

After our discussion of the proportions of ppd-elements in classical groups in Section 8 we shall return to our discussion of the recognition algorithm. We shall see that the algorithm can be completed by simply seeking a few more ppd-elements of a special kind which, if found, will rule out all but one possibility for $G$, enabling us to conclude that $G$ contains $\Omega$.

### 8 Proportion of ppd-elements in classical groups.

The questions we wish to answer from our discussion in this section are the following. If $\Omega \leq G \leq \Delta \leq GL(d, q)$, and $G$ contains two different ppd-elements at least one of which is large and at least one of which is basic, then what is the probability of finding two such elements after a given number $N$ of independent random selections of elements from $G$? In particular, for a given positive real number $\varepsilon$, is it true that the probability of failing to find such elements after $N$ selections is less than $\varepsilon$ provided $N$ is sufficiently large? And if so just how large must $N$ be?
These questions can be answered using simple probability theory provided that we can determine, for a given $e$ (where $d/2 < e \leq d$), the proportion $\text{ppd}(G, e)$ of elements of $G$ which are $\text{ppd}(d, q; e)$-elements. This proportion may depend on the nature of the classical group $\Delta$: that is, on whether $\Delta$ is a linear, symplectic, orthogonal or unitary group. In particular $\text{ppd}(G, e) = 0$ if $\Delta$ is a symplectic or orthogonal group and $e$ is odd, or if $\Delta$ is a unitary group and $e$ is even, or if $\Delta$ is of type $O^+$ and $e = d$. This can be seen easily by examination of the orders of these groups. In all other cases, provided that $d$ and $q$ are not too small, any subgroup of $\Delta$ which contains $\Omega$ will contain $\text{ppd}(d, q; e)$-elements.

So suppose now that $\Omega \leq G \leq \Delta$, that $d/2 < e \leq d$, and that $G$ contains a $\text{ppd}(d, q; e)$-element $g$. It is not difficult (see [32, Lemma 5.1]) to show that $V$ has a unique $e$-dimensional $g$-invariant subspace $W$ and that $g$ acts irreducibly on $W$. Moreover, if $\Delta$ is a symplectic, orthogonal, or unitary group, then $W$ must be nonsingular with respect to the bilinear, quadratic, or sesquilinear form defining $\Delta$.

Next (see [32, Lemma 5.2]) we observe that the group $G$ acts transitively on the set of all nonsingular $e$-dimensional subspaces of $V$ (or all $e$-dimensional subspaces if $\Delta = \text{GL}(d, q)$). Thus the proportion of $\text{ppd}(d, q; e)$-elements in $G$ is the same as the proportion of such elements which fix a particular nonsingular $e$-dimensional subspace $W$. Therefore we need to determine the proportion of $\text{ppd}(d, q; e)$-elements in the setwise stabiliser $G_W$ of $W$ in $G$.

Now consider the natural map $\varphi : g \mapsto g|_W$ which sends $g \in G_W$ to the linear transformation of $W$ induced by $g$. Then $\Omega(W) \leq \varphi(G) \leq \Delta(W) \leq \text{GL}(W)$, and $\Delta(W)$ has the same type (linear, symplectic, orthogonal, or unitary) as $\Delta$. If $g \in G_W$ and $g$ is a $\text{ppd}(d, q; e)$-element, then every element of the coset $g\text{Ker}\varphi$ is also a $\text{ppd}(d, q; e)$-element, since all elements in the coset induce the same linear transformation $g|_W$ of $W$. Moreover in this case $g|_W$ is a $\text{ppd}(e, q; e)$-element in $\varphi(G)$ and all such elements arise as images under $\varphi$ of $\text{ppd}(d, q; e)$-elements in $G_W$. It follows that $\text{ppd}(G, e)$ is equal to the proportion $\text{ppd}(\varphi(G), e)$ of $\text{ppd}(e, q; e)$-elements in $\varphi(G)$.

Thus it is sufficient for us to determine $\text{ppd}(G, d)$ for each of the possibilities for $\Delta$ which contain $\text{ppd}(d, q; d)$-elements. This was done already by Neumann and the author in [30, Lemmas 2.3 and 2.4] in the case where $\Delta = \text{GL}(d, q)$. The techniques used there work also in the other cases although some care is needed. The basic ideas are as follows.

