EXAMPLES OF RANK 3 PRODUCT ACTION TRANSITIVE DECOMPOSITIONS

GEOFFREY PEARCE

Abstract. A transitive decomposition is a pair $(\Gamma, \mathcal{P})$ where Γ is a graph and $\mathcal{P}$ is a partition of the arc set of Γ such that there is a subgroup of automorphisms of Γ which leaves $\mathcal{P}$ invariant and transitively permutes the parts in $\mathcal{P}$. In an earlier paper we gave a characterisation of G-transitive decompositions where Γ is the graph product $K_m \times K_m$ and G is a rank 3 group of product action type. This characterisation showed that every such decomposition arose from a 2-transitive decomposition of K_m via one of two general constructions. Here we use results of Sibley to give an explicit classification of those which arise from 2-transitive edge-decompositions of K_m.

1. Introduction

A G-transitive decomposition is a pair $(\Gamma, \mathcal{P})$ where Γ is a graph, $\mathcal{P}$ is a partition of its arc set $A\Gamma$, and G is a subgroup of $\mathrm{Aut}\Gamma$ such that

(i) for all $P \in \mathcal{P}$ and $g \in G$ we have $P^g \in \mathcal{P}$; and
(ii) for all $P, P' \in \mathcal{P}$, there exists $g \in G$ with $P^g = P'$.

Usually we require that $|\mathcal{P}| > 1$; however we may sometimes allow $|\mathcal{P}| = 1$, in which case we call the decomposition *degenerate*. We say that $\mathcal{P}$ is *symmetric* if for any $P \in \mathcal{P}$ and $(\alpha, \beta) \in P$ we have $(\beta, \alpha) \in P$ also. In this case we may view $\mathcal{P}$ as an *edge*-decomposition of Γ by identifying the pair $(\alpha, \beta), (\beta, \alpha)$ of arcs with the edge $\{\alpha, \beta\}$.

Transitive decompositions generalise a number of other mathematical structures, including homogeneous factorisations [10, 11], line transitive partial linear spaces [6], and 2-transitive 1-factorisations of complete graphs [4]; and they are related to 2-transitive symmetric graph designs [3] and 2-transitive symmetric association schemes [2]. Explanations of several of these relationships can be found in [13], [14] and [15]. The last of these papers ([15]) is a characterisation by Sibley of all G-transitive decompositions where G is a 2-transitive (rank 2) permutation group. In [1] we extended Sibley's work to the rank 3 case; in particular, we gave a characterisation of G-transitive decompositions where G is a primitive rank 3 group of product action type. In doing so we generalised a classification of rank 3 product action partial linear spaces by Devillers [6].

This paper concerns the G-transitive decompositions studied in [1]. We may assume that such a rank 3 group G of product action type is contained in $H \wr S_2$ where H is a 2-transitive group of almost simple type (see for example [1, Lemma 3.4]). The characterisation in [1] amounted to showing that any such transitive decomposition can be obtained using one of several explicit 'product' constructions. These constructions involved an H-transitive decomposition $(K_m, \mathcal{Q})$, and all such $(K_m, \mathcal{Q})$ with a symmetric partition $\mathcal{Q}$ are classified in [15]. However, [1, Construction 2.10] (which we re-state in Construction 1.3) also involved an H-invariant refinement $\mathcal{R}$ of the partition $\mathcal{Q}$, and a

2000 *Mathematics Subject Classification.* Primary 20B15, 05C25, 05E20.

Key words and phrases. transitive decomposition, rank 3 primitive group, product action type.

The author gratefully acknowledges advice and assistance from his Ph.D. supervisors, Cheryl E. Praeger and John Bamberg, in the writing of this paper. The author was supported by an Australian Postgraduate Award.

'twisting' function φ. The purpose of this paper is to find all possible $\mathcal{R}$ and φ when $\mathcal{Q}$ is symmetric, and thereby give a more explicit description of this class of rank 3 product action transitive edge-decompositions.

Throughout the paper we use the following notation.

Notation 1.1.
(a) Γ *is the graph product* $\Delta \times \Delta$, *where* $\Delta = K_m$ *with vertex set* Ω_0 *and* $|\Omega_0| = m$. *Here* $V\Gamma = \Omega_0 \times \Omega_0$ *and* $((\alpha,\gamma),(\beta,\delta)) \in A\Gamma$ *whenever* (α,β) *and* (γ,δ) *are both arcs of* K_m *(that is, whenever both* $\alpha \neq \beta$ *and* $\gamma \neq \delta$*).*
(b) $G \leq H \wr S_2 \leq \mathrm{Aut}\Gamma$ *in product action on* $\Omega_0 \times \Omega_0$ *where* H *is almost simple and* 2*-transitive on* Ω_0. *We let* $T = \mathrm{P\Gamma L}(2,8)$ *if* $(H, |\Omega_0|) = (\mathrm{P\Gamma L}(2,8), 28)$, *and otherwise we let* $T = Soc(H)$, *the unique minimal normal subgroup of* H. *Note that* T *is* 2*-transitive on* Ω_0.
(c) $(\Gamma,\mathcal{P})$ *is a* G*-transitive decomposition and* $\mathcal{P} = \mathcal{P}(\mathcal{T},\mathcal{R},\varphi)$ *where* $\mathcal{T} = (\Delta,\mathcal{Q})$ *is an* H*-transitive decomposition,* $\mathcal{R}$ *is a proper* H*-invariant refinement of* $\mathcal{Q}$*, and* φ *is a 'twisting' homomorphism as in Construction* 1.3*.*

Our main result is the following.

Theorem 1.2. *Let* G, Γ, m, $\mathcal{P}(\mathcal{T},\mathcal{R},\varphi)$ *and* $\mathcal{Q}$ *be as in Notation* 1.1*, and let* (α,β) *be an arc in* $Q_0 \in \mathcal{Q}$. *Then*
(i) *there exist subgroups* L *and* M *with* $T_{(\alpha,\beta)} \leq M \lhd L \leq T$ *and* $T_{\{\alpha,\beta\}} \leq L$, *and* $\varphi_0 \in \mathrm{Aut}(L/M)$ *such that* L, M, φ_0 *determine* $\mathcal{T},\mathcal{R},\varphi$*; and*
(ii) L, M *are as in Table* 1 *or* 2*.*

Remark. Lemma 3.1 describes explicitly how L, M and φ_0 determine $\mathcal{T}$, $\mathcal{R}$ and φ.

	T	m	L	M
(i)	Any 2-t group	-	$T_{\{\alpha,\beta\}}$	$T_{(\alpha,\beta)}$
(ii)	A_7	15	Line stabiliser (induces S_3)	Induces A_3, $T_{(\alpha,\beta)}$
(iii)	$\mathrm{PSL}(2,7)$	8	1-factor stabiliser $\cong S_4$	A_4
(iv)	$\mathrm{PSL}(2,5)$	6	1-factor stabiliser $\cong A_4$	$T_{\{\alpha,\beta\}} \cong V_4$
(v)	$\mathrm{PSU}(3,3)$	28	T_Q from Table 3, Case 8	$T_{\{\alpha,\beta\}}$
(vi)	$\mathrm{PSL}(a,2)$, $a \geq 3$	$2^a - 1$	Line stabiliser (induces S_3)	Induces A_3, $T_{(\alpha,\beta)}$
(vii)	$\mathrm{PSL}(a,3)$, $a \geq 3$	$\frac{3^a-1}{2}$	T_Q from Table 3, Case 6	$T_{\{\alpha,\beta\}}$

TABLE 1. $T \neq \mathrm{P\Gamma L}(2,8)$

Below is a version of [1, Construction 2.10]. Given subsets R and R' of $A\Delta$ we write $R \times_{\mathrm{graph}} R'$ to denote the subset

$$\{((\alpha,\gamma),(\beta,\delta)) \,|\, (\alpha,\beta) \in R, (\gamma,\delta) \in R'\}$$

of $A(\Delta \times \Delta)$. A transitive permutation group is called *regular* if each point stabiliser is trivial.

Construction 1.3. Let $\mathcal{T} = (\Delta,\mathcal{Q})$ be a (possibly degenerate) H-transitive decomposition, let $\mathcal{R}$ be a proper H-invariant refinement of $\mathcal{Q}$, and let $\Gamma = \Delta \times \Delta$.

