Optomechanical Physics in the Design of Gravitational Wave Detectors

Yiqiu Ma

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Abstract

Einstein’s General Theory of Relativity predicts the existence of gravitational waves, which are the ripples of spacetime. Recently, ground-based kilometer scale Fabry-Perot Michelson interferometers detected the gravitational waves radiated from a pair of black holes as they coalesced and merged. This event represented a gravitational explosion of 3 solar mass of gravitational wave energy. This was the most powerful transient astronomical event ever observed. However the gravitational wave detectors required to detect this event are the most sensitive instruments ever created, able to detect a mechanical energy of $\sim 10^{-32}$ J.

The operation of gravitational wave detectors is based on the optomechanical coupling that converts tiny differential mechanical motion of the test masses to measurable optical signals. These detectors are based on the concept of a simple Michelson interferometer. However, to turn this concept into a practical design requires the creation of the most beautiful and intricate optomechanical devices. Understanding the optomechanical physics of gravitational wave detectors, and using this understanding to design methods for improving sensitivity, is the main motivation of the research work presented in this thesis.

After the background introduction in Chapter 1 and 2, Chapter 3 discusses the energy exchange between circulating optical fields and the test masses by analysing a simple optomechanical model of gravitational wave detector,. This chapter reveals the fact that the gravitational wave energy can be directly absorbed by gravitational wave detectors using a process of detuning. This draws the connection between interferometric detectors with bar detectors, and gives a better understanding of laser interferometer detectors as transducers of gravitational wave energy.

In Chapter 4, a new classical noise source in gravitational wave detectors is introduced. It arises from the optomechanical coupling between thermal excitations of mirror acoustic modes and the intra-cavity optical fields. The results show that the new noise source will not significantly affect the sensitivity of the detectors within the
current target frequency range of advanced detectors, but could set limits on future low frequency detectors that aim to exceed the quantum noise limit.

Chapter 5 to 8 in this thesis are devoted to the study of quantum noise in gravitational wave detectors. This noise comes from optomechanical interactions at the quantum level. These chapters focus on two quantum limits that constrain the sensitivity of the detectors: (1) the Standard Quantum Limit due to the trade-off between the shot noise and radiation pressure noise; (2) the Mizuno Limit due to the trade-off between the detector’s detection bandwidth and its peak shot-noise-limited sensitivity. Both of these limits can be surpassed.

To beat the Standard Quantum Limit over the detection band, it has been proposed to inject squeezed vacuum into the interferometer through a very narrowband filter cavity. Such narrowband cavities which ideally ought to be tunable, are difficult to realise. Chapter 5 shows that optomechanics can create an extremely narrow filter cavity bandwidth comparable to the mechanical bandwidth, which can be realised by an optomechanical cavity pumped by red-detuned laser light.

Optomechanical devices are very sensitive to the thermal noise. For solving this issue, optomechanics allows us to increase the resonance frequency and the Q-factor of the mechanical resonator thereby diluting the thermal noise. This is called optical dilution. Chapter 5 also presents a novel optical dilution method which suppresses both quantum noise and potential optomechanical instabilities associated with the optical spring.

Optomechanics can also create negative dispersion. Negative dispersion allows the creation of a white light cavity, which can be used in the interferometer design to circumvent the Mizuno Limit. In Chapter 6, an interferometer configuration with an optomechanical filter cavity operating in the dynamically unstable blue-detuned region is studied. It is shown that, using feedback control to stabilise the system, this configuration can in-principle broaden detector bandwidth without sacrificing its peak sensitivity, thereby surpassing the Mizuno limit.

The optomechanical approach to white light cavity is very different from previously discussed methods, which proposed the use of stable atomic gaseous media as to create white light cavities. Quantum noise analysis had not been done before for such
systems. Chapter 7 shows that the sensitivity of an interferometer configuration with a double-gain atomic filter is strongly constrained by stability requirements and an additional quantum noise. This noise is associated with the parametric amplification process arising in atomic media. Due to these constraints, some designs are not able to surpass the Mizuno limit.

Optomechanical systems are concrete examples of general linear optical measurement devices. Chapter 8 discusses the quantum sensitivity limit of these general measurement devices. Using the Heisenberg uncertainty principle, this chapter presents a new derivation of the upper bound of the displacement sensitivity of these linear optical measurement devices. The result reveals that fact that the fundamental quantum limit of a linear optical measurement device is determined by the quantum fluctuation of optical power.
Preface

This preface summarises provides a chapter by chapter summary and details of publications. The research were completed in collaboration with many researchers where their contributions are also listed in this preface.

Chapter 3 was published in Classical and Quantum Gravity:


This work was motivated by the analogy between the microwave readout system in bar detectors and laser interferometers. Blair proposed to study the interferometer energy absorption issue by considering the interferometer as a parametric transducer similar to what was done in the studies of bar detectors. By developing a quantum analysis on the Manley-Rowe relation, I attempted to solve it by analysing the energy interaction between the optical field and the center of mass (CoM) motion of the test masses. I found that interferometer with unbalanced sideband field will extract gravitational wave energy through optomechanical modification of the test mass CoM motion dynamics. The energy absorption cross section can be dramatically higher than the balanced interferometer, which absorbs gravitational wave energy through Doppler friction first discussed by P. Saulson. W. Kells provided a more accurate classical derivation of Doppler friction. C. Zhao read the paper and gave many important suggestions.

Chapter 4 discusses the effect of an amplified noise in gravitational wave detectors due to the modification of mechanical dynamics from optomechanical interaction. This work was published as:
Li Ju, Chunnong Zhao, Yiqiu Ma, David Blair, Stefan L Danilishin and Slawek Gras, *Three mode interaction noise in interferometric gravitational wave detectors*, Class. Quantum Grav. **31** 145002, (2014).

Blair noticed that the three mode parametric process can convert the thermal fluctuation of mirror acoustic modes to random fluctuation of intra-cavity power in the arm cavities, and hence he proposed to study this three mode interaction noise. Ju and Zhao estimate the effect of this noise to the sensitivity of gravitational wave detectors, which is actually very close to the result shown in exact calculation. Together with Danilishin, I developed a theory for this noise and derived the exact formula of the noise spectrum. Notably, I found that this high order noise effect actually be amplified by the optomechanical modification of acoustic mode dynamics. The spectrum formula I developed in the simulation performed by Ju. I wrote the theory part of this paper and was responsible for communicating with referees.

Chapter 5-8 in this thesis are related to the quantum noise in gravitational wave detectors. We discuss the topics related to two quantum sensitivity limits (as mentioned in the Abstract), the Standard Quantum Limit and the Mizuno limit. The basic concepts of these two limits are introduced in Chapter 1 and 2.

**Chapter 5** is based on two published papers:


In the first paper, I showed theoretically how optomechanics can help us rotate the squeezed light in a compact and tunable way. This theoretical design could be used to improve the broadband sensitivity of future gravitational wave detectors by surpassing the Standard Quantum Limit. This idea was initially proposed by Zhao and Korth. I theoretically explored the feasibility of this idea under the supervision of Miao and Zhao. I found that the squeezed light is greatly affected by thermal noise when filtered by this optomechanical device. To solve this problem, optomechanics motivated me to study the optical dilution idea and its related problems such as potential dynamical instability and additional radiation pressure noise. Then I theoretically proposed a novel optical dilution scheme based on the principle of dissipative optomechanical dynamics, which can in principle avoid these problems. I wrote the initial draft of the first paper which was then revised by Miao and Ward.

In the second paper, I participated the design of the experiment and wrote the theoretical part of the paper. This experiment was done by Qin under the supervision of Zhao. The paper was revised by Ju and Blair.

Chapter 6 shows that the optomechanical effect can also be used to surpass the Mizuno limit by creating a white light cavity. This work was published as:


In this paper, we found that an optomechanical device that works in a dynamically unstable region has a negative dispersion behaviour, which can be used for surpassing the Mizuno limit. However, the unstable dynamics of the system must be controlled. Under the supervision of Miao, I performed the theoretical analysis to the stability of this system using Nyquist diagram, which is important in the design of feedback stabilisation control loop. This research topic is initiated by Miao and the sensitivity curves are computed by Miao and confirmed by me. Zhao and Chen gave many in-
sightful suggestions to this project.

Chapter 7 shows that the designed configurations in previous literatures for creating the white light cavity effect by using double-pumped atomic gain media are strongly constrained by the stability criteria and quantum noise brought by atomic gain media. This work was published as:


Under the supervision of Miao and Chen, I developed an input-output formalism for atomic media which includes the exact form of the additional noise contribution brought by atomic gain media. This input-output formalism is very useful in computing the sensitivity of the interferometer configuration containing atomic media. I analysed the stability by using Nyquist diagram and figured out the stable region by numerically surveying the system parameters. Furthermore, I computed the sensitivity using the input-output formalism and concluded that the previously proposed design cannot surpass the Mizuno limit.

Furthermore, during the studying of the atomic media, Miao and I found that in the linear region, the double-pumped atomic gain media can be mapped to an optomechanical system. Chapter 7 also presented an experiment on showing the optomechanically generated anomalous dispersion phenomenon. This work was published as:

Jiayi Qin, Chunnong Zhao, Yiqiu Ma, Xu Chen, Li Ju, and David G. Blair, Linear anomalous dispersion with a gain doublet via optomechanical interactions, Optics Letters 40, 10, pp. 2337-2340,(2015)

In this work, I participated the experiment design and helped Qin conduct the theoretical analysis of the experiment. The experiment was done by Qin under supervision of Zhao. I revised the theoretical part of the paper. Ju and Blair revised the paper
further and gave many important advices on experimental details.

Chapter 8 extended the work in Chapter 7 which identified the bound for the ultimate quantum noise limited sensitivity for a large class of optical parametric measurement devices. This draft is still in preparation. This is an ongoing project in collaboration with Yanbei Chen and Haixing Miao. I contributed in the second part of this chapter about the calculation of the sensitivity bound. The proof of Mizuno limit and the use of Heisenberg uncertainty principle for limiting the variance of optical amplitude quadrature is provided by Chen.
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Chapter 1

Introduction to Gravitational Wave Detectors

1.1 Preface

In this Chapter, we present a brief introduction to the gravitational waves and its detection and we give a general overview of the research background. More details can be found in several review papers [1, 2, 3, 4].

The outline of this Chapter is the following: In Section 1.1, we describe the background of the gravitational wave research and then in Section 1.2, we introduce the basic principle of a current, second generation ground-based laser-interferometer gravitational wave detector. Section 1.3 provides an overview of the noise sources currently affecting the sensitivity of a ground-based interferometer. Section 1.4 is a discussion about the interferometer from the aspect of energy circulation.

1.2 The detection of gravitational waves

Gravitational waves are “the ripple of space-time” predicted by Einstein’s theory of General Relativity [5], which describes the gravity as space-time curvature. This curvature implies that the space-time can be viewed effectively as an elastic medium, with ultra-high stiffness \( \sim \frac{c^4}{(8\pi G)} \sim 10^{43} \), where \( c \) and \( G \) are the speed of light and Newton’s gravitational constant respectively [7]. The space-time’s ultra-high stiffness implies the difficulty of generating a gravitational excitation and explains why the gravitational waves usually have an ultra-small amplitude unless they are generated from the extremal astrophysical/cosmological processes.
1.2.1 Gravitational wave sources

Here we list the main astrophysical/cosmological sources of gravitational waves [3, 4, 5, 7]:

- **Inspiral/Coalesing binary neutron star/black hole systems.** — Binary systems radiate gravitational waves by three continuously connected stages: First, the two objects of a binary system orbit each other and radiate gravitational waves. The radiated waves cause back-action to the source system, and make it lose energy and angular momentum, hence the two objects gradually spiral inwards. During this process, the gravitational wave signal strength and frequency increase, resulting in a chirp signal. Secondly, the distance between the two objects is so close that the tidal interaction between them becomes very important and the system starts to coalesce. At this stage, the motion of the source objects is no longer Keplerian and finally they will merge together, forming a more compact object (typically a new black hole). The gravitational wave emitted from this dynamical process can be predicted by Numerical Simulation involving the joint evolution of the space-time and matter. Finally, this new-born compact object starts to radiate gravitational waves and ring-down itself. Detection of the gravitational waves at this ring-down stage allows us to probe the characters of this new-born compact object.

- **Spinning compact objects** — A new born spinning compact object with large degree of asymmetry (i.e., large quadrupole moments) can radiate gravitational waves and lose the energy of angular motion. For example, after the coalescing of the binary system, a fast-spining black-hole can be formed with big quadrupole moments. Another examples are the non-symmetric pulsars and neutron stars.

- **Stochastic gravitational wave background** — multiple effects of different single-sources and the cosmological gravitational waves can create a stochastic background of gravitational waves. The cosmological gravitational wave, which is expected to be the remnants of the inflationary stage of the universe, is of particular interests [9, 10]. This cosmological gravitational wave background carries the information of the physical processes near the big bang. Recent observation of the polarization of Cosmic Microwave Background opens the possibility to detect those gravitational waves [11].
1.2. The detection of gravitational waves

- **Gravitational wave bursts generated by supernovae explosion**—Short-duration gravitational wave burst can be generated through the non-symmetry collapse of massive stars [12, 13].

The above brief summary of the gravitational wave sources clearly demonstrates the importance of detecting gravitational waves in astronomy research. By combining the observations of astrophysical events using different methods, namely, through gravitational waves, neutrinos and electromagnetic waves, it is possible to obtain a more complete understanding of the properties of the astrophysical events. It is of particular importance for the astrophysical processes where the nonlinearity of gravitational field plays an crucial role. This is known as multi-messenger astronomy which strongly motivates the detection of gravitational waves [14].

The detection of gravitational waves is also motivated by the experimental test of the fundamental theory of our space-time: Einstein’s theory of General Relativity. Despite of its mathematical and logical elegance, a direct test of General Relativity is difficult because of the smallness of the relativistic gravitational effect. Lots of work has been done in this field in the early 20th century [15], such as the observation of the deflection of light by the sun led by Arthur Eddington [16] and the excellent match between the theory and the observation of perihelion precession of Mercury [15]. However, it was not until 1959, that a program of precision tests of General Relativity was started. The first precision test carried out was the Pound-Rebka experiment on measuring the gravitational red-shift of photons [17]. Modern technology also allows us to observe the Shapiro delay [18, 19] and test the equivalency principle. Recent space research even allows us to test the General Relativity effect due to the rotation of the Earth (Gravity Probe B program) [20, 21]. Observation of the gravitational lensing [22] and binary pulsar system [23] tested the General Relativity through astronomical methods. Currently, none of these experiments shows any signal deviated from the predictions of the General Relativity. Still, a successful direct detection of gravitational waves will be a key support for Einstein’s General Relativity.
1.2.2 Gravitational wave detectors

Though understanding of the gravitational wave physics is important, the experimental detection of the gravitational waves is very difficult. It is due to the huge space-time stiffness and the large distance between those strong gravitational sources and our Earth. For example, the strain amplitude of an audio-band gravitational wave generated by a typical binary black hole merger which is 100Mpc far from us, is around $10^{-21}$. This strain will shorten/lengthen the distance of two test masses with an initial distance $= 4$km by $\sim 10^{-18}$m, which is 1000 times smaller than the diameter of a nucleus! Therefore, extremely sensitive detectors must be built to detect the effect of gravitational waves. Since the 1960s, several experimental methods have been proposed for detecting this tiny effect:

- **Resonant bar detector**—The first detector was built by Joesph Weber in the 1960s [24]. It was designed to sense the excitation of the fundamental longitudinal vibrational mode of a cylinder bar by the driving of gravitational waves. In such a detector, the acoustic signal created by the gravitational wave would be transduced and amplified by converting it to an electromagnetic signal. This was the conceptual foundation of the bar (resonant mass/acoustic) detectors. Although numerous improvements of the bar detectors have been made since Weber’s work, for instance the improvement of the amplitude sensitivity by several hundred times, these bar detectors were gradually replaced by the laser interferometer detectors which had achieved significantly higher sensitivity and larger bandwidth [7, 25]. However, as we will discuss later, the insights built up during the construction of bar detector turns out to be very illuminating for the understanding and proposing of other kinds of detectors.

- **Terrestrial laser interferometer detectors** [26]—Early in 1960s, not only the above mentioned resonant bar detectors were proposed, detecting gravitational wave by harnessing the direct interaction between the gravitational wave and the electromagnetic field was also suggested [27]. Later on, the idea of using Michelson interferometer was proposed [28, 29] and gradually conducted. The original configuration is prevented from having arbitrary length, by the curvature of the Earth surface, cost and vacuum requirements. This difficulty was overcome by the arm cavity design, which allows the light field to undergo multiple reflections within each arm [31].
The gravitational wave induced test masses motion would modulate the carrier light and the cavity design amplifies the signal. This design of Michelson interferometer configuration with kilometer scale arm cavities is called Laser Interferometer Gravitational Wave Observatory (LIGO) \[^{30,31}\]. The LIGO-type design was upgraded to have dual recycling mirrors, increased laser power and more advanced seismic isolation, mirrors, coatings, and suspension technologies \(^{1}\). These upgrades have led to a second generation of laser interferometer detectors \(^{1}\): the Advanced LIGO, currently located in each of the two USA sites (Livingston and Hanford). The designed sensitivity is 10 times better and the observation volume is 1000 times larger than the initial LIGO. Similar interferometric gravitational wave detectors are also being built in Europe (Virgo, Advanced Virgo) \[^{32}\], Japan (KAGRA) \[^{33}\] and India. It is worthy to mention that the third generation ground-based detectors are under design such as Einstein telescope and 3rd generation LIGO. These detectors are used to detect gravitational waves around $10^{-10} - 10^{+4}\text{Hz}$.

- **Space laser interferometry \[^{26}\]** — The interferometer strategies can also be used to detect the low frequency gravitational waves. The Laser interferometer Space Antenna (LISA) is a proposed NASA/ESA project, which will be used to detect the gravitational waves with frequencies within $10^{-4}\text{Hz} - 10\text{Hz}$. Three spacecrafts, each carrying a pair of test masses and laser source, will be placed at the vertices of an equilateral triangle of size $L \sim 5 \times 10^{6}\text{km}$. The laser injected from one of the spacecrafts will suffer strong diffraction loss. Therefore the return beam will be generated by locking the laser on the other spacecraft to the weak incoming beam (transponding). Gravitational wave (strain amplitude is $h \sim 10^{-20}$) with wavelength roughly equal to $L$ will induce a phase shift around $\Delta \phi \sim k_0 h L \sim 1.5 \times 10^{-4}$, this corresponds to a frequency drift $\Delta \omega \sim \Omega \Delta \phi \sim 10^{-6}\text{Hz}$. Current ground-based test experiment has demonstrated that the frequency drift of a properly designed laser source at $10^{-3}\text{Hz}$ can be as small as $10^{-7}\text{Hz} \[^{7}\].

- **Pulsar timing \[^{34}\]** — For extremely low frequency gravitational waves with frequencies $\sim 10^{-9} - 10^{-8}\text{Hz}$ (wavelength $\lambda_{gw} \sim 3 \times 10^{16} - 3 \times 10^{17}\text{meters}$), the pulsar timing technique which uses pairs of “test masses” consisting of the Earth and a millisecond pulsar, is proposed for detecting the gravitational wave by measuring the
arrival time of the pulse using radio telescopes (arrays). The typical distance between
the pulsar and the Earth are around \( d \sim 200\text{pc} \), which is roughly \( 20\lambda_{gw} - 200\lambda_{gw} \). In
this short-wavelength case \((\lambda \ll d)\), when a pulse travels through this distance, the
gravitational wave effect will cancel out except the last \( \sim \lambda_{gw} \) length. Therefore the
gravitational wave induced arrival time latency can be estimated as \( \Delta t_{gw} \sim h\lambda_{gw}/c = hT_{gw} \) where \( h \) and \( T_{gw} \) are the strain amplitude and the gravitational waves period.
Current timing accuracy is about \( \Delta t_{accuracy} \sim 100\text{ns} \), which leads to \( h \sim 10^{-16} - 10^{-15} \).
Although the gravitational waves have not been detected yet, recent analysis of the
Pulsar Timing Array data sets have already placed some upper limits on \( h \). Various
methods for the source localization were also proposed for pulsar timing gravitational
wave detection [35, 36].

Besides, various other methods are also proposed (even used) to detect the grav-
itational waves, such as using atomic interferometer [37], and observation of CMB
polarizations [38]. In this thesis, we will mainly focus on the ground-based LIGO
type interferometer detector, discussing its related physics and possible upgrading
strategies.

### 1.3 Basic principle of laser interferometer GW detectors

The basic Michelson interferometer is shown in Fig.1.1(a). The static configuration
will have zero output since the light wave will be exactly canceled out at the photo-
detector, named as “dark port”, and all the light field, after a round-trip propagation
inside the interferometer, will go back to the laser source, named as “bright port”.
A passing continuous gravitational wave with “+” polarization will “squeeze” the
interferometer plane, induce a differential motion of the two end mirrors and create
the length difference of the two arms. In this case, the interference at the dark port
will not be perfect and the leaked light field will encode the information of test mass
motion \(^1\). Let us suppose the gravitational wave has strength \( h \) and the distance

\(^1\)notice that a “common motion” of the two mirrors will equally increase/decrease both East-
West and North-South light path, thereby no light will leak out from dark port as a response of the
1.3. Basic principle of laser interferometer GW detectors

![Diagram of a basic Michelson interferometer and an interferometer with arm cavities.](image)

**Figure 1.1** – (a) Basic Michelson interferometer. A gravitational wave with “+” polarization passes the interferometer, displaces the test masses and creates a tiny difference between the light pathes of the two arms. (b) Interferometer with the arm cavities. For further details see the Section 1.3.