Let $g$ be a $\text{ppd}(d, q; d)$-element in $G$, and let $C := C_G(g)$. Then $C$ is a cyclic group, called a Singer cycle for $G$, and has order $n$ say, where $n$ divides $q^d - 1$ and $n$ is divisible by some primitive prime divisor of $q^d - 1$. The group $C$ is self-centralising in $G$. Further each $\text{ppd}(d, q; d)$-element in $G$ lies in a unique $G$-conjugate of $C$. The number of $G$-conjugates of $C$ is $|G : N_G(C)|$, and so the number of $\text{ppd}(d, q; d)$-elements in $G$ is equal to $|G : N_G(C)|$ times the number of such elements in $C$. It follows that

$$\text{ppd}(G, d) = |G : N_G(C)| \cdot \text{ppd}(C, d) \cdot \frac{|C|}{|G|} = \frac{\text{ppd}(C, d)}{u},$$

where $\text{ppd}(C, d)$ is the proportion of $\text{ppd}(d, q; d)$-elements in $C$, and $u := |N_G(C) : C|$. In
the linear, symplectic and unitary cases $u = d$, while in the orthogonal case $u$ is either $d$ or $d/2$ depending on which intermediate subgroup $G$ is $(\Omega \leq G \leq \Delta)$. In the orthogonal case we certainly have $u = d$ if $G$ contains $O(V)$.

Thus we need to estimate $\text{ppd} (C, d)$. Let $\Phi$ denote the product of all the primitive prime divisors of $q^d - 1$ (including multiplicities), so that $(q^d - 1)/\Phi$ is not divisible by any primitive prime divisor of $q^d - 1$. In all cases $\Phi$ divides $n = |C|$. Moreover an element $x \in C$ is a $\text{ppd} (d, q; e)$-element if and only if $x^{n/\Phi} \neq 1$, that is if and only if $x$ does not lie in the unique subgroup of $C$ of order $n/\Phi$. Hence

$$\text{ppd} (C, d) = \frac{n - n/\Phi}{n} = 1 - \frac{1}{\Phi},$$

and therefore

$$\text{ppd} (G, d) = \frac{1}{u} \left( 1 - \frac{1}{\Phi} \right) < \frac{1}{u}.$$ 

Since each primitive prime divisor of $q^d - 1$ is of the form $kd + 1 \geq d + 1$, the quantity $\Phi$ is at least $d + 1$, and hence

$$\text{ppd} (G, d) \geq \frac{1}{u} \left( 1 - \frac{1}{d + 1} \right)$$

so we have

$$\frac{1}{u} \left( \frac{d}{d + 1} \right) \leq \text{ppd} (G, d) < \frac{1}{u}.$$ 

Putting all of this together we see that in almost all cases $\text{ppd} (G, d)$ lies between $1/(d + 1)$ and $1/d$, with the exception being some orthogonal cases where $\text{ppd} (G, d)$ lies between $2/(d + 1)$ and $2/d$.

To pull back this result to the general case where $d/2 < e \leq d$, we need to have some particular information about the group $\varphi(G)$ in the orthogonal case in order to know which of the bounds apply. It turns out (see [32, Theorem 5.7]) that for all cases, and all $e$ for which $d/2 < e \leq d$ and $\Delta$ contains $\text{ppd} (d, q; e)$-elements, we have

$$\frac{1}{e + 1} \leq \text{ppd} (G, e) < \frac{1}{e}$$

except if $\Delta$ is an orthogonal group of minus type, $e = d$ is even, and $G \cap \Omega^-(d, q)$ is either $\Omega^-(d, q)$ (for any $q$) or $\text{SO}^-(d, q)$ (for $q$ odd), in which case $2/(d + 1) \leq \text{ppd} (G, d) < 2/d$.

Further (see [32, Theorem 5.8]), the proportion of large $\text{ppd} (d, q; e)$-elements in $G$ and the proportion of basic $\text{ppd} (d, q; e)$-elements in $G$, whenever such elements exist, also lie between the lower and upper bounds we have above for $\text{ppd} (G, e)$.

