Detection of Gravitational Waves: from Nanohertz to the Audio Band

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School of Physics
June 2015
致
佳骏和莉媛

To
Jiājùn and Lìyuán
Abstract

The existence of gravitational waves was first predicted by Einstein’s general theory of relativity a century ago. However gravitational waves have not yet been directly detected\(^1\). Their direct detection will open a new observational window allowing one to listen to the Universe for the first time. Einstein’s gravitational spectrum spans from extremely low frequencies, with wave periods comparable to the age of the Universe, to the audio band, with frequencies roughly between 10 Hz and several kilohertz. The subject of this thesis is the direct detection of gravitational waves in two different frequency bands: a) the audio band observed by terrestrial laser interferometers; b) the nanohertz band surveyed by pulsar timing arrays.

Ground-based interferometers, such as the Laser Interferometer Gravitational-wave Observatory (LIGO), are primarily searching for signals from binary coalescences of neutron stars and black holes. As the two objects orbit around each other, they lose energy through the emission of gravitational waves. As the orbital period decreases, the gravitational wave frequency and amplitude increase until the binary merges, emitting copious amounts of gravitational waves. Advanced LIGO detectors, which came online in September 2015, are expected to detect tens of such events each year after they reach design sensitivity in around 2019.

In addition to those individually detectable events, a gravitational wave background exists as a result of the combined emission of numerous sources throughout the Universe. This provides another promising target for advanced detectors. Modelling such a background signal and predicting its detection prospects is the subject of the first part of this thesis. We present a comprehensive study on the gravitational wave background from compact binary coalescences throughout the Universe.

\(^1\)Note added.–It was announced on 11 February 2016 that a signal from the coalescence of two black holes was detected by two LIGO detectors on 14 September 2015 (Abbott et al. 2016).
It improves on previous work by using observation-based source distributions and realistic gravitational waveforms. We show that advanced detectors are likely to detect this signal assuming a realistic coalescence rate. We also investigate if subtracting the contribution of individually detectable events can potentially unmask the highly sought primordial signals.

Pulsar timing arrays target gravitational waves at the lower nanohertz band. This is achieved through performing long-term radio timing observations of a spatial array of millisecond pulsars. In the second part of this thesis we focus on detection and sky localization of single sources by pulsar timing arrays, in particular signals expected from individual supermassive binary black holes. We have developed two independent coherent methods for this purpose.

The first technique is fast and robust, which is adapted from network analysis methods used by ground-based detectors. We demonstrate its effectiveness for three types of sources: circular binaries, eccentric binaries and bursts. To test its robustness, we apply it to realistic synthetic data sets that include effects such as uneven sampling, heterogeneous data spans and measurement precision. With the second technique, we perform an all-sky search for continuous waves in the recent Parkes Pulsar Timing Array data set. Although no statistically significant signals were detected, the quality of the data allows us to set the best limit on the gravitational wave amplitude in the nanohertz regime. With this data set we could detect gravitational waves from supermassive binary black holes with masses higher than one billion solar mass out to a luminosity distance of about 100 Mpc.
Declaration

This thesis presents studies carried out between August 2011 and June 2015 at the School of Physics, The University of Western Australia (Crawley, WA), and between March 2013 and June 2015 at the CSIRO Astronomy and Space Science (Marsfield, NSW). It contains no material that has been submitted for a degree at this or any other institution. To the best of my knowledge, the material presented in this thesis is original, except where due reference is made in the customary manner.

Publications arising from this thesis

The following published journal articles have arisen from the work in the thesis.

  This study was conducted by myself, in consultation with supervisors and a collaborator (Zhu Z.-H., who was my MSc supervisor). I wrote the paper; comments from coauthors improved its presentation.

  I conducted this research and wrote the paper. Supervisor Wen and I initiated the original plan for the work. Supervisor Hobbs helped with simulations of realistic data sets. Undergraduate student Zhang from China carried out a preliminary study on the detection of monochromatic signals using idealized simulations (under the guidance of Wen and myself), when visiting UWA between July and August 2013. All authors commented on various draft versions of the paper.

I conducted this research and wrote the paper. Supervisor Hobbs helped with setting up codes to implement the method (especially the TEMPO2 findCWs plugin) and gave extensive advice on doing simulations with TEMPO2. Supervisor Wen and collaborator Coles helped with designing the maximum likelihood detection statistic. Hobbs, Coles and a few other PPTA members developed the original TEMPO2 global fitting routine; I tested its effectiveness for continuous gravitational waves. All authors contributed to timing observations with the Parkes radio telescope that led to the data set used in this work and commented on various draft versions of the paper.

Additionally, Zhu et al. (2011b) was published in the very early stage of the thesis period, but it was not included since the work had been substantially completed prior to the commencement of this thesis. Some text in section 3.3.2.2 and Figure 3.3 are adapted from this paper.

Xingjiang Zhu
Perth, Australia
June 2015
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you to Shin Kee for tremendous help with computer-related stuff, such as installing TEMPO2, making programs running on Fornax, etc. Thank you to Dai Shi, Jingbo, Matthew, Ryan and Xiaopeng; you made my stays in Marsfield unforgettable. Thank you to other colleagues Dustin, Pablo, Vikram, and (both) Wang Yan for useful discussions. Special thanks go to Ron Burman at UWA and to ‘academic brother’ Fan Xi-Long for carefully reading this thesis.

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Thank you to my friends in Perth, (in addition to some already listed above), Wan Zhijian, Ge Xintian, Cui Weiguang, Liu Yufang and many unnamed others; you made my PhD life colorful and enjoyable.

Last but definitely not least, many thanks go to my family. To my parents, thank you for your longstanding and continued support. To my wife Liyuan, thank you for always being by my side, providing encouragement whenever necessary, and listening to me whenever I was excited or frustrated with my research. Undoubtedly, the completion of this thesis could not have been possible without your support. To my son Jiajun, thank you for adding a new dimension to my life.

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List of Abbreviations

**AIGO** Australian International Gravitational Observatory.

**ATNF** Australia Telescope National Facility.

**BBH** binary black hole.

**BH** black hole.

**BNS** binary neutron star.

**CBC** compact binary coalescence.

**CDF** cumulative distribution function.

**CSFR** cosmic star formation rate.

**CW** continuous wave.

**DM** dispersion measure.

**EPTA** European Pulsar Timing Array.

**ET** Einstein Telescope.

**FAP** false alarm probability.

**FAST** Five-hundred-meter Aperture Spherical Telescope.

**GRB** gamma-ray burst.

**GW** gravitational wave.

**GWB** gravitational wave background.

**IPTA** International Pulsar Timing Array.
LIGO Laser Interferometer Gravitational-wave Observatory.

LISA Laser Interferometer Space Antenna.

LSO last stable orbit.

MJD Modified Julian Date.


NS neutron star.

PN post-Newtonian.

PPTA Parkes Pulsar Timing Array.

PTA pulsar timing array.

rms root mean square.

SKA Square Kilometre Array.

SMBBH supermassive binary black hole.

SVD singular value decomposition.

TOA time of arrival.
One century ago, Albert Einstein completed his general theory of relativity (Einstein 1915, 1916), radically revolutionizing our understanding of gravity. In this theory, gravity is no longer a force, but instead an effect of spacetime curvature. The mass and energy content of spacetime creates curvature, and the curvature in turn dictates the behavior of objects in spacetime. Generally speaking, when a massive object is accelerating, it produces curvature perturbations that propagate at the speed of light. Such ripples in the fabric of spacetime are called gravitational waves (GWs).

Forty years ago, Hulse and Taylor discovered the binary pulsar system PSR B1913+16 (Hulse & Taylor 1975). Subsequent observations in the following years showed that its orbital period was gradually decreasing, at a rate that agreed remarkably well with that predicted by general relativity as a result of gravitational radiation (Taylor & Weisberg 1982, Weisberg et al. 2010). This provided the most convincing evidence of the existence of GWs and earned Hulse and Taylor the Nobel Prize in Physics in 1993.

At the beginning of the 21st century, several km-scale laser interferometers were built across the world, such as the Laser Interferometer Gravitational-wave Observatory (LIGO; Abramovici et al. 1992) in the US and the French/Italian Virgo detector in Italy (Caron et al. 1997). These detectors aim to directly detect GWs in the audio band (from 10 Hz to a few kHz), e.g., those from binary neutron stars – systems such as PSR B1913+16. The Hulse-Taylor binary pulsar won’t be detectable for LIGO-like detectors because the waves being radiated are at frequencies much lower than the sensitive band of these instruments. However, it
is doomed to merge in hundreds of millions of years, as \( \text{GWs} \) radiate more and more orbital kinetic energy away. This does not mean that LIGO has to wait for millions of years to observe a binary neutron star coalescence. Analysis based on the handful of detected binary pulsar systems in our Galaxy and its neighborhood indicates that there should be many more double neutron stars in the Galaxy and beyond, some of which may be in the final stage of their coalescences (e.g., Lorimer 2008).

The chance of detecting a neutron star coalescence event with LIGO/Virgo detectors at their initial configurations was very low. These instruments, decommissioned in around 2010, were only sensitive to volumes encompassing 20 Mpc (Abadie et al. 2012b). The expected event rate is quite low within such volumes (Abadie et al. 2010c) — most likely of order \( \lesssim \) once in one hundred years albeit with an uncertainty of 1-2 orders of magnitude. However, the prospects are expected to dramatically change for these detectors at their advanced versions. Advanced LIGO detectors started scientific observations on 18 September 2015 and recently achieved a sensitive range of about 80 Mpc (as of December 2015; Aasi et al. 2013c). They are designed to be ten times more sensitive and thus will increase the accessible volume by a factor of one thousand compared to initial detectors (Aasi et al. 2015b). Careful estimates of the binary neutron star coalescence rate in the local Universe, coupled with the expected detector sensitivity, show that they are likely to detect tens of neutron star coalescence events per year when design sensitivity is met in \( \sim 2019 \) (Aasi et al. 2013c). Similar detection rates for binary black holes and black hole-neutron star systems are also expected (see, e.g., Abadie et al. 2010c and references therein).

Such compact binary coalescences are the most promising sources for ground-based LIGO detectors. While individually detectable coalescence events are expected within distances of hundreds of Mpc, the combined emission from these sources over cosmological volumes could form a stochastic background (Phinney 2001). This signal represents another promising target for the upcoming advanced instru-
ments. Modelling such a background signal and making realistic predictions on its detection prospects is the focus of the first part of this thesis.

Quite interestingly, while timing observations of the Hulse-Taylor binary and similar binary pulsar systems have convinced us that gravitational waves (GWs) exist, an array of millisecond pulsars in the Milky Way also provides us with a natural detector that are sensitive to nanohertz signals. These millisecond pulsars are fantastic celestial clocks with exceptional stability. GWs passing across the pulsar-Earth baseline will induce variations in the pulse times of arrival that are otherwise extremely stable. Such arrival time variations are correlated between different pulsars with a unique quadruple signature (Hellings & Downs 1983; Foster & Backer 1990). This forms the basic concept of pulsar timing array (PTA) experiments. The primary sources detectable for these experiments are inspiralling binary black holes with very high masses, ranging from around a million to ten billion solar mass. Such supermassive black holes are thought to be ubiquitous in the centers of galaxies (Magorrian et al. 1998), including our own Galaxy (Schödel et al. 2002). When pairs of galaxies merge, their central black holes are expected to form binary systems (Begelman et al. 1980). Whereas these supermassive binary black holes are difficult to observe in the electromagnetic domain, they are the strongest emitters of gravitational radiation (Rajagopal & Romani 1995; Jaffe & Backer 2003; Wyithe & Loeb 2003; Sesana et al. 2008).

The Parkes Pulsar Timing Array (PPTA) is one of three major PTA projects around the world (Manchester et al. 2013; Hobbs 2013). Building on earlier timing projects with the Australian Parkes 64-m radio telescope, PPTA started regular timing observations for \( \sim 20 \) millisecond pulsars in 2005. Jenet et al. (2005) suggested that it is possible to have a detection of the stochastic background from the cosmic population of supermassive binary black holes if 20 pulsars are timed with a precision of \( \sim 100 \) ns over \( \gtrsim 5 \) years. Although no detection has been made, analysis of the latest PPTA data placed the most stringent upper limit on the

\[ \text{http://www.atnf.csiro.au/research/pulsar/ppta/} \]
Chapter 1. Introduction

...strength of such a background signal (Shannon et al. 2013).

While PTA experiments have traditionally focused on the detection of a stochastic background, it was recently appreciated that a single bright source may provide another promising target given ever improving sensitivities (e.g., Sesana et al. 2009; Ravi et al. 2015). Such a prospect is even more relevant as the detection of GWs by carrying out pulsar timing observations is an important goal for more powerful future radio telescopes, such as the Chinese Five-hundred-meter Aperture Spherical Telescope (FAST) which is expected to be online in 2016 (Nan et al. 2011; Hobbs et al. 2014), and the planned Square Kilometer Array (SKA) and its pathfinders (see Lazio 2013 and references therein). The second part of this thesis is concerned with data analysis methods for the detection and sky localization of single sources using PTAs, and presents a search for signals from individual supermassive binary black holes using a recent PPTA data set published by Manchester et al. (2013).

Thesis outline

In this thesis we present studies on three topics related to the detection of GWs, from the audio band to the nanohertz frequency range. They are detailed in three central chapters that are based on published refereed journal articles:

• Chapter 3: the GW background from the cosmological population of compact binary coalescence events;

• Chapter 4: a coherent analysis method for the detection and sky localization of single sources using PTAs;

• Chapter 5: an all-sky search for signals from individual supermassive binary black holes in circular orbits with PPTA.

Chapter 2 provides an introductory overview, including basics of GWs and various efforts aiming for their direct detection. In particular we review the detection
principles, noise sources and GW sources for both ground-based laser interferometers and PTA. We also describe the basic principles of GW data analysis. This chapter will lay down some useful background information for the specific studies presented in the following chapters.

Chapter 3 contains a comprehensive study on the background signal formed by compact binaries. We make use of inspiral-merger-ringdown waveforms and observation-based black hole and neutron star mass distributions to produce realistic signal models. We systematically examine effects such as mass distributions, cosmic star formation rate and the delay time between binary formation and the final coalescence. We then evaluate the chance for a detection by the upcoming advanced detectors. In particular we discuss the relative benefit of various options for detector configurations and the sensitivities of different pairs of detectors. We also investigate if the subtraction of individually detectable events could unmask the highly-sought primordial inflationary background.

In Chapter 4 we develop a general coherent method for the detection, sky localization and waveform estimation of single sources with PTA. Applications of this method to monochromatic waves, eccentric binaries and GW bursts are illustrated using synthetic data sets. Although the method works in the frequency domain, we demonstrate that it works equally well as time-domain methods for realistic data sets. We also compare the sensitivities to detect eccentric binaries using a monochromatic statistic and a harmonic summing technique.

In Chapter 5 we report on an all-sky search in a recent PPTA data set (with observations dated until 2011) for signals that could be produced by individual black hole binaries in circular orbits. We first summarize our observations and introduce a new coherent data analysis method used to perform the search. We investigate the use of the maximum likelihood statistics in degenerate multivariate Gaussian noise. We develop and test a fully functional pipeline for detecting and limiting single sources in real PTA data. Finally we present results of the search – not surprisingly a non-detection – and compute upper limits on the signal amplitude
and on the coalescence rate of supermassive binary black holes.

In Chapter 6 we review the key results obtained in this thesis and discuss possible directions for future research. This chapter is followed by various appendices that include some codes and computation programs implemented in this thesis.
Chapter 2

Detection of Gravitational Waves

In this chapter, we first give an overview in section 2.1. We introduce the basics of GWs, summarize various direct detection efforts, and describe Einstein’s gravitational spectrum and how observations across this spectrum will revolutionize our understanding of the Universe. We then introduce the detection principles, noise processes and GW sources for ground-based laser interferometers in section 2.2 and for pulsar timing arrays in section 2.3. In section 2.4 we review the basic principles of GW data analysis. These three sections provide the necessary framework for the studies presented in the remaining chapters.

2.1 Overview

2.1.1 Gravitational waves

In 1915, Albert Einstein completed his theory of general relativity, which revolutionized humanity’s understanding of the Universe. General relativity describes gravity as a manifestation of the spacetime curvature – the presence of matter curves spacetime and the curvature in turn tells matter how to move. This theory enjoyed some great successes soon after its publication, e.g., 1) it perfectly explained the anomalous perihelion advance of Mercury that could not be accounted for with Newtonian gravity (Le Verrier [1859]); 2) its prediction of light deflection was confirmed by observing the light of distant stars deflected by the Sun during a total solar eclipse in 1919 (Dyson et al. [1920]). Since then various experiments have all verified general relativity with ever increasing accuracy (see Will [2014] for a review on experimental tests of general relativity).
Chapter 2. Detection of Gravitational Waves

Figure 2.1: Deformation of a ring of free-falling test particles due to a GW propagating in the $z$ direction, for the “+” polarization (left) and the “×” polarization (right). The black and red ellipses are separated by half a wave period. The principal axes of the deformation for two polarizations are rotated by 45°. The effect shown in this picture is greatly exaggerated.

One of the most important predictions of general relativity is the existence of GWs. Just like accelerated electrically-charged particles produce electromagnetic waves, (non-symmetrically) accelerated masses generate GWs – ripples in the fabric of spacetime that travel at the speed of light. Because of the weakness of the gravitational interaction, only massive objects moving at relativistic speeds can produce detectable signals. The effect of a passing GW is to stretch and squeeze the local spacetime transverse to the direction of wave propagation (see Figure 2.1). In general relativity, GWs have two independent modes of polarization – the “+” mode and the “×” mode, as illustrated in Figure 2.1.

The existence of GWs was confirmed by long-time radio observations of binary pulsars. In 1974, R. Hulse and J. Taylor discovered for the first time a pulsar in a binary system, PSR B1913+16 (Hulse & Taylor 1974, 1975). Measurements of the pulse arrival times in the following years made it possible to determine the mass of this binary system: both components were found to have almost equal mass of around 1.4 times the solar mass ($M_\odot$). Such observations and theoretical considerations suggest that it is a binary neutron star system (Taylor & Weisberg

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1Additional polarization modes are possible in some alternative theories of gravity (see, e.g., section 7.2 of Will 2014, and references therein).
According to general relativity, a binary system should radiate energy in the form of gravitational radiation and hence the orbital period should be decreasing. The measured orbital decay rate of the Hulse-Taylor binary agreed remarkably well with that predicted by general relativity, providing compelling evidence for the existence of gravitational waves (GWs) (see Figure 2.2). Because of this beautiful test of general relativity and its profound impact on gravitational research, Hulse and Taylor were awarded the Nobel Prize in Physics in 1993. The latest measurements show the ratio between the observed decay rate and the predicted value is $0.997 \pm 0.002$ (Weisberg et al. 2010).

The properties of GWs can be compared with those of electromagnetic waves.

- Electromagnetic waves are oscillations of the electromagnetic field propagating through spacetime, whereas GWs are propagating oscillations of the spacetime curvature itself.

- Electromagnetic waves are produced by the incoherent dipole motions of charged particles, while GWs are caused by coherent quadrupole motions of massive objects such as black holes and neutron stars.

- Electromagnetic waves interact strongly with matter; they can be absorbed and scattered. GWs couple extremely weakly to matter. This implies, on the one hand that detecting GWs is a very challenging task, and on the other hand that GWs may carry information about astrophysical processes that are inaccessible to electromagnetic observations. For example, in principle GW observations will make it possible to probe the earliest moments of the Big Bang and the dynamics of core collapse supernovae.

### 2.1.2 A brief history of detection efforts

Although GWs were seen as a direct consequence of general relativity, it was not until the late 1950s that a consensus on their physical reality was reached. Bondi et al. (1959) showed that these waves transport energy; soon afterwards it was
Figure 2.2: The cumulative shift in the periastron time in the binary pulsar system PSR B1913+16. The parabolic curve depicts the prediction by general relativity for the energy loss due to gravitational radiation. Data points are measurements with error bars which may be too small to see. Figure from Weisberg et al. (2010), copyright © 2010 The American Astronomical Society.
realized that detectors could be built. Joseph Weber pioneered in experimental
detection by constructing the first detectors – resonant bar detectors (Weber 1960).
The working principle of such bar detectors is as follows. A passing GW may
excite mechanical oscillations of the bar; oscillations are recorded using highly
sensitive sensors and converted to electric signals; such signals are analyzed to
search for GWs. In 1969 Weber reported that coincidences were observed in two
detectors located some 1000 km apart (Weber 1969). This was very controversial;
if they were truly due to GWs as the report suggested, the event rate would be
orders of magnitude higher than any theoretical predictions. Nevertheless, Weber’s
experiments caused huge waves of enthusiasm on the detection of GWs. Many bar
detectors were built across the world in the following years, but none of them could
repeat Weber’s discovery even with greater sensitivities (see Tyson & Giffard 1978,
and references therein).

In the late 1990s and early 2000s, there were five major bar detectors operating
around the world (Astone et al. 2007, 2010), including one in The University of
Western Australia (Blair et al. 1995). These detectors typically operate within a
very narrow band (about a few tens Hz) centering around their resonant frequencies
(ranging from \(\sim 700\) to \(\sim 900\) Hz). They were designed to target burst signals from
nearby supernovae. These bar detectors achieved greatly improved sensitivities
(by up to three orders of magnitude) compared to Weber’s bar detectors, with
the use of cryogenic techniques and advanced transducers and vibration isolation
system. While it is appreciated that many of these technologies are useful for
laser interferometers, bar detectors are constrained by their narrow bandwidth and
limited sensitivities – they can only detect supernovae events within the Galaxy.
Indeed, soon after major ground-based interferometers came online they achieved
broadband sensitivities better than the narrow-band sensitivities of bar detectors.

As Thorne (1987) put it, “the germ of the idea of a laser interferometer gravi-
tational wave detector can be found in Pirani (1956).” Gertsenshtein & Pustovoït
(1963) made the first explicit suggestion of such a detector and proposed that it
could be much more sensitive than Weber’s bar detectors. In 1970, Rai Weiss carried out a detailed design and feasibility study, laying out the foundation for the first generation ground-based detectors (Weiss 1972). Robert Forward, Weber’s former student, made the first experimental attempt by building a small interferometer with decent sensitivities (Forward 1978). In the 1990s, some prototype interferometers of 10 to 40 m in arm length were built and full-scale detectors were funded (see, e.g., Ju et al. 2000 and references therein).

In the early 2000s, several large-scale laser interferometers were constructed and put into science operation, which are collectively called the first generation ground-based interferometers. These include LIGO in the US, the Italian/French Virgo detector, the German/British GEO600 project (Lück & GEO600 Team 1997), and TAMA300 in Japan (Ando et al. 2001). LIGO and Virgo detectors were decommissioned in 2010 and 2011 respectively, when design sensitivities were met or slightly surpassed, making way for installations of their advanced versions – Advanced LIGO (aLIGO) and Advanced Virgo (aVirgo; Acernese et al. 2015). It is proposed that one of the aLIGO detectors will be relocated in India (Iyer et al. 2011), primarily aiming for a network of detectors with greater performance of source localization (Fairhurst 2014). Construction for an advanced detector (KAGRA) in Japan is also underway (Somiya 2012). These advanced detectors are expected to come online in the coming years as shown in Table 2.1. Additionally, research groups in Australia have been looking for funds for a detector – Australian International Gravitational Observatory (AIGO) – to be built in Gingin, Western Australia (Barriga et al. 2010). Design studies for a third generation detector – the Einstein Telescope (ET; Punturo et al. 2010) – have also started in recent years.

At the other front, it was realized in late 1970s that precision timing observations of pulsars can be used to search for very low frequency GWs (Sazhin 1978).
Table 2.1: A list of ground-based laser interferometric GW detectors, including those have been decommissioned, currently operational, soon to be operational or have been proposed.

<table>
<thead>
<tr>
<th>generation</th>
<th>detector</th>
<th>location</th>
<th>arm length</th>
<th>date</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>LIGO H1</td>
<td>Hanford, USA</td>
<td>4 km</td>
<td>2002-2010</td>
</tr>
<tr>
<td></td>
<td>LIGO H2</td>
<td>Hanford, USA</td>
<td>2 km</td>
<td>2002-2010</td>
</tr>
<tr>
<td></td>
<td>LIGO L1</td>
<td>Livingston, USA</td>
<td>4 km</td>
<td>2002-2010</td>
</tr>
<tr>
<td></td>
<td>Virgo</td>
<td>Cascina, Italy</td>
<td>3 km</td>
<td>2007-2011</td>
</tr>
<tr>
<td></td>
<td>GEO600</td>
<td>Hannover, Germany</td>
<td>600 m</td>
<td>2001-</td>
</tr>
<tr>
<td></td>
<td>TAMA300</td>
<td>Tokyo, Japan</td>
<td>300 m</td>
<td>1999-2004</td>
</tr>
<tr>
<td>2nd</td>
<td>aLIGO H1</td>
<td>Hanford, USA</td>
<td>4 km</td>
<td>2015</td>
</tr>
<tr>
<td></td>
<td>aLIGO L1</td>
<td>Livingston, USA</td>
<td>4 km</td>
<td>2015</td>
</tr>
<tr>
<td></td>
<td>aVirgo</td>
<td>Cascina, Italy</td>
<td>3 km</td>
<td>2016</td>
</tr>
<tr>
<td></td>
<td>KAGRA</td>
<td>Japan</td>
<td>3 km</td>
<td>2017-18</td>
</tr>
<tr>
<td></td>
<td>LIGO India</td>
<td>India</td>
<td>4 km</td>
<td>~ 2018b</td>
</tr>
<tr>
<td></td>
<td>AIGOc</td>
<td>Australia</td>
<td>4 km</td>
<td>2020+?</td>
</tr>
<tr>
<td>3rd</td>
<td>ET</td>
<td>Europe</td>
<td>10 km</td>
<td>2025+?</td>
</tr>
</tbody>
</table>

*Notes:*  
- a: A question mark means funding is being sought.  
- b: Funding for LIGO India is pending for final approval.  
- c: A recent design study (Miao H., private communication) has demonstrated that a $4 \times$ better sensitivity than that of second generation detectors can be achieved with an arm length of 8 km and modest technical improvements.
Detweiler (1979). In particular Detweiler (1979) explicitly showed that[a] “a gravitational wave incident upon either a pulsar or the Earth changes the measured frequency and appears then as an anomalous residual in the pulse arrival time”. Hellings & Downs (1983) used timing data from four pulsars to constrain the energy density of any stochastic background to be $\lesssim 10^{-4}$ times the critical cosmological density at $\lesssim 10^{-8}$ Hz (see, Romani & Taylor 1983; Bertotti et al. 1983, for similar results). Almost simultaneously to these works, Backer et al. (1982) discovered the first millisecond pulsar PSR B1937+21. Because of its far better rotational stability than any previously known pulsars and relatively narrower pulses, timing observations in the following years improved the limit on stochastic backgrounds very quickly and by orders of magnitude (Davis et al. 1985; Rawley et al. 1987; Stinebring et al. 1990; Kaspi et al. 1994).

Measurements with a single pulsar cannot make definite detections of GWs whose effects may be indistinguishable from other noise processes such as irregular spinning of the star itself. By continued timing of a spatial array of millisecond pulsars, i.e., by constructing a pulsar timing array (PTA) Foster & Backer (1990), GWs can be searched for as correlated signals in the timing data. Hellings & Downs (1983) first did such a correlation analysis to put limits on stochastic backgrounds. Romani (1989) and Foster & Backer (1990) explored the greater scientific potential of PTA experiments: (1) searching for GWs; (2) providing a time standard for long time scales; and (3) detecting errors in the Solar System ephemerides.

In 2005, three major PTA projects were set up and regular timing observations of a number of millisecond pulsars have been made since then. They are PPTA based in Australia, the European Pulsar Timing Array (EPTA; Kramer & Champion 2013), and NANOGrav in North America (McLaughlin 2013). Basic information about these PTAs, including the radio telescopes used and the number of pulsars monitored, is summarized in Table 2.2. Recently the three PTA collabor-
2.1. Overview

Table 2.2: Information about pulsar timing array projects.

<table>
<thead>
<tr>
<th>Project</th>
<th>Telescope</th>
<th>Diameter (m)</th>
<th>Country</th>
<th>Number of pulsars$^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PPTA</td>
<td>Parkes</td>
<td>64</td>
<td>Australia</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>Effelsberg</td>
<td>100</td>
<td>Germany</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Lovell</td>
<td>76.2</td>
<td>UK</td>
<td></td>
</tr>
<tr>
<td>EPTA</td>
<td>Nancay</td>
<td>94$^b$</td>
<td>France</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>WSRT</td>
<td>96$^b$</td>
<td>Netherlands</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sardinia</td>
<td>64</td>
<td>Italy</td>
<td></td>
</tr>
<tr>
<td>NANOGrav</td>
<td>Arecibo</td>
<td>305</td>
<td>USA</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>GBT</td>
<td>100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes:  
$^a$ We present the number of pulsars that have been timed for more than 5 years for each project here. It is worth pointing out that a number of pulsars were recently added to the timing arrays. The total number of pulsars that are currently being observed for IPTA is 50, of which 11 are timed by two pulsar timing arrays and 8 by all three arrays (see table 3 in Manchester 2013 for more information on IPTA pulsars).

$^b$ Values of circular-dish equivalent diameter (Ferdman et al. 2010).

orations were combined to form the International Pulsar Timing Array\footnote{http://www.ipta4gw.org/} (IPTA; Hobbs et al. 2010a; Manchester 2013; McLaughlin 2014). Looking into the future, GW detection with pulsar timing observations is one of major science goals for next generation radio telescopes, such as the Chinese FAST (which is expected to be operational in 2016), and for the planned SKA and its pathfinders.

Space-based interferometers have long been proposed to detect millihertz GWs (Weiss 1979; Faller et al. 1985). Among many proposed missions, the Laser Interferometer Space Antenna (LISA) is the most established one, operating at $10^{-4} - 10^{-1}$ Hz (Danzmann et al. 1993; Jafry et al. 1994). LISA is designed to consist of three spacecraft arranged in an equilateral triangle with sides of five million km. The constellation would be placed in an Earth-like orbit at the same distance as the Earth from the Sun but 20° behind the Earth, and inclined at 60° to the ecliptic (see Figure 2.3 for an illustration). Each satellite would send a laser to the other two
and laser interferometry is performed with drag-free test masses in the spacecraft. Until early 2011, LISA was considered as a joint ESA/NASA mission. Funding constraints in the US forced NASA to pull out of the project. Now a descoped version of LISA, called the “evolved” LISA12 (Amaro-Seoane et al. 2012), is proposed as a joint effort of eight European countries and supported by the original LISA team in the US. Its Pathfinder was launched on 3 December 2015, while the final GW observatory may have an opportunity to be launched in 2034. Other space-based detectors that have been proposed include (1) the Japanese project DECIGO (DECi-hertz Interferometer Gravitational-wave Observatory) which would fill the frequency gap between LISA and ground-based interferometers, i.e., 0.1–10 Hz (Kawamura et al. 2011) and (2) the US Big Bang Observer, operating at 0.1–10 Hz, which was proposed as a follow-on mission for LISA (Harry et al. 2006).

2.1.3 Gravitational wave astronomy

Undoubtedly, the direct detection of GWs will mark the beginning of a new era in Astronomy. Looking back the history of Astronomy in the 20th century, the opening of new windows (radio, X-ray, γ-ray) in the electromagnetic wave spectrum

12https://www.elisascience.org/
has always revolutionized our understanding of the Universe. The unique properties of
\textit{GW}s make the wealth of information they carry distinct from that of photons.
In particular, \textit{GW} astronomy is expected to advance our understanding of the
Universe in the following aspects (see, e.g., Sathyaprakash & Schutz \citeyear{2009}, for an
excellent review):

- insights on the earliest moments of the Big Bang; observational test of inflation
theory;

- observational test of exotic physical phenomena, such as cosmic strings and
early-Universe phase transitions;

- test of general relativity and alternative theories of gravity in the strongest-
field regime;

- direct confirmation of the existence of black holes; test of the black hole
no-hair theorem;

- insights on the neutron star interior; detailed information of the neutron star
equation of state;

- formation and evolution of supermassive black holes in galactic nuclei;

- the progenitors of and engines driving gamma ray bursts;

- physics of core collapse supernovae.

Given recent developments of ground-based detectors and PTAs, it is believed
that we are on the threshold of \textit{GW} astronomy. Scientific observations with Advanced
\textit{LIGO} detectors started on 18 September 2015; both detectors have been
taking science data for months and recently reached an all-sky range of \( \sim 80 \) Mpc
for binary neutron stars (as of December 2015; Aasi et al. \citeyear{2013}). Other second
generation interferometers are expected to join the network within a few years.
In the meanwhile, PTAs have essentially achieved their design sensitivities that
were previously thought to be sufficient to detect a stochastic background from the
cosmic population of supermassive binary black holes. Although no detection has been made, the latest PPTA upper limit on the amplitude of such a background signal lies well in the most plausible region of theoretical predictions (Shannon et al. 2015).

Ground-based interferometers and PTA\textsuperscript{s} probe a different part of Einstein’s gravitational spectrum, and therefore are complementary to each other. Before we present studies on specific topics of GW detection using these two techniques in the following chapters, it is useful to take a look at the bigger picture of this new spectrum.

For astrophysical sources, the emitting GW frequencies are related to their sizes and masses (Schutz 1999; Flanagan & Hughes 2005). The source’s size is its dynamical length scale, e.g., the radius of a neutron star or the separation of two components in a binary. The typical GW frequency for an astrophysical source of size $R$ and mass $M$ is

$$f_{GW} \sim \left(\frac{1}{2\pi}\right)\sqrt{\frac{GM}{R^3}}.$$

Given that a lower limit for the size is the Schwarzschild radius ($R \gtrsim 2GM/c^2$), there exists an upper limit for the frequency of astrophysical GWs:

$$f_{GW}(M) < \frac{1}{4\sqrt{2\pi}} \frac{c^3}{GM} \approx 10^4 \text{ Hz} \left(\frac{M_\odot}{M}\right). \quad (2.1)$$

The above equation provides a rough guide for the GW frequency ranges of astrophysical emitters. For instance, stellar mass compact objects emit waves primarily in the audio band, whereas massive black hole binaries are strong emitters of (very) low frequency signals.

Sources of GW\textsuperscript{s} can be broadly classified into four categories by the frequency band in which signals are generated (Cutler & Thorne 2002). Figure 2.4 illustrates such a classification and the detection experiments that probe different parts of the spectrum. We give a brief description below.

1. The high frequency band, $\sim 1$–$10^4$ Hz, in which ground-based laser interferometers and resonant bar detectors operate. It also corresponds roughly to the audio band for human ear. Indeed, it is a common practice in the

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[1999: Schutz]
[2002: Cutler & Thorne]
community to convert \textit{GW} signals or detector data to audio files. Important sources in this band include: binary inspirals of stellar mass compact objects such as neutron stars and black holes, known pulsars and undetected fast spinning neutron stars, core collapse supernovae and stochastic backgrounds of cosmological or astrophysical origins.

2. The low frequency band, $\sim 10^{-4}–1$ Hz, in which space-based interferometers operate. Important sources in this band are: binary systems consisting of massive ($10^6–10^8M_\odot$) or intermediate mass ($10^2–10^6M_\odot$) black holes, Galactic binaries of white dwarfs and neutron stars, extreme/intermediate mass ratio inspirals – stellar mass compact objects that are captured by massive or intermediate mass black holes.

3. The very low frequency band, $\sim 10^{-9}–10^{-7}$ Hz, in which signals are being searched for with \textit{PTAs}. The most promising sources are supermassive ($\sim 10^8–10^{10}M_\odot$) binary black holes. Other sources include cosmic strings, and the stochastic background from amplification of primordial fluctuations during inflation and early-Universe phase transitions. These latter sources are expected to emit signals in a wide range of frequencies, e.g, also in the first two bands mentioned above. In particular, inflationary \textit{GW}s span all four frequency bands described here.

4. The extremely low frequency band, $\sim 10^{-18}–10^{-14}$ Hz, where primordial \textit{GW}s from the early Universe may contribute to the B-mode polarization of the cosmic microwave background (Seljak & Zaldarriaga 1997; Kamionkowski et al. 1997). This band is probed by experiments such as the ESA space observatory Planck (Ade et al. 2014a) and BICEP2 (Ade et al. 2014b).
Figure 2.4: The GW spectrum: sources and detection experiments. The color bar in the middle is used to show the whole frequency range in a logarithmic scale. The ‘spectrum’ is divided roughly into four bands, for which the corresponding detection experiments (bottom) and major sources (top) are also shown. The horizontal length of text for sources is drawn to approximately represent the frequency range (except the top two cosmological sources, see text). Pictures shown at the bottom are (from left to right): the cosmic microwave background (credit: ESA and the Planck Collaboration); an artist’s illustration of a pulsar timing array (credit: David Champion); an artist’s conception of LISA spacecraft (credit: NASA); aerial view of the LIGO detector in Livingston (credit: LIGO).
2.2 Detection of gravitational waves using laser interferometers

In this section we review the basics of GW detection using ground-based detectors. We first describe the detection principle in section 2.2.1. We introduce the Michelson interferometer – the core component of ground-based laser interferometric GW detectors – and describe its response to a passing GW. In section 2.2.2 we describe various important noise sources that affect the sensitivities of these detectors. Finally section 2.2.3 contains a summary of GW sources and the corresponding astrophysical results obtained using first generation detectors. We refer interested readers to Saulson (1994) for a detailed description of how modern interferometric GW detectors work, and to Blair (2012) for recent developments in various aspects.

2.2.1 The detection principle

The present-day laser interferometric GW detectors are based on Michelson interferometers. Michelson invented his interferometer in 1881. It is very precise at measuring length and differential length changes and thus very useful for a wide range of physical applications. For example, it was used to (a) disprove the existence of aether in the famous Michelson-Morley experiment (Michelson & Morley 1887), thus providing experimental evidence of Einstein’s special theory of relativity, and (b) to perform the first experimental calibration of the standardized meter in terms of the wavelength of light (Michelson & Benoit 1895).

Figure 2.5 illustrates the effect of a passing GW on the arms of a simple Michelson interferometer. As we discuss in 2.1.1, a GW changes the proper distance between freely falling test masses. Such an effect can be parameterized by the strain amplitude of a GW

\[ h = \frac{\Delta L}{L}, \]  

(2.2)

where \( L \) and \( \Delta L \) is the proper distance between two test masses and its change
Figure 2.5: Panel (a) – a sinusoidal GW panel (b) – illustration of its effect on the arms of a simple Michelson interferometer. The wave propagates in the direction perpendicular to the plane of (b). The interference pattern is monitored at the photodetector, which is denoted by the green semi-circle. Figure taken from Abbott et al. (2009b).

due to a passing GW respectively.

A Michelson interferometer consists of a light source (e.g., a laser), a beamsplitter, two mirrors and a photodetector. Two mirrors and the beamsplitter form two arms along which the laser light is running. Here we assume that the beamsplitter and two mirrors are free masses and the two L-shaped arms have equal length. A linearly polarized GW propagating in the direction perpendicular to the plane of an interferometer causes a phase shift between two beams of light after each completes a round trip in the arm:

$$\Delta \phi(t) = \frac{4\pi L}{\lambda} h(t), \quad (2.3)$$

where $\lambda$ is the wavelength of the light. Such a phase shift can be detected as a change in the interference pattern. Therefore, we can measure the amplitude of a passing GW by monitoring the intensity of the output light with a photodetector. Equation 2.3 is appropriate when $L$ is much smaller than the gravitational wavelength, which is the case for ground-based interferometers ($L \sim \text{km}$ and grav-
2.2. Detection of gravitational waves using laser interferometers

Gravitational wavelengths $\sim 10^3$ km).

Let us now consider a general case where a gravitational wave (GW) originates from an arbitrary direction on the sky, e.g., as described by the right ascension $\alpha$ and declination $\delta$, and its polarization axis are rotated at an angle $\psi$ with respect to the constant-$\alpha$ plane. In this case the GW strain measured by the detector is:

$$h(t) = F_+ (\alpha, \delta, \psi) h_+ (t) + F_\times (\alpha, \delta, \psi) h_\times (t),$$

where $F_+$ and $F_\times$ are the antenna pattern functions, giving the detector response to the $h_+$ and $h_\times$ components respectively. They are given by (see, e.g., Thorne 1987):

$$F_+ (\alpha, \delta, \psi) = \frac{1}{2} (1 + \sin^2 \delta) \cos 2\alpha \cos 2\psi - \sin \delta \sin 2\alpha \sin 2\psi,$$

$$F_\times (\alpha, \delta, \psi) = \frac{1}{2} (1 + \sin^2 \delta) \cos 2\alpha \sin 2\psi + \sin \delta \sin 2\alpha \cos 2\psi.$$  \hspace{1cm} (2.4)

From equation (2.3), one can see that the GW induced phase shift is proportional to the arm length of the interferometer. But this does not imply that longer arms always lead to more sensitive interferometers. It turns out that for $L = \lambda_{GW}/4$ with $\lambda_{GW}$ being the gravitational wavelength, in which case the round-trip light travel time equals half a wave cycle as illustrated in Figure 2.5, one obtains the maximum phase shift. For a frequency of 100 Hz, this implies an optimal arm length of 750 km, which is obviously impractical for a ground-based detector. There are ways to overcome this difficulty by increasing the effective arm length (see, e.g., Pitkin et al. 2011 for details).