From the end mirror to the beam splitter is \( L \). The total light path difference will be \( \delta L = 2h(t)L \), equivalently, the phase shift for the light field will be \( \Delta \phi(t) = 2k_0h(t)L \) where \( k_0 = \omega_0/c \) is the wave-vector of the laser field. For a typical gravitational strain amplitude \( h \sim 10^{-21} \), and typical infra-red laser wave vector \( k_0 \sim 3 \times 10^6 \text{m}^{-1} \) and a 4 km length arm, we have the phase shift \( \Delta \phi \sim 10^{-11} \) which is too small to be measurable. One natural way for solving this problem is to increase the length of the delay line, but this method holds other problems such as the light scattering and the requirement for large area mirrors [39]. This problem was finally solved by adding input test mirrors on the interferometer arms to form resonating optical cavities as shown in Fig.1.1(a) [30]. For illustration purpose, we give an order of magnitude estimation here. Let us suppose the light field travels \( N \) round trips inside the cavities, then the total phase shift for the light field can be estimated as \( \Delta \phi \sim 2Nk_0hL \). On the other hand, the “phase-resolution” limit of an interferometer is determined by the number-phase uncertainty relation \( \Delta \phi \sim 1/\Delta N_p \) where \( \Delta N_p = \sqrt{N_p} \) is the variance of the intra-cavity photon number \( N_p \) for a coherent carrier light. For an interferometer with input test mass transmissivity \( \sim 10^{-2} \) (\( N \sim 10^2 \)), this requires the \( N_p \sim 10^{18} \) mirrors’ common motion.
and the input power $\sim 200$ Watt which is achievable. In real design, this could be even smaller.

Although the arm cavity design successfully increases the signal strength through amplifying the light phase induced by the test masses motion, this design decreases the signal bandwidth. As we know, optical cavity has a series of finite-bandwidth resonance point at $\omega_n = n c \pi / L$. Once the frequency of the pumping field is on resonance with these resonance points, the intra-cavity field will build up to its maximum. The bandwidth $\gamma$ of these resonant peaks depends on 1) the transmissivity of the mirrors and cavity losses, which represents the interaction strength of the intra-cavity field and external electromagnetic baths (for our ideal case, we assume the cavity loss and the end mirror transmissivity to be zero), and 2) the length of the cavity $L$. The inverse of $\gamma$ is the time scale $\tau \gamma$ for a photon to escape out of the cavity. Intuitively, the longer the cavity length $L$ is, the more time the photon will spend inside the cavity; the smaller the front mirror transmissivity $T$ is, the photon will bounce more round-trips inside the cavity. Therefore, the bandwidth should be proportional to $T$ and inversely proportional to $L$. By dimensional analysis, $\gamma \sim c T / L$ (rigorous result: $\gamma = c T / (4 L)$).

The test masses motion under the driving of gravitational wave will modulate the carrier light and create sideband fields with frequency $\omega_0 \pm \Omega$ where $\Omega$ is the frequency of the gravitational wave. The sideband field with frequency $\Omega \gg \gamma$ can not be enhanced by the circulation inside the arm cavities due to the finite bandwidth. Therefore, the detection bandwidth for the detector is roughly around $\sim \gamma$. For those sidebands with $\Omega \ll \gamma$, the multi-reflection of the cavity mirrors will significantly enhance them by a factor of $\sim 1/T \propto 1/\gamma$. Therefore, it is clear that the detectability of our interferometer based on the arm-cavity design has a trade off: the enhancement of the signal is associated with the decreasing of detection bandwidth and vice versa. This is the so-called Mizuno hypothesis[41, 40]. As we shall discuss later, this hypothesis can be proved for many configurations including the second-generation detectors.

For initial LIGO, in order to further increase the intra-cavity power further (since the minimum phase detection limit of the detector is $\propto 1/\sqrt{N_p}$), a power-recycling
1.3. Basic principle of laser interferometer GW detectors

Figure 1.2 – (a) Dual-recycling laser interferometer (b) Effective single-cavity model

mirror is placed at the bright port of the interferometer. This design was further improved in the advanced detector by using dual recycling configuration as shown in Fig. 1.2. An additional mirror (named as signal recycling mirror) is placed at the dark port of the interferometer, which increases the tunability of the detector [42]. Moreover, this additional signal-recycling mirror also modifies the dynamical behavior of the interferometer [43].

The differential mode of a dual-recycling interferometer can be effectively treated as a single Fabry-Perot cavity with a tunable compound front mirror consists of an input test mirror, a signal recycling mirror and a movable end mirror [44]. The transmissivity and the position of the signal recycling mirror can be tuned to change the optical response of the whole compound mirror. For example, 1) the resonance frequency of this effective mirror can be detuned from the carrier light frequency, which leads to the optical spring effect [44, 45], or 2) the position of the SRM can be tuned in such a way that perfect destructive interference can happen between the intra-cavity sideband field directly reflected from the input test mass mirror and the sideband field first circulate inside the compound mirror and then transmit out from the input test mass mirror (See Fig. 1.3). In this case, the compound mirror is almost transparent from the sideband field’s viewpoint. Therefore, the detection bandwidth is increased. This design is the called “Resonant Sideband Extraction”, which is being used in the Advance LIGO detector. Detailed analysis of these configuration will be elaborated in the next chapter.
Chapter 1. Introduction to Gravitational Wave Detectors

Figure 1.3 – Resonant Sideband Extraction mode: the signal-recycling cavity is effectively transparent to the sideband fields. This design is being used in the advanced LIGO detector to broaden the detection bandwidth.

Figure 1.4 – A weak force measurement process flow chart: Signal force drives the motion of the probe $x$, which couples to the detector quantity $F$. The detector transduce the signal force to its output quantity $Z$, which will be measured.

The detectability of these large-scale interferometers are limited by various noise sources, or equivalently, by the interaction between these detectors and the mechanical and electromagnetic environments. These noise issue will be discussed in Section 1.5.

In this section, we have given a brief summary of basic physical principle of a interferometric gravitational wave detector. At first glance, the interferometric gravitational wave detector looks very different from the resonant mass detector. However, as we shall see, the physics of these two designs are closely connected [46].

These interferometric detector designs and the develop of the resonant mass detector have several conceptual connections, although they appear to be quite different. In this section, we first summarize the main character of a weak-force measurement device, and then we highlight out the similarities between the resonance mass detector and the interferometric detector.

The model of a typical measurement process is schematically shown in Fig 1.4. A measurement process consists of the following steps:

1) the weak force acts on the probe and create a signal.

2) the driven probe interacts with the detector (sensor), transduces the signal into a measurable dynamical quantity of the detector and at the same time, be back-acted by the detector.
3) The readout of the signal. Both the resonance mass detector and the interferometric detector fall into this category.

The initial Weber’s design of resonant mass detector used passive PZT crystal transducers. Later on, the modern resonant mass detector design uses parametric transducer, as first discussed by Braginsky and Panov [47]. This type of transducer requires external power source (a pumped oscillator), transfers the gravitational wave signal with $10^3 - 10^4$Hz acoustic frequency to much higher frequency, generally microwave frequency. This idea of active transducer is the key conceptual connection between the resonant mass detector design and the interferometer design.

A simplified model of a typical resonant mass detector with a parametric transducer is shown in the left panel of Fig. 1.5 [48], in which the gravitational wave drives the probe, which is the resonant mass with elastic eigenfrequency $\omega_m$ and damping factor $\gamma_m$. The tiny gravitational wave induced displacement of the test mass is transduced into electromagnetic signal by a LCR circuit with capacity depending on the test mass motion. The apparent similarity can be figured out from a Fabry-Perot cavity since the optical cavity, as an oscillator, can be mapped to the LCR circuit in the optical frequency region.
1.4 Noise sources of interferometric gravitational wave detectors

Advanced Gravitational wave detectors are supposed to measure an extremely tiny displacement signal $\sim 10^{-20}$ meters. Therefore, the interaction between the detector and the noise sources need to be carefully considered. In this section, we list the possible noise sources and estimate their contribution to the Advanced LIGO sensitivity. In general, the noises can be classified into two broad classes\cite{1}: phase noise and displacement noise. The phase noise is the fluctuations of the phase of the light field, which is used to measure the gravitational wave signal. The typical phase noises sources are photon arrival time fluctuation due to the quantum vacuum fluctuations of electromagnetic field; the residual gas’s scattering of the light field and scattering of light field by the imperfections of the test mass mirrors. The displacement noise is the random fluctuation of the positions of the test masses due to their dynamical interaction with the environment. Typical displacement noise includes: quantum radiation pressure noise mirror thermal noise; seismic noise; Gravity gradient noise and other noise sources such as cosmic rays.

1.4.1 Phase noise

• Quantum shot noise— Classical electrodynamics tell us that the electromagnetic field can be treated as an infinite collection of oscillators. The ground state of this oscillators ensemble contributes to the zero-point random fluctuation of electromagnetic fields. These fluctuations can cause a phase fluctuation of the light field traveling out of the dark port, which corresponds to a photon arrival time fluctuation. This noise is the so-called quantum shot noise. In quantum mechanics, time fluctuation is associated with energy fluctuation through: $\Delta E \Delta t \geq \hbar$. For a monochromatic light field with frequency $\omega_0$, we have the number-phase uncertainty relation $\Delta N_p \Delta \phi \geq 1$ since $\Delta E \sim \Delta N_p \hbar \omega_0$ and $\Delta \phi \sim \omega_0 \Delta t$. For a coherent laser whose the number of photons follows a Poisson distribution, we have $\Delta N_p \sim \sqrt{N_p}$, which leads to $\Delta \phi \sim 1/\sqrt{N_p}$. Therefore the stronger the laser power is, the smaller
the shot-noise is. Using $\Delta \phi \sim k_p \Delta x$, the corresponding displacement noise spectrum can be estimated as $S_{x}^{sh} \sim \hbar c^2 / (I_0 \omega_0)$ where $I_0$ is the intra-cavity laser power, which is a frequency-independent white noise spectrum [49]. Recalling that the signal field is decreasing with the increase of detection frequency due to the limited bandwidth of the arm cavity. Therefore the sensitivity curve (or equivalently, the noise spectrum normalized by the signal amplitude) due to the quantum shot noise will rise up with the increasing frequency.

• Residue gas scattering — Although the arm of AdvLIGO is a highly pumped vacuum, there will still be some residue gas inside the arm. The fluctuations of the gas density will scatter the light field and create noise in the optical phase. This noise can be estimated in the following way: The refraction index of the tube filled with residue gas with molecule’s number density $\rho$ is roughly $n = 1 + \alpha \rho$, where $\alpha$ is the polarizability for a single molecule. The total number of molecules encountered by the light beam will be $N = \rho AL$, where $A \sim \pi w^2$ is the effective area of light spot. Let us suppose the molecules obey the Poisson statistics, then the variance of the molecule number fluctuations will be $\sim \sqrt{N}$. Therefore, the variance of phase fluctuation is around $\Delta \phi \sim k_p \sqrt{N} \alpha / A \sim k_p \alpha \sqrt{\rho L / A}$. This phase fluctuations can be converted into the strain fluctuation by using $h(f) \sim \Delta \phi / (k_p L) \sqrt{\tau} \sim \alpha \sqrt{\rho / L w v}$. Here $\tau = w / v$ is the time scale for a molecule to spend inside the beam, whose inverse represents the frequency range. For example, for H$_2$ gas, the strain noise will be $\sim 10^{-25} / \sqrt{Hz}$ if the residual gas pressure is $\sim 10^{-9}$ torr [39].

• Back scattering by the imperfections of mirror — Imperfections in the mirror shape (sometimes called “figure error”) with the size larger than 1mm can scatter the incident light into a small angle. This scattering can create phase noise (See. Fig. 1.6).

1.4.2 Displacement noise

• Quantum radiation pressure noise — As we have already discussed, the variance of fluctuation of photon numbers for a coherent light is $\sqrt{N_P}$. This photon number fluctuation will induces a random fluctuation of the momentum kick, thereby a random radiation pressure force over a duration $\tau$ with variance $\Delta F_{rp} \sim \sqrt{N_P} \hbar k_p / \tau$. 
The mechanical response function of the test masses is $-1/(m\Omega^2)$. Therefore, the displacement noise spectrum caused by this fluctuating radiation pressure noise is:

$$S^{rp}_{xx} \sim \Delta F_{rp}^2 \tau/(m^2\Omega^4) = I_0 \hbar \omega_0 / m^2 c^2 \Omega^4 \ [49].$$

Unlike the quantum shot noise, this radiation pressure noise is small when the detection frequency is large or the laser power is small. The sensitivity curve due to this noise will be risen up with the decrease of the detection frequency.

The different power dependence of the quantum shot noise and the quantum radiation pressure noise imply a very important concept: the **standard quantum limit** of a detector [50] [49]. The total displacement noise spectrum is given by:

$$S^{tot}_{xx} = S^{sh}_{xx} + S^{rp}_{xx} = \frac{\hbar c^2}{I_0 \omega_0} + \frac{\hbar}{c^2 m^2 \Omega^2} \geq \frac{2\hbar}{m \Omega^2}. \ (1.1)$$

which has a minimum value called **standard quantum limit**. The existence of this sensitivity limit is due to the Heisenberg uncertainty principle: if you want to measure phase more accurately (practically means to increase the power), then you introduce stronger disturbance to the probe (stronger radiation pressure force noise). This disturbance will back-act on the measurement of the phase and cause measurement errors. This trade-off creates the standard quantum limit. Though being introduced from the example of the laser-interferometer gravitational wave detector, standard quantum limit is a general concept of the physics of quantum measurement. This important concept was discovered and developed by V. B. Braginsky et.al.

**Seismic noise**— the laboratory will suffer seismic vibrations at relatively low frequency region. The current vibration spectral density for the ground motion of LIGO sites are around $\sim 10^{-8} \text{m/\sqrt{Hz}}$ at 1Hz and this value decreases as $1/f^2$ due to the response function of free mass $\propto -1/(mf^2)$. In order to reach a strain sensitivity $\sim 10^{-21}/\sqrt{\text{Hz}}$, it is necessary to suppress the isolation within the detection band by $10^8$ at 10Hz and $10^6$ at 100Hz. Sophisticated vibration isolation systems have been designed and deployed, which can suppress the seismic noise well within the detection band. Another way of suppress the seismic noise is to build an underground detector such as KAGRA built in Japan [33].

**Gravity gradient noise**— This noise is due to the fluctuations of the local gravitational field caused by moving massive bodies, e.g. cars, people, trains, animals and the change of local environment densities. The strain noise can be estimated...
using

\[ \delta h_{gg}(f) \sim \left(\frac{G \rho L x_{\text{GND}}(f)}{f^2} \right). \] (1.2)

- **Thermal noise**— Thermal noise in the interferometric gravitational wave detector can be generally classified as mirror thermal noise and suspension thermal noise. Suspension thermal noise is due to the fact that the mirror should be suspended in the LIGO type detector and thereby the mirrors are coupled to the thermal bath through the suspension system. The mirror thermal noise is due to the mirror’s optical and mechanical properties. For example, the large amount of acoustic modes of the mirror substrate will do thermal-driven Brownian motion, since the temperature of the mirror is non-zero. This Brownian motion will generate mirror Brownian thermal noise.

Besides, the mirror also has coating surfaces for increasing its reflectivity. These coating surfaces, though having very ideal optical properties, are very poor from the aspect of the internal friction. This mechanical dissipation will be associated with mirror coating thermal noise. Moreover, the thermal-elastic effect of the mirror can also introduce noise: let us suppose the heating temperature of the cavity laser power to the mirrors has a fluctuation \( \delta T \), then this temperature fluctuation will induce a displacement fluctuation \( \delta x \sim \alpha T \) where \( \alpha \) is the thermal expansion coefficient. This noise is known as thermal elastic noise. Not only the displacement fluctuation will be induced by \( \delta T \), but it will also induce the fluctuation of the refractive index \( \delta n \). Therefore the effective light propagation length and thereby the phase will have a fluctuation \( \Delta \phi \sim \delta nk_0L \). This noise is called thermal refractive noise.

In summary, we give a noise source budget in terms of gravitational wave strain sensitivity limit for advanced LIGO detector in Fig.1.6.

### 1.5 Conclusions

In this chapter, we have introduced the motivation and the basic concepts of gravitational wave detection. We gave a brief summary of the gravitational wave sources and a brief summary of the gravitational wave detectors and how they work. In particular, we gave an qualitative description of how the laser interferometers work and
Figure 1.6 – Noise budget of advanced LIGO gravitational wave detector. The horizontal axis is the frequency and the vertical axis is the strain sensitivity of the AdvLIGO detector limited by various noise sources. See the text for the details of these noise sources.
estimate the effect of some of the possible noise sources.
Bibliography


Chapter 2

Quantum noise of optomechanical systems

2.1 Preface

As we have discussed briefly in Chapter 1, the quantum noise will be one of the dominant noise sources for the gravitational wave detectors. This chapter reviews the quantum noise theory of gravitational wave detectors. This chapter first discusses two fundamental concepts, namely

i) the quantization of light and different state of light which is the basis of the quantum optical dynamics;

ii) the dynamics of a basic optomechanics model—a pumped optical cavity with movable end mirror, which is the basic structure of a typical laser interferometer gravitational wave detector.

The discussion of these two concepts leads to the optomechanical linear-control theory, which is the systematical way of analyzing the quantum sensitivity curve of more complicate laser interferometer configuration. Several examples are given for illustration purpose. Based on these sensitivity curves, we will discuss

i) the origin of the Standard Quantum Limit (SQL) for GW sensitivity.

ii) Mizuno theorem which limits the integrated signal-to-noise ratio.

iii) the origin of the energetic quantum limit.

These three topics will lead to the proposed way of surpassing the SQL and the Mizuno theorem, which will be the basis of Chapter 3, 4 and 5. Finally, the mathematical tools and physics concepts presented in this chapter will be useful for understanding other chapters.
2.2 Quantization of light field

2.2.1 Second quantization of electromagnetic field

Classical Electrodynamics teaches us that the source-free electromagnetic fields can be decomposed as a collection of infinite harmonic oscillators [1]. On the other hand, quantum mechanics teaches us that the ground state of a harmonic oscillator with eigenfrequency $\omega$ is $\hbar\omega/2$. These leads to the conclusion that the ground state of a quantized electromagnetic field is not static but fluctuating. Mathematically, the above physical statement can be formulated in the following way [1]:

The electromagnetic potential (say, vector potential) can be expanded in the form of

$$ A(t, x) = \sum_k A_k(t)e^{ikx}, \quad (2.1) $$

where $A_k$ is the expansion parameter and $A(t, x)$ is real (implies that $A_k = A^*_k$).

Substituting it into the wave equation for the vector potential: $\nabla^2 A - \ddot{A}/c^2 = 0$ gives:

$$ \ddot{A}_k(t) + \omega_k^2 A_k(t) = 0, \quad (2.2) $$

where $\omega_k = ck$. Apparently, this is an equation for the harmonic oscillation of quantity $A_k$. The total electromagnetic energy can be written as:

$$ E = \frac{1}{8\pi} \int (E^2 + H^2)dV = \frac{V}{8\pi} \sum_k \frac{1}{c^2} \dot{A}_k \cdot \dot{A}_k^* + (k \times A_k) \cdot (k \times A_k^*), \quad (2.3) $$

Using the fact that $(k \times A_k) \cdot (k \times A_k^*) = k^2 A_k \cdot A_k^*$ gives:

$$ E = \frac{V}{8\pi c^2} \sum_k \dot{A}_k \cdot \dot{A}_{-k} + \omega_k^2 A_k \cdot A_{-k}, \quad (2.4) $$

this form of Hamiltonian reminds us of the Hamiltonian of an ensemble of harmonic oscillators. These analogues motivate us to write the “generalized coordinate” $A_k$ and “generalized momentum” $\dot{A}_k$ in terms of:

$$ A_k = \sqrt{\frac{2\pi c^2}{\omega_k}} [\hat{a}_k + \hat{a}_{-k}^*] $$

$$ \dot{A}_k = -ick \sqrt{\frac{2\pi c^2}{\omega_k}} (\hat{a}_k - \hat{a}_{-k}^*). \quad (2.5) $$
In terms of these \( \hat{a}_k \)s, the vector potential can be written as:

\[
A(t, x) = \sum_k \sqrt{\frac{2\pi e^2}{\omega_k}} [\hat{a}_k e^{ikx} + \hat{a}_k^* e^{-ikx}],
\]

and the energy can be written as:

\[
E = \frac{1}{2} \sum_k E_k = \frac{1}{2} \sum_k \omega_k [a_k a_k^* + a_k^* a_k].
\]

In classical Electrodynamics, the vector potential and its associated “coefficients” \( a_k \) are functions. However, in quantum mechanics, they become operators. The canonical quantization condition for the vector potential and its time derivatives will lead to \([\hat{a}_k, \hat{a}_k^\dagger] = \hbar \delta_{kk'}\). Redefining these operators by normalizing the \( \hbar \) gives:

\[
E = \sum_k \hbar \omega_k \left( \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right).
\]

This is just the energy of an ensemble of harmonic oscillators with different frequency \( \omega_k \). The \( \hat{a}_k^\dagger \) is the annihilation (creation) operator of an electromagnetic excitation with momentum \( k \) and energy \( \omega_k \). Notice that the operators \( \hat{a}_k^\dagger \) is non-Hermitian thereby not a measurable quantity. The \( \sum_k \omega_k / 2 \) term in the energy \( E \) represents the ground state energy of the electromagnetic field, which is the source of the quantum noise of our gravitational wave detector.

Experimentally, the physical quantity measured for a light field is its electric and magnetic field. For a polarized field propagating in \( z \) direction with electric and magnetic field align to \( x \) and \( y \) direction respectively, the representations of these fields in terms of creation and annihilation operators are:

\[
\hat{E}_x(x, t) = i \sum_k \sqrt{\frac{2\pi \hbar \omega_k}{\omega_k}} [\hat{a}_k(t)e^{ikz} - \hat{a}_k^\dagger(t)e^{-ikz}]
\]

\[
\hat{H}_y(x, t) = -i \sum_k \sqrt{\frac{2\pi \hbar c^2}{\omega_k}} k [\hat{a}_k e^{ikx} - \hat{a}_k^\dagger e^{-ikx}].
\]

Straightforward calculation shows that these operators obey the following commutation relation:

\[
[\hat{E}_x(z, t), \hat{E}_x(z', t')] = 0, \forall z, z'
\]

\[
[\hat{H}_y(z, t), \hat{H}_y(z', t')] = 0, \forall z, z'
\]

\[
[\hat{E}_x(z, t), \hat{H}_y(z', t')] = -i\hbar c^2 \frac{\delta(z - z')}{\partial z}
\]

\[
[\hat{E}_x(z, t), \hat{E}_x(z, t')] = i\hbar \frac{\delta(t - t')}{\partial t}.
\]
These relations indicate us that the quantum measurement of these quantities is not a trivial issue: 1) Simultaneously measuring the electric and magnetic fields at two space-like separated points are impossible; 2) Measurement of an electric field at one spacetime point \((x, t)\) will introduce disturbance to the measurement result of the electric field at a time-like separated spacetime point.

However, in most experiments, the measurement devices has a resolution limit. As shown later, these resolution limit makes the above mentioned quantities be simultaneously measurable under the corresponding approximation.

