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Vortex-induced vibration of four cylinders in an in-line square configuration

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This paper presents a numerical study of vortex-induced vibration of four rigidly connected and four separately mounted circular cylinders in an inline square configuration at a Reynolds number of 150, a low mass ratio of 2.5, and a range of spacing ratio L from 1.5 to 4, where the spacing ratio is defined as the centre-to-centre distance of two adjacent cylinders normalized by the cylinder diameter. For the rigidly connected cylinder array, the maximum and minimum response amplitudes occur at $L = 1.5$ and $L = 2.0$, respectively, for the range of spacing ratio covered in this study and the maximum response amplitude at $L = 1.5$ is accompanied by a wide lock-in range. The large response amplitude at a small spacing ratio $L = 1.5$ is because the cylinder array responds to the flow as a single cylinder with an overall size that is much larger than the diameter of the single cylinder while the small response amplitude observed at $L = 2.0$ is attributed to the strong interaction of the vortices through the gap between the top and bottom rows of the cylinder and also in the wake of the cylinder array. For spacing ratio $L \geq 2.5$, the lock-in regime of four rigidly connected cylinders is similar to that of a single cylinder and the response amplitudes in the lock-in regime are slightly higher than that of a single cylinder. The energy transfer analysis between fluid flow and individual cylinders in the array shows that the hydrodynamic forces on individual cylinders either excite or damp the vibration, depending on the reduced velocity. An interesting flow feature observed at $L = 2, 2.5$, and 3 is the biased vortex street in the wake of four rigidly connected cylinders. The biased vortex street leads to a shift of the mean position of the cylinder array with the largest mean position shift being observed at $L = 3$. Four response modes are identified for four separately mounted cylinders. These are the in-phase mode, the anti-phase mode, the correlated out-of-phase mode, and the uncorrelated mode. It is found that the response mode chosen by the cylinders is dependent not only on the spacing ratio but also on the initial condition of the flow. The response amplitude under the in-phase mode is generally higher than that under the anti-phase mode at identical spacing ratios. This is attributed to the interaction of vortices in the wake of the cylinders. © 2016 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4941774>]

I. INTRODUCTION

Vortex-induced vibration (VIV) of a group of cylinders in a fluid flow is of importance in engineering applications. VIV of a circular cylinder has been studied extensively in the past decades

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and most of the studies are focused on the VIV of an elastically mounted rigid cylinder in steady flows. Sarpkaya,¹ Sumer and Fredsoe,² Bearman,³ Williamson and Govardhan^{4,5} amongst others have provided extensive reviews on the topic.

The response of an elastically mounted cylinder in a steady flow depends on the Reynolds number, the mass ratio, the damping ratio, and the reduced velocity. The Reynolds number is defined as $Re = UD/\nu$, with U , D , and ν being the free stream velocity, the cylinder diameter, and the kinematic viscosity of the fluid, respectively. The mass ratio m^* is the ratio of the cylinder mass to the displaced fluid mass. For a circular cylinder, $m^* = m/(\rho\pi D^2/4)$, where m is the cylinder mass per unit length, and ρ is the density of the fluid. The reduced velocity is defined as $V_r = U/f_n D$ with f_n being the structural natural frequency of the cylinder.

The response frequency and the vortex shedding frequency synchronize (or lock-in) in a range of reduced velocities and the lock-in range of the reduced velocity depends on the mass ratio and the damping ratio. Experimental studies of one-degree-of-freedom (1-dof) vibration of a cylinder in the cross-flow direction show that for low mass ratios, the lock-in range of the reduced velocity can be divided into three branches: the initial branch, the upper branch, and the lower branch.⁶⁻⁹ The vortex shedding is in the 2S mode in the initial branch and 2P mode in the upper and lower branches. The 2S and 2P stand for two single and two pairs of vortices that are shed from the cylinder in one vibration period, respectively. If the cylinder is allowed to vibrate in both the in-line and the cross-flow directions, the maximum response amplitude is increased significantly compared with that in the case of cross-flow 1-dof VIV and a 2 T vortex shedding mode (two triplets of vortices are shed from the cylinder in one period of vibration) was found when the response reaches its maximum.¹⁰

Many two-dimensional numerical studies on VIV of circular cylinders have been conducted at Reynolds numbers in the laminar flow regime.¹¹⁻¹⁶ Different aspects of the VIV were examined by those studies. For instance, Singh and Mittal¹² focused their study on the hysteresis behavior of the cylinder, Borazjani and Sotiropoulos¹⁴ focused their study on the interference between two tandem vibrating cylinders, Zhao¹⁵ found the existence of VIV and galloping for two cylinders in side-by-side arrangements, and Zhao *et al.*¹⁶ studied the VIV of a square cylinder. Although the maximum response amplitudes at low values of Reynolds number are generally much lower than those found in the high Reynolds number flows, the numerical studies with low values of Reynolds number have been very effective in revealing fundamental flow mechanisms responsible for VIV of cylinders.

Two-dimensional numerical models based on the Reynolds-averaged Navier-Stokes (RANS) equations were also used to simulate the VIV of circular cylinders at high Reynolds numbers in the subcritical Reynolds number regime. It has been demonstrated that the RANS equations provide reasonable good results of the response amplitude and frequency. By solving the RANS equations, Pan *et al.*¹⁷ and Guilmineau and Queutey¹⁸ obtained good results of 1-dof VIV except in the upper branch and Zhao and Cheng¹⁹ predicted the super upper branch of the response in the 2-dof VIV very well.

Many studies investigated the VIV of two cylinders in fluid flow in order to understand the effects of the interference between the two cylinders on the response.²⁰ Most of those studies were focused on the wake interference between two cylinders in tandem arrangements. Vibration of an elastically mounted circular cylinder in the wake of a fixed upstream cylinder can be a VIV, a galloping or a combination of VIV and galloping, depending on the gap between the two cylinders.^{21,22} The studies of the VIV of two side-by-side cylinders in fluid flow are fewer than those of two tandem cylinders. Huera-Huarte and Gharib²³ found that the interference between the responses of two cylinders in a side-by-side configuration was very weak if the centre-to-centre gap exceeded 3.5 times the cylinder diameter. Wang *et al.*²⁴ focused their study on the effects of turbulence on the response of two side-by-side cylinders in steady flows and found that the enhancement of the response by the turbulence is significant when the two cylinders are very close to each other. Numerical studies on VIV of two cylinders of different configurations are also conducted but mainly at low Reynolds numbers.^{15,25-27}

While studies on flow past four cylinders are many, few studies have been reported for VIV of four cylinders. When four cylinders are placed in a fluid flow in an inline square arrangement, the vortex shedding from them can be either in-phase or anti-phase when the centre-to-centre spacing

ratio is greater than 2.²⁸ The vortex shedding is generally dominated by the anti-phase flow pattern at small spacing ratios and the dominance of the anti-phase flow weakens with the increase of spacing ratio.^{28,29} It has also been found that the in-phase vortex shedding may lead to the eventual formation of a binary-vortex street.³⁰ A number of experimental studies have been conducted to understand the wake flow patterns for flow past four cylinders. Sayers³¹ measured vortex shedding frequency for flow past four cylinders and found that a small change in the flow incident angle may lead to a sudden change in the vortex shedding frequency. Lam and Lo³² conducted Particle Image Velocimetry (PIV) tests of steady flow around four cylinders in an inline square arrangement and discovered a biased flow pattern in the wake of four cylinders for the spacing ratio less than 1.7. Lam and Fang³³ conducted extensive measurements of the forces on four cylinders in fluid flow. The PIV technique was also used in some experimental studies of flow past four cylinders to identify the wake flow features.^{34–37} It was found that the wake flow for four cylinders in an inline square arrangement can be generally classified into following three regimes: the shielding regime for spacing ratio less than 2, the shear layer reattachment regime for spacing ratios between 2.5 and 3.5, and the vortex impinging regime for spacing ratios greater than 4.

Flow past four cylinders in an inline arrangement is also well studied by numerical simulations. Farrant *et al.*³⁸ and Han *et al.*³⁹ simulated flow past four cylinders in an inline square arrangement at a low Reynolds number of 200. The in-phase and out of phase vortex shedding patterns were well predicted in these numerical simulations. Lam *et al.*,⁴⁰ Lam and Zou,⁴¹ and Tong *et al.*⁴² conducted three-dimensional numerical simulations to investigate the three-dimensionality of the flow. Oscillatory flow past four cylinders in a square arrangement was also studied numerically due to its engineering importance.^{43,44} While flow past four cylinders has been studied extensively, the studies of the VIV of four cylinders in fluid flow are very rare. Zhao and Cheng⁴⁵ studied response of four cylinders in a square arrangement with a constant spacing ratio of 3 and various flow approaching angles numerically. The lock-in regime of the four cylinders was found to be affected significantly by the flow approaching angle.