A straightforward solution is to increase the number of round-trips via multiple reflections of laser beams (Weiss 1972). GEO600 employs such a scheme to achieve a folded arm length of 1.2 km. Another way is to use Fabry-Perot cavities, which are formed by adding two additional mirrors near the beamsplitter. This is the case for LIGO, Virgo and TAMA300 detectors. When the cavities are on resonance (i.e., the separation of two mirrors is very close to an integer number of half-wavelengths
of the light in the arm), the laser light is trapped for hundreds of round trips before exiting the arms. This results in an effective arm length of the order of 1000 km for LIGO and Virgo detectors as needed for maximizing the detectable phase shift in around 100 Hz.

Modern interferometers are much more sophisticated than a simple Michelson interferometer. Because GWs are extremely weak: a binary neutron star coalescence located 20 Mpc away from the Earth produces a signal with amplitude \( h \sim 10^{-21} \) at 100 Hz (Thorne 1987). For a detector with km-scale arm length, detection of such signals requires measuring the distance change between two test masses with a precision better than \( \Delta L \sim 10^{-18} \) m, i.e., one thousandth of the diameter of a single proton! This indicates the astonishing sensitivities of interferometers, probably the most sensitive instruments ever built by human beings.

### 2.2.2 Sensitivities of ground-based interferometers

In the previous subsection we discuss the response of a laser interferometer to GWs. In the actual interferometer data, signals are buried by various noise sources. These noise sources essentially determine the sensitivities of ground-based interferometers. In Figure 2.6, we plot the representative noise spectra of LIGO detectors during its fifth (S5; Abbott et al. 2009b) and sixth (Abadie et al. 2012b, S6;) science runs along with the design goal sensitivity curve for H1 and L1. To date, LIGO detectors have completed six science runs between 2002 and 2010. The S5 run lasted from November 2005 to September 2007, during which, LIGO acquired one year of data coincident among three detectors (H1, H2 and L1) at their design sensitivities. The S6 run lasted from July 2009 until October 2010, during which sensitivities were enhanced by about a factor of 2 at frequencies above \( \sim 200 \) Hz.

Sensitivity curves shown in Figure 2.6 are generally dominated by a few noise processes: (1) seismic noise at below 40 Hz, (2) thermal noise between 40 and 200 Hz and (3) shot noise above 200 Hz. We briefly describe these important noise sources below; for a more detailed description we refer the interested reader to...
2.2. Detection of gravitational waves using laser interferometers

Figure 2.6: Representative noise spectra of the LIGO H1 detector during the fifth (S5) and sixth science runs (S6). Noise spectra for L1 are similar and thus not shown here. Also shown is the design sensitivity curve for LIGO 4 km detectors. The theoretical curve is dominated by three major noise sources: the seismic “wall” below about 40 Hz, the suspension thermal noise between 40 and 200 Hz, and shot noise above 200 Hz. Data used to make this plot were taken from the LIGO Document T0900499 (S5) and T1100338 (S6), and the LIGO Science Requirements Document E950018-02 (design goal). Data may be downloadable from LIGO Laboratory Home Page for Interferometer Sensitivities at http://www.ligo.caltech.edu/~jzweizig/distribution/LSC_Data/
Blair (2012).

Seismic noise

Ground-based detectors inevitably suffer from seismic vibrations of the ground, such as earthquakes, ocean waves and human activities. The typical seismic noise at a reasonably quiet site on the Earth is, as represented by the displacement spectral density, $\sim 10^{-7} f^{-2} m/\sqrt{\text{Hz}}$ at frequencies above $\sim 1$ Hz. One can see that seismic noise is a more serious problem at low frequencies. All test masses of the interferometer are suspended on pendula, giving seismic attenuation by a factor of $f^2/f_p^2$ with $f_p$ being the natural oscillation frequency of the pendulum ($\sim 0.76$ Hz). This alone is not enough. To achieve a strain sensitivity of $2 \times 10^{-23}/\sqrt{\text{Hz}}$ at 100 Hz (as has been done for the initial LIGO detectors H1 and L1), seismic noise should be reduced by at least 8 orders of magnitude. Such a demanding requirement is accomplished by a sophisticated seismic isolation system.

With the use of more advanced seismic isolation technologies, the seismic “wall” will be reduced to about 10 Hz for Advanced LIGO from 40 Hz for initial LIGO detectors. Similarly, the initial Virgo detector had much better sensitivities than initial LIGO detectors at low frequencies ($\lesssim 50$ Hz). Another way to greatly suppress seismic noise is to build underground detectors. The Japanese KAGRA (KAmioka GRAvitational wave telescope) is an underground interferometer, to be built in tunnels in the Kamiokande mountain near the famous Super-Kamiokande neutrino detector. The proposed third generation detector Einstein Telescope will be of a similar design.

Thermal noise

At frequencies where seismic vibrations have been sufficiently attenuated, thermal noise dominates the strain sensitivities of the first generation interferometers. Thermal noise is caused by thermal Brownian motions of the mirrors and the suspending pendulum. Mirrors and suspension systems are – actually the whole interferometer
2.2. Detection of gravitational waves using laser interferometers

is – placed in vacuum chambers. They are coupled to the external heat reservoir. Increasing the mechanical quality factor of the mirrors and suspension systems can reduce thermal noise because (1) a high quality factor means less energy exchange between mirrors, pendula and their environment, and therefore less noise fluctuations are transferred to the mechanical system, and (2) thermal energy is more concentrated within a narrow band around the resonant frequency, implying less thermal noise outside this frequency region. Indeed, many spiky peaks shown in a typical plot of LIGO/Virgo sensitivities are due to the violin modes of the suspension wires (see Figure 2.6). Alternatively, thermal noise can be suppressed by using cryogenically cooled optics and suspension systems, as planned for KAGRA.

**Shot noise**

Shot noise arises from the random fluctuations in the light intensity or equivalently the number of photons measured at the interferometer output. It is the dominant source of noise at frequencies above a few hundred Hz. The number of photons at a given time follows a Poisson distribution. The fractional error in the mean number of photons goes as $1/\sqrt{N}$ with $N$ being the mean photon count. Therefore shot noise can be reduced by increasing the laser power.

Fabry-Perot cavities and the so-called *power-recycling mirror* also reduce the shot noise. The power recycling mirror is placed between the laser and the beam-splitter and it reflects returning “wasted” light back to the beamsplitter. (Here note that the interference is normally kept at dark so most light is returned to the laser.) All LIGO and Virgo detectors are equipped with power recycling mirrors and Fabry-Perot cavities, so they are usually called power-recycled Fabry-Perot Michelson interferometers.

**Radiation pressure noise**

Photons carry momentum and exert pressure on the mirror when striking or reflecting from it. The quantum fluctuations of the radiation pressure directly cause
displacement noise of the mirrors. Radiation pressure noise is not a dominant source of noise even for advanced detectors but it is important in its relation to shot noise. In contrast to shot noise, radiation pressure noise is proportional to the square root of the laser power.

Therefore, rather than arbitrarily increasing the laser power to reduce the shot noise, one should choose a particular laser power to minimize the total quantum noise – shot noise plus radiation pressure noise. This sensitivity limit is called the standard quantum limit, which is related to the Heisenberg uncertainty principle. Quantum squeezing techniques can be used to get sensitivities considerably greater than the standard quantum limit, as was demonstrated first in GEO600 (Abadie et al. 2011a) and later in LIGO detectors (Aasi et al. 2013b).

Other noises

Other sources of noise that are not as dominant as the aforementioned ones but are still important to ground-based detectors include: (1) **Gravity gradient noise**, which is caused by the direct gravitational coupling of the suspended mirrors to the environmental mass density fluctuations. Its noise spectrum falls steeply with increasing frequencies and its amplitude is well below that of seismic noise. Similar to seismic noise, gravity gradient noise can also be reduced by building an underground detector; (2) **glitches**. Real-world interferometer data usually contain a large number of transient noise events of environmental or instrumental origins. There are no theoretical models for such kind of non-Gaussian noise. However, many of them can be identified and removed from the data with the help of various auxiliary monitors. Others can be dealt with using data-based vetoing techniques. Coincident analysis of data from different detectors can also dramatically reduce the impact of glitches on the detection of GWs.
2.2.3 Gravitational wave sources in the audio band

In this subsection, we give a brief overview of gravitational wave sources in the audio band and the astrophysical results from searches conducted with ground-based detectors. There are broadly four types of sources that have been searched for using interferometers: (1) binary coalescences of neutron stars and stellar mass black holes; (2) continuous wave sources, such as spinning neutron stars; (3) bursts, e.g., core collapse supernovae; (4) stochastic backgrounds, of either cosmological or astrophysical origins.

Compact binary coalescences

Compact binary coalescences are the most promising sources for ground-based interferometers. In the audio band, these binary systems are in the late stage of inspiral, followed by the merger and ringdown phase. During the final minutes before the merger, the frequency and amplitude increase monotonically with time, leading to chirp-like signals (e.g., Cutler et al. 1993; Finn & Chernoff 1993). The optimal method for detecting such signals is matched filtering (Helstrom 2013; Jaranowski & Królak 2012), which requires accurate models of gravitational waveforms. While the inspiral and ringdown phases can be analytically modelled by post-Newtonian formalisms and perturbation theories respectively, numerical relativity simulations are necessary to understand the physics during the merger.

Extensive searches have been performed for coalescing compact binaries with various masses (see, e.g. Abadie et al. 2012b; Aasi et al. 2013d; 2014b for latest attempts), with the last science run of initial LIGO detectors (H1 and L1) getting the furthermost – for a canonical 1.4–1.4 $M_\odot$ binary neutron star system they reached a horizon distance of $\sim 40$ Mpc (The LIGO Scientific Collaboration & The Virgo Collaboration 2012). Searches with initial detectors result in upper limits on the coalescence rate of compact binaries, which remain 2–3 orders of magnitude above realistic expectations and ten times higher than optimistic estimates (Abadie et al. 2012b).

Mergers of compact binaries with at least one companion being a neutron star
are thought to be progenitors of short gamma-ray bursts \( \text{GRBs; Paczynski 1986, Eichler et al. 1989; Narayan et al. 1992; Rezzolla et al. 2011} \). There were two short \( \text{GRB} \) events coincident with the \( \text{LIGO} \) S5 run: \( \text{GRB} \) 051103 (Hurley et al. 2010) and \( \text{GRB} \) 070201 (Perley & Bloom 2007), which were suggested to be occurring within the nearby galaxies M81 (with a distance of 3.6 Mpc) and M31 (distance \( \simeq 770 \text{ kpc} \) respectively. With these sky directions and distances assumed, \text{LIGO} observations ruled out the possibilities of these two events being caused by binary neutron star or black hole-neutron star mergers with very high confidence (Abbott et al. 2008b; Abadie et al. 2012d), favoring the models of soft gamma repeater giant flares.

**Continuous waves**

Continuous waves are long-lasting and quasi-monochromatic \( \text{GWs} \). Such signals can be generated by spinning neutron stars in a few ways, e.g., non-axisymmetric deformations of the star, unstable oscillation modes and free precession (Andersson et al. 2011). The initial generation interferometers have conducted various searches for this type of signal, either targeting known Galactic pulsars (Abadie et al. 2010b; Aasi et al. 2014a, 2015c) or in an all-sky blind fashion (Abadie et al. 2012a; Aasi et al. 2013a). These have led to some astrophysically interesting results, such as beating the spin-down limits for the Crab and Vela pulsars. Pulsars are generally spinning down because of loss of energy in a variety of mechanisms including the emission of \( \text{GWs} \). Their observed spin-down rates imply upper limits on the amplitudes of \( \text{GWs} \), which are named the spin-down limits. Latest observations with \text{LIGO} and Virgo detectors show that gravitational radiation contributes to less than 1% and 10% of the spin-down rates of the Crab and Vela pulsars respectively (Abbott et al. 2008a; Abadie et al. 2011b).
2.2. Detection of gravitational waves using laser interferometers

Gravitational wave bursts

A burst signal is generally defined as a short-duration signal with unknown waveform. Supernovae that experience nonspherical core collapse are potential sources of such signals. It is an extremely challenging task to predict the resulting gravitational waveform from the core collapse process because of incomplete understanding of the explosion mechanism and the complexity of the physics involved (see Ott [2009] for a review). Much work has gone into detailed simulations of the supernova process, providing the energy and spectral characteristics of the emitted signals (see, e.g., Fryer et al. [2001] Ott et al. [2006] Müller et al. [2013]). It is suggested that the signal probably has a duration within hundreds of milliseconds, with the total energy emitted in gravitational radiation being in the range $\sim (10^{-12} - 10^{-4}) M_\odot c^2$ (Baiotti & Rezzolla [2006] Ott [2009]). As a comparison, the total energy of a GW burst event (assuming a sine-Gaussian signal with a center frequency of 153 Hz) that would be detectable with the initial LIGO-Virgo detectors is $1.8 \times 10^{-8} M_\odot c^2$ and $4.6 \times 10^{-3} M_\odot c^2$ for a typical Galactic distance (10 kpc) and the Virgo cluster (16.5 Mpc) respectively (Abadie et al. [2010a]).

Stochastic backgrounds

A stochastic background may have two different origins. It may result from a number of early-Universe processes, such as amplification of quantum vacuum fluctuations during inflation, phase transitions, cosmic strings etc. (see, e.g. Maggiore [2000] Buonanno [2003] for reviews). An astrophysical background may be produced by the superposition of a large number of unresolved sources that were formed since the beginning of star formation (see, Regimbau [2011] for a review). Interesting sources for ground-based detectors include compact binaries (e.g., Regimbau & de Freitas Pacheco [2006b] Zhu et al. [2011b] Rosado [2011] Wu et al. [2012]), core-collapse supernovae (e.g., Blair & Ju [1996] Ferrari et al. [1999a] Buonanno et al. [2005] Zhu et al. [2010]) and spinning neutron stars (e.g., Ferrari et al. [1999b] Regimbau & de Freitas Pacheco [2001] [2006a] Zhu et al. [2011a] Rosado [2012]).
Analysis of data from initial LIGO detectors have constrained the energy density of the stochastic background to be \(< 6.9 \times 10^{-6}\) times the closure energy density of the Universe at around 100 Hz (Abbott et al. 2009a). Such a result improved for the first time upon the indirect limits from Big Bang nucleosynthesis (Maggiore 2000) and cosmic microwave background (Smith et al. 2006), both of which only apply to backgrounds generated before the Big Bang nucleosynthesis and the cosmic microwave background decoupling respectively.

In Chapter 3 we will present a comprehensive study on the background signal formed by compact binaries, showing that such a signal could be detectable with advanced detectors and it could also mask the highly-sought primordial signals.

### 2.3 Detection of gravitational waves using pulsar timing arrays

In this section, we introduce pulsars, pulsar timing techniques, and noise sources in pulsar timing data. We also describe how PTAs can be used to detect GWs in the nanohertz band and give an overview of the corresponding sources.

#### 2.3.1 Pulsars

Neutron stars are born as compact remnants of core collapse supernovae during the death of massive main sequence stars with masses of around 8–25 $M_\odot$ (Fryer 1999, Smartt 2009). A pulsar is a highly magnetized, spinning neutron star. It can be detected by us as it emits beams of electromagnetic radiation along its magnetic axis which is misaligned with its rotational axis. Such beams of radiation sweep over the Earth in the way lighthouse beams sweep across an observer (see Figure 2.7 for an illustration), leading to pulses of radiation received at the observatory. Because of their exceptional rotational stability, pulsars are powerful probes for a wide range of astrophysical phenomena. For example, long-term timing observations of PSR B1913+16 provided the first observational evidence of the existence of GWs as...
2.3. Pulsar timing arrays

Timing observations of PSR B1257+12 led to the first confirmed discovery of planets outside our solar system (Wolszczan & Frail 1992). The double pulsar system PSR J0737−3039A/B, with both neutron stars having been detected as radio pulsars (Burgay et al. 2003a; Lyne et al. 2004), enabled very stringent tests of general relativity and alternative theories of gravity in the strong-field regime (Kramer et al. 2006).

The first pulsar was discovered by J. Bell and A. Hewish in 1967 (Hewish et al. 1968). Since then over 2500 pulsars have been discovered with spin periods ranging from about 1 millisecond to 10 seconds. It is generally believed that pulsars are born with periods of order tens of milliseconds but quickly (∼10 Myr) spin down (because of the loss of rotational energy) to periods of order seconds (Lorimer & Kramer 2005). This energy loss could be due to a variety of mechanisms, such as the magnetic dipole radiation, emission of relativistic particle winds and even

---

13See the ATNF Pulsar Catalogue website [http://www.atnf.csiro.au/research/pulsar/psrcat/] for up-to-date information. Nearly two thirds of known pulsars were first detected by the Australian Parkes radio telescope (Manchester 2012).
GWs (as discussed in the previous section). As pulsars spin down, they eventually reach a point where there is insufficient energy to power electromagnetic radiation. However, for pulsars in binary systems, it is very likely that pulsars are spun up as mass and angular momentum are accreted from their stellar companions. Such an accretion process is observed in X-ray binaries. These pulsars, usually named as millisecond pulsars, have spin periods of about several milliseconds and much lower spin-down rates. Figure 2.8 shows the distribution of all known pulsars in the period-period derivative diagram. To date pulsars have been observed in radio, infrared, optical, X-ray and $\gamma$-ray wavelengths. In this thesis we are concerned with radio timing observations of millisecond pulsars.

2.3.2 Pulsar timing techniques

Much of the science based on pulsar observations makes use of the “pulsar timing” technique (Lorimer & Kramer 2005, Edwards et al. 2006, Lommen & Demorest 2013), which involves measurement and prediction of pulses’ times of arrival (TOAs). Individual pulses are generally not useful in this regard as they are unstable and mostly too weak to observe. The average pulse profile over a large number of pulses is very stable for a particular pulsar at a given observing wavelength, and therefore very suitable for timing experiments.

The first step in pulsar timing is to measure the topocentric pulse arrival times with clocks local to the radio observatories. This is done as follows. Data collected with the telescope are de-dispersed to correct for frequency-dependent dispersion delays due to the ionized interstellar medium. These data are then folded with the period derived from previous observations to form the mean pulse profile. This profile is then correlated with a standard template, either an analytic function or simply a very high signal-to-noise ratio observation, to record the pulse arrival time at the observatory.

The measured TOAs are further transformed to the pulse emission time via a timing model, from which the pulse phase of emission is computed. The rotational
Figure 2.8: The period ($P$) vs. period derivative ($\dot{P}$) diagram for all known pulsars (black dots; data taken from the ATNF Pulsar Catalogue version 1.53). Red/grey circles represent the millisecond pulsars currently being timed by the IPTA project. Also shown are lines of constant characteristic age $\tau = P/2\dot{P}$ (dash-dotted) assuming that pulsars are spinning down solely because of magnetic dipole radiation (e.g., Lorimer & Kramer 2005), and of constant inferred surface magnetic field $B_0 = 3.2 \times 10^{19} \sqrt{\dot{P}}$ Gauss (dash; Manchester & Taylor 1977). Two distinct populations are apparent in this diagram: (a) normal pulsars, with $P \sim 0.1$-4 seconds and $B_0 \sim 10^{11} - 10^{13}$ Gauss; (b) millisecond pulsars, with $P \sim 3$ milliseconds and $\dot{P} \sim 10^{-20}$ ss$^{-1}$.
phase $\phi(t)$ of the pulsar as a function of time $t$ (measured in an inertial reference frame of the pulsar) can be represented as a Taylor series:

$$\phi(t) = \phi(t_0) + f(t - t_0) + \frac{1}{2} \dot{f}(t - t_0)^2 + \cdots,$$

where $t_0$ is an arbitrary reference time, $f = d\phi/dt$ is the spin frequency, and $\dot{f}$ is the frequency derivative. A number of corrections are applied when converting the topocentric TOAs to the pulsar frame. Such corrections include:

1. Clock corrections, which account for differences in the observatory time and a realization of Terrestrial Time (e.g., the International Atomic Time).

2. Pulse delay induced by Earth’s troposphere.

3. The Einstein delay, i.e., the time dilation due to changes in the gravitational potential of the Earth, the Earth’s motion, and the secular motion of the pulsar or that of its binary system.

4. The Roemer delay, i.e., the vacuum light travel time between the observatory and the solar system barycenter, and for pulsars in binaries between the pulsar and the binary system’s barycenter.

5. Shapiro delays, i.e., gravitational time delays due to the solar system objects and if applicable the pulsar’s companion.

6. Dispersion delays caused by the interstellar medium, the interplanetary medium and the Earth’s ionosphere.

The timing model, which describes the above corrections and the pulsar’s intrinsic rotational behavior, predicts the rotational phase of the pulsar at any given time as observed from the solar system barycenter. Basic parameters of a pulsar timing model include the spin period, spin-down rate, right ascension and declination of the pulsar, the dispersion measure (to be discussed later), and five Keplerian
orbital parameters if the pulsar is in a binary system. Measured TOAs are compared with predictions based on the timing model, and the differences are called timing residuals. The (pre-fit) timing residual for the $i$-th observation is calculated as (Hobbs et al. 2006b):

$$ R_i = \frac{\phi(t_i) - N_i}{f_0}, $$

where $N_i$ is the nearest integer\textsuperscript{14} to each $\phi(t_i)$. One can see the key point in pulsar timing is that every single rotation of the pulsar is unambiguously accounted for over long periods (years to decades) of time.

A linear least-squares fitting procedure is carried out to obtain estimates of timing parameters, their uncertainties and the post-fit timing residuals. In practice, this is done iteratively: one starts from a small set of data and only includes the most basic parameters (with values derived from previous observations) so that it is easier to coherently track the rotational phase. Parameter estimates are then improved by minimizing the timing residuals and additional parameters can be included for a longer data set.

The fitting to a timing model and analysis of the timing residuals can be performed with the pulsar timing software package TEMPO2 (Hobbs et al. 2006b; Edwards et al. 2006; Hobbs et al. 2009), which is freely available on the internet for download\textsuperscript{15}. More recently, an alternative method based on Bayesian inference was also developed (Lentati et al. 2014; Vigeland & Vallisneri 2014).

\subsection{2.3.3 Noise sources in pulsar timing data}

Timing residuals generally come from two groups of contribution: (1) un-modelled deterministic processes such as an unknown binary companion or a single-source GW and (2) stochastic processes such as the intrinsic timing noise and a GW background. In Figure 2.9 we show the post-fit timing residuals of 4 pulsars for the PPTA Data Release 1 (DR1) data set (Manchester et al. 2013). This data

\textsuperscript{14}Here note that $\phi(t)$ is measured in turns equal to $2\pi$ radians.

\textsuperscript{15}www.sf.net/projects/tempo2/
set will be discussed in detail in Chapters 4 and 5. Here we use this plot as an example to discuss various noise processes in pulsar timing data. Note that most of these processes are astrophysically interesting in themselves, but we call them noise processes as we focus on the detection of GWs.

**Radiometer noise**

Radiometer noise arises from the observing system and the radio sky background (including the atmosphere, the cosmic microwave background and synchrotron emission in the Galactic plane). It can be quantified as (Lorimer & Kramer 2005):

\[
\sigma_{\text{rad}} \approx \frac{W}{S/N} \approx \frac{W S_{\text{sys}}}{S_{\text{mean}} \sqrt{2 \Delta \nu t_{\text{int}}} \sqrt{P - W}},
\]

(2.9)

where \(W\) and \(P\) are the pulse width and period respectively, \(S/N\) is the profile signal-to-noise ratio, \(S_{\text{sys}}\) is the system equivalent flux density which depends on the system temperature and the telescope’s effective collecting area, \(S_{\text{mean}}\) the pulsar’s flux density averaged over its pulse period, \(\Delta \nu\) and \(t_{\text{int}}\) are the observation bandwidth and integration time respectively. Radiometer noise can be reduced by using low-noise receivers, observing with larger telescopes, and increasing observing time and bandwidth. One reason to fold individual pulses to obtain an average profile is to reduce the radiometer noise; the reduction is equal to the square root of the number of pulses folded. Radiometer noise is an additive Gaussian white noise and is formally responsible for TOA uncertainties. In Figure 2.9 only PSR J1909–3744 shows timing residuals that are largely consistent with Gaussian white noise. From equation (2.9), one can see that bright, fast spinning pulsars with narrow pulse profiles allow the highest timing precision.

**Pulse jitter noise**

Pulse jitter manifests as the variability in the shape and arrival phase of individual pulses. The mean pulse profile is an average over a large number of single pulses.
Figure 2.9: Post-fit timing residuals of 4 pulsars for the PPTA DR1 data set (Manchester et al. 2013). The dash line marks zero residual and the value shown below the pulsar name gives the range of the residuals (i.e., the vertical extent of each subplot). The PPTA DR1 data set is publicly available online (DOI: 10.4225/08/534CC21379C12).
Although it is stable for most practical purposes, there always exists some degree of stochasticity in the phase and amplitude of the average pulse profile. Pulse jitter noise is intrinsic to the pulsar itself, and thus can only be reduced by increasing observing time, i.e., averaging over more single pulses. Pulse jitter is also a source of white noise, which has been found to be a limiting factor of the timing precision for a few very bright pulsars (Shannon et al. 2014; Dolch et al. 2014). For future telescopes such as FAST and SKA, jitter noise may dominate over the radiometer noise for many millisecond pulsars (Hobbs et al. 2014). Improvement in the timing precision for the brightest pulsar PSR J0437−4715 was recently demonstrated with the use of some mitigation methods for pulse jitter noise (Oślowski et al. 2011, 2013).

“Timing noise”

It has long been realized that timing residuals of many pulsars show structures that are inconsistent with TOA uncertainties (Blandford et al. 1984; Hobbs et al. 2006a). Such structures are collectively referred to as timing noise. Timing noise is very commonly seen in normal pulsars and has also become more prominent in a number of millisecond pulsars as timing precision increases and the data span grows. The exact astrophysical origins of timing noise are not well understood. It is mostly suggested to be related to rotational irregularities of the pulsar and therefore it is also usually called as spin noise (Hobbs et al. 2010b; Shannon & Cordes 2010). For millisecond pulsars, power spectra of timing residuals can typically be modelled as the sum of white noise and red noise. For red noise, a power law spectrum with a low-frequency turnover appears to be a good approximation (see, e.g., figure 11 in ref. (Manchester et al. 2013) for analyses of 20 PPTA pulsars).

The presence of red noise results in problems to the pulsar timing analysis as the standard least-squares fitting of a timing model assumes time-independent TOA errors. Blandford et al. (1984) analytically showed the effects of timing noise on the estimates of timing model parameters and suggested the use of the noise
covariance matrix to pre-whiten the data for improved parameter estimation. More recently, Coles et al. (2011) developed a whitening method that uses the Cholesky decomposition of the covariance matrix of timing residuals to whiten both the residuals and the timing model. By doing so, noise in the whitened residuals is statistically white and the ordinary least-squares solution of a timing model can be obtained. van Haasteren & Levin (2013) developed a Bayesian framework that is capable of simultaneously estimating timing model parameters and timing noise spectra.

**Dispersion measure variations**

Because of dispersion due to the interstellar plasma, pulses at low frequencies arrive later than at high frequencies. Specifically, this dispersion delay is given by (Lorimer & Kramer 2005):

\[
\Delta_{\text{DM}} \approx (4.15 \text{ ms}) \, \text{DM} \, \nu_{\text{GHz}}^{-2},
\]

(2.10)

where \(\nu_{\text{GHz}}\) is the radio frequency measured in GHz, and dispersion measure (DM, measured in pc cm\(^{-3}\)) is the integrated column density of free electrons between an observer and a pulsar. Because of the motion of the Earth and the pulsar relative to the interstellar medium, the DM of a pulsar is not a constant in time. Such DM variations introduce time-correlated noise in pulsar timing data.

Pulsars in a timing array are usually observed quasi-simultaneously at two or more different frequencies. For example, the PPTA team observes pulsars at three frequency bands – 50 cm (~700 MHz), 20 cm (~1400 MHz), and 10 cm (~3100 MHz) – during each observing session (typically 2-3 days). This makes it possible to account for DM variations and thus reduce the associated noise with various methods (Demorest et al. 2013; Keith et al. 2013; Lee et al. 2014). The method currently being used by the PPTA is described in Keith et al. (2013), which was built on a previous work by You et al. (2007). In this method timing residuals are modelled
as the combination of a (radio-)wavelength-independent (i.e., common-mode) delay and the dispersion delay. Both components are represented as piecewise linear models and can be estimated through a standard least-squares fit. A key feature of this method is that GW signals are preserved in the common-mode component. Ultra wide-band receivers are in development by various collaborations in order to better correct for noise induced by DM variations (along with other benefits such as increasing timing precision, studies of interstellar medium, and etc.).

**Interstellar scintillation**

Interstellar scintillation refers to strong scattering of radio waves due to the spatial inhomogeneities in the ionized interstellar medium (Narayan 1992), analogous to twinkling of stars due to scattering in the Earth’s atmosphere. There are multiple effects associated with interstellar scintillation that cause time-varying delays in measured TOAs, with the dominant one being pulse broadening from multipath scattering (Stinebring 2013; Cordes & Shannon 2010). Various mitigation techniques have been developed for this type of noise (Cordes & Shannon 2010; Demorest 2011; Liu et al. 2014). Generally speaking, noise induced by interstellar scintillation is a Gaussian white noise, and can be reduced by increasing observing time and bandwidth. For millisecond pulsars that are observed at current radio frequencies by PTAs, the effects of scintillation are predicted to be small. However, when pulsars are observed at lower frequencies, or more distant (and more scintillated) pulsars are observed, these effects can become more important (Cordes & Shannon 2010; Cordes et al. 2015).

**Correlated noise among different pulsars**

The above noise processes are generally thought to be uncorrelated among different pulsars. In a PTA data set that we hope to detect GWs, some correlated noise may be present. For example, (1) instabilities in Terrestrial Time standards affect TOA measurements of all pulsars in exactly the same way, i.e., clock errors result in a
monopole signature in a PTA data set; (2) the solar system ephemerides, which provide accurate predictions of the masses and positions of all the major solar system objects as a function of time, are used to convert pulse arrival times at the observatory to TOAs referenced at the solar system barycenter. Imperfections in the solar system ephemerides induce a dipole correlation in a PTA data set. Indeed, it has been demonstrated that PTAs can be used (1) to search for irregularities in the time standard and thus to establish a pulsar-based timescale (Hobbs et al. 2012), and (2) to measure the mass of solar system planets (Champion et al. 2010).

2.3.4 Pulsar timing arrays

The effects of GWs in a single-pulsar data may be indistinguishable from those due to noise processes as discussed in the previous subsection. Indeed, even without any such noise, GWs that have the same features as those due to uncertainties in the timing model parameters would still be very difficult to detect with only one pulsar. Therefore analysis of single-pulsar data sets would only lead to constraints on the strength of potential GWs (e.g. Jenet et al. 2004; Yi et al. 2014), and when the timing precision is sufficiently high to evidence of GWs.

A PTA is a Galactic-scale GW detector. If one wishes to have an analog to a laser interferometer, pulsars in the timing array are “test masses”; pulses of radio waves act as the laser; and the pulsar-Earth baseline is a single “arm”. Millisecond pulsars in our Galaxy, typically ∼kpc away, emit radio waves that are received at the telescope with extraordinary stability. A GW passing across the pulsar-Earth baseline perturbs the local spacetime along the path of radio wave propagation, leading to an apparent redshift in the pulse frequency that is proportional to the GW strain amplitude. As in section 2.2.1 let us first consider the special case where a linearly polarized GW propagates in a direction perpendicular to the pulsar-Earth baseline, the resulting timing residual is given by:

\[ r(t) = \int_0^{L/c} h(t - \frac{L}{c} + \tau)d\tau, \] (2.11)
where $L$ is the pulsar distance and we adopt the plane wave approximation\footnote{For sources that are close enough ($\lesssim 100\,\text{Mpc}$), it may be necessary to consider the curvature of the gravitational wavefront. This, in principle, would allow luminosity distances to GW sources to be measured via a parallax effect (Deng & Finn 2011).} With the definition of $dA(t)/dt = h(t)$, the timing residual takes the following form:

$$r(t) = \Delta A(t) = A(t) - A(t - \frac{L}{c}). \quad (2.12)$$

Here we can see that $A(t)$ results from the induced spacetime perturbation incident on the Earth (i.e., the Earth term), and $A(t - \frac{L}{c})$ depends on the strain at the time of the radio wave emission (i.e., the pulsar term). Note that in the long-wavelength approximation that is appropriate for ground-based interferometers, equation (2.12) is reduced to:

$$r(t) = h(t)\frac{L}{c}. \quad (2.13)$$

Multiplied by a factor of 2 to account for the round-trip light travel time for interferometers, this leads exactly to equation (2.3). Typical PTA observations have a sampling interval of weeks and span over $\sim 10$ yr, implying a sensitive frequency range of $\sim 1$–$100\,\text{nHz}$. Therefore PTAs are sensitive to GWs with wavelengths of several light years, much smaller than the pulsar-Earth distance.

In the general case where a GW originates from a direction $\hat{\Omega}$, the induced timing residuals can be written as:

$$r(t, \hat{\Omega}) = F_+(\hat{\Omega})\Delta A_+(t) + F_\times(\hat{\Omega})\Delta A_\times(t), \quad (2.14)$$

where $F_+(\hat{\Omega})$ and $F_\times(\hat{\Omega})$ are analogous to the antenna pattern functions as given in equations (2.5-2.6). These angular factors are given by (Wahlquist 1987):

$$F_+(\hat{\Omega}) = \frac{1}{4(1 - \cos \theta)} \left\{ (1 + \sin^2 \delta) \cos^2 \delta_p \cos[2(\alpha - \alpha_p)] - \sin 2\delta \sin 2\delta_p \cos(\alpha - \alpha_p) + \cos^2 \delta (2 - 3 \cos^2 \delta_p) \right\} \quad (2.15)$$

$$F_\times(\hat{\Omega}) = \frac{1}{2(1 - \cos \theta)} \{ \cos \delta \sin 2\delta_p \sin(\alpha - \alpha_p) - \sin \delta \cos^2 \delta_p \sin[2(\alpha - \alpha_p)] \} \quad (2.16)$$
where \( \cos \theta = \cos \delta \cos \delta_p \cos (\alpha - \alpha_p) + \sin \delta \sin \delta_p, \) \( \theta \) is the opening angle between the GW source and pulsar with respect to the observer, and \( \alpha (\alpha_p) \) and \( \delta (\delta_p) \) are the right ascension and declination of the GW source (pulsar) respectively. Note that we separate the polarization angle from \( F_+ (\hat{\Omega}) \) and \( F_\times (\hat{\Omega}) \) and insert it into \( \Delta A_{+,\times}(t) \) for convenience of later analyses in Chapters 4 and 5. The source-dependent functions \( \Delta A_{+,\times}(t) \) in equation (2.14) are given by:

\[
\Delta A_{+,\times}(t) = A_{+,\times}(t) - A_{+,\times}(t_p)
\]

\[
t_p = t - (1 - \cos \theta) \frac{L}{c}.
\]

The forms of \( A_+(t) \) and \( A_\times(t) \) depend on the type of source that we are looking for. This will be discussed in detail in section 4.2.

The Hellings-Downs curve

An isotropic stochastic background is expected to produce a correlated signal in PTA data sets. Such a correlation uniquely depends on the angular separation between pairs of pulsars, as given by (Hellings & Downs 1983):

\[
\zeta(\theta_{ij}) = \frac{3}{2} \frac{(1 - \cos \theta_{ij})}{2} \ln \left[ \frac{(1 - \cos \theta_{ij})}{2} \right] - \frac{1}{4} \frac{(1 - \cos \theta_{ij})}{2} + \frac{(1 + \delta_{ij})}{2},
\]

where \( \theta_{ij} \) is the angle between pulsars \( i \) and \( j \), and \( \delta_{ij} \) is 1 for \( i = j \) and 0 otherwise. Figure 2.10 shows the famous Hellings-Downs curve as given by equation (2.19) – it is a factor of 3/2 larger than the original result of Hellings & Downs (1983). This is because \( \zeta(\theta_{ij}) \) is normalized to 1 for the autocorrelation of the stochastic background induced timing residuals for a single pulsar. In Figure 2.10 the correlation function takes a value of 0.5 at zero angular separation as the autocorrelation due to pulsar terms is neglected. It is worth mentioning that the function \( \zeta(\theta_{ij}) \) is essentially the correlation of the response functions \( F_+ (\hat{\Omega}) \) and \( F_\times (\hat{\Omega}) \) for pairs of pulsars averaged over the whole sky. This is analogous to the overlap reduction function used in cross-correlation searches with pairs of ground-based interferometers, as we
2.3.5 Gravitational wave sources in the nanohertz band

In this subsection, we give a brief overview of sources in the nanohertz band and the astrophysical results from searches conducted with current PTA experiments. Potential signals that could be detectable for PTAs include: (1) stochastic backgrounds. The primary target is that formed by the combined emission from numerous supermassive binary black holes distributed throughout the Universe. Cosmological backgrounds generated in the early Universe are also accessible to PTAs and stringent constraints have been placed on various models (Lentati et al. 2015; Arzoumanian et al. 2015b); (2) continuous waves, which can be produced by individual bright binaries; (3) bursts with memory associated with supermassive black hole binary mergers; (4) other bursts.
Stochastic backgrounds

The stochastic background from the cosmic population of supermassive binary black holes has been the most popular target for PTA efforts. Generally speaking the signal amplitude depends on how frequently these binaries merge in cosmic history and how massive they are. Both of these quantities are poorly constrained observationally. Assuming that all binaries are in circular orbits and evolve through gravitational radiation only, the characteristic amplitude spectrum of this background is given by (Rajagopal & Romani 1995; Jaffe & Backer 2003; Wyithe & Loeb 2003; Wen et al. 2009; Sesana 2013b; Ravi et al. 2012; McWilliams et al. 2014):

\[
h_c(f) = A_{yr} \left(\frac{f}{f_{yr}}\right)^{-2/3},
\]

where \(A_{yr}\) is the amplitude at a reference frequency \(f_{yr} = yr^{-1}\). Various models predict a similar range of \(A_{yr}\), most likely to be \(\sim 10^{-15}\) (see, e.g., Wen et al. 2009; Sesana 2013b; Ravi et al. 2012). An exception is the recent model by McWilliams et al. (2014) whose prediction is two to five times higher. Recent studies that include the effects of environmental coupling and orbital eccentricities indicate a reduced signal at below \(\sim 10\) nHz (Sesana 2013a; Ravi et al. 2014).

Three major PTAs have searched for such a background signal, leading to increasingly more stringent upper limits on the background strength (Jenet et al. 2006; Yardley et al. 2011; van Haasteren et al. 2011; Demorest et al. 2013; Shannon et al. 2013; Lentati et al. 2015; Arzoumanian et al. 2015b). The most constraining limit published to date (\(A_{yr} < 1 \times 10^{-15}\)) comes from the PPTA collaboration (Shannon et al. 2015). Such a limit ruled out the aforementioned standard models that assume circular and GW driven binaries with \(> 90\%\) confidence.

Recently methods have also been proposed to search for a more general anisotropic background signal (Cornish & Sesana 2013; Mingarelli et al. 2013; Taylor & Gair 2013; Gair et al. 2014). Using the 2015 EPTA data, Taylor et al. (2015) placed constraints on the angular power spectrum of the background from circular, GW driven supermassive black hole binaries and found that the data could not update
the prior knowledge on the angular distribution of a GW background.

**Continuous waves**

Individual supermassive binary black holes, especially the most nearby and/or massive ones, could provide good opportunities for detection of continuous waves. Unlike compact binaries in the audio band, supermassive binary black holes detectable for PTAs are mostly in the early stage of inspiral and therefore emit quasi-monochromatic waves. Details of the expected signal will be described in section 4.2.

In recent years growing efforts have gone into investigating the detection prospects (e.g., Sesana et al. 2009; Lee et al. 2011; Ravi et al. 2015) of, and designing data analysis methods (e.g., Babak & Sesana 2012; Ellis et al. 2012) for, continuous waves. Current estimates show that a timing array based on the future SKA telescopes may be able to detect such sources, with timing residuals of tens ns (see, e.g., Sesana et al. 2009; Ravi et al. 2015).

Yardley et al. (2010) calculated the first sensitivity curve of a PTA to single sources using an earlier PPTA data set presented in Verbiest et al. (2009). Recently both PPTA (Zhu et al. 2014) and NANOGrav (Arzoumanian et al. 2014) conducted searches for continuous waves in their corresponding real data sets. Thanks to the best data quality, PPTA has achieved by far the best sensitivity for continuous waves. For example, it reached a horizon distance of about 100 Mpc for a $10^9 - 10^9 M_\odot$ binary black hole. Details of this study will be described in Chapter 5.

**Gravitational wave memory**

A GW memory is a permanent distortion in the spacetime metric (Braginskii & Thorne 1987; Favata 2009). Such effects can be produced in mergers of supermassive binary black holes and cause a step change in pulse frequency. For a single pulsar, this is indistinguishable from a glitch event. With a timing array, GTW

\footnote{More recently, comparable results were obtained by the EPTA group (Babak et al. 2016).}
memory effects can be searched for as simultaneous pulse frequency jumps when the burst reaches the Earth. It has been suggested that memory signals are in principle detectable with current PTAs for black hole mass of $10^8 M_\odot$ within a redshift of 0.1 (see, e.g., Seto 2009, van Haasteren & Levin 2010, Pshirkov et al. 2010). The event rate is highly uncertain. Current estimates are very pessimistic, predicting only 0.03 to 0.2 detectable events every 10 years for a future PTA based on the SKA (Cordes & Jenet 2012, Ravi et al. 2015). Actual searches in existing PTA data sets – see Wang et al. (2015) for PPTA and Arzoumanian et al. (2015a) for NANOGrav – have set upper limits on the memory event rate, which remain orders of magnitude above theoretical expectations.