In the classical recognition algorithm in [32] we are not especially interested at first in particular values of $e$. We simply wish to find $\text{ppd}$-elements for some $e$ between $d/2$ and $d$. 

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The proportion of such elements in $G$ is

$$\text{ppd} (G) := \sum_{d/2 < e \leq d} \text{ppd} (G, e).$$

In the linear case, where $\Delta = \text{GL} (d, q)$, this is approximately equal to $\sum_{d/2 < e \leq d} e^{-1}$ which, in turn, is approximately

$$\int_{d/2}^{d} \frac{dx}{x} = \log 2 = 0.693\ldots$$

while in the other cases $\text{ppd} (G)$ is approximately equal to the sum of $e^{-1}$ either over all even $e$, or all odd $e$ between $d/2$ and $d$; this is approximately equal to $(\log 2)/2$. These computations can be done carefully resulting in very good upper and lower bounds for $\text{ppd} (G)$ which differ by a small multiple of $d^{-1}$, see [32, Theorem 6.1]. Moreover, except for small values of $d$, these upper and lower bounds for $\text{ppd} (G)$ are also upper and lower bounds for the proportions of large ppd-elements and of basic ppd-elements in $G$.

We can model the process of random selection of $N$ elements from $G$, seeking ppd-elements, as a sequence of $N$ binomial trials with probability of success on each trial (that is, each selection) being $\text{ppd} (G)$. Using this model we can compute the probability of finding (at least) “two different ppd-elements” after $N$ independent random selections. The extent to which this computed probability measures the true probability in a practical implementation depends on whether the assumptions for the binomial model hold for the implementation. In particular the binomial model will give a good fit if the selection procedure is approximately uniform, that is the probability of selecting each element of $G$ on each selection is approximately $|G|^{-1}$, and if the selections are approximately independent. For any small positive real number $\varepsilon$, under the binomial model the probability of failing to find “two different ppd-elements” after $N$ independent uniform random selections is less than $\varepsilon$ provided that $N$ is greater than a small (specified) multiple of $\log \varepsilon^{-1}$, see [32, Theorem 6.4 and Lemma 6.5].

The same approach (under the same assumptions about uniformity and independence of the random selections) gives good estimates for the number $N = N(\varepsilon)$ of selections needed in order that the probability of failing to find “two different ppd-elements”, at least one of which is large and at least one of which is basic, after $N$ random selections is less than $\varepsilon$. Namely $N(\varepsilon)$ is a small (specified) multiple of $\log \varepsilon^{-1}$. For example, if $\Delta = \text{GL} (d, q)$ with $40 \leq d \leq 1000$ and $\varepsilon = 0.1$, then $N(\varepsilon) = 5$.

9 Classical recognition algorithm: an outline

Suppose that $G \subseteq \Delta$ for some classical group $\Delta$ in $\text{GL} (d, q)$, with $G$ irreducible on the underlying vector space $V$, and that we have complete information about $G$-invariant forms
(so that \( G \) is not contained in the class \( C_8 \) of subgroups of \( \Delta \)). We wish to determine whether or not \( G \) contains the corresponding classical group \( \Omega \). Our algorithm is a Monte Carlo algorithm which may occasionally fail to detect that \( G \) contains \( \Omega \). The probability of this happening is less than a predetermined small positive real number \( \varepsilon \).

First we make a number \( N \) of independent uniform random selections of elements from \( G \), where \( N \geq N(\varepsilon/3) \) as in Section 8. If we fail to find two different ppd-elements in \( G \), with at least one of them large and at least one basic, then we report that \( G \) does not contain \( \Omega \). There is a possibility that this response is incorrect, but if in this case \( G \) does contain \( \Omega \) then from Section 8 the probability of failing to find suitable elements is less than \( \varepsilon/3 \). Thus the probability of reporting at this stage that \( G \) does not contain \( \Omega \), given that \( G \) does contain \( \Omega \), is less than \( \varepsilon/3 \).