In this study, VIV of four cylinders in an inline square arrangement at a low Reynolds number of 150 and a low mass ratio of 2.5 is studied numerically. Previous studies have shown that flow past four cylinders becomes three-dimensional at lower Reynolds number than that past a single cylinder.⁴⁶ Lam *et al.*⁴⁰ reported the flow in the wake of four cylinders is three-dimensional at $Re = 200$. Tong *et al.*⁴² reported that when the Reynolds number is in the range of $100 \leq Re \leq 220$, the flow between the downstream and the upstream cylinders is two dimensional and the three-dimensionality behind the downstream cylinders is very weak. Zhao *et al.*⁴⁶ found that the wake flow behind a vibrating cylinder in the lock-in regime transits to 3D at a higher Reynolds number ($Re = 250$) than that of a stationary cylinder. To ensure the accuracy of the 2D numerical model, the Reynolds number for 2D simulations should be in the laminar flow regime. The Reynolds number of 150 chosen in this study ensures that the VIV can be simulated accurately at a wide parametric space using the 2D numerical model.

The flow around the cylinders and the response of the cylinders are predicted by solving the two-dimensional Navier-Stokes (NS) equations and the equation of motion of the cylinders, respectively. The cylinders are allowed to vibrate in the cross-flow direction only. Simulations are conducted for four rigidly connected cylinders and four separately mounted cylinders. Rigidly connected cylinders are often found in offshore engineering applications. For example, the main structure of a tension leg platform is comprised of four vertical cylinders that are connected by horizontal pontoons. In this case, the four cylinders respond approximately as a single object to various environmental loadings. Since the vertical cylinders are often closely spaced, the flow interference around the cylinders has a significant effect on the response of the cylinders. A group of separately mounted cylinders are found in offshore oil and gas engineering applications where risers are sometimes arranged in close proximity and subjected to VIV caused by ocean currents. It is expected that the flow characteristics and VIV responses of four separately mounted cylinders are different from those of four rigidly coupled cylinders.

The remaining of this paper is organized as follows. The method used in this study is described briefly in Section II. The numerical model validation and flow past four stationary cylinders in a square arrangement are discussed in Sec. III. The numerical results and discussions about four

rigidly connected cylinders and those about four separately mounted cylinders are presented in Sections IV and V, respectively. The conclusions are drawn in Sec. VI.

II. NUMERICAL METHOD

The four rigidly connected and four separately mounted cylinders in an inline square arrangement are illustrated in Figs. 1(a) and 1(b), respectively. For the convenience of discussion, the cylinders are numbered from 1 to 4 as shown in Fig. 1. In the subsequent discussions, we will refer the cylinders as C1 to C4 correspondingly. In the numerical simulations, the nondimensional diameter of each cylinder is taken as 1.0 and the nondimensional spacing (spacing to diameter ratio) between two adjacent cylinders is denoted as L . In this study, only 1-dof vibration of the cylinders in the cross-flow direction is considered. The governing equations for the flow around the cylinders are the two-dimensional incompressible NS equations. The Arbitrary Lagrangian Eulerian (ALE) scheme is implemented to account for the moving boundaries of the cylinder surfaces. The velocity (u, v) , the time t , the coordinate (x, y) , and the pressure p are nondimensionalized as $(u, v) = (\tilde{u}, \tilde{v})/(f_n D)$, $t = \tilde{t} f_n$, $(x, y) = (\tilde{x}, \tilde{y})/D$, $p = \tilde{p}/(f_n^2 D^2)$, respectively, where the tildes denote the dimensional parameters. By using the above nondimensionalization method, the NS equations in the ALE are expressed as

$$\frac{\partial u_i}{\partial t} + (u_j - \hat{u}_j) \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = \frac{V_r}{\text{Re}} \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad (1)$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (2)$$

where $x_1 = x$ and $x_2 = y$ are the Cartesian coordinates in the in-line and the transverse directions of the flow as defined in Fig. 1, respectively, u_i is the fluid velocity component in the x_i -direction, and $\hat{u}_i$ is the moving velocity of the mesh nodes. For four separately mounted cylinders, the motion of

FIG. 1. (a) Sketch for VIV of four rigidly connected cylinders in an in-line arrangement; (b) sketch for VIV of four separated mounted cylinders; (c) computational mesh for a spacing ratio of $L=2$; (d) computational mesh for $L=2$ when the two downstream cylinders are very close to each other.

each cylinder is predicted by solving the equation of the motion of this cylinder,

$$\frac{\partial^2 Y_k}{\partial t^2} + 4\pi\zeta \frac{\partial Y_k}{\partial t} + 4\pi^2 Y_k = \frac{1}{2\pi} \frac{V_r^2 C_{Lk}}{m^*}, \quad (3)$$

where Y_k is the displacement of the k th cylinder; $m^* = m/m_d$ is the mass ratio, with m being the mass of the k th cylinder and $m_d = \rho\pi D^2/4$ is the mass of the displaced fluid by the k th cylinder; $\zeta = \frac{c}{2\sqrt{Km}}$ is the damping ratio with c and K being the damping constant and spring constant of the system, respectively, and C_{Lk} is the lift coefficient of the k th cylinder. The drag and lift coefficients of a cylinder are defined as $C_{Dk} = F_{Dk}/(\rho DU^2/2)$ and $C_{Lk} = F_{Lk}/(\rho DU^2/2)$ with F_{Dk} and F_{Lk} being the drag and lift forces in the in-line and the cross-flow directions, respectively.

For four rigidly connected cylinders, the equation of the motion for the whole system is

$$\frac{\partial^2 Y}{\partial t^2} + 4\pi\zeta \frac{\partial Y}{\partial t} + 4\pi^2 Y = \frac{1}{2\pi} \frac{V_r^2 C_{LA}}{m^*}, \quad (4)$$

where Y is the displacement of the cylinder group, and C_{LA} is the average lift coefficient of the four cylinders. Because the cylinders only move in the cross-flow direction, the mesh nodes are moved only in the cross-flow direction after each computational time step. The governing equation for calculating the displacements of the nodes of the FEM mesh is¹⁹

$$\nabla \cdot (\gamma \nabla S_y) = 0, \quad (5)$$

where S_y represents the displacement of the nodal points in the y -direction and γ is a parameter that controls the mesh deformation. To avoid excessive deformation of the near-wall elements, the parameter γ in a finite element is set to be $\gamma = 1/A$, with A being the area of the element. The displacement of the mesh nodes is the same as the displacement of the cylinder on the cylinder surface and zero on other boundaries. By specifying the displacements at all the boundaries, Eq. (5) is solved by a Galerkin finite element method. Two initial conditions are used in this study to identify the effects of the initial condition on the solution. Initial condition 1 (IC1) corresponds to zero initial velocity and pressure of the flow, and zero initial velocity and displacement of the cylinder. Initial condition 2 (IC2) corresponds to an asymmetric condition that will be discussed later on in Sections IV and V, respectively.

When two cylinders are very close to each other, the use of Eq. (5) can lead to an undesirable distorted mesh in the gap between two adjacent cylinders in the same column, resulting in a breakdown of simulations. To prevent this from happening, the finite element sizes in the y -direction in a small area within the gap (marked by dashed lines in Fig. 1(c)) are forced to be the same when the gap between two cylinders is smaller than 0.2 diameters. Fig. 1(d) shows an example of the computational mesh when the two downstream cylinders are about to collide with each other. It can be seen that good mesh quality is retained even when the two downstream cylinders are very close to each other.

For four separately mounted cylinders, the collision between the two upstream cylinders or between the two downstream cylinders is possible. To avoid simulation breakdowns induced by cylinder collision, the two cylinders are considered to collide with each other and forced to separate from each other if the gap between two cylinders becomes less than 0.02 diameters during any stage of the simulation. The collision between two cylinders is treated to be elastic in the numerical simulations, i.e., both the momentum and the kinetic energy are conserved during the collision. Because this study only focuses on the maximum possible response amplitude, the effect of the energy loss during the collision is not discussed. After the collision, the velocities of the two cylinders are calculated according to the conservation law of the momentum.⁴⁷ Collision is only found in a very small range of the reduced velocity with spacing ratios of 1.5 and 2 in the present study.

