Neumann Series Expansion Based LMMSE Channel Estimation for OFDM Systems

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Abstract—In an orthogonal frequency division multiplexing (OFDM) system, the linear-minimum-mean-square-error (LMMSE) based channel estimator often requires a matrix inversion with cubic complexity. In this letter, by employing a K terms Neumann series expansion to approximate the matrix inversion, the computational complexity is reduced to $\mathcal{O}(N \log L)$ per channel realization where N is the number of subcarriers and L is the number of non-zero time domain channel taps. Extensive simulation results show that even with small K ($K \leq 2$), the performance loss caused by the proposed approximation is marginal.

Index Terms—Channel Estimation, OFDM, Low Complexity, LMMSE, Neumann Series Expansion

I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) can transform a frequency selective fading channel into a set of parallel frequency flat fading channels and thus can greatly reduce the complexity of equalization. The frequency selective channel is often assumed time invariant within one OFDM symbol and the frequency correlation among different subcarriers is often employed to reduce the computational complexity of the channel estimator. If the channel power delay profile (PDP) is available, the linear-minimum-mean-square-error (LMMSE) estimation is typically employed with the aid of pilot signals (and/or data fed back from the detector or the channel decoder). However, directly implementing such an estimator typically involves a matrix inversion with cubic complexity of channel length. To reduce the cubic complexity, windowed discrete Fourier transform (WDFT) methods were proposed in [1] [2] [3] to achieve a complexity of $\mathcal{O}(N \log N)$ (N is the total number of subcarriers) but with significant performance loss. In [4], using the law of large numbers, an approximation to the matrix inversion was proposed to reduce the complexity to $\mathcal{O}(N \log N)$. However, this incurs a mean-square error (MSE) floor in the high signal-to-noise ratio (SNR) region. In [5], a Dual-Diagonal LMMSE channel estimation for OFDM systems was proposed and the corresponding MSE was analyzed. With this method, the channel estimation can be achieved with complexity of $\mathcal{O}(N \log N)$ and the MSE performance is close to the exact LMMSE algorithm from low to medium SNR. But for high SNR, both the simulation results and MSE analysis showed that there is still some performance loss. Recently, basis expansion model (BEM) algorithms based on discrete prolate spheroidal (DPS) sequences have attracted much interests as they need no channel statistics but the

knowledge of the maximum delay spread and the maximum Doppler spread. Assuming that the CIR is invariant within one OFDM symbol, a two dimensional DPS for time-frequency variant MIMO OFDM channel estimation was firstly proposed in [6], and simplified in [7] by replacing the two-dimensional Slepian-basis expansion with two serially concatenated one-dimensional Slepian-basis expansions. Then in [8], the time variant CIR within one OFDM symbol was taken into account and algorithms with complexity of $\mathcal{O}(N^2)$ were proposed. These DPS based algorithms were well designed for fast-fading environment. For block fading channels, [9] proposed several DPS based algorithms with low complexity.

In this letter, we focus on how to reduce the complexity of the traditional LMMSE channel estimator for slow fading channels. The proposed algorithm can be used in data-aided channel estimation in an iterative system where the estimated data or decoded data are employed as the virtual pilot. A preamble-type of pilots based frame structure is employed to provide the initial LMMSE channel estimation. For the channel estimation, we firstly reformulate the conventional LMMSE channel estimation to a form that has small sized $L \times L$ matrix inversion where L is the number of non-zero time domain channel taps. Then by employing the fact that this small sized matrix is diagonally dominant due to the law of large numbers, we propose to use a K terms Neumann series expansion to approximate its inversion. In this way, the LMMSE estimator can be achieved by a cascade of matrix vector products, which can be implemented with Fast Fourier Transform (FFT) or Inverse Fast Fourier Transform (IFFT) operations with L inputs or L outputs, thus has the complexity of $\mathcal{O}(N \log L)$. Simulation results show that with small K ($K \leq 2$), the performance of the proposed approximation is close to the exact LMMSE implementation from low to high SNR for both pilot-aided channel estimation and data-aided channel estimation.

The remainder of this letter is organized as follows. Section II describes the OFDM system model. Then the conventional LMMSE channel estimation algorithm is presented in Section III. In Section IV, we propose to use Neumann series expansion to perform LMMSE channel estimation with low complexity. Simulation results are shown in Section V and Section VI concludes this letter.