Suppose now that \( G \) contains two different ppd-elements, say a ppd \( (d, q; e) \)-element \( g \) and a ppd \( (d, q; e') \)-element \( h \), where \( d/2 < e < e' \leq d \), and that at least one of \( g, h \) is large and at least one is basic. As discussed in Section 7, the possibilities for \( G \) are that (i) \( G \supseteq \Omega \), or that (ii) \( G \) is conjugate to a subgroup of \( \text{GL}(d/b, q^b) \) for some prime \( b \) dividing \( d \), or that (iii) \( G \) is one of a very restricted set of nearly simple groups. In order to distinguish case (i) from cases (ii) and (iii) it turns out that essentially we need to find a few extra ppd-elements.

The “extension field groups” in case (ii) are the most difficult to handle. The basic idea here can be illustrated by considering the linear case where \( \Delta = \text{GL}(d, q) \). For a prime \( b \) dividing \( d \), the only values of \( e \) such that \( \text{GL}(d/b, q^b) \) contains a ppd \( (d, q; e) \)-element are multiples of \( b \) (apart from the exceptional case where \( b = d \) and \( d \) is a primitive prime divisor of \( q^{d-1} - 1 \)). Thus finding in \( G \) a ppd \( (d, q; e) \)-element for some \( e \) which is not a multiple of \( b \) will prove that \( G \) is not conjugate to a subgroup of \( \text{GL}(d/b, q^b) \). If \( G \supseteq \Omega \), then the proportion of such elements in \( G \) is ppd \( (G) - \sum_{d/2 < \text{i}b \leq d} \text{ppd}(G, \text{i}b) \) which is approximately equal to ppd \( (G) - (\sum_{d/(2b) < \text{i}b \leq d} \text{ppd}(G, \text{i}b)^{-1}) \). This in turn is approximately equal to \( \log 2 - b^{-1} \log 2 = (\log 2)(b-1)/b \). By [34, Theorem 8.30], the number \( \mu(d) \) of distinct primes dividing \( d \) is \( O(\log d / \log \log d) \). Arguing as in Section 8 there is an integer \( N_b(\varepsilon) \) such that, if \( G \supseteq \Omega \), then the probability of failing to find a ppd \( (d, q; e) \)-element in \( G \) with \( e \) coprime to \( b \) after \( N_b(\varepsilon) \) independent random selections is less than \( \varepsilon/3 \mu(d) \). If \( G \supseteq \Omega \), then we may need to find up to \( \mu(d) \) extra ppd-elements to eliminate case (ii) as a possibility, and the probability of failing to eliminate it after \( N \) random selections from \( G \), where \( N \) is the maximum of the \( N_b(\varepsilon) \), is less than \( \varepsilon/3 \). If we fail to find the required set of elements after these \( N \) further random selections then we report that \( G \) does not contain \( \Omega \). Thus the probability of reporting at this second stage that \( G \) does not contain \( \Omega \), given that \( G \) does contain \( \Omega \), is less than \( \varepsilon/3 \). The number \( N \) of selections we need to make for this second stage is \( O(\log \varepsilon^{-1} + \log \log d) \). Eliminating possibility (ii) for the other classical groups is done using these basic ideas, but the details are considerably more complicated for the symplectic and orthogonal groups when \( b = 2 \).
For each of the nearly simple groups which contain two different ppd-elements $g,h$ as above, there are in fact only two values of $e$ for which the group contains ppd $(d,q;e)$-elements, namely the values corresponding to the elements $g$ and $h$. To distinguish groups $G$ containing $\Omega$ from this nearly simple group we simply need to find in $G$ a ppd $(d,q;e)$-element for a third value of $e$. For each pair $(d,q)$ there is only a small number of possible nearly simple groups (usually at most 1, and in all cases at most 3). As before there is some $N_{\text{sim}}(\varepsilon)$ such that, if $G \supseteq \Omega$, then the probability of failing to find suitable elements to eliminate these nearly simple groups after $N_{\text{sim}}(\varepsilon)$ random selections from $G$ is less than $\varepsilon/3$. If we fail to find the required elements after $N_{\text{sim}}(\varepsilon)$ further random selections then we report that $G$ does not contain $\Omega$. Thus the probability of reporting at this third and final stage that $G$ does not contain $\Omega$, given that $G$ does contain $\Omega$, is less than $\varepsilon/3$.

Once we have found all the required elements to remove possibilities (ii) and (iii) we may report with certainty that $G$ does contain $\Omega$.