A rectangular computational domain with a height of $40D$ in the cross-flow direction and a width of $60D$ in the flow direction is used, corresponding to a blockage ratio of 0.05. Previous studies showed that the domain width has little effects on the result of the cylinder response if the blockage ratio is less than 0.05.^{48,49} The computational domain is divided into four-node quadrilateral bi-linear finite elements as shown in Fig. 1(c). The number of the finite element nodes varies from 49 725 to 50 083 for $L = 1.5$ –4. Refined elements are used close to the cylinder surfaces in order to capture the strong variations of the flow field. A total of 96 elements are distributed along the

surface of each cylinder. The minimum nondimensional mesh size at the cylinder surface is 0.002. On the inlet boundary, the horizontal velocity is set equal to V_r and vertical velocity component is set to zero. On the cylinder surfaces, the flow velocity equals to the vibration speed of the cylinders. At the outflow boundary, the pressure is set to be zero and the gradient of the velocity in the flow direction is zero. At the two lateral boundaries, the gradient of the pressure and the velocity in the cross-flow direction are zero.

III. FLOW PAST FOUR STATIONARY CYLINDERS IN AN IN-LINE SQUARE ARRANGEMENT

To validate the numerical model, simulations of flow past four stationary circular cylinders in an inline square arrangement at $Re = 200$ are carried out and the numerical results are compared with the published data. Fig. 2 shows the instantaneous vorticity contours for flow past four stationary cylinders for a range of spacing ratio at $Re = 200$. The nondimensional vorticity shown in Fig. 2 is defined as $\omega = (\partial \tilde{v} / \partial \tilde{x} - \partial \tilde{u} / \partial \tilde{y}) / (U/D)$. It is seen from Fig. 2 that the flow around the cylinder array can be considered as the interaction of flows around two rows of tandem cylinders. At relatively small spacing values, say $L \leq 2.0$, the interaction between the shear layers formed inside the gap between two cylinder rows is strong and this appears to have weakened their

FIG. 2. Instantaneous vorticity contours for flow past four stationary cylinders at $Re = 200$.

interactions with the corresponding shear layers from the two outer sides of the cylinder rows, leading to a reduced lift coefficient on each cylinder compared with that of a single isolated cylinder. The vortex shedding from the upstream cylinders is observed only when the spacing ratio $L \geq 3$ for the cases considered in this study. Based on instantaneous flow snapshots shown in Fig. 2, the vortex shedding processes from the two upstream cylinders C1 and C2 are almost in phase with each other for $L = 3, 3.5$, and 5 and nearly in anti-phase with each other for $L = 4$. It is noted that the flow features observed at $L = 3, 3.5$, and 4 largely remain their corresponding states over the entire period of simulation with very small phase shifts and no transient mode swapping is observed in those cases. However mode swapping is observed for $L = 5$ where the in-phase and anti-phase flow features alternate in time as shown in Figs. 2(g) and 2(h). A distinct flow feature observed in Fig. 2 is the asymmetrical wake structures about x axis behind the cylinder array. For example, the wake structure ($x > 4.0$) is clearly biased downwards for $L = 3$, where the vortices shed from the top row of cylinders occupied a majority of the wake space behind the cylinder array. For $L = 3.5, 4$, and 5, the wake structure behind the top row of cylinders is not exactly the same as that from the bottom row of cylinders. The asymmetrical vortex shedding flow in the wake of the four cylinders are similar to that observed in Fig. 5 in Ref. 40.

The flow features shown in Fig. 2 can be better understood by observing the transient lift coefficient on the cylinders as shown in Fig. 3, where the time histories of the lift coefficient on C3 and C4 are presented. For $L \leq 2.5$, the time histories of the lift coefficient on C3 and C4 are very irregular. This is due to (1) the strong interaction between the attached wakes of upstream cylinders

FIG. 3. Time histories of the lift coefficients of C3 and C4 for flow past four stationary cylinders at $Re = 200$.

and the downstream cylinders and (2) the interaction of shear layers from the top and bottom rows of cylinders in the cylinder array. The time histories of the lift coefficient on C3 and C4 become rather regular for $L \geq 3$. This is related to the onset of vortex shedding from the upstream cylinders. For $L = 3$, the higher amplitude of lift coefficient observed on C4 than C3 is attributed to the biased wake structure towards the direction of C3. For $L = 4$, the amplitudes of the lift coefficients on C3 and C4 are similar and fluctuate periodically. Whenever the lift coefficient on C3 reaches its maximum, the lift coefficient on C4 reaches its minimum and vice versa. This is a direct indication of the interaction of shear layers and vortices in the wakes of tow cylinder rows and suggests that the vortices from C3 and C4 dominate the wake flow alternatively. At $L = 5$, the lift coefficients of the two downstream cylinders becomes in-phase and anti-phase with each other alternatively, indicating the alternate change between the in-phase and the anti-phase vortex shedding pattern as shown in Figs. 2(g) and 2(h).

Fig. 4 shows the comparison between the calculated mean drag coefficients (averaged over 40 vortex shedding period) and RMS lift coefficient on individual cylinders with the 2D numerical results in Ref. 40. For the flow past four cylinders in a square arrangement, vortex shedding does not occur behind the two upstream cylinders until the spacing ratio L exceeds a critical value. Because the vortex shedding flow in the wake of the four cylinders is asymmetric, i.e., the vortex shedding from one downstream cylinder dominates that from the other downstream cylinder as shown in Fig. 2, resulting in different force coefficients on the two downstream cylinders.⁴⁰ It appears that the difference in the mean drag coefficient between the two upstream cylinders is smaller than that between the two downstream cylinders. The computed variations of the mean drag coefficients with the spacing ratio agree with the results in Ref. 40 well. For $L = 4$, the present mean drag coefficients on C3 and C4 are close to each other, because the amplitude of the lift coefficients of both cylinders vary with time as shown in Fig. 3(d) and the long period average evens out the difference in the lift coefficients on C3 and C4. It is observed from Fig. 4 that the mean drag and RMS lift coefficients on all four cylinders experience sudden changes at approximately $L = 2.6$. The sudden changes observed at $L = 2.6$ actually correspond to the onset of vortex shedding from the upstream cylinders. Therefore $L = 2.6$ is identified as the critical spacing ratio for the vortex shedding from the upstream cylinders in this study. Once the vortex shedding occurs behind the two upstream cylinders, the mean drag coefficient of the downstream cylinders increases significantly

FIG. 4. Variations of the mean drag and the RMS lift coefficients with the spacing ratio for flow past four stationary cylinders for $Re = 200$.

FIG. 5. Variation of the force coefficient with the spacing ratio for flow past four stationary cylinders at $Re = 150$.

but is still much smaller than those on the two upstream cylinders. Present results of the RMS lift coefficient agree very well with those in Ref. 40. Due to the strong influence from the upstream cylinders, the RMS lift coefficient on the downstream cylinders is much greater than that on the upstream cylinders if the vortex shedding occurs ($L \geq 2.6$).

Flow past four stationary cylinders at $Re = 150$ is also simulated to obtain reference flow structures and force coefficients for VIVs of the four cylinder systems. To encourage the onset of the asymmetrical wake structures observed at $Re = 150$, each of the four cylinders is initially rotated in the anti-clockwise directions with a surface tangential speed of $u_T = U \sin(2\pi t/10)$ until $Ut/D = 10$, after which the rotation is stopped. The force coefficients for flow past four stationary cylinders at $Re = 150$ are shown in Fig. 5. When the spacing ratio is increased from 2.5 to 3, the vortex shedding from the two upstream cylinders occurs, evidenced by the sharp increase in the RMS lift coefficient. The vortex shedding flow patterns for $Re = 150$, although are not presented here, are similar to those for $Re = 200$. The occurrence of the vortex shedding from the two upstream cylinders results in big changes in the mean and RMS force coefficients on all of the cylinders. The biased wake vortex flows are observed for $3.5 \leq L \leq 5$. The biased vortex flow leads to the difference between the forces on the two downstream cylinders. The repulsive lift coefficients (negative mean lift coefficient of C1 and C3 and positive mean lift coefficient of C2 and C4) at small spacing ratios were also observed by Farrant *et al.*³⁸ The repulsive mean lift coefficients of the two upstream cylinders are stronger than those of the two downstream cylinders.

The present numerical model has been validated against VIVs of a single cylinder and two cylinders at low Reynolds numbers¹⁵ previously and will not be further validated in the present study. It should be pointed out that the computational meshes used in the present study are selected based on a mesh dependency study for both cases with stationary and oscillating cylinders, although these results are not detailed here.