**Gravitational wave bursts**

Potential burst sources of interest to PTAs include the formation or coalescence of supermassive black holes, the periapsis passage of compact objects in highly elliptic or unbound orbits around a supermassive black hole (Finn & Lommen 2010), cosmic (super)string cusps and kinks (Vilenkin 1981, Damour & Vilenkin 2000, Siemens et al. 2006), and triplets of supermassive black holes (Amaro-Seoane et al. 2010). Finn & Lommen (2010) developed a Bayesian framework for detecting and characterizing burst GWs (see also Deng 2014, for a more recent work). No specific predictions have been made for detecting GW bursts with PTAs in the literature. This type of signal has not been searched for in real PTA data. In Chapter 4, we will describe a general coherent (frequentist) method that can be used to search for GW bursts with PTAs.

## 2.4 Gravitational wave data analysis

In this section we review the basic principle of GW data analysis – the practice of extracting weak signals that are buried in detector noise. A thorough description on this topic may be found in Jaranowski & Królak (2009).
There are generally two questions that we wish to answer by analyzing data collected with ground-based interferometers or PTA’s: (a) is there a GW signal present? (b1) in the case of a detected signal, what are the properties of the source that generated such a signal? or (b2) in the case of a non-detection, what is the largest signal that could be present in the data? These correspond to the problems of signal detection, parameter estimation and setting upper limits. Below we briefly discuss the frequentist approach to such problems assuming stationary Gaussian noise\(^{18}\) providing context for data analysis methods used in the following chapters. In recent years an alternative Bayesian approach to these problems is becoming more widely used in the field. We refer interested readers to Sivia (1996) and Gregory (2010) for useful textbooks on Bayesian data analysis.

### 2.4.1 Signal detection

Signal detection is essentially a statistical hypothesis testing problem. Given the observational data \(d(t)\), we wish to choose between two mutually exclusive hypotheses:

1. the **null hypothesis** \(\mathcal{H}_0\) that the signal is absent, \(d(t) = n(t)\),

2. the **alternative hypothesis** \(\mathcal{H}_1\) that the signal is present, \(d(t) = n(t) + s(t)\).

Here \(n(t)\) and \(s(t)\) denotes noise and signal respectively. Two types of errors exist in choosing one hypothesis against the other. They are *type I error* – \(\mathcal{H}_1\) is chosen when \(\mathcal{H}_0\) is true (a “false positive”) and *type II error* – \(\mathcal{H}_0\) is chosen when \(\mathcal{H}_1\) is true (a “false negative”). The probability of making a type I error is called the false alarm probability, whereas the probability of making a type II error is called false dismissal probability.

Simply speaking, a hypothesis test is to make a decision between \(\mathcal{H}_0\) and \(\mathcal{H}_1\) given the data. The problem is to find a test that is “optimal” in some sense. It is

---

\(^{18}\)The actual GW data are likely to be non-stationary and non-Gaussian, which could pose a serious challenge for data analysis and thus requires additional care in designing signal-processing algorithms (see, e.g., Kassam 2012, Allen et al. 2002, 2003).
understood that “optimality” is a relative concept, depending on different criteria used in different experiments. For the detection of GWs, we wish to choose a decision rule that minimize the false dismissal probability for a fixed value of false alarm probability. This is known as the Neyman-Pearson criterion, in which the optimal decision is to reject $\mathcal{H}_0$ in favor of $\mathcal{H}_1$ when:

$$\Lambda = \frac{\mathcal{L}(s|d)}{\mathcal{L}(0|d)} = \frac{p(d|s)}{p(d|0)} \geq \lambda_0, \quad (2.21)$$

where $\Lambda$ is likelihood ratio, i.e., the ratio between the likelihood for the presence of a signal and that for the absence of a signal given the data, and $\lambda_0$ is the threshold that can be calculated from the desired false alarm probability (or in the terminology of statistics, the significance of the test). Equation (2.21) is also called the likelihood ratio test. The false alarm probability associated with the threshold $\lambda_0$ is given by:

$$\alpha(\lambda_0) = \int_{\Lambda > \lambda_0} p(d|0), \quad (2.22)$$

and the false dismissal probability is given by:

$$\beta(\lambda_0) = \int_{\Lambda < \lambda_0} p(d|s). \quad (2.23)$$

The detection strategy depends on our knowledge of the signals that we are searching for. Those GW signals that have been introduced in sections 2.2.3 and 2.3.5 can be broadly classified as either a) deterministic or b) stochastic. For a) some are of known waveforms, such as chirp signals in the audio band and continuous waves in the PTA band, whereas others are of poorly known or unknown waveforms, such as GW bursts. We describe here the matched filtering technique (Helstrom 2013) which is optimal for signals of known waveforms (relevant to Chapters 3 and 5) and the cross correlation statistic (Allen & Romano 1999) which is optimal for stochastic searches (relevant to Chapter 3).
Matched filter

Let us start with the characterization of noise. We assume that the noise $n(t)$ can be described by a Gaussian stochastic process. The probability distribution function of the noise vector $n$ is given by:

$$p(n) = \frac{1}{\sqrt{2\pi\det\Sigma_n}} e^{-\left(n^T \Sigma_n^{-1} n\right)/2}, \quad (2.24)$$

where samples in $n(t)$ are assumed to have zero mean, $\Sigma_n$ is the noise covariance matrix defined as $\Sigma_n = \langle nn^T \rangle$ with the brackets $\langle \ldots \rangle$ denote the ensemble average of the random process. We can therefore write the likelihood of the presence of a signal $s$ given the data $d$ as:

$$L(s|d) \equiv p(d|s) = \frac{1}{\sqrt{2\pi\det\Sigma_n}} e^{\left((d-s)^T \Sigma_n^{-1} (d-s)\right)/2}. \quad (2.25)$$

One can define the noise weighted scalar product for two time vectors $a$ and $b$:

$$(a, b) = a^T \Sigma_n^{-1} b. \quad (2.26)$$

Then the likelihood ratio defined in equation (2.21) is given by

$$\Lambda = \frac{e^{-(d-s, d-s)/2}}{e^{-(d, d)/2}} = e^{(d, s)}e^{-(s, s)/2}. \quad (2.27)$$

It turns out to be more convenient to work with the log likelihood ratio:

$$\ln \Lambda = (d, s) - \frac{1}{2}(s, s). \quad (2.28)$$

Here we can see that the likelihood ratio depends on the data only through the scalar product $(d, s)$, which can be calculated if the functional form for $s$ is known before the analysis. The quantity $(d, s)$ is essentially a noise weighted correlation of the expected signal with the data and thus called the *matched filter*.

The matched filter is an optimal detection statistic following the Neyman-
Pearson criterion. In the frequency domain, it can be written as:

\[(d, s) = 4 \text{Re} \int_0^\infty \frac{\tilde{d}(f)\tilde{s}^*(f)}{P_n(f)} df, \quad (2.29)\]

where \(\tilde{d}(f)\) and \(\tilde{s}(f)\) are the Fourier transforms of \(d(t)\) and \(s(t)\) respectively, the symbol \(*\) denotes the complex conjugate, and \(P_n(f)\) is the one-sided power spectral density of the noise.

**Cross-correlation statistic**

For the detection of stochastic backgrounds, signals are indistinguishable from noise in a single detector. However, they are correlated in the outputs of different detectors, whereas noise is generally uncorrelated. Such a correlation provides a frequentist detection statistic for stochastic backgrounds, which has been widely used for ground-based detectors (Michelson 1987; Christensen 1992; Flanagan 1993; Allen & Romano 1999) and for PTAs (Hellings & Downs 1983; Jenet et al. 2005; Anholm et al. 2009). Below we briefly discuss the cross correlation statistic which will be used in Chapter 3 to evaluate the detectability of the background formed by compact binary coalescence events.

Given two detectors with outputs \(d_1(t)\) and \(d_2(t)\), the generalized correlation is

\[
\mu_Q(d_1, d_2) = \int_0^T dt \int_0^T dt' d_1(t)d_2(t')Q(t, t'), \quad (2.30)
\]

where \(Q(t, t')\) is a filter function, and \(T\) is the observation time. When there is no signal present in the data and the noise is uncorrelated between two detectors, the quantity \(\mu_Q\) has vanishing mean. The presence of a stochastic background changes the statistical properties of \(\mu_Q\). Now let us define the signal-to-noise ratio

\[
\rho_Q = \frac{\mu_Q}{\sigma_Q}, \quad (2.31)
\]

where \(\sigma_Q^2 = \langle \mu_Q^2 \rangle\) is the variance of \(\mu_Q\) in the absence of a signal. Allen & Romano
Chapter 2. Detection of Gravitational Waves

(1999) showed that an optimal decision rule is to reject the null hypothesis if \( \rho_Q \) exceeds a fixed threshold, following the Neyman-Pearson criterion. The filter function \( Q(t, t') \) can be chosen to maximize the signal-to-noise ratio, depending on a) our knowledge of the spectrum of the stochastic background, b) noise spectral properties and c) the relative position and orientations of two detectors. The factor c) comes into play because two physically distinct detectors respond differently to a background signal and the correlation is generally smaller than that for two co-located and co-aligned detectors\(^{19}\) (e.g., \textsc{LIGO} H1 and H2). This effect is described by the overlap reduction function as we will discuss in section 3.5.1.

2.4.2 Parameter estimation

For data \( d(t) \) consisting of a signal \( s(t) \) which is described by some true parameters \( \vec{\theta}_0 \) and stationary Gaussian noise: \( d(t) = s(\vec{\theta}_0; t) + n(t) \), the log likelihood of the data given a signal model \( s(\vec{\theta}) \) is:

\[
\ln L(\vec{\theta}|d) \propto (d, s(\vec{\theta})) - \frac{1}{2}(s, s(\vec{\theta})).
\]

The maximum likelihood estimator \( \hat{\vec{\theta}}_{ML} \) is the one that maximizes the quantity \( \ln L(\vec{\theta}|d) \). The maximum likelihood estimator is unbiased for high signal-to-noise ratio detections, which means the ensemble average \( \hat{\vec{\theta}}_{ML} \) recovers the true parameters \( \vec{\theta}_0 \).

If we assume that the global maximum of \( \ln L(\vec{\theta}) \) is also a local extreme, then \( \hat{\theta}_{ML} \) satisfies

\[
\frac{\partial \ln L(\vec{\theta})}{\partial \theta_i} = 0.
\]

When an analytical solution to the above equation does not exist, one could use a grid search or other numerical methods to find \( \hat{\theta}_{ML} \). For the first option, an exhaustive search over the possible parameter space is performed to find \( \hat{\theta} \) that produces

\(^{19}\)However, correlated noise could be an issue in this case. Indeed, huge efforts are required to identify and mitigate correlated noise in performing cross correlation searches with two \textsc{LIGO} Hanford detectors (Aasi et al. [2015a]).
the largest likelihood, such as a template bank search for individual inspiral events (e.g., Cutler & Flanagan 1994, Owen 1996). In Chapter 5 we also employ such a strategy for continuous waves in the nanohertz band. A grid search is straightforward to implement and works well when the number of unknown parameters is small. However, when the number of parameters increases, the grid search may become computationally impractical. In this case, more powerful numerical methods are required.

**Upper limits**

In the case of a non-detection, we wish to set upper limits on the source parameters that determine the properties of potential GW sources. The establishment of upper limits, which has been the subject of all actual searches performed so far in observational data, gives us new insights on the physical processes that are expected to generate GWs (see sections 2.2.3 and 2.3.5).

In a frequentist framework, a simple detection statement is “if the data consisted of noise only, a detection statistic above the threshold $\lambda_0$ would have only been observed with probability $\alpha$ (e.g., $10^{-4}$)”. An upper limit statement is somewhat reversed to that, along the lines of “had the signal amplitude been larger than $A_*$, a higher detection statistic than the measured value would have been observed with at least 95% probability”. Here $A_*$ is the frequentist upper limit on the signal amplitude at 95% confidence. This will be discussed in detail for an analysis of real PTA data later in Chapter 5.
Chapter 3

The Gravitational Wave Background from Compact Binary Coalescences


Some text in section 3.3.2.2 and Figure 3.3 are adapted from Zhu et al. (2011b).

3.1 Introduction

Compact binary coalescences (CBCs) are the most promising sources of GWs for ground-based interferometers such as Advanced LIGO and Advanced Virgo. They include binary neutron stars (BNSs), stellar mass binary black holes (BBHs) and black hole-neutron star (BHNS) systems. While individually detectable CBC events are expected within distances of hundreds of Mpc, the superposition of the gravitational radiation from these sources over cosmological volumes can form a GW background (GWB). Such a background signal represents another promising target for the upcoming advanced instruments (see, e.g. Zhu et al. 2011b; Marassi et al. 2011b; Rosado 2011; Wu et al. 2012; Kowalska-Leszczynska et al. 2015, for the most recent studies). A detection of this GWB will confirm the existence of a population of coalescing binaries throughout the Universe.
A gravitational wave background (GWB) can be characterized by the dimensionless energy density parameter $\Omega_{GW}(f)$, which represents the present-day fractional energy density in GWs as a function of frequency $f$. In general, assuming Newtonian energy spectra and circular binary orbits for all sources, the CBC background can be described by a power law function $\Omega_{GW}(f) = \Omega_\alpha f^\alpha$, with $\alpha = 2/3$ and an amplitude $\Omega_\alpha$ determined by system masses, coalescence rates and their evolution over cosmic time. Such power law models have been widely used in searches for stochastic backgrounds using LIGO/Virgo data (Abbott et al. 2009a; Abadie et al. 2012c), in mock-data challenges (Regimbau et al. 2012) for the planned Einstein Telescope (ET), and in parameter estimation of a stochastic background (Mandic et al. 2012).

In this work we investigate two issues of importance for stochastic searches with ground-based interferometers. Firstly we refine the power law model for the CBC background by using analytical inspiral-merger-ringdown waveforms that include post-Newtonian (PN) amplitude corrections, and observation-based parameterized models of NS and BH mass distributions. The aim is to investigate what information can be extracted from a potential detection of the CBC background and to provide a ready-to-use $\Omega_{GW}(f)$ model for CBC background searches. Another motivation to study the properties of an astrophysical GWB is that it could act as a foreground masking the primordial signals generated in the very early Universe. As the spatial distribution of individual sources produces time series of varying GW amplitudes, the strongest signals, which would be detected as single events, can be subtracted from the data. Therefore, as demonstrated for the BNS population using the proposed Big Bang Observer (Cutler & Harms 2006), a detector with high enough sensitivity, could remove a foreground entirely by subtracting all individually identified transient signals. We show in this work that there is a significant residual foreground in the (1–500) Hz frequency range from sub-threshold BNS merger events. Such a foreground should be considered in future stochastic searches for primordial GWs and other astrophysical backgrounds.

The organization of this chapter is as follows. In section 3.2 we review the the-
3.1 Astrophysical backgrounds

Theoretical framework for calculating $\Omega_{gw}(f)$ and other quantities of an astrophysical background used in the literature. We also present a practical power law model of astrophysical backgrounds. In section 3.3 we extend this model to the case of three CBC populations by considering the effects of cosmic star formation rates (CSFRs) and the delay time between binary formation and its final coalescence. Then using complete waveforms we calculate semi-analytically $\Omega_{gw}(f)$ of this background signal. We describe in section 3.4 a Monte-Carlo approach to calculate $\Omega_{gw}(f)$ which allows NS and BH mass distributions to be included and then show how the information of mass distributions is encoded in background energy spectra. In section 3.5 we evaluate carefully the detectability of the CBC background signal and further investigate the construction of $\Omega_{gw}(f)$ templates for future detectors. In section 3.6 we simulate the residual foreground noise for ET through the subtraction of the individually detectable events. In section 3.7 we discuss the unique time-frequency statistical properties of the CBC background and possible implications. Finally we present our conclusions in section 3.8.

3.2 Astrophysical gravitational wave backgrounds

In this section we summarize the broad range of formalisms used by different authors to calculate $\Omega_{gw}(f)$ of an astrophysical background. We start from Phinney’s practical theorem (Phinney 2001) and compare it with various versions found in the literature. Then we derive a practical model in the general case of astrophysical backgrounds. The reader who is only interested in signal models and the detectability of the CBC background can skip this section and go straight to section 3.3.

Firstly recall that $\Omega_{gw}(f)$ is defined as the GW energy density per logarithmic frequency interval at observed frequency $f$, divided by the critical energy density required to close the Universe today $\rho_c = 3H_0^2c^2/8\pi G$ with $H_0$ the Hubble constant. Assuming a homogeneous and isotropic Universe, it is straightforward to compute
this dimensionless function as (see Phinney 2001 for details):

$$\Omega_{\text{GW}}(f) = \frac{1}{\rho c} \int_{z_{\text{min}}}^{z_{\text{max}}} \frac{N(z)}{(1+z)} \left( \frac{dE_{\text{GW}}}{d\ln \nu_r} \right) \bigg|_{\nu_r = f(1+z)} dz,$$

(3.1)

where $N(z)$ is the spatial number density of GW events at redshift $z$; the factor $(1+z)$ accounts for redshifting of GW energy since emission; $f_r = f(1+z)$ is the GW frequency in the source frame and $dE_{\text{GW}}/d\ln \nu_r$ is the single source energy spectrum. The limits of the integral over $z$ are given by

$$z_{\text{min}} = \max(0, f_r^{\text{min}}/f - 1),$$

(3.2)

$$z_{\text{max}} = \min(z_*, f_r^{\text{max}}/f - 1),$$

(3.3)

where $z_*$ marks the beginning of source formation and $f_r^{\text{min}}$ and $f_r^{\text{max}}$ are the minimum and maximal source rest-frame GW frequency respectively. Note that $f_r^{\text{min}}$, $f_r^{\text{max}}$, and $dE_{\text{GW}}/d\ln \nu_r$ depend on the source parameters (e.g., system mass). This has been mostly neglected in previous studies and should be taken into account in order to fully characterize the background signal.

It is convenient to replace $N(z)$ in equation (3.1) with the differential GW event rate (per unit of cosmic time in the source frame) $d\dot{N}/dz = N(z)c4\pi r_z^2$, where $r_z$ is the comoving distance related to the luminosity distance through $dL = r_z(1+z)$. We then obtain another version:

$$\Omega_{\text{GW}}(f) = \frac{f}{\rho c} \int_{z_{\text{min}}}^{z_{\text{max}}} \frac{1}{4\pi r_z^2} \left( \frac{dE_{\text{GW}}}{df_r} \right) \bigg|_{\nu_r = f(1+z)} d\dot{N} dz.$$

(3.4)

The quantity given by the integration in the above equation, in erg cm$^{-2}$ Hz$^{-1}$ s$^{-1}$, is called the spectral energy density (e.g. Ferrari et al. 1999a; Marassi et al. 2009) or the integrated flux (e.g. Regimbau 2011; Wu et al. 2012). Its dimension shows that it can be related to the specific intensity by integrating the latter over the solid angle. The first two terms inside the integral give the locally measured energy flux per unit frequency (or simply fluence) emitted by a source at redshift $z$ (Flanagan
3.2. Astrophysical backgrounds

\& Hughes [1998]:

\[
\frac{dE_{GW}}{dS df} = \frac{1}{4\pi r_z^2} \left( \frac{dE_{GW}}{df} \right) \bigg|_{f_r=f(1+z)},
\]

(3.5)

while \(d\dot{N}/dz\) can also be written as:

\[
\frac{d\dot{N}}{dz} = \frac{R(z)}{(1+z)} \frac{dV}{dz},
\]

(3.6)

with the comoving volume element \(dV/dz\) given by:

\[
\frac{dV}{dz} = 4\pi c r_z^2 H(z),
\]

(3.7)

where the Hubble parameter is

\[
H(z) = H_0 \sqrt{\Omega_\Lambda + \Omega_m (1+z)^3},
\]

(3.8)

and the comoving distance is

\[
r_z = \int_0^z \frac{c}{H(z')} dz'.
\]

(3.9)

Throughout this thesis, we assume a standard ΛCDM cosmology with parameters \(H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}, \Omega_m = 0.27\) and \(\Omega_\Lambda = 0.73\) (see, e.g. Jarosik et al. 2011).

In equation (3.6) we define \(R(z) = r_0 e(z)\) (see, e.g. Coward et al. 2001; Howell et al. 2004), which gives the rate density measured in cosmic time local to the event. The parameter \(r_0\) is the local rate density, usually used to estimate detection rates for different detectors, and \(e(z)\) is a dimensionless factor which models the source rate evolution over cosmic time. The later is usually associated with the CSFR for stellar catastrophic events.

The factor \((1+z)\) in equation (3.6) converts \(R(z)\) to an earth time based quantity. The statement that such a factor does not exist given in de Araujo \& Miranda (2005) does not change the calculation of \(\Omega_{GW}(f)\) as the factor appears additionally in their equation for \(dE/dSdf\). This caveat has also appeared in other
publications (e.g. Regimbau & Mandic 2008; Zhu et al. 2010, 2011a,b; Howell et al. 2011). We correct it with equations (3.5) and (3.6) since they provide physically correct estimates of the corresponding quantities.

Combining equations (3.4)-(3.7) results in a compact form:

$$\Omega_{GW}(f) = \frac{f}{\rho_c H_0} \frac{r_0}{z_{\text{min}}} \int_{z_{\text{min}}}^{z_{\text{max}}} \frac{e(z)}{(1+z)\sqrt{\Omega_{\Lambda} + \Omega_m (1+z)^3}} \frac{dE_{GW}}{df} \, dz.$$  

(3.10)

Alternatively $$\Omega_{GW}(f)$$ can be calculated through the single-source characteristic amplitude $$h_c(f) = f \langle |\tilde{h}(f)| \rangle$$ with $$\langle |\tilde{h}(f)| \rangle$$ denoting the frequency-domain GW amplitude (in Hz$^{-1}$) averaged over source orientations. In this case one can use the following relation to replace equation (3.5):

$$\frac{dE_{GW}}{dS df} = \frac{\pi c^3}{2G} h_c^2(f).$$  

(3.11)

The average over all source orientations for an inspiraling binary, a rotating NS or a ringing BH is given by (Sathyaprakash & Schutz 2009):

$$\int_0^1 d(\cos \iota) \left[ \left( \frac{1 + \cos^2 \iota}{2} \right)^2 + \cos^2 \iota \right] = \frac{4}{5},$$  

(3.12)

where $$\iota$$ is the inclination angle of the characteristic direction of the source, determined by the orbital or spin angular momentum, with respect to the line of sight.

The one-sided spectral density of a GWB, $$S_h(f)$$, can be conveniently compared with detector sensitivities and is related to $$\Omega_{GW}(f)$$ through (Maggiore 2000):

$$S_h(f) = \frac{3 H_0^2}{2\pi^2 f^{-3}} \Omega_{GW}(f).$$  

(3.13)

Note that assuming an isotropic GWB, a factor of 1/5 should be included to account for the average detector response over all source locations in the sky, when the above

---

1Note that in Phinney (2001) $$h_c$$ is used to represent the characteristic amplitude of the background.
3.2. Astrophysical backgrounds

The equation is used directly to compare $S_h(f)$ with noise power spectral densities of L-shaped interferometers. For instruments with non-perpendicular arms, it becomes $\sin^2 \zeta/5$ with $\zeta$ being the opening angle between the two arms.

Another important quantity of an astrophysical background is the (dimensionless) duty cycle, $\xi$, which describes the degree of overlap of individual signals in the time domain. It can be computed as (see, e.g. Coward & Regimbau 2006):

$$
\xi = \int_0^{z_*} \Delta \tau \frac{d\dot{N}}{dz} dz, \quad (3.14)
$$

where $\Delta \tau$ is the average observed signal duration. A value of $\xi \geq 1$ generally implies a continuous background. Here $\Delta \tau$ is assumed to be frequency independent; if a dependence exists, the upper limit of the integration should be changed to $z_{\text{max}}$ given in equation (3.3). The above defined duty cycle may not be useful if there is significant frequency evolution of $\Delta \tau$, e.g., for CBC sources. This will be further discussed in section 3.7.

3.2.1 A practical model

We now derive a practical model for astrophysical backgrounds formed by sources for which the gravitational energy spectrum can be approximated by a power law function of frequency. Such a case is of particular interest because $\Omega_{\text{GW}}(f) \sim f^\alpha$ is naturally obtained when $dE_{\text{GW}}/df_t = Af_t^{\alpha-1}$ with $A$ being the overall amplitude. Then equation (3.10) has a simple form within the frequency range $f_t^{\text{min}} \leq f \leq f_t^{\text{max}}/(1 + z_*)$:

$$
\Omega_{\text{GW}}(f, \alpha) = \frac{A}{\rho_c H_0} f^\alpha J(\alpha), \quad (3.15)
$$

where we have defined a dimensionless function:

$$
J(\alpha) = \int_0^{z_*} \frac{e(z)(1 + z)^{\alpha-2}}{\sqrt{\Omega_\Lambda + \Omega_m (1 + z)^3}} dz. \quad (3.16)
$$
Figure 3.1: The dimensionless rate evolution factor $e(z)$ based on different parameterized models of CSFR. SF2 is taken from Porciani & Madau (2001), and we use the “low rate” gamma ray bursts derived model of Robertson & Ellis (2012).

Note that for $f_{\text{min}} \leq f \leq f_{\text{max}}/(1 + z_*)$, the lower and upper limit of the above integral become 0 and $z_*$ respectively according to equations (3.2-3.3). In this frequency range equation (3.15) strictly holds.

We define $e(z) = \dot{\rho}_*(z)/\dot{\rho}_*(0)$, where $\dot{\rho}_*(z)$ is the CSFR density (in $M_\odot \text{yr}^{-1} \text{Mpc}^{-3}$). The assumption made here is that the GW event rate closely tracks the CSFR, e.g., in core collapse supernovae related mechanisms; otherwise effects of delay times should be included as we show in section 3.3.1 for CBC events. Note that $r_0$ is equivalent to the parameter $\lambda$ used in some studies to represent the fraction of stellar mass converted to GW source progenitors (Regimbau & Mandic 2008; Wu et al. 2012). As estimates of $r_0$ do not normally rely on measurements of $\dot{\rho}_*(0)$, rather they can be based on independent observations or theoretical calculations, we choose to treat $r_0$ as a free parameter, independent on the CSFR models throughout this work.
In this work we consider five parameterized forms of $\dot{\rho}_*(z)$ derived from various observations (see Porciani & Madau 2001, Hopkins & Beacom 2006, Fardal et al. 2007, Wilkins et al. 2008, Robertson & Ellis 2012, for details). The corresponding models of $e(z)$ are shown in Figure 3.4. We therefore set $z_*$ as the maximal redshift for which the CSFR model is applicable: $z_* = 15$ for the recent study of Robertson & Ellis (2012) which is derived from gamma ray burst observations and $z_* = 6$ for the other four models.

Figure 3.2 (upper panel) shows $J(\alpha)$ calculated for $\alpha = [0, 5]$ using the five models of $e(z)$. Since all current predictions of astrophysical backgrounds in the frequency band of terrestrial detectors indicate that $\Omega_{GW}(f)$ increases from about 10 Hz to several hundred Hz (see, e.g., Figure 6 of Regimbau 2011), the chosen range of $\alpha$ is adequate for most of possible scenarios, e.g., $\alpha = 2/3$ for inspiraling compact binaries as mentioned earlier, $\alpha = 2$ for NS r-mode instabilities (Owen et al. 1998, Ferrari et al. 1999b, Zhu et al. 2011a), and $\alpha = 4$ for magnetars (Regimbau & de Freitas Pacheco 2006a, Marassi et al. 2011a, Wu et al. 2013). The five curves of $J(\alpha)$ are within a factor of 2 around the average, for which a least-squares fit is $\log[J(\alpha)] = 0.04\alpha^2 + 0.3\alpha + 0.35$.

For a power law energy spectrum, the total GW energy emitted in the frequency range $(f_r^{\min}, f_r^{\max})$ is $\Delta E_{GW} = A \int f^{\alpha-1}df$. We further define a dimensionless function:

$$K(\alpha) = A (100 \text{ Hz})^\alpha \frac{J(\alpha)}{\Delta E_{GW}},$$

(3.17)

to obtain a practical form:

$$\Omega_{GW}(f, \alpha) = 10^{-9} \left( \frac{r_0}{1 \text{ Mpc}^{-3}\text{Myr}^{-1}} \right) \left( \frac{\Delta E_{GW}}{0.01 \text{ M}_\odot c^2} \right) \left( \frac{f}{100 \text{ Hz}} \right)^\alpha K(\alpha).$$

(3.18)

The function $K(\alpha)$ obtained while arbitrarily setting $f_r^{\min} = 10$ Hz and $f_r^{\max} = 1000$ Hz is shown in the lower panel of Figure 3.2. The least-squares fit of the average over the five models of $e(z)$ is given by $K(\alpha) = (1.2 - 0.04\alpha)/(\alpha^2 - 1.1\alpha + 2.4)$. Note that the chosen values of $(f_r^{\min}, f_r^{\max})$ correspond to a frequency band to which
ground-based detectors are most sensitive; one can calculate \( K(\alpha) \) for a specific type of source using equation (3.17). Figure 3.2 implies that: a) as \( \alpha \) increases, the high redshift (\( z \gtrsim 4 \)) sources contribute more to the background; b) effects of the CSFR introduce uncertainties in the overall amplitudes of \( \Omega_{GW}(f) \) within a factor of about 2 for \( \alpha \lesssim 3 \) and up to 5 for larger \( \alpha \).

Combining equations (3.18) and (3.13) gives \( S_h(f) \) in a convenient form:

\[
S_h^{1/2}(f, \alpha) = 1.3 \times 10^{-26} \text{ Hz}^{-\frac{1}{2}} \left( \frac{f}{100 \text{ Hz}} \right)^{\frac{\alpha-3}{2}} [K(\alpha)]^{\frac{1}{2}} \left( \frac{r_0}{1 \text{ Mpc}^{-3} \text{ Myr}^{-1}} \right)^{\frac{1}{2}} \left( \frac{\Delta E_{GW}}{0.01 M_{\odot} c^2} \right)^{\frac{1}{2}}.
\]

Similarly, a convenient relation between the duty cycle \( \xi \) of an astrophysical background and \( \Delta \tau \) and \( r_0 \) can be obtained by combining equations (3.6), (3.7)
3.2. Modelling the CBC background: analytical approaches

and (3.14) and averaging over the five models of $e(z)$ shown in Figure 3.1:

$$\xi = 0.2 \left( \frac{\Delta \tau}{1 \text{ sec}} \right) \left( \frac{r_0}{1 \text{ Mpc}^{-3} \text{Myr}^{-1}} \right).$$ (3.20)

Such a relation shows whether or not a continuous GWB is formed by one particular type of sources.

Equations (3.15)-(3.19) represent our practical power law model for astrophysical backgrounds. The power law relation holds for the frequency range $[f_{\text{min}}^r, f_{\text{max}}^r/(1+z_*)]$, where changes in rate evolutionary histories only affect the overall amplitude of the background. The model allows quick evaluation of the background signal strength and its uncertainty using estimates of $r_0$ and $\Delta E_{\text{GW}}$ (which are also essential for back-of-the-envelope predictions of single-source detection prospects). As our knowledge improves the model can be easily modified to provide templates for future stochastic background searches. In the following sections we will develop a ready-to-use model for the CBC background by considering additional issues that have not been considered here.

3.3 Modelling the gravitational wave background from compact binaries: analytical approaches

In this section we extend the derivation in subsection 3.2.1 to obtain models for CBC events analytically.

3.3.1 A simple power law model

Previous calculations of $\Omega_{\text{GW}}(f)$ for the CBC background have employed the Newtonian inspiral energy spectrum, with the exception of the BBH population (Zhu et al. 2011b; Marassi et al. 2011b; Wu et al. 2012). Following the previous derivation, we present here a simple power law model generalized for three CBC populations.
In the Newtonian limit, the energy spectrum for an inspiralling circular binary of component masses $m_1$ and $m_2$ is given by (see, e.g. Thorne 1987):

$$\frac{dE_{GW}}{df_r} = \frac{(\pi G)^{2/3} M_c^{5/3}}{3} f_r^{-1/3}, \quad (3.21)$$

where $M_c$ is the chirp mass defined as $M_c = M_\eta^{5/3}$, with $M = m_1 + m_2$ the total mass and $\eta = m_1 m_2 / M^2$ the symmetric mass ratio. Inserting this into equation (3.10) and combining the expression of $\rho_c$ gives ($f_r^{\text{min}} \leq f \leq f_r^{\text{max}}/(1 + z_*)$):

$$\Omega_{GW}(f) = \frac{8}{9} \frac{1}{c^2 H_0^2 r_0} (\pi GM_c)^{5/3} f^{2/3} J_{2/3}, \quad (3.22)$$

where we have defined a dimensionless quantity:

$$J_{2/3} = \int_0^{z_*} \frac{e(z)(1 + z)^{-4/3}}{\sqrt{\Omega_\Lambda + \Omega_m(1 + z)^3}} dz. \quad (3.23)$$

To determine the applicable frequency range of the above power law relation, one has $f_r^{\text{min}}$ well below 1 Hz and $f_r^{\text{max}}$ given by the frequency at the last stable orbit (LSO) during inspiral $f_{\text{LSO}} \simeq 4400\,\text{Hz}/M$ with $M$ in units of $M_\odot$.

The newly defined quantity $J_{2/3}$ differs from $J(2/3)$ as given in equation (3.16) in the definition of $e(z)$: for CBC events, effects due to the delay time $t_d$ between the formation and the final merger of binaries should be taken into account. By assuming compact binary formation closely tracks the cosmic star formation, we define $e(z) = \dot{\rho}_{s,c}(z)/\dot{\rho}_{s,c}(0)$ by introducing a $\dot{\rho}_s$-related quantity:

$$\dot{\rho}_{s,c}(z) = \int_{t_{\text{min}}}^{t_*} \dot{\rho}_s(z_f) \frac{dt_f}{dz} P(t_d) \, dt_d, \quad (3.24)$$

where $P(t_d)$ and $t_{\text{min}}$ denote the probability distribution for and minimum value of $t_d$ respectively. The upper limit of the integral $t_*$ corresponds to $z_*$. For CBC events, $P(t_d)$ follows a $1/t_d$ form as suggested by latest population-synthesis stud-

---

\footnote{We note that $P(t_d)$ of type Ia supernovae was observationally found to be consistent with the $1/t_d$ predictions for progenitors of the GW induced binary white dwarf mergers (Graur et al. 2011; Maoz et al. 2012).}
3.3. Modelling the CBC background: analytical approaches

ies on compact binary evolution (Dominik et al. 2012). The parameters \(z\) and \(z_f\) are the redshifts when a GW event occurred and the system was initially formed respectively, with corresponding time coordinates \(t_z\) and \(t_f\). In our fiducial cosmology, \(t_d\) is given by the lookback time between \(z\) and \(z_f\), integrating \(dz' / [(1 + z')H(z')]\) from \(z\) to \(z_f\). The term \(dt_f / dt_z = (1 + z) / (1 + z_f)\) is included to convert a rate at \(t_f (\dot{\rho}_*(z_f))\) to the one local to \(t_z (\dot{\rho}_*, s(z))\).

It should be mentioned that our equation (3.24) is equivalent to equation (2) of Regimbau & Hughes (2009) by noting that the \((1 + z)\) factor cancels with the one in equation (3.6). The additional \((1 + z)\) term in equation (9) of Zhu et al. (2011b) was an error, and led to a factor of 2 underestimate of the BBH background signal.

For the five considered CSFR models and for a minimum delay time \(t_{\text{min}}\) in the range of \(10 - 100\) Myr, \(J_{2/3}\) is well constrained within \((1.3 - 2.6)\). It is roughly a factor of 2 smaller than \(J(2/3)\) as given by equation (3.16) and shown in Figure 3.2 where no time delay is assumed. Our selected range of \(t_{\text{min}}\) is largely consistent with results presented in Dominik et al. (2012); see their figures 14–17 for details. We note, however, that in some extreme cases \(t_{\text{min}}\) for BBHs could be much higher, e.g., 500 Myr. This does not change our results significantly as we will show below.

For the commonly used CSFR of Hopkins & Beacom (2006), \(J_{2/3}\) as a function of \(t_{\text{min}}\) (in Myr) can be expressed as:

\[
J_{2/3}(t_{\text{min}}) = 3.67 - 0.85 (t_{\text{min}})^{0.165},
\]

(3.25)

for \(10\) Myr \(\leq t_{\text{min}} \leq 500\) Myr; increasing \(t_{\text{min}}\) from 100 to 500 Myr reduces \(J_{2/3}\) from 1.85 to 1.3.

Replacing the constants with their numerical values, equation (3.22) becomes:

\[
\Omega_{\text{GW}}(f) = 9.1 \times 10^{-10} \left( \frac{r_0}{1\text{Mpc}} \right) \left( \frac{M_c^{5/3}}{1\text{M}_\odot} \right) \frac{J_{2/3}}{2} \left( \frac{f}{100\text{Hz}} \right)^{2/3}.
\]

(3.26)

Here we have replaced \(M_c^{5/3}\) in equation (3.22) with \(\langle M_c^{5/3} \rangle\) to account for a distribution of system masses – the consideration of \(\langle M_c^{5/3} \rangle\) rather than \(\langle M_c \rangle^{5/3}\) is based
on the fact that $\Omega_{GW}(f)$ is an average over individual energy spectra characterized by $M_c^{5/3}$. As the differences between the two quantities are very small (as we will show in Table 3.2), we do not attempt to distinguish between them and will use the term average chirp mass. Note that the CBC background signal contains information about the physical chirp mass\footnote{although its effect is indistinguishable from variations in $r_0$.} while single event detections normally measure the redshifted chirp mass $M_c(1+z)$ (Cutler & Flanagan 1994).

We have reviewed calculations of $\Omega_{GW}(f)$ for the CBC background through a simple power law model. The model extends that of Phinney (2001) by considering different rate evolutionary histories and by combining uncertainties associated with CSFRs and delay times into a single parameter $J_{2/3}$. In the next subsection we will introduce some additional inputs to produce more accurate estimates.

### 3.3.2 Beyond a simple power law

We consider new information in two aspects to refine previous estimates:

1. Observation-based parameterized models of NS and BH mass distribution – through a Monte-Carlo simulation (section 3.4); we will in subsection 3.4.1 investigate how the spectral shape of the background depends on the mass distributions;

2. Up-to-date complete waveforms for populations of BNS, BBH and BH-NS systems – these will show how well a CBC background can be approximated by a simple power law model in the audio band.

The main parameters are summarized in Tables 3.1 and 3.2, for which we provide an overview below. Unless we otherwise specify, we will use the information contained in these two tables, and the CSFR of Hopkins & Beacom (2006) in the following sections.
3.3. Modelling the CBC background: analytical approaches

3.3.2.1 Observational inputs

We consider the parameterized models of NS and BH mass distribution recently derived from observational mass measurements. For NS, that are observed in double NS systems (with one or two pulsars), high-precision mass measurements are available (see Table 1 in Özel et al. 2012 and references therein), indicating a very narrow distribution. Using the observational data for the 6 double NS systems, Özel et al. (2012) found that the NS mass distribution can be well described by a Gaussian with a mean $\mu = 1.33 M_\odot$ and a standard deviation $\sigma = 0.06 M_\odot$.

In contrast to the consensus on the narrowness of the NS mass distribution, the BH mass measurements are subject to much larger uncertainties, leading to a greater range in inferred distribution. Utilizing the maximal amount of observational information available for 16 BH in transient low-mass X-ray binaries, Özel et al. (2010) concluded that the underlying mass distribution can be best described by a Gaussian with $\mu = 7.8 M_\odot$ and $\sigma = 1.2 M_\odot$. More recently, Farr et al. (2011) considered a broad range of parameterized models; using a Bayesian model selection analysis, they found that a Gaussian and a power law distribution are preferred for low-mass X-ray binaries, whereas an exponential distribution and a two-Gaussian model are favored if 5 high-mass, wind-fed X-ray binary systems were included (see Farr et al. 2011 for details).

Unless stated explicitly our considered mass (in $M_\odot$) interval in Table 3.1 for NS (BH) is $[1, 2]$ $([4, 40])$. Given the adopted models of distributions, it is highly unlikely to obtain masses outside these intervals. We note that the existence of a “gap” between the maximum NS mass and the lower bound of observationally inferred BH masses has been suggested in Özel et al. (2010) and Farr et al. (2011). Such a “gap” can not be attributed to observational selection effects as concluded in the former paper.

The BH spin distribution is highly uncertain – currently there have been only about 10 stellar mass BHs with (model dependent) spin estimates available (Miller et al. 2009; McClintock et al. 2011). Considering recent results on the determination
of the extreme spin of the BH in Cygnus X–1 (Gou et al. 2011, Fabian et al. 2012), we assume a uniform distribution with spin parameter \( \chi = S_a/m^2 \) between −0.95 and 0.95, where \( S_a \) is the spin angular momentum and \( m \) is the BH mass and positive or negative sign of \( \chi \) implies alignment or anti-alignment between component spin and orbital angular momentum. As most NSs are observed to be weakly spinning (Manchester et al. 2005), and the fastest spinning NS in binary pulsar systems, PSR J0737–3039A, has a spin period of 22.70 ms (Burgay et al. 2003b) and equivalently \( \chi \sim 0.05 \) (Brown et al. 2012), we neglect the spin of NSs in our analysis.

Observational NS and BH mass measurements were also used as inputs or calibrations in the population-synthesis simulations adopted by Marassi et al. (2011b) and Kowalska-Leszczynska et al. (2015). Results of these studies are based on chirp mass distributions of some simulated populations of CBC sources. We assume in this work that components of coalescing compact binaries follow the observational mass/spin distributions. Note that: a) for BNS, simulated chirp mass distribution presented in Dominik et al. (2012) is also very narrow and should give similar results to what we will obtain in the following sections; b) our adopted BH mass/spin distributions only apply to BHs in X-ray binaries and may not be representative for BBH and BH-NS systems.