Let the parts in $\mathcal{Q}$ be denoted by $Q_0, Q_1, \ldots, Q_{s-1}$, and for each $Q_i \in \mathcal{Q}$ let $\mathcal{R}_{Q_i}$ denote the set $\{R \in \mathcal{R} \,|\, R \subset Q_i\}$. Assume that the permutation group $H_{Q_0}^{\mathcal{R}_{Q_0}}$ induced by H_{Q_0} on $\mathcal{R}_{Q_0}$ is regular, and let φ be an element of $\mathrm{Sym}(\mathcal{R}_{Q_0})$ such that φ normalises $H_{Q_0}^{\mathcal{R}_{Q_0}}$. Let $W := \{w_0, w_1, \ldots, w_{s-1}\}$ be a transversal for H_{Q_0} in H such that $Q_0^{w_i} = Q_i$ for each i. For a fixed $R_0 \in \mathcal{R}_{Q_0}$, let $V := \{v_1, \ldots, v_t\}$ be a transversal for H_{R_0} in H_{Q_0}.

	L	M
(i)	$T_{\{\alpha,\beta\}} = \mathbb{Z}_2^2$	$T_{(\alpha,\beta)} = \mathbb{Z}_2$
(ii)	$\mathrm{P\Gamma L}(2,8)$	$\mathrm{PSL}(2,8)$
(iii)	$\mathrm{A\Gamma L}(1,8)$	$\mathrm{AGL}(1,8)$, $\mathbb{Z}_2^3$
(iv)	$\mathrm{AGL}(1,8)$	$\mathbb{Z}_2^3$
(v)	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$, $T_{\{\alpha,\beta\}}$, $T_{(\alpha,\beta)}$
(vi)	$T_\ell \cong A_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$, $T_{(\alpha,\beta)}$
(vii)	$A_4 \times \mathbb{Z}_2$	A_4, $\mathbb{Z}_2^3$, $T_{\{\alpha,\beta\}}$
(viii)	A_4	$T_{\{\alpha,\beta\}}$
(ix)	$A_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$

TABLE 2. $T = \mathrm{P\Gamma L}(2,8)$. (The groups L in lines (vi), (vii) and (ix) are conjugate in T but not equal.)

Let $Q_i, Q_j \in \mathcal{Q}$, and let $k \in \{1, \ldots, t\}$. Define

$$P(Q_i, Q_j, k) = (\bigcup_{R \in \mathcal{R}_{Q_0}} R^{w_i} \times_{\mathrm{graph}} R^{v_k \varphi w_j}) \subset Q_i \times_{\mathrm{graph}} Q_j$$

and let $\mathcal{P}(\mathcal{T}, \mathcal{R}, \varphi)$ denote the set of all $P(Q_i, Q_j, k)$ for all $0 \leq i, j \leq s-1$, $1 \leq k \leq t$.

2. 2-TRANSITIVE EDGE-DECOMPOSITIONS OF K_m

Table 3 gives a rough summary of the classification in [15, Theorem 6] of all T-transitive edge decompositions $\mathcal{T} = (K_m, \mathcal{Q})$ where T is a 2-transitive non-abelian simple group. (We examine the case with $T \cong \mathrm{P\Gamma L}(2,8)$ of degree 28 in Section 2.2.) Sibley's classification draws on and extends classifications of a number of closely related structures, including linear spaces (see Lemma 2.3) and also 1-factorisations of K_m. (A 1-*factorisation* of K_m is a partition $\mathcal{F}$ of the edge set such that for each $F \in \mathcal{F}$, the subgraph of K_m induced by F has valency 1 and is incident with every vertex of K_m. The 1-factorisations of K_m preserved by a 2-transitive group were classified in [4].) In Table 3 we refer to some of these connections, and also to Constructions 2.1 and 2.2 which are paraphrased from [15].

The numbering of the cases in Table 3 corresponds to the numbering of the Examples in [15]; so for a more detailed description of Case n, see Example n of [15].

Construction 2.1. (see [15, Example 5]) Let $T = \mathrm{PSL}(a, 2)$ and let K_m be the complete graph with vertex set $\mathrm{PG}(a-1, 2)$. For each $\gamma \in VK_m$, let $Q(\gamma)$ be the set of all edges $\{\alpha, \beta\}$ of K_m such that α, β and γ are co-linear in $\mathrm{PG}(a-1, 2)$ and $\gamma \neq \alpha$ or β. Let $\mathcal{Q} = \{Q(\gamma) \,|\, \gamma \in VK_m\}$.

Construction 2.2. (see [15, Examples 6,7 and 8]) Let $T \leq \mathrm{PSL}(a, q)$ and let K_m be the complete graph with vertex set $\mathrm{PG}(a-1, 2)$. Let $\mathcal{Q}'$ be the partition of AK_m corresponding to the line set of $\mathrm{PG}(a-1, 3)$ (see Lemma 2.3), and assume that for each $Q' \in \mathcal{Q}'$, the (complete) subgraph of K_m corresponding to Q' admits a $T_{Q'}$-invariant 1-factorisation $\mathcal{F}_{Q'}$. Let $\mathcal{Q} = \bigcup_{Q' \in \mathcal{Q}'} \mathcal{F}_{Q'}$.

In order to prove Theorem 1.2 we need to give some more detailed information about certain classes of almost simple 2-transitive decompositions of K_m.

2.1. 2-transitive decompositions corresponding to 2-transitive linear spaces.

A linear space $\mathcal{D}$ is a set $\mathcal{V}$ of points together with a set $\mathcal{L}$ of lines (subsets of points) such that each pair of points lies in exactly one line. The automorphism group of $\mathcal{D}$, denoted by

Case	T	m	Description of $\mathcal{Q}$
1	-	-	Each part in $\mathcal{Q}$ contains exactly one edge.
2	$\mathrm{PSL}(a,q)$ $\mathrm{PSU}(3,q)$ ${}^2G_2(q)$ A_7	$\sum_{i=0}^{a} q^i$ q^3+1 q^3+1 15	Constructed from a linear space (see Lemma 2.3).
3	$\mathrm{PSL}(2,q)$ $q=5,7$ or 11	$q+1$	1-factorisation (see [4]).
5	$\mathrm{PSL}(a,2)$	$\sum_{i=0}^{a} 2^i$	Construction 2.1
6	$\mathrm{PSL}(a,3)$	$\sum_{i=0}^{a} 3^i$	Construction 2.2
7	$\mathrm{PSL}(a,5)$	$\sum_{i=0}^{a} 5^i$	Construction 2.2
8	$\mathrm{PSU}(3,q)$ $q=3$ or 5	q^3+1	Construction 2.2
9	$\mathrm{Sp}(2l,2)$	$2^{2l-1} \pm 2^{l-1}$	See Section 2.3.
10	$\mathrm{PSU}(3,3)$	28	Each part in $\mathcal{Q}$ consists of 6 vertex-disjoint edges.
11	$\mathrm{PSL}(2,9)$	10	Each part in $\mathcal{Q}$ consists of 3 vertex-disjoint edges.

TABLE 3. The T-transitive edge-decompositions where T is a non-abelian simple 2-transitive group.

Aut$\mathcal{D}$, is the group of all permutations of $\mathcal{V}$ which preserve $\mathcal{L}$, and $\mathcal{D}$ is called 2-transitive if Aut$\mathcal{D}$ is 2-transitive on $\mathcal{V}$. Every 2-transitive linear space corresponds to a 2-transitive decomposition of a complete graph into complete subgraphs. This correspondence is given in the following lemma (which is essentially a special case of [14, Lemma 2.1] concerning partial linear spaces). Given a graph Γ and a partition $\mathcal{P}$ of $A\Gamma$, for each $P \in \mathcal{P}$ we write Γ_P for the subgraph of Γ with $A\Gamma_P = P$ and $V\Gamma_P$ the set of all vertices incident with arcs in P.

Lemma 2.3.

(i) *Let $\mathcal{D} := (\mathcal{V}, \mathcal{L})$ be a 2-transitive linear space, and suppose that G is a 2-transitive subgroup of* Aut$\mathcal{D}$. *Let Γ be the complete graph with vertex set $\mathcal{V}$. For each $\ell \in \mathcal{L}$, let P_ℓ be the set of all unordered pairs of distinct elements of ℓ, and let $\mathcal{P} = \{P_\ell \mid \ell \in \mathcal{L}\}$. Then $(\Gamma, \mathcal{P})$ is a G-transitive decomposition, and each Γ_{P_ℓ} is a complete subgraph of Γ.*

(ii) *Let $(\Gamma, \mathcal{P})$ be a G-transitive decomposition where G is 2-transitive and Γ is a complete graph such that for each $P \in \mathcal{P}$, the subgraph Γ_P is a complete subgraph of Γ. Let $\mathcal{V} = V\Gamma$, and let $\mathcal{L} = \{V\Gamma_P \mid P \in \mathcal{P}\}$. Then G is a 2-transitive subgroup of* Aut$\mathcal{D}$ *and hence $(\mathcal{V}, \mathcal{L})$ is a 2-transitive linear space.*

The 2-transitive linear spaces were classified in [9]. Theorem 2.4 lists those preserved by a 2-transitive almost simple group.