IV. VIV OF FOUR RIGIDLY CONNECTED CYLINDERS

A. Response amplitude and frequency

VIV of four rigidly connected cylinders in an in-line, square arrangement is simulated for a constant Reynolds number of 150, a constant low mass ratio of 2.5, and spacing ratios of 1.5, 2, 2.5, 3, and 4. To achieve the highest possible response amplitudes, the damping ratio is set at

zero in the simulations.^{12,50} When the VIV of four rigidly connected cylinders are simulated, the initial displacement and velocity of the cylinder are zero and the initial fluid velocity is zero in the whole fluid domain, unless otherwise specified. Simulations are carried out over a range of reduced velocity from 1 to 25. This range of reduced velocity was found wide enough to cover the lock-in regimes for all the spacing ratios investigated in this study. The four rigidly connected cylinders vibrate in the cross-flow direction as a single body. In all the simulations the length (streamwise direction) and the width (cross-flow direction) of the computational domain is taken as $60D$ and $40D$, respectively. The four cylinders are placed $20D$ from the inlet boundary. The computational meshes used for simulating VIV are the same as the ones used for flow past four stationary cylinders shown in Fig. 1.

Fig. 6 shows the variations of the response amplitude and frequency with the reduced velocity. The nondimensional amplitude in the cross-flow direction (y -directions) is defined as $A_y = (Y_{\max} - Y_{\min})/2$, where the subscripts “max” and “min” stand for the maximum and minimum displacements, respectively. The response frequencies are determined by the Fast Fourier Transform (FFT) of the cylinders’ displacement. The frequency corresponding with the highest response amplitude is deemed as the response frequency in this study. The nondimensional frequency f_y is defined as the ratio of the response frequency to the structural natural frequency. The lock-in regime for a single cylinder is identified as $3.5 \leq V_r \leq 8$ based on relatively large response amplitudes and the closeness between the natural frequency and the vibration frequency. For $L = 1.5$, the maximum response amplitude is about twice that of the single cylinder and the lock-in regime is also much wider than that of the single cylinder. The response frequency for $L = 1.5$ increases with increasing reduced velocity until $V_r = 7$, after which the response frequency changes little until $V_r = 25$. It appears that the four cylinder array responses in a similar way (but not exactly the same) to a single cylinder with a larger dimension than the cylinder diameter. To demonstrate this point, the response amplitudes and frequency for the case with $L = 1.5$ normalized by $2.5D$ rather than D are shown in Fig. 6(a), where $A_{y,2.5D}$ and $V_{r,2.5D}$ are defined as $A_{y,2.5D} = A_y/(2.5D)$ and $V_{r,2.5D} = U/(2.5f_n D)$, respectively. It is found that the variation of $A_{y,2.5D}$ with $V_{r,2.5D}$ for $L = 1.5$ is similar to the variation of A_y with V_r for a single cylinder. The smaller response amplitude of $A_{y,2.5D}$ for the case with $L = 1.5$ than A_y of the single cylinder over the lock-in regime is largely due to the gap flow through the cylinder array.

FIG. 6. Variations of response amplitude and frequency with the reduced velocity for VIV of four rigidly connected cylinders. $A_{y,2.5D}$ and $V_{r,2.5D}$ are defined as based on $A_{y,2.5D} = A_y/(2.5D)$ and $V_{r,2.5D} = U/(2.5f_n D)$, respectively.

The response amplitudes for $L = 2$ are the smallest among all the spacing ratios considered in this study, although the response frequency for $L = 2$ is close to the natural frequency over a wide range of reduced velocity of $3 \leq V_r \leq 25$. The low response amplitudes observed for $L = 2$ is believed to be due to the flow transition from a near single body wake to multi-body interacting wakes. The strong interactions of the shear layers passing through the gap between the top and bottom rows of cylinders and the merging of the vortex shedding from the top and bottom rows of the cylinders into a single global wake influence the lift force on the cylinders. Since these processes occur at different frequencies, the lift coefficient spectrum at $L = 2$ is characterized by multiple peaks at different frequencies as shown in Fig. 7. In Fig. 7, the cylinder displacement and lift coefficient spectra are plotted as functions of both V_r and f_y . To maintain uniformity, each spectrum shown in Fig. 7 is normalized by the amplitude of the highest peak. It is seen from Fig. 7(d) that for $L = 2$, the spectra of the lift coefficient are broad-banded over a wide range of reduced velocity, leading to the reduction in the response amplitude. The response

FIG. 7. Normalized FFT amplitude spectra of the displacement and the total lift coefficient for VIV of four rigidly connected cylinders. The amplitudes of the displacement and the lift coefficient are normalized by their peak amplitudes, respectively.

is generally very irregular when the lift is broad-banded because the component of the lift force that synchronizes with the vibration frequency is not necessarily the one with the highest peak value. This explains why the response amplitude for $L = 2$ is the lowest among other spacing ratios. For $L = 1.5$, the sudden reduction of response amplitude observed at V_r between 20 and 23 is also because the cylinder oscillation did not synchronize with the dominant lift component as shown in Fig. 7(b). Zhao and Cheng⁴⁵ also reported that the peak response frequency is different from the peak frequency of the lift force for irregular response of four cylinders in the turbulent flow regime. For a broadband lift force spectrum, it is the lift component with the closest frequency to the natural frequency that excites the vibration regardless whether it is the dominant lift component. A response is defined as in the lock-in regime in this study if the difference between the dominant displacement and lift frequencies is less than 5%. This is the reason why the cases such as $V_r \geq 9$ for $L = 2$ and $V_r \geq 21$ for $L = 1.5$ are not classified as in the lock-in regime, even though the response frequency is close to the natural frequency.

It is found the VIV response of the cylinder array at $L = 3$ and 4 is hysteretic, i.e., the response amplitudes at the same V_r are different, depending on the ways the simulations are initiated. For example, the response amplitude for $L = 4$ becomes absolutely zero as $V_r \geq 9$ when the simulation is initiated with zero fluid velocity and pressure and zero displacement and velocity of the cylinder. It will be shown later on that the zero response amplitude for $V_r \geq 9$ is because the vortex shedding from the top row of the cylinders is in anti-phase with the vortex shedding from bottom row of the cylinders at $L = 4$. For the same case, the response amplitude for $V_r \geq 9$ is no longer zero if the simulations are initiated with an asymmetric flow field. To quantify this hysteretic effect, the simulations for $L = 3$ and 4 are repeated with the equilibrium asymmetric flow field calculated from $V_r = 7$ as the initial conditions. In the subsequent discussions, we denote the initial conditions with zero velocity and displacement of the cylinder, and zero velocity of the fluid flow as initial condition 1 (IC1) and the equilibrium asymmetric flow field and the displacement and the velocity of the cylinders calculated from $V_r = 7$ as initial condition 2 (IC2). The VIV responses obtained with IC1 and IC2 for $L = 3$ and 4 are compared in Fig. 6(b). Apart from the zero response amplitude for $V_r \geq 9$ with IC1, the response amplitudes for $L = 3$ and 4 in the range of $V_r \leq 3$ with IC1 are also zero due to the anti-phase vortex shedding patterns. The non-zero response amplitudes with IC2 are due to the asymmetric vortex shedding patterns from the top and bottom rows of cylinders. It is observed from Fig. 6(b) that if the vibration amplitudes with both IC1 and IC2 are non-zeros, the response amplitude and frequency under IC1 are the same as their counterparts under IC2, respectively. It should be pointed out that although the lock-in regimes for $L = 3$ and 4 are similar to that of the single cylinder, the responses are strongly influenced by the interaction of the vortices in the wake of the four cylinder array at these two gap ratios, which will be discussed later on.