The probability that the algorithm reports that $G$ does not contain $\Omega$, given that $G$ does contain $\Omega$, is less than $\varepsilon$. The requirements to bound the probability of error at the three stages of the algorithm are such that the complete algorithm requires us to make $O(\log \varepsilon^{-1} + \log \log d)$ random selections from $G$.

10 Computing with polynomials

In this section we describe how we process an element $g$ of a classical group $\Delta \leq \text{GL}(d,q)$ to decide if it is a ppd-element. This is a central part of the algorithm.

The first step is to compute the characteristic polynomial $c_g(t)$ of $g$, and to determine whether or not $c_g(t)$ has an irreducible factor of degree greater than $d/2$. If no such factor exists then $g$ is not a ppd-element. So suppose that $c_g(t)$ has an irreducible factor $f(t)$ of degree $e > d/2$.

Thus we know that there is a unique $g$-invariant $e$-dimensional subspace $W$ of $V$ and that the linear transformation $g|_W$ induced by $g$ on $W$ has order dividing $q^e - 1$; $g$ will be a ppd $(d,q;e)$-element if and only if the order of $g|_W$ is divisible by some primitive prime divisor of $q^e - 1$. By an argument introduced in Section 8, this will be the case if and only if $(g|_W)^{(q^e-1)/\Phi} \neq 1$, where $\Phi = \Phi(e,q)$ and $\Phi(e,q)$ denotes the product of all the primitive prime divisors of $q^e - 1$ (including multiplicities). Determining whether or not this is the case can be achieved by computing within the polynomial ring $\text{GF}(q)[t]$ modulo the ideal $\langle f(t) \rangle$, namely $(g|_W)^{(q^e-1)/\Phi}$ will be a non-identity matrix if and only if $t^{(q^e-1)/\Phi} \neq 1$ in this ring.

We can test whether not $g$ is a large or basic ppd $(d,q;e)$-element by the same method with $\Phi(e,q)$ replaced by $\Phi_l(e,q)$ or $\Phi_b(e,q)$ respectively. Here $\Phi_l(e,q)$ and $\Phi_b(e,q)$ are the products of all the large and basic primitive prime divisors of $q^e - 1$ (including multiplicities).
respectively.

The idea for checking the ppd-property by determining whether a single power of \( g \) is the identity comes from the special linear recognition algorithm in [30], while the idea of deciding this by a computation in the polynomial ring is due to Celler and Leedham-Green [7].

11 Complexity of the classical recognition algorithm

In [31, Section 4] an analysis of the running cost for the classical recognition algorithm was given based on “classical” algorithms for computing in finite fields. For example the cost of multiplying two \( d \times d \) matrices was taken to be \( O(d^3) \) field operations (that is, additions, multiplications, or computation of inverses). We take this opportunity to re-analyse the algorithm in terms of more modern methods for finite field computations. These methods can lead to improvements in performance over the classical methods. However efficient implementation of the modern methods is a highly nontrivial task requiring substantial effort, see for example the paper of Shoup [39] which addresses the problem of efficient factorisation of polynomials over finite fields. I am grateful to Igor Shparlinski for some interesting and helpful discussions concerning such algorithms.

The *exponent of matrix multiplication* is defined as the infimum of all real numbers \( x \) for which there exists a matrix multiplication algorithm which requires no more than \( O(d^x) \) field operations to multiply together two \( d \times d \) matrices over a field of order \( q \). It is denoted by \( \omega \) or \( \omega(d, q) \). Thus, for all positive real numbers \( \varepsilon \), there exists such an algorithm which requires \( O(d^{\omega+\varepsilon}) \) field operations, that is matrix multiplication can be performed with \( O(d^{\omega+o(1)}) \) field operations. In [6, Sections 15.3, 15.8] an algorithm is given and analysed for which \( O(d^x) \) field operations are used with \( x < 2.39 \) (and hence \( \omega < 2.39 \)), and it was shown there also that \( \omega \) can depend (if at all) only on the prime \( p \) dividing \( q \) rather than on the field size \( q \). Moreover the cost of performing a field operation depends on the data structure used to represent the field and is \( O((\log q)^{1+o(1)}) \) for each field operation, that is, the cost is \( O((\log q)^{1+\varepsilon}) \) for each \( \varepsilon > 0 \).