### 2.2.2 Cavity field quantization and Quadrature operators

- **Amplitude and phase quadratures**—The optical modes supported by the arm-cavity of the interferometer are described by the Hermite-Gauss mode \([2]\). The intra-cavity electric field is described by:

\[
E(x, y, z, t) = i x \sum_j \sqrt{2\pi \hbar \omega_j} u_j(x, y, z) [\hat{a}_j e^{-i(\omega_j t - k_j x)} - \hat{a}_j^\dagger e^{i(\omega_j - k_j x) t}], \tag{2.11}
\]

where the \(u(x, y, z)\) is the slowly changing (compare to light wavelength) spatial profile of the intra-cavity field. Suppose the external pumping field excites a single spatial mode with \(j = 0\) and \(\hat{a}_0 = -i A_0 / \sqrt{2}\), then:

\[
E(x, y, z, t) = x \sqrt{2\pi \hbar \omega_0} u_0(x, y, z) \hat{A}_0 \cos(\omega_0 t - k_0 z) \equiv E_0 \cos(\omega_0 t - k_0 z). \tag{2.12}
\]

For the high-precision experiment using laser interferometers, the main carrier light is perturbed by the signal and noise sources \([3]\), which can be represented as:

\[
E(z, t) = (E_0 + \delta A) \cos(\omega_0 t - k_0 z + \delta \phi). \tag{2.13}
\]

The \(\delta A \sim \epsilon\) and \(\delta \phi \sim \epsilon\) here represent the amplitude and phase perturbation of the carrier light field. To the first order of perturbation:

\[
E(z, t) \approx E_0 \cos(\omega_0 t - k_0 z) + \delta A \cos(\omega_0 t - k_0 z) - \delta \phi \sin(\omega_0 t - k_0 z) + O(\epsilon^2). \tag{2.14}
\]

On the other hand, these small perturbation, represented in terms of second-quantization operators, can be written as

\[
E(z, t) = E_0 \cos(\omega_0 t - k_0 z) + i u_0(x, y, z) \int_0^\infty \frac{d\omega}{2\pi} \sqrt{2\pi \hbar \omega} [\hat{a}_\omega e^{-i(\omega t - k_0 z)} - \hat{a}_\omega^\dagger e^{i(\omega t - k_0 z)}]. \tag{2.15}
\]
Suppose $\omega = \omega_0 + \Omega$, $k \to k_0 + k$ and assuming $\Omega \ll \omega_0$ (narrow band approximation), in this case:

\[
E(z, t) = E_0 \cos(\omega_0 t - k_0 z) + \delta \hat{E}(z, t) \tag{2.16}
\]

\[
\delta \hat{E}(z, t) = u_0(x, y, z) \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \sqrt{2\pi \hbar \omega} [i\hat{a}_\Omega e^{-i\Omega t + ikz} - i\hat{a}_\Omega^\dagger e^{i\Omega t - ikz}] \cos(\omega_0 t - k_0 z)
\]

\[
+ u_0(x, y, z) \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \sqrt{2\pi \hbar \omega} [\hat{a}_\Omega e^{-i\Omega t + ikz} + \hat{a}_\Omega^\dagger e^{i\Omega t - ikz}] \sin(\omega_0 t - k_0 z). \tag{2.17}
\]

Comparing Eq.(2.16b) and Eq.(2.14), this gives the second quantization representation of the phase and amplitude perturbation as:

\[
\delta \hat{A} = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \sqrt{2\pi \hbar \omega} [i\hat{a}_\Omega e^{-i\Omega t + ikz} - i\hat{a}_\Omega^\dagger e^{i\Omega t - ikz}] \tag{2.18}
\]

\[
\delta \hat{\phi} = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \sqrt{2\pi \hbar \omega} [\hat{a}_\Omega e^{-i\Omega t + ikz} + \hat{a}_\Omega^\dagger e^{i\Omega t - ikz}] \tag{2.19}
\]

The gravitational wave community uses the symbol conventions 1) absorbing the $i$ and $e^{ikz}$ into $\hat{a}_\Omega$; 2) setting the $\Omega$ to be positive. Therefore:

\[
\delta \hat{A} = \sqrt{2\hbar \omega} \int_0^{\infty} d\Omega [\hat{a}_1 e^{-i\Omega t + ikz} + \hat{a}_1^\dagger e^{i\Omega t - ikz}], \tag{2.20}
\]

\[
\delta \hat{\phi} = \sqrt{2\hbar \omega} \int_0^{\infty} d\Omega [\hat{a}_2 e^{-i\Omega t + ikz} + \hat{a}_2^\dagger e^{i\Omega t - ikz}], \tag{2.21}
\]

where $\hat{a}_1 = (\hat{a}_\Omega + \hat{a}_\Omega^\dagger)/\sqrt{2}$ and $\hat{a}_2 = (\hat{a}_\Omega - \hat{a}_\Omega^\dagger)/\sqrt{2i}$.

These creation and annihilation operator combinations $\hat{a}_1$ and $\hat{a}_2$ are named “amplitude” and “phase” quadrature operators respectively. Notice that exciting an “amplitude (phase)” quanta requires the creation and annihilation of a pair of sideband photons with frequency $\omega_0 \pm \Omega$, respectively.

These quadrature operators obey the non-commutative relation:

\[
[\hat{a}_1(\Omega), \hat{a}_2^\dagger(\Omega')] = 2\pi i \delta(\Omega - \Omega'). \tag{2.22}
\]

- **Probing the phase of the weak light: Homodyne detection**—In interferometer experiment, the above mentioned phase quadrature usually carries the displacement information of the test masses. This phase-readout can be achieved by the so-called homodyne scheme which mix the output signal $\hat{b}(t)$ with a local oscillator (in our case, is a strong laser field) $L(t)$:

\[
E_{\text{measured}}(t) = L(t) + \hat{b}(t) \tag{2.23}
\]

with $L(t) = L_1 \cos \omega_0 t + L_2 \sin \omega_0 t$ and $\hat{b}(t) = \hat{b}_1(t) \cos(\omega_0 t) + \hat{b}_2(t) \sin(\omega_0 t)$. 

Then the photocurrent can be written as:

\[ i(t) \propto \langle |L(t) + \hat{b}(t)|^2 \rangle = \langle L_1 \hat{b}_1(t) + L_2 \hat{b}_2(t) \rangle + \text{DC terms and high oscillating terms} \]

(2.24)

Defining the homodyne angle as \( \tan \xi = L_1/L_2 \) gives:

\[ \hat{i}(t) \propto \sqrt{L_1^2 + L_2^2} \langle \hat{b}_\xi(t) \rangle = \sqrt{L_1^2 + L_2^2} \langle [\hat{b}_1(t) \sin \xi + \hat{b}_2(t) \cos \xi] \rangle. \]

(2.25)

Therefore as long as \( \xi \neq \pi/2 \), the photocurrent will contain the phase signal.

### 2.3 Quantum state of light

The formalism established above allows a quantitatively study of the quantum state of the optical field. This section mainly focuses on three different optical field state which are very important in the design of an interferometric gravitational detector: **vacuum state, coherent state and squeezed state**.

- **Vacuum state**—Electromagnetic field can be treated as an infinite set of harmonic oscillators. Suppose all these oscillators are on their ground states \( \hat{a}_k \langle 0 \rangle_k = 0 \), then it can be said that the electromagnetic field is on its ground state:

\[ |0\rangle = \prod_{\otimes k} \langle 0 \rangle_k. \]

(2.26)

The spectral densities for the amplitude and phase fluctuations are \( S_{\hat{a}_1\hat{a}_1}(\Omega) = S_{\hat{a}_2\hat{a}_2}(\Omega) = 1 \) with zero cross-correlation. In a gravitational wave detector, this vacuum optical state will enter into the dark port of the interferometer and contribute a quantum noise to the sensitivity of the detector.

- **Coherent state**—The above vacuum state can be viewed as an eigenstate of annihilation operator with zero eigenvalue. However, it is interesting that the annihilation operator also has eigenstate with non-zero eigenvalue, namely, \( \hat{a}_\Omega |\alpha\rangle_{\alpha\Omega} = |\alpha\rangle \). These states are called coherent state with a general multiple-mode expression:

\[ |\alpha\rangle = \hat{D}(\alpha)|0\rangle = \exp \left[ \int \frac{d\omega}{2\pi} (\alpha_\omega \hat{a}_\omega - \alpha_\omega^* \hat{a}_\omega^*) \right] |0\rangle. \]

(2.27)

Annihilating a photon from the coherent state does not affect the state since in quantum mechanics, \( C|\psi\rangle (\forall C), |\psi\rangle \) are the same state. Therefore, the particle number
of this state is not conservative. For example, for a single-mode coherent state, it is straightforward to prove that the variance of the particle number is $\Delta N \sim \sqrt{N}$ where $N = \alpha^* \alpha$ is the expectation value of $\hat{a}^\dagger \hat{a}$ on the $|\alpha\rangle$. Correspondingly, the variance of phase is around $1/\sqrt{N}$. The main carrier field in the laser interferometer can be approximated as a coherent state. The particle number distribution of the main carrier field is a Poisson distribution:

$$P(N) = \langle N|\alpha\rangle\langle\alpha|N\rangle = \frac{(\alpha^*\alpha)^N e^{-\alpha^*\alpha}}{N!}.$$ (2.28)

There is another way to treat the coherent state. The single-mode coherent state (where $\alpha_\omega \propto \alpha \delta(\omega - \omega_0)$) can be understood as a boosted vacuum state by the “displacement operator” $\hat{D}(\alpha)$. Redefining $\delta \hat{a} = \hat{a} - \alpha$ gives the result $\delta \hat{a}|\alpha\rangle = 0$. Therefore, the coherent state, as viewed from $\delta \hat{a}$, is a “vacuum state”. Note however that we should notice that only the photon state with frequency $\omega_0$ is at this “vacuum state”, all the other states, are exactly in the real vacuum state. To some extent, the single-mode coherent state can be viewed as a constant, “classical” monochromatic pumping field $\alpha$ superimposed by a fluctuating broadband vacuum fluctuation $\delta a$. This viewpoint is not entirely correct, but can capture the main character of the coherence state. For example, the particle number fluctuation is $\sim \alpha (\delta \hat{a} + \delta \hat{a}^\dagger) \sim \sqrt{N}$ where $N \sim \alpha^2$ while the phase fluctuation normalized by the classical amplitude ($\propto \alpha$) is roughly $\sim (\delta \hat{a} - \delta \hat{a}^\dagger)/\alpha \sim 1/\sqrt{N}$. Later on, it is shown that the difference between the scaling of particle number fluctuation and phase fluctuation on $N$ will lead to the Standard Quantum Limit.

Lastly, the fluctuation of the quadrature operators for a coherent state is the same as the vacuum state. Therefore in the phase-space $(A, \phi)$ of the light field, the fluctuation can be schematically shown as in Fig 2.1

- **Squeezed state**— The other important quantum optical state is the squeezed state. The phase and amplitude fluctuations of the squeezed light have different error variance as shown in Fig 2.1. The squeezed state is of particular interests in the high-precision measurement experiment. The reason is that if one of the light quadratures contains the signal (say the phase quadrature), then a phase-squeezed light allows us to perform measurement with lower quantum noise.
Figure 2.1 – A schematic plot of the electric field $E(t)$ and the amplitude and phase quadrature fluctuation. The left panel shows the time evolution of the electric field and the right panel shows the error ellipse of the quadrature $(A, \phi)$. Notice that all these states are Gaussian states.
A general multimode squeezed coherent state is described by:
\[ |ζ⟩ = \hat{S}(ξ) |α⟩ = \exp \left[ \int_0^∞ \frac{dΩ}{2\pi} \left( ζ^* Ω^{\dagger} a_{Ω}^{\dagger} a_{Ω} - ζ_{Ω} a_{Ω}^{\dagger} a_{Ω} \right) \right] |α⟩ \] (2.29)
while particularly, when \( α = 0 \), the state is called “squeezed vacuum state” \( |ζ⟩_0 \).
Notice that the “squeezed vacuum” state is not a vacuum state since the occupation number is not zero. The above multi-mode squeezed state can reduce to single-mode squeezed state in the case that the sideband frequency \( Ω \) is zero.

Using the above definition, it is straightforward to show that the two quadratures \( \hat{a}_{1,2}(Ω) \) will be transformed under the squeezing operator \( \hat{S}(ξ) \) to be:
\[
\hat{S}^\dagger(ζ) \hat{a}_1 \hat{S}(ζ) = (\cosh ζ + \sinh ζ \cos 2ξ) \hat{a}_1 - \sinh ζ \sin 2ξ \hat{a}_2 \] (2.30)
\[
\hat{S}^\dagger(ζ) \hat{a}_2 \hat{S}(ζ) = (\cosh ζ - \sinh ζ \cos 2ξ) \hat{a}_2 - \sinh ζ \sin 2ξ \hat{a}_1 \] (2.31)
where the \( ξ \) is the phase angle of the complex coefficient \( ζ_{Ω} \). From these relations, it is clear that (1) if \( ξ = 0 \), then \( \hat{a}_1 \rightarrow \hat{a}_1 e^{i|ζ|} \) and \( \hat{a}_2 \rightarrow \hat{a}_2 e^{-|ζ|} \); (2) if \( ξ = π/2 \), then \( \hat{a}_1 \rightarrow \hat{a}_1 e^{-|ζ|} \) and \( \hat{a}_2 \rightarrow \hat{a}_2 e^{i|ζ|} \); (3) if \( ξ = π/4 \), then: \( \hat{a}_1 \rightarrow \cosh |ζ| \hat{a}_1 - \sinh |ζ| \hat{a}_2 \) and \( \hat{a}_2 \rightarrow \cosh |ζ| \hat{a}_2 - \sinh |ζ| \hat{a}_1 \). These situations are schematically shown in Fig.2.2. The physical reason for these “squeezing” effect is the establishing of correlation between the phase and amplitude quadrature of light field during the squeezing generation process. Interestingly, as we shall discuss later on, the laser interferometer itself is a squeezed light generator in the sense that the pondermotive interaction between the
light field and the test masses establishes the correlation between phase and amplitude quadrature. Intuitively, the radiation pressure force associated with the amplitude quadrature of light drives the test mass motion, the information of which will be carried by the phase quadrature of light through modulation. Therefore, the vacuum electromagnetic field injected into the dark port will be converted to a squeezed vacuum state when it travels out of the dark port [3]. This squeezing is literally called “pondermotive squeezing” and has been experimentally demonstrated [5, 6].

2.4 Fabry-Perot cavity

2.4.1 Optical dynamics of a Fabry-Perot cavity

- Propagating field description— The different quantum state of light, when they are injected into the detector, will circulate inside the Fabry-Perot arm cavity. Understanding of the light propagating inside a Fabry-Perot cavity is one of the key ingredients for understanding the optomechanical dynamics of the whole interferometer. This section discusses the light propagation inside a Fabry-Perot cavity and an effective one-mode description.

The interaction between the optical field and cavity consists of the following ingredients:

1) The free-propagating of the optical field in the vacuum.

\[ E(t) = E(t - \frac{x}{c}). \]  

2) The transmission (reflection) of the optical field through (on) the front/end mirror (See Fig.2.3):

\[ E_2(t) = -\sqrt{R}E_1(t) + \sqrt{T}E_4(t) \]
\[ E_3(t) = \sqrt{T}E_1(t) + \sqrt{R}E_4(t). \]  

Here the \( E_i \) are defined at the same space-time point because the mirror can be approximated as a geometrically thin object located at \((x, t)\). The \( R \) and \( T \) represents the power reflectivity and transmissivity of the mirror. For a lossless mirror, we have \( R + T = 1 \). The two quadratures of field \( E_i(t) \) also satisfy the above relations. Then
it is clear that for a cavity configuration (for simplicity, the end mirror reflectivity is assumed to be zero):

\[
E_2(t) = -\sqrt{R}E_1(t) + \sqrt{T}E_4(t), \quad E_3(t) = \sqrt{T}E_1(t) + \sqrt{R}E_4(t),
\]

\[
E_4(t) = E_6(t - \frac{L}{c}) \quad E_5(t) = E_3(t - \frac{L}{c}), \quad E_5(t) = E_6(t).
\] (2.34)

In the case of the optomechanics, the frequency scale of the various perturbation to the light field is normally smaller than \(\omega_0\), thereby allowing the following decomposition:

\[
E_i(t) = \tilde{E}_i(t)e^{-i\omega_0 t} + \tilde{E}_i^*(t)e^{i\omega_0 t},
\] (2.35)

where \(\omega_0\) is the center frequency of the light field which is detuned from cavity resonance by \(\Delta = \omega_0 - \omega_c\) and \(\tilde{E}_i(t)\) is the slowly varying amplitude compare to the time scale \(1/\omega_0\) (schematically shown in Fig.2.4).

Here \(\omega_c = n\pi c/\lambda\) is the resonant frequency of the cavity field. Then the Eq. (2.34) can be represented as (in the frequency domain):

\[
\tilde{E}_2(\Omega) = \frac{e^{2i(\omega_0 + \Omega)L/c} - \sqrt{R}}{1 - \sqrt{R}e^{2i(\omega_0 + \Omega)L/c}} \tilde{E}_1(\Omega), \quad \tilde{E}_3(\Omega) = \frac{\sqrt{T}}{1 - \sqrt{R}e^{2i(\omega_0 + \Omega)L/c}} \tilde{E}_1(\Omega),
\] (2.36)

where \(\tilde{E}_2\) and \(\tilde{E}_3\) are the reflected field and intra-cavity field respectively. Under the approximation that \(\Omega L/c, \Delta L/c \ll 1\) and \(R \approx 1\), to the first order approximation:

\[
\tilde{E}_3(\Omega) = \sqrt{\frac{c}{2L}} \frac{\sqrt{2\gamma}}{\gamma - i(\Delta + \Omega)} \tilde{E}_1(\Omega), \quad \tilde{E}_2(\Omega) = \frac{i(\Delta + \Omega) + \gamma}{-i(\Delta + \Omega) + \gamma} \tilde{E}_1(\Omega),
\] (2.37)
where $\gamma = cT/L$.

Notice that the approximations used above are: 1) narrowband approximation in the sense that $\Omega \ll \omega_0$; 2) a further approximation: single-mode approximation, which means that we only consider those slowly varying field components with frequency $\Omega + \Delta$ smaller than one free-spectral range $\omega_{fsr} = \pi c/\lambda$ (see Fig. 2.3). If the frequency of the external injecting field $E_1(\Omega)$ matches the cavity resonance frequency (i.e. $\Delta + \Omega = 0$), the intra-cavity field strength will be amplified by a factor of $1/T$. The cavity bandwidth $\gamma$, as explained intuitively in Chapter 1, has been derived here. As can be seen from the above discussion, the interaction between the cavity field and external field behaves like a harmonic oscillator interacting with the driving force. This analogy motivates us to treat the optical dynamics of the cavity in the effective Hamiltonian way.

- **Hamiltonian description of a Fabri-Perot cavity** The analogy between the optical dynamics of a Fabri-Perot cavity and a driven harmonic oscillator implies the following Hamiltonian:

$$\hat{H}_o = \hbar \omega_c \hat{a}^\dagger(t) \hat{a}(t) + i\hbar \sqrt{2\gamma} [\hat{a}_\text{in}(t) \hat{a}^\dagger(t) - \hat{a}_\text{in}^\dagger(t) \hat{a}(t)],$$

(2.38)

in which the first term is the free-Hamiltonian of the cavity field (the harmonic oscillator) and second term is the interaction between the cavity field and the external electromagnetic field (the driving source). The $\hat{a}/\hat{a}^\dagger$ here is the annihilation/creation operator of the cavity field with resonance frequency $\omega_c$, and the $\hat{a}_\text{in}/\hat{a}_\text{in}^\dagger$ is the annihilation/creation operator of the external field. Notice they have similar but different commutation relation: $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$ and $[\hat{a}_\text{in}(t), \hat{a}_\text{in}(t')] = \delta(t - t')$. The dimensions of these two sets of operators are also different: $\hat{a}^{(t)}$ is dimensionless and $\hat{a}_\text{in}^{(t)}$ represents the intra-cavity photon numbers. However, $\hat{a}_\text{in}^{(t)}$ has dimension $[\text{time}]^{1/2}$ and
\( \hat{a}_\text{in}^\dagger \hat{a}_\text{in} \) represents the external photon numbers per unit time. For a pumped cavity with pumping field at frequency \( \omega_0 \), the annihilation operator is \( \hat{a}_\text{in}(t) = [\bar{a}_\text{in} + \delta \hat{a}_\text{in}(t)]e^{-i\omega_0 t} \)

where \( \bar{a}_\text{in} \) is the amplitude of the steady pumping field and \( \delta \hat{a}_\text{in} \) is the perturbative field around it (e.g., the vacuum fluctuations). The factor of \( \sqrt{2\gamma} \) describes the interaction strength between them. The intra-cavity field dynamics is determined by the Heisenberg equation of motion \( i\hbar \dot{\hat{a}} = [\hat{a}, \hat{H}] \):

\[
\dot{\hat{a}} = -i\omega_c \hat{a} - \gamma \hat{a} + \sqrt{2\gamma} [\bar{a} + \delta \hat{a}_\text{in}(t)]e^{-i\omega_0 t} \tag{2.39}
\]

where the damping term describes the dissipation of the cavity field due to the finite transmissivity of the front mirror. (the derivation of the damping term \(-\gamma \hat{a}\) will be explained in the appendix.) Working in the rotating frame with frequency \( \omega_0 \) transforms \( \hat{a} \) as \( \hat{a} \rightarrow \hat{a}^{-i\omega_0 t} \) and the above equation will be rewritten accordingly as:

\[
\dot{\hat{a}} = i\Delta \hat{a} - \gamma \hat{a} + \sqrt{2\gamma} (\bar{a}_\text{in} + \delta \hat{a}_\text{in}) \tag{2.40}
\]

Perturbatively solving Eq. (2.40) leads to the zeroth-order and the first order equations:

\[
\bar{a} = \frac{\sqrt{2\gamma}}{\gamma - i\Delta} \bar{a}_\text{in}, \quad \hat{a}(\Omega) = \frac{\sqrt{2\gamma}}{\gamma - i(\Delta + i\Omega)} \hat{a}_\text{in}(\Omega) \tag{2.41}
\]

It is easy to figure out that Eq. (2.41) is just another version of Eq. (2.37).