It is seen from Fig. 6 that the vibration amplitudes outside the lock-in range $V_r \geq 9$ with $L = 2.5$ and 3 are about 0.2 diameters, which are still significantly greater than their single cylinder counterpart. This is believed to be due to the greater RMS lift coefficient for $L = 2.5$ and 3 cases than that of a single cylinder. The cylinder response outside the lock-in range has similar frequencies as the ones based on the Strouhal law. The vibration amplitude increases with the increase in the RMS lift force. It has been shown previously that if the cylinders are stationary, the RMS lift coefficient increases significantly when the spacing ratio exceeds 3 (inclusive) because of the occurrence of vortex shedding from the upstream cylinders.^{39,40} The increase of the vibration amplitude with the RMS lift force coefficient can be illustrated by a simplified 1-dof forced vibration system. If the lift coefficient is represented by a sinusoidal function of the time, the vibration amplitude of the cylinder system without any damping can be determined analytically as

$$A_{y,\text{sinusoidal lift}} = C_{LN} = \frac{\sqrt{2}C'_{LA}V_r^2}{2\pi^3m^*[1 - (f/f_n)^2]}, \quad (6)$$

where C'_{LA} is the RMS averaged lift coefficients of the four cylinder system. The $A_{y,\text{sinusoidal lift}}$ is the same as the normalized lift coefficient C_{LN} . Outside the lock-in range, the response frequency is the same as the vortex shedding frequency and follows the Strouhal law. Fig. 8 shows the variation of the normalized lift coefficient C_{LN} with the reduced velocity. It is seen from Fig. 8 that the variation trends of the normalized lift coefficient C_{LN} with the reduced velocity are similar to those of the response amplitude shown in Fig. 6 at the corresponding spacing ratios. An obvious exception to

FIG. 8. Variation of the normalized lift coefficient with the reduced velocity for VIV of four rigidly connected cylinders.

this occurs at $V_r = 16$ and $L = 1.5$, where the normalized lift coefficient C_{LN} assumes a very large value because the nondimensional vibration frequency is close to 1 (the denominator of Eq. (5) is close to zero). The normalized lift coefficient C_{LN} for $L = 2.5$, 3, and 4 (IC2) are significantly greater than that of a single cylinder when the reduced velocity is greater than 10.

Another interesting phenomenon observed in Fig. 6 is that the response amplitudes for $L = 2$ are significantly smaller than those for $L = 1.5$ over $7 < V_r < 22$, despite that the lock-in seems to have had occurred for the case with $L = 2.0$ in a similar way to the case with $L = 1.5$ by examining the response frequencies shown in Fig. 6(c). Although the physical reasons for this was explained qualitatively in relation to the lift and displacement spectra shown in Fig. 7, a quantitative explanation is offered by examining the correlation between the response displacement and the lift coefficient as shown in Fig. 9. In this study, the phase between the lift coefficient and the displacement is calculated based on the cross correlation coefficient. The cross correlation coefficient between two signals $a(t)$ and $b(t)$ is defined as $C_{ij}(\tau)$ and calculated using n periods of data by $C_{ab}(\tau) = \frac{1}{nT} \int_0^{nT} [a(t) - \bar{a}] \cdot [b_j(t + \tau) - \bar{b}] dt / (\sigma_a \sigma_b)$, where T is the vibration period, τ is the time lag, σ_a and σ_b are the standard deviation of $a(t)$ and $b(t)$, respectively, and $\bar{a}$ and $\bar{b}$ are the mean values of $a(t)$ and $b(t)$, respectively. It is seen from the definition that the cross correlation coefficient is also time dependent with a period that is the same as the period of cylinder vibration. The cross correlation coefficient $C_{ab}(\tau)$ for $0 \leq \tau \leq T$ is calculated and the maximum value of $C_{ab}(\tau)$ is denoted by $C_{m,ab}$. For signals (a and b) with identical frequencies, we have $C_{m,ab} = 1$. The phase between a and b is defined as $\varphi_{ab} = \varphi_a - \varphi_b$, where φ_a and φ_b are the phase of the vibrations of a and b , respectively. The time lag corresponding to the maximum value of $C_{m,ab}$ is defined as τ_m and the time lag is converted to the phase difference by $\varphi_{ab} = 2\pi\tau_m/T$. In this study, 30 periods of data (i.e., $n = 30$) are used to calculate the phase difference.

FIG. 9. The correlation coefficient and the phase difference between the lift coefficient and the vibration displacement for VIV of four rigidly connected cylinders.

It can be seen that excellent correlations between the lift coefficient and the displacement are observed in the cases with $L = 3$ and $L = 4$ under IC2 and poor correlations are observed for $V_r \geq 10$ in the cases with $L = 2$ and 2.5 and for $V_r \geq 19$ in the case with $L = 1.5$. These are consistent with the qualitative explanations offered from the observations of power spectra of the lift coefficient and displacement in Fig. 7. The phase differences between the lift coefficient and the displacement (φ) are only calculated when the correlation coefficient is greater than 0.9 in this study and shown in Fig. 9. Govardhan and Williamson⁹ found that for a freely vibrating cylinder in a fluid flow at low mass ratios, the phase between the lift coefficient and the displacement jumps through 90° at the boundary between the upper and lower branches. The variations of φ for the cases with $L = 3, 4$ under IC2 are similar to that of a single cylinder reported by Govardhan and Williamson.⁹ It can be seen that for the case with $L = 1.5$, the phase difference between the lift coefficient and the displacement remains less than 90° until $V_r = 18$, leading to a vibration frequency that is consistently smaller than the structural natural frequency as shown in Fig. 6(c). The phases between the displacement and the lift coefficient for $L = 1.5$ and $V_r \geq 19$ are not calculated because the correlation is very poor.

B. Vortex shedding patterns

Vortex shedding flow patterns are investigated in order to explain the VIV responses of the four cylinder array presented in Subsection IV A. First of all, instantaneous vorticity contours for $L = 1.5$ and $V_r = 15$ are shown in Fig. 10 in order to explain the large amplitude oscillations and the wide lock-in regime observed in the case with $L = 1.5$. The vortex shedding patterns behind the cylinder array are qualitatively similar to the 2P vortex shedding mode of a single cylinder observed in the experiments.⁶⁻⁸ Two negative vortices are shed from the cylinder when the cylinder is moving downwards and two positive vortices are shed when the cylinder is moving upwards, forming a 2P wake. The response amplitude corresponding to Fig. 10, which is normalized by the cylinder diameter, is much higher than that of a single cylinder because of the larger dimension of the four-cylinder structure. When the response amplitude and frequency are normalized by 2.5 cylinder diameters, the response curve is rather similar to that of a single cylinder case as shown in Fig. 6, suggesting the similarities between the two. It is observed that vortex shedding flow patterns shown in Fig. 10 do not follow the temporal-spatial symmetry strictly as in the case of the single cylinder. This is because the VIV response at $V_r = 15$ is not perfectly periodic.

An interesting phenomenon is found in the case with $L = 2.5$ at $V_r = 5$ that the wake vortex shedding flow biases towards the bottom side as shown in Fig. 11. The vortex shedding processes from the two upstream cylinders for $L = 2.5$ and $V_r = 5$ appears to synchronize as shown in Fig. 11. The vortex shedding processes from the two downstream cylinders also appears to synchronize. The synchronization of the vortex shedding leads to high response amplitudes in the lock-in regime. However, the wake flow is asymmetric because all the vortices generated from the gap sides of the cylinders (bottom side of the top right cylinder and top side of the bottom right cylinder) are convected towards the bottom right direction instead of the horizontal direction. The negative vortices that are shed from the top-right cylinder are lined regularly in a row without much disturbance from other vortices in the street. The positive vortices from the bottom side of the top downstream cylinder and the negative vortices from the top side of the bottom right cylinder interact with each other strongly in the near

FIG. 10. Instantaneous vorticity contours for $L = 1.5$ and $V_r = 15$ for VIV of four rigidly connected cylinders.

FIG. 11. Instantaneous vorticity contours for $L = 2.5$ and $V_r = 5$ for VIV of four rigidly connected cylinders.

wake for the cylinder array and are largely cancelled out after $x > 10$. A row of positive vortices, mainly from the bottom side of the bottom right cylinder of the array, survive for $x > 10$, forming a near parallel vortex street with the row of negative vortices above in the range of $10 < x < 18$. The biased near-wake flow structures were also found for flow past two side-by-side stationary cylinders^{28,30} and flow past stationary four cylinders.⁴⁰ In this study, the biased flow is found to develop immediately after the vortices are shed from the cylinders instead of developing slowly as in the case of four stationary cylinders. In the cases where the vibration amplitudes are not zero, the response displacements and the flow patterns under IC1 and IC2 are the same. If the initial velocity and pressure are set to zero in the whole fluid domain, direction towards which the vortex street is biased is random. However once a bias direction is chosen, it remains in that direction and no intermittent change of the bias direction is observed in this study.

Fig. 12 shows the instantaneous vorticity contours for $V_r = 5$ at different spacing ratios. The vortex shedding for $L = 1.5$ and $V_r = 5$ is similar to that of four stationary cylinders because the response

FIG. 12. Instantaneous vorticity contours at $V_r = 5$ and different spacing ratios for VIV of four rigidly connected cylinders when $Y = Y_{\max}$.