### 3.3.2.2 Up-to-date analytical complete waveforms

The evolution of a CBC source can be generally divided into three phases: inspiral, merger and ringdown. While the early inspiral and ringdown phases can be approximated analytically by PN expansion and perturbation theory respectively, to model the late inspiral and merger process requires a numerical solution of the Einstein field equations. In the last few years breakthroughs in numerical relativity have enabled the inspiral-merger-ringdown evolution of BBHs to be modelled with high accuracy for a broad space of parameters (see, e.g. Hughes 2009, Hannam 2009, Hinder 2010, for reviews).
Table 3.1: NS/BH mass ($m$) and spin ($\chi$) distribution.

<table>
<thead>
<tr>
<th></th>
<th>$\chi$</th>
<th>$m$</th>
<th>$\langle m \rangle$</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS</td>
<td>$\cdots$</td>
<td>$N(1.33, 0.06)$</td>
<td>1.33</td>
<td>Özel et al. (2012)</td>
</tr>
<tr>
<td>BH</td>
<td>$U(-0.95, 0.95)$</td>
<td>$N(7.8, 1.2)$</td>
<td>7.8</td>
<td>Özel et al. (2010)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Power law</td>
<td>7.35</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exponential</td>
<td>10</td>
<td>Farr et al. (2011)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Two-Gaussian</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Notes: All values of mass are in $M_\odot$. $N(\mu, \sigma)$ implies a Gaussian distribution with a mean $\mu$ and a standard deviation $\sigma$; $U(a, b)$ is a uniform distribution between $a$ and $b$. The upper/lower bound of BH spin corresponds to the recently determined extreme spin of the BH in Cygnus X–1 (Gou et al. 2011; Fabian et al. 2012); positive or negative $\chi$ implies alignment or anti-alignment between component spin and orbital angular momentum. References for mass distributions: (1) Özel et al. (2012); (2) Özel et al. (2010) – Gaussian BH mass, which is used as our fiducial model; (3) Farr et al. (2011) – for the other three models of BH mass distribution, and we use the median values of parameters given in the paper: Power law – $P(m) \sim m^{-6.4}$ for $6 \leq m \leq 23$; Exponential – $P(m) \sim e^{-m/m_0}$, with $m_0 = 4.7$ for $m \geq 5.33$; Two-Gaussian – $N(7.5, 1.3)$ and $N(20.4, 4.4)$ with weights 0.8 and 0.2 respectively. The power law model in Farr et al. (2011) has a slightly lower mean $\mu = 7.35M_\odot$ due to the exclusion of one low-mass X-ray binary system in their analysis as compared to Özel et al. (2010).
Table 3.2: Information about CBC populations used in this work.

<table>
<thead>
<tr>
<th>Waveform</th>
<th>( r_0 ) (Mpc(^{-3})Myr(^{-1}))</th>
<th>( t_{\text{min}} ) (Myr)</th>
<th>( \langle M_{c}^{5/3} \rangle ) (( M_{\odot}^{5/3} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>BNS</td>
<td>TaylorT4</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BH-NS</td>
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<td>30</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BBH</td>
<td>IMR</td>
<td>0.005</td>
<td>50</td>
</tr>
<tr>
<td></td>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>

Notes: IMR – the phenomenological inspiral-merger-ringdown waveform for non-precessing spinning BBHs presented in Ajith et al. (2011); we also use this model for BH-NS as an approximation to the type-II spectrum found in numerical simulations (Shibata & Taniguchi 2011). For BNS waveform we adopt the TaylorT4 formula with 3.0 PN amplitude accuracy given in Blanchet et al. (2008). Values of \( r_0 \) correspond to the realistic estimates in Abadie et al. (2010c). \( t_{\text{min}} \) given here is used as the fiducial value, based on the standard Submodel A for solar metallicity \( Z_{\odot} \) in Dominik et al. (2012) – see Figure 8 therein; we also consider a range of 10-100 Myr to account for uncertainties. The quantities \( \langle M_{c}^{5/3} \rangle \) are calculated using mass distributions presented in Table 3.1 and assuming component masses are uncorrelated and follow the same distribution for BNS and BBH; four values for BH-NS and BBH are given in order from top to bottom as for a Gaussian, Power law, Exponential and Two-Gaussian BH mass distribution. We note that the quantity \( \langle M_{c}^{5/3} \rangle \) is smaller than \( \langle M_{c}^{5/3} \rangle \) by < 1% for BNS and the first two entries of BH-NS and BBH, and about 2% (4%) for the other BH-NS (BBH) values – we go with the latter quantity throughout this Chapter, but also use the former when comparing with other studies (in which case we neglect their differences).
3.3. Modelling the CBC background: analytical approaches

For the complete evolution history of coalescing BBHs, phenomenological waveforms can be constructed by frequency domain matching of post-Newtonian inspiral waveforms with coalescence waveforms from numerical simulations (Ajith et al. 2007; Buonanno et al. 2007; Pan et al. 2008; Santamaría et al. 2010). Such waveforms share a common feature, in that the Fourier amplitude is approximated to a leading order as a power law function of frequency $f^{-7/6}$ for the inspiral phase, followed by $f^{-2/3}$ for the merger stage and a Lorentzian function around the quasi-normal mode frequency for the ringdown stage.

Here we briefly describe some features of such phenomenological waveform models. First we convert the Fourier amplitude given by equation (4.13) in Ajith et al. (2008) to an energy spectrum $dE_{GW}/df_r$ for non-spinning BBHs. As we expect the inspiral spectrum to equal the Newtonian approximation (see, e.g. Cutler et al. 1993; Finn & Chernoff 1993), the inspiral-merger-ringdown spectrum for a BBH with component masses $m_1$ and $m_2$ is given by:

$$dE_{GW} \equiv \frac{(G\pi)^{2/3} M_\odot^{5/3}}{3} \begin{cases} f_r^{-1/3} & \text{if } f_r < f_1 \\ \omega_1 f_r^{2/3} & \text{if } f_1 \leq f_r < f_2 \\ \omega_2 \left[ \frac{f_r}{1 + \left( \frac{f_r - f_3}{\sigma/2} \right)^2} \right]^{2} & \text{if } f_2 \leq f_r < f_3 \end{cases}$$

(3.27)

Here $\omega_1 = f_1^{-1}$ and $\omega_2 = f_1^{-1} f_2^{-4/3}$ are constants chosen to make $dE/df$ continuous across $f_1$ and $f_2$. The set of parameters $(f_1, f_2, \sigma, f_3)$ can be determined by the two physical parameters (the total mass $M$ and the symmetric mass ratio $\eta$) in terms of $(a\eta^2 + b\eta + c)/\pi M$, with coefficients $a, b, c$ given in Table 1 of Ajith et al. (2008), producing (404, 807, 237, 1153) Hz for a $10M_\odot-10M_\odot$ BBH ($M_\odot = 8.7M_\odot$).

The waveform presented in Ajith et al. (2011) includes spin effects through a single spin parameter $\chi = (1 + \delta)\chi_1/2 + (1 - \delta)\chi_2/2$, with $\delta = (m_1 - m_2)/M$. 


Chapter 3. Gravitational Wave Background from Binaries

Figure 3.3: GW energy spectra for a $10M_\odot - 10M_\odot$ coalescing BBH in the non-spinning case and cases for non-precessing spins with three values of the single spin parameter $\chi$.

and $\chi_i = S_i/m_i^2$. The parameter $S_i$ represents the spin angular momentum of the $i$th BH. The corresponding Fourier amplitude includes a minor correction (related to $\chi$ and $\eta$) for non-spinning BBHs. We construct energy spectra for BBHs with non-precessing spins based on equation 1 in Ajith et al. (2011).

Figure 3.3 shows the GW energy spectra for a $10M_\odot - 10M_\odot$ BBH assuming the non-spinning case, $\chi = 0.85$, $\chi = 0$ and $\chi = -0.85$. The two extreme values for $\chi$ are set by the numerical simulations of Ajith et al. (2011), corresponding to both binary components having maximal spins aligned or anti-aligned with the orbital angular momentum. The radiation efficiencies for these energy spectra are 6.7%, 9.74%, 5.15% and 4.28% respectively. We note that the radiated GW energy mainly depends on $M_c$ and $\chi$, and that the energy spectra for $f \lesssim 100$ Hz are largely comparable.
3.3. Modelling the CBC background: analytical approaches

For this work, we use the phenomenological inspiral-merger-ringdown waveforms for non-precessing spinning BBH presented in Ajith et al. (2011). In this model the TaylorT1 waveform is adopted for the inspiral phase, with 1.5\textsuperscript{PN} order amplitude corrections to the Newtonian waveform (Arun et al. 2009). We note that the waveform model is calibrated against numerical relativity simulations in the parameter range of mass ratios between 1 and 4 and $\chi$ between -0.85 and 0.85, but we employ it for slightly broader parameter space. Our calculations can be improved once more accurate and general models become available.

As no phenomenological complete waveforms are currently available for BNS and BH-NS systems, we consider analytical models that approximate the waveforms given by numerical relativity simulations. For BH-NS, we use the same model as that for BBH. The justification for our choice is two-fold. Firstly, the type-II spectrum found in numerical simulations is similar to that of a BBH with the same mass ratio (Shibata et al. 2009; Shibata & Taniguchi 2011), showing a clear signature of inspiral, merger and ringdown. This happens primarily for larger mass ratios ($\gtrsim 3 - 5$) when the smaller NS is simply swallowed by the BH. For the NS and BH mass distribution used in this work this condition is largely fulfilled. Secondly, PN amplitude corrections and effects of BH spins can be included by using the adopted BBH model.

For BNS, we use the TaylorT4 point-particle waveform with 3.0\textsuperscript{PN} order amplitude accuracy (Blanchet et al. 2008). We apply the waveform up to 5000 Hz to account for a realistic cutoff of the complete spectrum. Comparisons between the TaylorT4 waveform and numerical relativity results generally indicate that the former underestimates the post-merger emission (Kiuchi et al. 2009; Faber & Rasio 2012). Recently Bauswein et al. (2012) found that the generic outcome of two $1.35M_\odot$ NS mergers is the formation of a deformed differentially rotating massive NS and that violent oscillations of the merger remnants lead to a pronounced peak in the GW spectra. We note that the peaks shown in this work are sharper than results obtained in full general relativistic simulations (see, e.g. Kiuchi et al. 2009).
Figure 3.4: The GW energy spectrum for a BNS of equal mass $1.33M_\odot$ calculated using the TaylorT4 formula. The post-merger signal is represented (optimistically) by a Gaussian spectrum centered at around 2 kHz. Also shown are curves of $f^{-1/6}$ and $f^{-1/3} - f^{-1/6}$ corresponds to the Newtonian inspiral spectrum given by equation (3.21) and $f^{-1/3}$ shows the gradient at around 1 kHz. The curves are displayed as the square root of $dE_{GW}/df$ in order to be directly comparable to the quantity $h_{\text{eff}} = f|\tilde{h}(f)|$ commonly used in the numerical relativity community.

Rezzolla et al. (2010), largely due to a different numerical treatment. Our codes to compute the waveform models considered in this work are included in Appendix B.

Figure 3.4 shows the energy spectra for a BNS of equal mass $1.33 M_\odot$. We consider a simple Gaussian spectrum to investigate the possible contribution from the post-merger emission to the GWB. Following Zhu et al. (2010), we take the form of $dE_{GW}/df = A \exp[-(f - f_{\text{peak}})^2/2\Delta^2]$ where $A$ arbitrarily set to be twice that of TaylorT4 waveform at 1000 Hz, $f_{\text{peak}} = 1840$ Hz and $\Delta = 250$ Hz; $f_{\text{peak}}$ corresponds to the lowest value given in Table 2 of Bauswein et al. (2012) and we use a much higher width $\Delta$. The chosen parameters give a optimistic representation
of post-merger emission because: a) depending on the equation of state, the peak frequency can be higher (up to about 4 kHz), together with narrower peaks, making it harder to detect (in terms of both single events and the contribution to a GWB); b) for the case of prompt BH formation (mainly for larger binary masses) the peaks are much smaller. Due to these uncertainties, the above mentioned Gaussian spectrum will be used only for semi-analytical calculations presented in the next subsection.

### 3.3.3 Semi-analytical results

Figure 3.5 compares three models of $\Omega_{GW}(f)$, using a NS (BH) mass of $1.33 (7.8) M_\odot$ and zero BH spin, and assuming that sources of each population have the same mass/spin values:

1. A semi-analytical model calculated using equation (3.10) with complete waveforms described in subsection 3.3.2.2.

2. A Newtonian model based on equations (3.10) and (3.21), and assuming $f_r^{\text{max}} = f_{\text{LSO}}$.

3. A simple power law model based on equations (3.25) and (3.26) with an upper frequency cutoff $f_{\text{LSO}}/5$. Note that: a) an exact power law relation applies only for $f \leq f_{\text{LSO}}/(1+z_*)$ with $z_* = 6$; we empirically set the cutoff at $f_{\text{LSO}}/5$ since the function inside of the integral in equation (3.23) has negligible values for $z \geq 4$; b) As mass distributions are not considered here, $\langle M_c^{5/3} \rangle$ in equation (3.26) becomes $M_c^{5/3}$ with $M_c$ determined by two component masses mentioned above.

The following features can be observed from Figure 3.5: a) Newtonian models can be perfectly described by simple power law models up to $f_{\text{LSO}}/5$, about 300 Hz, 80 Hz and 60 Hz for BBH, BHNS and NS respectively, as suggested in the previous paragraph; b) Semi-analytical models start to drop slightly below a $f^{2/3}$ power law from a few tens Hz due to PN amplitude corrections; c) Newtonian...
Figure 3.5: The energy density parameter $\Omega_{GW}(f)$ of the GWBs formed by three CBC populations (BNS, BH–NS, and BBH) calculated using complete waveforms (semi-analytical), compared with Newtonian models and simple power law models $f^{2/3}$ (see text). For BNS, the bump at around 1 kHz corresponds to the (optimistic) contribution from the post-merger emission represented as a Gaussian spectrum shown in Figure 3.4. The BBH curves are scaled up by a factor of 4 to separate the three groups of curves, which is also the case in Figures 3.6-3.8. Note that the relative amplitudes of the GWBs are mainly determined by the local coalescence rate density (see Table 3.2).
models give incorrect peaks and the followed abrupt decline because of the exclusion of post-inspiral emission.

For BNS we specifically show that: a) the power law index of $\Omega_{GW}(f)$ drops from $2/3$ ($f \lesssim 100$ Hz) to $1/3$ before peaking at around 1-2 kHz; b) if the post-merger emission is included in the form of Gaussian spectra, the peak of $\Omega_{GW}(f)$ can be considerably enhanced while the low frequency part ($\lesssim 300$ Hz) stays at the same level. We will show, however, in section 3.5 that the contribution from post-merger emission to the background is unlikely to be detectable even with ET.

Figure 3.6 shows $\Omega_{GW}(f)$ of the semi-analytical models using 10 different forms of $e(z)$ based on the five CSFR models and two minimum delay times $t_{min} = 10, 100$ Myr. Two main results are: a) for each population, different curves follow the same gradient up to 100–200 Hz. This is in agreement with our derivation in subsection 3.3.1 where we show that within this frequency range effects of the CSFR and delay times are linear. However, their effects would be indistinguishable from that of $r_0$ and average chirp mass. Furthermore, there is a degeneracy between the CSFR and $t_{min}$: to break this degeneracy a fully reconstructed $e(z)$ and precise CSFR measurements are required; the former could become possible if we can efficiently detect most of the individual events out to high redshift, e.g., as we will show in section 3.6 for the BBH population; b) the only distinguishable feature comes from the adoption of CSFR model of Robertson & Ellis (2012); the relatively high CSFR from $z = 3$ up to $z = 15$ shifts the peaks of $\Omega_{GW}(f)$ to lower frequencies and suppresses the post-peak amplitudes.

### 3.4 Monte-Carlo simulations

In this section we describe a Monte-Carlo simulation approach to calculate $\Omega_{GW}(f)$ of an astrophysical background. This will allow us to investigate two important aspects of the CBC background in the next two sections:

1. The dependency of $\Omega_{GW}(f)$ on NS and BH mass distributions;
Figure 3.6: As in Figure 3.5, but only shows the semi-analytical models calculated for 10 different rate evolution models based on the five CSFRs shown in Figure 3.1 and two minimum delay times $t_{\text{min}} = 10, 100$ Myr. There is a degeneracy between CSFR model and $t_{\text{min}}$ below 100 Hz and the unique signature at around the peaks is due to the CSFR model in Robertson & Ellis (2012).
2. How much of the [CBC] background can be removed through single-source
detections to allow greater accessibility to primordial [GWB]s from the Big
Bang.

Combining equations (3.4)-(3.7) and (3.11) yields:

$$\Omega_{GW}(f) = \frac{1}{\rho_c H_0} \frac{2\pi^2 c^3}{G} \int_{z_{\text{min}}}^{z_{\text{max}}} f^3 \langle |\tilde{h}(f)| \rangle^2 g(z) \, dz,$$

where we have defined

$$g(z) = \frac{r_z^2 e(z)}{(1 + z)\sqrt{\Omega_\Lambda + \Omega_m (1 + z)^3}}$$

assuming that there is no correlation between the rate evolution and source intrinsic
parameters.

The discrete version of the integration in equation (3.28) is a sum over events
distributed in redshift, leading to:

$$\Omega_{GW}(f) = \frac{1}{\rho_c H_0} \frac{2\pi^2 c^3}{G} \frac{1}{N_{\text{mc}}} \sum_{i=1}^{N_{\text{mc}}} f^3(\Theta_i, z_i) |\langle \tilde{h}(f; d_L, \Theta_i) \rangle|^2 \frac{g(z_i)}{P(z_i)},$$

where $i$ denotes the $i$-th event; $\Theta_i$ contains the intrinsic source parameters, which in
our case includes binary component masses $m_1^i$ and $m_2^i$, and BH spin parameters
$\chi_1^i$ and $\chi_2^i$; the parameter $d_L$ is the luminosity distance, given by $r_z(1 + z)$; the
function $P(z)$ is the probability distribution function of source redshift $z$; and $N_{\text{mc}}$ is the number of events in our Monte-Carlo simulations, chosen to be $10^6$
– the approximate expected number of [BNS] merger events within $z_*$ in one-year
observation. Note that: a) $\Omega_{GW}(f)$ does not depend on $N_{\text{mc}}$ or an observation
time; b) $1/[N_{\text{mc}}P(z)]$ plays the role of $dz$ in the integration of equation (3.28) –
$1/P(z)$ is essentially a weight used when calculating the average over individual
sources; it simply becomes the length of integration ($z_{\text{max}} - z_{\text{min}}$) without any prior
knowledge of source redshift distribution, e.g., for semi-analytical integrations of
section 3.3 and other similar studies.
Chapter 3. Gravitational Wave Background from Binaries

To investigate the detectability of individual events and show how much of the CBC background can be removed through the subtraction of detected events (in section 3.6), we adopt the so-called effective distance $D_{\text{eff}}$, which is related to $d_L$ through (Allen et al. 2012):

$$D_{\text{eff}} = d_L \left[ \frac{F_+^2 \left( \frac{1 + \cos^2 \iota}{2} \right)^2 + F_x^2 \cos^2 \iota}{1/2} \right], \quad (3.31)$$

where $F_+$ and $F_x$ are the antenna pattern functions for + and × polarized GWs respectively, depending on source position with respect to the detector (described by the right ascension $\alpha$ and declination $\delta$ of the source) and the polarization angle $\psi$; $\iota$ is the inclination angle (see equations 2.5 and 2.6). When averaging over uniformly distributed $\alpha$, $\delta$ and $\psi$, one obtains $\langle F_+^2 \rangle = \langle F_x^2 \rangle = \sin^2 \zeta/5$ with $\zeta$ being the opening angle between the two arms of the laser interferometer (see, e.g. Maggiore 2000).

An initial step in performing a Monte-Carlo simulation is to construct probability distribution functions of parameters. Here we use the NS-BH mass and BH spin distributions given in Table 3.1 and further assume that $m_1$ and $m_2$, $\chi_1$ and $\chi_2$ are uncorrelated. The parameters $\cos \alpha$, $\delta/\pi$, $\psi/\pi$ and $\cos \iota$ are all uncorrelated and uniformly distributed over $[-1, 1]$, where the consideration with $\cos \alpha$ and $\cos \iota$ ensures that individual sources and the direction of their orbital angular momentum are uniformly distributed on a spherical surface. The function $P(z)$ is obtained by normalizing the differential event rate given in equation (3.6).

Our final results are obtained using an average of 10 independent realisations of the Monte-Carlo simulation. Numerical error in our simulation, defined as the relative variation (between each realisation and the average) of reference values of $\Omega_{GW}(f)$ at 100 Hz, are within a few percent. As individual sources contribute to the background through a $f^{2/3}$ power law below 100 Hz, the outputs of different realisations vary by a small linear factor in magnitudes of $\Omega_{GW}(f)$. Note that our results represent the average background energy spectra. The actual background
3.4. Monte-Carlo simulations

signal can deviate considerably from the average depending on the time-frequency properties as we will discuss in section 3.7.

We shall present our results as both: a) a full background – calculated using equation (3.30) for each population, which is the background signal we will be searching for; b) a residual foreground – in the summation of equation (3.30) individual events above a given detection threshold (see section 3.6) are discarded, which represents the residual noise due to sub-threshold sources. Result a) will be presented in section 3.4.1 and a) and b) are compared in section 3.6. For completeness, we also present an example of a simulated time series due to the BNS population in section 3.7.

3.4.1 Backgrounds encoded with information of mass distributions

We compare the numerical results of $\Omega_{GW}(f)$ for each CBC population, with the semi-analytical model presented in subsection 3.3.3 and a simple power law model described in subsection 3.3.1.

Figure 3.7 shows such a comparison in the case of Gaussian mass distributions. The very narrow distribution of NS masses has negligible influence on the BNS background – the numerical result perfectly matches the semi-analytical model except a slightly broader shape around the peak, while effects of a Gaussian BH mass distribution are moderately noticeable for BH-NS and BBH – the reduction from the $f^{2/3}$ curve is partly alleviated due to the contribution from the merger-ringdown emission of more massive systems. Unless otherwise stated we adopt the numerical models assuming Gaussian mass distributions in the following sections.

Figure 3.8 illustrates the numerical models of BH-NS and BBH for different BH mass distributions. Curves are scaled according to the individual values of $\langle M^5_5 \rangle$ as given in Table 3.2 for each distribution to ensure that all have the same value as that of a Gaussian distribution. Two groups are clearly distinguishable above 200 Hz – one containing low mass BHs only (Gaussian and power law) and
Figure 3.7: The CBC background energy spectra calculated numerically assuming Gaussian mass distributions are compared with the semi-analytical models and simple power law models $f^{2/3}$ (up to 200 Hz) shown in Figure 3.5.
Figure 3.8: As in Figure 3.7, but showing only the numerical results of BH-NS and BBH for 4 models of BH mass distribution. Curves are scaled linearly to the same $\langle M_e^{5/3} \rangle$ as that of a Gaussian distribution (see Table 3.2).
another for a broader distribution when high mass BHs are also included – different peak frequencies are due to variations of average total masses and slightly distinct spectral width comes from the different degree of concentration of the distribution. The amplitudes of $\Omega_{GW}(f)$ up to 100 Hz are very similar to each other (within numerical errors), agreeing with the expected dependency on $\langle M_5^{5/3} \rangle$.

Two main conclusions from Figures 3.7 and 3.8 are: a) mass distribution plays a role only through $\langle M_5^{5/3} \rangle$ in the low-frequency ($\lesssim 100$ Hz) power law part of the background energy spectrum. This was mentioned in Wu et al. (2012), and also confirmed independently in Kowalska-Leszczynska et al. (2015) where a power law relation was obtained using mass distributions derived from population-synthesis simulations; b) it could become possible to probe mass distributions through stochastic background measurements, e.g., peaks shown in Figure 3.8 (once measured) will provide information about the average total mass and the degree of concentration of the distribution.

In our calculations we do not consider more sophisticated distributions of BH spin and mass ratio. These two parameters play a minor role (compared with $M_c$) above a few tens Hz in our adopted waveforms. Therefore, our results will not be affected significantly as long as their true distributions are not highly asymmetrical. An additional effect due to orbital eccentricity is not relevant as the orbits of coalescing compact objects are expected to circularize before their GW signals enter the ground-based frequency window (Brown & Zimmerman 2010). We note, however, that dynamically formed BBHs, of which the population is not considered in the current work, may be highly eccentric and could merge before their orbits are circularized (Benacquista & Downing 2013). As mentioned in Zhu et al. (2011b), such a population, possibly with much higher average masses, could provide considerable contribution to a GWB.
3.5 Issues on the detection

In this section we first update previous estimates on the detectability of the CBC background for second and third generation terrestrial detectors, using improved background models. By considering practical issues in detection and parameter estimation, we further discuss the choice of $\Omega_{GW}(f)$ templates for data analysis.

We consider five advanced detectors – Advanced LIGO (aLIGO) at Hanford (H) and Livingston (L), Advanced Virgo (V) in Italy, KAGRA (K) in Japan, and AIGO in Australia, as well as ET for which we consider two configurations – ET-B (Hild et al. 2008) and ET-D (Hild et al. 2011). Note that the inclusion of LIGO India should have similar contribution as AIGO. Unless otherwise stated, we use the sensitivity curve of aLIGO for the zero detuning, high laser power configuration (see the public LIGO document T0900288 for details), and of KAGRA for the broadband configuration. We assume that AIGO has the same sensitivity as aLIGO. The target sensitivities of these detectors are shown later in Figure 3.14.

3.5.1 Signal-to-noise ratios

As discussed in section 2.4, the optimum detection strategy for a stochastic background is to cross-correlate the outputs of two or more detectors (see, e.g. Allen & Romano 1999). Strictly speaking, the CBC background is not a stochastic background in the sense that individual signals do not sufficiently overlap in time-frequency space. This was suggested in Rosado (2011) and will be further discussed in section 3.7. Nevertheless it has been shown, both theoretically (Drasco & Flanagan 2003) and experimentally (Regimbau et al. 2012), that the cross correlation method works nearly optimally in the non-Gaussian regime, because through long time integration it is always possible to obtain a sufficiently large number of signals in a frequency interval and “form” a Gaussian background for which the cross
correlation statistic applies. Therefore, we consider this standard method to assess the detectability of the CBC background for future detectors.

The optimal signal-to-noise ratio \((S/N)^2\) obtainable by two-detector cross correlation is given by (e.g., equation 3.75 in Allen & Romano 1999):

\[
(S/N)^2 = 2T \int_0^\infty \frac{\gamma^2(f) S^2_h(f)}{P_{n1}(f)P_{n2}(f)} df ,
\]

where \(\gamma(f)\) is the normalized overlap reduction function, which accounts for the sensitivity loss due to the separation and relative orientation of the two detectors (Flanagan 1993). For co-located and co-aligned detectors, \(\gamma(f) = 1\). The one-sided noise power spectral densities of the two detectors are given by \(P_{n1}(f)\) and \(P_{n2}(f)\), and \(T\) is the integration time (set to be one year). Note that we have substituted \(\Omega_{GW}(f)\) with the spectral density \(S_h(f)\) through equation (3.13) to obtain a more intuitive format. We use the \(\gamma(f)\) for the 10 pairs of advanced detectors presented in Nishizawa et al. (2009) and adopt the form of ET for two V-shaped detectors separated by 120° (see figure 8 of Regimbau et al. 2012).

As we will be observing a GWB due to all possible contributions of CBC sources, we calculate the signal-to-noise ratio of the total background from the three CBC populations without considering other types of sources. The background spectrum of the total CBC background is simply the sum of that of each population, as shown later in Figure 3.13.

We present in Table 3.3 values of signal-to-noise ratios calculated for advanced detectors. We consider three cases:

1. Cross correlation between pairs of advanced detectors using real \(\gamma(f)\) and the individual sensitivities of each detector; we additionally assume all detectors have the same sensitivity as aLIGO to evaluate the effect of \(\gamma(f)\);

2. Assuming \(\gamma(f) = 1\) for pairs of detectors with the sensitivity of either aLIGO or KAGRA or Advanced Virgo;

3. An optimal combination of cross correlation statistics for 10 pairs of advanced
3.5. Issues on the detection

Table 3.3: Signal-to-noise ratios of the CBC background for advanced detectors.

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<td>a</td>
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<tr>
<td>b</td>
<td>(0.43)</td>
<td>(0.17)</td>
<td>(0.46)</td>
<td>(0.18)</td>
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</tr>
</tbody>
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<table>
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<th>H-V</th>
<th>K-L</th>
<th>K-V</th>
<th>L-V</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
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<td>0.12</td>
<td>0.02</td>
<td>0.12</td>
<td>0.14</td>
</tr>
<tr>
<td>b</td>
<td>(1.05)</td>
<td>(0.23)</td>
<td>(0.03)</td>
<td>(0.29)</td>
<td>(0.25)</td>
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<td>1.26</td>
<td>1.34</td>
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Notes: Assuming that the background is contributed by the three populations of CBC sources (BNS, BH-NS, and BBH), we present signal-to-noise ratios of cross correlating different interferometer pairs from a worldwide network. Results are shown for: a – individual detector sensitivities (but note that A, H and L have the same aLIGO sensitivity); b – assuming all detectors have aLIGO sensitivity. The motivation for calculating b is to investigate effects of the overlap reduction function \( \gamma(f) \); HH, KK, and VV assume \( \gamma(f) = 1 \) for aLIGO, KAGRA and Advanced Virgo respectively; Values of \( Ca \) and \( Cb \) are just the square roots of the quadratic sum of the 10 values in a and b respectively, which can in principle be achieved by optimally combining measurements from multiple pairs of detectors. The improvement of \( Ca \) and \( Cb \) on H-L is not appreciable due to the suppressing effects of \( \gamma(f) \).

detectors, for which \( (S/N)^2 \) is simply the sum of those calculated in case (1) (see, e.g., equation 5.46 in Allen & Romano 1999).

Note that case (3) is mathematically simple but requires 5 advanced detectors to be simultaneously online.

Our results show that: a) among the 10 pairs, H-L performs the best in terms of detecting a CBC background, giving a signal-to-noise ratio of 1, while the lowest value of signal-to-noise ratio is only 0.02. Assuming the aLIGO sensitivity for all detectors only increases the lowest value to 0.03; b) the improvement from combining the network of advanced detectors is only \( \sim 30\% \) on the best performing pair H-L, while assuming \( \gamma(f) = 1 \) for aLIGO increases the signal-to-noise ratio by nearly 3 folds. This is well below the expectation that these two should give similar improvement (Wu et al. 2012, Kowalska-Leszczynska et al. 2015). Such a
pessimistic prospect is mainly due to effects of \( \gamma(f) \). This has been pointed out in our previous studies (Zhu et al. 2011a,b) and is discussed in detail below.

The property of \( \gamma(f) \) is mainly described by its characteristic frequency \( f_{\text{char}} \), given by \( f_{\text{char}} = c/(2|\Delta X|) \) with \( |\Delta X| \) being the distance between two detectors, above which \( \gamma(f) \) decays rapidly towards zero. Among the 10 pairs of detectors, H-L has the smallest separation (\( |\Delta X| = 3000 \text{ km} \)), resulting in the highest \( f_{\text{char}} \) of 50 Hz, while values of \( f_{\text{char}} \) for other pairs vary from 10 Hz to 20 Hz (Nishizawa et al. 2009). This, combined with the fact that advanced detectors have a low frequency seismic wall at about 10 Hz, can easily explain the very small 30% improvement. Note that the overlap reduction function also depends on the relative orientation of the two detectors, and we refer interested readers to Nishizawa et al. (2009) for discussion about the optimal configurations of (geographically separated) detector pairs.

For ET, the CBC background can be easily detected, with signal-to-noise ratios of 178 (350), 19 (38), and 15 (30) assuming ET-B (ET-D) sensitivity for the BNS, BH-NS and BBH population respectively – the factor of 2 increment from ET-D is due to greater sensitivity at frequencies below \( \sim 20 \text{ Hz} \). This implies that the detection prospects benefit significantly from improvement of low-frequency sensitivities (as also shown later in Figure 3.11).

We note that the (optimistic) BNS post-merger contribution to the GWB as shown in Figure 3.5 results in a signal-to-noise ratio of only 0.43 (0.46) for ET-B (ET-D), implying that detecting the imprint of BNS post-merger emission on a GWB requires a coalescence rate at least 5 times higher than the realistic value adopted in this study. On the other hand, we quantify PN effects with the difference in signal-to-noise ratios between a simple power law model and the numerical model shown in Figure 3.7 (both are cutoff at 200 Hz). We find that PN amplitude corrections cause a reduction of signal-to-noise ratio in the range of (5.5% – 8.5%) and (2.1% – 3.2%) for ET-B and ET-D respectively. ET-D is less sensitive to these effects as its best sensitivity is more concentrated at lower frequencies.
3.5. Issues on the detection

3.5.2 Detection prospects for advanced detectors

As the operation of advanced detectors is expected to start imminently (see Table 2.1), it is now important to carefully assess the detection prospects of the CBC background, which represents one of the most (if not the most) promising background sources. We look at this issue in much more depth by considering variations in both source parameters and detector configurations.

For each population, $\Omega_{GW}(f)$ scales linearly with $r_0$, for which the uncertainties are generally of orders of magnitude – much larger than those of other parameters. To determine what possible combinations of $r_0$ the total CBC background will be accessible to advanced instruments, we simply scale the numerical models for different values of $r_0$, keeping all other parameters fixed. The considered range of $r_0$ (in $\text{Mpc}^{-3}\text{Myr}^{-1}$) is 0.1 – 10 (pessimistic to optimistic) for BNS and 0.005 – 0.30 (realistic to optimistic) for BBH while the BH-NS rate is set to be the realistic 0.03 (all values taken from Table 4 of Abadie et al. 2010c).

The motivation of our choice regarding BBH and BH-NS is two-fold: a) it was recently predicted, through population synthesis studies (Belczynski et al. 2010; Dominik et al. 2012) and empirical estimation based on two observed BH-Wolf-Rayet star systems (Bulik et al. 2011), that $r_0$ for BBH can plausibly be at the optimistic value adopted above; b) while the same population synthesis studies gave similar realistic rates of BH-NS (see, e.g., Tables 2 and 3 in Dominik et al. 2012), a negligible coalescence rate for BH-NS was recently empirically determined by following the future evolution of Cyg X-1 (Belczynski et al. 2011). In the current analysis for advanced detectors, the contribution of the BH-NS population is nearly negligible at the chosen rate.

Figure 3.9 shows the detectable “rate space” for advanced detectors: a signal-to-noise ratio threshold of 3 is used to indicate a detection, which corresponds to 95% detection rate and 5% false alarm rate. Note that: a) we have taken the integer 3 for convenience, while the accurate number is 3.29, given by $\sqrt{2[\text{erfc}^{-1}(2\beta) - \text{erfc}^{-1}(2\varsigma)]}$, assuming a false alarm rate $\beta = 5\%$ and a detection rate $\varsigma = 95\%$. Here $\beta$ and $\varsigma$...
Figure 3.9: The detectable “rate space” of a CBC background, assumed to be contributed by three populations (BNS, BH-NS, and BBH), for ground-based advanced detectors. The local coalescence rate $r_0$ is in Mpc$^{-3}$ Myr$^{-1}$ with $r_0$ for BH-NS fixed at 0.03 (see text). A signal-to-noise ratio greater than 3 for one year observation can be obtained for rates above each curve assuming: a) cross correlation with aLIGO H-L; b) an ideal case of $\gamma(f) = 1$ for two aLIGO detectors; c) an optimal combination of measurements of 10 pairs advanced detectors (advanced network). Note that this background will be easily detectable for ET i.e., the corresponding curve is well below the origin of the two axes.
3.5. Issues on the detection

need not sum to 1 (see Allen & Romano 1999, for details); b) one can lower the threshold on signal-to-noise ratio with the cost of increasing $\beta$, or decreasing $\varsigma$ or both. For example, a threshold of 2.56 may be used if one chooses $\beta = 10\%$ and $\varsigma = 90\%$.

In the case of $\gamma(f) = 1$ for aLIGO, our results are consistent with Wu et al. (2012), implying that the BNS population alone may produce a detectable background signal at the realistic coalescence rate. However, an important point here is that for a worldwide network of advanced detectors, the required rate for a detection is more than twice higher. This motivates us to consider all three populations as a whole as they will be observed in reality. Figure 3.9 shows that some combinations of the BNS population and the BBH population can form a detectable GWB, while both of their individual contributions alone are not sufficient for detection. In practice, if a $\Omega_{\text{GW}}(f) \sim f^{2/3}$ power law background has been detected, one would certainly wish to determine the relative contribution from every possible population.

The possible variation in $\langle M^{5/3}_c \rangle$ (for BBH and BH-NS systems) and the effects of CSFR and delay times (which can be represented using the parameter $J_{2/3}$) are not considered in Figure 3.9. Combining the simple power law model given by equation (3.26), which will be shown to be a good approximation in the next subsection, with equations (3.13) and (3.32) we have a simple relation:

$$S/N = C_{2/3} \left( \sum_{k=1}^{3} r_0^k \langle M^{5/3}_c \rangle^k J_{2/3}^k \right)$$

(3.33)

where the indices $k = 1, 2, 3$ denote the three CBC populations and $C_{2/3}$ is a constant depending only on detector sensitivity and $\gamma(f)$ for different detector pairs used in cross correlation. For convenience we have omitted the division by the corresponding reference values as in equation (3.26) for the three parameters.

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\footnote{In practice, this may be impossible for present detectors that are limited at frequencies above a few hundred Hz. One needs to measure the spectral peaks to break the degeneracy among the contribution from different CBC populations.}
Note that one needs to set the upper frequency limit of the integration in equation (3.32) at 100 Hz so that the above equation is representative of results obtained using numerical models of $\Omega_{GW}(f)$. In fact, our Figure 3.9 can be easily reproduced by using equation (3.33) together with values of $\langle M_{5/3}^c \rangle$ and $t_{\text{min}}$ (to obtain the parameter $J_{2/3}$ through equation (3.25)) given in Table 3.2. Varying the values of $\langle M_{5/3}^c \rangle$ for BBH and BH-NS from those of a Gaussian distribution (as assumed in Figure 3.9) to the highest entries in Table 3.2 increases the total signal-to-noise ratio by 5% (40%) assuming a BBH coalescence rate of $r_0 = 0.005 (0.3) \text{Mpc}^{-3} \text{Myr}^{-1}$ and realistic values of $r_0$ for both BNS and BH-NS. Such an increment of signal-to-noise ratio is smaller than 40% for higher coalescence rates of BNS and BH-NS. Therefore, our Figure 3.9 does not change appreciably for variations of $\langle M_{5/3}^c \rangle$ given the current observational BH mass estimates. Meanwhile, effects of CSFR and delay times could moderately degrade (i.e., no more than a factor of 2) the detection prospects, as our current choice gives $J_{2/3} = 2.3$ for the dominant BNS population, which is close to the high end of the range $(1.3 - 2.6)$ obtained in section 3.3.1.

Note that for a putative population of dynamically formed BBHs in dense stellar clusters (see, e.g. Sadowski et al. 2008) or for the same field population (as considered in this study) but assumed to be formed in low metallicity environments (Dominik et al. 2012), a much larger average chirp mass $\langle M_c \rangle$ up to about $\sim 20 M_\odot$ (in comparison to $\langle M_c \rangle \simeq 7 M_\odot$ for the mass distribution used in this work) was suggested to be possible.\footnote{In the study of Dominik et al. (2012), the authors found that a larger average chirp mass is associated with a higher coalescence rate for BBH in low metallicity environments, resulting in BBH dominating the whole CBC population.} To allow for these possibilities, a plot of detectable $r_0 - \langle M_c \rangle$ space is useful. Such illustrative studies, which only apply to a single population, have been presented for the BBH population in Zhu et al. (2011b), and for each of the three CBC populations in Wu et al. (2012) – for a given signal-to-noise ratio threshold, the scaling relation $r_0 \sim \langle M_c \rangle^{-5/3}$ was shown to be a good approximation.\footnote{Curves for advanced detectors in Figure 5 of Zhu et al. (2011b) underestimate the detectability of BBH signals as compared to our results.} At a coalescence rate of the order $10^{-3} \text{Mpc}^{-3} \text{Myr}^{-1}$ (see, e.g.
Miller & Lauburg (2009), dynamical formation scenarios should have similar contributions to a GWB as the field population of BBHs considered in this study (at the realistic rate) and thus will not improve considerably the detection prospects for advanced detectors. Such a back-of-the-envelope argument also applies to binaries involved with one or two intermediate mass BHs - much lower rates cancel out the advantage of higher masses (see, e.g., Tables 8-10 of Abadie et al. 2010c). Despite the involved uncertainties, these systems should be a more interesting source for single event searches/detections (at least for advanced detectors), from which their very existence will be tested or the associated coalescence rates can be stringently constrained.

Looking forward to the advanced detector era, it is now crucial to investigate how the detection prospects of the CBC background (which could be the first to be detected) can be enhanced. For co-located detectors, techniques to remove correlated environmental and/or instrumental noises will be required (Fotopoulos & LIGO Scientific Collaboration 2008; Aasi et al. 2015a). Provided that no co-located instruments are available (as currently planned), detection of a GWB from CBC events will require higher coalescence rates than what are presently thought to be realistic, i.e., $r_0 = 3(0.2)\,\text{Mpc}^{-3}\text{Myr}^{-1}$ for BNS (BBH), or alternatively given the realistic rates an integration time of 4 years to obtain a signal-to-noise ratio of 2, which was assumed as a threshold in Wu et al. (2012). We note that a single-detector auto-correlation approach was recently proposed to be comparable in signal-to-noise ratio to what is achievable by cross correlation of two co-located and co-aligned detectors (Tinto & Armstrong 2012), whose usefulness needs to be further tested in realistic data analysis experiments.