### 2.4.2 Generalized Hamiltonian for Cavity fields

- **Free-propagating fields** The Hamiltonian Eq. (2.38) based on the single-mode assumption, which implies that a more generalized Hamiltonian could be written out so that Eq. (2.36) can be derived as Heisenberg equation of motion. Moreover, the Hamiltonian Eq. (2.38) can also be derived through the single-mode approximation.

Let us start from free-field electromagnetic Hamiltonian:

\[
\hat{H} = \hbar c \int_{-\infty}^{\infty} |k| \hat{a}_k^\dagger \hat{a}_k dk = \hbar c \int_0^\infty k \hat{a}_k^\dagger \hat{a}_k dk + \hbar c \int_0^\infty k \hat{a}_{-k}^\dagger \hat{a}_{-k} dk. \tag{2.42}
\]

\(^1\)Here and after, we will use \( \hat{a} \) and \( \hat{a}_\text{in} \) to represent \( \delta \hat{a} \) and \( \delta \hat{h}_\text{ata}\text{in} \) respectively as long as no confusion will be introduced.
in which the first(second) term after the second equality represents the right(left)-propagating fields. Suppose the right(left) propagating fields has the central wavevector \( k_p (- k_p) \), the above Hamiltonian can be rewritten as:

\[
\hat{H} = \hbar c \int_{-k_p}^{k_p} (k_p + k') \hat{a}_{k_p+k}^\dagger \hat{a}_{k_p+k'} \, dk' + \hbar c \int_{-k_p}^{k_p} (k_p - k') \hat{a}_{-k_p+k}^\dagger \hat{a}_{-k_p+k'} \, dk'.
\] (2.43)

By doing the narrow band approximation \((k' \ll k_p)\), we can extend \( k_p \) to infinity thereby reduce the above Hamiltonian to be:

\[
\hat{H} = \hbar c \int_{-\infty}^{\infty} (k_p + k') \hat{a}_{k_p+k}^\dagger \hat{a}_{k_p+k'} \, dk' + \hbar c \int_{-\infty}^{\infty} (k_p - k') \hat{a}_{-k_p+k}^\dagger \hat{a}_{-k_p+k'} \, dk'.
\] (2.44)

Defining the Fourier transformation of \( \hat{a}_k \) (obviously also under narrow band approximation) as:

\[
\hat{a}(x) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \hat{a}_{-k_p+k} e^{-ik'x}, \quad \text{(left-propagating mode)}
\]

\[
\hat{b}(x) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \hat{a}_{k_p+k} e^{-ik'x}, \quad \text{(right-propagating mode)}
\] (2.45)

Notice that \( [\hat{a}(x), \hat{a}^\dagger(x')] = \delta(x - x') \) but \( [\hat{a}(x), \hat{b}(x)] = 0 \).

Using these transformation to re-write Eq.(2.44):

\[
\hat{H} = \hbar c k_p \int_{-\infty}^{\infty} dx [\hat{a}^\dagger(x) \hat{a}(x) + \hat{b}^\dagger(x) \hat{b}(x)] - i\hbar c \int_{-\infty}^{\infty} dx \left[ \hat{a}^\dagger(x) \frac{\partial \hat{a}(x)}{\partial x} - \hat{b}^\dagger(x) \frac{\partial \hat{b}(x)}{\partial x} \right]
\] (2.46)

In this case, the equation of motion for \( \hat{a}(x) \) will be:

\[
\frac{1}{c} \frac{\partial \hat{a}(x)}{\partial t} = -i k_p \hat{a}(x) - \frac{\partial \hat{a}(x)}{\partial x}
\] (2.47)

If we assume \( \hat{a}(x) \to \hat{\alpha}(x)e^{-ik_p x} \), we have:

\[
\frac{1}{c} \frac{\partial \hat{\alpha}(x)}{\partial t} + \frac{\partial \hat{\alpha}(x)}{\partial x} = 0
\] (2.48)

which is exactly the wave propagation (rightward) equation. From Eq.(2.48) and (2.47), we can see that the electromagnetic field defined in Eq.(2.45) is the \textit{spatially slowly varying amplitude} and the \( \hat{a}(x) \) in Eq.(2.48) is the field without splitting the spatially and temporary main-oscillating part \((e^{-ik_p x} \text{ and } e^{i\omega_p t}, \text{ respectively})\). The solution of the above equation naturally describes the propagation of rightward-propagating fields. This generalized Hamiltonian will be very useful in our discussion of quantum noise in white light cavity system (See Chapter 6).
2.4. Fabry-Perot cavity

![Diagram](image)

Figure 2.5 – Phenomenological model for describing the interaction between a mirror and the light field. The mirror is effectively modeled as a degree of freedom with detuning \( \Delta \) and couples to the leftward and rightward propagating waves with strength \( g \).

It is important to comment a bit on the physical meaning of \( \hat{a}(\hat{b})(x) \) (which will be useful for the future Chapter 5). The meaning of \( \hat{a}(x) \) should be understood as the annihilation of a \textit{spatially slowly varying amplitude}. Therefore the minimum spatial resolution is determined by the frequency \( \Omega \) we are interested in. For example, for \( \Omega \sim 2\pi \times 10^2 \text{rad/s} \) gravitational wave detection, the spatial resolution is: \( 2\pi c/\Omega = 30\text{km} \).

Interestingly, this spatial resolution value justifies the validity of the single-cavity approximation we used in analyzing the gravitational wave detector. The typical length scale of an interferometer (the length of a round-trip light path) is around \( 2L \sim 8\text{km} \), which is smaller than the above minimum spatial resolution. However, if the frequency we are interested in is higher, say larger a free-spectral range \( \pi c/L \), then the minimum length scale is smaller than \( 2L \). In this case, the single-mode approximation for the cavity mode is no longer correct and we can not assume that \( \Omega L/c \ll 1 \). In Fig.2.14, we can see the multi-mode effect to the sensitivity of a gravitational wave detector.

- **Field interacts with one mirror**— In this case, the mirror can be treated as an extra degree of freedom couple to the leftward and rightward propagating fields. Then the Hamiltonian can be written as (See Fig.2.5):

\[
\hat{H} = -i\hbar c \int_{-\infty}^{\infty} dx \left[ \hat{a}^\dagger(x) \frac{\partial \hat{a}(x)}{\partial x} - \hat{b}^\dagger(x) \frac{\partial \hat{b}(x)}{\partial x} \right] + ig[\hat{a}(0)\hat{A}^\dagger + \hat{b}(0)\hat{A}^\dagger] + \hbar \Delta \hat{A}^\dagger \hat{A}.
\]

(2.49)
The corresponding equations of motion can be written as:

\[
\hat{a}_t + c \hat{a}_x = -cg\hat{A}, \quad \hat{b}_t - c\hat{b}_x = -cg\hat{A} - i(\Omega + \Delta)\hat{A} = g\hat{a}_0 + g\hat{b}_0
\]  

(2.50)

Integrating the equations of motion for the electromagnetic fields around the interaction point leads to:

\[
\hat{a}_{\text{out}} = \hat{a}_{\text{in}} - g\hat{A}, \quad \hat{b}_{\text{out}} = \hat{b}_{\text{in}} - g\hat{A}.
\]  

(2.51)

Notice that in deriving the above relation, we have used the fact that the \(\hat{b}_{\text{out}}\) is the field on the left of the interaction point since \(\hat{b}\) is a leftward propagating field. The junction condition of the optical field here is taken to be: \(\hat{a}(\hat{b}(0)) = \hat{a}(\hat{b}_{\text{out}}) + \hat{a}(\hat{b}_{\text{in}})/2\).

With this junction condition, the equation of motion for \(\hat{A}\) can be written as:

\[
[-i(\Omega + \Delta) + g^2]\hat{A} = g(\hat{a}_{\text{in}} + \hat{b}_{\text{in}})
\]  

(2.52)

where the \(g^2\) can be interpreted as the loss rate of the mirror internal mode \(\hat{A}\), the physical reason is clear since the loss of the internal mode must happen through the interaction with the environmental EM bath. Using the above equation under the large-detuned approximation \(\Delta \gg \Omega\), the \(\hat{A}\) can be adiabatically eliminated, which gives an input-output relation as

\[
\hat{a}_{\text{out}} = \frac{i\Delta}{g^2 - i\Delta}\hat{a}_{\text{in}} - \frac{g^2}{g^2 - i\Delta}\hat{b}_{\text{in}},
\]  

(2.53)

in which the coefficient before \(\hat{a}_{\text{in}}\) can be viewed as the “transmissivity” \(t\) and the coefficient before \(\hat{b}_{\text{in}}\) can be viewed as the “reflectivity” \(r\). It is clear that if the coupling is very strong: \(g \gg \Delta\), then we have \(r \approx 1\) while \(t \approx 0\), corresponding to the total reflective case.

Similarly, adiabatic elimination of \(\hat{A}\) can also gives the effective interaction Hamiltonian:

\[
\hat{H}_{\text{eff}}^{\text{int}} = t(\hat{b}_{\text{in}}\hat{a}_{\text{out}} + \hat{a}_{\text{in}}\hat{b}_{\text{out}}) + r(\hat{a}_{\text{in}}\hat{b}_{\text{out}} + \hat{b}_{\text{in}}\hat{a}_{\text{out}}) + h.c
\]  

(2.54)

If the in- and out- fields are defined as new independent fields, the free Hamiltonian can now be written as:

\[
\hat{H}_{\text{free}} = -i\hbar c \int_{-\infty}^{0} dx\hat{a}_{\text{in}}(x)\frac{\partial \hat{a}_{\text{in}}(x)}{\partial x} - i\hbar c \int_{0}^{\infty} dx\hat{a}_{\text{out}}(x)\frac{\partial \hat{a}_{\text{out}}(x)}{\partial x} + i\hbar c \int_{0}^{\infty} dx\hat{b}_{\text{in}}(x)\frac{\partial \hat{b}_{\text{in}}(x)}{\partial x} + i\hbar c \int_{-\infty}^{0} dx\hat{b}_{\text{out}}(x)\frac{\partial \hat{b}_{\text{out}}(x)}{\partial x}
\]  

(2.55)
This Hamiltonian $\hat{H} = \hat{H}_\text{free} + \hat{H}_\text{int}$ Eqs. (2.55), (2.54) is the most general form of a one-dimensional mirror-light interaction Hamiltonian. This Hamiltonian can be easily extend for describing an optical cavity. For example, for a perfectly reflective cavity with length $L$, we have:

$$\hat{H}_\text{free} = -i\hbar c \int_0^0 dx \hat{a}_\text{in}^\dagger(x) \frac{\partial \hat{a}_\text{in}(x)}{\partial x} - i\hbar c \int_0^L dx \hat{a}_\text{out}^\dagger(x) \frac{\partial \hat{a}_\text{out}(x)}{\partial x}$$

$$\hat{H}_\text{int} = t_1 [\hat{b}_\text{in}(0) \hat{b}_\text{out}^\dagger(0) + \hat{a}_\text{in}(0) \hat{a}_\text{out}^\dagger(0)] + r_1 [\hat{a}_\text{in}(0) \hat{b}_\text{out}^\dagger(0) + \hat{b}_\text{in}(0) \hat{a}_\text{out}^\dagger(0)] + \hat{b}_\text{in}^\dagger(L) \hat{a}_\text{out}(L) + \hbar c \tag{2.56}$$

Solving out the Heisenberg equation can lead to the input-output relation given in Eq. (2.34).

Single mode approximation will reduce the above complicated general Hamiltonian to the single mode Hamiltonian Eq. (2.38). The two Hamiltonian describing the free-evolution of the field inside the cavity can be reduced as:

$$H_\text{cav} = -i\hbar c \int_0^L dx \hat{a}_\text{out}^\dagger(x) \frac{\partial \hat{a}_\text{out}(x)}{\partial x} + i\hbar c \int_0^L dx \hat{b}_\text{in}^\dagger(x) \frac{\partial \hat{b}_\text{in}(x)}{\partial x} \approx \hbar \Delta \int_0^L dx [\hat{a}_\text{out}^\dagger(x) \hat{a}_\text{out}(x) + \hat{b}_\text{in}^\dagger(x) \hat{b}_\text{in}(x)] \approx \hbar \Delta \hat{a}^\dagger \hat{a} \tag{2.57}$$

where $\Delta = c \Delta k$ and $\hat{a} = \hat{a}_\text{out}(\hat{b}_\text{in})/\sqrt{2L}$. The interaction Hamiltonian can be reduced as:

$$H_\text{int} = \sqrt{\frac{T_1}{L}} [\hat{b}_\text{out}^\dagger \hat{a} + \hat{a}_\text{in} \hat{a}^\dagger] - [\hat{a}_\text{in}(0) \hat{b}_\text{out}^\dagger(0) + \hat{b}_\text{in}(0) \hat{a}_\text{out}^\dagger(0)] \tag{2.58}$$

The first term describes the interaction between the cavity mode and the external mode while the second term describes the field directly reflected by the front mirror. Notice that the interaction Hamiltonian describes the reflection of the end mirror is encoded in the $\hat{H}_\text{cav}$ since its corresponding Heisenberg equation of motion $\hat{a}_\text{out}(L) = \hat{b}_\text{in}(L)$ ensures the validity of the condition $\hat{a}_\text{out}(x) = \hat{b}_\text{in}(x)$ in deriving the $\hat{H}_\text{cav}$. 
2.4.3 Rotation of optical field in the phase-space

For a lossless cavity with perfect reflective end mirror, the input-output relation for the cavity field can be written as:

\[ \hat{a}_{\text{out}}(\Omega) = \frac{\gamma + i(\Delta + \Omega)}{\gamma - i(\Delta + \Omega)} \hat{a}_{\text{in}}(\Omega) = e^{i\alpha_+} \hat{a}_{\text{in}}(\Omega) \]
\[ \hat{a}_{\text{out}}^\dagger(-\Omega) = \frac{\gamma - i(\Delta - \Omega)}{\gamma + i(\Delta - \Omega)} \hat{a}_{\text{in}}(\Omega) = e^{-i\alpha_-} \hat{a}_{\text{in}}^\dagger(-\Omega) \]  

(2.59)

These two sideband fields have different phase-delay when they are reflected from the cavity, which creates a rotation (and an unimportant phase-shift) in the phase space of light:

\[ \begin{pmatrix} \hat{a}_{\text{out}1}(\Omega) \\ \hat{a}_{\text{out}2}(\Omega) \end{pmatrix} = e^{i\alpha_+ - \alpha_-} \begin{pmatrix} \cos \frac{1}{2}(\alpha_+ + \alpha_-) & -\sin \frac{1}{2}(\alpha_+ + \alpha_-) \\ \sin \frac{1}{2}(\alpha_+ + \alpha_-) & \cos \frac{1}{2}(\alpha_+ + \alpha_-) \end{pmatrix} \begin{pmatrix} \hat{a}_{\text{in}1}(\Omega) \\ \hat{a}_{\text{in}2}(\Omega) \end{pmatrix} \]  

(2.60)

The rotation angle can be figured out from the above formula as:

\[ \theta(\Omega) = \frac{\alpha_+(\Omega) + \alpha_-(\Omega)}{2} = \arctan \left[ \frac{2\Delta \gamma}{\gamma^2 - \Delta^2 + \Omega^2} \right] \]  

(2.61)

If there is no detuning, then these two sideband field will have the same phase-delay which keep the initial quadrature combination. In this case, the \( \theta = 0 \) and there is only a phase-delay.

Notice that this rotation is frequency-dependent, which will be very useful for creating frequency-dependent squeezing to beat the quantum noise of a gravitational wave detector [3]. Imaging that if we injected a phase-squeezed vacuum into a(or a series) Fabri-Perot cavity(cavities), then the outgoing field of these Fabri-Perot cavities will be rotated frequency-independently, create a squeezed state with a frequency-dependent squeezing angle (See Fig 2.6).

2.5 Basic model of an optomechanical system

After the discussion about the quantum state of light and Fabry-Perot cavity, this section discusses the basic model of an optomechanical system. The simplest example of an optomechanical system is the interaction between a propagating light field with a movable mirror.
2.5. Basic model of an optomechanical system

The optical field can be represented as:

$$\hat{E}(t) = u(x, y, z) \sqrt{\frac{4\pi \hbar \omega_0}{Ac}} \left[ \sqrt{\frac{I_0}{\hbar \omega_0}} + \hat{a}_1(t) \right] \cos(\omega_0 t) + \hat{a}_2(t) \sin \omega_0 t,$$

(2.62)

where $\hat{a}_{1,2}$ obeys the commutation relation: $[\hat{a}_1(t), \hat{a}_2(t')] = i\delta(t - t')$. The reflected field is (defining $\tau = L/c$):

$$\frac{\hat{E}(t-2\tau-\frac{2x}{c})}{u(x, y, z)\sqrt{\frac{4\pi \hbar \omega_0}{Ac}}} = \left[ \sqrt{\frac{I_0}{\hbar \omega_0}} + \hat{a}_1(t-2\tau-\frac{2x}{c}) \right] \cos \omega_0(t-\frac{2x}{c}) + \hat{a}_2(t-2\tau-\frac{2x}{c}) \sin \omega_0(t-\frac{2x}{c}),$$

(2.63)

in which $2\omega_0 x/c$ is a small quantity. Therefore, to the first order:

$$\hat{E}(t-2\tau-\frac{2x}{c}) \approx u(x, y, z) \sqrt{\frac{4\pi \hbar \omega_0}{Ac}} \left[ \left( \sqrt{\frac{I_0}{\hbar \omega_0}} + \hat{b}_1(t) \right) \cos \omega_0 t + \hat{b}_2(t) \sin \omega_0 t \right].$$

(2.64)
where the amplitude/phase fluctuations of the reflected field is related to that of the incoming field as:

\[
\hat{b}_1(t) = \hat{a}_1(t - 2\tau) \\
\hat{b}_2(t) = \hat{a}_2(t - 2\tau) + \sqrt{\frac{I_0}{\hbar\omega_0}} \frac{2\omega_0 x(t - \tau)}{c}.
\] (2.65)

The \(x\) here represents the motion of the mirror, which is determined by the mirror response function and the driving force. It is important to note that the light also exerts a force on the mirror: the radiation pressure force, which is given by:

\[
F_{\text{rp}}(t) = 2 \frac{A}{4\pi} \langle |\hat{E}_{\text{in}}|^2 \rangle = 2 \frac{I_0}{c} \left[ 1 + \sqrt{\frac{\hbar\omega_0}{I_0}} \hat{a}_1(t - \tau) \right],
\] (2.66)

in which the first term is a trivial DC effect and can be safely ignored. With this radiation pressure force, the equation of motion for the mirror (suppose it is a free test mass) is:

\[
m\ddot{x} = 2\sqrt{\frac{\hbar\omega_0 I_0}{c^2}} \hat{a}_1(t - \tau) + G(t)
\] (2.67)

where \(G(t)\) represents other external forces. These equations (Eq.2.65, 2.66, 2.67) determines the optomechanical dynamics of the system shown in Fig. 2.7.

Suppose the above system is used to detect a small displacement signal of the mirror due to the driving of \(G(t)\). Then the above radiation pressure force is the back-action force of the detector (optical field) to the probe (mirror). If \(\hat{a}_1(t - \tau)\) is due to the quantum vacuum fluctuation, then this back-action will impose a fluctuating force on the mirror, which is the radiation pressure force. For the quadrature \(\hat{b}_2\) which contains the displacement information of the mirror, not only the above mentioned radiation pressure noise will enter the final result, but also the “shot noise” contributed by the \(\hat{a}_2(t - 2\tau - 2x/c)\).
In the frequency domain, the above result can be written as:

\[
\begin{pmatrix}
\hat{b}_1(\Omega)
\hat{b}_2(\Omega)
\end{pmatrix}
= e^{2i\Omega \tau}
\begin{pmatrix}
1 & 0
-\mathcal{K} & 1
\end{pmatrix}
\begin{pmatrix}
\hat{a}_1(\Omega)
\hat{a}_2(\Omega)
\end{pmatrix}
+ e^{i\Omega \tau}
\begin{pmatrix}
0
\sqrt{2\mathcal{K}}
\end{pmatrix}
\frac{G(\Omega)}{\sqrt{\hbar m \Omega^2}}
\]  

(2.68)

where \( \mathcal{K} = 4I_0\omega_0/m\Omega^2c^2 \)

The error of the \( G(\Omega) \) signal by measuring the \( \hat{b}_2 \) quadrature can be written as:

\[
\Delta b_2(\Omega) = -\mathcal{K}e^{2i\Omega \tau}\hat{a}_1(\Omega) + e^{2i\Omega \tau}\hat{a}_2(\Omega)
\]  

(2.69)

The first term is the radiation pressure noise while the second term is the shot noise, normalized by the signal value, we have the noise spectral for \( G(\Omega) \) is:

\[
S_{GG}(\Omega) = \frac{\hbar m \Omega^2}{2}
\left[ \mathcal{K} + \frac{1}{\mathcal{K}} \right] \geq \frac{\hbar m \Omega^2}{\sqrt{2}}
\]  

(2.70)

When the shot noise is equal to the radiation pressure noise, the total quantum noise takes its lowest value and this lowest value is called **Standard Quantum Limit (SQL)**.

After introducing the dynamics and the quantum sensitivity of the above simplest optomechanical system, now there are several points related to surpassing the standard quantum limit and the pondermotive squeezing:

### 2.5.2 Surpassing the standard quantum limit

- If we do not only measure the phase quadrature \( \hat{b}_2 \) of the outgoing field, but some smartly combined quadrature of light field as:

  \[
  \hat{b}_\xi = \hat{b}_1 \cos \xi + \hat{b}_2 \sin \xi
  \]

  \[
  = e^{2i\Omega \tau}\hat{a}_1(\Omega) \cos \xi
  - \mathcal{K}e^{2i\Omega \tau}\hat{a}_1(\Omega) + e^{2i\Omega \tau}\hat{a}_2(\Omega)
  + e^{i\Omega \tau}\sqrt{2\mathcal{K}}\frac{G(\Omega)}{\sqrt{\hbar m \Omega^2}} \sin \xi.
  \]  

(2.71)

Suppose the \( \xi \) satisfies the frequency-dependent relation:

\[
\tan \xi_v(\Omega) = \frac{1}{\mathcal{K}(\Omega)}.
\]  

(2.72)

the \( \hat{a}_1 \) term will be completely removed:

\[
\hat{b}_{\xi_v} = (e^{2i\Omega \tau}\hat{a}_2(\Omega) + e^{i\Omega \tau}\sqrt{2\mathcal{K}}\frac{G(\Omega)}{\sqrt{\hbar m \Omega^2}}) \sin \xi_v(\Omega).
\]  

(2.73)
In this case, the noise spectrum for $G(\Omega)$ is:

$$S_{GG}(\Omega) = \frac{\hbar m \Omega^2}{2K}. \quad (2.74)$$

This method—removing the radiation pressure noise by carefully designing a homodyne detector with specific frequency-dependent homodyne angle, is called “variational readout”, which was first discussed by Sergey. P. Vyatchanin [3, 7]. The key technology in realizing the variational readout method is to obtain a frequency-dependent homodyne angle. This can be achieved using the Fabri-Perot cavity since it can rotate the light field in a frequency-dependent way [3].