FIG. 13. Instantaneous vorticity contours at $V_r = 15$ and different spacing ratios for VIV of four rigidly connected cylinders when $Y = Y_{\max}$.

amplitude outside the lock-in regime is very small. The biased flow is also observed for $L = 2$ and 3 at $V_r = 5$, and the biasness of the flow for $L = 3$ appears much stronger than that for $L = 2$. For the biased flows shown in Figs. 11 and 12, strong vortex interactions occur immediately behind the cylinder array and the wake further downstream is characterized by parallel rows of vortices.

It can be seen that the critical spacing ratio for the vortex shedding to occur from the upstream cylinders is much smaller than that for flow past four stationary cylinders. The vortex shedding from the two upstream cylinders synchronizes in the lock-in regime. This is also different from the anti-phase vortex shedding patterns observed in Fig. 5. The biased flow disappears when the spacing ratio reaches 5 as shown in Fig. 12(d), due to the weakening of the interactions of flow around individual cylinders in the array. The four rows of vortices are very regularly aligned in the wake of the cylinders for the case with $L = 4$ at $V_r = 5$ as shown in Fig. 12(d).

Fig. 13 shows the instantaneous vorticity contours for different spacing ratios at $V_r = 15$, which is outside the lock-in regimes. The vortex shedding for $L = 2$ and $V_r = 15$ is similar to that for flow past four stationary cylinders. The vortex shedding from the two upstream cylinders disappears outside of the lock-in regime as shown in Fig. 13(a). The in-phase vortex shedding leads to non-zero, but small response amplitudes outside the lock-in regime. For $L = 4$ and $V_r = 15$ with IC1, the vortex shedding from the two top cylinders is perfectly out of phase with that from the bottom two cylinders, leading to an absolutely zero response amplitude as shown in Fig. 6(b). For $L = 3$ and $L = 4$ with IC2, the merging of the vortices are found in the wake of the cylinders and the vortex street becomes asymmetric.

C. Lift coefficient and its correlation with the vibration

Fig. 14 shows the variation of the RMS lift coefficient on individual cylinders with the reduced velocity. C_{L1} to C_{L4} stand for lift coefficients on C1 to C4, respectively. It is believed that the

FIG. 14. Variation of the RMS lift coefficient on individual cylinders with the reduced velocity for VIV of four rigidly connected cylinders.

biased vortex flows shown in Figs. 11 and 12 are largely responsible for the different lift coefficients measured on the two top cylinders from those measured on the two bottom cylinders. For $L = 1.5$, where the biased vortex flow is not observed and for $L = 2$, where the biasness of the vortex flow is very weak, the RMS lift coefficients for cylinders 1 and 3 are the same as those for cylinders 2 and 4, respectively. For $L = 2.5$, the difference in the lift coefficient mainly occurs in the range of $4 \leq V_r \leq 7$, where the response amplitude is high. The largest difference between the lift coefficient of cylinders 1 and 3 and that of cylinder 2 and 4 occur in the intermediate spacing ratio for $L = 3$ and $L = 4$ with IC2, and these differences are obvious both inside and outside the lock-in range.

The asymmetry in the lift coefficients shown in Figs. 14(d) and 14(e) lead to the non-zero mean cross-flow displacements as shown in Fig. 15, where the variation of the mean cross-flow displacement with the reduced velocity is shown. Since the mean deflection of the cylinder array's displacement can be either positive or negative due to the randomness nature of the biased wake flow, the absolute value of the mean displacements is shown in Fig. 15. Small non-zero mean displacements can be seen for $L = 2.5$ due to the weak biased flow. The maximum mean displacement for $L = 2.5$ is about 0.02 occurring at $V_r = 8$. However, for $L = 3$, the mean cross-flow displacement increases with increasing reduced velocity because the wake flow remains biased over the entire range of reduced velocity covered in this study. At $V_r = 25$, the mean deflection of the cylinders is about 0.12. The deflection of the cylinder displacement is expected to continue to increase with V_r because the dimensional mean lift force increases with the increase in the incoming velocity. When $V_r \geq 11$, the deflection of the mean displacement for $L = 3$ is strongest among other calculated spacing ratios.

FIG. 15. Variation of the mean cross-flow displacement with the reduced velocity for VIV of four rigidly connected cylinders.

To examine the contribution of the lift force on each of the cylinders to the vibration of cylinder array, the lift force that is in-phase with the velocity of the cylinder, which excites the vibration, is separated from the total lift force. In this study, the component of the force that is in-phase with the velocity of the cylinder is defined as the excitation force, because it transfers energy from the fluid flow to the cylinder. The excitation lift force is defined as F_{LE} and it is calculated by⁵¹

$$F_{LE} = \frac{\sqrt{2}(F_L - \bar{F}_L)\dot{Y}}{\sqrt{\dot{Y}^2}}. \quad (7)$$

Then, the excitation lift coefficient C_{LE} is calculated based on the excitation lift force. A positive excitation force excites the vibration and a negative excitation force damps the vibration. The contribution of the forces on each individual cylinder to the vibration of the cylinder array is examined by the time averaged excitation lift coefficient as shown in Fig. 16. It can be seen that the forces

FIG. 16. Variation of the time averaged excitation coefficients of individual cylinders with the reduced velocity for VIV of four rigidly connected cylinders.

on two of the four cylinders excite the vibration and those on the other two damp the vibration in most of the cases. For $L = 1.5, 2$, and 4 , whenever the excitation lift coefficients of the two upstream cylinders become positive, those on the two downstream cylinders become negative. The contribution of the lift force on each cylinder to the VIV of the cylinder array depends on the reduced velocity. For $L = 1.5$, the lift forces on the two upstream cylinders 1 and 2 excite the vibration and those on the two downstream cylinders damp the vibration for $1 \leq V_r \leq 9$ and $21 \leq V_r \leq 25$. The opposite is true for $10 \leq V_r \leq 18$. The variation of $\bar{C}_{LE}$ on two downstream cylinders (C3 and C4) with the reduced velocity for $L = 2.5$ is irregular because the strong interaction among the vortices in the wake of the cylinder system. For the cases (e.g., $L = 3$ and $L = 4$ with IC2) where the vortex shedding flow from the gaps is biased towards one side of the cylinder system as discussed above, the situations are slightly different from those observed at small spacing ratios. For examples, for $V_r \geq 11$ in the case with $L = 3$, the lift forces on the two upstream cylinders equally excite the vibrations, while the lift force on one of the downstream cylinder damp the vibrations and that on the other downstream cylinder has little contribution to the vibration. For $V_r \geq 11$ in the case with $L = 4$ under IC2, C4 damps the vibration, while the other cylinders excite the vibration. The excitation lift coefficients for the cases where the response amplitude is zero ($L = 3$ and $V_r = 3$) are not shown in Fig. 16(d).

V. VIV OF FOUR SEPARATELY MOUNTED CYLINDERS

A. Response modes

In this section, VIV response of four separately mounted cylinders as shown in Fig. 1(b) is investigated. The parameters used here are the same as those used in Section IV, i.e., $Re = 150$, $m^* = 2.5$, $V_r = 1 - 30$. The spacing ratio varies from 1.5 to 5 with an increment of 0.5. To identify the effects of the initial condition (IC) on the responses of the cylinders, two initial conditions are used. Initial condition 1 (IC1) represents the situations where the pressure and fluid velocity in the fluid, the displacement and velocity of the cylinder are initially zero. After the simulations under IC1 were completed, it was found that the vibrations of the four cylinders are generally in a symmetric pattern, namely, the vibrations of the two bottom cylinders are in anti-phase with respect to those of the two top cylinders, respectively, with a few exceptions. For example, the vibrations of the top and bottom rear cylinders at $L = 1.5$ and $V_r = 12$ are found to be in an in-phase pattern under IC1. In order to identify the potential hysteretic effects, simulations were also conducted with an asymmetric initial condition (IC2). To trigger the in-phase response pattern in IC2, initial displacements of all the four cylinders are specified as one diameter in the positive y -direction and the initial velocities of the cylinders are set to zero. The initial fluid velocity in the fluid domain is zero in IC2. By using both IC1 and IC2, it is expected that all the possible response modes will be discovered. The responses of the cylinders are found to be indeed dependent on the initial conditions in many cases.

Fig. 17 shows the time histories of the displacement of the four separately mounted cylinders under some representative conditions. Based on the comparison of the results obtained under IC1 with those under IC2, it can be seen that the initial condition has a significant effect on the response. The responses of the cylinders in the top and bottom rows for $L = 1.5$ and $V_r = 10$ are in an anti-phase pattern under IC1 but in an in-phase pattern under IC2. For $L = 3$ and $V_r = 10$, while the response is in the in-phase mode under IC2, the amplitude of response of C4 varies with time. For $L = 4$ and $V_r = 10$, the responses of C3 and C4 under IC1 are generally in an anti-phase pattern, although the response amplitudes slightly vary with time. However, for $L = 4$ and $V_r = 10$ under IC2, the correlation between C3 and C4 appears to be poor with a transient variation of phase difference between the responses of these two cylinder.