Theorem 2.4 (Kantor)**.** *Let $\mathcal{D}$ be a linear space and suppose that $T \leq$* Aut$\mathcal{D}$ *where T is the socle of a 2-transitive almost simple group. Then one of the following holds*

(i) $T = \mathrm{PSL}(a,q)$ *where $a \geq 3$ and* $\mathcal{D} = \mathrm{PG}(a-1,q)$

(ii) $T = \mathrm{PSU}(3,q)$ *with $q \geq 3$ and $\mathcal{D}$ is an Hermitian unital. That is, for a 3-dimensional vector space V over* $\mathrm{GF}(q^2)$ *with a non-degenerate Hermitian form,* *the points of $\mathcal{D}$ are the totally isotropic 1-subspaces of V, and each line is the set of points contained in a non-degenerate 2-space.*

(iii) $T = {}^2G_2(q)$ *and $\mathcal{D}$ is the same linear space as in (ii).*

(iv) $T = A_7$ *and* $\mathcal{D} = \mathrm{PG}(3, 2)$.

We now give a lemma concerning line stabilisers for almost simple 2-transitive linear spaces.

Lemma 2.5. *Suppose that $\mathcal{D}$ is a linear space and T is a non-abelian simple 2-transitive subgroup of* $\mathrm{Aut}\mathcal{D}$*. Then for any line ℓ of $\mathcal{D}$, either*

(a) *the permutation group induced on ℓ by T_ℓ is* $\mathrm{PGL}(2, q)$*, or*

(b) *we are in case (iii) of Theorem* 2.4 *and the permutation group induced on ℓ by T_ℓ contains* $\mathrm{PSL}(2, q)$*, and if $q = 3$ it is equal to* $\mathrm{PSL}(2, q) \cong A_4$.

Proof. We consider each of the cases in Theorem *2.4*. In case (i) the linear space is $\mathrm{PG}(a-1, q)$, with $T = \mathrm{PSL}(a, q)$. The points of $\mathcal{D}$ are the 1-spaces of an a-dimensional vector space V over $\mathrm{GF}(q)$, and each line is the set of 1-spaces contained in some 2-space of V. Hence the induced action of T_ℓ on ℓ is that of $\mathrm{PGL}(2, q)$.

In Case (ii) the result follows from [12, Proof of Lemma 2.8]. (More details can be found in [7, p. 132].)

If we are in case (iii) of Theorem 2.4, then $T = {}^2G_2(q)$ and according to the proof of Theorem 1 in [9], T_ℓ^ℓ contains $\mathrm{PSL}(2, q)$ acting 2-transitively on ℓ, and is equal to $\mathrm{PSL}(2, 3) \cong A_4$ if $q = 3$.

In case (iv) we have $T_\ell^\ell = \mathrm{PGL}(2, 2)$. □

2.2. **2-transitive decompositions preserved by** $\mathrm{P\Gamma L}(2, 8)$ **of degree** 28**.** In [15], Sibley identifies and describes most of the T-transitive decompositions $(K_{28}, \mathcal{Q})$ where $T = \mathrm{P\Gamma L}(2, 8)$ of degree 28. In recomputing these decompositions we discovered a further three examples that had been overlooked in [15, Theorem 7]. We give here the complete classification. The existence of these decompositions was discovered through computation with Magma, and we refer to Magma computations in the proof of Theorem 2.6.

Theorem 2.6. *Let* $T = \mathrm{P\Gamma L}(2, 8)$ *of degree* 28*, and suppose that* $(K_{28}, \mathcal{Q})$ *is a (possibly degenerate) T-transitive decomposition. Let* $\{\alpha, \beta\} \in EK_{28}$*, and let $Q \in \mathcal{Q}$ be the part containing* $\{\alpha, \beta\}$*. Then the stabiliser T_Q appears in Table* 4.

	T_Q	Q
(i)	$T := \mathrm{P\Gamma L}(2, 8)$	K_{28}
(ii)	$T_{\{\alpha,\beta\}}$	$\{\alpha, \beta\}$
(iii)	$T_\ell \cong A_4 \times \mathbb{Z}_2$	K_4
(iv)	$\mathrm{AGL}(1, 8)$	1-factor of K_{28}
(v)	$\mathrm{PSL}(2, 8)$	9-factor of K_{28}
(vi)	$S \cong \mathbb{Z}_2^3$, the 8 translations from $\mathrm{AGL}(1, 8)$	2 disjoint edges
(vii)	$\mathrm{A\Gamma L}(1, 8)$	3-factor of K_{28}
(viii)	$C_1 \cong A_4 \times \mathbb{Z}_2$	6 disjoint edges
(ix)	$D \cong A_4$ $(D \leq C_1)$	3 disjoint edges
(x)	$C_2 \cong A_4 \times \mathbb{Z}_2$	6 disjoint edges

Table 4. Transitive decompositions preserved by $\mathrm{P\Gamma L}(2, 8)$ of degree 28.

Remark. Lines (i)-(vii) of Table 4 are numbered to correspond with [15, Theorem 7], while lines (viii)-(x) contain new examples. (Note that in the proof of [15, Theorem 7] on p 131, $\mathrm{AGL}(2, 8)$ and $\mathrm{A\Gamma L}(2, 8)$ should read $\mathrm{AGL}(1, 8)$ and $\mathrm{A\Gamma L}(1, 8)$ respectively.)

Proof. Lines (i)-(vii) of Table 4 correspond to possibilities (i)-(vii) of [15, Theorem 7]. We now explain how lines (viii)-(x) arise.

By [9], T preserves a $(28, 4, 1)$ linear space $\mathcal{D} = (\mathcal{V}, \mathcal{L})$. Let $\ell \in \mathcal{L}$ be the unique line of $\mathcal{D}$ containing the points α, β. Then $T_{\{\alpha,\beta\}} \leq T_\ell$, and hence T_ℓ yields a T-transitive decomposition $(K_{28}, \mathcal{Q})$ where $VK_{28} = \mathcal{V}$ and where $\mathcal{Q} = (\{\alpha,\beta\}^{T_\ell})^T$ (line (iii) of Table 4). By Lemma 2.5 (b), T_ℓ^ℓ is permutationally isomorphic to A_4. Since T_ℓ has order 24, it follows that the kernel K of the action of T_ℓ on ℓ is isomorphic to $\mathbb{Z}_2$, and that T_ℓ has a unique Sylow 2-subgroup S containing K. Moreover since $|T : T_\ell| = 63$ is odd, S is a Sylow 2-subgroup of T and hence $S \cong \mathbb{Z}_2^3$. Thus $T_\ell \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_3 = K \times (\mathbb{Z}_2^2 \rtimes \mathbb{Z}_3)$. Since $T_\ell/K \cong A_4$ it follows that $T_\ell \cong K \times A$ where $A \cong A_4$. Then we have $T_{\{\alpha,\beta\}} = K \times (A)_{\{\alpha,\beta\}} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. There is exactly one proper subgroup of A containing $(A)_{\{\alpha,\beta\}}$, namely $S \cap A \cong \mathbb{Z}_2^2$, and hence the only subgroup of T_ℓ containing $T_{\{\alpha,\beta\}}$ is the one in line (vi) of Table 4.

We used Magma to determine the following information:

(a) T_ℓ has three orbits on VK_{28}. These are ℓ, which has 4 points; and two orbits O_1 and O_2 each of length 12.
(b) $T_\ell^{O_i}$ is non-regular for each i.
(c) $T_\ell^{O_1}$ is not permutationally isomorphic to $T_\ell^{O_2}$.

Thus, for each i, $T_\ell^{O_i}$ may be represented by the coset action of T_ℓ on some core-free subgroup L_i of index 12 in T_ℓ (so in other words $L \neq K$). Let τ be an involution in A, and let σ be the generator of K. Then L_i is conjugate in T_ℓ to either $\langle\tau\rangle$ or $\langle\tau\sigma\rangle$ (and L_1 is not conjugate to L_2). Assume without loss of generality that $L_1 = \langle\tau\rangle$ and $L_2 = \langle\tau\sigma\rangle$. We will show that for each i, there exist $\gamma_i, \delta_i \in O_i$ such that $T_{\{\gamma_i,\delta_i\}} \leq T_\ell$.