- What about if the ingoing field $\hat{a}$ is in the squeezed state? Then the noise spectrum in Eq.(2.61) becomes:

$$S_{GG}(\Omega) = \frac{\hbar m \Omega^2 e^{-2r}}{2} \left[ K + \frac{1}{K} \right] \quad (2.75)$$

when the input squeezing angle $\xi(\Omega)$ satisfy:

$$\tan \xi(\Omega) = \frac{1}{K}. \quad (2.76)$$

This means that the squeeze injection can indeed surpass the SQL. There are two issues here that need to be noticed: (1) the squeezed light needs to be rotated in a frequency-dependent way—this can be realized again by passing the squeezed light through a series of Fabri-Perot cavities. (2) Interestingly, the requirement for the squeezing angle is exactly the same as the requirement for the homodyne angle—this is not surprising. The reason for this equivalency is that both of these two methods need to align a specific optical quadrature, either squeeze it (squeezing injection), either remove it (variational readout).

### 2.5.3 Pondermotive squeezing

In the above discussion, it is assumed that the ingoing optical fluctuation $\hat{a}_1/\hat{a}_2$ takes the vacuum expectation. After the optomechanical interaction, the outgoing field $\hat{b}_1/\hat{b}_2$ is no-longer vacuum, but a squeezed state. This fact can be checked by calculating the variance:

$$\begin{pmatrix} S_{b_1 b_1} & S_{b_1 b_2} \\ S_{b_2 b_1} & S_{b_2 b_2} \end{pmatrix} = \begin{pmatrix} 1 & -K \\ -K & K^2 + 1 \end{pmatrix} = S^T \begin{pmatrix} \frac{2 + K^2 + \sqrt{K^4 + 4K^2}}{2} & 0 \\ 0 & \frac{2 + K^2 - \sqrt{K^4 + 4K^2}}{2} \end{pmatrix} S \quad (2.77)$$
The second equality is the diagonalization transformation) which indicates that the squeezing parameters are:

$$e^{\pm 2r} = \frac{2 + K^2 \pm \sqrt{K^4 + 4K^2}}{2},$$  

and it is easy to verify that the determination of this diagonalized matrix is equal to one. Therefore, the outgoing field \( \hat{b} \) is a squeezed vacuum state with a frequency-dependent squeezing angle determined by the matrix \( S(S^T) \). This squeezing process is the **pondermotive squeezing** we have briefly discussed previously. It is now clear from Eq.(2.59) that the reason for this pondermotive squeezing is the mix-up of the phase and amplitude quadrature due to the mechanical motion induced by the radiation pressure force.

## 2.6 Mapping between an interferometer to a single-cavity system

After the introduction of the above basic optomechanical system, we now switch to the cavity-optomechanical system which is the basic ingredient of a laser interferometer detector. Before detailed discussion of the dynamics of cavity-optomechanical system, in this section, we first give a brief summary of the mapping relation between the interferometer and the cavity optomechanical system.

The elegant mapping between a second generation interferometric gravitational wave detector and a single cavity optomechanical system is established by Buonanno and Chen in 2004 [8].

- **The simplest Michelson interferometer**—The simplest Michelson interferometer is shown in Fig.2.8(A). The mapping between Fig.2.8(A) and Fig.2.8(a) is described as following: 1) As discussed in Chapter 1, the detection port is locked on a dark fringe in the unperturbed situation when the two arms are exactly identical. Any field entering into the bright(dark) port will return to the bright(dark) port because of the destructive interference between the light propagated through two different arms. This fact actually “separates” the strong carrier laser from the
Figure 2.8 – Laser interferometers and their effective cavity mapping. (A-a) the delay-line configuration where the dynamics can be mapped to the interaction between a mirror and a propagating light; (B-b) the initial LIGO design can be mapped to a tuned Fabry-Perot cavity with movable end mirror; (C-c) the dual-recycling interferometer, can be mapped to a three-mirror cavity. The SRM and ITM can be treated as an effective mirror with a tunable frequency response, which enriches the optomechanical dynamics of this configuration.
2.6. Mapping between an interferometer to a single-cavity system

weak “differential signal” emerging in the dark port. Therefore in its effective description Fig.2.8(a), the light field interacts with the mirror contains both the carrier light and the weak signal/noise field. However, in the detection port, the carrier light is “separated” and there is only the signal/noise field.

2) Suppose the field in each arm has power $I_0$, then due to the 50 : 50 beam splitter, then the laser source power should be $2I_0$. But the incoming carrier power in the effective model Fig.2.8 should still be $I_0$. 3) Notice that the detectable motion of the interferometer mirrors are differential motion $x_d(t) = x_N(t) - x_E(t)$ where $x_N(t)$ and $x_E(t)$ are motion of north-south arm mirror and east-west arm mirror, respectively. This means in the effective model Fig.2.8, the mass for the end mirror should be the reduced mass: $m/2$ if we set the driving force to be the radiation pressure force of a single arm, that is, the $F_{GW} = mL\ddot{h}(t)/2$ and $F_{np}$ depends only on the vacuum fluctuation entering from the dark port which reflects/transmits to one arm (notice that the vacuum fluctuation enter from the bright port only affects the common motion). 4) Notice that only half (in terms of power) of the vacuum fluctuation enters one arm, therefore the dark port field amplitude should be $\hat{a}_1/\sqrt{2}$.

In summary, when we “fold” the two arms are folded into one arm, this will 1) half the laser power; 2) half the mirror mass; 3) half the vacuum fluctuation power. Therefore, the equation of the differential motion can be written as:

\[ \frac{1}{2}m\ddot{x_d}(t) = \frac{1}{2}mL\ddot{h}(t) + \frac{2I_0}{c} \frac{\hbar\omega_0}{I_0} \frac{\hat{a}_1(t - \tau)}{\sqrt{2}}. \]  

(2.79)

The input-output relation would be:

\[ \hat{b}_1(t) = 2 \frac{\hat{a}_1(t - 2\tau)}{\sqrt{2}} \]  

(2.80)

\[ \hat{b}_2(t) = 2 \frac{\hat{a}_2(t - 2\tau)}{\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{2\omega_0}{c} \frac{I_0}{\hbar\omega_0} (\hat{x}_E(t) - \hat{x}_N(t)) \]  

(2.81)

The first and second square root in $\hat{a}_1, \hat{a}_2$ terms are due to the fact that the field entering into dark port needs to pass the beam splitter twice before leaving the interferometer. The 2-factor in these terms is due to the fact that the field detected at the dark port consists of both the field propagating in the east-west arm and north-south arm. The same logic can lead to the differential mode motion term in $\hat{b}_2(t)$.

- **The Power-recycling interferometer**— In this case, the intra-cavity power gets enhanced due to the power-recycling mirror. The Power-recycling interferometer
shown in Fig. 2.8(B) can actually be mapped into two effective cavities: common mode cavity and differential mode cavity. The common mode cavity consists of the power recycling mirror and the arm cavities. The field enters in/ reflected from the common mode cavity all through the bright port and the common mode cavity does not sense the differential motion of the end mirrors. The differential mode consists of the dark port and the arm cavity, which is only sensitive to the differential motion. In Fig. 2.8(b), we only shows the differential mode cavity which is relevant to gravitational wave detection. In this case, both of the two effective cavities are tuned on resonance. Apparently, the enhanced intra-cavity power leads to the enhancement of the signal, which is in principle beneficial for gravitational wave detection, at least in the high-frequency region.

- **The dual recycling interferometer**— For dual recycling interferometer design, there is another mirror on the dark port to feedback the signal into the system, as shown in Fig. 2.8(c). It is clear that there are two effective cavities in this system: the common mode cavity which is on resonance with the laser frequency as before for enhancing the intra-cavity power and the differential mode cavity which consists of a signal recycling mirror and arm cavities. However, the signal recycling cavity does not have to be tuned on resonance with respect to the arm cavity—and this detuning is actually the source of rich dynamical behavior of this dual recycling configuration. The discussion of the dynamics of this configuration will be postponed to next section, here, we focus on how to effectively describe this three-mirror differential cavity using a two-mirror effective cavity.

This two-mirror effective cavity was first discussed by Buonanno and Chen [8]. The effective front mirror for this effective cavity is actually the signal-recycling cavity. The reflectivity and transmissivity of both sides of this effective mirror is given by [8]:

\[
\begin{align*}
    r'_{\text{eff}} &= \frac{\sqrt{R} + r e^{2i\phi}}{1 + \sqrt{R} r e^{2i\phi}}, \\
    r_{\text{eff}} &= -\frac{r + \sqrt{R} r e^{2i\phi}}{1 + \sqrt{R} r e^{2i\phi}}, \\
    t_{\text{eff}} &= t_{\text{eff}} = \frac{t \sqrt{T} e^{i\phi}}{1 + \sqrt{R} r e^{2i\phi}},
\end{align*}
\]

(2.82) (2.83)

where \( \phi = \omega_0 L/c \) is the phase delay by lights with carrier frequency \( \omega_0 \) upon
2.6. Mapping between an interferometer to a single-cavity system

one trip across the signal recycling cavity, $r$, $\sqrt{R}$ are the amplitude reflectivity of the signal recycling mirror and the input test mirror. It is clear that if $r = \pm \sqrt{R}$, the signal recycling cavity behaves like a normal mirror (the reflectivity and transmissivity magnitude are the same on both sides of the mirror). The tunability of the $r'_{\text{eff}}$ and $t_{\text{eff}}$ allows the interferometer to work in differential mode and it is also possible for the carrier light (which is tuned to the arm cavity) to be detuned with respect to the compound cavity consists of the arm cavity and the signal recycling mirror.

In this case, the resonant point of the whole interferometer configuration is no longer the same as the resonant point of the arm cavity. The new resonant point $\Omega_{\text{res}}$ is determined by:

$$r'_{\text{eff}} e^{2i\Omega_{\text{res}} \tau} = 1,$$

(2.84)
in which it is assumed that the distance between the signal recycling mirror and the input test mirror is negligible compared to the arm length. Expanding the exponential to the first order gives the resonance point as:

$$\Omega_{\text{res}} = \Delta + i\gamma_{\text{eff}},$$

(2.85)
where:

$$\Delta = \frac{2r\gamma \sin 2\phi}{1 + r^2 + 2r \cos 2\phi}, \quad \gamma_{\text{eff}} = \frac{(1 - r^2)\gamma}{1 + r^2 + 2r \cos 2\phi},$$

(2.86)
with $\gamma = cT/(4L)$. Now the differential mode of the interferometer can be mapped to a detuned cavity with the above detuning $\Delta$ and effective bandwidth $\gamma_{\text{eff}}$.

For example, as just mentioned, if $\phi = \pi/2$, then $r_{\text{eff}} = -r'_{\text{eff}}, t_{\text{eff}} = t'_{\text{eff}}$. The signal recycling mirror behaves just like an mirror with tunable reflectivity and transmissiv-
ity. The detuning is zero and the bandwidth is:

\[ \gamma_{RSE} = \frac{1 + r}{1 - r} \gamma = \frac{\gamma_{\text{eff}}(\phi = \pi/2)}{4L}. \]  

(2.87)

This configuration is called "Resonant Sideband Extraction" mode. In the extremal case that \( r \to \sqrt{R} \), the \( \gamma_{RSE} \to \infty \), which implies that:

\[ r'_{\text{eff}} \approx -r, \]  

(2.88)

in this case, 1) the input test mass mirror is almost transparent to the sideband field; 2) the signal recycling cavity and the end mirror form a compound mirror which is anti-resonant. The system bandwidth will be almost infinity. Notice that the Resonant Sideband Extraction mode always broadens the bandwidth since \( r > 0 \), but at the sacrifice of the signal strength. This fact can be understood as the destructive interference between the light field directly reflected from input test mass and the field transfers out of the the signal recycling mirror as we have shown in Fig.1.3 since the one round-trip phase is \( 2\phi = \pi \).

Another example is the situation when \( \phi = 0 \). In this case:

\[ \gamma_{SR} = \frac{1 + r}{1 - r} \gamma. \]  

(2.89)

This is so-called Signal Recycling Mode, in which the bandwidth \( \gamma_{SR} \) is always narrowed but the signal get enhanced.

There are also signal-recycling cavity parameter regions when \( \Delta \neq 0 \). In these case, there will be an optical spring effect, which is discussed in the next section.

In summary, a general dual-recycling interferometer can be mapped to a detuned effective Fabri-Perot cavity, with tunable front mirror. Different choices of the parameters of this effective cavity can lead to different interferometer behaviors. In the next section, we will focus on the physics of such a detuned cavity.

### 2.7 Cavity optomechanics

The last section showed that understanding the dynamics of the cavity optomechanics is the key to understand the dynamical behavior and sensitivity of the interferometer. There are many excellent reviews containing the discussion of cavity optomechanics
2.7. Cavity optomechanics. This section discusses the dynamics of cavity optomechanics in detail using two different approaches.

2.7.1 Transfer matrix method

• Sensitivity calculation of an optomechanical device—Firstly, regarding the cavity optomechanical dynamics by establishing a systematic analyzing method—transfer matrix method—this method was firstly discussed by Corbitt et al. [12].

The input-output relation for light transmitting/reflecting through a mirror can be written (in the two-photon formalism) as:

\[
\begin{pmatrix}
\hat{c}_1 \\
\hat{c}_2
\end{pmatrix} = \sqrt{T} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{pmatrix} + \sqrt{R} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\hat{d}_1 \\
\hat{d}_2
\end{pmatrix}
\] (2.90)

(in the following, the \( \hat{c} \) is used to represent the column \((\hat{c}_1, \hat{c}_2)^T\)). Similarly, \( \hat{b} = \sqrt{T}\hat{d} - \sqrt{R}\hat{a} \).

The light field propagate through the vacuum with distance \( L \) (cavity length) can be represented as:

\[
\begin{pmatrix}
\hat{e}_1 \\
\hat{e}_2
\end{pmatrix} = e^{i\Omega\tau} \begin{pmatrix}
\cos \Delta \tau & \sin \Delta \tau \\
-\sin \Delta \tau & \cos \Delta \tau
\end{pmatrix} \begin{pmatrix}
\hat{c}_1 \\
\hat{c}_2
\end{pmatrix}
\] (2.91)

(or \( \hat{e} = e^{i\Omega\tau} M_{\text{rot}} \hat{c} \)) where \( \Delta \) is the detuning of the pumping carrier light with respect to the cavity resonance.

The matrix representation of optomechanical interaction is given by Eq.(2.59). For completeness, we re-write it using the symbol convention in Fig.2.10:

\[
\begin{pmatrix}
\hat{f}_1 \\
\hat{f}_2
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
-\mathcal{K} & 1
\end{pmatrix} \begin{pmatrix}
\hat{e}_1 \\
\hat{e}_2
\end{pmatrix} + \begin{pmatrix}
0 \\
\sqrt{2\mathcal{K}}
\end{pmatrix} \frac{R_{xx}(\Omega)}{\hbar} G(\Omega),
\] (2.92)

where \( \mathcal{K} = 4\omega_0 I_0 R_{xx}(\Omega)/c^2 \) for the mechanical oscillator with a general response function \( R_{xx}(\Omega) \) \((R_{xx}^{-1}(\Omega)\hat{x}(\Omega) = F(\Omega))\). This relation now can be simply written as \( \hat{f}(\Omega) = M_{\text{opt}}(\Omega)\hat{e}(\Omega) + DG(\Omega) \).

The optical dynamics of the cavity can be understood as a linear feedback control problem: the open-loop of this control system is one round-trip propagation of light field inside the cavity (including the optomechanical interaction) and the feedback
kernel is the reflection of light on the front mirror. Then the open-loop transfer matrix can be written as:

\[
G_o(\Omega) = e^{2i\Omega\tau} M_{\text{rot}}(\Omega) M_{\text{opt}}(\Omega) = T e^{2i\Omega\tau} M_{\text{opt}}(\Omega),
\]

while the closed-loop transfer matrix is:

\[
G_c(\Omega) = \frac{G_o(\Omega)}{1 - \sqrt{R} G_o(\Omega)} = T e^{2i\Omega\tau} M_{\text{opt}}(\Omega) \left[1 - \sqrt{R} T e^{2i\Omega\tau} M_{\text{opt}}(\Omega)\right]^{-1},
\]

\[
b(\Omega) = M_{\text{refl}}(\Omega) a(\Omega) + D_c(\Omega) G(\Omega),
\]

where \(D_c = e^{\Omega \tau} M_{\text{rot}} D(\Omega) [1 - \sqrt{R} T e^{2i\Omega\tau} M_{\text{opt}}(\Omega)]^{-1}\) and \(M_{\text{refl}}(\Omega) = 1 + G(\Omega)_c\). It is clear that the \(G_c(\Omega) a(\Omega)\) is the noise part where \(D_c(\Omega) G(\Omega)\) is the signal part. Finally, the (force)-sensitivity (normalized noise spectrum) can be written as:

\[
S_{GG}(\Omega) = \frac{v_\xi [M_{\text{refl}}(\Omega) M_{\text{refl}}^\dagger(\Omega) + M_{\text{refl}}^\dagger(\Omega) M_{\text{refl}}(\Omega)] v_\xi^\dagger}{v_\xi [D_c(\Omega) D_c^\dagger(\Omega) + D_c^\dagger(\Omega) D_c(\Omega)] v_\xi^\dagger},
\]

where \(v_\xi = (\cos \xi, \sin \xi)\) is the homodyne detection matrix with \(\xi\) be the homodyne angle. If \(G(t) = \frac{1}{2} m \ddot{h}(t)\) which is the gravitational tidal force, then it is easy to obtain the strain sensitivity for gravitational wave.

This transfer-matrix method will be used in the calculation of the quantum noise limited sensitivity of the interferometers in this thesis.

- **Single-mode approximation** — Under the single-mode approximation, the simple analytical result of the input-output relation can be written down:

\[
b(\Omega) = \frac{1}{C} [M(\Omega) a(\Omega) + D_c(\Omega) h(\Omega)].
\]
where

\[
C = \Omega^2[(\Omega + i\gamma)^2 - \Delta^2] + \Delta t
\]

\[
\mathbb{M}(\Omega) = \begin{pmatrix}
-\Omega^2(\Omega^2 + \gamma^2 - \Delta^2) - \Delta t_c & 2\gamma \Delta \Omega^2 \\
-2\gamma \Delta \Omega^2 + 2\gamma t_c & -\Omega^2(\Omega^2 + \gamma^2 - \Delta^2) - \Delta t_c
\end{pmatrix}
\]

\[
D(\Omega) = \begin{pmatrix}
\Delta \Omega \\
(\gamma + i\Omega)\Omega
\end{pmatrix} \frac{2\sqrt{\gamma t_c}}{\hbar \text{SQL}}.
\]

with \( t = 4\omega_0 I_0 / mLc \). The gravitational wave strain sensitivity can then be written as:

\[
S_{hh}(\Omega) = \frac{\mathbf{v}_x^T \mathbf{M}_x T \mathbf{v}_x^T}{\mathbf{v}_x^T \mathbf{D} \mathbf{D}^T \mathbf{v}_x^T}.
\]

Later on, we will show that these results under single-mode approximation can be recovered using the Hamiltonian approach.

**• non detuned case**— If the carrier light has no detuning with respect to the cavity resonance, then the input-output relation has a very simple expression:

\[
\mathbb{M}(\Omega) = \begin{pmatrix}
e^{2i\phi} & 0 \\
-K & e^{2i\phi}
\end{pmatrix}, \quad D(\Omega) = e^{i\phi(\Omega)} \begin{pmatrix} 0 \\ \sqrt{2K} \end{pmatrix}
\]

where:

\[
\phi = \arctan(\Omega / \gamma), \quad K(\Omega) = \frac{2\gamma t_c}{\Omega^2(\Omega^2 + \gamma^2)}.
\]

The strain sensitivity is given by:

\[
S_{hh}(\Omega) = \frac{4\hbar}{m\Omega^2 L^4} \left[ \frac{1}{K(\Omega)} + K(\Omega) \right].
\]

**• Sampling sensitivity curves**— Using the parameters for advanced LIGO, we plot some sample sensitivities here. Figs 2.11, 2.12 and 2.13 correspond to the previously mentioned signal recycling mode, resonant-sideband extraction mode and the detuned case. From these figures, it is clear that the signal recycling mode increases the sensitivity peak but decreases the detection bandwidth, while the resonant sideband extraction mode broaden the detection bandwidth but decrease the sensitivity peak. This trade-off phenomenon is discussed later.

From Fig 2.13 it is interesting that some parameter choice of the signal recycling mirror can lead to two sensitivity dips, which surpass the free-mass standard quantum
Figure 2.11 – Strain sensitivity plot: $\sqrt{S_{hh}(\Omega)/S_{\text{SQL}}(\gamma)}$ versus $\Omega/\gamma$ for the dual-recycling interferometer working in the signal recycling mode. Here the Advanced LIGO parameters are assumed and the intra-cavity power is set to be the SQL power. We also assume that only the phase quadrature is measured. The reflectivity of the signal recycling mirror takes the value of $r = 0.15, 0.35, 0.75, 0.95$ and the phase is zero. The magenta and read curve are the $\sqrt{S_{hh}(\Omega)/S_{\text{SQL}}(\gamma)}$ for the conventional interferometer configuration and the free-mass Standard quantum limit of the strain sensitivity respectively.

Figure 2.12 – Strain sensitivity curve for the dual recycling interferometer working in the resonant sideband extraction mode. The reflectivity of the signal recycling mirror takes the value of $r = 0.7, 0.8, 0.9$ and its detuning phase is $\pi/2$. 
limit while the squeezing input and variational readout scheme are not applied here. This interesting phenomenon is due to the modification of test mass dynamics in the detuned interferometer, which will be fully discussed in the next section.