In addition to the time histories of the displacement, the cross correlation coefficient is also used to examine the similarity between the vibrations of two cylinders in a large parameter space. The cross correlation coefficient between the displacements of two cylinders i and j , defined as $C_{ij}(\tau)$, is calculated using the displacements of the two cylinders in 30 periods of cylinder vibration. The maximum cross correlation coefficients C_{m34} over the 30 periods are shown in the legends of

FIG. 17. Time histories of the displacement of the cylinders for some representative cases of VIV of four separately mounted cylinders.

the left column in Fig. 17. It can be seen that C_{m34} is greater than 0.9 in most cases except in Fig. 7(h), where phase difference between the displacements of C3 and C4 varies with time. The degree of correlation between the vibrations of the two cylinders is evaluated by using the maximum correlation coefficient C_{mij} . The slightly irregular vibrations of the two downstream cylinders in Fig. 17(f), where $C_{m34} = 0.926$, are also treated to be correlated, because the phase between the two vibrations remains constant most of the time. In this study, the vibrations of two cylinders i and j are defined to be correlated if $C_{mij} > 0.9$. For the cases with $C_{mij} > 0.9$, the phase angle φ_{ij} between the displacements of the two downstream cylinders are calculated. A constant phase difference between the responses of two cylinders only exists only if the responses are correlated to each other. If $C_{m34} < 0.9$, the vibrations of the two downstream cylinders are deemed to be uncorrelated and the phase between them are not calculated. In summary, four response modes are revealed based on C_{m34} and φ_{34} using the following criteria.

- (1) In-phase mode: $C_{m34} > 0.9$ and $|\varphi_{34}| < 5^\circ$.
- (2) Anti-phase mode: $C_{m34} > 0.9$ and $|\varphi_{34} - 180^\circ| < 5^\circ$.
- (3) Correlated out-of-phase-mode: $C_{m34} > 0.9$ and $|\varphi_{34} - 180^\circ| \geq 5^\circ$.
- (4) Uncorrelated mode: $C_{m34} < 0.9$.

It should be noted that the correlation between the two upstream cylinders is found to be the same as that between the two downstream cylinders. If the responses of the two downstream

FIG. 18. Variation of the maximum correlation coefficient and the phase between the two downstream cylinders with the reduced velocity for VIV of four separately mounted cylinders.

cylinders are in the in-phase mode, the responses of the two upstream cylinders are also in the in-phase mode and vice versa.

The response modes of all the simulated cases are identified with the aid of Fig. 18, where the variations of the maximum cross correlation coefficient and the phase difference between the two downstream cylinders with the reduced velocity are shown. It can be seen in Figs. 18(a) and 18(b) that for small spacing ratios of $L = 1.5$ and 2, the correlation coefficients between the two downstream cylinders are close to 1, except at very small reduced velocities of 1 and 2. It is interesting to see that in a wide range of reduced velocity ($6 \leq V_r \leq 20$ and $7 \leq V_r \leq 18$ for $L = 1.5$ and 2, respectively), the response is dominated by the anti-phase mode under IC1 and by the in-phase mode under IC2. The vibrations of the two downstream cylinders are dominated by the anti-phase mode at large reduced velocities ($V_r \geq 12$) under both IC1 and IC2, except for $L = 4$, where the correlation coefficient C_{m34} is very small. From Figs. 18(g) and 18(h) it can be seen that for large spacing ratios and large reduced velocities, the in-phase mode does not exist and the anti-phase mode only occurs under IC1, while the response under IC2 is dominated by the uncorrelated mode. The vibrations of the two downstream cylinders are uncorrelated with each other, probably because of the weak interaction between them.

The response modes identified based on Fig. 18 are mapped on the V_r - L plane as shown in Fig. 19. The possible correlated response modes (in-phase, anti-phase, and correlated out-of-phase modes) under both IC1 and IC2 are plotted in Fig. 19. In general, the anti-phase mode occurs much more frequently than the in-phase and the correlated out-of-phase modes. The in-phase mode

FIG. 19. Distribution of the response modes on the V_r - L plane for VIV of four separately mounted cylinders.

mainly occurs at small spacing ratios of $L = 1.5$, 2, and 2.5 and occasionally at $L = 5$. The correlated out-of-phase modes are sporadically scattered on the V_r - L plane. When the two downstream cylinders are not too far away from each other ($L \leq 3.5$), each of the three correlated response modes can occur in continuous ranges of reduced velocity. Two vibrations become correlated only if there is interferences between them. When the spacing ratio is relatively small, the interference between the two downstream cylinders makes the two vibrations correlated. If the spacing ratio is very large, the vibrations of the top row and the bottom row of the cylinders are less correlated because the wake interaction becomes weak.

B. Vibration amplitude and frequency

Fig. 20 shows the variation of the response amplitude with the reduced velocity under both IC1 and IC2. It can be seen that the difference between the response amplitude of each cylinder and that of a single cylinder decreases with the increase of the spacing ratio. The response amplitudes of the two upstream cylinders (C1 and C2) are very close to that of the single cylinder when $L \geq 3.5$. However, the amplitude of each of the downstream cylinders still differs significantly from that of the single cylinder at the largest simulated spacing ratio $L = 5$, mainly because of the influence of the vortex shedding from the two upstream cylinders. The oscillation amplitude of the lift coefficients of the downstream cylinders is significantly increased compared with that of the single cylinder if the spacing ratio exceeds a critical value.³⁴ The increased lift coefficient results in the increase in the response amplitude.

The experimental studies by Assi *et al.*^{52,53} and Huera-Huarte and Gharib⁵⁴ found that when a cylinder is elastically mounted in the wake of a stationary cylinder, the wake-induced vibration (WIV) of the downstream cylinder remains high amplitude for high reduced velocities. In this study, although the amplitudes of the two downstream cylinders at high reduced velocities are increased compared with that of the single cylinder, they are significantly lower than those observed in the laboratory tests of WIV of a cylinder in the wake of a stationary cylinder. This is mainly because the Reynolds number used in this study is much smaller than those used in the experiments. Bao *et al.*⁵⁵ and Zhao¹⁵ found that the vibration amplitude of an elastically mounted cylinder in the wake of a stationary cylinder for $Re = 150$, $m^* = 2.55$, and $\zeta = 0$ reduced with increasing reduced velocity after it peaks at about $V_r = 6$, instead of continuously increasing with the reduced velocity.

The initial conditions are observed to influence the response amplitudes. By observing Figs. 20 and 18, it can be seen that the response amplitude in the in-phase mode, which predominantly occurs under IC2, is higher than that in the anti-phase mode, which generally occurs under IC1. The difference between the response amplitudes in the in-phase and anti-phase modes are more obvious at small spacing ratios of $L = 1.5$ and 2. For large spacing ratios, the differences between the response amplitudes under IC1 and those under IC2 are very small. The results shown in Fig. 20 suggest that the four cylinders should be separated from each other as far as possible in order to minimize the response amplitudes. When $L = 1.5$ and 2, the maximum vibration amplitudes of the two downstream cylinders are over 2.5 times that of a single cylinder. When $L = 1.5$, the high

FIG. 20. Variation of the response amplitude with the reduced velocity under both IC1 and IC2 for VIV of four separately mounted cylinders.

amplitude range of the reduced velocities of all the four cylinders is widened significantly compared with that of a single cylinder, apart from the increase in the vibration amplitudes.

Fig. 21 shows the contours of the maximum amplitudes of the two upstream cylinders and the two downstream cylinders on the V_r - L plane. The maximum amplitudes for the upstream and downstream pairs of cylinders are defined as the maximum amplitude of C1 and C2 and the maximum amplitude of C3 and C4, respectively, regardless of initial conditions. It can be seen that the maximum amplitudes of both upstream and downstream cylinders occur at the smallest spacing ratio of 1.5. When the spacing ratio is greater than 3, the variation of the maximum amplitude with the spacing ratio becomes very weak. The maximum amplitude of the two downstream cylinders occurs in the range of $6 \leq V_r \leq 8$ for all the simulated spacing ratios greater than 2.5.