The notations used in this letter are as follows. Lower and upper case letters denote scalars. Bold lower and upper case letters represent column vectors and matrices, respectively. The superscripts “ T ” and “ H ” denote transpose and conjugate transpose, respectively. Let $\mathbf{I}_N$ denote an $N \times N$ identity matrix, $\mathbb{E}[\cdot]$ the expectation operation and $\text{tr}\{\cdot\}$ the

trace operation. The function of $\text{diag}\{\mathbf{a}\}$ returns a diagonal matrix with vector $\mathbf{a}$ being the main diagonal and $\{\mathbf{M}\}_{\text{diag}}$ returns $\mathbf{M}$ with the off-diagonal elements of $\mathbf{M}$ set to be zero.

II. SYSTEM MODEL

We consider a coded OFDM system with N subcarriers. The data vector of the n th OFDM symbol $\mathbf{x}(n) = [x_1, x_2, \dots, x_N]^T$, which is mapped from an interleaved code sequence $\mathbf{c}$, i.e., each $x_i \in \mathcal{A} = \{\alpha_1, \alpha_2, \dots, \alpha_{2^Q}\} (|\mathcal{A}| = 2^Q)$ corresponds to a length- Q subsequence of $\mathbf{c}$ denoted by $\mathbf{c}_i = [c_{i,1}, c_{i,2}, \dots, c_{i,Q}]^T$, is stacked into one OFDM symbol. The cyclic prefixes (CP) are inserted before the IFFT of $\mathbf{x}(n)$ to ensure the orthogonality among the subcarriers and prevent inter-symbol interference (ISI) between consecutive OFDM symbols. Considering a quasi-static channel which is constant during one OFDM symbol, this OFDM system can be described as a set of parallel additive white Gaussian noise (AWGN) channels. After dropping the CP and performing FFT, the received frequency domain signal for OFDM symbol n is given by

$$\mathbf{y}(n) = \mathbf{X}(n)\boldsymbol{\eta}(n) + \mathbf{w}(n) \quad (1)$$

where $\mathbf{y}(n)$ denotes a length- N observation vector, $\mathbf{X}(n) \equiv \text{diag}\{\mathbf{x}(n)\}$ denotes an $N \times N$ diagonal matrix with $\mathbf{x}(n)$ on its diagonal, $\boldsymbol{\eta}(n)$ is the frequency domain channel coefficients and $\mathbf{w}(n)$ denotes a length- N circularly symmetric AWGN vector with PDF $\mathcal{CN}(\mathbf{w}; 0, \sigma^2 \mathbf{I})$. For notation simplicity, from now on we omit the time index n .

The time domain channel coefficients $\mathbf{h} = [h_1, h_2, \dots, h_L]^T$ is related to the frequency domain channel coefficients $\boldsymbol{\eta}$ with

$$\boldsymbol{\eta} = \mathbf{F}_L \mathbf{h} \quad (2)$$

where $\mathbf{F}_L$ is a truncated DFT matrix (sized $N \times L$) with the (k, l) -th element given by $\mathbf{F}_L(k, l) = \exp(-j \frac{2\pi k l}{N}) / \sqrt{N}$ with $j = \sqrt{-1}$. We assume that the power delay profile (PDP) of a multipath channel is known¹, which can be exploited by the channel estimator. The channel coefficients $[h_i]$ have zero mean and the covariance $\mathbb{E}[\mathbf{h}\mathbf{h}^H] = \text{diag}\{p_1, \dots, p_L\} \equiv \mathbf{P}$ is regarded as the PDP, where p_i is the average power of the i -th delay path.

III. LMMSE CHANNEL ESTIMATION

From (1), the LMMSE estimation of frequency domain channel coefficients $\hat{\boldsymbol{\eta}}$ can be computed by [11]

$$\hat{\boldsymbol{\eta}} = \mathbf{C}_{\boldsymbol{\eta}\mathbf{y}} \mathbf{C}_{\mathbf{y}\mathbf{y}}^{-1} \mathbf{y} \quad (3)$$

where $\mathbf{C}_{\boldsymbol{\eta}\mathbf{y}}$ and $\mathbf{C}_{\mathbf{y}\mathbf{y}}$ are the covariance matrix of $\boldsymbol{\eta}$ and $\mathbf{y}$, and the auto-covariance matrix of $\mathbf{y}$, respectively. Based on the definition of the covariance matrix of two vectors $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{C}_{\mathbf{ab}} = \mathbb{E}[(\mathbf{a} - \mathbb{E}[\mathbf{a}])(\mathbf{b} - \mathbb{E}[\mathbf{b}])^H], \quad (4)$$