Last but not least, the above transfer matrix method also can be used to calculate the sensitivity beyond one free spectral range. For a typical interferometer, the $\Omega_{fsr}/\gamma \sim 4\pi/T \sim 200$. The frequency range of the previously given sample sensitivity curves are within $\Omega_{fsr}$ where the single-mode approximation is valid. Fig.2.14 also gives a sample example of the sensitivity curve beyond $\Omega_{fsr}$: Some proposals are discussed for using those narrow band dips around $n\Omega_{fsr}, n \in \mathbb{N}$ for detecting the high-frequency gravitational waves.

### 2.7.2 Hamiltonian method and optical spring effect

- **Hamiltonian and equations of motion**—Within $\omega_{fsr}$, the single-mode approximation is valid, therefore a Hamiltonian approach can be used to describe this general detuned cavity optomechanical system:

\[
\hat{H} = \hbar \omega_c \hat{a}^\dagger \hat{a} + i\hbar \sqrt{2\gamma} (\hat{a} \hat{a}^\dagger_{in} - h.c) + \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_m^2 \hat{x}^2 - \hbar G_0 \hat{a}^\dagger \hat{a} \hat{x}.
\]  

\[
(2.103)
\]
in which the first term is the free Hamiltonian for the cavity field and the second term describes the interaction between the external field and the intra-cavity field. The third and fourth terms are the free-Hamiltonian for the harmonic oscillator while the last term is the radiation pressure force term. $G_0 = \omega_0/L$ is the optomechanical coupling strength.

The corresponding equations of motion can be written as:

\begin{align}
\dot{x}(t) &= \dot{p}(t)/m, \\
\dot{p}(t) &= -\gamma_m \dot{p}(t) - m\omega_m^2 \dot{x}(t) + F_{\text{th}}(t) - \hbar G_0 \dot{a}(t) \dot{a}, \\
\dot{a}(t) &= -(\gamma - i\Delta) \dot{a}(t) iG_0 \dot{a}(t) \dot{x}(t) + \sqrt{2}\gamma \dot{a}_{\text{in}}(t),
\end{align}

in which the $\dot{F}_{\text{th}}$ is the thermal force and $\gamma_m$ is the mechanical damping rate associated with $\dot{F}_{\text{th}}$, $\gamma$ is the optical damping rate (cavity bandwidth) associated with electromagnetic bath $\dot{a}_{\text{in}}(t)$. The detuning is given by $\Delta = \omega_0 - \omega_c$ For simplicity, we ignore the thermal fluctuation temporarily. The mutual interaction between the cavity field and the mechanical oscillator shows in the last term of Eq. (2.105)(b), which represents the radiation pressure force, and the second term on the right hand side of Eq. (2.105)(c), which represents the motion-induced modulation.

The above equations are in general nonlinear. To linearize the system, we simply replace the operator with the sum of a zero-th order steady operator with a perturbed...
This linearization gives the following equation of motion:

\[
m \ddot{x}(t) + \gamma m \dot{x}(t) + m \omega_m^2 \dot{x}(t) = -\hbar G_0 \tilde{a}[\tilde{a}^\dagger(t) + \tilde{a}(t)], \tag{2.105a}
\]

\[
\dot{\tilde{a}}(t) + (\gamma - i \Delta) \tilde{a}(t) = i G_0 \tilde{a} \tilde{x}(t) + \sqrt{2\gamma} \tilde{a}_{in}(t), \tag{2.105b}
\]

where \( \tilde{a} \) is the zeroth-order solution given by:

\[
\tilde{a} = \frac{\sqrt{2\gamma}}{\gamma - i \Delta} \tilde{a}_{in}. \tag{2.106}
\]

The phase in this complex amplitude can be in principle moved out by re-setting the phase of \( \tilde{a}_{in} \). Here and after, it is assumed that \( \tilde{a} \) is real.

The corresponding linearized equation of motion in the Fourier domain can be written as:

\[
\chi_{xx}(\Omega) \tilde{x}(\Omega) = -\hbar G_0 \tilde{a}[\tilde{a}^\dagger(\Omega) + \tilde{a}(\Omega)], \tag{2.107a}
\]

\[
\chi_{opt}(\Omega) \tilde{a}(\Omega) = i G_0 \tilde{a} \tilde{x}(\Omega) + \sqrt{2\gamma} \tilde{a}_{in}(\Omega). \tag{2.107b}
\]

\[
\chi^*_{opt}(\Omega)[\tilde{a}(-\Omega)]^\dagger = -i G_0 \tilde{a} \tilde{x}(\Omega) + \sqrt{2\gamma} [\tilde{a}_{in}(-\Omega)]^\dagger, \tag{2.107c}
\]

with \( \chi_{xx}(\Omega) = -m(\Omega^2 + i\Omega \gamma_m - \omega_m^2) \) and \( \chi_{opt}(\Omega) = \gamma - i(\Delta + \Omega) \). Notice that the modulation of the oscillator motion will create two sidebands with frequency \( \omega_0 + \Omega \) and \( \omega_0 - \Omega \), described by Eq.(2.107)(b) and (c), respectively. The right hand side of Eq.(2.107)(a) represents the radiation pressure force contributed by the beating between the main carrier beam and the weak perturbed fields given in Eq.(2.107)(b)(c).

Interesting physics emerges by considering the fact that these weak perturbed fields depend on the mirror oscillation \( x(\Omega) \): this means the optomechanical interaction will modify the dynamical character of the mirror oscillator—the so called “optical spring effect”.

\[\bullet\] Optical spring effect— Substituting Eq. Eq.(2.107)(b)(c) into Eq.(2.107)(a), we have the modified mirror oscillation equation:

\[
\tilde{\chi}_{xx}(\Omega) \tilde{x}(\Omega) = \tilde{F}_{rad}(\Omega) \tag{2.108}
\]

in which \( \tilde{F}_{rad}(\Omega) \) and \( \tilde{\chi}_{xx}(\Omega) \) are the radiation pressure noise force and the modified mechanical susceptibility:

\[
\tilde{F}_{rad}(\Omega) = 2\hbar G_0 \tilde{a} \sqrt{7} \frac{(\gamma - i\Omega) \tilde{a}_{in1}(\Omega) - \Delta \tilde{a}_{in2}(\Omega)}{(\Omega + \Delta + i\gamma)(\Omega - \Delta + i\gamma)} \tag{2.109a}
\]

\[
\tilde{\chi}_{xx}(\Omega) = \chi_{xx}(\Omega) - \frac{2\hbar G_0^2 \tilde{a}^2 \Delta}{(\Omega + \Delta + i\gamma)(\Omega - \Delta + i\gamma)}. \tag{2.109b}
\]
For convenience, we define the real part of the second term of Eq. (2.109) (b) (which is the optomechanical modification of mechanical susceptibility) as $\chi_{om}(\Omega)$. Suppose the $\Omega$ is small compared to other frequency scales, then by expanding over $\Omega$ to the first order, we have:

$$\tilde{\chi}_{xx}(\Omega) \approx -m \left( \Omega^2 + i \left[ \gamma_m - \frac{4hG_0^2\bar{a}^2\gamma\Delta}{m(\gamma^2 + \Delta^2)^2} \right] \Omega - \left[ \omega_m^2 + \frac{2hG_0^2\bar{a}^2\Delta}{m(\gamma^2 + \Delta^2)} \right] \right). \quad (2.110)$$

As shown in the above formula, the resonance frequency and the damping rate of the mirror are modified. The optomechanical part of $\omega_m$ and $\gamma_m$ are named as “optical rigidity” and “optical damping”.

In the following, we give several comments about this effect:

1. A simple and quasi-static way to explain the optical spring effect is the following. Suppose the mechanical motion $x$ is quasi-static, then the intra-cavity amplitude and the intra-cavity power can be written as:

$$\bar{a} = \frac{\sqrt{2\gamma}}{\gamma - i\Delta - iG_0\bar{a}x(t)} \bar{a}_{in}, \quad I_c = \frac{2\gamma}{\gamma^2 + (\Delta + G_0\bar{a}x)^2} I_{in}. \quad (2.111)$$

Since the radiation pressure force exerted on the mirror is roughly $F_{rp} \propto I_c/c$, therefore the radiation pressure force is depended on the mirror position. Expanding the above formula in terms of $x$ gives:

$$F_{rp} \propto I_c \approx \frac{2I_{in}\gamma}{\gamma^2 + \Delta^2} - \frac{4G_0\bar{a}I_{in}\gamma\Delta}{(\gamma^2 + \Delta^2)^2} x + \mathcal{O}(x^2). \quad (2.112)$$

It is clear that the radiation pressure force is proportional to $x$ if $\Delta \neq 0$. The damping exists because the mechanical motion is only quasi-static. In reality, the response of the intra-cavity power to the test mass motion has a delay, which induced the friction component of the optical spring.

2. The optical damping has a microscopic explanation. The linear parametric optomechanical dynamics can be understood as the interaction between the photons and phonons as shown in Fig. 2.15. In this figure, the Stokes process emits a phonon and transfer the energy from the optical degree of freedom to the mechanical degree of freedom while the anti-Stokes process absorbs a phonon and thereby causing the mechanical degree of freedom to lose energy. If the carrier light is blue-detuned such that the Stokes sideband is higher than the anti-Stokes sideband, there will be a net energy flow from the optical degree of freedom to the mechanical degree of freedom,
2.7. Cavity optomechanics

Figure 2.15 – Photon-phonon interaction: (a) The anti-Stokes process, which causes a cooling effect by drawing a phonon out of the mechanical degrees of freedom (mirrors), creating an upper sideband photon with higher energy $\hbar(\omega_0 + \Omega)$; (b) The anti-Stokes process, creates a heating effect by emitting a phonon to the mechanical degrees of freedom, creating a lower sideband photon with lower energy $\hbar(\omega_0 - \Omega)$. (c) Red-detuned case where the rate of Stokes process is higher than that of the anti-Stokes process.
thus creating an anti-damping effect. Alternatively there is a net energy flow from
the optical degrees of freedom to the mechanical degree of freedom if the carrier light
is red-detuned, thus creating a damping effect.

- **Effect on the quantum noise limited sensitivity curve**— As shown in [13, 14], the effect of optical spring on the quantum sensitivity curve is to modify
the original free-mass standard quantum limit to the oscillator’s standard quantum limit.

Take the simplest example shown in Section 2.5 as an example, if the equations
of motions for the mechanical degree of freedom becomes:

\[
\chi_{xx}(\Omega)\ddot{x}(\Omega) = 2\sqrt{\frac{\hbar \omega_0 I_0}{c^2}} \dot{a}_1(\Omega) e^{2\Omega \tau} + G(\Omega). \tag{2.113}
\]

This leads to (for detecting the gravitational wave strains):

\[
S_{hh}(\Omega) = \frac{8\hbar}{m^2\Omega^4L^2} \sqrt{\left|\chi_{xx}(\Omega)\right|^2}. \tag{2.114}
\]

Apparently, this **Standard Quantum Limit** for a harmonic oscillator does have a
dip at the mechanical resonance and this corresponds to the lower frequency dip of
the sensitivity curves in Fig.[2.13] while the higher frequency dip is due to the optical
resonance of the interferometer given in Eq.(2.76).

Therefore, the optical spring effect makes it possible to surpass the **free mass SQL**
in the way that it modify the free mass SQL to be the oscillator SQL. However, the
detection bandwidth decreases.

- **Pondermotive cooling**— When the pumping light is red-detuned, the anti-
Stokes sideband is dominate over the Stokes sideband. Therefore the mechanical
energy will be transfered into the optical degrees of freedom and the mechanical
bandwidth gets increased. This is called the pondermotive cooling (or optical cooling).
The pondermotive cooling can be understood using the feedback flow chart: Fig.[2.16]

The mechanical oscillator in this case is damped through two channels: the opti-
cal channel and the thermal channel. These two damping processes \(\gamma_{\text{opt}}\) and \(\gamma_m\)
are associated with two random force: the thermal random force and the radiation
pressure force noise according to the fluctuation-dissipation theorem:

\[
S_{FF} = \sum_j 4m\gamma_j'\hbar\omega_m \coth \left( \frac{\hbar \omega_m}{2k_B T_j} \right) = \sum_j 8m\gamma_j \hbar \omega_m \left( \bar{n}_j + \frac{1}{2} \right), \tag{2.115}
\]
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Figure 2.16 – Block diagrams for the pondermotive cooling/heating effect. In the cooling regime. The mechanical oscillator couples both to the low temperature optical bath with effective coupling coefficient $\gamma_{\text{opt}}$ and the high temperature thermal bath with coupling coefficient $\gamma_m$. However, in the heating regime, the energy is transferred from the optical bath (including the pumping) to the mechanical oscillator which induces instability. The feedback control kernel is a force acting on the mechanical oscillator, based on the measurement result.
where we have neglect the modification of these bath to the mechanical resonant
frequency $\omega_m$, and $n_j$ is the mean occupation number contributed by the $j$th channel, defined as:

$$\bar{n}_j = \left( e^{\frac{\hbar \omega_m}{k_B T_j}} - 1 \right)^{-1}. \tag{2.116}$$

The effective occupation number of the mechanical oscillator and effective temperature are given by:

$$\bar{n}_{\text{eff}} = \frac{\sum_j \gamma_j \bar{n}_j}{\sum_j \gamma_j}, \quad T_{\text{eff}} \approx \frac{\sum_j \gamma_j \bar{T}_j}{\sum_j \gamma_j}. \tag{2.117}$$

Notice that the $T_{\text{eff}}$ is obtained approximately by assuming that $\hbar \omega_m < k_B T_j$. For the system with $\gamma_\text{opt} \gg \gamma_m$, we have $n(T)_{\text{eff}} = n(T)_{\text{opt}}$ which means that the mean occupation number and the effective temperature is dominate by the radiation pressure noise. Interestingly, this pondermotive cooling allows us to prepare the mechanical oscillator onto its quantum ground state when $\bar{n}_{\text{eff}} < 1 \ [15, 16]$.

- **Dynamical Instability** — On the other hand, from Eq. (2.110), it is clear that the optical rigidity and optical damping are always opposite to each other, namely, the positive optical rigidity is accompanied with a negative optical damping, vice versa. This means there is a potential instability in this system, especially when the test mass is a free mass so that there is no mechanical spring to cancel the negative damping/rigidity.

  The instability due to the negative damping coefficient has been demonstrated by the table-scale experiment, which reappears with the name of “optical heating”, “parametric instability” or “phonon lasing” [17, 19]. The increasing amplitude of the mechanical motion can boost the system into an interesting non-linear coupling region, which has been discussed by Marquardt et.al [20].

  For a interferometric gravitational wave detector, this kind of instability has two different effects: 1) It will bring us an instability for the center-of-mass motion of the test mass, which can be suppressed by adding a feedback control loop [14]; 2) The internal acoustic modes of the test mass will also suffer from the instability when the intra-cavity power is high enough, which will be a challenge for future high-power interferometer [18]. The next subsection discusses these two effects in more details.
2.7.3 Parametric instability

- **Center of mass motion instability**—For a red-detuned cavity, the mirror’s center of mass motion will suffer instability. However, this instability can be suppressed by feedback control and most importantly, this feedback control will not affect the sensitivity of the optomechanical devices (or interferometric gravitational wave detectors). This was first pointed out by Buonanno and Chen. Here is a sketch of mathematical proof given by Buonanno and Chen [14].

The Hamiltonian for the optomechanical system with feedback can be schematically written as:

$$\hat{H} = \hat{H}_m + H_o - \hat{x}G - \hat{x}\hat{F} - \hat{x}\hat{C}$$

(2.118)

where $H_{m/o}$ is the test mass/optical free Hamiltonian, $G/F$ are the external force/radiation pressure force, while $\hat{C}$ is the feedback force. The measurement-based feedback force is given by:

$$\hat{C}(t) = \int_{-\infty}^{t} dt' K(t - t') \hat{Z}(t')$$

(2.119)

where $\hat{Z}^{(0)}(t)$ is the measurement result. This Hamiltonian is a time-non-local Hamiltonian. The equation of motion for $x$ is then given by:

$$\chi_{xx}(\Omega) \ddot{x}(\Omega) = F(\Omega) + G(\Omega) + K(\Omega) Z(\Omega)$$

(2.120)

where we have substituted the Fourier transformation of the definition of $C(t)$ (Eq. [2.119]). The output $\hat{Z}(t)$ and the radiation pressure force (or equivalently the amplitude quadrature of the optical field) $\hat{F}$ evolute as (in the Fourier domain):

$$\hat{F}(\Omega) = \hat{F}^{(0)}(\Omega) + \chi_{FF}(\Omega) \hat{x}(\Omega)$$

$$\hat{Z}(\Omega) = \hat{Z}^{(0)}(\Omega) + \chi_{ZF}(\Omega) \hat{x}(\Omega)$$

(2.121)

which leads to:

$$Z(\Omega) = f(\Omega) \left( \frac{Z^{(0)}(\Omega) + \chi_{ZF}(\Omega) \chi_{xx}^{-1}(\Omega) [G(\Omega) + \hat{F}^{(0)}(\Omega)]}{1 - \chi_{xx}^{-1}(\Omega) \chi_{ZF}(\Omega)} \right)$$

(2.122)

with

$$f(\Omega) = \frac{1 - \chi_{xx}^{-1}(\Omega) \chi_{FF}(\Omega)}{1 - \chi_{xx}^{-1}(\Omega) \chi_{ZF}(\Omega) + \chi_{ZF}(\Omega) K_c(\Omega)}.$$

Notice that the feedback kernel only exists as an over-all normalization factor which does not affect the signal to noise ratio! This is due to the fact that both the signal
and the noise are fed back onto the test masses, therefore the signal and noise are affected in the same way by the control loop.

- **Acoustic mode instability**— The LIGO test mass mirrors have many acoustic (elastic) modes, these modes are excited through its coupling to the thermal baths. When the pumping carrier light (fundamental spatial mode) with frequency $\omega_0$ interacts with these thermally excited acoustic modes (say, with frequency $\omega_m$), it will be scattered into two sidebands with frequencies $\omega \pm \omega_m$. If the lower Stokes sideband field matches with a high-order cavity mode both in frequency and in spatial profile, then the beating between this lower sideband and the fundamental mode will exert a radiation pressure force back onto the acoustic mode and further excite it, thus the
acoustic mode becomes unstable. The instability gain is given by:

\[ R = \frac{2\Lambda I_0 \Omega_1}{m \omega_m L^2 \gamma_0 \gamma_1 \gamma_m}, \]  

(2.123)

which can be derived through multi-mode Hamiltonian given in [21].

This phenomenon was first predicted by Braginsky et al. [18]. This instability will be a challenge for the interferometer with high optical power. Recently, in LIGO and Gingin 80 meter cavity, this effect has been observed [23, 22]. Schemes for suppressing this instability were also proposed [24], such as using thermal modulation of the test masses to reduce this effect [22].

### 2.7.4 Modifying the dynamics of the test masses

Although the optical spring effect brings us some instability issues as listed above, it also allows us to modify the test masses dynamics for better response to the gravitational wave and pondermotive force, which may useful in improving the sensitivity of the detector.

One example is the so-called double-spring interferometer where double detuned injection is used to achieve a stable optomechanical dynamics of the test masses [25]. Properly design the parameters of the double-spring system can also allows us to cancel both the optical rigidity and the optical (anti-)damping. In this case, the pondermotive effect will leave only a negative inertia, which effectively increases the response of the probe to the gravitational waves [26].

Multiple optical springs also allows us to broaden the detection bandwidth. As we know, a single optical spring can only surpass the free mass SQL within a limited bandwidth around the optical spring frequency. However, if multiple optical springs are used, then the interferometer will have multiple mechanical resonances with the sensitivity dips below the free-mass SQL. The broadband sensitivity will be the locus of all these sensitivity dips. The flexibly shaping of the quantum noise spectrum of this kind of multi-carriers (springs) gravitational wave detectors has been studied recently [27].
2.8 General discussion of quantum measurement

The previous sections of this Chapter calculated the dynamics of optomechanical system and the sensitivity of these optomechanical devices. As we can see from these discussions, these sensitivities are always limited by the quantum noise and smart methods can be applied to surpass the quantum noise limited sensitivity. In this section, regardless of the details of the detector dynamics, we give a general discussion of quantum measurement process. The materials in this section based on the literatures [14] [28] [29].

As already briefly mentioned in the Introduction, a general measurement process consists of a probe which is directly interact with the external force and a readout device. The interaction between the readout device and the probe will convey the information of the probe to the detector and get measured, while at the same time, impose a back-action force to the probe. This back-action can cause the dynamical modification of the probe and an additional back-action noise. For example, the optical spring and the radiation pressure noise are the back-action dynamical modification and the back-action noise, respectively.

For a general displacement measurement process, the output quantity of the detector can be written as:

\[ \hat{Z}(t) = \hat{x}_0(t) + \hat{N}(t) + \int_{-\infty}^{t} dt' \chi_{xx}(t-t')[F(t') + G(t')] \]  

(2.124)

where the \( \hat{x}_0(t) \) describes the free-evolution of the probe displacement, \( \chi_{xx} \) is the response function of the probe to the external force environment and \( \hat{N}(t) \) is the sensing noise (e.g: shot noise in an optomechanical system).

Since \( \hat{Z}(t) \) is the measurement result, therefore it should satisfy the following commutation relation:

\[ [\hat{Z}(t), \hat{Z}(\tilde{t})] = 0 \]  

(2.125)

which leads to:

\[ [\hat{x}_0(t), \hat{x}_0(\tilde{t})] + [\hat{N}(t), \int_{-\infty}^{\tilde{t}} \chi_{xx}(\tilde{t} - t'')[F(t'')dt''] + \int_{-\infty}^{\tilde{t}} \chi_{xx}(\tilde{t} - t')F(t')dt' \hat{N}(\tilde{t})] \]

\[ + [\hat{N}_0(t), \hat{N}_0(\tilde{t})] + \int_{-\infty}^{\tilde{t}} \chi_{xx}(\tilde{t} - t')F(t')dt' \int_{-\infty}^{\tilde{t}} \chi_{xx}(\tilde{t} - t'')[F(t'')]dt'' = 0 \]  

(2.126)
where \([x(t), x(t')] = -i\hbar\chi_{xx}(t - t')\) and the commutation relations of \(\hat{N}, \hat{F}\) are determined as following:

Assuming that the devices has some tunable parameters \(\eta\)s such as the intra-cavity power in the optomechanical system, then \(\hat{N}\) and \(\hat{F}\) has some scaling relation with the parameter \(\eta\). It is of importance to notice that the scalings of \(\hat{N}\) and \(\hat{F}\) are always inverse to each other for satisfying the above commutation relation since there will be no term canceling \([\hat{x}_0(t), \hat{x}_0(\tilde{t})]\) if both \(\hat{N}, \hat{F} \propto \eta\) (or \(\eta^{-1}\)). In the optomechanical system, a good example of \(\eta\) is the intra-cavity power since the shot noise and radiation pressure force are proportional and inversely proportional to the intra-cavity power.