Now let \( \mu \) be the cost of producing a single random element from the given subgroup \( G = \langle X \rangle \) of GL \((d, q)\). As discussed in [36, p. 190], theoretical methods for producing approximately random elements from a matrix group are not good enough to be translated into practical procedures for use with algorithms such as the classical recognition algorithm. For example, Babai [3, Theorem 1.1 and Proposition 7.2] produces, from a given generating set \( X \) for a subgroup \( G \leq \text{GL}(d,q) \), a set of \( O(d^2 \log q) \) elements of \( G \) at a cost of \( O(d^{10}(\log q)^5) \) matrix multiplications, from which nearly uniformly distributed random elements of \( G \) can be produced at a cost of \( O(d^2 \log q) \) matrix multiplications per random element. The practical implementation of the classical recognition algorithm uses an algorithm developed in [9] for producing approximately random elements in classical groups which, when tested on
a range of linear and classical groups was found to produce, for each relevant value of \( e \), \( \text{ppd}(d, q; e) \)-elements in proportions acceptably close to the true proportions in the group. This procedure has an initial phase which costs \( O(d^{2+o(1)}) \) field operations, and then the cost of producing each random element is \( O(d^{2+o(1)}) \) field operations (see also \[31\] Section 4.1)). Further analysis of the algorithm in \[9\] may be found in \[10\, 13\, 14\].

Testing each random element \( g \in G \) involves first finding its characteristic polynomial \( c_g(t) \). The cost of doing this deterministically is \( O(d^{2+o(1)}) \) field operations (see \[24\] or \[6\] Section 16.6)). Next we test whether \( c_g(t) \) has an irreducible factor of degree greater than \( d/2 \). This can be done deterministically at a cost of \( O(d^{2+o(1)} + d^{1+o(1)} \log q) \) field operations, see \[24\]. (Although the full algorithm in \[24\] for obtaining a complete factorisation of \( c_g(t) \) is non-deterministic, we only need the first two parts of the algorithm, the so-called square-free factorisation and distinct-degree factorisation procedures, and these are deterministic.) Suppose now that \( c_g(t) \) has an irreducible factor \( f(t) \) of degree \( e > d/2 \). We then need to compute \( \Phi(e, q) \), the product of all the primitive prime divisors of \( q^e - 1 \) (counting multiplicities). A procedure for doing this is given in \[30\] Section 6]. It begins with setting \( \Phi = q^e - 1 \) and proceeds by repeatedly dividing \( \Phi \) by certain integers. The procedure runs over all the distinct prime divisors \( c \) of \( e \), and by \[34\] Theorem 8.30 there are \( O(\log e / \log \log e) = O(\log d / \log \log d) \) such prime divisors. For each \( c \), the algorithm computes twice the greatest common divisor of two positive integers where the larger of the two integers may be as much as \( q^e \), and then makes up to \( d \log q \) greatest common divisor computations for which the larger of the two integers is \( O(d) \). By \[1\] Theorem 8.20 and its Corollary] (or see \[6\] Note 3.8], the cost of computing the greatest common divisor of two positive integers less than \( 2^n \), is \( O(n(\log n)^{O(1)}) \) bit operations. It follows that the cost of computing \( \Phi(e, q) \) is \( O(d(\log d)^{O(1)}(\log q)^2) \) bit operations. Having found \( \Phi(e, q) \), we need to determine whether \( t(q^e - 1)^{\left(\frac{1}{\Phi(e, q)}\right)} \mod \Phi \) is equal to 1 in the polynomial ring \( \text{GF}(q)[t] \) modulo the ideal \( \langle f(t) \rangle \). This involves \( O(d \log q) \) multiplications modulo \( f(t) \) of two polynomials of degree less than \( d \) over \( \text{GF}(q) \). Each of these polynomial multiplications costs \( O(d \log d \log d \log d) \) field multiplications, (see \[6\] Theorem 2.13 and Example 2.6]). Thus this test costs \( O(d^2 \log d \log d \log d \log q) \) field operations. Therefore the cost of testing whether a random element \( g \) is a \( \text{ppd} \)-element is \( O(d^{2+o(1)} + d^2 \log d \log d \log d \log q) \) field operations plus \( O(d(\log d)^{O(1)}(\log q)^2) \) bit operations, and hence is