Let $\psi_i : [T_\ell : L_i] \longrightarrow O_i$ be the bijection defining the permutational equivalence between the action of T_ℓ on $[T_\ell : L_i]$ and on O_i. Let $\tau' \neq \tau$ be an involution in A, and let $\psi_1(\langle\tau\rangle) = \gamma_1$ and $\psi_1(\langle\tau\rangle\tau') = \delta_1$. Then the stabiliser in T_ℓ of $\{\gamma_1, \delta_1\}$ is equal to $\langle\tau, \tau'\rangle \cong \mathbb{Z}_2^2$. Since $|T_{\{\gamma_1,\delta_1\}}| = 4$ it follows that $T_{\{\gamma_1,\delta_1\}} = \langle\tau,\tau'\rangle \leq T_\ell$. On the other hand let $\psi_2(\langle\tau\sigma\rangle) = \gamma_2$ and $\psi_2(\langle\tau\sigma\rangle\sigma) = \delta_2$. Then the stabiliser in T_ℓ of $\{\gamma_2, \delta_2\}$ is equal to $\langle\tau\sigma, \sigma\rangle \cong \mathbb{Z}_2^2$. Since $|T_{\{\gamma_2,\delta_2\}}| = 4$ it follows that $T_{\{\gamma_2,\delta_2\}} = \langle\tau\sigma, \sigma\rangle \leq T_\ell$.

In each case, the index of $T_{\{\gamma_i,\delta_i\}}$ in T_ℓ is 6. Since $|O_i| = 12$, it follows that $\{\gamma_i, \delta_i\}^{T_\ell}$ consists of 6 disjoint pairs.

Now, observe that the stabiliser $(T_\ell)_{\{\gamma_1,\delta_1\}} = \langle\tau,\tau'\rangle$ is contained in the subgroup A of T_ℓ. The index of $(T_\ell)_{\{\gamma_1,\delta_1\}}$ in A is 3, and since $\{\gamma_1,\delta_1\}^A \subset \{\gamma_1,\delta_1\}^{T_\ell}$, it follows that the orbit $\{\gamma_1,\delta_1\}^A$ consists of 3 disjoint pairs.

Now, since T acts transitively on ordered pairs of points, there exist elements $t_1, t_2 \in T$ with $\{\gamma_i,\delta_i\}^{t_i} = \{\alpha,\beta\}$. Writing $C_i := T_\ell^{t_i}$ and $D := A^{t_1}$, we have $T_{\{\alpha,\beta\}} < C_i < T$, and $T_{\{\alpha,\beta\}} < D < T$, where $\{\alpha,\beta\}^{C_i}$ consists of 6 disjoint edges and $\{\alpha,\beta\}^D$ consists of 3 disjoint edges. This gives lines (viii)-(x) of Table 4. □

2.3. **2-transitive decompositions for** $\mathrm{Sp}(2l, 2)$**.** In this section we give some results pertaining to the 2-transitive actions of $\mathrm{Sp}(2l, 2)$, in preparation for the proof of Theorem 1.2. We first explain the notation used in [8, Section 7.7] to describe these actions of $\mathrm{Sp}(2l, 2)$.

Let $T = \mathrm{Sp}(2l, 2)$ with $l \geq 3$ and let V be a $2l$-dimensional vector space over $\mathrm{GF}(2)$. Let

$$e = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{0}$ and $\mathbf{I}$ denote the $l \times l$ zero and identity matrices respectively. Define a symmetric bilinear form ϕ by $\phi(u, v) := ufv^\top$ and for each $c \in V$ define $\theta_a : V \longrightarrow \mathrm{GF}(2)$ by $\theta_a(u) = ueu^\top + ufa^\top$. (Note that [8] uses φ to denote the form ϕ; however this conflicts with our usage of φ in Construction 1.3.) For each $c \in V$, define a *transvection* t_c by $t_c : u \longmapsto u + \phi(u,c)c$. Then $t_c \in \mathrm{Sp}(2l,2)$ and $x^{-1}t_a x = t_{ax}$ for all $x \in \mathrm{Sp}(2l,2)$. For each a and c and $u \in V$ we have $\theta_a^{t_c}(u) = \theta_a(ut_c^{-1})$. This leads to the following result, which is taken directly from [8].

Lemma 2.7. (i) *For all $a, c \in V$ we have*

$$\theta_a^{t_c} = \begin{cases} \theta_a & \text{if } \theta_a(c) = 1 \\ \theta_{a+c} & \text{if } \theta_a(c) = 0 \end{cases}$$

(ii) *For all $a, b \in V$ there is at most one $c \in V$ such that t_c maps θ_a onto θ_b. Such a c exists if and only if $\theta_0(a) = \theta_0(b)$ (and then $c = a + b$).*

The group $\mathrm{Sp}(a,2) = \langle t_c \,|\, c \in V\rangle$, and has two orbits on the set $\{\theta_a \,|\, a \in V\}$. These orbits are

$$\Omega^+ := \{\theta_a \,|\, \theta_0(a) = 0\} \quad \text{and} \quad \Omega^- := \{\theta_a \,|\, \theta_0(a) = 1\}.$$

It is shown in [8, Theorem 7.7A] that $\mathrm{Sp}(2l,2)$ acts 2-transitively on each of Ω^+ and Ω^- for each $l \geq 2$.

The proof of Theorem 1.2 in the case $T = \mathrm{Sp}(2l,2)$ of degree $2^{2l-1} \pm 2^{l-1}$ involves several key steps which we prove here as separate lemmas. First we give an explanation of the family of transitive decompositions in [15, Example 9].

Let Ω' equal either Ω^+ or Ω^-, and let K_m be the complete graph with vertex set Ω'. The T-transitive decomposition $(K_m, \mathcal{Q})$ in [15, Example 9] is such that for an edge $\{\theta_a, \theta_b\}$, the part of $\mathcal{Q}$ containing $\{\theta_a, \theta_b\}$ is the set of all edges $\{\theta_c, \theta_d\}$ such that $c + d = a + b$. For each vector v in $V \backslash \{0\}$, define Q_v to be the part in $\mathcal{Q}$ (if one exists) consisting of all edges $\{\theta_a, \theta_b\}$ with $a + b = v$.

Lemma 2.8. *Let t_c and t_d be transvections in T. Then $t_c = t_d$ if and only if $c = d$.*

Lemma 2.9. *Let $v, c \in V\backslash\{0\}$. Then t_c fixes Q_v setwise if and only if t_c fixes v (in the action of T on $V\backslash\{0\}$).*

Proof. Assume first that t_c fixes v, and let $a, b \in V$ be such that $v = a+b$. Then $v = v^{t_c} = v+\phi(v,c)c$, which implies that $\phi(v,c) = 0$. This means that $\phi(a+b,c) = \phi(a,c)+\phi(b,c) = 0$ and hence that $\phi(c,a) = \phi(c,b)$. Now we have $\theta_a(c) = cec^\top + cfa^\top = \theta_0(c) + \phi(c,a)$ and $\theta_b(c) = \theta_0(c) + \phi(c,b)$. Hence, since $\phi(c,a) = \phi(c,b)$, either $\theta_a(c) = 0$ and $\theta_b(c) = 0$, or $\theta_a(c) = 1$ and $\theta_b(c) = 1$. It follows from Lemma 2.7 that $\{\theta_a, \theta_b\}^{t_c}$ equals either $\{\theta_a, \theta_b\}$ or $\{\theta_{a+c}, \theta_{b+c}\}$, both of which are contained in Q_v. So the transvection t_c fixes Q_v setwise. Conversely, suppose that t_c fixes Q_v setwise. Then $\{\theta_a, \theta_b\}^{t_c} = \{\theta_{a+d}, \theta_{b+d}\}$ for some d. Lemma 2.7 implies that d is either 0 or c, and that, in either case, $\theta_a(c) = \theta_b(c)$. This means that $\theta_0(c) + \phi(a,c) = \theta_0(c) + \phi(b,c)$ and hence that $0 = \phi(a,c) + \phi(b,c) = \phi(v,c)$. Hence t_c fixes v. □

Lemma 2.10. $T_{Q_v} = T_v$.

Proof. Let S_v denote the set of all transvections in T fixing v and let B denote the set of vectors in V fixed by every transvection in S_v. Recall that for any transvection t_c and any $x \in T$ we have $t_c^x = t_{cx}$. From this it follows that T acts transitively by conjugation on the set of all non-trivial transvections. Since $S_v^x = S_{vx}$ for any $x \in T$, we find that $|S_u| = |S_w|$ for all $u, w \in V$. Suppose that $u, w \in B$. Then by the definition of B, each element of S_v fixes both u and w; so $S_v \subseteq S_u \cap S_w$. Hence $S_u = S_w = S_v$. If for some $x \in T$ we have $u^x \notin B$, then $S_u^x \neq S_v = S_u$ and so $S_w^x \neq S_w$. This means that $w^x \notin B$, which implies that $B^x \cap B = \varnothing$ and hence that B is a block of imprimitivity for T. But T acts primitively on $V\backslash\{0\}$, and so B must be $\{v\}$ (since no non-trivial transvection fixes every vector in $V\backslash\{0\}$). Now, by Lemma 2.9, T_{Q_v} contains S_v and no other transvections.