Bearing in mind that the above commutation relation should be satisfied to all the orders of \(\eta\), it is easy to see that the last two terms in Eq.(2.126) should be zero since they are \(\propto \eta^{\pm 2}\) while the second and third terms should cancel \([\hat{x}_0(t), x_0(\tilde{t})]\).

It is easy to check that for canceling \([\hat{x}_0(t), \hat{x}_0(\tilde{t})]\), we have to have 1) \(\chi_{xx}(t)\) is an odd function which is normally satisfied, 2) the commutation relation:

\[
[\hat{N}(t), \hat{F}(\tilde{t})] = i\hbar\delta(t - \tilde{t}).
\]  

(2.127)

Using the Heisenberg-Uncertainty relation by the Robertson-Schrodinger relation:

\[
\sigma_A^2 \sigma_B^2 \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}]_+ \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \right|^2 + \frac{1}{2\hbar} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2,
\]

where \([...]+\) is the anti-commutator, then Eq.(2.127) leads to:

\[
S_{NN}(\Omega)S_{FF}(\Omega) - |S_{NF}(\Omega)|^2 \geq \hbar^2.
\]  

(2.129)

On the other hand, the noise spectrum of \(\hat{x}\) is given by:

\[
S_{xx}(\Omega) = S_{NN}(\Omega) + 2\text{Re}[S_{NF}(\Omega)R_{xx}(\Omega)] + |R_{xx}(\Omega)|^2 S_{FF}(\Omega)
\]

(2.130)

Therefore, if \(S_{NF} = 0\), then the commutation relation between \(\hat{N}, \hat{F}\) does impose a constraint on the \(S_{xx}\):

\[
S_{xx}(\Omega) = S_{NN}(\Omega) + |R_{xx}(\Omega)|^2 S_{FF}(\Omega) \geq 2\hbar|R_{xx}(\Omega)|,
\]

(2.131)

for strain sensitivity of the gravitational wave detection, the above result leads to Eq.(2.114). However, if \(S_{NF} \neq 0\), this sensitivity limit does not apply. This gives
the hint that the way to surpass the SQL is to establish the correlation between $\hat{N}$ and $\hat{F}$—this is why the squeezed light and the signal-recycling mirror can be used to surpass the SQL. Alternatively, if the goal is only to surpass the free-mass SQL, another solution is to change the dynamics of the test mass through the back-action.

In summary, for surpassing the free-mass SQL, we have the following methods: 1) Squeezing injection with input filter cavity, 2) variational readout of the output field passing through the output filter cavity, 3) adding signal-recycling mirror to build up correlation and create pondermotive modification to the test mass dynamics.

### 2.9 Speed meter configurations

Current interferometers are all displacement sensors, that is, the readout signal is the displacement of the test masses under the driven of gravitational waves. However, modification to the original interferometer configuration allows us to measure the speed of the test mass motion.

The original motivation of measuring the test mass speed is due to the Standard quantum limit in the displacement measurement. Measuring the speed will not be limited by the standard quantum limit. The reason is obvious: the momentum (thereby the speed) of the test mass is a conserved dynamical quantity with zero commutator of momentum at different times. The measurement of the speed is called a quantum...
2.9. Speed meter configurations

The pioneering work by Braginsky and Khalili (for microwave system) [30, 31] and then by Purdue and Chen (for laser interferometer) [32, 33] proposed the realistic design of the speed meter. The key idea is to add an additional cavity to the main detector which allows the signal light sloshing between the additional cavity and the main detector. For example, in Fig. 2.19(a), the signal light at \( t_0 \) transfers out of the dark port will be reflected into the additional cavity, stored in the additional cavity for a time \( \tau \) and then reflected back to the dark port, combine with the signal at the later time \( t_0 + \tau \) in a destructive way, leaves a sensitivity to \( x(t_0 + \tau) - x(t) \propto v\tau \).

This idea was later on generalized to another kind of speed meter configuration—Sagnac interferometer [34, 35, 36], whose configuration topology is different from traditional Michelson interferometer since it encircles a closed area. The physical principle of this Sagnac interferometer speedmeter is fairly simple. The laser beam is separated by the Beam Splitter into two channels. Each channel obtains two displacement signals \( x_{EW} \) on the east-west arm and \( x_{NS} \) on the north-south arm at different times. When they propagate back to the beam splitter and combined at the dark port, they destructively interfere and lead to the final phase shift:

\[
\phi(t) \propto x_{EW}(t) + x_{NS}(t + \tau) - x_{NS}(t) - x_{EW}(t + \tau) \propto (v_{NS} - v_{EW})\tau. \tag{2.132}
\]

(Notice that the gravitational wave drives in the way that \( v_{NS} \) and \( v_{EW} \) has different direction).

A Hamiltonian approach to analyze the speed meter system can be as follows\(^3\). The speed meter consists of two resonant cavities and one of them is coupled with the test mass motion. Therefore the Hamiltonian can be written as:

\[
\hat{H} = \hbar \omega_0 (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) + \hbar \omega_s (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) + \hbar G_0 \hat{a}^\dagger \hat{x} + i\hbar \sqrt{2\gamma} (\hat{a}^\dagger \hat{a}_{in} - h.c)
\]

\[
+ \frac{\hat{p}^2}{2m}. \tag{2.133}
\]

Then the equations of motion can be derived as:

\[
\dot{\hat{a}} + \gamma \hat{a} = i\omega_s \hat{b} - iG_0 \hat{x}(t) + \sqrt{2\gamma} \hat{a}_{in},
\]

\[
\dot{\hat{b}} = -i\omega_s \hat{a}, \tag{2.134}
\]

\(^3\)there is also a very similar model in the next Chapter and for the analyzing using slowly varying amplitude method, see [33].
where $\bar{G}_0 = G_0\hat{a}$. Then the intra-cavity mode solved as:

$$\hat{a}(\Omega) = \frac{\bar{G}_0\Omega x(\Omega) + i\sqrt{2}\gamma\Omega\hat{a}_{in}(\Omega)}{\Omega^2 - \omega_s^2 + i\Omega\gamma}. \tag{2.135}$$

Therefore the detector output is given by:

$$\hat{a}_{out}(\Omega) = \left[-1 + \frac{2\gamma\Omega}{\Omega^2 - \omega_s^2 + i\Omega}\right] \hat{a}_{in}(\Omega) + \frac{\sqrt{2}\gamma\bar{G}_0\Omega x(\Omega)}{\Omega^2 - \omega_s^2 + i\Omega\gamma}. \tag{2.136}$$

The $\Omega\hat{x}(\Omega)$ is just the Fourier transformation of the velocity. The equation of motion for $x$ is given by:

$$m\ddot{x} = \frac{1}{2}mL\ddot{h}(t) - \bar{G}_0(\hat{a} + \hat{a}^\dagger). \tag{2.137}$$

Substituting the $G_0$ and $\bar{a}$ in terms of interferometer parameters $I_c$, $L$, $m...$, finally gives (suppose we measure the phase quadrature):

$$S_{hh}(\Omega) = \left[\frac{1}{K_{sm}} + K_{sm}\right] \frac{\hbar^2}{2} SQL \tag{2.138}$$

where $K_{sm} = 16\omega_0\gamma I_c/mcL[(\Omega^2 - \omega_s^2)^2 + \gamma^2\Omega^2]$.

Note that though the original motivation of design speed meter is to get rid of the limitation set by quantum mechanics, the speed meter is still also affected by quantum noise as shown in Fig. 2.19(d). This is due to the fact that the conserved momentum is no longer the mechanical momentum when the mechanical object is coupled to the electromagnetic field. The conserved momentum is actually the canonical momentum.

### 2.10 Conclusions

This chapter introduced the basic concepts of optomechanical system and the associated quantum noise of an interferometric gravitational wave detector. Basic formalism for the quantum noise calculation was established from both the input-output approach and the Hamiltonian approach. The transfer matrix method, as a systematic way of studying the transfer of optical field in the optomechanical system was discussed. The interesting dynamical behavior of optomechanical systems such as parametric instability and optical spring effect were also discussed. Most importantly, the concept of standard quantum limit and the method to surpass this limit, which motivates the design of different interferometer configuration were also discussed. This chapter is a conceptual basis and theoretical tools for the other chapters of this thesis.
Figure 2.19 – Speed meter configurations. (a) Purdue-Chen configuration; (b) Sagnac speed meter configurations; (c) effective cavity model; (d) sensitivity curve with squeezing factor: $e^{-2q} = 0.1$ using advanced LIGO parameters: Blue (magenta) curve is the sensitivity curve without (with) squeezed input.
Bibliography


Chapter 3

Extraction of energy from gravitational waves by laser interferometer detectors

3.1 Preface

This Chapter discusses the energy interaction between gravitational waves and the laser interferometer gravitational wave detectors. We show that the widely held view that the laser interferometer gravitational wave detector absorbs no energy from gravitational waves is only valid under the approximation of a frequency-independent optomechanical coupling strength and a pump laser without detuning with respect to the resonance of the interferometer. For a strongly detuned interferometer, the optical-damping dynamics dissipates gravitational wave energy through the interaction between the test masses and the optical field. For a non-detuned interferometer, the frequency-dependence of the optomechanical coupling strength causes a tiny energy dissipation, which is proved to be equivalent to the Doppler friction raised by Braginsky et al. This Chapter is based on the collaborative work with David. G. Blair, Chunnong Zhao, and William Kells. The relevant publication is: Class. Quantum Grav. 32 015003.

3.2 Introduction

Astronomically large fluxes of gravitational waves are expected to be detected by advanced laser interferometer gravitational wave detectors such as Advanced LIGO and Advanced Virgo now being commissioned [1][2]. For example, a binary black
hole coalescence at 1Gpc distance, which has peak luminosity $\sim 10^{23} L_\odot$, has a flux at the Earth of $\sim 10 \text{Wm}^{-2}\text{s}^{-1}$, which vastly exceeds the flux of all electromagnetic astronomical sources except the Sun. Clearly a large amount of energy is available in the signals. However the extremely weak interaction of gravitational wave detectors with gravitational waves makes detection very difficult and is the main reason that gravitational wave detection has not yet been accomplished.

This chapter addresses the question of how much energy can be extracted from gravitational waves. Traditionally only resonant mass gravitational wave detectors have been understood from an energy interaction viewpoint. Following Weber \cite{3}, the sensitivity of resonant mass detectors was estimated by considering the work done by an incident gravitational wave. However for laser interferometer gravitational wave detectors estimation has normally been based on considering the test masses as free masses that experience the gravitational wave spatial strain $h$ of a passing wave. If the masses are truly free, no energy is extracted from the wave. The free-mass approximation naturally leads to an approach that neglects the energy interaction. However, as emphasised by Saulson: “an important kind of understanding is lost in the neglect of such an essential physical concept as energy” \cite{8}.

The above discussion recalls the debate about the existence of gravitational waves that occurred from 1916 to 1957 \cite{4}. The proof of the reality of gravitational waves was eventually clarified by the rubbing sticks gedanken experiment presented by Feynman at the 1957 Chapel Hill Conference \cite{5}. Feynman showed that gravitational waves are able to deposit frictional energy and therefore cannot be a mathematical artifact. This leads to the viewpoint that a practical detector must be a transducer for gravitational waves, converting wave energy into electromagnetic energy, and amplifying it to enable it to be resolved against the inevitable background of instrument noise.

In this Chapter we discuss the fundamental question of energy absorption in relation to laser interferometer gravitational wave detectors by studying the energy flow. Our discussion is designed to illuminate fundamental principles in the context of new and more general interferometer designs, and to present results consistent with the concept of energy absorption cross section. Because laser interferometers operate in the quantum regime it is necessary to use a full quantum optomechanical analysis.
We will begin our analysis with a quantum analysis of the Doppler friction effect which arises from the frequency change of photons on reflection from a moving mirror. This was first discussed by Einstein in a thought experiment [6] and then rediscovered by Braginsky et al. [7] and Saulson [8]. Saulson showed that this effect indeed provides a viscous coupling to gravitational waves. While it is like the friction between Feynman’s sticks, the effect is small because it is a second order relativistic effect (∼(v/c)²), in which v is the speed of test mass motion and c is the speed of light. In this Chapter, we derive this friction from a quantum mechanical viewpoint and give a classical derivation. For a typical predicted wave and a LIGO-like interferometer, Saulson showed that the power absorbed is ∼ 10⁻⁴⁰W. If the primary gravitational wave signal was provided by this mechanism, the detector would need to have power gain that scales as the square of the ratio of optical frequency to gravitational wave frequency. However, we show here that the Doppler friction is not the primary signal source but a small and generally negligible additional term that can also be interpreted as a result of unbalanced Stokes and anti-Stokes sidebands. Our analysis of a toy model reveals that the power gain of the detector follows the usual form for parametric transducers, scaling linearly with the frequency ratio.

Through analysis of interferometers we will show that the free mass approximation is indeed an excellent approximation for detectors constructed to date, which all use a balanced pair of sidebands, but that it is not valid for more general detector configurations such as detuned [13] or double optical spring [18] interferometers in which the sidebands are unbalanced. In this case there can be strong optical damping which give rise to much stronger absorption of gravitational wave energy. Finally having demonstrated how energy absorption is related to sideband unbalance, and with view to stimulating new thinking about detectors, we mention the tilt interferometer as an example of a detector which has a single sideband and hence maximal sideband imbalance. [9]

The purpose of this Chapter is to address the single issue of energy absorption in the context of modern interferometer concepts, and to point out that gravitational wave energy absorption can be engineered into detector designs. We begin by considering a toy model to show that Doppler friction appears naturally as long as the
frequency dependence of the optomechanical coupling strength is included. In Section 3, we extend the discussion to general interferometer configurations, giving a rigorous derivation of energy dissipation through optical damping, which allows us to derive an energy absorption cross section which is analogous to that of resonant bar detectors.

### 3.3 Energy dissipated by Doppler friction

#### 3.3.1 Derivation of Doppler friction by Lorentz transformation

Doppler friction can be derived in several ways. For completeness, in this subsection, we give an exact derivation of Doppler friction using Lorentz transformations of the electromagnetic wave field.

We consider the same toy model shown in Fig. 2.7. For perfectly conducting surface with boundary position $X = x \cos \Omega t$, the expression for the reflective electric field $E_{\text{ref}}$ can be written as:

$$E_{\text{ref}}(X = x \cos \Omega_{gw} t, t) = -E_0 e^{ik_0 x + i\omega_0 t} e^{-2ik_0 x \cos \Omega_{gw} t}. \quad (3.1)$$

It could be more rigorous to take boundary conditions of Maxwell equations in the boundary’s rest frame. In the case of inertial mirror motion $x = vt$, imposing this rest frame boundary conditions, we have:

$$E_{\text{ref}} = \frac{1 - \beta}{1 + \beta} E_0 e^{i(k_0 x + \omega_0 t)\frac{1 - \beta}{1 + \beta}}, \quad (3.2)$$

with $\beta = v/c$. This is exactly what would be expected physically: The reflected wave is Doppler shifted in frequency by $\sim (1 - 2\beta)$, and the reflected waves’s Poynting vector is reduced by $\sim (1 - 4\beta)$. In this case, the reflected light has lost power, which means the light field does work on the mirror at rate $2c|E_0|^2 \beta$. Second, the receding mirror in the laboratory frame leaves a growing path of "stored" beam energy in its wake, effectively absorbing power $2|E_0|^2 \beta c$. The factor $(1 - \beta)/(1 + \beta)$ in the above equation accounts for these losses.

In case of periodic motion at frequency $\Omega$ which is of interest to gravitational wave detector, these power flows, to the order of $\beta$, will average to zero. Thereby expansion
to order $\beta^2$ is needed. For slow periodic motion, during the first half-cycle motion, the reflected beam energy passing a fixed reference plane is:

$$U_1 = P_0 \left( \frac{1 - \beta}{1 + \beta} \right) \left( \frac{x}{c\beta} + \frac{x}{c} \right),$$

(3.3)
during the second half cycle motion when the velocity of the mirror changes direction, it is given by:

$$U_2 = P_0 \left( \frac{1 + \beta}{1 - \beta} \right) \left( \frac{x}{c\beta} - \frac{x}{c} \right).$$

(3.4)

Then the correct power flow per cycle (the period is equal to $2x/c\beta$) is then:

$$P_{\text{cycle}} = \frac{U_1 + U_2}{2x/(c\beta)}.$$  

(3.5)

Substitute Eq. (3.3) and (3.4) into Eq. (3.5), keeping terms to the order $\beta^2$, and taking the average over one cycle, we have:

$$P_{\text{cycle}} = P_0(1 + 4\langle\beta^2\rangle_{\text{cycle}}) = P_0(1 + 2\frac{\Omega_{\text{gw}}^2}{c^2}x^2),$$

(3.6)

where $x = hL$. For better illustrating this effect, in the following, we will try to describe this phenomenon from the viewpoint of phonon-photon interactions.

### 3.3.2 The phonon-photon interaction interpretation

- **Optomechanical interaction model**—In laser interferometer gravitational wave detectors the basic physical process is the interaction between the light beam and the center of mass (CoM) degrees of freedom of the mirrors. The CoM motion of the mirrors, driven by gravitational waves, modulates the light beam and creates anti-Stokes (upper) and Stokes (lower) sidebands with frequency $\omega_c \pm \Omega$. Here, $\omega_c$ is the frequency of the carrier beam and $\Omega$ is the frequency of gravitational waves. This modulation process can also be treated in the quantum picture as generation of Stokes and anti-Stokes photons by scattering between the carrier photon and mechanical phonon, which can be described by the Feynman diagrams shown in Chapter 2, Fig. 2.15.

From these Feynman diagrams, it is clear that if the rate of the anti-Stokes process is higher than that of the Stokes process, more phonons will be absorbed through the anti-Stokes process than emitted through the Stokes process. In this case, there is a net flow of mechanical energy into the light field, and vice versa.
A widely-held view of this modulation process is that the Stokes and anti-Stokes sidebands are ‘balanced’. This means that the creation of an anti-Stokes sideband photon must be accompanied by the creation of a Stokes sideband photon. In other words, the Stokes and anti-Stokes photon generation rates are equal, implying that there is no net energy transfer between the mirror and the optical field when a free propagating laser field is modulated by mirror motion.

In this Section, by carefully analyzing a toy model, we will show that this viewpoint is only approximately correct. What has been neglected here is the Doppler friction discussed by Braginsky and Saulson in [7] [8]. We will derive the dynamics of the model, then give an intuitive interpretation of the result.

- **System Dynamics**— First, we review the derivation of the optomechanical coupling Hamiltonian using a toy model consisting of a light beam reflected by a mirror. The light beam, which is accompanied by quantum fluctuations, is given by:

\[
E_{in} = 2\sqrt{\frac{2\pi}{\omega_0}} E_0 \cos(\omega_0 t) + e^{-i\omega_0 t} \int_0^\infty \frac{d\Omega}{2\pi} \left( \sqrt{\frac{2\pi}{\omega_0}} \frac{\hbar}{S c} \hat{a}_+ e^{-i\Omega t} + \sqrt{\frac{2\pi}{\omega_0}} \frac{\hbar}{S c} \hat{a}_- e^{i\Omega t} \right) + e^{i\omega_0 t} \int_0^\infty \frac{d\Omega}{2\pi} \left( \sqrt{\frac{2\pi}{\omega_0}} \frac{\hbar}{S c} \hat{a}^\dagger_+ e^{i\Omega t} + \sqrt{\frac{2\pi}{\omega_0}} \frac{\hbar}{S c} \hat{a}^\dagger_- e^{-i\Omega t} \right).
\]  

(3.7)

Here, \(\omega_0\) is the pumping frequency of the steady part of the incoming light beam, \(E_0\) is the amplitude of steady field, \(S\) is its transverse cross-section and \(\hat{a}_\pm (\hat{a}^\dagger_\pm)\) are the annihilation (creation) operators of the optical field at the sideband frequencies \(\omega_\pm = \omega_0 \pm \Omega\). The first term here is the steady part of the optical field while the second part is the fluctuating part which has a continuous frequency distribution.

This optical field exerts a radiation pressure force \(F = 2|E_{in}|^2 S/4\pi\) and does work on the mirror. Therefore the interaction Hamiltonian is: \(H = -F \cdot x\). After substituting (1) and keeping the first order terms, we have:

\[
H_{int} = -\frac{\hbar E_0}{c} \int_0^\infty \frac{d\omega}{2\pi} \sqrt{\omega_0\omega} \left( \hat{a}_\omega e^{-i(\omega-\omega_0)t} + \hat{a}^\dagger_\omega e^{i(\omega-\omega_0)t} \right) \cdot x. 
\]

(3.8)

We will neglect the non-interesting steady part \(\propto |E_0|^2 \dot{x}\) since it can be balanced by exerting an external constant force. We also rewrite \(\hat{a}_{\omega_0\pm\Omega}\) to be \(\hat{a}_\omega\). This form of Hamiltonian can also be found in [13] [14].

It is important to notice that the coupling strength at frequency \(\omega\) is now proportional to \(\sqrt{\omega_0\omega}\), a factor that comes from the beating between the steady and
fluctuating optical amplitude. Usually, we treat $\omega \sim \omega_0$, thereby approximating the optomechanical coupling strength as a frequency independent constant. However clearly the coupling strength is not frequency independent.