The above results have assumed standard versions of design sensitivities for advanced detectors. In practice, detectors can be tuned to different configurations for various purposes, e.g., to allow optimization for different searches. As the aLIGO H-L pair gives the majority of contribution to the network signal-to-noise ratio for a by a factor of 4 due to the use of an old version of aLIGO sensitivity and one additional factor of $(1 + z)$ in the calculation of $\Omega_{\text{GW}}(f)$. 


Figure 3.10: Anticipated sensitivity curves using different tuning options for aLIGO. This figure is taken from LIGO public document T0900288 (link: https://dcc.ligo.org/LIGO-T0900288/public).

CBC background, apart from the standard zero-detuning and high laser power configuration we consider here four additional tuning options of aLIGO (data for the corresponding sensitivities are available publicly at the link given in the beginning of this section, and we refer interested readers to the LIGO document T0900288 therein for descriptions and technical details): a) Zero-detuning, low laser power; b) Optimal BNS which is optimized to the BNS inspiral search; c) Optimal BBH which is optimized for 30-30 solar mass BBH inspirals; d) High frequency, which has a narrowband tuning at 1 kHz. Figure 3.10 shows the anticipated sensitivity curves for these tuning options.

We re-calculate the signal-to-noise ratio of the total CBC background for H-L using the additional four sensitivity curves, and obtain 1.18, 0.83, 1.49 and 1.23 for a), b), c) and d) respectively, in comparison to 1.05 for the standard configuration. This shows that modest improvement of low-frequency sensitivity provide considerable enhancement in signal-to-noise ratio, which is comparable to
or even greater than that due to the combination of multiple detector pairs (again for the currently proposed network). The largest value of signal-to-noise ratio (for one year observation), which comes from the adoption of c), implies that $S/N = 3$ is achievable with an integration time of 4 years in comparison to 8 years for the standard option.

To extend the above comparison to generic power law GWB models – $\Omega_{GW}(f) = \Omega_\alpha (f/100\text{Hz})^\alpha$, we show in Figure 3.11 the minimum detectable energy density $\Omega_{\alpha\text{min}}$ for aLIGO L-H considering different tuning options. The values of $\Omega_{\alpha\text{min}}$ can be easily obtained by setting a threshold on signal-to-noise ratio and solving the equality given by equation (3.32). We take the integration range from 10 Hz to 1 kHz, and consider the range $(0 – 5)$ for $\alpha$. The curves in Figure 3.11 represent the upper limits obtainable by aLIGO which apply to primordial GWBs (in addition to astrophysical backgrounds), e.g., $\alpha = 0$ in many early-Universe scenarios (see, e.g., Figure 2 in Abbott et al. 2009a).

### 3.5.3 The construction of templates for the background energy density

In previous sections, we have shown that:

1. for $f \lesssim 100\text{ Hz}$, the power law model given by equation (3.26) is a good approximation and requires only three parameters. The power law relation holds for three populations and thus for the total background as well;

2. above 100 Hz, PN corrections become more notable, and different behaviors are expected from other effects such as CSFR and mass distributions, making it difficult to predict the background spectral properties.

In this subsection we show that the power law model is sufficient to be used as search templates for a CBC background and is also useful for parameter estimation of the coalescence rate and average chirp mass (information other than these two quantities can only be extracted from measurements of high-frequency peaks).
Figure 3.11: The minimum detectable energy density $\Omega_{\text{GW}}^{\min}$ for generic power law GWB models (with indices $\alpha$) - $\Omega_{\text{GW}}(f) = \Omega_{\alpha}(f/100\text{Hz})^{\alpha}$, for one year observation using the aLIGO H-L pair. We assume a signal-to-noise ratio threshold of 3, and consider five tuning options for aLIGO: zero-detuning with high/low laser power, optimized for searches of BNS/BBH inspirals and one with high frequency narrow-band tuning. We refer interested readers to the public aLIGO document T0900288 for descriptions and technical details about aLIGO tunings.
Figure 3.12: The fractional signal-to-noise ratio (S/N) as a function of upper cut-off frequency for 10 pairs of advanced detectors for the BNS background. The “fastest” and “slowest” to accumulate 99% of the total S/N are highlighted with thick lines, corresponding to cross-correlating LIGO H-L and KAGRA-Advanced Virgo (K-V) respectively.
Figure 3.12 illustrates the fractional signal-to-noise ratio as a function of upper frequency limits for the 10 pairs of advanced detectors for the BNS background: the signal-to-noise ratio calculated by applying an arbitrary frequency upper limit in equation (3.32) divided by that integrated to the higher frequency limit of detector sensitivity. We see the signal-to-noise ratio has saturated below 100 Hz due to the suppressing effects of $\gamma(f)$ and the fact that the background is “red”, i.e., $S_h(f) \sim f^{-7/3}$. Quantitatively, 99% of the total signal-to-noise ratio can be obtained up to 51 Hz and (at most) 98 Hz by cross-correlating H-L and K-V respectively. Such upper frequencies are slightly higher for ET-B (133 Hz) or assuming $\gamma(f) = 1$ for LIGO (128 Hz), and could be even lower for ET-D (47 Hz), as also noted in Kowalska-Leszczynska et al. (2015). The exactly same values are obtained in the cases of BH-NS and BBH due to the similarity in $\Omega_{GW}(f)$ below 200 Hz.

We then quantify the effectiveness of the proxy of a power law model to the CBC background below 100 Hz by looking at the following quantity:

$$\langle \mathcal{S} \rangle = \frac{T}{2} \lambda \int_0^{100} \frac{\gamma^2(f)S_h(f)S'_h(f)}{P_{n1}(f)P_{n2}(f)} df,$$

which gives the mean value of the cross-correlated signal, with $\lambda$ the normalization constant to ensure $\langle \mathcal{S} \rangle = \Omega_{\alpha T}$ for a power GWB with $\Omega_{GW}(f) = \Omega_{\alpha f^a}$ (Allen & Romano 1999). Here $S_h(f)$ is the “true” spectral density of the background, assumed to be that given by our numerical models; $S'_h(f)$ corresponds to the template adopted in stochastic background searches. A simple power law template results in an overestimation of $\langle \mathcal{S} \rangle$ within 2% – 5% for the three CBC populations for 10 pairs of advanced detectors. This can be further reduced by up to 1% by decreasing the upper cutoff frequency from 100 Hz to 50 Hz.

Overall, we suggest that a simple power law model for the CBC background as given by equation (3.26) with an appropriate upper frequency cutoff at 50-100 Hz is sufficient for detection and the followed-by parameter estimation of average masses and coalescence rates using ground-based interferometers. For third-generation detectors like ET, however, more accurate models, such as those presented in sub-
sections 3.3.3 and 3.4.1 will be required to extract information such as CSFR, PN effects, and mass distributions.

### 3.6 A foreground formed by sub-threshold events

When searching for primordial GWs from the early Universe, astrophysical backgrounds formed by more recent sources could act as contaminating foregrounds. One resolution to this problem is to subtract individually detected signals from the data. This has been demonstrated for the proposed Big Bang Observer, which has a sufficiently good sensitivity that it can resolve and thus subtract away almost all BNS inspirals in the Universe from the overall background (Cutler & Harms 2006).

We refer interested readers to Cutler & Harms (2006) for details of the method and related practical issues, and we simply apply this method to ET to estimate the “residual” foreground from sub-threshold CBC events.

As discussed in section 2.4, the optimal method to detect signals with known waveforms is through matched filtering. To examine the detectability of single events in our simulations, we calculate the signal-to-noise ratio $\rho$, defined as the ratio between the expectation value of the optimal detection statistic in the presence of a signal and its root mean square value in the absence of signals. Directly following equation (2.29), one obtains:

$$\rho^2 = 4 \int_0^{f_{\text{max}}} \left| \tilde{h}(f) \right|^2 \frac{P_n(f)}{D_{\text{eff}}(f)} \, df,$$

where $f_{\text{max}}$ is the maximum observed frequency, depending on source redshift, component masses and spins (if applicable). We use the ET antenna pattern function, which goes to equation (3.31) for $D_{\text{eff}}$ and determines the overall amplitudes of $|\tilde{h}(f)|$, for a triangle configuration including three V-shaped detectors (see, e.g., equation (24) of Regimbau et al. 2012). Note that the spectral density of the CBC

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10 Our calculations show that the subtraction of detectable single events reduces the foreground by less than a few percent, as the detection horizon of advanced detectors such as LIGO is at most $z \sim 0.4$ for BBH systems (Abadie et al. 2010c).
Table 3.4: Detection rates ($N_{\text{det}}$) of CBC sources for ET.

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**Notes:** All values are in $10^4$ and have assumed the realistic coalescence rates (see Table 3.2) adopted from Abadie et al. (2010c). $N_{\text{tot}}$ is the total event rate up to $z = 6$ or $z = 15$ for CSFR models of HB (Hopkins & Beacom 2006) or RE (Robertson & Ellis 2012) respectively. We scale the number of events above the detection threshold ($\rho \geq 8$) in the Monte-Carlo simulation (see section 3.4) according to $N_{\text{tot}}/N_{\text{mc}}$ to obtain the detection rate $N_{\text{det}}$. We have considered NS/BH mass distributions as described in Table 3.1 – Gaussian (G), Power law (P), Exponential (E) and Two-Gaussian (TG); and adopted ET sensitivities of two configurations – ET-B and ET-D (values are given in parentheses). ET-D gives slightly higher detection rates due to a greater low-frequency ($f \lesssim 20$ Hz) sensitivity.

The background is below that of the instrumental noise of ET even at optimistic rate estimates. Therefore, we do not need to consider $S_h(f)$ as an additional contribution to $P_n(f)$ in equation (3.35), whereas one must do so in the case of the Big Bang Observer (Cutler & Harms 2006).

We calculate $\rho$ for each of the simulated CBC events in our Monte-Carlo simulation as described in section 3.4, those loudest events resulting in $\rho \geq \rho_{\text{th}} = 8$ are discarded (termed with “subtraction”) to estimate a residual noise. Before moving forward to discussions of ET’s potential in removing the CBC background
through a subtraction process, we present in Table 3.4 ET detection rates (which are conveniently obtained in our simulations) of CBC sources for completeness. The calculations improve the approximation (for advanced detectors) used in Abadie et al. (2010c) with the following considerations: a) cosmic evolution of coalescence rates, the standard ΛCDM cosmology and cosmological redshifts; b) observational NS/BH mass distributions; c) complete waveforms that include PN amplitude corrections. While these effects may not be important for detection rate predictions for advanced detectors, they must be considered for ET due to its 1000 times larger accessible volume. It is important to mention that the detection rates obtained from our simulations are fully consistent with those presented in Abadie et al. (2010c) for aLIGO.

Based on results presented in Table 3.4, we find that the realistic CBC detection rates for ET are $10^5$ (BNS), $10^4$ (BH-NS) and $10^4$ (BBH) given the current realistic coalescence rate predictions. Note that ET will have an overall detection efficiency (defined as $N_{\text{det}}/N_{\text{tot}}$) of $\sim 10\%$ (BNS), $\sim 40\%$ (BH-NS) and $\sim 85\%$ (BBH), which is independent of $r_0$ and weakly dependent on coalescence rate evolution and sensitivity models as shown in Table 3.4.

Figure 3.13 compares the results of $\Omega_{GW}(f)$ calculated using equation (3.30) without and with the subtraction of individually detectable events. We see that ET will be able to reduce the CBC background energy densities by a factor of about 2, 10 and 200 from the BNS, BH-NS and BBH population respectively through a subtraction scheme. The total residual foreground is overwhelmingly due to sub-threshold BNS merger events and is insensitive to rate evolutionary histories and BH mass distributions. It is worth mentioning that the residual background is still detectable using ET with a high signal-to-noise ratio ($\gtrsim 100$). The possibility that $r_0$ for BBH could be much higher than the value used here, e.g., $r_0 = 0.36\, \text{Mpc}^{-3}\, \text{Myr}^{-1}$ found in Bulik et al. (2011), does not significantly change the level of such a residual foreground because most of these BBH systems would be detectable and therefore can be subtracted. Additionally the contribution from
Figure 3.13: The energy density parameter $\Omega_{GW}(f)$ of the total CBC background and its contributions by the BNS, BH-NS, and BBH populations, without and with (labeled as sub) the subtraction of individually detectable sources for ET assuming ET-B sensitivity (the results are essentially the same for ET-D sensitivity). Note that: a) the total residual “noise” is dominated by sub-threshold BNS merger events; b) ET will be able to reduce the foreground level by a factor of about 2, 10 and 200 from the BNS, BH-NS and BBH population respectively; c) the reduction shown here is optimistic because of the assumption that every theoretically detectable single signal will be perfectly subtracted (see Cutler & Harms (2006) for detailed discussion about potential subtraction errors).
3.7. Time-frequency properties

A possible population of dynamically formed BBHs to such a residual foreground is negligible, as ET will be able to detect these sources out to much larger distances than the field population of BBHs considered in this work due to significantly higher chirp masses (Sadowski et al. 2008).

Figure 3.14 compares the noise power spectral densities of future terrestrial detectors with the spectral densities of the CBC residual foreground and a range of primordial GWBs from the very early Universe which could be described by a flat energy spectrum in the frequency band of ground-based interferometers. Examples of such primordial GWBs include inflationary, cosmic strings, and pre-Big-Bang models (see Figure 2 in Abbott et al. 2009a, and references therein for details). Considering significant uncertainties associated with model predictions, we show a shaded region formed by $\Omega_{0}^{\text{UP}} = 10^{-9}$ and $\Omega_{0}^{\text{LOW}} = 10^{-14}$ with $\Omega_{0}^{\text{UP}}$ corresponding to the upper limit achievable by aLIGO (a level that could be reached or surpassed in cosmic strings and pre-Big-Bang models) and $\Omega_{0}^{\text{LOW}}$ for the likely level of inflationary GWBs (which is below the ET stochastic sensitivity). Figure 3.14 implies that, without considering other types of astrophysical sources, the contribution to a foreground from sub-threshold CBC events could be a challenging issue for future stochastic searches for primordial GWBs because these signals are beyond the capability of current data analysis methods and always add up to act as an additional “noise” component in the data. Note that: a) the CBC curve shown in Figure 3.14 only applies to ET and the foreground level for advanced detectors is $2\times$ higher (as the original pre-subtraction background signal); b) in the case of different coalescence rates and average chirp masses, the total residual foreground level can be estimated in a fashion similar to equation (3.33).

3.7 Time-frequency properties

The analysis in the previous section is based on the expectation that individual CBC events contributing to a background have different amplitudes in the data because of a distribution over source distances, orientations and sky positions.
Figure 3.14: The spectral densities, $S_h(f)$, of the residual foreground formed by sub-threshold (for ET) CBC events and some putative primordial GWB from the very early Universe – the shaded region is encompassed by two flat energy spectra $\Omega_0^\text{UP} = 10^{-9}$ and $\Omega_0^\text{LOW} = 10^{-14}$ (note that they are not upper or lower limit, see text for details), are compared against noise power spectral densities, $P_n(f)$, of second (aLIGO, KAGRA, Advanced Virgo) and third generation (two possible configurations for ET, ET-B and ET-D, are considered) ground-based GW interferometers.
3.7. Time-frequency properties

Rigorously one needs to track the number of sources contributing in the relevant frequency intervals. For CBC events, the duty cycle is frequency dependent – individual signals stay much longer at low frequencies, leading to much smaller duty cycle for higher frequencies, as pointed out in recent studies (e.g., Rosado 2011; Wu et al. 2012).

Practically we want to know the critical frequency, $f_c$, above which individual signals do not simultaneously occupy the same frequency interval. Without going into specific details of the calculations, we provide the following relation:

$$
\left( \frac{f_c}{15 \text{ Hz}} \right)^{11/3} = \left( \frac{\Delta f}{1 \text{ Hz}} \right) \left( \frac{r_0}{1 \text{ Mpc}} \right)^{3} \left( \frac{M_c}{1 \text{ M}_\odot} \right)^{-5/3},
$$

where $\Delta f$ is the size of frequency interval relevant to the analysis. We consider a reference value of 1 Hz for $\Delta f$, while in practice it could be as small as the frequency resolution of an experiment. For ground-based interferometers such a resolution is given by $1/\Delta T$, where $\Delta T$ is the time duration of short data segments which are used in cross correlation analysis and is typically of orders of seconds (Allen & Romano 1999; Abadie et al. 2012c). In equation (3.36) we neglect the effect of mass distribution.

Once we know the critical frequency $f_c$, the duty cycle function $\xi(f, \Delta f)$ is exclusively determined through:

$$
\xi(f, \Delta f) = \left( \frac{\Delta f}{1 \text{ Hz}} \right) \left( \frac{f}{f_c} \right)^{-11/3}.
$$

Note that: a) the duty cycle function is comparable to the overlap function in Rosado (2011) and the duty cycle parameter in Wu et al. (2012); b) the contribution of post-inspiral emission to $\xi(f, \Delta f)$ is negligible due to much shorter durations; c) the above two convenient relations are applicable to three CBC populations up to a few hundreds Hz.

The physical meaning of the duty cycle function is the (statistically average) number of intersections of the tracks of individual inspirals in time-frequency plane.
Figure 3.15: The spectrograms for a simulated BNS background signal (upper panel) and for the same background signal plus an arbitrary amount of Gaussian white noise (lower panel). In the simulation we increase the coalescence rate by a factor of 50 and scale down the final amplitude by the same factor. The colors or intensities in the spectrogram effectively tell the relative signal/noise power spectral densities at given time and frequency instances. The lower panel is only for illustration and should not be related to actual detection prospects.

at a given frequency interval. We demonstrate this in the upper panel of Figure 3.15 by plotting the spectrogram of a simulated time series of BNS inspirals up to $z = 6$. This allows one to visualize the unique time-frequency properties of the BNS background: a) above a few tens Hz, individual chirps are only occasionally present, separated in both time and frequency; b) a large number of overlapping signals below $\sim 20$ Hz create a continuous and stochastic background, with the colors showing the “redness” of the background. The simulation follows the same procedure as described in Regimbau et al. (2012) except that we increase $r_0$ by a factor of 50 and scale down the final amplitude by the same factor.

For illustration we add an arbitrary amount of Gaussian white noise to the
simulated background, and the spectrogram is shown in the lower panel of Figure 3.15. As most of the signals are deeply buried in detector noise, one can only resolve the strongest events which are above the detection threshold, just like a few “chirping” structures apparent in the noisy spectrogram. Note that the contrast of signals and noise in this plot are not representative of the actual detection prospects since the added noise is not comparable to (and obviously weaker than) the instrumental noise of current and future ground-based detectors.

The conclusion from equation (3.36) and Figure 3.15 is that at each 1 Hz frequency bin above 15 Hz, we will be observing individual BNS inspirals rather than a background. The equivalent critical frequency is of order 4 (2) Hz for the BH-NS (BBH) population. Furthermore, when an array of detectors with moderate angular resolution is used to observe the CBC background, the background must be considered as a set of discrete transient sources randomly distributed across the sky. However, this does not necessarily mean the GW emission due to various CBC populations as a whole can not be detected as a background by the standard cross correlation method (as we have pointed out in the beginning of subsection 3.5.1). In practice, the underlying numerous individual signals are averaged when cross-correlating data of length from months to years, and it is possible to recover the theoretical expected spectral density as demonstrated in a recent mock-data challenge study for ET (Regimbau et al. 2012). In the same way, sub-threshold transient signals remain in the data as an additional noise component which could obscure the primordial GWs and other astrophysical backgrounds.

3.8 Conclusions

In this chapter we first reviewed the formalism of the calculation of $\Omega_{\text{GW}}(f)$ and developed a practical model for astrophysical backgrounds – a power law energy spectrum $dE_{\text{GW}}/df \sim f^{\alpha-1}$ naturally leads to $\Omega_{\text{GW}}(f) = \Omega_\alpha f^\alpha$ where $\Omega_\alpha$ depends almost exclusively on the local rate density $r_0$ and total amount of radiated energy $\Delta E_{\text{GW}}$. Such a model allows one to quickly evaluate uncertainties in estimates
of the background strength and the associated detectability.

We have provided updated estimates of the spectral properties of the CBC background formed by populations of BNS, BH-NS, and BBH systems. By systematically investigating effects of CSFRs, delay times, NS/BH mass distributions, and using up-to-date analytical complete waveforms including PN amplitude corrections, we showed that:

1. Effects of CSFRs and delay times are linear below 100 Hz and can be represented by a single parameter $J_{2/3}$ with an uncertainty $\sim 2$;

2. PN effects cause a small reduction of $\Omega_{GW}(f)$ from a $f^{2/3}$ power law function above a few tens Hz;

3. Below 100 Hz, $\Omega_{GW}(f)$ can be approximated by a $f^{2/3}$ power law function, with the magnitude determined by three parameters – the local coalescence rates $r_0$, the average chirp mass $\langle M_5^{5/3} \rangle$ plus $J_{2/3}$. In particular, within this frequency range $\Omega_{GW}(f)$ does not depend on chirp mass distributions. This finding, which was also obtained independently in a recent study using a population-synthesis approach (Kowalska-Leszczynska et al. 2015), is important for extracting astrophysical information from future stochastic background measurements. Future individual CBC detections are required to break the degeneracy among these parameters;

4. A variety of features at high frequencies ($\gtrsim 200$ Hz), e.g., different peak frequencies and widths of $\Omega_{GW}(f)$, are expected from different CSFRs, delay times, and mass distributions. Measurements of the peaks will be rewarding although challenging because of the small contribution (less than 1%) to signal-to-noise ratio by the high frequency signal;

5. The post-merger emission of BNS coalescences could considerably enhance the peak of the BNS background at around 1-2 kHz, but will not alter the background spectrum below 300 Hz. While this contribution to a GWB may be too weak to be detectable even for ET, the latter fact is advantageous for
parameter \((r_0 \text{ and } \langle M_e^{5/3} \rangle)\) estimation by measuring only the low-frequency power law spectrum.

Using updated estimates of \(\Omega_{GW}(f)\), we revisited the issue on the detectability of this background signal. Assuming a detection target of the total background contributed by three CBC populations for a worldwide network of advanced detectors, we showed in Figure 3.9 the accessible “rate space” of the local coalescence rates \(r_0\) (in Mpc\(^{-3}\) Myr\(^{-1}\), with the value of BH-NS fixed at 0.03), implying:

1. A combination of a BNS population at the realistic rate of \(r_0 = 1\) and a BBH population at a rate of \(r_0 = 0.1\) will give rise to a detectable background signal;

2. Either a BNS rate of \(r_0 = 2.7\) or a BBH \(r_0 = 0.16\) will be necessary for detection, when BBH or BNS has very low coalescence rate (note that the chosen \(r_0\) for BH-NS ensures a negligible contribution).

In both cases, recent optimistic rate estimates for BBHs provide interesting detection prospects for a CBC background. The above quoted values are for optimally combining a network of 5 advanced detectors. Such an optimal combination gives 30% improvement in detectability over the aLIGO H-L pair. This is way below the common expectation (Wu et al. 2012; Kowalska-Leszczynska et al. 2015) that such a network could perform as well as two co-located and co-aligned aLIGO detectors, which gives a 3-fold improvement on H-L. In the latter case our results are consistent with those presented in Wu et al. (2012), showing it is likely that at the realistic rate a BNS background may be detected within one year observation using two co-located aLIGO interferometers.

We emphasize that the somewhat “disappointing” performance of a network of detectors is due to effects of the overlap reduction functions for the current configurations of the advanced detector array – the large separations between pairs of detectors, of the orders of \(10^4\) km (except H-L, 3000 km), result in very modest correlation of background signals above 20 Hz (50 Hz for H-L). This further implies
that stochastic background searches can benefit significantly from a pair of closely spaced detectors, with separation chosen to be both within one reduced wavelength (about 300 km for 150 Hz) and relatively large to ensure that their noise sources are largely uncorrelated.

We found that 99% of the signal-to-noise ratio can be obtained by considering only the contribution up to 50 Hz (a\textup{LIGO} H-L) or at most 100 Hz (KAGRA-Advanced Virgo). Two main implications for advanced detectors are:

1. Only the low frequency part is important for detection;

2. Improvement on the sensitivity below 50 Hz is beneficial for detection.

We conclude that a simple power law model as given by equation (3.26) with an upper frequency cutoff of 50-100 Hz is sufficient for background searches. Since the model is generalized to three CBC populations and only requires three parameters, it could prove useful to constrain or estimate these parameters with future stochastic searches – particularly one can marginalize over a uniform distributed $J_{2/3}$ to obtain confidence levels of $r_0$ and $\langle M_e^{5/3} \rangle$. In addition, our generalized model can also be used to identify the relative contribution from different populations in the case of a likely detection of the CBC background. This will further require combination of stochastic background measurements with CBC single event detections (Mandic et al. 2012). Regarding the above point 2), we specifically showed that for the CBC background the a\textup{LIGO} tuning configuration offering the best low-frequency sensitivity (which is optimized for BBH inspiral searches) will provide a 50% enhancement in the achievable signal-to-noise ratio against the standard sensitivity (zero detuning with high laser power), and such an improvement is even better than that due to an optimal combination of the currently proposed detector network (which comprises 5 advanced detectors). We further compared the sensitivities of stochastic searches using different a\textup{LIGO} tuning options to generic power law GWB models in Figure 3.11. The results show that a\textup{LIGO} H-L will be able to detect a GWB with a signal-to-noise ratio above 3 for $\Omega_0 \geq 1.87 \times 10^{-9}$ (assuming
3.8. Conclusions

a flat energy spectrum) with one year observation at the standard sensitivity, and this limit could be reduced down to $1.24 \times 10^{-9}$ using the optimal [BBH] option.

For third generation detectors like [ET], the background will be easily detectable, with a signal-to-noise ratio from tens up to hundreds contributed by individual populations. The high achievable signal-to-noise ratio will open up new possibilities to: a) enable different populations to be disentangled; b) probe mass distributions and rate evolutionary histories by measuring the peaks of the background energy spectra. To gain more insights about how this information can be extracted from background measurements, models presented in this study can be further improved in the following ways:

1. Contribution from possible populations of dynamically formed [BBH] and/or binaries involved with one or two intermediate-mass should be considered. Due to significantly higher masses of such systems, their contribution could peak at a few tens Hz and might affect the power law relation for the three normal [CBC] populations;

2. More accurate complete waveforms are required. In this regard, the three types of [BHNS] waveforms corresponding to different merger processes (Shibata & Taniguchi 2011) are of particular interest and can be used to investigate how the information of [NS] equation of state is encoded in the background signal.

We demonstrated that [ET] could potentially reduce the contributions to a [GWB] from the [BNS], [BHNS] and [BBH] populations respectively by a factor of 2, 10 and 200 through the subtraction of individually detectable events but there is a strong residual foreground dominated by sub-threshold [BNS] merger events. Such a foreground, at the level of $\Omega_{GW} \sim 10^{-10}$ in the (1~500) Hz frequency range, can hardly be removed and should be considered in future terrestrial searches of primordial [GW] and other astrophysical backgrounds.

We finally discussed the unique properties of the [CBC] background – well defined continuously rising tones, localized directions and well defined average spectral den-
sity. These have not so far been fully exploited by stochastic background searches. We believe that new algorithms could exploit these properties to go beyond the standard cross correlation limit that applies only to true stochastic backgrounds.
4.1 Introduction

The primary sources in the PTA frequency band are inspiralling supermassive binary black holes (SMBBHs). It is widely considered that a stochastic background due to the combined emission from a large number of individual SMBBHs over cosmological volume (see, e.g. Sesana 2013b; Ravi et al. 2014 for recent work) provides the most promising target; indeed, some studies suggested that a detection of this type could occur as early as 2016 (Siemens et al. 2013). Analyses of actual PTA data have previously focused on a search for such a background signal. However, individual resolvable sources that are sufficiently close and/or massive may provide chances for the detection of continuous waves (CWs; Sesana et al. 2009; Ravi et al. 2012). Over the past few years interest has grown substantially regarding the prospects of detecting single-source GWs using PTAs, for example,
for individual SMBBHs (Sesana et al. 2009; Lee et al. 2011; Mingarelli et al. 2012; Ravi et al. 2015), for GW memory effects associated with SMBBH mergers (Seto 2009; Cordes & Jenet 2012; Madison et al. 2014), for GW bursts (Pitkin 2012) and for unanticipated sources (Cutler et al. 2014). In the meantime, many data analysis methods have been proposed in the context of PTA for single-source detection, for example, for monochromatic signals emitted by SMBBHs in circular orbits (Yardley et al. 2010; Babak & Sesana 2012; Ellis et al. 2012; Ellis 2013; Taylor et al. 2014; Wang et al. 2014; Zhu et al. 2014), for memory effects (van Haasteren & Levin 2010; Wang et al. 2015), and bursts (Finn & Lommen 2010; Deng 2014). Searches in real PTA data for GWs from circular binaries produced steadily improved upper limits on the GW strain amplitude (Yardley et al. 2010; Arzoumanian et al. 2014; Zhu et al. 2014). While circular binaries emit waves at the second harmonic of the orbital frequency, eccentric binaries radiate at multiple harmonics. Jenet et al. (2004) first derived the expression of pulsar timing signals produced by eccentric binaries and developed a framework in which pulsar timing observations can be used to constrain properties of SMBBHs.

In this work we adapt the network analysis method used in the context of ground-based interferometers (e.g. Pai et al. 2001; Wen & Schutz 2005; Wen & Schutz 2012; Wen 2008; Klimenko et al. 2008; Sutton et al. 2010) as a general method for detection and localization of single-sources using PTA. In particular, we consider the following types of sources: (1) SMBBHs in circular orbits; (2) eccentric binaries; and (3) GW bursts. We demonstrate the effectiveness of this method using synthetic data sets that contain both idealized and realistic observations.

The organization of this chapter is as follows. In section 4.2 we discuss the signal models of single sources considered in this work. In section 4.3 we present the mathematical framework of our method and propose practical detection statistics. Using idealized simulations we show examples of detection, sky localization and waveform estimation in section 4.4. We demonstrate the implementation of the method in realistic data sets in section 4.5. In section 4.6 we present sensitivities...
to eccentric binaries. In particular we compare two detection strategies towards the detection of eccentric binaries – a monochromatic search and a harmonic summing technique. Finally, we summarise in section 4.7.

4.2 Models of single-source gravitational waves

We have discussed a PTA’s response to single-source GWs in section 2.3.4. Here we describe signal models that are considered in this study. Throughout this chapter, we only consider the correlated Earth-term signals. In fact for the case of CWs as expected from SMBBHs, pulsar terms act as an extra source of uncorrelated noise for different pulsars. For GW bursts whose duration is smaller than the data span, it is very unlikely that pulsar terms and Earth terms are simultaneously present in timing residuals for one particular pulsar unless the source sky direction is very close to that pulsar.

4.2.1 Supermassive binary black holes in circular orbits

At the leading Newtonian order, \( A_+ (t) \) and \( A_\times (t) \) take the following forms for GWs emitted by non-spinning SMBBHs in circular orbits (Babak & Sesana 2012; Ellis et al. 2012; Zhu et al. 2014):

\[
A_+ (t) = \frac{h_0}{2\pi f(t)} \left\{ (1 + \cos^2 \iota) \cos 2\psi \sin[\Phi(t) + \Phi_0] + 2 \cos \iota \sin 2\psi \cos[\Phi(t) + \Phi_0] \right\} \quad (4.1)
\]

\[
A_\times (t) = \frac{h_0}{2\pi f(t)} \left\{ (1 + \cos^2 \iota) \sin 2\psi \sin[\Phi(t) + \Phi_0] - 2 \cos \iota \cos 2\psi \cos[\Phi(t) + \Phi_0] \right\},
\]

(4.2)

where \( \iota \) is the inclination angle of the binary orbit with respect to the line of sight, \( \psi \) is the polarization angle, \( \Phi_0 \) is a phase constant, and the intrinsic GW strain amplitude \( h_0 \) is given by

\[
h_0 = 2 \left( \frac{GM_c}{c^4} \right)^{5/3} \left( \frac{\pi f}{d_L} \right)^{2/3},
\]

(4.3)
where $d_L$ is the luminosity distance of the source, and $M_c$ is the chirp mass. It should be noted that we have neglected effects of cosmological redshift as the current data set is only sensitive to SMBBHs up to $z \sim 0.2$ even for the most massive sources. To account for the effects of cosmological redshift, the following transformation (from the source-rest frame to the observer frame) applies: $(f_r, M_c, r_z, t_r) \rightarrow (f_r/(1+z), M_c(1+z), d_L, t_r(1+z))$, where $d_L = r_z(1+z)$ with $r_z$ being the comoving distance as defined in equation (3.9). Therefore for cosmological sources, equation (4.3) is rewritten as (Cutler & Flanagan 1994):

$$h^z_0 = 2\left(\frac{GM^z_c}{c^4}\right)^{5/3} \frac{(\pi f)^{2/3}}{d_L},$$

(4.4)

where $M^z_c = M_c(1+z)$ is the so-called redshifted chirp mass.

The GW phase and frequency in equations (4.1–4.2) are given by:

$$
\Phi(t) = \frac{1}{16} \left(\frac{GM_c}{c^3}\right)^{-5/3} \left\{ (\pi f_0)^{-5/3} - [\pi f(t)]^{-5/3} \right\},
$$

(4.5)

$$
f(t) = \left[ f_0^{-8/3} - \frac{256}{5} \pi^{8/3} \left(\frac{GM_c}{c^3}\right)^{5/3} t \right]^{3/8},
$$

(4.6)

where $f_0$ is the GW frequency at the time of our first observation. For quasi-monochromatic sources the above two equations are reduced to $\Phi(t) \simeq 2\pi f_0 t$ and $f(t) \simeq f_0$ (hereafter $f$ is used to denote GW frequency).

It is clear that signals emitted by individual non-spinning SMBBHs in circular orbits can be divided into two categories – quasi-monochromatic waves and chirps – depending on the frequency evolution over an observation span. Note that black hole spins are known to introduces post-Newtonian corrections to both the amplitude and phase of the Newtonian waveforms used here, through the spin-orbit and spin-spin couplings (Blanchet 2014). Mingarelli et al. (2012) considered such effects and showed that masses and spins of SMBBHs could be measurable with PTAs if both the pulsar terms and Earth terms are detected. However, such corrections are insignificant for our work since we consider only the Earth-term signals. Here
we briefly discuss the two distinctive detection regimes for circular binaries. The frequency evolution is given by

$$\Delta f \simeq 3.94 \text{nHz} \left( \frac{M_c}{10^9 M_\odot} \right)^{5/3} \left( \frac{f_0}{10^{-7} \text{Hz}} \right)^{11/3} \left( \frac{T_{\text{obs}}}{10 \text{yr}} \right),$$  \hspace{1cm} (4.7)

It should be emphasized that the above approximation only holds for the frequency evolution over an observation span (of order 10 yrs), and does not apply for pulsar-Earth light-travel time of order $10^3$ yrs.

The GW signals from inspiralling SMBBHs are well modelled by the PN approximation up to the LSO, which corresponds to a frequency of $f_{\text{LSO}} = 4.4 \times 10^{-7} \text{Hz}/M_{10}$ where $M_{10} = (m_1 + m_2)/10^{10} M_\odot$. It is also interesting to note that there exists a minimum frequency above which the binary evolution is mainly driven by radiation of GWs. Assuming that 1) the binary can reach a separation of $\sim 1$ pc so that dynamical friction becomes ineffective; and 2) the binary hardens through the repeated scattering of stars in the core of the host, such a minimum frequency is given by (see Quinlan 1996, for details):

$$f_{\text{min}} = 2.7 \text{nHz} \left[ \frac{m_1 m_2}{(10^8 M_\odot)^2} \right]^{-0.3} \left( \frac{m_1 + m_2}{2 \times 10^8 M_\odot} \right)^{0.2}.$$  \hspace{1cm} (4.8)

We show in Figure 4.1 two regions in the frequency-mass plane where circular equal-mass SMBBHs are in the quasi-monochromatic and chirping regimes. The lower and upper limits of the frequency axis roughly corresponds to the accessible GW frequency range for PTAs – from $1/(30 \text{ yr})$ to $0.5/(2 \text{ weeks})$. Figure 4.1 shows that most SMBBHs observable to PTAs are in the quasi-monochromatic regime. The fact that binary sources spend most of their lifetime at low frequencies further strengthens this statement.

It should be mentioned that in Figure 4.1 the two unshaded regions are not necessarily inaccessible to PTAs. For example, the merger and ringdown phases of the most massive systems may be probed in the upper-right corner of the parameter space. During the early inspiral stage of coalescences where the orbital frequency
Figure 4.1: The parameter space (GW frequency and BH mass) for circular equal-mass SMBBHs that are expected to emit quasi-monochromatic GWs (light grey) and to experience significant frequency evolution (dark grey) over a typical PTA observation span of 10 yrs. The two regions are divided by the line that marks $\Delta f = 3/(10 \text{ yr})$ (i.e., frequency evolved by 3 frequency bins in 10 yrs). The upper bound of the dark shaded area is determined by $f_{\text{LSO}}$ and the lower bound of the light area is given by the minimum frequency above which the binary evolution is dominated by radiation of GWs.
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Evolution is negligible for typical observation spans of $\sim 10$ yrs. SMBBHs in circular orbits are expected to emit monochromatic GWs. Sufficiently massive systems that are in their late stage of inspirals may experience significant frequency evolution over an observation span.

Unless otherwise specified, we assume throughout this chapter that the source is strictly monochromatic and neglect pulsar terms. In Chapter 5 which deals with real PTA data we further assume that pulsar terms are in the same frequency bin as Earth terms (which is termed as ‘non-evolving’ sources), as also assumed in Yardley et al. (2010). As the detection method targets only the Earth-term signals, pulsar terms only matter in the establishment of upper limits by acting as a “self-generated” source of noise. We use the pulsar DM distance estimates provided in the ATNF Pulsar Catalogue – the actual choice of pulsar distance has negligible effects on our results.

We note that GW signals were under-represented by a factor of $\sqrt{2}$ in Yardley et al. (2010) as we can see below. In that paper, the root mean square (rms) inclination-averaged strain amplitude was used:

$$h_{z \text{rms}} = \sqrt{\frac{32}{5}} \frac{(GM)^{5/3}}{c^4} \frac{(\pi f)^{2/3}}{d_L}.$$  \hspace{1cm} (4.9)

Here we denote it by $h_{\text{rms}}$ since it gives the rms strain amplitude averaged over binary orientations (see, e.g., equation 24 in Jaffe & Backer 2003). Explicitly, one can find $h_{\text{rms}}^z = 2h_0^z \times \sqrt{4/5} \times \sqrt{1/2}$ where $\sqrt{4/5}$ comes from the average over inclination angle as given by equation (3.12), and the $\sqrt{1/2}$ factor corresponds to the difference between the maximum amplitude of a sinusoid and its rms amplitude. The later factor was incorrectly included in Yardley et al. (2010).

4.2.2 Eccentric binaries

Although it is well known that radiation of GWs circularizes the binary orbits (Peters & Mathews 1963), the assumption of circular orbits is not always appro-
appropriate. Recent models for the SMBBH population including the effects of binary environments on orbital evolution suggest that eccentricity is important for GW frequencies $\lesssim 10^{-8}$ Hz (Sesana 2013a; Ravi et al. 2014).

In this work we use the expressions of GW-induced timing residuals for eccentric SMBBHs as given in Jenet et al. (2004), which followed the derivations of Estabrook & Wahlquist (1975) and Wahlquist (1987). For completeness, we also present the key equations below.

\[ A_+ (t) = \alpha (t) [B_1 (t) \cos(2\phi) - B_2 (t) \sin(2\phi)], \]  
\[ A_\times (t) = \alpha (t) [B_1 (t) \sin(2\phi) + B_2 (t) \cos(2\phi)], \]  
\[ \alpha (t) = \left( \frac{GM_z^5}{c^4 d_L \omega(t)^{1/3}} \right) \frac{\sqrt{1 - e(t)^2}}{[1 + e(t) \cos(\theta(t))]}. \]

where $\phi$ is the orientation of the line of nodes on the sky, $\omega(t)$ is the orbital frequency, $e(t)$ is the eccentricity and $\theta(t)$ is the orbital phase. The functions $B_1(t)$ and $B_2(t)$ are given by:

\[ B_1 (t) = 2e(t) \sin\theta(t) \left\{ \cos[\theta(t) - \theta_n]^2 - \cos^2 \iota \sin[\theta(t) - \theta_n]^2 \right\} \]  
\[ - \frac{1}{2} \sin\{2[\theta(t) - \theta_n]\} \{1 + e(t) \cos[\theta(t)]\}[3 + \cos(2\iota)], \]  
\[ B_2 (t) = 2 \cos \iota \cos\{2[\theta(t) - \theta_n]\} + e(t) \cos[\theta(t) - 2\theta_n], \]

where $\iota$ and $\theta_n$ are the orbital inclination angle and the value of $\theta(t)$ at the line of nodes respectively. The functions $e(t)$ and $\theta(t)$ can be obtained by solving the following coupled differential equations.

\[ \frac{d\theta}{dt} = \omega(t) \frac{1 + e(t) \cos[\theta(t)]}{[1 - e(t)^2]^{3/2}}, \]  
\[ \frac{de}{dt} = -\frac{304}{15} \frac{M_c^{5/3}}{\omega_0^{8/3}} \chi_0^{-4} \frac{e(t)^{-29/19}}{[1 + (121/304)e(t)^2]^{1181/2299}}, \]

where $\omega_0$ is the initial orbital frequency and $\chi_0$ is a constant related to the initial
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Here the function \(\omega(t)\) is given by

\[
\omega(t) = a_0 e(t)^{-18/19} \left[ 1 - e(t)^2 \right]^{3/2} \left[ 1 + \frac{121}{304} e(t)^2 \right]^{-1305/2299},
\]

where \(a_0\) is determined by \(\omega(t = 0) = \omega_0\). The above equations are accurate to first order in \(v/c\) (where \(v\) is the orbital velocity) and valid only when both \(e(t)\) and \(\omega(t)\) vary slowly with time.