Hence for any $x \in T_{Q_v}$ we have $S_v^x = S_{v^x} = S_v$. So x must fix v, and hence $T_{Q_v} \leq T_v$. On the other hand, given that each part $Q \in \mathcal{Q}$ corresponds to a unique vector $v \in V \backslash \{0\}$, the size of $\mathcal{Q}$ is at most $|V| - 1$. Hence the index of T_{Q_v} in T cannot exceed $|V|-1$, and so $T_{Q_v} = T_v$. □

Now we describe the structure of the group T_v. Although this information is well-known in the theory of classical groups, it does not appear to be covered explicitly in a convenient reference. We outline a proof of Lemma 2.11, omitting routine computations, and we acknowledge unpublished lecture notes by David Vogan for the notation and method of proof.

Since the form ϕ is non-degenerate and v is non-zero, we may choose a vector $u \in V$ with $\phi(v,u) = 1$. Let $W = \{w \in V \mid \phi(v,w) = \phi(u,w) = 0\}$. Then W is a $(2l-2)$-dimensional subspace of V and $V = \langle v, u\rangle \oplus W$. We define three types of linear transformations of V by specifying their actions on v, u and W. For $x \in GF(2)$, $w_1 \in W$ and $g \in Sp(W)$, define maps z_x, n_{w_1} and s_g, each from V to V, by

$$\begin{aligned} z_x \ &: v \longmapsto v \\ &: u \longmapsto u + xv \\ &: w \longmapsto w \end{aligned} \qquad \begin{aligned} n_{w_1} \ &: v \longmapsto v \\ &: u \longmapsto u + w_1 \\ &: w \longmapsto w + \phi(w_1, w)v \end{aligned} \qquad \begin{aligned} s_g \ &: v \longmapsto v \\ &: u \longmapsto u \\ &: w \longmapsto w^g \end{aligned}$$

for all $w \in W$. It is easily verified that each such linear transformation preserves ϕ and hence is contained in T_v.

Lemma 2.11. *T_v has normal subgroups Z and N where $Z < N$, $|Z| = 2$, and $N/Z \cong \mathbb{Z}_2^{2l-2}$. Furthermore, T_v has a subgroup P isomorphic to $Sp(2l-2,2)$, such that $T_v/Z = N/Z \rtimes PZ/Z \cong \mathbb{Z}_2^{2l-2}.\mathrm{Sp}(2l-2,2)$. In particular, N/Z is the unique minimal normal subgroup of T_v/Z.*

Proof. Let z_x, n_{w_1} and s_g be as defined above. Then it is routine to verify that the sets $Z = \{z_x \mid x \in GF(2)\}$ and $N = \{z_x n_{w_1} \mid w_1 \in W, x \in GF(2)\}$ are subgroups of T_v, and that $Z < N$ and $|Z| = 2$. Also, we have that $Z \lhd N$ with $N/Z \cong W \cong \mathbb{Z}_2^{2l-2}$. Furthermore, $P = \{s_g \mid g \in Sp(W)\} \cong Sp(W)$ is a subgroup of T_v which normalises N and Z, with $(z_x n_{w_1})^{s_g} = n_{w_1^g} z_x$ for all $z_x n_{w_1} \in N$ and $s_g \in P$. Using this fact together with the orders of T_v, P and N, and the fact that $N \cap P$ is trivial, we deduce that $T_v = N \rtimes P$, whence we obtain the result. □

To prove the next result, we note that the binary operation of N is given by

$$(z_{x_1} n_{w_1})(z_{x_2} n_{w_2}) = z_{x_1 + x_2 + \phi(w_1, w_2)} n_{w_1 + w_2}.$$

Lemma 2.12. *Let T_v, N and Z be as in Lemma* 2.11*, and suppose that $K \leq N$ with $|N : K| = 2$ and $K \lhd T_v$. Then $Z \leq K$.*

Proof. Let ψ denote the homomorphism $K \longrightarrow W : n_w z_x \longmapsto w$. By Lemma 2.11, $\ker\psi$ is either trivial or Z. In the latter case $Z \leq K$ as required, so assume that $\ker\psi$ is trivial. Then $\psi(K) = W$ since $|N : K| = 2$. Now recall that W is a vector space over GF(2), and fix $i \in$ GF(2). Suppose that for all $n_w z_x \in K$ with w non-trivial we have $x = i$. There exist $w_1, w_2 \in W$ with $w_1 \neq w_2$ and $\phi(w_1, w_2) = i - 1$, and so $(n_{w_1} z_i)(n_{w_2} z_i) = n_{w_1+w_2} z_{i+i+i-1} = n_{w_1+w_2} z_{i-1}$. That is to say, K contains a non-trivial element $n_{w_1+w_2} z_x$ with $x \neq i$ which is a contradiction; hence there exist elements $n_{w_1} z_0$ and $n_{w_2} z_1$ in K. Now P acts transitively as the symplectic group on W, and so there exists $s_g \in P$ with $w_1^g = w_2$. Since $K \lhd T_v$, the group P normalises K, and so we have $(n_{w_1} z_0)^{s_g} = n_{w_2} z_0 \in K$. But then $n_{w_2} z_0 n_{w_2} z_1 = n_{w_2+w_2} z_{0+1+\phi(w_2,w_2)} = z_1 \in K$. Thus $Z = \langle z_1 \rangle \leq K$ which contradicts the assumption that $\ker\psi$ is trivial. Hence $Z \leq K$. □

3. Proof of Theorem 1.2.

First we give a lemma which essentially proves part (i) of Theorem 1.2.

Lemma 3.1. *Let G, Γ, $\mathcal{P}(\mathcal{T},\mathcal{R},\varphi)$ and $\mathcal{Q}$ be as in Notation* 1.1*, and let (α,β) be an arc in $Q_0 \in \mathcal{Q}$. Let R_0 be the part in $\mathcal{R}$ containing (α,β), and let $L := T_{Q_0}$ and $M := T_{R_0}$. Then $T_{(\alpha,\beta)} \leq M \lhd L \leq T$ and $T_{\{\alpha,\beta\}} \leq L$; and we have $\mathcal{Q} = Q_0^T$ with $Q_0 = (\alpha,\beta)^L$, and $\mathcal{R} = {R_0}^T$ with $R_0 = (\alpha,\beta)^M$. Moreover, the homomorphism φ is determined by an automorphism φ_0 of L/M.*

Proof. Note that T is 2-transitive on Ω_0, so both $\mathcal{Q}$ and $\mathcal{R}$ are systems of imprimitivity for T in its action on $A\Delta$. Thus $T_{(\alpha,\beta)} \leq M$, and since $\mathcal{Q}$ is symmetric (and therefore essentially an edge-partition) we have $T_{\{\alpha,\beta\}} \leq L$. Since T is 2-transitive we have $\mathcal{Q} = Q_0^T$ with $Q_0 = (\alpha,\beta)^L$, and $\mathcal{R} = R_0^T$ with $R_0 = (\alpha,\beta)^M$.

Now since $\mathcal{R}$ refines $\mathcal{Q}$ and $R_0 \subset Q_0$, we have $M \leq L$. By assumption (see Construction 1.3), $H_{Q_0}^{\mathcal{R}_{Q_0}}$ is regular, implying that $H_{R_0} \lhd H_{Q_0}$. Now $L = T \cap H_{Q_0}$ and $M = T \cap H_{R_0}$, and so $M \lhd L$. Thus L/M is regular and permutationally isomorphic to $H_{Q_0}^{\mathcal{R}_{Q_0}}$, and the element φ of $N_{\mathrm{Sym}(\mathcal{R}_{Q_0})}(H_{Q_0}^{\mathcal{R}_{Q_0}})$ may be identified with an element φ_0 of $\mathrm{Aut}(L/M)$. □

From Sibley's classification [15] we can determine all possibilities for the subgroup L. Note that we need to consider the possibility $L = T$ (in which case the decomposition $\mathcal{Q}$ is degenerate) since as long as $|\mathcal{R}| > 1$, the partition $\mathcal{P}(\mathcal{Q},\mathcal{R},\varphi)$ will still be non-degenerate.

Before proving Theorem 1.2 we make some further observations about L and M. First, if both L and M contain the edge stabiliser $T_{\{\alpha,\beta\}}$, then the transitive decompositions corresponding to L and M are both described in Table 3 (and in greater detail in [15]). If $M = T_{(\alpha,\beta)}$ then the corresponding transitive decomposition is such that each part in the arc partition contains exactly one arc. The only remaining situation has M properly containing $T_{(\alpha,\beta)}$ but not containing $T_{\{\alpha,\beta\}}$. The following lemma shows what happens in this case.