\[
O(d^{2+o(1)}(\log q)^{1+o(1)} + d^2 \log d \log d \log (\log q)^{2+o(1)})
\]

bit operations. This is at most \( O(d^{2+o(1)}(\log q)^{2+o(1)}) \) bit operations. The cost of checking whether \( g \) is a large \( \text{ppd} \)-element is the same as this. To check if \( g \) is a basic \( \text{ppd}(d, q; e) \)-element involves computing \( \Phi_b(e, q) = \Phi(ae, p) \) (where \( q = p^a \)) instead of \( \Phi(e, q) \). Arguing as above, the cost of computing \( \Phi_b(e, q) \) is \( O(ad(\log(\log d))^{O(1)}(\log p)^2) = O(d(\log d)^{O(1)}(\log q)^2) \) bit operations, and hence the cost of testing whether \( g \) is a basic \( \text{ppd} \)-element is also at most \( O(d^{2+o(1)}(\log q)^{2+o(1)}) \) bit operations.
Since we need to test $O(\log \varepsilon^{-1} + \log \log d)$ elements of $G$, the total cost of the algorithm is as follows.

**Theorem 11.1** Suppose that $G \subseteq \Delta$ for some classical group $\Delta$ in $GL(d, q)$, with $G$ irreducible on the underlying vector space $V$, and that we have complete information about $G$-invariant forms (so that $G$ is not contained in the class $C_8$ of subgroups of $\Delta$). Assume that $d$ is large enough that $\Omega$ contains two different ppd-elements with at least one of them large and at least one basic. Further let $\varepsilon$ be a positive real number with $0 < \varepsilon < 1$. Assume that we can make uniform independent random selections of elements from $G$ and that the cost of producing each random element is $\mu$ bit operations. Then the classical recognition algorithm in [32] uses $O(\log \varepsilon^{-1} + \log \log d)$ random elements from $G$ to test whether $G$ contains $\Omega$, and in the case where $G$ contains $\Omega$, the probability of failing to report that $G$ contains $\Omega$ is less than $\varepsilon$. The cost of this algorithm is

$$O((\log \varepsilon^{-1} + \log \log d)(\mu + d^{\omega+o(1)}(\log q)^{2+o(1)}))$$

bit operations, where $\omega$ is the exponent of matrix multiplication.

### 12 Classical recognition algorithm: final comments

The classical recognition algorithm in [32] has been implemented and is available as part of the matrix share package with the GAP system [37], and is also implemented in MAGMA [5]. In the MAGMA implementation rather large groups have been handled by the algorithm without problems: John Cannon has informed us that, on a SUN Ultra 2 workstation with a 200 MHz processor, recognising $SL(5000, 2)$, for example, took 3214 CPU seconds averaged over five runs, while recognising $SL(10000, 2)$ was possible in 14334 CPU seconds, again averaged over five runs of the algorithm.

The algorithm as described in this paper relies on the presence in the classical group $\Omega$ of two different ppd-elements, where at least one is large and at least one is basic. However, for some small values of the dimension $d$, depending on the type of the classical group and the field order $q$, $\Omega$ may not contain such elements. In these cases a modification of the algorithm has been produced in [33] which makes use of elements which are similar to ppd-elements. The results in [33] demonstrate that, with some effort, it is possible to extend the probability computations in Section 8.

An alternative algorithm to recognise classical groups in their natural representations has been developed by Celler and Leedham-Green in [8]. This algorithm also uses the Aschbacher classification [2] of subgroups of $GL(d, q)$ as its organisational principle. Like the algorithm in [32], it makes use of a search by random selection for certain elements. Although no analysis of the complexity of the algorithm is given in [8], the analysis we give in Section 11...
gives a reasonable measure of the complexity of this algorithm also. Finally, as with the algorithm in [32], the algorithm in [8] does not work for certain families of small dimensional classical groups (notably the groups of type $O^+(8, q)$), and the methods of [33] are required to deal with these groups.

References