The variations of the response frequency with the reduced velocity are shown in Fig. 22. It is found that for $L = 1.5$ and 2 and when the reduced velocity exceeds 8, the response frequency in

FIG. 21. Contours of the maximum amplitude of (a) the two upstream cylinders and (b) the two downstream cylinders on the V_r - L plane for VIV of four separately mounted cylinders.

FIG. 22. Variation of the response frequency of cylinder 3 in both IC1 and IC2 for VIV of four separately mounted cylinders.

FIG. 23. Instantaneous vorticity contours for $L = 1.5$ and $V_r = 11$ in the anti-phase mode for VIV of four separately mounted cylinders.

all the cases is lower than that of a single cylinder, regardless the initial condition. The vibration frequency of all the four cylinders is close to that of a single cylinder at large spacing ratios because the interference between the cylinders is very weak.

C. Wake flow patterns

Fig. 23 shows an example of flow structures around the four cylinders in the anti-phase mode at a very small spacing ratio of $L = 1.5$, where the flow structure is always symmetric with respect

FIG. 24. Instantaneous vorticity contours for $L = 1.5$ and $V_r = 12$ in the in-phase mode for VIV of four separately mounted cylinders.

to the x -axis. In each vibration period, two vortices are shed from the two sides of the system and two vortices are shed from the gap between the top and bottom cylinders. The shear layers from the two upstream cylinders do not form vortices due to the very small gaps among the cylinders. The two vortices that are shed from the gap between the two downstream cylinders are very weak because they are not shed naturally and are shed due to the collision between the two downstream cylinders. The weak vortices from the gap weaken the vortex interaction and further reduce the vibration frequency, compared with that of a single cylinder. Based on the directions of the cylinder velocity marked inside the cylinders in Fig. 23, it can be seen that the vibrations of the two downstream cylinders lag those of the two upstream cylinders. This can be explained by the motion of the vortices. When the two upstream cylinders come to close to each other (Fig. 23(a)), the two vortices behind the gap grow to their maximum strength and attract the two downstream cylinders together. When the two upstream cylinders are separating from each other, the vortices from the outer sides of the two cylinder attract the two downstream cylinders away from each other.

FIG. 25. Instantaneous vorticity contours of representative cases for VIV of four separately mounted cylinders.

Fig. 24 shows a typical vortex flow structure of the in-phase response mode. Only one vortex street is observed in the wake of the cylinders, which is similar to that when the four cylinders are rigidly connected with each other shown in Fig. 10. Although the four cylinders vibrate separately, the velocity of the four cylinders is in the same direction over the majority of a vibration period, resulting in a vortex structure similar to that in the wake of a single structure. Although there is a phase difference between the vibrations of the upstream and downstream cylinders (about 90°), the gaps among the cylinders always remain very small. When the four cylinders are moving up as shown in Fig. 24(a), the shear layers from the bottom of the two downstream cylinders merge, forming a larger vortex. The vortex merging also happens when the four cylinders are moving down (Fig. 24(c)). The merging of vortices makes the four cylinders behave as a single object with a dimension greater than the cylinder diameter. This leads to a lower reduced vortex shedding frequency and vibration frequency than that of a single isolated cylinder. The positive and negative vortices are aligned alternatively in one row in the wake, instead of two rows as in the single cylinder case.

Fig. 25 shows the instantaneous vorticity contours for some representative cases of $L \geq 2$. The vortex shedding flow pattern under the two initial conditions (IC1 and IC2) are found to be different from each other except in Figs. 25(g) and 25(h). It can be seen that in the anti-phase mode, four rows of vortices are formed in the wake of the cylinder array, and the vortices from the gap between the two downstream cylinders become stronger with increasing spacing ratio as shown in Figs. 25(a), 25(c), and 25(e). The merging of vortices of the same sign is not found in the anti-phase mode, regardless of the spacing ratio. In the in-phase mode, strong vortex interaction and merging are observed in the wake of the cylinder array. In the in-phase mode, two vortices with the same sign are shed from the two downstream cylinders almost simultaneously, and they merge into one larger vortex after they are shed from the downstream cylinders. The closeness in the timing of their shedding from the cylinders increases the chance of their merging with each other. In the in-phase mode, the vortex merging in the wake of the cylinders occurs even when the spacing ratio is $L = 5$ (Fig. 25(j)). In the anti-phase mode, merging of vortices is not observed because the two vortices shed simultaneously from the downstream cylinders are of the opposite signs. The two vortices of different signs from the gap sides of the cylinders interact with each other as they are convected downstream.

VI. CONCLUSIONS

VIV of four circular cylinders in an in-line square configuration at a low Reynolds number of 150 and a low mass ratio of 2.5 is investigated numerically. Numerical simulations were conducted for two scenarios: (1) four rigidly connected cylinders and (2) four separately mounted cylinders. The conclusions are summarized as follows.

A. Four rigidly connected cylinders

1. The response for the case with $L = 1.5$ is characterized by large response amplitudes and a wide lock-in region. This is largely because the four cylinders behave as a single body with a larger dimension than a single cylinder. The response amplitudes in the lock-in regime for $L = 2$ are the smallest among those in other spacing ratios, because of the strong interactions between the vortices and shear layers. For $L = 2.5, 3$, and 4 , the lock-in regime is similar to that of a single cylinder and the response amplitudes in the lock-in regime is slightly higher than that of a single cylinder.
2. An interesting phenomenon found for $L = 4$ is that the response in the range of $8 \leq V_r \leq 25$ depends on the initial condition. The vortex shedding in this range of reduced velocity is in a symmetric pattern, leading to a zero lift force and no vibration if the velocity and the pressure of the flow field and the displacement and the velocity of the cylinder are initially set to zero in the whole fluid domain. If the initially condition is asymmetric, the vortex shedding from the four cylinders is in an in-phase pattern and vibration is observed. The dependence of the response on the initial condition is also found for $L = 4$ with $V_r \leq 3$ and $L = 3$ with $V_r \leq 2$.

3. For the intermediate spacing ratio of $L = 2$, the response frequency is found to be close to the natural frequency in a wider range of reduced velocity of $3 \leq V_r \leq 25$. However, the response amplitude for $L = 2$ is smaller than those in other spacing ratios. The reason for this is that the lift coefficient at $L = 2$ is broad-banded and the component of the lift coefficient that excites the vibration is not the component with the highest energy.
4. The vortex shedding for VIV of four cylinders is different from that for flow past four stationary cylinders mainly in two aspects. First, the critical spacing ratio for vortex shedding from the two upstream cylinders is between 1.5 and 2, which is much smaller than that for flow past four stationary cylinders. Second, combination of the vortices is found in the wake of the four vibrating cylinders for $L = 2, 2.5$, and 3. The strongest biasness of the flow is found when $L = 3$, leading to the increase of the mean cross-flow position with the increasing reduced velocity.
5. The energy transfer between the fluid and structure is examined by analyzing the force coefficient that is in-phase with the velocity of the cylinder. It is found that the lift force of the two upstream cylinders and those of the two downstream cylinders play different roles in terms of exciting or damping the vibration in most of the cases in the lock-in regimes with regular vibrations, i.e., whenever the lift forces of the two upstream cylinders excite the vibration, those of the downstream cylinders damp the vibration and vice versa.

B. Four separated mounted cylinders

6. Simulations are conducted under two initial conditions: symmetric IC1 and asymmetric IC2. The most interesting finding is that the vibrations of the cylinders depend on the initial condition in many cases. Four response modes are identified based on the correlation and the phase between the vibrations of the two downstream cylinders. These are the in-phase mode, the anti-phase mode, the correlated out-of-phase mode, and the uncorrelated mode. When the spacing ratio is small ($L = 1.5$ and 2), the vibrations are generally in the anti-phase mode under IC1 and in-phase mode under IC2, but with exceptions, for example, $V_r = 12$ at $L = 1.5$. The correlation between the vibrations of the two top cylinders and that of the two bottom cylinders is generally good in small spacing ratios because the interaction between the cylinders is in favor of the correlation. By analyzing all the possible vibration modes, it is found that the chance of the correlated anti-phase mode is much higher than the in-phase mode.
7. The response amplitude in the anti-phase mode is generally higher than that in the in-phase mode for the same spacing ratio and same reduced velocity. Vortex merging, which is a common phenomenon in the wake of the cylinders in the anti-phase mode, creates strong vortices and leads to the increase in the response amplitude, especially at small spacing ratios.
8. Collision between the two downstream cylinders or between the two upstream cylinders only occurs at small spacing ratios of $L = 1.5$ and 2 and in the anti-phase response mode. When the two downstream cylinders collide with each other, the vortices from the gap between them become very weak, because they are not shed naturally and instead are cut off the cylinders by the collision.

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