Lemma 3.2. *Suppose that T is a 2-transitive group and that $T_{\{\alpha,\beta\}} \leq L \leq T$ with $T_{\{\alpha,\beta\}}$ maximal in L. Suppose also that $M \lhd L$ such that $T_{(\alpha,\beta)} < M$ and $T_{\{\alpha,\beta\}} \not\leq M$. Then $|L : M| = 2$ and $(\alpha,\beta)^M$ is a 'directed copy' of the undirected $(\alpha,\beta)^L$; that is, for every pair (γ,δ),(δ,γ) of arcs in $(\alpha,\beta)^L$, exactly one of (γ,δ) and (δ,γ) is in $(\alpha,\beta)^M$.*

Proof. First, observe that $T_{\{\alpha,\beta\}} < \langle T_{\{\alpha,\beta\}}, M\rangle \leq L$ and so by the maximality of $T_{\{\alpha,\beta\}}$ in L, we have $\langle T_{\{\alpha,\beta\}}, M\rangle = L$. Since $T_{\{\alpha,\beta\}}$ normalises M, we have $L = MT_{\{\alpha,\beta\}}$ and hence $T_{\{\alpha,\beta\}}/(M \cap T_{\{\alpha,\beta\}}) \cong T_{\{\alpha,\beta\}}M/M = L/M$. Since $T_{\{\alpha,\beta\}} \not\leq M$ we have $M \cap T_{\{\alpha,\beta\}} = T_{(\alpha,\beta)}$ and so $|L : M| = |T_{\{\alpha,\beta\}}|/|T_{(\alpha,\beta)}| = 2$. This implies that $|(\alpha,\beta)^M| = |(\alpha,\beta)^L|/2$. If $(\alpha,\beta)^M$ contained (β,α), then M would have to contain an element x swapping α and β, in which case $\langle T_{(\alpha,\beta)}, x\rangle = T_{\{\alpha,\beta\}}$ would be a subgroup of M, which is not the case. It follows that $(\alpha,\beta)^M$ has the form described in the statement. □

We need one more lemma before proving Theorem 1.2.

Lemma 3.3. *Suppose that $\mathcal{T} = (\Delta,\mathcal{Q})$ is a T-transitive decomposition, and let $Q \in \mathcal{Q}$. Let $V\Delta_Q$ be the set of all vertices of Δ incident with arcs in Q, and let $\alpha,\beta \in V\Delta_Q$. Assume that $T_{(\alpha,\beta)} \leq M \trianglelefteq L \leq T_Q$. If $M^{V\Delta_Q} = L^{V\Delta_Q}$, then $M = L$.*

Proof. Since $\alpha,\beta \in V\Delta_Q$, $T_{(\alpha,\beta)}$ contains the kernel K of the action of T_Q on $V\Delta_Q$. Suppose that $M \neq L$. Then since $M^{V\Delta_Q} \cong M/K$ and $L^{V\Delta_Q} \cong L/K$, we have $M^{V\Delta_Q} \neq L^{V\Delta_Q}$, by Lemma 3.2. Hence if $M^{V\Delta_Q} = L^{V\Delta_Q}$, then $M = L$. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Part (i) follows immediately from Lemma 3.1.

We now prove part (ii). For each L with $T_{\{\alpha,\beta\}} \leq L \leq T$ we need to find all M with $T_{(\alpha,\beta)} \leq M \lhd L \leq T$. We begin with two observations. The first is that if $T_{(\alpha,\beta)}$ or $T_{\{\alpha,\beta\}}$ were normal in T, then $T_{(\alpha,\beta)}$ would be trivial, meaning that T would be sharply 2-transitive. By [8, p 238], every sharply 2-transitive group is of affine type; hence, since T is almost simple, we cannot have $L = T$ with M equal to either $T_{(\alpha,\beta)}$ or $T_{\{\alpha,\beta\}}$. Second, we note that $T_{(\alpha,\beta)}$ is normal in $T_{\{\alpha,\beta\}}$ for any group T, and so we may take $M = T_{(\alpha,\beta)}$ and $L = T_{\{\alpha,\beta\}}$, whence we obtain Line (i) of Table 1.

We will assume at this point that T is simple (and we will treat the case $T \cong \mathrm{P\Gamma L}(2,8)$ of degree 28 later). Assume also that $T_{(\alpha,\beta)} < M \lhd L \leq T$ where $T_{\{\alpha,\beta\}} < L$ (and so $M \neq L$ and $T_{\{\alpha,\beta\}} \neq L$). For each 2-transitive simple group T, we will refer to Table 3 to determine all possibilities for L. Then for each L, either we will show that M and L must occur in some line of Table 1, or we will derive a contradiction (usually with the assumption that $M \neq L$).

If T is one of $\mathrm{PSL}(2,11)$ of degree 11, A_n of degree n, HS, Co_3, ${}^2B_2(q)$, or one of the Mathieu groups, then according to [15], $T_{\{\alpha,\beta\}}$ is maximal in T, and so $L = T$. Hence L is simple, which contradicts the assumption that $1 \neq M \neq L$.

We will examine the remaining 2-transitive simple groups T in roughly the order in which they appear in [9, Section 2]. For each T, we work through the possible cases in Table 3.

Case $T = \mathrm{PSL}(a,q)$, $m = (q^a-1)/(q-1)$ with $a \leq 2$, $q > 3$: Here L corresponds to a transitive decomposition described in Case 3 or 11 of Table 3. In Case 3, T is one of $\mathrm{PSL}(2,5)$, $\mathrm{PSL}(2,7)$ or $\mathrm{PSL}(2,11)$, and for each of these groups the subgroup L (which is the stabiliser of a 1-factor) is specified in [4] as follows. When $T = \mathrm{PSL}(2,5)$, the subgroup L is permutationally isomorphic to A_4 acting on the cosets of a subgroup of order 2, which we may assume is $\langle(12)(34)\rangle$. The setwise stabiliser of the two cosets $\langle(12)(34)\rangle$ and $\langle(12)(34)\rangle(13)(24)$ is V_4 (the Klein 4-group) which is the only proper non-trivial normal subgroup of A_4, and hence (taking $M \cong V_4$) we obtain Line (iv) of Table 1. When $T = \mathrm{PSL}(2,7)$, the subgroup $L = S_4$ in its action on the cosets of, say, $\langle(123)\rangle$. In this case the stabiliser of an edge is contained in A_4 (but not in V_4), and hence (taking $M \cong A_4$) we obtain Line (iii) of Table 1. When $T = \mathrm{PSL}(2,11)$, the subgroup $L = A_5$ which is simple, and so we have a contradiction with $1 \neq M \neq L$. Now suppose that we are in Case 11. Here $T = \mathrm{PSL}(2,9)$ and L is maximal of order 24; and hence by [5], $L \cong S_4$. The order of $T_{(\alpha,\beta)}$ is 4, and so either $T_{(\alpha,\beta)} \cong \mathbb{Z}_2^2$ or $T_{(\alpha,\beta)} \cong \mathbb{Z}_4$. Assume that α is the 1-space $\langle(1,0)\rangle$, and let Z denote the centre of $\mathrm{SL}(2,9)$. Let ω be a primitive element of the multiplicative group of $\mathrm{GF}(9)$, and let

$$A := \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \leq \mathrm{SL}(2,9).$$

Then $X := \langle ZA\rangle \cong \langle A\rangle/(Z \cap \langle A\rangle)$ is a subgroup of T_α, and since $|Z \cap \langle A\rangle| = |\langle A^4\rangle| = 2$, it follows that X is cyclic of order 4. Furthermore, since $|T_\alpha| = 2^2.3^2$, X is a Sylow 2-subgroup of T_α. This means that any order 4 subgroup of T_α is cyclic, and since $T_{(\alpha,\beta)} \leq T_\alpha$, we have $T_{(\alpha,\beta)} \cong \mathbb{Z}_4$. But then since $L \cong S_4$, we have $N_L(T_{(\alpha,\beta)}) = L$, implying that $M = L$, which is a contradiction.

Case $T = \mathrm{PSL}(a,q)$, $m = (q^a-1)/(q-1)$ with $a \geq 3$: Here L corresponds to a transitive decomposition occurring in one of Cases 2, 5, 6 or 7 of Table 3.