It is interesting to note that eccentric binaries emit gravitational waves at multiple harmonics of the binary orbital frequency. At low eccentricities the emission is dominated by the second harmonic, while for high eccentricities the orbital frequency itself will dominate. The induced timing residuals from an eccentric binary for a sample of 4 of the 20 PPTA pulsars are shown Figure 4.2.

4.2.3 Gravitational wave bursts

Generally a gravitational wave burst is defined as a transient signal with a duration smaller than the observation span. This may be the only information we know about the source. For this reason we use a simple but general sine-Gaussian model for timing residuals induced by bursts:

\[
A_+ (t) = A \exp \left( -\frac{(t-t_0)^2}{2\tau^2} \right) \{ (1 + \cos^2 \iota) \cos 2\psi \cos [2\pi f_0 (t - t_0) + \phi_0] 
- 2 \cos \iota \sin 2\psi \sin [2\pi f_0 (t - t_0) + \phi_0] \} \]

\[
A_\times (t) = A \exp \left( -\frac{(t-t_0)^2}{2\tau^2} \right) \{ (1 + \cos^2 \iota) \sin 2\psi \cos [2\pi f_0 (t - t_0) + \phi_0] 
+ 2 \cos \iota \cos 2\psi \sin [2\pi f_0 (t - t_0) + \phi_0] \},
\]

(4.19)

(4.20)
Figure 4.2: GW induced timing residuals from an eccentric SMBBH located in the sky direction of the Virgo cluster for a sample of 4 of 20 PPTA pulsars: J0437−4715 (solid blue), J1713+0747 (dash green), J1909−3744 (dash-dotted red) and J1939+2134 (solid black). Only the Earth-term signals are shown. The simulated source is an equal-mass ($10^9$−$10^9 M_\odot$) binary, having an eccentricity of 0.5, an orbital frequency of 5 nHz, an initial orbital phase of 1, and an inclination angle of 0 (i.e., face-on).
where $A$ is the signal amplitude (in seconds), $\tau$ is the Gaussian width, $\iota$ is the source inclination angle, $f_0$ is the central frequency, $t_0$ and $\phi_0$ are the time and phase at the midpoint of the burst respectively. The sine-Gaussian model used here can represent qualitatively the signals for parabolic encounters of two massive black holes as studied in Finn & Lommen (2010) and more recently in Deng (2014). The detection method that we will describe in the next section should apply equally well to other burst sources such as cosmic (super)string cusps and kinks and triplets of supermassive black holes (see discussion in section 2.3.5).

4.3 A coherent method for detection, localization and waveform estimation of single-sources

In this section, we describe how the singular value decomposition (SVD) can be used in a general method for the detection and localization of single-source GWs using PTA. The method is adapted from the coherent network analysis method used in the context of ground-based GW interferometers. There are some important features that are unique to PTA observations, e.g., (1) PTA data are irregularly sampled in contrast to nearly continuous sampling for ground-based experiments; and (2) a least-squares fitting process is performed to obtain estimates of timing parameters such as the pulsar’s spin period and its first time derivative, pulsar position and proper motion, etc (see section 2.3.2 for details). As we show later in this section, the SVD method proposed here relies on transforming the timing residuals of each pulsar to the frequency domain. For the idealized simulations (assuming even sampling and white Gaussian noise with equal error bars) used in section 4.4, a discrete Fourier Transform was used. In section 4.5 for more realistic data sets we adopt a maximum-likelihood-based method to estimate Fourier components of the timing residuals making use of the noise covariance matrix (see, e.g., section 5.3.2). This is equivalent to the least-squares spectral analysis method (see, e.g. Coles et al. 2011). Our method works with post-fit timing residuals, i.e., after fitting
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TOAs for timing models of individual pulsars. The effects of the fitting process on our results will be discussed in section 4.5. We note that a similar approach was used in Lentati et al. (2013) where a stochastic background is modeled as a sum of a number of frequency components.

For a given source direction, the timing residuals from an array of $N_p$ pulsars can be generally written in the frequency domain as:

$$d_k = F_k A_k + n_k,$$

(4.21)

where the index $k$ denotes the $k$-th frequency bin, $d_k$ are timing residual data, $F_k$ is the response matrix, $A_k$ are gravitational waveforms and $n_k$ is the timing noise. The data are whitened\(^1\) so that

$$d_k = \begin{bmatrix} d_{1k}/\sigma_{1k} \\ d_{2k}/\sigma_{2k} \\ \vdots \\ d_{Npk}/\sigma_{Npk} \end{bmatrix}, A_k = \begin{bmatrix} A_{+k} \\ A_{\times k} \end{bmatrix}, n_k = \begin{bmatrix} n_{1k}/\sigma_{1k} \\ n_{2k}/\sigma_{2k} \\ \vdots \\ n_{Npk}/\sigma_{Npk} \end{bmatrix},$$

(4.22)

where $\sigma_{ik}^2$ is the noise variance of the $i$-th pulsar at the $k$-th frequency bin. Here $d_k$, $A_k$ and $n_k$ are all complex vectors, while the real whitened response matrix $F_k$ is defined as

$$F_k = \begin{bmatrix} F_{1+}/\sigma_{1k} & F_{1\times}/\sigma_{1k} \\ F_{2+}/\sigma_{2k} & F_{2\times}/\sigma_{2k} \\ \vdots & \vdots \\ F_{Np+}/\sigma_{Npk} & F_{Np\times}/\sigma_{Npk} \end{bmatrix}.$$

(4.23)

Data for all frequencies can be stacked (e.g., in order of increasing frequency) to preserve the format of equation (4.21), in which case $F$ is a block-diagonal matrix (with a dimension of $N_p N_k \times 2N_k$ where $N_k$ is the number of frequency bins) of $F_k$. For simplicity we will hereafter suppress the index $k$ while keeping in mind

\(^1\)Here it is assumed that noise for different pulsars is uncorrelated. If not the full covariance matrix should be used in a whitening process.
that equations (4.21, 4.23) apply to each frequency bin in the analysis.

The SVD of the response matrix $F$ can be written as

$$ F = USV^*, S = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, $$

(4.24)

where $U$ and $V$ are unitary matrices with dimensions of $N_p \times N_p$ and $2 \times 2$ respectively, and the symbol $*$ denotes the conjugate transpose. Singular values in $S$ are ranked such that $s_1 \geq s_2 \geq 0$. We can then construct new data streams as follows:

$$ \tilde{d} = U^*d, \ \tilde{A} = V^*A, \ \tilde{n} = U^*n. $$

(4.25)

One can find that $\tilde{d} = S\tilde{A} + \tilde{n}$, and explicitly

$$ \tilde{d} = \begin{bmatrix} s_1 (V^*A)_1 + \tilde{n}_1 \\ s_2 (V^*A)_2 + \tilde{n}_2 \\ \tilde{n}_3 \\ \vdots \\ \tilde{n}_{N_p} \end{bmatrix}. $$

(4.26)

In the absence of GW signals, the real and imaginary parts for each element of $\tilde{d}$ independently follow the standard Gaussian distribution. Here $\tilde{d}_1$ and $\tilde{d}_2$ are referred to the two signal streams since they contain all information about present signals, while the remaining terms are ‘null streams’ (denoted by $\tilde{d}_{\text{null}}$) since they have a null response to GWs. It has been shown that, in the context of ground-based interferometers, null streams can be used as a consistency check on whether a candidate event is produced by detector noise or by a real GW (Wen & Schutz)

\footnote{Here the indices 1 and 2 refer to the first two terms in the new data streams $\tilde{d}$. More indices are used for time and frequency when necessary.}
and ‘semi-null streams’ (i.e., \( \vec{d}_2 \) if \( s_1 \gg s_2 \)) can be included to improve the angular resolution (Wen 2008; Wen et al. 2008). In this work we only consider Earth terms in our signal model so the null streams (as constructed in the current way) are indeed ‘null’ but in reality they would have some response to the pulsar-term signals. In a future study pulsar terms will be incorporated in the detection framework with the addition of \( N_p \) free parameters for pulsar distances in the response matrix \( F \).

It is straightforward to show that the maximum likelihood estimator for the gravitational waveform is

\[
\hat{A} = \bar{F} \vec{d}, \quad \text{where } \bar{F} = VSU^*, \quad S = \begin{bmatrix}
1/s_1 & 0 & 0 & \ldots & 0 \\
0 & 1/s_2 & 0 & \ldots & 0
\end{bmatrix}.
\]  

(4.27)

The matrix \( \bar{F} \) is the Moore-Penrose pseudoinverse of the response matrix \( F \). The covariance matrix for the estimated waveform is

\[
\text{var}(\hat{A}) = (VS^T SV^*)^{-1}. 
\]  

(4.28)

It is interesting to note that the statistical uncertainties of the estimated waveforms are a linear combination of \( s_1^{-2} \) and \( s_2^{-2} \). Estimation of physical parameters can be obtained based on the estimated waveforms \( \hat{A} \) and we leave this to a future study.

Now we propose our detection statistics for the three types of signals discussed in the previous section. For monochromatic waves, the detection statistic can be written as:

\[
P_{\text{mon}} = |\vec{d}_1|^2 + |\vec{d}_2|^2. 
\]  

(4.29)

Note that such a statistic is optimal under the Neyman-Pearson criterion for a monochromatic signal (Sutton et al. 2010). It is important to note that: (1) in the absence of signals, \( P_{\text{mon}} \) follows a \( \chi^2 \) distribution with 4 degrees of freedom; and (2) the detection statistic applies to a given frequency and source sky location. In practice when such information is unknown, a search is usually performed to
4.3. A SVD-based method for single-source data analysis

find the maximum statistic.

For signals produced by eccentric binaries, we use a harmonic summing technique for which the detection statistic is given by:

$$P_{\text{ecc}} = \sum_{j=1}^{N_h} \left( |\tilde{d}_{1,jk_0}|^2 + |\tilde{d}_{2,jk_0}|^2 \right),$$

(4.30)

where $N_h$ corresponds to the highest harmonic considered in the search, $k_0$ is the bin number for the binary orbital frequency. Note that $P_{\text{ecc}}$ is sub-optimal because it involves an incoherent sum of signal power distributed in different harmonics. In the absence of signals, $P_{\text{ecc}}$ follows a $\chi^2$ distribution with $4N_h$ degrees of freedom. In practice $N_h$ should be determined as the one that gives the lowest false alarm probability (FAP). When the orbital frequency is unknown one should search over all possible frequencies to find the maximum statistic.

For GW bursts with unknown waveforms, we adopt a time-frequency strategy in which the PTA data are first divided into small segments and then SVD is applied to each segment to output the two signal streams. Note that it is usually necessary to allow overlap between successive segments to catch signals occurring in the beginning or end of the segment. The detection of GW bursts of unknown waveforms generally involves searching for any ‘tracks’ or ‘clusters’ of excess power in the time-frequency space (e.g. Anderson & Balasubramanian 1999, Wen & Gair 2005, Gair et al. 2008). So the detection statistic is designed as accumulating signal power in (a ‘box’ of) the time-frequency domain and can be written as:

$$P_{\text{GWb}}(i,k) = \sum_{a=-l/2}^{l/2} \sum_{b=-m/2}^{m/2} \left( |\tilde{d}_{1,(i+a)(k+b)}|^2 + |\tilde{d}_{2,(i+a)(k+b)}|^2 \right),$$

(4.31)

for the $i$-th segment and $k$-th frequency bin. Here $l$ and $m$ are the length of box in time and frequency respectively. Note that $P_{\text{GWb}}$ is optimal under the Neyman-Pearson criterion for burst GWs with unknown waveforms (Sutton et al. 2010). If the data consist of only Gaussian noise, $P_{\text{GWb}}$ follows a $\chi^2$ distribution with $4N_h$.
degrees of freedom (where \( N_b = l \times m \)).

The detection statistics proposed here all follow a noncentral \( \chi^2 \) distribution with their corresponding degrees of freedom when signals are present in the data. It is convenient to define the expected signal-to-noise ratio \( (\rho) \) as the noncentrality parameter

\[
\langle P \rangle = N_{\text{dof}} + \rho^2,
\]

where the brackets \( \langle ... \rangle \) denote the ensemble average of the random noise process, \( N_{\text{dof}} \) is the number of degrees of freedom for the ‘central’ distribution of \( P \). We use \( \rho \) to quantify the signal strength in our simulations. Note that \( \rho^2 \) equals the detection statistic calculated for noiseless data. In a frequentist detection framework, we are interested in the FAP of a measured detection statistic \( P_0 \), i.e., the probability that \( P \) exceeds the measured value for noise-only data. For the aforementioned methods the single-trial FAP is given by \( 1 - \text{CDF}(P_0; \chi^2) \) where \( \text{CDF}(; \chi^2) \) denotes the cumulative distribution function (CDF) for the central \( \chi^2 \) distribution in question.

When a search is performed over unknown source parameters, the total FAP is \( 1 - [\text{CDF}(P_{\text{max}}; \chi^2)]^{N_{\text{trial}}} \) for the maximum detection statistic \( P_{\text{max}} \) found in the search with \( N_{\text{trial}} \) being the trials factor, which is defined as the number of independent cells in the searched parameter space for a grid-based search.

### 4.3.1 Relation to the ‘\( A_+ A_\times \)’ method

In the pulsar timing software package TEMPO2, there exists a functionality with which two time series \( A_+^2(t) \) and \( A_\times^2(t) \) can be estimated for a given sky direction. These two time series correspond to two polarizations of the coherent Earth-term timing residuals. Here we call it the ‘\( A_+ A_\times \)’ method, which will be described in more detail in section 5.3.1. The ‘\( A_+ A_\times \)’ method was first illustrated for arbitrary GW bursts in fig. 5 of Hobbs [2013]. It will be used in Chapter 5 for an all-sky search for monochromatic signals in a recent PPTA data set. A complete presentation on this method and its applications to single-source detection can be found in Madison et al. [2016]. Here we briefly discuss the relation between
the ‘\(A_+ A_x\)’ method and the SVD method proposed in this chapter. Firstly the
principle of both methods is identical since both use the response matrix \(F\) in
a similar way and equation (4.27) essentially gives the least-squares solution to
the two polarization waveforms, i.e., \(\hat{A}\) in equation (4.27) is the frequency-domain
equivalent of \(A_{1,x}^2(t)\). The critical difference is in the implementation – the ‘\(A_+ A_x\)’
method works in the time domain whereas the SVD method works in the frequency
domain.

There are two advantages of the SVD method over the ‘\(A_+ A_x\)’ method:

1. The SVD method is much faster when an all-sky search is required. This is
because it works with post-fit residuals after data for each pulsar have been
fitted for a timing model and a search over unknown source sky location
only involves doing the SVD for the response matrix \(F\), while in the ‘\(A_+ A_x\)’
method a global fit is done for each searched sky location (as in its current
implementation);

2. A truncated SVD may be used to improve the detection sensitivity, which
applies to the case when \(s_1 \gg s_2\), e.g., when the PTA has very low response
to one of the two GW polarizations for some sky regions. This is possible
especially for current PTAs whose sensitivities are dominated by a few best-
timed pulsars.

The ‘\(A_+ A_x\)’ method has been fully tested and implemented in real data for CWs
in Zhu et al. (2014). We will demonstrate the effectiveness of the SVD method
with some examples using idealized data sets in section 4.4 and then more realistic
data sets in section 4.5.

4.4 Examples using idealized data sets

For the purpose of illustration of our method, we consider an idealized PTA
consisting of 20 PPTA pulsars. The simulated observations are evenly sampled once
every two weeks with a time span of 10 yr. All observations contain stationary
white Gaussian noise with a \textit{rms} of 100 ns and equal error bars. The simulated data sets are produced as a combination of realizations of white Gaussian noise and Earth-term timing residuals. When we apply our detection statistics to noise-only data sets, it is confirmed that they follow a $\chi^2$ distribution with their respective numbers of degrees of freedom.

Here, we illustrate how our method works in terms of detection, sky localization and waveform estimation for the three types of signals considered in this work. The analysis is simplified as follows: (1) first the detection problem is demonstrated by evaluating the detection statistics at the injected source sky direction; (2) then detection statistics are computed on a uniform grid of sky directions and for a range of frequencies to find the maximum; and (3) finally the frequency-domain waveforms $A_+(f)$ and $A_\times(f)$ are estimated at the actual source sky location. To show the correctness of the estimation process, a number of noise realizations were performed in order to obtain average estimates of $A_+,\times(f)$ which were then compared with the true waveforms. For simplicity, in the relevant figures we only plot the absolute values of $A_+(f)$ and $A_\times(f)$, which are called spectral signatures. For the detection problem we calculate the single-trial FAP to quantify the detection significance. This is only appropriate when source parameters (e.g., sky location and frequency) are known as we assume for the following examples. The simulated signals are weak to moderately strong with signal-to-noise ratios ranging from 5 to 10.

\subsection*{4.4.1 Monochromatic waves}

For monochromatic waves, the simulated signal is characterized with the following parameters: $h_0 = 1.16 \times 10^{-15}$, $f = 10$ nHz, $\cos \iota = 1$, $\psi = 0$, $\phi_0 = 0$, $(\alpha, \delta) = (0, 0)$. The expected signal-to-noise ratio is $\rho = 10$. Figure 4.3 shows the detection statistics $P_{\text{mon}}$ as a function of frequency evaluated at the injected sky location. The maximum statistic, which gives extremely strong evidence of a detection, is found at a frequency of 9.94 nHz. To localize this source, we calculate $P_{\text{mon}}$ for the
4.4. Examples using idealized data sets

Figure 4.3: Detection statistics ($P_{\text{mon}}$) as a function of frequency for a simulated data set that includes a strong monochromatic signal (solid black) and for the noise-only data set (dashed red). The vertical line marks the injected frequency (10 nHz), while the horizontal line corresponds to a single-trial FAP of $10^{-4}$.

same range of frequencies and for a grid of sky directions. At each sky direction, the maximum statistic over frequencies is recorded. Figure 4.4 shows that the source is successfully localized to where the signal is generated.

Having already detected and localized the source, we use equations (4.27–4.28) to infer the gravitational waveforms. Figure 4.5 shows the estimated spectral signatures along with the true spectra, indicating a very good estimation as one would expect since the injected signal is very strong. To check the correctness of our method, we overplot in Figure 4.5 the average estimates taken over 100 noise realizations – they are almost identical to the true spectra. We note that there appears to be a spectral leakage problem in our waveform reconstruction. This may affect the recovery of physical parameters using our method, which will be investigated in a future study.
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Figure 4.4: Sky map of detection statistics calculated for a simulated data set that includes a strong monochromatic signal. The signal is simulated at the center of the map and the maximum detection statistic is found at “○” (which is the same as the actual source location). Sky locations of the 20 PPTA pulsars are marked with “⋆”.

4.4.2 Eccentric binaries

The simulated source for this example is an eccentric SMBBH located in the Virgo cluster as described by the following parameters: $m_1 = m_2 = 2 \times 10^8 M_\odot$, $d_L = 16.5 \text{ Mpc}$, $\cos \iota = 1$, $(\alpha, \delta) = (3.2594, 0.2219)$, an orbital frequency of 5 nHz, an initial orbital phase of 1 and an eccentricity of 0.5. Here we choose such a moderate eccentricity to test the performance of the harmonic summation technique because for very high or very low eccentricities only one harmonic dominates the GW emission.

In order to ‘detect’ this simulated signal, we first apply our analysis at the injected sky location and experiment on the number of harmonics that should be considered. It turns out that a summation up to the second harmonic gives the lowest FAP. We will further discuss in section 4.6 the number of harmonics that should be included for binaries with different eccentricities. Figure 4.6 shows the detection statistics $\mathcal{P}_{\text{ecc}}$ as a function of orbital frequency. The maximum statistic 59.85, corresponding to a FAP of $5 \times 10^{-10}$, is found at 4.87 nHz. The
Figure 4.5: Estimated spectral signatures (thin solid black) for the same data set used in Figures 4.3 and 4.4, along with true spectra (solid blue) and average estimates taken over 100 noise realizations (dashed red).
Figure 4.6: Detection statistics ($P_{ecc}$) as a function of orbital frequency for a simulated data set that includes a moderately strong signal (solid black) and for the noise-only data set (dashed red). The signal is produced by an eccentric SMBBH with $e = 0.5$. Detection statistics are calculated as a harmonic summation up to the second harmonic. The vertical line marks the injected orbital frequency (5 nHz), while the horizontal line corresponds to a single-trial FAP of $10^{-4}$.

The expected signal-to-noise ratio for this injection is $\rho = 7$ when we perform a harmonic summing up to the second harmonic. Then we show in Figure 4.7 the detection statistics calculated for the whole sky (after maximizing over orbital frequencies for each sky direction). The maximum statistic is found at a grid point close to the injected sky location.

Figure 4.8 shows the estimated spectral signatures assuming that we know the actual source location. Since the detection statistic varies only slightly within an area of tens of square degrees (as shown in Figure 4.7), similar results should be obtained if we apply the spectral estimation analysis in the sky location where the maximum statistic is found. It is shown that reasonably good estimates are
4.4. Examples using idealized data sets

Figure 4.7: Sky map of detection statistics calculated for a simulated data set that includes a simulated signal produced by an eccentric SMBBH located in the Virgo cluster (marked by a white diamond). The maximum statistic is found at “o”. Sky locations of the 20 PPTA pulsars are marked with “⋆”.

Figure 4.8: Estimated spectral signatures (thin solid black) for the same data set used in Figures 4.6 and 4.7, along with true spectra (solid blue) and average estimates taken over 100 noise realizations (dashed red).
4.4.3 Gravitational wave bursts

In this example the signal parameters are ($\rho = 5$): $A = 100 \text{ ns}$, $\tau = 60 \text{ days}$, $\cos \iota = 0.5$, $f_0 = 50 \text{ nHz}$, $\psi = 1.57$, $\phi_0 = 0$, $(\alpha, \delta) = (0.9512, -0.6187)$, i.e., originating from the Fornax cluster. Such a signal may qualitatively represent GW produced by a parabolic flyby of two $10^9 M_\odot$ BHs (Finn & Lommen 2010). The signal occurs in the middle of our observations (MJD 55250). Waveforms of $A_{+\times}(t)$ are shown in the upper panel of Figure 4.9. For this burst source the amplitudes of timing residuals are $\lesssim 100 \text{ ns}$, which means the signal should not be visibly apparent in individual pulsar data set.

To dig out this burst signal from noisy data, we divide the 10-yr data set into segments of length of 300 days and calculate the detection statistic given by
4.4. Examples using idealized data sets

Figure 4.10: Sky map of detection statistics calculated for a simulated data set that includes a simulated GW burst originating from the Fornax cluster (marked by a white diamond). The maximum statistic is found at "○". Sky locations of the 20 PPTA pulsars are marked with "★".

Figure 4.11: Estimated spectral signatures (thin solid black) for the same data set used in Figures 4.9 and 4.10, along with true spectra (solid blue) and average estimates taken over 100 noise realizations (dashed red).
equation (4.31) in the time-frequency domain. The segment length is (roughly) chosen based on the knowledge of $f_0$, i.e., having $\gtrsim$ one cycle per segment. In practice since the time-frequency analysis for PTA data should not be limited by computational power, many trials of segment length can be performed to search for signals of different durations. The lower panel of Figure 4.9 shows results of the time-frequency analysis assuming known source sky location. The most significant statistic 38.36, corresponding to a FAP of $9 \times 10^{-8}$, is found at MJD 55257 and at a frequency of 51.67 nHz. Therefore our analysis has clearly detected the injected signal and correctly identified its occurrence time and central frequency. For this example, the signal is very ‘isolated’ in the time-frequency space, so it is sufficient to only look at the maximum statistic in the time-frequency map.

Figure 4.10 shows the detection statistics evaluated at the whole sky (after maximizing over frequencies for each sky direction). The maximum statistic is found at a grid point close to the injected source location. Figure 4.11 shows the inferred spectral signatures compared against the true spectra. As the injected signal is relatively weak for this example, the spectrum is not recovered as well as the previous two examples. For both plots only the central segment that contains the majority of signal power (as illustrated in Figure 4.9) was used.

4.5 More realistic data sets

Examples given in the previous section have assumed idealized PTA observations that are evenly sampled and contain only stationary white Gaussian noise with equal error bars. While this is unrealistic for actual PTA data, evenly sampled data can be constructed using interpolation. For example, when correcting the DM variations for multiple-band pulsar timing data, one obtains estimation of the common-mode signal that is independent of radio wavelengths (Keith et al. 2013). These common-mode data sets are evenly sampled and can be used for GW searches.

Realistic features typical to real PTA data include: (1) observations are ir-
4.5. More realistic data sets

regularly sampled and have varying TOA error bars. It is also common that the data span varies significantly among different pulsars; and (2) low-frequency (‘red’) timing noise may be present for some pulsars. To simulate realistic data sets, we make use of the actual data spans, sampling and error bars of the PPTA 6-yr Data Release 1 (DR1) data set that was published in Manchester et al. (2013) and used in Zhu et al. (2014) for an all-sky search for CWs (see Chapter 5 for details). As a check, in some simulations an uncorrelated red-noise process with a power-law spectrum is also included. This has no effect on the results since we assume the noise spectrum is known and thus include it explicitly in the noise covariance matrix. In actual analyses of real data, the noise estimation is a very important step and we do not attempt to address it in this work.

Figure 4.12 shows the distribution of the detection statistics $P_{\text{mon}}$ in the presence of signal along with the distribution of corresponding squared null streams ($|\tilde{d}_{\text{null}}|^2$) as defined in equation (4.26). The injected signal is characteristic of a circular SMBBH described by the following parameters: $h_0 = 1 \times 10^{-14}$, $f = 20$ nHz, $\cos \iota = 0.6172$, $\psi = 4.1991$, $\phi_0 = 1.9756$, $(\alpha, \delta) = (0, 0)$. The expected signal-to-noise ratio is 10, which is identical to the example shown in section 4.4.1. To obtain the distribution of the detection statistic and null streams, we perform $10^4$ realizations of white Gaussian noise and for each noise realization keep the search parameters (i.e., source frequency and sky direction) fixed at their injected values. We can see that both match the expected distributions perfectly. As mentioned in the previous section, null streams can be used as a consistency check in the case of a detected signal – if the detected signal is due to a GW null streams should follow the expected noise distribution while otherwise false alarms caused by any noise processes are very unlikely to exhibit such a property.

Figure 4.13 shows the detection statistics as a function of frequency calculated for realistic simulated data sets in the absence or presence of the same monochromatic signal as in Figure 4.12. We have considered two statistics, namely $P_{\text{mon}}$ proposed in this work and the $F_e$-statistic that was adapted from the $F$-statistic
Figure 4.12: (a) Probability distribution of the detection statistic $P_{\text{mon}}$ (solid blue) in the presence of a monochromatic signal compared to the expected noncentral $\chi^2$ distribution (dashed red) with 4 degrees of freedom and a noncentrality parameter $\rho^2 = 100$. The simulation was performed with search parameters fixed at the injected values and $10^4$ realizations of white Gaussian noise. (b) For the same simulation as panel (a) but instead showing the probability distribution of the squared null streams $|\tilde{d}_{\text{null}}|^2$ (solid blue) compared to the expected $\chi^2$ distribution with 2 degrees of freedom (dashed red).
Figure 4.13: Detection statistics as a function of frequency for realistic simulated data sets in the absence (thin curves) and presence (thick curves) of a strong monochromatic signal. Here we compare two statistics – $P_{\text{mon}}$ (solid red) proposed in this work and the $F_e$-statistic (black dash). All statistics are evaluated at the injected source sky direction. The vertical line marks the injected frequency (20 nHz), while the horizontal line corresponds to a single-trial FAP of $10^{-4}$. 
used in CW searches with ground-based interferometers (Jaranowski et al. 1998) and first proposed in the context of PTAs by Babak & Sesana (2012) and later strengthened by Ellis et al. (2012). Both statistics give nearly identical results. In the signal-present case, the maximum value (129.4) of $P_{\text{mon}}$ is found at 20.5 nHz, while the largest $F_{e}$-statistic (125.4) is measured at 20.6 nHz. It is also obvious that at some high frequencies there is a spectral leakage problem that applies to both methods. We leave the investigation of such problems to a future study. The nearly identical performance of both methods for this example is not surprising given the similar principle of the two statistics: (1) the $F_{e}$-statistic works fully in the time domain, but it also involves a process of maximum-likelihood estimation of the fourier components in individual pulsar data set; (2) in the derivation of the $F_{e}$-statistic, the four (time-dependent) basis functions of the monochromatic timing residuals are essentially equivalent to the two orthogonal sine-cosine pairs of $A_{+}(t)$ and $A_{\times}(t)$.

Figure 4.14 shows the results of an all-sky search using the SVD method for the same signal-present data set as used in Figure 4.13. The maximum detection statistic 130.1 is found at $(\alpha, \delta) = (6.185, -0.133)$ with a frequency of 20.5 nHz. The source is localized to a direction close to the injected location. However, compared with Figure 4.4, we can see that the angular resolution for realistic PTAs is much worse because the sensitivity is dominated by a few good ‘timers’ in the array. It is worth pointing out that the SVD method is also advantageous in terms of computational efficiency over that of the $F_{e}$-statistic, since for the latter the amount of computation is proportional to the total number of sky locations being searched.

Similar results were obtained if the simulated data sets used in Figures 4.12–4.14 have gone through the TEMPO2 fitting process for a full timing model for each pulsar. In fact, it has been shown that the fitting process can be approximated by multiplying the timing residuals by a data-independent and non-invertible linear operator matrix (e.g. Demorest 2007). This matrix can be explicitly included in
4.6. Sensitivities to eccentric supermassive binary black holes

Here we are interested in current PTA’s sensitivities to eccentric binaries. Given that previous published searches for signals from SMBBHs in circular orbits, we attempt to answer the following question – when is a monochromatic search suffi-
cient to detect eccentric binaries? For this purpose, we use simulated data sets that have the same sampling and error bars as the PPTA DR1 data set, and consider two detection statistics $P_{\text{mon}}$ and $P_{\text{ecc}}$. The sensitivity is parameterized by the luminosity distance within which 95% of sources are detectable at a FAP of $10^{-4}$.

To facilitate our calculations, we simulate signals due to eccentric binaries drawn from uniform distribution in $\cos \delta$ and $\alpha$ and evaluate both $P_{\text{mon}}$ and $P_{\text{ecc}}$ for noiseless data at the injected sky location. For the demonstration here, we consider sources with a chirp mass of $10^9 M_\odot$ and an orbital frequency of 10 nHz with other parameters such as $\cos \iota$, initial orbital phase and polarization angle randomized. Since here the data consist of signals only, the detection statistic equals the squared signal-to-noise ratio $\rho^2$ and scales inversely with $d_L^2$. We use this scaling relation to find the value of $d_L$ that corresponds to the given detection threshold. We perform $10^4$ Monte Carlo simulations and choose the 95% quantile as a point in the sensitivity plot.

Figure 4.15 shows the sensitivity to GWs from eccentric SMBBHs using two detection strategies – a monochromatic search and a harmonic summing technique. It is clearly demonstrated that for high and low eccentricities a monochromatic search is better than the harmonic summing search, while the latter is more sensitive in the moderate regime ($0.3 \lesssim e_0 \lesssim 0.55$). This is in good agreement with the fact that for high and low eccentricities gravitational radiation is dominated by the orbital frequency and its second harmonic respectively. In the high- and low-eccentricity regimes, adding incoherently power from secondary harmonics is not beneficial because when a harmonic summing technique is adopted the detection threshold should be increased for a given FAP. We note that a recent work by Taylor et al. (2016) found a qualitatively similar trend as Figure 4.15 when applying Bayesian searches for signals of varying eccentricity. We also find that the inclusion up to the third harmonic is sufficient for all possible eccentricities in this example.
4.6. Sensitivities to eccentric supermassive binary black holes

Figure 4.15: Luminosity distance ($d_L$), within which 95% of eccentric SMBBHs with a chirp mass of $10^9 M_\odot$ and an orbital frequency of 10 nHz could be detectable with current PTA, as a function of orbital eccentricity ($e_0$). Two detection strategies are considered: a monochromatic search (black dots) and a harmonic summation technique (blue open circles).
**4.7 Conclusions**

PTA experiments have steadily improved their sensitivities over the past few years and produced very stringent constraints on the evolution of ensembles of SMBBHs. While the traditional focus was a stochastic background from SMBBHs throughout the Universe, possibilities of detecting and studying single sources using PTAs have been explored in depth in recent years. Although it may be more challenging, detections of individual sources will provide rich information on source properties such as the orbital eccentricities, masses and even spins of SMBBHs. This prospect becomes increasingly important as FAST is expected to come online in 2016 and the planned SKA will be operational within a decade. Both FAST and SKA will provide a major step forward in terms of single-source detection with PTAs.

Based on the coherent network analysis method used in ground-based interferometers, we proposed a method that is capable of doing detection, sky localization and waveform estimation for various types of single-source GWs. We demonstrated the effectiveness of this method with proof-of-principle examples for SMBBHs in circular or eccentric orbits and GW bursts. We also demonstrated the implementation of this technique using realistic data sets that include effects such as uneven sampling and varying data spans and error bars. The new method is found to have the following features: (1) it is fast to run especially for all-sky blind searches; (2) it performs as well as published time-domain methods for realistic data sets; and (3) null streams can be constructed as a consistency check in the case of detected signals. Finally, we presented sensitivities to eccentric binaries and found that (1) a monochromatic search that is designed for circular binaries can efficiently detect SMBBHs with both high and low eccentricities; and (2) a harmonic summing technique provides better detection sensitivities for moderate eccentricities.
Chapter 5

An All-sky Search for Continuous Waves in the Parkes Pulsar Timing Array Data Set

This chapter is adapted from the following published paper (Zhu et al. 2014):

Section 5.2 has been expanded to include extra discussions on the measured white noise in PPTA pulsars (summarized in Table 5.1).

5.1 Introduction

In this chapter we develop a new method which conducts a global generalized least-squares fit for Earth-term timing residuals induced by single-source GWs (regardless of their specific waveforms) and outputs two time series which correspond to the two GW polarizations and their covariance matrix. A maximum likelihood detection technique is then applied to the two time series to form our detection statistics for CWs from different sky directions. Using this method we perform an all-sky search in the PPTA DR1 data set for CWs that could be produced by circular SMBBHs (as described in section 4.2). As the search did not find any statistically significant signals, we set upper limits on the intrinsic GW strain amplitude \( h_0 \) and present all-sky and directional sensitivity curves.
This chapter is organized as follows. In section 5.2 we provide a brief overview of our timing observations. In section 5.3 and 5.4 we describe respectively the data analysis method used in performing the search and the data analysis pipeline for detecting and limiting CWs using PTA observations. We present in section 5.5 results and discussions. Finally we conclude in section 5.6.

## 5.2 Observations

Based on the Parkes 64-m radio telescope, the [PPTA](https://www.ppta.org) project started in early 2005 routine timing observations (e.g., once every 2-3 weeks) of 20 millisecond pulsars in three radio bands (10, 20 and 50 cm) with a typical integration time of 1 h. With the development of new instrumentation, the timing precision has been steadily improved (see [Manchester et al. 2013](https://doi.org/10.1093/mnras/stt995) for details). We use in this search the DR1 data set reported in [Manchester et al. 2013](https://doi.org/10.1093/mnras/stt995), including observations made between 2005 March 1 (MJD 53430) and 2011 February 28 (MJD 55620). For each pulsar TOAs of the best band (i.e., where the lowest root mean square (rms) timing residuals are seen) have been selected after correcting for dispersion measure (DM) variations ([Keith et al. 2013](https://doi.org/10.1093/mnras/sts55)).

The data set that we use here is identical to that used for searching for GW memory events by [Wang et al. 2015](https://doi.org/10.1126/science.aab2368). Details on how we obtain the noise models for each pulsar are given in that paper and we only give a very brief description here. For about half the [PPTA](https://www.ppta.org) pulsars, low-frequency timing noise (“red noise”) is observed in the timing residuals. In the case of detected red noise, we first fit a power-law model to the power spectrum of timing residuals, then obtain estimates of noise covariance matrices iteratively, and finally use the Cholesky decomposition of this covariance matrix to transform the problem to an ordinary least-squares problem ([Coles et al. 2011](https://doi.org/10.1111/j.1365-2966.2010.17655.x)). Additionally, it is common that white noise of the timing residuals is underestimated by their TOA uncertainties. We

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1 A slight modification was made to the published DR1 data set to fix a small offset in early 10 cm (3 GHz) data for J1909−3744 (Shannon et al. 2013).
Table 5.1: The rms values of white noise measured for 20 PPTA pulsars in the DR1 data set.

<table>
<thead>
<tr>
<th>PSR</th>
<th>rms (µs)</th>
<th>PSR</th>
<th>rms (µs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>J0437−4715</td>
<td>0.118</td>
<td>J0613−0200</td>
<td>1.297</td>
</tr>
<tr>
<td>J0711−6830</td>
<td>4.511</td>
<td>J1022+1001</td>
<td>2.608</td>
</tr>
<tr>
<td>J1024−0719</td>
<td>2.947</td>
<td>J1045−4509</td>
<td>2.836</td>
</tr>
<tr>
<td>J1600−3053</td>
<td>0.725</td>
<td>J1603−7202</td>
<td>2.367</td>
</tr>
<tr>
<td>J1643−1224</td>
<td>2.119</td>
<td>J1713+0747</td>
<td>0.510</td>
</tr>
<tr>
<td>J1730−2304</td>
<td>2.602</td>
<td>J1732−5049</td>
<td>3.750</td>
</tr>
<tr>
<td>J1744−1134</td>
<td>1.038</td>
<td>J1824−2452A</td>
<td>1.861</td>
</tr>
<tr>
<td>J1857+0943</td>
<td>1.379</td>
<td>J1909−3744</td>
<td>0.348</td>
</tr>
<tr>
<td>J1939+2134</td>
<td>0.303</td>
<td>J2124−3358</td>
<td>3.299</td>
</tr>
<tr>
<td>J2129−5721</td>
<td>3.851</td>
<td>J2145−0750</td>
<td>3.946</td>
</tr>
</tbody>
</table>

therefore introduce the factors – “EFAC” and “EQUAD” in TEMPO2 – to rescale the TOA errors so that the observed scatter is represented. The scaled TOA uncertainty $\sigma_s$ is related to the original uncertainty $\sigma$ by:

$$\sigma_s^2 = \left( \sigma^2 + \text{EQUAD}^2 \right) \times \text{EFAC}^2. \quad (5.1)$$

Values of EFAC and EQUAD were estimated such that the rescaled TOA uncertainties best match the Gaussian probability distribution using a Kolmogorov–Smirnov test. Such an analysis for the PPTA DR1 data set can be found in Appendix B of Wang et al. (2015). Table 5.1 gives the rms values of white noise measured for the DR1 data set. Of these 20 PPTA pulsars, only J0437−4715 has a white noise rms approaching 100 ns, 4 have noise rms between 0.3 and 1 µs, and the remaining 15 have noise rms between 1 and 5 µs.

Since our analysis depends on accurate noise models, we have used simulations to show that the mean power spectra obtained from data sets simulated with those noise models agree with the power spectra obtained from the actual data sets. These simulations are carried out with the same data span and sampling as in the actual data. Throughout our analysis: 1) noise models are determined before the
A CW search is implemented and 2) we do not repeat the noise estimation process in the case of signal injections. We note, however, that some noise contributions (e.g., the intrinsic red noise) may be indistinguishable from monochromatic GWs and it may be desirable to estimate both noise and signal parameters simultaneously as discussed in Arzoumanian et al. (2014).

5.3 The data analysis method

5.3.1 Accounting for effects of single-source gravitational waves in TEMPO2 – the ‘$A_+A_\times$’ method

As shown in section 2.3.4, single-source GWs affect TOAs in a deterministic and quadrupolar way. For any source sky position, the coherent Earth-term signals $A_+(t)$ and $A_\times(t)$ can be considered as common signals existing in multiple pulsars. Using a method similar to Hobbs et al. (2012) who searched for a common signal in all the pulsar data sets, we fit for $A_+A_\times(t)$ as a set of equally spaced samples without specifying their functional forms but making use of the geometric factors given in equations (2.15-2.16). Linear interpolation is used in such a global fit as observations for different pulsars are usually unevenly sampled and not at identical times. We refer to the two time series estimated with TEMPO2 as $A_{t_2}^{+\times}(t)$, which could contain potential CW signals of the form given by equations (4.1-4.2).

Below we briefly discuss some features of this method. Its TEMPO2 usage will be described in Appendix C.1. In the next section we outline a maximum likelihood technique for CW detection in $A_{t_2}^{+\times}(t)$.

The $A_+A_\times$ fit has the following properties: a) it allows one to simultaneously fit for single-source GWs and normal pulsar timing parameters; b) it significantly reduces the computation because the number of data points in $A_{t_2}^{+\times}(t)$ is typically $\sim 100$ in comparison to thousands of TOAs for data spans of $\lesssim 10$ yrs; c) it can be used to check the correctness of the noise models. For example, if our pulsar noise models are correct and the covariance matrix estimation is reliable, the whitened
5.3. The data analysis method

$A_{t_2}^{m,x}(t)$ time series independently follow the standard Gaussian distribution in the absence of signals. The property a) also requires that constraints must be set on $A_{t_2}^{m,x}(t)$ to avoid the covariance between a global fit for $A_{t_2}^{m,x}(t)$ and the fit for timing model parameters of individual pulsars. Currently three kinds of constraints are implemented in TEMPO2, namely, 1) quadratic constraints that correspond to pulsar spin parameters, 2) annual sinusoids for pulsar positions and proper motions, 3) biannual sinusoids for pulsar parallax. These were first introduced and implemented in Keith et al. (2013) where the DM variations were modelled as linear interpolants. Details on the constrained least-squares fitting can be found in appendix A of Keith et al. (2013).