we have

$$\mathbf{C}_{\boldsymbol{\eta}\mathbf{y}} = \mathbf{C}_{\boldsymbol{\eta}\boldsymbol{\eta}} \mathbf{X}^H = N \mathbf{F}_L \mathbf{P} \mathbf{F}_L^H \mathbf{X}^H \quad (5)$$

and

$$\mathbf{C}_{\mathbf{y}\mathbf{y}} = \mathbf{X} \mathbf{C}_{\boldsymbol{\eta}\boldsymbol{\eta}} \mathbf{X}^H + \sigma^2 \mathbf{I}_N = N \mathbf{X} \mathbf{F}_L \mathbf{P} \mathbf{F}_L^H \mathbf{X}^H + \sigma^2 \mathbf{I}_N. \quad (6)$$

¹PDP can also be estimated with low complexity of $\mathcal{O}(L^2)$, see [10].

From (3), (5) and (6), it can be seen that directly computing frequency domain channel coefficients needs an $N \times N$ matrix inversion with $\mathcal{O}(N^3)$ complexity.

Considering that the number of non-zero time domain channel delay taps L can be much less than the number of subcarriers N , using matrix inversion lemma, (3) can be reformulated to

$$\begin{aligned} \hat{\boldsymbol{\eta}} &= N \mathbf{F}_L \mathbf{P} \mathbf{F}_L^H \mathbf{X}^H (N \mathbf{X} \mathbf{F}_L \mathbf{P} \mathbf{F}_L^H \mathbf{X}^H + \sigma^2 \mathbf{I}_N)^{-1} \mathbf{y} \\ &= N \mathbf{F}_L \sqrt{\mathbf{P}} (N \sqrt{\mathbf{P}}^H \mathbf{F}_L^H \mathbf{X}^H \mathbf{X} \mathbf{F}_L \sqrt{\mathbf{P}} + \sigma^2 \mathbf{I}_L)^{-1} \sqrt{\mathbf{P}}^H \mathbf{F}_L^H \mathbf{X}^H \mathbf{y}. \end{aligned} \quad (7)$$

As $\mathbf{P}$ is a positive definite diagonal real-valued matrix, it is easy to get that $\sqrt{\mathbf{P}}^H = \sqrt{\mathbf{P}} = \text{diag}\{\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_L}\}$. As a result, (7) involves a matrix inversion (sized $L \times L$) and FFT (IFFT) with computational complexity of $\mathcal{O}(N \log L + L^3)$. At the same time, by the law of large numbers, it is easy to see that matrix $\mathbf{F}_L^H \mathbf{X}^H \mathbf{X} \mathbf{F}_L$ is diagonally dominant, which can be exploited to enable Neumann series expansion to approximate the matrix inversion, thereby reducing the complexity.

Algorithm 1 LMMSE Channel Estimation for OFDM

Input: $\mathbf{y}, \mathbf{P}, \mathbf{X}$ ▷ $\mathbf{X}$ is the pilot or feedback data
Output: $\hat{\boldsymbol{\eta}}$ ▷ Frequency domain channel coefficients
1: $\mathbf{D} = N \mathbf{P} \text{tr}\{\mathbf{X}^H \mathbf{X}\} + \sigma^2 \mathbf{I}_L$
2: $\mathbf{v} = \sqrt{\mathbf{P}} \mathbf{F}_L^H \mathbf{X}^H \mathbf{y}$
3: $\mathbf{v}_0 = \mathbf{D}^{-1} \mathbf{v}$ ▷ Initialize Neumann series expansion
4: $\mathbf{s}_0 = \mathbf{v}_0$
5: **for** $i = 1$ to K **do**
6: $\mathbf{v}_i = \mathbf{v}_{i-1} - N \mathbf{D}^{-1} \sqrt{\mathbf{P}}^H \mathbf{F}_L^H \mathbf{X}^H \mathbf{X} \mathbf{F}_L \sqrt{\mathbf{P}} \mathbf{v}_{i-1} - \sigma^2 \mathbf{D}^{-1} \mathbf{v}_{i-1}$
7: $\mathbf{s}_i = \mathbf{s}_{i-1} + \mathbf{v}_i$
8: **end for**
9: $\hat{\boldsymbol{\eta}} = N \mathbf{F}_L \sqrt{\mathbf{P}} \mathbf{s}_K$