Suppose we are in Case 2; so L is the stabiliser of the unique line ℓ of $\mathrm{PG}(a-1,q)$ containing α and β. Suppose that $q > 3$. Then Lemma 2.5 shows that the group L^ℓ

induced by L on ℓ is almost simple with a 2-transitive socle, meaning that $T^\ell_{(\alpha,\beta)} \neq 1$. Hence M^ℓ is a non-trivial normal subgroup of L^ℓ, which means that M^ℓ is 2-transitive on ℓ. But then by Lemma 3.3, M must equal L, which is a contradiction. Assume now that $q = 2$. Then M^ℓ is a proper normal subgroup of $L^\ell = \mathrm{PSL}(2,2) \cong S_3$, meaning that we can take either $M^\ell \cong A_3$ or $M^\ell = T^\ell_{(\alpha,\beta)}$ (both of which contain $T^\ell_{(\alpha,\beta)}$); this gives us Line (vi) of Table 1. Finally, assume that $q = 3$. Then $L^\ell = \mathrm{PGL}(2,3) \cong S_4$, of which the only proper non-trivial normal subgroups are A_4 and V_4, neither of which contains a stabiliser in S_4 of two points. This contradicts the assumption that $T_{(\alpha,\beta)} \leq M$.

Suppose now that we are in Case 5 of Table 3. Here $T = \mathrm{PSL}(a,2)$ with $a \geq 3$, which we view as $\mathrm{SL}(a,2)$ acting on an a-dimensional vector space V over $\mathrm{GF}(2)$. The group $L = T_Q$ where $Q = Q(\gamma)$ as in Construction 2.1 for some $\gamma \in V\backslash\{0\}$; that is, Q consists of all edges $\{\alpha,\beta\}$ with $\alpha,\beta \neq \gamma$ such that α and β lie in a 2-subspace together with γ. Now L is 2-transitive on the set of lines incident with γ, and hence it is 2-transitive on the set $\{\{\alpha',\beta'\} \mid (\alpha',\beta') \in Q\}$. So L^Q has a set $\mathcal{Q}'$ of $|Q|/2$ blocks of imprimitivity of size 2, namely all pairs of the form $\{(\alpha',\beta'),(\beta',\alpha')\}$. Let $R \in \mathcal{R}$ with $R \subset Q$ and $(\alpha,\beta) \in R$. If $(\beta,\alpha) \in R$, then R is a union of blocks in $\mathcal{Q}'$, and since $L^{\mathcal{Q}'}$ is primitive and $R \neq Q$ we obtain $R = \{(\alpha,\beta),(\beta,\alpha)\}$, implying that $M = T_R = T_{\{\alpha,\beta\}}$. But then M is not normal in L, which is a contradiction. So assume instead that $(\beta,\alpha) \notin R$, and suppose that $|R| > 1$. Then R contains (α',β') where α',β' lie together in a 2-space with γ and $\{\alpha',\beta'\} \cap \{\alpha,\beta\} = \varnothing$. Now $T_{\gamma\alpha}$ fixes the arc (α,β) and contains an element swapping α' and β'. Hence R must contain both (α',β') and (β',α'), and it follows that R contains (β,α), which is a contradiction. Hence $R = \{(\alpha,\beta)\}$, implying that $M = T_{(\alpha,\beta)}$, which is also a contradiction since $T_{(\alpha,\beta)}$ is not normal in L.

We next examine Cases 7 and 6 of Table 3. In each of these cases, the transitive decomposition refines a decomposition corresponding to a 2-transitive linear space $\mathcal{D}$, and we have

$$T^\ell_{(\alpha,\beta)} \leq M^\ell \trianglelefteq L^\ell \leq T^\ell_\ell$$

where ℓ is the line of $\mathcal{D}$ containing α and β.

Suppose that we are in Case 7 of Table 3. Here $T = \mathrm{PSL}(a,5)$ and $\mathcal{D} = \mathrm{PG}(a-1,5)$. We know from Lemma 2.5 that $T^\ell_\ell = \mathrm{PGL}(2,5)$ and the description of Case 7 in [15] that L^ℓ is the subgroup of $\mathrm{PGL}(2,5)$ fixing a 1-factor of K_6. By [4], this subgroup is permutationally isomorphic to S_4 acting on the cosets of, say, $\langle(1234)\rangle$. The stabiliser of two points in this action is generated by a 4-cycle in S_4, and hence is not contained in any proper normal subgroup of S_4, and so $M^\ell = L^\ell$. Lemma 3.3 then implies that $M = L$, which is a contradiction.

Suppose now that we are in Case 6. Then $T^\ell_\ell = \mathrm{PGL}(2,3) \cong S_4$. As shown in [15, Figure 3], $Q := \{\alpha,\beta\}^L$ consists of exactly two disjoint edges. It follows that $T_{\{\alpha,\beta\}}$ has index 2 in L, making it a normal subgroup of L. Hence if $M = T_{\{\alpha,\beta\}}$ we obtain Line (vii) of Table 1. Now suppose that $\{\gamma,\delta\}$ is the other edge in Q. The normal subgroup M^ℓ of L^ℓ must contain both $T^\ell_{(\alpha,\beta)}$ and its conjugate $T^\ell_{(\gamma,\delta)}$. Since $T^\ell_\ell = S_4$, $T^\ell_{(\alpha,\beta)}$ transposes γ and δ, and $T^\ell_{(\gamma,\delta)}$ transposes α and β, and so we have $\langle(\alpha\beta),(\gamma\delta)\rangle = T^\ell_{\{\alpha,\beta\}} \leq M^\ell$. Since $T_{\{\alpha,\beta\}}$ is maximal in L, it follows that $T_{\{\alpha,\beta\}}$ is the only possibility for M, since otherwise M^ℓ would equal L^ℓ, giving a contradiction by way of Lemma 3.3.

Case $T = \mathrm{PSU}(3,q)$, $m = q^3+1$ with $q \geq 3$: Here L corresponds to a transitive decomposition occurring in one of Cases 2,8 or 10 of Table 3. Again, the transitive decomposition refines a decomposition corresponding to a 2-transitive linear space $\mathcal{D}$, and we have

$$T^\ell_{(\alpha,\beta)} \leq M^\ell \trianglelefteq L^\ell \leq T^\ell_\ell$$

where ℓ is the line of $\mathcal{D}$ containing α and β. In Case 2 we can apply Lemma 2.5 and argue as we did for $T = \mathrm{PSL}(a, q)$ to find that $M^\ell = L^\ell$, which contradicts Lemma 3.3. Suppose we are now in Case 8. When $T = \mathrm{PSU}(3, 3)$ we have $T_\ell^\ell = PGU(2, 3) = \mathrm{PGL}(2, 3)$, and L^ℓ is as in Case 6. Hence, by our treatment of Case 6 we obtain Line (v) of Table 1. When $T = \mathrm{PSU}(3, 5)$ we have $T_\ell^\ell = PGU(2, 5) = \mathrm{PGL}(2, 5)$, and L^ℓ is as in Case 7; again giving a contradiction with Lemma 3.3. Now assume we are in Case 10. Then L is a maximal subgroup of T of order 96. A consequence of [15, Theorem 6] is that $T_{\{\alpha,\beta\}}$ is maximal in L, and so by Lemma 3.2, a proper normal subgroup of L containing $T_{(\alpha,\beta)}$ is either $T_{(\alpha,\beta)}$ or an index 2 subgroup of L. We checked using Magma that neither of these possibilites can occur. Hence $M = L$, which is a contradiction.

Case $T = {}^2G_2(q)$, $m = q^3 + 1$ with $q = 3^{2c+1} > 3$: Here we are in Case 2 of Table 3. Once again, applying Lemma 2.5 and arguing as we did for $T = \mathrm{PSL}(a, q)$ we obtain a contradiction with Lemma 3.3. (We examine $T = {}^2G_2(3) \cong \mathrm{P\Gamma L}(2, 8)$ separately at the end of the proof.)

Case $T = \mathrm{Sp}(2l, 2) = Sp(2l, 2)$, $m = 2^{2l-1} \pm 2^{l-1}$: Recall from Section 2.3 the description of the T-transitive decomposition $(K_m, \mathcal{Q})$ from Example 9 of [15]. Assume that $L = T_{Q_v}$ for some $v = a + b \in V\backslash\{0\}$, and recall that by Lemma 2.10, $T_{Q_v} \leq T_v$. Suppose that M is a normal subgroup of T_v, and assume that $2l \geq 8$. By Lemma 2.11, T_v contains normal subgroups Z and N where $|Z| = 2$ and N/Z is the unique minimal normal subgroup of T_v/Z. Thus MZ/Z either is trivial or contains N/Z. Since $T_{(\theta_a,\theta_b)} \leq M$ we must have $N/Z \leq MZ/Z$. Now $(MZ/Z)/(N/Z)$ is normal in $(T_v/Z)/(N/Z)$ which, by Lemma 2.11, is isomorphic to $Sp(2l - 2, 2)$ and is therefore simple since $2l \geq 8$. So $(MZ/Z)/(N/Z)$ is either $(T_v/Z)/(N/Z)$ or trivial. In the former case $MZ/Z = T_v/Z$ and so either $M = T_v = L$ (which is a contradiction), or $|T_v : M| = 2$ (and $Z \not\leq M$). But then $|N : M \cap N| = 2$, with $(M \cap N) \lhd T_v$ and $Z \not\leq (M \cap N)$, contradicting Lemma 2.12. Hence $(MZ/Z)/(N/Z)$ must be trivial, meaning that $M \leq N$.