For the purpose of illustration of our method, we inject to the PPTA DR1 data set a CW signal specified with the following parameters: $M_c = 7.35 \times 10^8 M_\odot$, $d_L = 16.5$ Mpc, $f = 10$ nHz, $\cos \iota = 1$, $\psi = \Phi_0 = 0$, $\alpha = 3.2594$ and $\delta = 0.2219$ (located in the Virgo cluster). Note that: 1) the chirp mass is chosen such that $h_0 = 10^{-14}$, which is below our upper limit ($1.7 \times 10^{-14}$) at 10 nHz as will be presented in section 5.5; 2) the frequency separations between pulsar terms and Earth terms for different pulsars are determined physically using equations (2.18) and (4.6). In this new data set we then fit for a full timing model for each pulsar and globally for $A_{t_2}^{m,x}(t)$ at the injected sky location and for evenly spaced times between MJD 53430–55620 with a sampling interval of 30 d. The resulting estimates of $A_{t_2}^{m,x}(t)$ are displayed as open circles with error bars in panel (a) of Figure 5.1, while the injected waveforms are shown as solid red lines. Because of the constraints applied in the global fit, data points in $A_{t_2}^{m,x}(t)$ are known to be degenerate and thus it is necessary to perform a maximum likelihood estimation of two sinusoidal waves using the noise covariance matrix (e.g., as we outline in the next section). In this case, the frequency is estimated to be 10.6 nHz and the reconstructed waveforms are plotted as black dash lines. One can see the phase and amplitude are biased because of pulsar terms. To more clearly see this effect, we remove the

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2This functionality is available with the TEMPO2 addCGW plugin, for which usage will be described in Appendix C.2.
Figure 5.1: (a) The $A_{t2x}(t)$ time series (open circles with error bars) estimated with a global least-squares fit in the PPTA DR1 data set that has included a CW signal injection (see text). The injected waveforms are plotted as solid red lines and the reconstructed waveforms based on a maximum likelihood technique are depicted by black dash lines. (b) As panel (a) but for the case of without pulsar terms in the signal injection.
pulsar terms in the signal injection and redo the same analysis. As shown in panel (b) of Figure 5.1, the match-up becomes much better. In the Earth-term-only case, the maximum likelihood frequency is 10.2 nHz. We leave the investigation of parameter estimation to a future study as in the current work we focus on detection and sensitivities.

5.3.2 Searching for continuous waves in $A_+ (t)$ and $A_x (t)$ time series

Now the search for CWs is ready to be performed in $A_+ (t)$ and $A_x (t)$ time series. In general, these two time series data can be written as:

$$d = \begin{bmatrix} d_+ \\ d_x \end{bmatrix} = \begin{bmatrix} s_+ \\ s_x \end{bmatrix} + \begin{bmatrix} n_+ \\ n_x \end{bmatrix},$$

(5.2)

where $s_{+,x}$ and $n_{+,x}$ are column vectors of signal and noise respectively. The noise covariance matrix is:

$$\Sigma_n = \begin{bmatrix} \Sigma_{++} & \Sigma_{+x} \\ \Sigma_{x+} & \Sigma_{xx} \end{bmatrix},$$

(5.3)

where $\Sigma_{kk} = \langle n_k n_k^T \rangle$ with the index $k \in \{+, \times\}$ and the brackets $\langle \ldots \rangle$ denote the ensemble average of the random noise process. (It is understood that $\Sigma_{++} = \Sigma_{++}^T$, $\Sigma_{+x} = \Sigma_{x+}^T$ and $\Sigma_{xx} = \Sigma_{xx}^T$.)

As discussed in section 2.3 we define the noise-weighted scalar product of two time vectors $x$ and $y$ using the noise covariance matrix $\Sigma_n$ as:

$$(x, y) = x^T \Sigma_n^{-1} y.$$  

(5.4)

Then the log likelihood ratio, $\ln \Lambda$, can be conveniently written as:

$$\ln \Lambda = \ln \frac{\mathcal{L}(s|d)}{\mathcal{L}(0|d)} = (d, s) - \frac{1}{2} (s, s).$$

(5.5)
Chapter 5. Searching for Continuous Waves with PPTA

It is worth noting that equation (5.5) is essentially correlating data \( \mathbf{d} \) with some signal templates \( \mathbf{s} \) and comparing the outputs to a threshold. Regarding \( \mathbf{s} \) we rewrite equations (4.1-4.2) in a more straightforward form:

\[
\mathbf{s}_{+,x}(t) = a_{+,x} \cos \Phi(t) + b_{+,x} \sin \Phi(t),
\]

(5.6)

where the four amplitude parameters are related to physical parameters through:

\[
\begin{align*}
    a_+ &= \frac{h_0}{2\pi f} [(1 + \cos^2 \iota) \cos 2\psi \sin \Phi_0 + 2 \cos \iota \sin 2\psi \cos \Phi_0] \\
    b_+ &= \frac{h_0}{2\pi f} [(1 + \cos^2 \iota) \cos 2\psi \cos \Phi_0 - 2 \cos \iota \sin 2\psi \sin \Phi_0] \\
    a_\times &= \frac{h_0}{2\pi f} [(1 + \cos^2 \iota) \sin 2\psi \sin \Phi_0 - 2 \cos \iota \cos 2\psi \cos \Phi_0] \\
    b_\times &= \frac{h_0}{2\pi f} [(1 + \cos^2 \iota) \sin 2\psi \cos \Phi_0 + 2 \cos \iota \cos 2\psi \sin \Phi_0].
\end{align*}
\]

(5.7)

If we write the inverse of the noise covariance matrix in the form of a block matrix:

\[
\Sigma_n^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},
\]

(5.8)

and define two column vectors \( \mathbf{x} = \cos \Phi(t) \) and \( \mathbf{y} = \sin \Phi(t) \), then the maximum likelihood estimators (\( \partial \ln \Lambda / \partial a_{+,x} = 0 \) and \( \partial \ln \Lambda / \partial b_{+,x} = 0 \), see section 2.4.2) for the four amplitudes are obtained by solving the following linear equation:

\[
\begin{bmatrix} x^T S_{11} x & x^T S_{11} y & x^T S_{12} x & x^T S_{12} y \\ x^T S_{11} y & y^T S_{11} x & y^T S_{21} y & y^T S_{21} y \\ x^T S_{12} x & x^T S_{21} x & x^T S_{22} x & x^T S_{22} y \\ x^T S_{12} y & x^T S_{21} y & x^T S_{22} y & y^T S_{22} y \end{bmatrix} \begin{bmatrix} \hat{a}_+ \\ \hat{b}_+ \\ \hat{a}_\times \\ \hat{b}_\times \end{bmatrix} = \begin{bmatrix} d^T S_{11} x + d^T x S_{21} x \\ d^T S_{11} y + d^T x S_{21} y \\ d^T S_{12} x + d^T x S_{22} x \\ d^T S_{12} y + d^T x S_{22} y \end{bmatrix},
\]

(5.9)

where we have applied the properties of \( S_{11} = S_{11}^T, S_{21} = S_{12}^T \) and \( S_{22} = S_{22}^T \).

Estimates of \( a_{+,x} \) and \( b_{+,x} \) obtained by solving equation (5.9) are used to calculate the log likelihood ratio with equations (5.5-5.6). Defining a column vector \( \lambda \) that contains maximum likelihood estimates of the four amplitudes \( [\hat{a}_+; \hat{b}_+; \hat{a}_\times; \hat{b}_\times] \) and
multiplying $\lambda^T$ to both sides of equation (5.9), one finds that $(\mathbf{d}|\mathbf{s}) = (\mathbf{s}|\mathbf{s})$. Our detection statistic is taken as $\mathcal{P} = 2 \ln \Lambda$ and thus the signal-to-noise ratio as defined by $\rho = \sqrt{(\mathbf{s}|\mathbf{s})}$ is related to $\mathcal{P}$ through $\rho = \sqrt{\mathcal{P}}$. Note that $\mathcal{P}$ in the case of Gaussian noise-only data follows a $\chi^2$ distribution with 4 degrees of freedom. Figure 5.2 shows the probability distribution of $\mathcal{P}$ in simulated noise-only data as compared against the expected distribution.

It is worth mentioning that the derivation of $\mathcal{P}$ follows that of the $F$-statistic in the context of CW search using ground-based interferometers (Jaranowski et al. 1998). The $F$-statistic was also adapted in Ellis et al. (2012) for PTA data to derive the coherent $F_c$-statistic (see also Babak & Sesana 2012) and incoherent $F_p$-statistic (equivalent to a power spectral summing technique). Our statistic is similar to the $F_c$-statistic in the following ways: 1) it is implemented fully in the time domain; 2) it targets the coherent Earth-term signals; 3) it has been maximized over extrinsic amplitude parameters. The main difference is that the $F_c$-statistic applies directly on timing residual data whereas $\mathcal{P}$ works with the reduced $A_{t^2+x}(t)$ data.

### 5.3.2.1 Maximum likelihood problem in degenerate multivariate Gaussian noise

The constraints set on $A_{t^2+x}(t)$ result in some degeneracy in the data, and therefore their covariance matrix $\Sigma_n$ is no longer a full-rank matrix. Specifically the rank $r$ of $\Sigma_n$ is given by $n - m$ where $n$ is the total number of data points in the stacked $[\mathbf{A}_+; \mathbf{A}_x]$ data and $m$ is the number of constraints that have been applied.

The above problem corresponds to finding the maximum likelihood solution in the case of degenerate multivariate Gaussian noise. One needs to replace $\Sigma_n^{-1}$ in equations (5.4) and (5.8) with the generalized inverse which can be obtained via eigen-decomposition ($\Sigma_n = \mathbf{E}\mathbf{D}\mathbf{E}^T$):

$$
\Sigma_n^{-1} = \mathbf{E}\mathbf{D}^{-1}\mathbf{E}^T, \tag{5.10}
$$

where $\mathbf{E}$ is $n \times r$ of full rank $r$, and has columns that are the eigenvectors cor-
Figure 5.2: Probability distribution of the detection statistics $\mathcal{P}$ in simulated noise-only data (solid blue). We perform 1000 noise realizations using noise models estimated for the PPTA DR1 data and for each realization we search over the same sky direction and 36 independent frequencies. Also shown is the expected $\chi^2$ distribution with four degrees of freedom (red dash).
5.3. A data analysis pipeline for continuous waves responding to the positive eigenvalues of $\Sigma_n$ as given in the diagonal elements of the $r \times r$ diagonal matrix $D$. It is straightforward to show that $\Sigma_n = UU^T$ where $U = E\sqrt{D}$ and we can use $U^{-1}_{\text{left}}$ (defined as $U^{-1}_{\text{left}}U = I_r$ with $I_r$ being the $r \times r$ identity matrix) to whiten the $A_{t_2}^\pm(t)$ time series. If we write $A = [A_+; A_\times]$, then elements in $A_w = U^{-1}_{\text{left}}A$ independently follow the standard Gaussian distribution.

5.4 A data analysis pipeline for detecting and limiting continuous waves

In a detection problem, we are concerned with the false alarm probability (FAP) of a measured detection statistic $P_{\text{obs}}$, which is the probability that $P$ exceeds $P_{\text{obs}}$ for noise-only data. So the single-trial FAP is given by $1 - \text{CDF}(P_{\text{obs}}; \chi^2_4)$ where CDF($\cdot; \chi^2_4$) denotes the cumulative distribution function (CDF) for a $\chi^2$ distribution with 4 degrees of freedom. Without prior knowledge of GW frequency and source sky location, a search should be performed in the 3-dimensional parameter space $(\delta, \alpha, f)$. This introduces a trials factor $N_{\text{trial}}$ defined as the number of independent cells in the searched parameter space. We are interested in the total FAP as given by $1 - \left[\text{CDF}(P_{\text{max}}; \chi^2_4)\right]^{N_{\text{trial}}}$ for the maximum detection statistic $P_{\text{max}}$ found in the search.

In a standard TEMPO2 generalized least-squares fit, in addition to a full timing model for each pulsar we globally fit for $A_{t_2}^\pm(t)$ for evenly spaced times between MJD 53430 and 55620 with a sampling interval of 30 days for a set of sky positions (with the number determined below). The sampling we choose for the $A_+A_\times$ fit is slightly lower than our observing cadence to ensure that there are $\gtrsim 20$ observations for each sampled epoch. It is well understood that for evenly sampled data independent frequencies are defined as a set of harmonics of $\Delta f = 1/T_{\text{span}}$ (with $T_{\text{span}}$ being the data span) up to the Nyquist frequency. Determined by the sampling for the $A_+A_\times$ fit, the frequency range of our search $5 \times 10^{-9} - 2 \times 10^{-7}$ Hz consists of 36 independent frequency channels (denoted as $N_f = 36$). In this work
we choose to analyze 141 GW frequencies from \(0.5\Delta f\) to \(36\Delta f\) with an interval of \(0.25\Delta f\) (because of the constraints set on \(A_{r,x}^2(t)\), two frequencies close to 1 yr\(^{-1}\) are excluded in the analysis).

Regarding the number of sky directions that need to be searched, one should consider that:

1. it must not be too small otherwise one could miss a potential signal due to a mismatch in \((\delta, \alpha)\).

2. there is an upper limit for the number of independent sky positions (denoted as \(N_{\text{sky}}\)) – the PPTA DR1 data set contains about 4000 TOAs (i.e., \(N_{\text{trial}} < 4000\)), which implies \(N_{\text{sky}} \lesssim 100\) if we assume \(N_{\text{trial}} = N_f N_{\text{sky}}\) by neglecting the correlation between GW frequency and source location\(^3\). Note that the purpose of this assumption is only to have a rough estimate of \(N_{\text{sky}}\) which is used in determining the detection threshold for all-sky sensitivities.

For the present search we choose to use a uniform sky grid consisting of 1000 points. We find that using a finer sky grid of 4000 points results in \(\ll 1\%\) increase in the maximum detection statistic for our data set. (However, for better visual quality of the figures sky maps shown in this Chapter consist of 4000 pixels.)

We set a total FAP of 1\% as the detection threshold for the all-sky blind search. Because \(N_{\text{trial}}\) is not precisely known, we choose to estimate the FAP for the most significant “event” found in the search by simulations as later described in section 5.5.

Our steps towards detection are as follows:

1. For each grid point on the sky, we use the TEMPO2 software package to simultaneously determine timing parameters for a full timing model of 20 PPTA pulsars and to form estimates of \(A_{r,x}^2(t)\) and their covariance matrix \(\Sigma_n\) through a global generalized least-squares fit.

\(^3\)As the search targets only the Earth terms and we are in a weak-signal limit such correlation is not important.
2. For each frequency $f_j$, we calculate and record values of “observed” detection statistic $P_{\text{obs}}(f_j)$. The first two steps can be accomplished using the TEMPO2 findCWs plugin, whose usage will be described in Appendix C.3.

3. Repeat steps 1-2 for other sky directions. If anywhere we find a $P_{\text{obs}}$ that corresponds to a FAP of less than 1%, we claim a detection.

In the absence of a detection, our steps towards setting upper limits on $h_0$ as a function of GW frequencies $f$ are as follows:

1. At each frequency $f_j$, we generate SMBBH signals for a given value of $h_0$ and other parameters ($\cos \delta$, $\alpha$, $\cos \iota$, $\psi$ and $\Phi_0$) randomly chosen from uniform distributions over their applicable ranges and add such signals to simulated noise. For completeness we also consider the case that CW signals are injected to real data.

2. The detection statistic is evaluated at the injected source sky location and frequency, and recorded as the “simulated” detection statistic $P_{\text{sim}}(f_j)$, which is then compared against $P_{\text{obs}}(f_j)$ found in the same frequency and sky position.

3. Perform 1000 simulations with each having source parameters randomized and independent noise realizations, and adjust $h_0$ until 95% of the injections lead to a value of $P_{\text{sim}}(f_j) > P_{\text{obs}}(f_j)$. Record the value of $h_0(f_j)$ as the frequentist 95% confidence upper limit. Since we use 1000 simulations, the 1–σ uncertainty of the confidence is only 0.1% given a binomial statistic.

While upper limits tell us the maximum amplitude of a GW at a particular frequency that is consistent with our observations, we are also interested in the sensitivities achieved by the PPTA DR1 data set, i.e., the minimum values of $h_0$ that would produce a detectable signal in our data set with 95% probability.

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4It could have been more appropriate to search over a sky patch centred around the randomly chosen sky location. We choose to search only at the injected source position because of limited computing resources.
Chapter 5. Searching for Continuous Waves with PPTA

The procedure of producing an all-sky sensitivity curve is the same as that for bounding as outlined above except that $P_{\text{sim}}$ should now be compared against a fixed detection threshold. For all-sky sensitivities, the detection threshold is chosen as $P_{\text{th,all-sky}} = 20.9$ using $N_{\text{sky}} = 30$ – the reduced trials factor as we have specified GW frequencies to obtain sensitivities. This rough estimate of $N_{\text{sky}}$ is simply taken as $N_{\text{trial}}/36$ and we consider $N_{\text{trial}} = 1000$ for the DR1 data set as estimated through simulations in section 5.5.

As it is expected that the sensitivity varies significantly across the sky, we additionally present in this chapter directional sensitivity curves for the most, median and least sensitive sky direction given our data set. In this case the detection threshold is $P_{\text{th,dir}} = 23.5$ corresponding to a single-trial FAP of $10^{-4}$. This is appropriate for the case that in a targeted search, the source sky location and frequency are known, e.g., for SMBBH candidates from electromagnetic observations (see, e.g. Burke-Spolaor 2013 and references therein). Usually in a directional search the source orbital frequency is unknown, e.g., for the so-called GW hotspots (Rosado & Sesana 2014; Simon et al. 2014). Although we do not consider it in the following calculations, it is worth mentioning that the threshold in this case should be increased to 31.2 to account for a trials factor of $N_f = 36$ and the corresponding sensitivities would be decreased by 15%. Because of computational limitations the process for obtaining directional sensitivities is simplified as follows:

1. At each frequency and each one of the 1000 sky grid points we inject to the real data set signals with a fixed $h_0$, $\cos \iota = 1$, and random values for $\psi$ and $\Phi_0$ over $[0, 2\pi)$. Then we calculate the signal-to-noise ratio $\rho$.

2. Because for a given noise realization (in our case the PPTA DR1 data), $\rho$ scales linearly as $h_0$, values of $\rho$ obtained in the previous step can be scaled to the given detection threshold to obtain detection sensitivities on $h_0$. Here a multiplying factor of $\sqrt{5/2}$ is included to account for average binary orientations. This factor corresponds to the difference in the square root between the maximum value (2) for $[(1 + \cos^2 \iota)/2]^2 + \cos^2 \iota$ and its
5.4. A data analysis pipeline for continuous waves

average value (4/5) for a uniform distribution of \( \cos \iota \) in \((-1, 1]\).

Using the above method we can also generate a sky map of sensitivities at a given frequency. Note that the value of injected \( h_0 \) does not matter as long as it is large enough so that the injection at the least sensitive sky direction results in a detection. The purpose of these directional sensitivity plots is just to show what sensitivity is available with the current data set for directional/targeted searches.

5.4.1 Verification of the pipeline

The analysis presented here has undergone extensive checking to make sure the pipeline works as expected. The first test is to confirm that the whitened \( A_{t, x}^{12} \) data follow the standard Gaussian distribution. For each searched sky position, we whiten the derived \( A_{t, x}^{12} \) time series using the noise covariance matrix as described in section 5.3.2.1. Figure 5.3 shows the empirical CDF for the whitened \( A_{t, x}^{12} \) data, which agrees well with the standard Gaussian distribution.

The second verification process involves the correct reconstruction of signal injections. In Figure 5.4, we show the detection statistics as a function of frequencies.
Figure 5.4: Detection statistics ($P$) as a function of frequencies for the PPTA DR1 data set (red dash) and for the same data set but has included a signal injection (solid black). Both were evaluated at the sky location of the Virgo cluster where the signal was injected. The vertical line marks the injected frequency (20 nHz), while the horizontal line corresponds to a single-trial FAP of 1%.
in the case of a signal specified with the following parameters: \( M_c = 7 \times 10^8 M_\odot \), 
\( d_L = 16.5 \text{ Mpc}, \ f = 20 \text{ nHz}, \ \cos \iota = 0.5, \ \psi = \Phi_0 = 0, \ \alpha = 3.2594 \) and \( \delta = 0.2219 \) (the sky location of the Virgo cluster) was injected to the PPTA DR1 data set. 
For this case the search was performed at the injected sky location and we use a frequency interval of \( \Delta f/16 \) to increase frequency resolution. The maximum detection statistic was found at a frequency of 20.15 nHz. As a comparison, we also show the detection statistics measured for the real data set.

Figure 5.5 shows sky maps of signal-to-noise ratios (\( \rho \)) for two strong signal injections made to simulated noise. The source sky locations were chosen in the least and most sensitive sky region, as will be illustrated later in Figure 5.11. In both cases the maximum \( \rho \) is found at a grid point near the injected sky location, but the error box varies dramatically (a few tens square degrees in comparison to thousands of square degrees), generally in agreement with Figure 4.14.

The final test is that for a data set that contains an injected signal, the established upper limit on \( h_0 \) should be above the injected value. Figure 5.6 shows results of such a test for 100 signal injections made to real data. All signals are simulated at \( 10^{-8} \) Hz with \( h_0 \) uniformly distributed in \( 1-5 \times 10^{-14} \) and other parameters randomly chosen from uniform distributions over their applicable ranges. Our upper limit at this frequency is \( 1.7 \times 10^{-14} \). Note that 5 out of 100 injections have failed this test, which is expected as the upper limits are at a 95% confidence level.

## 5.5 Results and discussion

### 5.5.1 Search results

We show in Figure 5.7 a sky map of the detection statistic maximized over 36 frequency channels for each sky direction. The most significant value \( P_{\text{max}} = 22.06 \) across the sky is found at 85 nHz. To estimate the FAP of this “event”, we produce simulated data sets and treat them exactly the same as real data set, i.e., going
Figure 5.5: Sky map of signal-to-noise ratios ($\rho$) for simulated data set that includes a strong signal injection made in the least (a) or most (b) sensitive sky region. The signal is injected at the location indicated by a “□” and the maximum $\rho$ is found at “◦”. Sky locations of the 20 PPTA pulsars are marked with “⋆”.
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Figure 5.6: Upper limits on $h_0$ established for real data with signal injections, are compared against injected values (both in $10^{-14}$). Each point represents a separate injection. The horizontal dash line marks the PPTA DR1 upper limit ($1.7 \times 10^{-14}$) at $10^{-8}$ Hz where all injections are made. The solid black line is for upper limits equal the injected values.

Figure 5.7: Sky map of the detection statistics ($\mathcal{P}$) measured for the PPTA DR1 data set. The most significant statistic is found at a direction indicated by “◦”. Sky locations of the 20 PPTA pulsars are labeled with “⋆”. 
Figure 5.8: FAP as a function of the maximum detection statistic ($P_{\text{max}}$) as determined by 1000 simulations (solid black). The red dash line is for a $\chi^2$ distribution with four degrees of freedom assuming a trials factor $N_{\text{trial}} = 1000$. The vertical line marks $P_{\text{max}} = 22.06$ measured for the PPTA DR1 data set.
through the same fitting process and searching over exactly the same grid points in
the parameter space. For each noise realization we record the maximum value of
the detection statistic. With 1000 simulations we show in Figure 5.8 estimates of
FAP based on the empirical distribution of \( P_{\text{max}} \). The FAP for \( P_{\text{max}} = 22.06 \) found
in the PPTA DR1 data set is estimated to be 17\%, implying the search result is
consistent with a non-detection. From the empirical distribution of \( P_{\text{max}} \), we also
obtain an estimate of \( N_{\text{trial}} = 1000 \) for the DR1 data set.

### 5.5.2 Upper limits and sensitivities

Figure 5.9 shows all-sky upper limits on \( h_0 \) for two cases: signals are injected to a)
real data or b) simulated noise. One can see that case b) gives slightly worse upper
limits across the whole frequency band (most notably between 5 and 15 nHz) and
the noisy trend in frequency for both set of limits is almost identical. Also shown
in Figure 5.9 is an all-sky sensitivity curve for case b). This sensitivity curve is
roughly a factor of two above the upper limit curve. This is expected because 1) in
the process of setting an upper limit \( P_{\text{sim}} \) is compared against \( P_{\text{obs}} \) which has an
average of 4 (the distribution of \( P_{\text{obs}} \) varies significantly over frequency, resulting
in the noisiness in the upper limit curves); 2) the threshold \( P_{\text{th},\text{all-sky}} = 20.877 \) is a
factor of 5 higher than \( P_{\text{obs}} \) on average; 3) the factor of \( \sqrt{5} \) matches the difference
shown in Figure 5.9. The difference can be interpreted as follows: an upper limit
can be converted to the maximum signal amplitude measured in data; this limit is
about a factor of 2 below the detection threshold for the same data.

In order to test the effect of our assumption that pulsar terms are in the same
frequency bin as Earth terms, we calculate the sensitivity at the 5th bin (the
most sensitive bin, with a centre frequency 8 nHz) for evolving sources with \( M_c = 10^{10}M_\odot \). As indicated by the “plus” sign in Figure 5.9, the sensitivity at this
frequency bin is only increased by 7\%.

Our upper limits are about a factor of 4 better than the previously published
limits in Yardley et al. (2010). This improvement is mainly because the new data
Figure 5.9: All-sky upper limits on $h_0$ as a function of GW frequencies for two cases: signals injected to real data (dash blue) or simulated noise (solid pink). The all-sky sensitivity curve (solid black) is obtained for simulated noise. Two vertical lines correspond to frequencies of 1 and 2 yr$^{-1}$. The dash-dotted straight lines are strain amplitudes expected from SMBBHs with $M_c = 10^{10} M_\odot$ and $d_L = 400$ Mpc (upper), or $M_c = 10^9 M_\odot$ and $d_L = 30$ Mpc (lower). The point marked by a plus sign is the sensitivity calculated for evolving sources assuming $M_c = 10^{10} M_\odot$, which can be compared to the solid black curve.
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Figure 5.10: Sensitivities as a function of GW frequencies for the most (lower), median (middle) and least (upper) sensitive sky direction given the PPTA DR1 data. Straight lines are for SMBBHs with $M_c = 10^9 M_\odot$ and $d_L = 100$ Mpc (lower), $M_c = 10^{10} M_\odot$ and $d_L = 1.5$ Gpc (middle), $M_c = 10^{10} M_\odot$ and $d_L = 170$ Mpc (upper), that could produce CW signals at the level of the three sensitivity curves between $10^{-8}$ and $10^{-7}$ Hz (except two narrow bands centred around 1 and 2 yr$^{-1}$).
Chapter 5. Searching for Continuous Waves with PPTA

Figure 5.11: Sky map of luminosity distance \( d_L \) out to which our data set is sensitive at \( 10^{-8} \) Hz to GW signals from circular binaries of chirp masses \( 10^9 M_\odot \). Sky locations of the 20 PPTA pulsars are labeled with “⋆”. White diamonds mark the location of possible SMBBH candidates or nearby clusters (luminosity distances are 92 Mpc, 102 Mpc, 19 Mpc, 16.5 Mpc and 1.07 Gpc for 3C66B, Coma, Fornax, Virgo and OJ287 respectively). Note that the sensitivity in \( d_L \) scales as \( M_5^{5/3} \) and is roughly the same for the frequency band of \( 10^{-8} - 10^{-7} \) Hz except two narrow bands centred around 1 and 2 yr\(^{-1}\) (see Figure 5.10).

set has significantly improved timing precision and cadence over the earlier data set. Our limits also improve by a factor of 2 on those reported in the recent paper by NANOGrav (Arzoumanian et al. 2014), comparing to their results based on “fixed-noise” approaches. This improvement is mostly caused by the higher observing cadence, the slightly longer data span and the much larger number of independent observing sessions in the PPTA data set. It should be noted that: 1) there is a factor of \( \sqrt{8/5} \) difference in the definition of the GW strain amplitude being constrained in Yardley et al. (2010). Their upper limits were set on the inclination-averaged mean-square amplitude that is given by \( h_0 \times \sqrt{8/5} \), see equation (4.9); 2) As CW signals were under-represented in Yardley et al. (2010) by a factor of \( \sqrt{2} \), corresponding to the difference between the maximum amplitude of a sinusoid and its root mean square amplitude, we have divided upper limits presented in that paper by the same factor when making comparisons.
5.5. Results and discussion

In Figure 5.10 we show sensitivity curves for the most, median and least sensitive sky direction given the PPTA DR1 data. It should be noted that such sensitivities are to be used as a guide for targeted searches, i.e., for known source sky locations and frequencies. For the most sensitive sky direction, the current data set is sensitive to average-oriented SMBBHs of chirp masses $10^9 M_\odot$ up to about 100 Mpc in the frequency band of $10^{-8}–10^{-7}$ Hz (except two narrow bands centred around 1 and 2 yr$^{-1}$). For directional searches with unknown orbital frequencies sensitivities would be decreased by 15% to account for the trials factor involved in a search over frequency as discussed in section 5.4. The median sensitivity curve is a factor of four below the all-sky sensitivity curve because for 95% of the whole sky sources with $h_0$ above the latter can be detected while the former only applies to $\sim$ 50% of the sky. For both Figures 5.9 and 5.10 the huge sensitivity loss at around 1 and 2 yr$^{-1}$ is because GW power is absorbed in the fit for positions, proper motions and parallax of individual pulsars.

Figure 5.11 shows the distance at which a circular SMBBH of a certain chirp mass would produce a detectable signal at $10^{-8}$ Hz in our data set. In this plot the signal injections only include Earth terms because the inclusion of pulsar terms bias the sky localization and thus make the sky map very noisy. However, we emphasize that similar results should be obtainable if we search over the sky (rather than only at the injected location) for each signal injection when pulsar terms are included. The purpose of Figure 5.11 is just to illustrate the angular distribution of PPTA’s sensitivities and to gain some insights on how the addition of new pulsars to the timing array helps. As expected, we are most sensitive in the sky region where the best-timed PPTA pulsars are located and least sensitive in the opposite direction. Main findings from Figure 5.11 include:

1. The PPTA DR1 data set is sensitive to potential SMBBHs with chirp masses of $\gtrsim 2.3 \times 10^9 M_\odot$ in the Coma Cluster, and of $\gtrsim 7 \times 10^8 M_\odot$ for both the Fornax Cluster and Virgo Cluster.

2. With the current analysis we are unable to place meaningful constraints on
3C66B as it was proposed to have an orbital period of 1.05 yr (Iguchi et al. 2010) where our sensitivity is very low because of the biannual sinusoidal constraint set on $A_{t_2}^2(t)$ (see Figures 5.9 and 5.10).

3. For another SMBBH candidate OJ287, modelled with an orbital period of 12 yrs (Sundelius et al. 1997), an upper limit on the chirp mass with the current data set is $\sim 10^{10} M_\odot$ which is about an order of magnitude higher than the current mass estimate (Valtonen et al. 2010).

4. The possible SMBBH candidates and nearby clusters are all located in the insensitive sky region. This shows the benefits of adding new good pulsars to increase the PPTA’s astrophysical reach.

5.5.2.1 Upper limits on the coalescence rate of supermassive binary black holes

Given the absence of signals in our data set, upper limits on the coalescence rate of SMBBHs can be computed in a straightforward way. Following Wen et al. (2011), but rather than constraining the differential coalescence rate (with respect to chirp mass and redshift), we wish to set limits on the local coalescence rate density. The expected number of events can be written as $\mu = R \sum_i \epsilon V_i \Delta T(f_i)$ where $R$ is the coalescence rate per unit volume, $\epsilon$ is the detection efficiency (which is 95% for all points on the all-sky sensitivity curve shown in Figure 5.9), $V_i$ is the sensitive volume at frequency $f_i$ (simply taken as $4\pi d_{L,i}^3/3$ with $d_{L,i}$ being the luminosity distance out to which a SMBBH would produce a detectable CW signal at $f_i$ with 95% probability) and $\Delta T(f_i)$ is the time duration that a binary stays in the $i$-th frequency bin. Assuming Poisson-distributed events, the probability of no events being detected is $e^{-\mu}$. Therefore, the frequentist 95% confidence upper limit is $R_{95\%} = -\ln(1 - 0.95)/\sum_i \epsilon V_i \Delta T(f_i)$.

Making use of the all-sky sensitivities for 141 frequencies shown in Figure 5.9 together with the assumption that the sensitivity is a constant for each frequency bin with width 1.32 nHz, we find that $R_{95\%} = 4 \times 10^{-3}(10^{10} M_\odot/M_c)^{10/3} \text{Mpc}^{-3}\text{Gyr}^{-1}$.
for nearby \((z \lesssim 0.1)\) SMBBHs (with the maximum applicable redshift corresponding to the largest \(d_{L,i}\) for \(M_c = 10^{10}M_\odot\)). Note that our limit is about two orders of magnitudes above the current estimates of galaxy merger rate density in the local Universe (see, e.g., fig. 13 in Conselice 2014).

### 5.6 Conclusions

Over the past few years, PTTAs have been collecting pulsar timing data with steadily improving precision and are starting to set astrophysically interesting upper limits. However, most of the analyses of PTA data have focused on a stochastic background that could be produced by the combined emission by a large number of individual SMBBHs. In this work we have developed a new coherent method for detection of individual CW sources and have tested it extensively on both simulated and real pulsar timing data. The method was applied to the PPTA DR1 data set to perform an all-sky search for signals from individual nearby SMBBHs in circular orbits. Since no GWs were detected, we set upper limits on the intrinsic GW strain amplitude over a range of frequencies. For example, at \(10^{-8}\) Hz our analysis has excluded the presence of signals with \(h_0\) larger than \(1.7 \times 10^{-14}\) with 95\% confidence. These new limits are a factor of 4 better than those presented in Yardley et al. (2010), and a factor of 2 better than the recent NANOGrav limits reported in Arzoumanian et al. (2014). We also placed upper limits on the coalescence rate of nearby \((z \lesssim 0.1)\) SMBBHs, e.g., for very massive binaries \((M_c \geq 10^{10}M_\odot)\) the rate is constrained to be less than \(4 \times 10^{-3}\)Mpc\(^{-3}\)Gyr\(^{-1}\) with 95\% confidence.

We have also presented all-sky and directional sensitivity curves and find that for the frequency band of \(10^{-8}–10^{-7}\) Hz (except two narrow bands centred around 1 and 2 yr\(^{-1}\)):

1. With the current data set we are able to detect with 95\% probability very massive binary systems \((M_c = 10^{10}M_\odot)\) out to a luminosity distance of 400 Mpc regardless of their sky locations and orientations;
2. For the most sensitive sky direction, the current data set is sensitive to average-oriented SMBBHs of chirp masses $10^9 M_\odot$ up to about 100 Mpc. Furthermore, we show a PPTA sensitivity map in Figure 5.11 and find that the PPTA DR1 data set is sensitive to potential SMBBHs with orbital frequencies between 5 and 50 nHz and chirp masses of $\gtrsim 2.3 \times 10^9 M_\odot$ in the Coma Cluster, and of $\gtrsim 7 \times 10^8 M_\odot$ for both the Fornax Cluster and Virgo Cluster. Directional sensitivity curves and the sensitivity sky map presented here can be used as a guide for future directional/targeted searches. Constraints on specific individual SMBBH candidates will be investigated in a future work.
In this thesis we studied two topics of GW detection: (1) an audio-band GW background (GWB) formed by the cosmic population of compact binary coalescences (CBCs), which could be detectable by the upcoming advanced ground-based detectors; (2) single-source detection with PTA. Here we first summarize the results of this thesis, and then outline some possibilities for future work.

### 6.1 Review of results

In Chapter 3, we report on a comprehensive study on the GWB formed by the cosmic population of CBCs, including binary neutron stars (BNSs), stellar mass binary black holes (BBHs) and black hole-neutron star (BH-NS) systems. Within a couple of years before and during this thesis work, there had been a few papers that investigated a similar topic. Here we first review these previous studies and then give a summary on how this thesis has advanced our knowledge of this topic.

In Zhu et al. (2011b) (not included in this thesis), we calculated analytically the energy density spectrum of the BBH background by using complete inspiral-merging-ringdown waveforms. We examined the detectable parameter space for second and third generation terrestrial interferometers and showed that such a background signal could be detectable by advanced detectors if the BBH coalescence rate is as high as that predicted by some population synthesis models (see, e.g. Belczynski et al. 2010, Dominik et al. 2012). Marassi et al. (2011b) carried out a calculation of the GWB formed by BBHs through population synthesis simulations. Various models were proposed based on different inputs in the simulation. Similar conclusions
were made regarding the detectability of the \text{BBH} background. Rosado (2011) conducted thorough analytical calculations on the background signal formed by \text{CBCs} as well as white dwarf binaries (relevant to the space-based interferometer) and supermassive binary black holes (\text{SMBBHs}; relevant to \text{PTAs}). A major conclusion in this paper is that the \text{CBC} background is not stochastic in nature since individual sources contribute to the background are resolvable. Wu et al. (2012) systematically investigated the detectable parameter space of the \text{CBC} background models for second and third generation ground-based detectors. They concluded that there is a good chance of observing this signal with advanced detectors.

Highlights of the work presented in Chapter 3 include:

1. The energy density spectrum $\Omega_{GW}(f)$ of the \text{CBC} background can be well described by a power law model up to around 100 Hz. In this model, $\Omega_{GW}(f)$ depends almost exclusively on the local coalescence rate density and the average chirp mass, while effects such as the cosmic history of binary formation and evolution are linear and cause only moderate uncertainties within a factor of 2. Such a power law model is found to be sufficient for background searches with advanced detectors.

2. Information on the chirp mass distribution, the cosmic evolution of coalescence rate and merger dynamics are all encoded in the high frequency ($\gtrsim 200 – 300$ Hz) components of the background signal. Future observations with the third generation detectors such as the planned Einstein Telescope (ET) may enable such information to be constrained or even extracted.

3. Assuming realistic rate estimates, this background signal could be detectable after a few years of observations at design sensitivities with advanced detectors, which is in good agreement with previous studies. However, we point out that combining the current network of advanced detectors offers no considerable improvement ($< 30\%$) in terms of detectability (compared with the Advanced \text{LIGO} H-L pair), whereas tuning configurations with reduced low
6.1. Review of results

Frequency noise may provide $\sim 50\%$ enhancement in the achievable signal-to-noise ratio for this background.

4. Assuming perfect subtraction of individually detectable CBC events for ET, the background could be reduced by a factor of $\sim 100$ if its dominant contribution is from BBHs. However, if the BNS contribution dominates, there is a strong residual foreground at the level of $\Omega_{\text{GW}} \sim 10^{-10}$. Such a foreground may pose a problem for future terrestrial searches of cosmological GWBs.

In Chapter 4 we have developed a general method for the detection, sky localization and waveform estimation of single sources with PTA\(^s\). It is adapted from the coherent network analysis method for GW burst detection with ground-based interferometers. We have applied the method to various types of signals and tested it in realistic simulated data sets. Compared with previously published methods, our method provides much greater computational efficiency while still performs equally well when applied to realistic PTA data that include effects such as irregular sampling and time-varying error bars. One major finding from this chapter is that a monochromatic search that is designed for circular binaries can efficiently detect eccentric binaries with both high and low eccentricities, while a harmonic summing technique provides greater sensitivities only for binaries with moderate eccentricities.

In Chapter 5 we have presented an all-sky search for continuous waves from SMBBHs in circular orbits in the latest PPTA data set. In this work we outline a new method which can be used to extract signatures of point-like GW sources from PTA data. A paper detailing the mathematical framework and applications to various sources is in preparation and not included in this thesis. We have developed a fully functional data analysis pipeline which is ready to be used for searching for continuous waves, and with some reasonable extension for other types of signals, in real PTA data. Some key results from this work include:

- No detections were made. We placed upper limits on the intrinsic GW strain amplitude $h_0$ for a range of GW frequencies. For example, at $10^{-8}$ Hz our
analysis has excluded with 95% confidence the presence of signals with $h_0 \geq 1.7 \times 10^{-14}$. These new limits are about a factor of 4 more stringent than those of Yardley et al. (2010) based on an earlier PPTA data set and a factor of 2 better than those reported (almost simultaneously to our work) in Arzoumanian et al. (2014) by the NANOGrav collaboration.

- We also calculated PPTA's directional sensitivities and find that for the most sensitive region on the sky, the current data set is sensitive to GWs from circular SMBBHs with chirp masses of $10^9 \text{M}_\odot$ out to a luminosity distance of about 100 Mpc.

- We set an upper limit of $4 \times 10^{-3} \text{Mpc}^{-3} \text{Gyr}^{-1}$ at 95% confidence on the coalescence rate of nearby ($z \lesssim 0.1$) SMBBHs in circular orbits with chirp masses $\geq 10^{10} \text{M}_\odot$.

6.2 Future prospects

On the one hand, it is widely expected that advanced ground-based detectors, once having achieved their design sensitivities, are likely to detect dozens of CBC events each year. This would ultimately open a new observational window, allowing various source properties to be examined. For example, with the observed sample of CBC events, estimates on the local coalescence rate density and average chirp mass can be established. This information together with a detection (or non-detection) of the CBC background will put tight constraints on the average quantities of the whole CBC population throughout the Universe. Below we propose some possibilities for future research that are motivated by this thesis work.