IV. NEUMANN SERIES EXPANSION BASED CHANNEL ESTIMATION

A. Neumann Series Expansion

Neumann series expansion [12] can be employed to approximate matrix inversion with the summation of a series of matrix multiplications. For a diagonally dominant matrix $\mathbf{M}$, let $\mathbf{D} = \{\mathbf{M}\}_{\text{diag}}$. A K terms Neumann series expansion of $\mathbf{M}$ can be written as

$$\mathbf{M}^{-1} \approx \sum_{i=0}^K (\mathbf{I} - \mathbf{D}^{-1} \mathbf{M})^i \mathbf{D}^{-1}. \quad (8)$$

It is easy to see that the multiplication of $\mathbf{M}^{-1}$ and a vector $\mathbf{v}$ can be computed by K loops as

$$\begin{aligned} \mathbf{v}_0 &= \mathbf{D}^{-1} \mathbf{v}, \quad \mathbf{s}_0 = \mathbf{v}_0 \\ \textbf{for } i &= 1 \textbf{ to } K \textbf{ do} \\ \mathbf{v}_i &= (\mathbf{I} - \mathbf{D}^{-1} \mathbf{M}) \mathbf{v}_{i-1} \\ \mathbf{s}_i &= \mathbf{s}_{i-1} + \mathbf{v}_i \\ \textbf{end for} \end{aligned} \quad (9)$$

Fig. 1. MSE performance with different L at SNR of 14dB

with $\mathbf{M}^{-1}\mathbf{v} \approx \mathbf{s}_K$. With (9), only matrix-vector multiplications are required, which can greatly reduce the computational complexity.

Let $\mathbf{M} = N\sqrt{\mathbf{P}}^H \mathbf{F}_L^H \mathbf{X}^H \mathbf{X} \mathbf{F}_L \sqrt{\mathbf{P}} + \sigma^2 \mathbf{I}_L$ and $\mathbf{v} = \sqrt{\mathbf{P}}^H \mathbf{F}_L^H \mathbf{X}^H \mathbf{y}$ and plug them into (9), then (7) can be computed by **Algorithm 1**. In this algorithm, line 1 uses the fact that $\{\mathbf{F}_L^H \mathbf{N} \mathbf{F}_L\}_{\text{diag}} = \text{tr}\{\mathbf{N}\} \mathbf{I}_L$ for any diagonal matrix $\mathbf{N}$.

B. Computational Complexity Comparison

In **Algorithm 1**, the matrices $\mathbf{D}$, $\mathbf{X}$, $\mathbf{P}$ are all diagonal. So lines 1, 3, 4, and 7 have trivial complexity. For Line 2 and Line 9, an IFFT (with L outputs) and an FFT (with L inputs) can be employed, respectively. Then in Line 6, the computation of $N\sqrt{\mathbf{P}}^H \mathbf{F}_L^H \mathbf{X}^H \mathbf{X} \mathbf{F}_L \sqrt{\mathbf{P}} \mathbf{v}_{i-1}$ can be implemented by an FFT with L inputs, followed by an IFFT with L outputs. In summary, the total complexity of the proposed LMMSE channel estimation for OFDM system is dominated by $2(K+1)$ FFTs (or IFFTs) with L inputs or L outputs. Note that the complexity of FFT or IFFT with L inputs or L outputs can be computed with complexity of $\mathcal{O}(N \log L)$ [13].

In [5], the authors have proved that the WDFT based technique [1] [2] [3] can be treated as special cases of their proposed Dual-Diagonal algorithm. So we focus on Dual-Diagonal LMMSE algorithm for complexity comparison. For the dual-diagonal LMMSE algorithm, the frequency domain channel coefficients $\hat{\boldsymbol{\eta}} = \mathbf{F} \mathbf{B} \mathbf{F}^H \mathbf{A} \mathbf{y}$, where both $\mathbf{A}$ and $\mathbf{B}$ are diagonal matrices, and $\mathbf{F}$ is an $N \times N$ DFT matrix, therefore the number of FFT (IFFT) required is two. Based on the fast algorithm in **Appendix I** of [5], there are also two FFTs needed to compute the matrix $\mathbf{B}$. As a result, the total number of N point FFTs (IFFTs) of the Dual-Diagonal LMMSE algorithm [5] is four, which has the computational complexity of $\mathcal{O}(N \log N)$.