The size of $T_{(\theta_a,\theta_b)}$ is

$$\frac{|T|}{(2^{2l-1} \pm 2^{l-1})(2^{2l-1} \pm 2^{l-1} - 1)} = \frac{\prod_{i=1}^{l}(2^{2i} - 1)2^{2i-1}}{(2^{2l-1} \pm 2^{l-1})(2^{2l-1} \pm 2^{l-1} - 1)},$$

and it can be shown that this value is larger than $|N| = 2^{2l-1}$. Hence M is not large enough to contain an arc stabiliser. So M must equal L, which is a contradiction. We used Magma to check that the result also holds for $2l = 6$.

Case $T = A_7$, $m = 15$: Here L is the stabiliser of a line ℓ of $\mathrm{PG}(3, 2)$, and L^ℓ is permutationally isomorphic to S_3. Therefore M^ℓ must be either A_3 or $T_{(\alpha,\beta)}^\ell = 1$, and hence we obtain Lines (xvi) and (xvii) of Table 1.

Case $T = \mathrm{P\Gamma L}(2, 8)$, $m = 28$: We go through each line in turn of Table 4. When $L = \mathrm{P\Gamma L}(2, 8)$, $\mathrm{A\Gamma L}(1, 8)$, or $\mathrm{AGL}(1, 8)$, the possibilities listed for M in Lines (ii)-(iv) of Table 2 are well known to be the only non-trivial normal subgroups. That each possibility for M contains $T_{(\alpha,\beta)}$ follows from the fact that it contains $T_{\{\alpha,\beta\}}$. The unique minimal normal subgroup $S \cong \mathbb{Z}_2^3$ of $\mathrm{AGL}(1, 8)$ is abelian, and so taking $L = S$, the possibilites for the normal subgroup M are $T_{(\alpha,\beta)} \times \mathbb{Z}_2 \cong \mathbb{Z}_2^2$, $T_{\{\alpha,\beta\}} \cong \mathbb{Z}_2^2$, and $T_{(\alpha,\beta)}$, giving Line (v). Next we consider the three possibilities with $L \cong A_4 \times \mathbb{Z}_2$, namely T_ℓ, C_1 and C_2. Let τ be an involution in A_4 and σ the generator of the direct factor $\mathbb{Z}_2$. When $L = T_\ell$, $T_{(\alpha,\beta)}$ corresponds to the subgroup $\langle\sigma\rangle$. Hence the only possibilites for M are S (which corresponds to $V_4 \times \mathbb{Z}_2$) and $T_{(\alpha,\beta)}$, giving Line (vi). When $L = C_1$, $T_{(\alpha,\beta)}$ corresponds to $\langle\tau\rangle$ and M can be D ($\cong A_4$), S ($\cong V_4 \times \mathbb{Z}_2$) or $T_{\{\alpha,\beta\}}$ (corresponding to V_4), giving Line (vii). When $L = C_2$, $T_{(\alpha,\beta)}$ corresponds to $\langle\tau\sigma\rangle$ and the only possibility for M is S ($\cong V_4 \times \mathbb{Z}_2$), giving Line (viii). Finally, when $L = D \cong A_4$, $T_{(\alpha,\beta)}$ corresponds to $\langle\tau\rangle$ and the only possibility for M is $T_{\{\alpha,\beta\}}$ (corresponding to V_4), giving Line (ix).

This completes the proof of Theorem 1.2. □

4. Partial linear spaces

A *partial linear space* is a set $\mathcal{V}$ of points together with a set $\mathcal{L}$ of (at least two) lines. Each line is a subset of points, and every pair of points lies in at most one line. We denote the partial linear space by the pair $(\mathcal{V}, \mathcal{L})$. A partial linear space is *line transitive* if there is a group of permutations of the points which preserves and transitively permutes the lines.

Lemma 5.1 of [1] shows that line transitive partial linear spaces are in one-to-one correspondence with transitive decompositions in which the subgraphs are complete. Thus the following theorem (which constitutes part of a result from [1]) gives a characterisation of a particular class of line transitive partial linear spaces.

Theorem 4.1. *Let $(\Gamma, \mathcal{P})$ be a G-transitive decomposition where $|\mathcal{P}| \geq 2$, $\Gamma = K_m \times K_m$ and G is a primitive rank 3 group of product action type. Assume that the subgraphs Γ_P are complete. Then for some 2-transitive normal subgroup T of H there exists a T-transitive decomposition $\mathcal{T} := (K_m, \mathcal{Q})$ corresponding to a 2-transitive linear space such that $\mathcal{P} = \mathcal{P}(\mathcal{T}, \mathcal{R}, \varphi)$ (as in* [1, Construction 2.10]*) for some φ, where $\mathcal{R}$ is the partition of AK_m in which each part contains only one arc.*

We can read off the possibilities for T, $\mathcal{Q}$ and $\mathcal{R}$ from Table 1, yielding the following Corollary to Theorem 1.2. This gives a more explicit classification of the class of partial linear spaces described in Theorem 4.1. An equivalent result in proved by Devillers in [6].

Corollary 4.2. *Let T be a 2-transitive group which is either non-abelian and simple or* $\mathrm{P\Gamma L}(2,8)$ *of degree 28. Suppose that $\mathcal{T} := (K_m, \mathcal{Q})$ is a T-transitive decomposition corresponding to a linear space, and let $\mathcal{R}$ be the partition of AK_m in which each part contains only one arc. Assume that $\mathcal{T}$ and $\mathcal{R}$ satisfy the conditions of* [1, Construction 2.10]*. Then one of the following holds.*

(i) $T = \mathrm{P\Gamma L}(2,8)$, *$m = 28$ and each $Q \in \mathcal{Q}$ induces a copy of K_4, or*
(ii) $T = A_7$, *$m = 15$ and each $Q \in \mathcal{Q}$ induces a copy of K_3, or*
(iii) $T = \mathrm{PSL}(a,2)$ *with $a \geq 3$, $m = 2^a - 1$ and each $Q \in \mathcal{Q}$ induces a copy of K_3.*

References

[1] John Bamberg, Geoffrey Pearce, and Cheryl E. Praeger. Transitive decompositions of graph products: rank 3 grid type. *Journal of Group Theory.* To appear.

[2] Eiichi Bannai and Tatsuro Ito. *Algebraic combinatorics. I.* The Benjamin/Cummings Publishing Co. Inc., Menlo Park, CA, 1984. Association schemes.

[3] Peter J. Cameron. SGDs with doubly transitive automorphism group. *J. Graph Theory*, 32(3):229–233, 1999.

[4] Peter J. Cameron and Gábor Korchmáros. One-factorizations of complete graphs with a doubly transitive automorphism group. *Bull. London Math. Soc.*, 25(1):1–6, 1993.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of finite groups.* Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[6] Alice Devillers. A classification of finite partial linear spaces with a rank 3 automorphism group of grid type. In preparation.

[7] Leonard Eugene Dickson. *Linear groups: With an exposition of the Galois field theory.* with an introduction by W. Magnus. Dover Publications Inc., New York, 1958.

[8] John D. Dixon and Brian Mortimer. *Permutation groups*, volume 163 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996.

[9] William M. Kantor. Homogeneous designs and geometric lattices. *J. Combin. Theory Ser. A*, 38(1):66–74, 1985.

[10] Cai Heng Li and Cheryl E. Praeger. Constructing homogeneous factorisations of complete graphs and digraphs. *Graphs Combin.*, 18(4):757–761, 2002. Graph theory and discrete geometry (Manila, 2001).

[11] Tian Khoon Lim. Edge-transitive homogeneous factorisations of complete graphs. 2004. Ph.D. Thesis. The University of Western Australia.

[12] Michael E. O'Nan. Automorphisms of unitary block designs. *J. Algebra*, 20:495–511, 1972.

[13] Geoffrey Pearce. On transitive decompositions of disconnected graphs. Submitted.

[14] Geoffrey Pearce. Transitive decompositions of graphs and their links with geometry and origami. Submitted.

[15] Thomas Q. Sibley. On classifying finite edge colored graphs with two transitive automorphism groups. *J. Combin. Theory Ser. B*, 90(1):121–138, 2004.

School of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley, W.A., Australia 6009

E-mail address: geoffrey.pearce@graduate.uwa.edu.au