1. We have shown that the three sub-populations of CBCs produce a similar power law background at low frequencies ($\lesssim 100 \text{ Hz}$). For advanced detectors such a simple power law model should prove useful in parameter estimation of the background signal. The parameter estimation problem was demonstrated
analytically in Mandic et al. (2012) for the BNS and BBH populations separately. Our results suggest that it is necessary to consider all contribution by BNSs, BBHs and BH-NS binaries as a whole. Moreover, more sophisticated algorithms are required to account for multiple models of GWBs simultaneously. This is important, for example if one wishes to set more stringent constraints on the primordial GWBs in the presence of a CBC foreground.

2. Our results indicate that a wide variety of features are present around the peak of $\Omega_{GW}(f)$ of the CBC background due to mass distributions, post-inspiral emission and coalescence rate evolution. Such information may also be extracted from the large observed sample ($\gtrsim 10^4$) of single sources detectable by ET. Joint measurements of the background and individual sources will provide much improved constraints on these poorly known quantities.

3. There are some unique properties of the CBC background signal: well defined continuously rising tones, localized directions and well defined average spectral density. These have not so far been fully exploited by stochastic background searches. New methods could be developed to make use of this extra information to improve the astrophysical outputs from the standard cross correlation searches.

On the other hand, PTA experiments have started to test various models of SMBBH formation and evolution. With steadily improved timing precision, increased data span and the expected combination of individual PTA data sets through the IPTA project, the majority of parameter space of these models has been probed. Latest analysis by the PPTA collaboration has suggested that the SMBBH coalescence rate may be significantly lower than previously predicted or other effects such as orbital eccentricities and the coupling of SMBBHs to their environments play a more important role (Shannon et al. 2015). We expect that analysis of single-source GWs based on more sensitive data sets will provide new insights on these issues. Specifically our future work is planned as follows.
1. We have developed a new method that can be used to detect and localize eccentric SMBBHs and other single sources with PTA data sets. This method should be further tested in more extensive simulations, for example in future IPTA mock-data challenges. We should also make comprehensive comparisons, in terms of detection sensitivity, parameter estimation and computational efficiency, between methods developed in this thesis and other methods such as some Bayesian techniques and the $F_e$-statistic. Again this could be done along with a future IPTA data challenge.

2. We will apply our methods to the updated PPTA and the forthcoming IPTA data set. Currently the PPTA DR2 data set is being actively worked on and will soon become available. In addition to an upgraded all-sky search, we will also perform directional/targeted searches for GWs from individual SMBBHs as more and more candidates are being discovered by electromagnetic observations (see, e.g. Graham et al. 2015; Liu et al. 2015).

3. The construction and usefulness of null data streams should be further investigated. This will result in a powerful veto statistic on whether a detected signal is real or due to accidental unaccounted-for noise.

4. How can additional information of pulsar terms be used to improve source localization? A recent theoretical work by Boyle & Pen (2012) showed that even with limited knowledge of pulsar distances it is still possible to localize the GW source to a precision much better than using Earth terms only. More recently Taylor et al. (2014) developed new techniques to significantly accelerate the Bayesian search for signals from SMBBHs, which makes the search over pulsar distances computationally possible. We plan to extend the localization method described in Chapter 4 to include pulsar distances as extra parameters and examine how much improvement can be gained in terms of localization precision.

5. It was recently suggested that the GWB formed by the combined emission
of the cosmic population of SMBBHs is not stochastic in nature, and could even be dominated by a small number of single bright sources (see, e.g. Ravi et al. 2012; Sesana 2013a). This would require a different search strategy than the one currently being used which is optimal only for a truly stochastic and Gaussian background. There have been some attempts to address this issue (Cornish & Sesana 2013; Mingarelli et al. 2013; Taylor & Gair 2013; Gair et al. 2014), focusing on characterizing the anisotropy of the GWB. The single-source detection methods presented in this thesis could be extended to search for a collection of sources that may be individually below the detection threshold.
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For reference and convenience, here we list some common definitions and constants used throughout this thesis.

\begin{align*}
1 \text{ pc} & = 3.08568 \times 10^{16} \text{ m} \\
1 \text{ yr} & = 3.15576 \times 10^{7} \text{ s} \\
\text{Gravitational constant} & \quad G = 6.67384 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \\
\text{Hubble constant} & \quad H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1} = 2.2685 \times 10^{-18} \text{ Hz} \\
\text{Speed of light} & \quad c = 2.99792 \times 10^{8} \text{ m s}^{-1} \\
\text{Solar mass} & \quad M_\odot = 1.989 \times 10^{30} \text{ kg} \\
\Omega_{GW} & \quad \text{the dimensionless energy density of a gravitational wave background}
\end{align*}

\begin{align*}
f_{\text{yr}} & \quad 1/(1 \text{ yr}), \text{ which is approximately}\ 3.17 \times 10^{-8} \text{ Hz} \\
M_c & \quad \text{binary chirp mass, defined as } (m_1 m_2)^{3/5} (m_1 + m_2)^{-1/5} \\
& \text{ with } m_1 \text{ and } m_2 \text{ being two component masses}
\end{align*}
Appendix B

Some Useful Codes

This Appendix contains some useful Matlab codes to calculate the GW waveforms as described in section 3.3.2.2.

B.1 Energy spectra for non-precessing spinning binary black holes

The following function produces the energy spectra $dE_{GW}/df_r$ for non-precessing spinning BBHs (Ajith et al. 2011).

function dedf=BBHnpsdedfun(m1,m2,c1,c2)
% dE/df for non-precessing spinning BBHs, based on 0909.2867v3 ...
% (Ajith et al 2011)
% Output the energy spectrum dedf
% Inputs: component masses (m1, m2) and spins (c1, c2)
% Note that the frequency vector is dependent on the input parameters

C = 299792458;
solar_mass = 1.9818*10^30;
G = 6.67384.*10.^(-11);
d = 10 * 10^6 * 3.08568025 * 10^16; % 10 kPc, arbitrary value, ...
% not used

mm=m1*solar_mass;
m2=m2*solar_mass;
M=mm+m2;


\[ yita = \frac{mm \cdot m2}{M^2}; \]
\[ dd = \frac{(mm - m2)}{M}; \]
\[ \chi = \frac{(1 + dd) \cdot c1 + (1 - dd) \cdot c2}{2}; \]

\[ yy1 = \left[ 0.6437, 0.827, -0.2706, -0.05822, -3.935, -7.092 \right]; \quad \% \text{parameters given in Table 1 of 0909.2867} \]
\[ yy2 = \left[ 0.1469, -0.1228, -0.02609, 0.1701, 2.325 \right]; \]
\[ yy3 = \left[ -0.4098, -0.03523, 0.1008, 1.829, -0.02017, -2.87 \right]; \]
\[ yy4 = \left[ -0.1331, -0.08172, 0.1451, -0.2714, 0.1279, 4.922 \right]; \]

\[ \text{miu} = yy1(1) \cdot yita + yy1(2) \cdot yita \cdot \chi + yy1(3) \cdot yita \cdot (\chi)^2 + yy1(4) \cdot \ldots \]
\[ \cdot (yita)^2 \cdot yy1(5) \cdot (yita)^2 \cdot yy1(6) \cdot (yita)^3; \]
\[ \text{miu} = yy2(1) \cdot yita + yy2(2) \cdot yita \cdot \chi + yy2(3) \cdot yita \cdot (\chi)^2 + yy2(4) \cdot \ldots \]
\[ \cdot (yita)^2 \cdot yy2(5) \cdot (yita)^2 \cdot yy2(6) \cdot (yita)^3; \]
\[ \text{miu} = yy3(1) \cdot yita + yy3(2) \cdot yita \cdot \chi + yy3(3) \cdot yita \cdot (\chi)^2 + yy3(4) \cdot \ldots \]
\[ \cdot (yita)^2 \cdot yy3(5) \cdot (yita)^2 \cdot yy3(6) \cdot (yita)^3; \]
\[ \text{miu} = yy4(1) \cdot yita + yy4(2) \cdot yita \cdot \chi + yy4(3) \cdot yita \cdot (\chi)^2 + yy4(4) \cdot \ldots \]
\[ \cdot (yita)^2 \cdot yy4(5) \cdot (yita)^2 \cdot yy4(6) \cdot (yita)^3; \]

\[ f10 = 1 - 4.455 \cdot (1 - \chi)^{0.217} + 3.52 \cdot (1 - \chi) \cdot (0.26); \]
\[ f20 = (1 - 0.63 \cdot (1 - \chi)^{0.3})/2; \]
\[ \delta10 = (1 - 0.63 \cdot (1 - \chi)^{0.3}) \cdot ((1 - \chi)^{0.45})/4; \]
\[ f30 = 0.3236 + 0.04894 \cdot \chi + 0.01346 \cdot \chi^2; \]

\[ f1 = \frac{(C^3/G) \cdot \text{miu} + f10}{(\pi \cdot M)}; \quad \% \text{f1 in the paper} \]
\[ f2 = \frac{(C^3/G) \cdot \text{miu} + f20}{(\pi \cdot M)}; \quad \% \text{f2 in the paper} \]
\[ \delta = \frac{(C^3/G) \cdot \text{miu} + \delta10}{(\pi \cdot M)}; \]
\[ f3 = \frac{(C^3/G) \cdot \text{miu} + f30}{(\pi \cdot M)}; \quad \% \text{f3 in the paper} \]

\[ pm = \frac{(\pi \cdot M)}{(C^3/G)}; \]
\[ \alpha_2 = 451 \cdot yita / 168 \cdot (323/224); \]
\[ a2 = \alpha_2 \cdot (pm^{-2/3}); \]
\[ \alpha_3 = (27/8 - 11/24) \cdot \chi; \]
\[ a3 = \alpha_3 \cdot pm; \]
\[ b1 = (1.4547 \cdot \chi - 1.8897) \cdot (pm^{-1/3}); \]
\[ b2 = (1.6557 - 1.8153 \cdot \chi) \cdot (pm^{-2/3}); \]
B.2. The TaylorT4 waveform

The following function produces the energy spectra $dE_{GW}/df_r$ using the TaylorT4 waveform (Blanchet et al. 2008; Santamaría et al. 2010) with the addition of a Gaussian component to represent the post-merger emission for BNSs (Bauswein et al. 2012).

```matlab
function dedf=dEdfTaylorT4fun(m1,m2,c1,c2,f,fp,delta)
% TaylorT4 formula based on 1005.3306v3 Sec III
% Output the energy spectrum dedf
% Inputs: component masses (m1, m2) and spins (c1, c2), f - ...
    freq1=round(f);
    freq2=(round(f)+1):round(f);
    freq3=(round(f)+1):round(f);

    Const=(M^(5/6)*((round(f))ˆ(-7/6))/(d*piˆ(2/3)))*sqrt(5*yita/24);

    A1=Const.*(freq1./max(freq1)).ˆ(-7/6).*(1+a2*freq1.(2/3)+a3*freq1);
    c12=(1+a2*(min(freq2)).(2/3)+a3*(min(freq2)))/(1+b1*(min(freq2)));
    A2=c12*Const.*(freq2./min(freq2)).(2/3).*(1+b1*freq2.(1/3)...
         +b2*freq2.(2/3));
    c23=(c12*(min(freq3)/min(freq2)).(2/3)*(1+b1*min(freq3).(1/3)...
         +b2*min(freq3).(2/3)))/(2/(pi*delta));
    A3=c23*Const.*(delta/(2*pi))./(freq3-min(freq3)).ˆ2+(delta./2)ˆ2;

    freq=[freq1,freq2,freq3];
    A=[A1,A2,A3];
    dedf=1e7*(24/5)*(1/3)*pi^(2)*d^(2)*(G^(2/3))*abs((freq.*A).ˆ2);
    return
```

B.2 The TaylorT4 waveform

The following function produces the energy spectra $dE_{GW}/df_r$ using the TaylorT4 waveform (Blanchet et al. 2008; Santamaría et al. 2010) with the addition of a Gaussian component to represent the post-merger emission for BNSs (Bauswein et al. 2012).
Appendix B. Some Useful Codes

% fp - peak frequency of the post-merger emission, e.g., 2000 Hz
% delta - Gaussian width, e.g., 250 Hz
% reference for the above two parameters - Table 2 in 1204.1888v2

C = 299792458;
solar_mass = 1.989*10^30;
G = 6.67384*10^(-11);
dL = 10 * 10^(-6) * 3.08568025 * 10^16; % 10 MPc

m1=m1*solar_mass;
m2=m2*solar_mass;
M=m1+m2;
yita=m1*m2/(M^2);

% Mc=M.*(yita^(3/5))./solar_mass; % yi ge shu
dd=(m1-m2)./M;
chi=((1+dd).*c1+(1-dd).*c2)./2;

rE=0.5772; % Euler constant

x=((G*M/(C^3))*pi.*f).^(2/3); % (G*M/(C^3))*

A0=1;
A1=0;
A2=-(107/42)+yita*(55/42);
A3=2*pi-chi*(4/3)+yita*(2/3)*(c1+c2);
A4=-(2173/1512)-yita*(1069/216-2*c1*c2)+(2047/1512)*(yita^2);
A5=-pi*(107/21)+yita*(34*pi/21-24i);
A6=(27027409/646800)-(856*rE/105)+(428*pi/105)*1i+(2*pi*pi/3)+...
yita*(41*pi*pi/96-278185/33264)-20261*yita*yita/2772+114635*...
(yita^3)/99792-(428/105).*log(16.*x);

A22=(8*yita.*x.*sqrt(pi/5)./dL).*(A0+A1.*(x.^(1/2))+A2.*x+A3.*...
(x.^(3/2))+A4.*(x.^2)+A5.*(x.^(5/2))+A6.*(x.^(3)));

a0=1;
B.2. The TaylorT4 waveform

\[ a_1 = 0; \]
\[ a_2 = \frac{743}{336} - 11\gamma/4; \]
\[ a_3 = 4\pi - 113\chi/12 + 19\gamma(\chi_1 + \chi_2)/6; \]
\[ a_4 = \frac{34103}{18144} + 5(\chi^2) + \gamma(13661/2016 - \chi_1 \chi_2/8) + 59(\gamma^2)/18; \]
\[ a_5 = -\pi(4159/672 + 189\gamma/8) - \chi(31571/1008 - 1165\gamma/24) + \gamma(\chi_1 + \chi_2)(21863\gamma/1008 - 79(\gamma^2)/6) - 3(\chi^3)/4 + 9\gamma\chi\chi_1\chi_2/4; \]
\[ a_6 = 16447322263/139708800 - 1712\pi E/105 + 16(\pi^2)/3 - (856/105)\log(16.x) + \gamma(451(\pi^2)/48 - 56198689/217728) + 541(\gamma^2)/896 \]
\[ - 5605(\gamma^3)/329280 + \pi\gamma(2529407/27216 - 845827\gamma/1008) + 1580239\gamma(\gamma^2)/54432 \]
\[ - 451597\gamma(\gamma^3)/432 + 107\gamma(\chi_1 + \chi_2)(\gamma^3)/6 - 5(\gamma^2)/2592 - 80\pi\gamma^3/105 + (20\pi/3 - 1135\chi/36)\gamma(\chi_1 + \chi_2)(21863\gamma/1008 - 79(\gamma^2)/6) - 3(\chi^3)/4 + 9\gamma\chi\chi_1\chi_2/4; \]
\[ a_7 = -\pi(4415/4032 - 358675\gamma/6048) - \chi(2529407/27216 - 845827\gamma/1008) + 1580239\gamma(\gamma^2)/54432 \]
\[ - 451597\gamma(\gamma^3)/432 + 107\gamma(\chi_1 + \chi_2)(\gamma^3)/6 - 5(\gamma^2)/2592 - 80\pi\gamma^3/105 + (20\pi/3 - 1135\chi/36)\gamma(\chi_1 + \chi_2)(21863\gamma/1008 - 79(\gamma^2)/6) - 3(\chi^3)/4 + 9\gamma\chi\chi_1\chi_2/4; \]

\[ x_t = (64\gamma(x^5)/5) + (a_0 + a_1(x^1/2) + a_2x + a_3(x^3/2) + a_4(x^2) + a_5(x^5/2) + a_6(x^3) + a_7(x^7/2)); \]
\[ o_t = 3.(x^1/2) \times x_t/2; \]
\[ h_f = \text{abs}(A22.*sqrt(pi./o_t)); \]
\[ h_f T4 = C^3/2 \times G^(-5/6) \times (G^2) \times (M^2) \times (C^(-5)) \times (pi^(-1/2)) \times (sqrt(5/16)).*h_f; \]
\[ \text{dedf0} = 1e7 \times (24/5) \times (1/3) \times pi^2 \times (G^2) \times (G^2/3) \times \text{abs}((f.*h_f T4)^2); \]
\[ \text{xls} = \text{find}(f == 1000); \]
\[ \% \text{the amplitude at fp is set to be twice that at 1000 Hz} \]
\[ \text{dedf2} = 2.*\text{dedf0}(\text{xls}).*\exp(-((f-fp)^2)/(2*delta^2)); \% 1840 102 ... \]
\[ \text{and 3720 152} \]
\[ \text{dedf} = \text{dedf0} + \text{dedf2}; \]
\[ \text{return} \]
This Appendix contains scripts to make use of some TEMPO2 functionalities as well as the source code of the TEMPO2 findCWs plugin. With the help of George Hobbs, I developed this plugin in order to implement the search for continuous waves in real PTA data sets as described in Chapter 5.

C.1 How to use the ‘$A_+ A_\times$’ routine in TEMPO2

Here we provide details for how to make use of the ‘$A_+ A_\times$’ routine which is described in section 5.3.1. To include the $A_+,\times (t)$ time series as part of a global fit, the following parameters should be provided in a global parameter file (which we assume is called “global.par”):

```plaintext
# The following line sets (S) the quadrupolar (Q) interpolation
# function (IFUNC) for the plus (p) polarization. The 1 indicates
# the type of interpolation (linear) and the 2 indicates that this
# will be fitted globally to all pulsars
SQIFUNC_p 1 2
# The next line is the same for the cross (c) polarization
SQIFUNC_c 1 2
# We now need to define the direction (in radians) of the quadrupole
# for the plus and cross functions (usually they are the same)
QIFUNC_POS_P <ra> <dec>
```
QIFUNC_POS_c <ra> <dec>

# We now define the actual grid points for the plus polarization
QIFUNC_p1 <mjd> <val> <err>
QIFUNC_p2 <mjd> <val> <err>

# and also for the cross polarization
QIFUNC_c1 <mjd> <val> <err>
QIFUNC_c2 <mjd> <val> <err>

# We then ensure that the constraints are correctly applied
CONSTRAIN QIFUNC_p
CONSTRAIN QIFUNC_c

Assuming that four pulsars have been observed, their parameters are in individual files (psr1.par, psr2.par, psr3.par and psr4.par) and their TOAs are in psr1.tim, psr2.tim etc., then the TEMPO2 fit can be carried out using:

tempo2 -f psr1.par psr1.tim -f psr2.par psr2.tim -f psr3.par psr3.tim -f psr4.par psr4.tim -global global.par

To include red noise models (defined in model.dat), then use:

tempo2 -f psr1.par psr1.tim -f psr2.par psr2.tim -f psr3.par psr3.tim -f psr4.par psr4.tim -global global.par -dcf model.dat

The output parameters are displayed on the screen and also written to a file called “aplus_across.dat”. This file contains the parameter values and their uncertainties.
C.2 The TEMPO2 addCGW plugin

The TEMPO2 addCGW plugin enables one to inject continuous gravitational wave signals to PTA data sets. The basic usage is as follows:

\texttt{tempo2 -gr addCGW -f psr1.par psr1.tim -freq 1e-8 -ra 0 -dec 0 -h0 1e-13 -ep 54500 -cosinc 1 -angpol 0 -mc 1e10}

Here \texttt{psr1.par} and \texttt{par1.tim} are the files for timing model parameters and TOAs for pulsar “psr1” respectively. The remaining entries in this line define the parameters of the continuous wave source, followed by their values. In this particular example, a continuous wave characterized by the following parameters is simulated: a frequency (freq) of $10^{-8}$ Hz, sky direction of $(\alpha, \delta) = (0, 0)$, \( h_0 = 10^{-13} \), a GW epoch (ep) of MJD 54500 (which is equivalent to the phase constant defined in section 4.2), a cosine of the inclination angle (cosinc) of 1, a polarization angle of 0, and a chirp mass (mc) of \(10^{10}M_\odot\). This produces a correction file called “psr1.tim.addCGW”. Such a usage is similar to plugins such as addGaussian (simulating Gaussian white noise), addRedNoise (simulating red noise) and addGWB (simulating a gravitational wave background).

Then one can create a new set of arrival times that are affected by the simulated continuous wave signal using

\texttt{tempo2 -gr createRealisation -f psr1.tim -corr psr1.tim.addCGW}

This produces a new arrival time file called “psr1.tim.real” (note that the parameter file is not used with the \texttt{createRealisation} plugin). This can be plotted as normal using, e.g., the \texttt{plk} plugin:

\texttt{tempo2 -gr plk -f psr1.par psr1.tim.real}

It is worth noting that: 1) to simulate pulsar terms a pulsar distance should be specified in the .par file. The can be done by including “GW_PSR_DIST” followed by the distance in meters; 2) the chirp mass parameter is only used to
calculate the frequency evolution between pulsar terms and Earth terms. This is an optional parameter. When it is not defined, pulsar terms are simulated at the same frequency as Earth terms.

C.3 The TEMPO2 findCWs plugin

The TEMPO2 findCWs plugin makes use of the ‘$A_+ A_\times$’ routine to calculate the detection statistics as a function of frequency as described in section 5.3.2. The basic usage is as follows:

```
tempo2 -gr findCWs -f psr1.par psr1.tim -f psr2.par psr2.tim -f psr3.par psr3.tim -f psr4.par psr4.tim -global global.par
```

Here again one could append “-dcf model.dat” in the end to account for red noise models (as given in “model.dat”). This outputs a file called “DectionSts.dat”, which contains three columns – the first gives the index of frequency bin, the second gives the frequency, and the third gives the detection statistic $P$ defined in section 5.3.2.

We present below the source code for this plugin. It has been slightly edited for appearance from the original code. This plugin is included as part of the standard TEMPO2 distribution.

```c
/*
 * This file is part of TEMPO2.
 *
 * TEMPO2 is free software: you can redistribute it and/or modify
 * it under the terms of the GNU General Public License as
 * published by the Free Software Foundation, either version 3 of
 * the License, or (at your option) any later version.
 * TEMPO2 is distributed in the hope that it will be useful,
 * but WITHOUT ANY WARRANTY; without even the implied warranty of
```
C.3. The TEMPO2 findCWs plugin

* MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE. See the
* GNU General Public License for more details.
* You should have received a copy of the GNU General Public License
* along with TEMPO2. If not, see <http://www.gnu.org/licenses/>.
*/

/*
 * If you use TEMPO2 then please acknowledge it by citing
 * pp. 655-672 (bibtex: 2006MNRAS.369..655H)
 * pp. 1549-1574 (bibtex: 2006MNRAS.372.1549E) when discussing the
 * timing model.
 */

#include <stdio.h>
#include <string.h>
#include <stdlib.h>
#include <math.h>
#include "tempo2.h"
#include "TKspectrum.h"
#include <gsl/gsl_math.h>
#include <gsl/gsl_blas.h>
#include <gsl/gsl_vector.h>
#include <gsl/gsl_matrix.h>
#include <gsl/gsl_eigen.h>
#include <gsl/gsl_linalg.h>

using namespace std;
void help() /* Display help */
{
}

extern "C" int graphicalInterface(int argc,char *argv[],pulsar *psr,
int *npsr)
{
  char parFile[MAX_PSR][MAX_FILELEN];
  char timFile[MAX_PSR][MAX_FILELEN];
  char covarFuncFile[128];
  int i, j;
  double globalParameter;
  const char *CVS_verNum = "$Revision: 1.2 "$;
  FILE *fout;
  strcpy(covarFuncFile,"NULL");
  if (displayCVSversion == 1) CVSdisplayVersion((char *)"findCW.C",
    (char *)"plugin",CVS_verNum);

  *npsr = 0;
  printf("Graphical Interface: findCW\n");
  printf("Author: X. Zhu, G. Hobbs\n");
  printf("CVS Version: $Revision: 1.2 $\n");
  printf(" --- type 'h' for help information\n");
  /* Obtain all parameters from the command line */
  for (i=2;i<argc;i++)
  {
    if (strcmp(argv[i],"-f")==0)
    {
      
    
  
}
C.3. The TEMPO2 findCWs plugin

```c
strcpy(parFile[*npsr],argv[++i]);
strcpy(timFile[*npsr],argv[++i]);
(*npsr)++;
}
else if (strcmp(argv[i],"-dcf")==0)
strcpy(covarFuncFile,argv[++i]);
}
readParfile(psr,parFile,timFile,*npsr); /* Load the parameters */
readTimfile(psr,timFile,*npsr); /* Load the arrival times */
preProcess(psr,*npsr,argc,argv);

for (i=0;i<2;i++) /* Do two iterations for pre- and post-fit residuals*/
    // i=0;
{
    logdbg("Calling formBatsAll");
    formBatsAll(psr,*npsr); /* Form the barycentric arrival times */
    logdbg("Calling formResiduals");
    formResiduals(psr,*npsr,1); /* Form the residuals */
    logdbg("Calling doFit");
    if (i==0) doFitAll(psr,*npsr,covarFuncFile); /* Do the fitting */
    else textOutput(psr,*npsr,globalParameter,0,0,0,(char *)"")); /* Display the output */
}

// Print A+ and Ax into a file
fout = fopen("aplus_across.dat","w");
```
for (i=0;i<psr[0].quad_ifuncN_p;i++)
{
    fprintf(fout,"%.2lf %g %g %g %g\n",psr[0].quad_ifuncT_p[i],
    psr[0].quad_ifuncV_p[i],psr[0].quad_ifuncE_p[i],
    psr[0].quad_ifuncV_c[i],psr[0].quad_ifuncE_c[i]);
}
fclose(fout);
// return 0;

// Calculate the Detection Statistics as a function of frequencies
{
    double freq[1024];
    double DS[1024];
    double dt, Tspan, fmin, fmax;
    int nSpec; /* number of independent freq channels */
    int nSpecOS4;
    /* number of freq channels with an OverSampling factor of 4 */
    const int lp = psr[0].quad_ifuncN_p; /* length of A+ or Ax */
    const int lpc = 2*psr[0].quad_ifuncN_p; /* length of the
    stacked data A+,x assuming A+&Ax have the same length*/
    const int n_cst = 18; /* number of constraints set on A+Ax */
    const int mlen = lpc-n_cst;
    /* rank of the noise covariance matrix */
    // Assuming that the ifuncs are regularly sampled
    //
    dt = psr[0].quad_ifuncT_p[1]-psr[0].quad_ifuncT_p[0];
    Tspan = psr[0].quad_ifuncT_p[lp-1]-psr[0].quad_ifuncT_p[0];
    fmin = 1.0/Tspan/86400.0;
    fmax = 1.0/dt/86400.0/2;
The TEMPO2 findCWs plugin

\[ n\text{Spec} = \text{floor}(f_{\text{max}}/f_{\text{min}}); \]
\[ n\text{SpecOS4} = 4 \times n\text{Spec}; /* a factor of 4 oversampling */ \]

// check
/*
printf("%d	%d\t%d\n", lp, lpc, mlen);
printf("%g\t%g\t%d\n", fmin, fmax, nSpec);
*/
gsl_vector *time = gsl_vector_alloc (lp);
gsl_vector *Apn = gsl_vector_alloc (lp);
gsl_vector *Acn = gsl_vector_alloc (lp);
for (i = 0; i < lp; i++)
{
    gsl_vector_set (time, i, psr[0].quad_ifuncT_p[i]);
    gsl_vector_set (Apn, i, psr[0].quad_ifuncV_p[i]);
    gsl_vector_set (Acn, i, psr[0].quad_ifuncV_c[i]);
}
gsl_matrix *Sigma_n = gsl_matrix_alloc (lpc, lpc);
/* noise convariance matrix */
gsl_matrix *Sigma_n1 = gsl_matrix_alloc (lpc, lpc);
/* inverse noise convariance matrix */
for (i = 0; i < psr[0].globalNfit; i++)
{
    for (j = 0; j < psr[0].globalNfit; j++)
    {
        if ((psr[0].fitParamI[i] == param_quad_ifunc_p
            || psr[0].fitParamI[i] == param_quad_ifunc_c) &&
            (psr[0].fitParamI[j] == param_quad_ifunc_p
            || psr[0].fitParamI[j] == param_quad_ifunc_c))
        {
            "}
gsl_matrix_set (Sigma_n, i, j, psr[0].covar[i][j]);
}
}
}

// print the covariance matrix to screen
/*
for (j = 0; j < lpc; j++)
    printf("Sigma_n(%d,%d) = %g\n", 0, j,
        gsl_matrix_get (Sigma_n, 0, j));
*/

// eign-decomposition

gsl_vector *eval = gsl_vector_alloc (lpc);
gsl_matrix *evec = gsl_matrix_alloc (lpc, lpc);
gsl_eigen_symmv_workspace * w = gsl_eigen_symmv_alloc (lpc);
gsl_eigen_symmv (Sigma_n, eval, evec, w);
gsl_eigen_symmv_free (w);
gsl_eigen_symmv_sort (eval, evec, GSL_EIGEN_SORT_ABS_ASC);
gsl_matrix *D1 = gsl_matrix_alloc (mlen, mlen);
gsl_matrix *A1 = gsl_matrix_alloc (lpc, mlen);
gsl_matrix_set_zero (D1);
/* print the eign-values
for (j = 0; j < lpc; j++)
    printf("%d\t%g\n", j, gsl_vector_get (eval, j));
*/

for (i = 0; i < mlen; i++)
{
    double eval_i = gsl_vector_get (eval, i+n_cst);
    gsl_matrix_set (D1, i, i, 1.0/eval_i);
C.3. The TEMPO2 findCWs plugin

```c
  gsl_vector *evect_i = gsl_vector_alloc (lpc);
  gsl_matrix_get_col (evect_i, evect, i+n_cst);
  gsl_matrix_set_col (A1, i, evect_i);
  gsl_vector_free (evect_i);
}
gsl_vector_free (eval);
gsl_matrix_free (evect);
gsl_matrix *AD1 = gsl_matrix_alloc (lpc, mlen);
gsl_matrix_set_zero (AD1);
gsl_matrix_set_zero (Sigma_n1);
gsl_blas_dgemm (CblasNoTrans, CblasNoTrans, 1.0, A1, D1, 0.0, AD1);
  gsl_blas_dgemm (CblasNoTrans, CblasTrans, 1.0, AD1, A1, 0.0, Sigma_n1);
  /* Sigma_n1=A1*D1*A1' */
  // print the inverse covariance matrix to screen
  /* checked and agree with Matlab results */
  for (j = 0; j < lpc; j++)
    printf ("Sigma_n1(%d,%d) = %g
", 1, j+1,
      gsl_matrix_get (Sigma_n1, 0, j));
  /*
  gsl_matrix_free (Sigma_n);
  gsl_matrix_free (D1);
  gsl_matrix_free (A1);
  gsl_matrix_free (AD1);
  gsl_matrix *S11 = gsl_matrix_alloc (lp, lp);
  gsl_matrix *S12 = gsl_matrix_alloc (lp, lp);
  gsl_matrix *S21 = gsl_matrix_alloc (lp, lp);
  gsl_matrix *S22 = gsl_matrix_alloc (lp, lp);
```
for (i = 0; i < lp; i++)
{
    for (j = 0; j < lp; j++)
    {
        gsl_matrix_set (S11, i, j, gsl_matrix_get (Sigma_n1, i, j));
        gsl_matrix_set (S12, i, j, gsl_matrix_get (Sigma_n1, i, (lp+j)));
        gsl_matrix_set (S21, i, j, gsl_matrix_get (Sigma_n1, (lp+i), j));
        gsl_matrix_set (S22, i, j, gsl_matrix_get (Sigma_n1, (lp+i), (lp+j)));
    }
}
gsl_matrix_free (Sigma_n1);
/* print the submatrix
for (j = 0; j < lp; j++)
    printf ("S11(%d,%d) = %g\n", 1, j+1,
            gsl_matrix_get (S11, 0, j));
*/

// calculate the Detection Statistics for a set of frequencies
for (i = 0; i < nSpecOS4; i++)
{
    double fIndx = 0.5+0.25*i;
    double f = fmin*fIndx;
    double y11, y12, y21, y22, y31, y32, y41, y42;
    double r11, r12, r13, r14, r22, r23, r24, r33, r34, r44;
    double d1, d2, d3, d4, d5, d6, d7, d8;
    double y1, y2, y3, y4, r21, r31, r32, r41, r42, r43;
C.3. The TEMPO2 findCWs plugin

```c
   gsl_vector *Sc = gsl_vector_alloc (lp);
gsl_vector *Ss = gsl_vector_alloc (lp);
for (j = 0; j < lp; j++)
{
   gsl_vector_set (Sc, j, 
                   cos(2.0*86400.0*M_PI*f*psr[0].quad_ifuncT_p[j]));
gsl_vector_set (Ss, j, 
                   sin(2.0*86400.0*M_PI*f*psr[0].quad_ifuncT_p[j]));
}
gsl_vector *tempy1 = gsl_vector_alloc (lp);
gsl_vector *tempy2 = gsl_vector_alloc (lp);
gsl_vector *tempy3 = gsl_vector_alloc (lp);
gsl_vector *tempy4 = gsl_vector_alloc (lp);
gsl_vector *tempy5 = gsl_vector_alloc (lp);
gsl_vector *tempy6 = gsl_vector_alloc (lp);
gsl_vector *tempy7 = gsl_vector_alloc (lp);
gsl_vector *tempy8 = gsl_vector_alloc (lp);
gsl_vector *tempr1 = gsl_vector_alloc (lp);
gsl_vector *tempr2 = gsl_vector_alloc (lp);
gsl_vector *tempr3 = gsl_vector_alloc (lp);
gsl_vector *tempr4 = gsl_vector_alloc (lp);
gsl_vector *tempr5 = gsl_vector_alloc (lp);
gsl_vector *tempr6 = gsl_vector_alloc (lp);
gsl_vector *tempr7 = gsl_vector_alloc (lp);
gsl_vector *temp5 = gsl_vector_alloc (lp);
gsl_vector *temp6 = gsl_vector_alloc (lp);
gsl_vector *temp7 = gsl_vector_alloc (lp);
gsl_vector *temp8 = gsl_vector_alloc (lp);
```
gsl_blas_dgemv (CblasTrans, 1.0, S11, Apn, 0.0, tempy1);
gsl_blas_dgemv (CblasTrans, 1.0, S21, Acn, 0.0, tempy2);
gsl_blas_ddot (tempy1, Sc, &y11);
gsl_blas_ddot (tempy2, Sc, &y12);
y1 = y11 + y12;
gsl_blas_ddot (tempy1, Ss, &y21);
gsl_blas_ddot (tempy2, Ss, &y22);
y2 = y21 + y22;
gsl_blas_dgemv (CblasTrans, 1.0, S12, Apn, 0.0, tempy3);
gsl_blas_dgemv (CblasTrans, 1.0, S22, Acn, 0.0, tempy4);
gsl_blas_ddot (tempy3, Sc, &y31);
gsl_blas_ddot (tempy4, Sc, &y32);
y3 = y31 + y32;
gsl_blas_ddot (tempy3, Ss, &y41);
gsl_blas_ddot (tempy4, Ss, &y42);
y4 = y41 + y42;
gsl_blas_dgemv (CblasTrans, 1.0, S11, Sc, 0.0, tempr1);
gsl_blas_dgemv (CblasTrans, 1.0, S12, Sc, 0.0, tempr2);
gsl_blas_ddot (tempr1, Sc, &r11);
gsl_blas_ddot (tempr1, Ss, &r12);
gsl_blas_ddot (tempr2, Sc, &r13);
gsl_blas_ddot (tempr2, Ss, &r14);
gsl_blas_dgemv (CblasTrans, 1.0, S11, Ss, 0.0, tempr3);
gsl_blas_dgemv (CblasTrans, 1.0, S21, Sc, 0.0, tempr4);
gsl_blas_dgemv (CblasTrans, 1.0, S21, Ss, 0.0, tempr5);
gsl_blas_ddot (tempr3, Ss, &r22);
gsl_blas_ddot (tempr4, Ss, &r23);
gsl_blas_ddot (tempr5, Ss, &r24);
gsl_blas_dgemv (CblasTrans, 1.0, S22, Sc, 0.0, tempr6);
C.3. The TEMPO2 findCWs plugin

```c
#include <gsl/gsl_blas.h>
#include <gsl/gsl_matrix.h>
#include <gsl/gsl_permutation.h>
#include <gsl/gsl_vector.h>

void findCWs
{
    gsl_blas_dgemv (CblasTrans, 1.0, S22, Ss, 0.0, tempr7);
    gsl_blas_ddot (tempr6, Sc, &r33);
    gsl_blas_ddot (tempr6, Ss, &r34);
    gsl_blas_ddot (tempr7, Ss, &r44);
    r21 = r12;
    r31 = r13;
    r32 = r23;
    r41 = r14;
    r42 = r24;
    r43 = r34;
    double a_data[] = { r11, r12, r13, r14,
                        r21, r22, r23, r24,
                        r31, r32, r33, r34,
                        r41, r42, r43, r44 };
    double b_data[] = { y1, y2, y3, y4 };
    gsl_matrix_view m = gsl_matrix_view_array (a_data, 4, 4);
    gsl_vector_view b = gsl_vector_view_array (b_data, 4);
    gsl_vector *Cc = gsl_vector_alloc (4);
    int s;
    gsl_permutation * p = gsl_permutation_alloc (4);
    gsl_linalg_LU_decomp (&m.matrix, p, &s);
    gsl_linalg_LU_solve (&m.matrix, p, &b.vector, Cc);
    gsl_permutation_free (p);

    /*
     for (j = 0; j < 4; j++)
       printf ("Cc(%d) = %g\n", j+1, gsl_vector_get (Cc, j));
    */
    double Cc1 = gsl_vector_get (Cc, 0);
}
```

double Cc2 = gsl_vector_get (Cc, 1);
double Cc3 = gsl_vector_get (Cc, 2);
double Cc4 = gsl_vector_get (Cc, 3);
gsl_vector_free (Cc);
gsl_vector *Ap = gsl_vector_alloc (lp);
gsl_vector *Ac = gsl_vector_alloc (lp);
for (j = 0; j < lp; j++)
{
    gsl_vector_set (Ap, j,
    Cc1*cos(2.0*86400.0*M_PI*f*psr[0].quad_ifuncT_p[j])
    + Cc2*sin(2.0*86400.0*M_PI*f*psr[0].quad_ifuncT_p[j]));
    gsl_vector_set (Ac, j,
    Cc3*cos(2.0*86400.0*M_PI*f*psr[0].quad_ifuncT_p[j])
    + Cc4*sin(2.0*86400.0*M_PI*f*psr[0].quad_ifuncT_p[j]));
}
gsl_blas_dgemv (CblasTrans, 1.0, S11, Ap, 0.0, tempy5);
gsl_blas_dgemv (CblasTrans, 1.0, S21, Ac, 0.0, tempy6);
gsl_blas_dgemv (CblasTrans, 1.0, S12, Ap, 0.0, tempy7);
gsl_blas_dgemv (CblasTrans, 1.0, S22, Ac, 0.0, tempy8);
gsl_blas_dgemv (CblasTrans, 1.0, S11, Apn, 0.0, temp5);
gsl_blas_dgemv (CblasTrans, 1.0, S21, Acn, 0.0, temp6);
gsl_blas_dgemv (CblasTrans, 1.0, S12, Apn, 0.0, temp7);
gsl_blas_dgemv (CblasTrans, 1.0, S22, Acn, 0.0, temp8);
gsl_blas_ddot (tempy5, Ap, &d1);
gsl_blas_ddot (tempy6, Ap, &d2);
gsl_blas_ddot (tempy7, Ac, &d3);
gsl_blas_ddot (tempy8, Ac, &d4);
gsl_blas_ddot (temp5, Ap, &d5);
gsl_blas_ddot (temp6, Ap, &d6);
gsl_blas_ddot (temp7, Ac, &d7);
gsl_blas_ddot (temp8, Ac, &d8);
freq[i] = f;
DS[i] = 2.0*(d5+d6+d7+d8)-(d1+d2+d3+d4);
gsl_vector_free (tempy1);
gsl_vector_free (tempy2);
gsl_vector_free (tempy3);
gsl_vector_free (tempy4);
gsl_vector_free (tempy5);
gsl_vector_free (tempy6);
gsl_vector_free (tempy7);
gsl_vector_free (tempy8);
gsl_vector_free (temp5);
gsl_vector_free (temp6);
gsl_vector_free (temp7);
gsl_vector_free (temp8);
gsl_vector_free (tempr1);
gsl_vector_free (tempr2);
gsl_vector_free (tempr3);
gsl_vector_free (tempr4);
gsl_vector_free (tempr5);
gsl_vector_free (tempr6);
gsl_vector_free (tempr7);
gsl_vector_free (Sc);
gsl_vector_free (Ss);
gsl_vector_free (Ac);
gsl_vector_free (Ap);
}
gsl_matrix_free (S12);
gsl_matrix_free (S21);
gsl_matrix_free (S22);
gsl_vector_free (time);
gsl_vector_free (Apn);
gsl_vector_free (Acn);

fout = fopen("DectionSts.dat","w");
for (i=0;i<nSpecOS4;i++)
    fprintf(fout,"%d %g %g\n",i+1,freq[i],DS[i]);
fclose(fout);
}
return 0;
}

// char * plugVersionCheck = TEMPO2_h_VER;