Fig. 2. MSE performance for the 10-tap COST259_RA channel

V. SIMULATION RESULTS

We consider an OFDM system with $N = 128$ subcarriers, the carrier frequency is 2.4 GHz and the symbol duration is $0.25 \mu\text{s}$. The CP is set to be one-eighth of the number of subcarriers. The modulation is 64-QAM with Gray mapping. We constrain the total transmit power to one, and set the noise variance at receive side to σ^2 , then the average received signal-to-noise ratio (SNR) is given by $1/\sigma^2$.

A. Mean-Square Error (MSE) Performance for Time-Invariant Channels

In order to determine the required minimum K under different channel length, we select a channel model with the PDP given by $p_i = Qe^{-0.1(i-1)}$, $i = 1, \dots, L$ where Q is a normalization factor ($\sum_i p_i = 1$). Fig. 1 shows the MSE performance of the exact LMMSE, the Dual-Diagonal LMMSE [5], the DPS based LMMSE [7]-[9]² and the proposed algorithm with different L at SNR of 14dB. We treat all transmit data as pilot in order to get the lower bound of the MSE performance.

It is obvious that the proposed algorithm outperforms both the Dual-Diagonal LMMSE [5] and the DPS based LMMSE [9] even with $K = 1$. At the same time, for the proposed algorithm with $K = 2$, there is small MSE performance loss compared with the exact LMMSE when L is greater than 8, and with $K = 3$ the proposed algorithm has nearly the same performance as the exact LMMSE algorithm for all channel length. The DPS based LMMSE algorithm has the worst performance as it only requires the maximum normalized delay spread as the input, but all other algorithms require the exact channel power profile.

²For the application scenario of this letter, the DPS algorithm is reduced to one dimension. The complexity of the exact DPS is $\mathcal{O}(N^3)$. In [9], the complexity is reduced to $\mathcal{O}(IN)$ based on space-alternating expectation maximization (SAGE), where I is the number of DPS sequences and is in the order of L .

Fig. 3. BER performance for 10-tap COST259_RA channel at speed of 100 km/hour

Then we use the 10-tap COST259_RA [14] channel to compare the MSE performance in Fig. 2. It can be seen that the proposed method has nearly the same performance as the exact LMMSE with $K = 3$ while there is small performance loss in high SNR region with $K = 2$. It is also obvious that even with $K = 2$ the proposed algorithm outperforms the Dual-Diagonal LMMSE algorithm and the DPS based LMMSE algorithm from low to high SNR range.

B. Bit Error Rate (BER) Performance for Iterative Systems

As in Section V.A of [15], we consider an iterative channel estimation scheme, where the hard decision from the output of the channel decoder is employed as the virtual pilot. We employ a frame structure that every frame contains 25 OFDM symbols and the first symbol is the pilot symbol to provide the initial LMMSE channel estimation for the iterative channel estimation scheme³. With slow fading assumption, the channel coefficients of the last OFDM symbol are used for the current symbol detection, then the hard decision fed back from the decoder is mapped to a data symbol, which is exploited to update the channel estimation. Although we assume that the channel is static within one OFDM symbol in the design of the channel estimation, the channel coefficients generated in the simulations changed at every sampling time according to Jakes model [16] and the received signal was generated using the time-varying channel. For the Jakes model, the relative speed between the transmitter and the receiver is assumed to be 100 km/hour. In the simulations, we use the 10-tap COST259_RA channel model [14]. A rate-1/2, regular (3,6) low-density parity-check (LDPC) code with codeword length of 768 bits is also used. For the LDPC decoder, the maximum number of iterations between the variable nodes and the check nodes is 25.

³Besides the preamble-type pilot in our example, the proposed algorithm can be easily applied to other types of pilots.

Fig. 3 shows the BER performance of the system with exact LMMSE, Dual-Diagonal LMMSE [5], DPS based LMMSE [9] and the proposed LMMSE approximation. It is clear that the proposed algorithm with $K = 2$ nearly has the same BER performance as the exact LMMSE algorithm and always better than the Dual-Diagonal method and the DPS based LMMSE algorithm.

VI. CONCLUSION

In this letter, we have proposed a low complexity LMMSE channel estimation algorithm for OFDM systems by approximating the matrix inversion with Neumann series expansion. This enables the channel estimation to be implemented with L -point input FFT or L -point output IFFT, which have the complexity of $\mathcal{O}(N \log L)$. Extensive simulation results show that under different channel models, the proposed algorithm can achieve good MSE performance from low to high SNR range, and nearly the same BER performance as the exact LMMSE algorithm.

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