The Heisenberg equations describing the evolution of the mirror-field system are given by:

$$\frac{d\hat{a}_\omega}{dt} = i\frac{E_0}{c}\sqrt{\frac{\omega}{\omega_0}}\hat{x}(t)e^{i(\omega-\omega_0)t},$$  \hspace{1cm} (3.9a)

$$\frac{dp}{dt} = \frac{\hbar E_0}{c} \int_0^\infty \frac{d\omega}{2\pi} \sqrt{\frac{\omega}{\omega_0}}[\hat{a}_\omega^\dagger(t)e^{i(\omega-\omega_0)t} + \hat{a}_\omega(t)e^{-i(\omega-\omega_0)t}],$$  \hspace{1cm} (3.9b)

$$\frac{dx}{dt} = \frac{p}{m}. \hspace{1cm} (3.9c)$$

First we consider the steady optical field and the fluctuating component due to modulation by the mirror motion, neglecting the quantum fluctuation field. Then the generation of sideband field with frequency $\omega_0 + \Omega$ is due to the mirror oscillation at frequency $\Omega$. We also assume an initial condition that at $t = 0$, there is no light except the pumping field at $\omega_0$. Solving the above Heisenberg equations, we have:

$$\hat{a}_\omega(t) = i\frac{E_0}{c}\sqrt{\omega/\omega_0} \int_0^t x(t')e^{-i(\omega-\omega_0)t'}dt',$$  \hspace{1cm} (3.10a)

$$F_{rad} = \frac{m}{2}\frac{d^2x}{dt^2} = \frac{d}{dt}\left[\frac{2\hbar E_0^2\omega_0}{c^2} \int_0^t dt' \int_0^\infty \frac{d\omega}{2\pi} \omega x(t') \sin(\omega - \omega_0)(t' - t)\right]. \hspace{1cm} (3.10b)$$

Equation (4b) can be derived by substituting (4a) into (3b). Substituting $\omega = \omega_0 + \Omega$ into (4b), we can separate out a force term dependent on $\Omega$:

$$F_{rad}^\Omega = -\frac{2\hbar E_0^2\omega_0}{c^2} \int_0^t dt' \int_{-\infty}^\infty \frac{d\Omega}{2\pi} \Omega x(t') \sin\Omega(t' - t). \hspace{1cm} (3.11)$$

Integrating by parts, and picking up the velocity dependent term which is related to the energy absorption, we have:

$$F_v = -\frac{2\hbar E_0^2\omega_0}{c^2} \dot{x}(t). \hspace{1cm} (3.12)$$

in which $\dot{x}(t)$ is the velocity of the test mass motion. Following the logic of the above argument, we can easily see that if we impose the approximation $\omega \sim \omega_0$, we will not have the $\Omega$-dependent radiation pressure term as in Eq.(3.11), and hence no velocity dependent force.
Suppose the free test mass is driven by a monochromatic gravitational wave with
can frequency $\Omega_{gw}$ and strain $h$, then we have $\dot{x}(t) = \Omega_{gw} h L \cos(\Omega_{gw} t)$ in the steady state. Then Eq. (3.12) can be written as

$$F_v(t) = -\frac{2P\Omega_{gw} c^2}{hL} \cos(\Omega_{gw} t)$$

(3.13)

in which $L$ is the distance from the equilibrium position of the mirror to a reference
point. The power is given by $P = h\omega_0|E_0|^2$ and the motion is at frequency $\Omega_{gw}$. Therefore Eq. (3.12) is exactly the Doppler friction force given in [8]. For the real
interferometer with optical resonant cavities, we only need to multiply the above
formula by the folding factor $N_{fold}$ as in [8].

The above discussion shows that the Doppler friction factor given by Braginsky
and Saulson emerges naturally in a Hamiltonian formalism as long as we avoid the
approximation of frequency independent optomechanical coupling strength. Thus
Doppler dissipation is a general phenomenon in interferometers. We now want to give
a more intuitive explanation of this dissipation within the quantum phonon-photon
scattering picture.

- **Sideband photon generation rate** — A more transparent way to investigate
our toy model is to calculate the sideband photon generation rate explicitly. When
the external GW force with frequency $\Omega_{gw}$ drives the motion of the test mass (or the
end mirror in our toy model) the optical fields in the sideband $\omega_0 \pm \Omega_{gw}$ appear. The
sideband photon generation rates $R_{\omega_0 \pm \Omega_{gw}}(t)$ are given by:

$$R_{\omega_0 \pm \Omega_{gw}}(t) = \langle i(t)|\hat{a}_{\omega_0 \pm \Omega_{gw}}(t)\hat{a}_{\omega_0 \pm \Omega_{gw}}(t)|i(t)\rangle.$$  

(3.14)

Here $|i(t)\rangle$ represents the sideband photon states. For the sidebands with initial
vacuum states, we have:

$$|i(t)\rangle = \frac{1}{i\hbar} \int_0^t \hat{H}_{int}(t')dt'|0\rangle.$$  

(3.15)

Here, the $H_{int}$ is given in (2). Substituting into (7), after some simple algebra,
integrating out all the $\delta$–funtions and take the average over one cycle $2\pi/\Omega_{gw}$ we have:

$$R_\omega = \frac{E_0^2}{c^2} \omega_0 \omega x^2 = \frac{P}{\hbar c^2} \omega x^2,$$  

(3.16)
where $x$ is the amplitude of harmonic motion. Substituting the two sideband frequencies, we obtain the difference of $\omega_0 \pm \Omega_{gw}$ sideband photon generation rates are:

$$R_{\omega_0 + \Omega_{gw}} - R_{\omega_0 - \Omega_{gw}} = 2\frac{P \Omega_{gw}^2}{\hbar c^2} x^2.$$  \hfill (3.17)

This is also the mechanical dissipation rate according to particle number conservation. Clearly, the $\omega_0 \pm \Omega_{gw}$ sideband-photon generation rates are only balanced under the approximation $\omega = \omega_0$. Thus Doppler friction effect can be explained as the result of unbalance between the Stokes and anti-Stokes process rates due to the frequency dependence of the optomechanical coupling constant ($\propto \sqrt{\omega_0 \omega}$).

Multiplying Eq. (3.17) by unit phonon energy $\hbar \Omega_{gw}$, we can express the mechanical power dissipated averaged over one cycle from the test mass as:

$$\mathcal{P}_{\text{diss}}^m = \hbar \Omega_{gw} [R_{\omega_0 + \Omega_{gw}} - R_{\omega_0 - \Omega_{gw}}] = 2 \hbar \frac{P \Omega_{gw}^2}{\hbar c^2} x^2 = 2 \frac{\Omega_{gw}^2}{c^2} x^2.$$  \hfill (3.18)

Expressing the sideband photon generation rate $R_{\omega_0 \pm \Omega_{gw}}$ as the generated sideband power over the energy of a single sideband photon: $W_{\omega_0 \pm \Omega_{gw}}/(\omega_0 \pm \Omega_{gw})$, we can express Eq.(3.18) as:

$$\frac{W_{\omega_0 + \Omega_{gw}}}{\omega_0 + \Omega_{gw}} - \frac{W_{\omega_0 - \Omega_{gw}}}{\omega_0 - \Omega_{gw}} = \frac{\mathcal{P}_{\text{diss}}^m}{\Omega_{gw}},$$  \hfill (3.19)

This is the classical form for the Manley-Rowe equation that was originally derived [11] to describe a lossless parametric amplifier using electrical circuit theory. In Appendix B, for completeness, we give the formal derivation of the Manley-Rowe equations using a Hamiltonian formalism.

To compare our power dissipation result with the classical derivation, we recall that the frictional power dissipated is given by $F_v v$. Then using $x(t) = hL \sin(\Omega_{gw} t)$, we can substitute in Eq.(3.12) and averaged over one cycle to obtain:

$$\langle P_v \rangle = - \frac{2 P \Omega_{gw}^2}{c^2} \hbar^2 L^2.$$  \hfill (3.20)

This result exactly matches with (12). This can be seen as a result of energy conservation: the energy flow out of the test mass should flow into the optical field. That is: $\langle P_v + \mathcal{P}_{\text{diss}}^m \rangle = 0$.

The above calculation carries over to conventional interferometers. The detectors commonly considered to have balanced sidebands are [8], in reality, not precisely
balanced. As already emphasised, the imbalance arises because of the frequency-dependence of the optomechanical coupling constant, and is the cause of Doppler friction.

In interferometers, sideband fields carry the gravitational wave information. The sidebands leak into the dark port of the interferometer and are measured by photodetectors. Each sideband contains the usual displacement dependent term, plus a velocity dependent Doppler friction component. The total power of these sideband optical fields $P_{\text{total}} = P_+ + P_-$ is given by:

$$P_{\text{total}} = \frac{2P}{c^2} \omega_0^2 x(t)^2 + \frac{2P}{c^2} \Omega_{gw}^2 x(t)^2,$$

where $P$ is the total circulating power. Both of the sideband terms and Doppler friction term are needed to describe the output of the interferometer. From this analysis it is clear that the sideband power is not amplified Doppler friction power as previous analysis suggested [8].

### 3.4 Energy absorption in general interferometer configurations

So far we have discussed Doppler friction in a simple but fundamental light-mirror interaction model. The energy absorption through Doppler friction is extremely small even when arm cavities like those used in LIGO type detectors are used to enhance the intracavity power. However, in more general interferometer configurations the energy absorption can be much larger. Many new interferometer configurations have been proposed, mainly to allow the free-mass standard quantum limit to be beaten through modifying the dynamics of test masses by opto-mechanical interaction. [4, 13, 10]. In the following discussion we will show that these configurations actually increase the energy absorption from the gravitational wave signal through the creation of unbalanced sidebands. This is achieved by detuning the laser frequency with respect to the resonance of interferometer. We note that although this detuning induced sideband unbalance is different from the unbalance that occurs in Doppler friction in terms of tunability, it can be understood on equal footing as arising from a variation
of density of electromagnetic field modes with respect to frequency.

To formulate the problem, we start from the basic structure of a general tunable interferometer configuration consisting of an optical cavity with a movable end mirror. For example, the differential mode of a signal recycling laser interferometer operating on the dark port shown can be mapped to a detuned cavity. This mapping relation was exactly proved by Buonanno and Chen [13] through treating the signal recycling cavity as an effective mirror. Variation of the tuning of the signal recycling mirror accommodates a range of interferometer configurations that includes as a special case the one considered in [8].

We will calculate the sideband photon generation rate and its associated mechanical damping factor which may be positive or negative. Detuning of the laser frequency from the cavity resonance leads to unbalanced sidebands. The feedback loop diagram shown in Fig. 3.1 explains how the detuning creates positive or negative damping. A monochromatic gravitational wave at frequency Ω acting on the interferometer causes the test mass to oscillate. The modulation generates Stokes and anti-Stokes sidebands inside the system. Both sidebands beat with the main beam and induce an AC radiation pressure force. However, the Stokes sideband radiation pressure is in phase with the velocity of mechanical motion, while the anti-Stokes radiation pressure has π phase shift relative to the velocity of mechanical motion. Thus the Stokes sideband represents positive feedback and can cause heating effect in which the optical energy will be pumped into the test mass motion and create a tiny increase of gravitational wave strength, while the anti-Stokes causes cooling (or damping). When an interferometer is unbalanced, its Stokes sideband becomes higher than its anti-Stokes sideband. This causes the net feedback driving of mechanical modes to be non-zero. By changing the detuning, we change the relative strength of cooling and heating radiation pressure forces.

To analyze the system quantitatively, we start by writing the Hamiltonian of the system in the reference frame of the beam splitter in terms of the optomechanical coupling constant $G_0 = \omega_0/L$ and the cavity bandwidth $\gamma$. The bandwidth $\gamma$ is given by $cT/4L$, where $T$ is the transmission of the input mirror, $L$ is the cavity length and $c$ is the speed of light. In the Hamiltonian, $\hat{a}$ and $\hat{a}_{\text{in}}$ are the annihilation
Chapter 3. Extraction of energy from gravitational waves by laser interferometer detectors

Figure 3.1 – Flow chart showing radiation feedback effects: a gravitational wave modulates the cavity field by driving the motion of the test mass, creating Stokes and anti-Stokes sidebands. The beating of these sidebands with the main laser beam creates a radiation pressure force which acts back on the test mass. The radiation pressure force differs by a phase of $\pi$ for the anti-Stokes and Stokes sidebands so that one contributes to the cooling and the other to the heating of the test mass motion.

The Hamiltonian can be written as [12]:

$$H = \hbar \omega_c \hat{a}^\dagger \hat{a} + p^2/2m + \hbar G_0 x \hat{a}^\dagger \hat{a} + i\hbar \sqrt{2\gamma} (\hat{a}_{in} \hat{a}^\dagger e^{-i\omega_0 t} - h.c.) - F_{GW} \cdot x. \quad (3.22)$$

Here, $-F_{GW} \cdot x$ is the work done by a gravitational wave tidal force on the test mass and $F_{GW} = (1/2)\hbar L$. From the above Hamiltonian, we obtain the linearized equations of motion:

$$m\ddot{x}(t) = -\hbar G_0[\hat{a}^\dagger(t) + \hat{a}(t)] + F_{GW}(t), \quad (3.23a)$$

$$\dot{\hat{a}}(t) + (\gamma - i\Delta)\hat{a}(t) = -iG_0 x(t) + \sqrt{2\gamma} \hat{a}_{in}(t). \quad (3.23b)$$

Here, $\tilde{G}_0 = G_0 \tilde{a}$ while $\tilde{a}$ is the steady amplitude of the cavity mode and $\Delta$ is the
3.4. Energy absorption in general interferometer configurations

The detuning factor defined as $\Delta = \omega_0 - \omega_c$. This detuning can be experimentally realized by tuning the reflectivity and phase of the signal-recycling mirror.

Taking a Fourier transform of the above equations, we obtain the following relations:

$$m\Omega^2 x(\Omega) = \hbar G_0 (\hat{a}(\Omega) + \hat{a}^\dagger(\Omega)) - F_{GW}(\Omega), \quad (3.24a)$$

$$\hat{a}(\omega) = \frac{\bar{G}_0 x(\omega)}{\omega + \Delta + i\gamma} + \frac{i\sqrt{2}\gamma a_n(\omega)}{\omega + \Delta + i\gamma}. \quad (3.24b)$$

The feedback processes shown in Fig.3.1 are described by the above equations. The first term on the right hand side of Eq.(3.24)(b) describes the effect of the external force driven mechanical motion on the optical field which in turn is fed back to the mechanical motion through the radiation pressure force given by the first term on the right hand side of Eq.(3.24)(a).

Now we can derive the sideband photon generation rate using perturbation methods. We divide the Hamiltonian into two parts: a) the unperturbed part consisting of the optical cavity and the mechanical oscillator, and b) the interaction term $\hbar G_0 x \hat{a}^\dagger \hat{a}$ representing the perturbed part. We use the Fermi Golden Rule, which states that the transition rate is proportional to the square of expectation value of perturbed Hamiltonian. Then we follow a method given by Marquardt et.al [14]. We define the (anti-)Stokes process rate as $R^{(a)}_S$. Since the mechanical (anti-)damping rate $\Gamma^{aS}$ measures the relative mechanical energy (gain) loss per unit time ($dE_m/E_m dt$) where $dE_m/dt = h\Omega R^{aS}$, it follows that:

$$\Gamma^S = -\frac{h\Omega R^S}{E_m}. \quad (3.25)$$

The energy change per unit time is just the unit phonon energy times the rate of the Stokes process. The same applies for anti-Stokes process. Then we have:

$$\Gamma^S = \frac{h\Omega R^S}{E_m} = \frac{h\Omega}{\hbar \Omega n_m} \left(\frac{\hbar G_0}{\hbar^2}\right)^2 |\langle f|x|i\rangle|^2 \langle \hat{a}\hat{a}^\dagger \rangle_{\omega=\Omega} = \bar{G}_0^2 \frac{\hbar}{2m\Omega (\Omega - \Delta)^2 + \gamma^2}. \quad (3.25)$$

In deriving this formula, we should substitute (17b) into $\langle aa^\dagger \rangle$ and the free evolution solution of (17a) into $|\langle f|x|i\rangle|^2$.

For the anti-Stokes process:

$$\Gamma^{aS} = \frac{h\Omega R^{aS}}{E_m} = \frac{h\Omega}{\hbar \Omega n_m} \left(\frac{\hbar G_0}{\hbar^2}\right)^2 |\langle f|x|i\rangle|^2 \langle \hat{a}\hat{a}^\dagger \rangle_{\omega=-\Omega} = \bar{G}_0^2 \frac{\hbar}{2m\Omega (\Omega + \Delta)^2 + \gamma^2}. \quad (3.26)$$
According to the particle conservation law, we have:

\[ \Gamma^m = \Gamma^{aS} - \Gamma^S. \]  

(3.27)

This tells us that the mechanical damping rate \( \Gamma^m \) is given by the difference between Eq. (3.25) and Eq. (3.26), which can be simplified to:

\[ \Gamma^m = -\tilde{G}_0^2 \frac{\hbar}{m} \frac{2\Delta \gamma}{[\Omega - \Delta]^2 + \gamma^2}[\Omega + \Delta]^2 + \gamma^2]. \]  

(3.28)

This result is equivalent to the optical damping factor given as the imaginary part of optical rigidity in [13, 12]. For a typical interferometer cavity used in gravitational wave detection, we plot the form of \( \Gamma^m \) as a function of cavity detuning \( \Delta \) in Fig. 3.2. When \( \Delta > 0 \), the optical damping factor is positive and the radiation pressure force fed back to mechanical motion has the form of \( F = m|\Gamma^m|\Omega x(\Omega) \). This force is in-phase with the velocity of mechanical motion, and induces the heating effect shown in Fig. 3.1. However when \( \Delta < 0 \), the radiation pressure force has the form of \( F = -m|\Gamma^m|\Omega x(\Omega) \). It differs by a \( \pi \) phase shift, thereby driving the mechanical motion in anti-phase which induces cooling. When there is no detuning (i.e. \( \Delta = 0 \)), then the transition rates become equal such that the total optomechanical damping rate is zero:

\[ \Gamma^S = \Gamma^{aS} = \tilde{G}_0^2 \frac{\hbar}{2m\Omega} \frac{2\gamma}{\Omega^2 + \gamma^2}. \]  

(3.29)

In this case the anti-Stokes sideband and Stokes-sideband rates are exactly canceled with each other and \( \Gamma^m = 0 \) (under the frequency-independent coupling strength approximation). This corresponds to the case illustrated in Fig. 3.1, in which the cooling and heating terms cancel each other. Under these circumstances the test masses can be treated as free masses except for the negligible Doppler friction. These results are not only correct for the near-resonance case, but also correct for more general cavity field structures such as the case discussed further below in which a single sideband is resolved and resonant with a high order mode. An analogous single sideband device that manifests the above behavior has recently been experimentally demonstrated by Chen et al. [21].

Note that the optical damping factor \( \Gamma^m \) is always associated with the optical spring effect, and for a system with optical-damping, the associated optical spring
3.4. Energy absorption in general interferometer configurations

Figure 3.2 – The relationship between the optical damping rate $\Gamma_m$ and the detuning $\Delta$. When $\Delta < 0$, the optical damping is positive and corresponds to optomechanical cooling while $\Delta > 0$, the optical damping is negative and corresponds to optomechanical heating. Here, we take the typical interferometer cavity parameters: cavity bandwidth is 100Hz, mirror mass is 40kg, intracavity photon number is $10^{20}$ and arm length $\sim 4$km.

The constant is always negative and hence can lead to instability. These relations were discussed by Buonanno et.al [13]. However, this instability problem can be solved using the double-optical spring configuration proposed by Rehbein et.al [18] or by an electronic feedback loop as proposed by Buonanno et.al [13]. Once stabilised, an optical spring interferometer operates like a resonant mass gravitational wave detector. The mechanical stiffness of this detector is contributed by the optical field.

For the optical spring interferometer, we can calculate the energy in the gravitational wave detector following results already derived long ago by Misner Thorne and Wheeler [22] for resonant bar detectors. The steady state vibration energy of the test masses is given by:

$$E_{kin} = \frac{mL^2h^2\Omega_{gw}^6}{16[\Omega_{gw}^2 - \omega_{opt}^2 + \Omega_{gw}^2\Gamma_{eff}^2]}.$$  \hspace{1cm} (3.30)

Here, the interferometer is treated as a mechanical quadrupole resonator with a resonant frequency $\omega_{opt}$ due to optical rigidity and $L$, $h$ and $\Omega_{gw}$ are the length of the arm cavity, strain and frequency of the gravitational wave. The effective test mass damping rate $\Gamma_{eff}$ is the resonator bandwidth. For the double optical spring interferometer the damping is contributed by the superposition of two optical springs (
Chapter 3. Extraction of energy from gravitational waves by laser interferometer detectors

Figure 3.3 – The average absorbed power in a double-spring interferometer as a function of optical spring frequency due to monochromatic gravitational waves of frequency 100Hz, 200Hz and 300Hz and $h \sim 10^{-23}$. We assume a LIGO type interferometer. The graphs show the absorbed power as a function of optical spring frequency for three different optical damping values. The red-dotted, blue-solid and black-dashed lines represent total optical damping $\Gamma_{\text{eff}}$ of 0.4, 40 and 400 s$^{-1}$ respectively.

$\Gamma_{\text{eff}} = \Gamma^m_1 + \Gamma^m_2$ \cite{18}. For a feedback stabilised optical spring interferometer the damping is contributed by the sum of optical anti-damping factor and damping rate contribute by the electronic feedback loop ($\Gamma_{\text{eff}} = \Gamma^m + \Gamma^{\text{feedback}}$) \cite{13}.

The steady state kinetic energy (assuming a continuous gravitational wave source) is dissipated internally at a rate $E_{\text{kin}} \cdot \Gamma_{\text{eff}}$, which is the energy absorption rate of the detector, or the average absorbed power from the gravitational waves by the detector \cite{22}. The average absorbed power, derived from Eq.\(3.30\) is shown in Fig.\(3.3\) for a sinusoidal gravitational wave of amplitude $10^{-23}$ in a typical advanced interferometer. From this, we can see that when the gravitational wave frequency is resonant with the mechanical resonant frequency of the mass-spring system, the absorbed power is much higher. It can be $10^{15}$ times higher than the Doppler friction power. Since the energy absorption cross section $\sigma = E_{\text{kin}} \cdot \Gamma_{\text{eff}}/\mathcal{F}$ with $\mathcal{F}$ is the gravitational wave energy flux, the $\sigma$ is also relatively high ($\sim 10^{-22}$m$^2$) when the mechanical resonant frequency of the interferometer is resonant with the gravitational wave frequency.

We should point out that the Doppler friction power also gets amplified when the detector is resonantly driven by gravitational waves, compared with the power levels
given in the last section. The reason is that the Doppler friction power, as shown in Eq. (3.12), depends on the oscillation amplitude. Estimates show that the Doppler friction power is still \( \sim 10^{12} - 10^{15} \) time smaller, and hence is still negligible.

Since detuning induced sideband imbalance leads to strong energy absorption from gravitational waves, it is interesting to consider other detector designs that can be dominated by a single sideband. One is the tilt interferometer [9]. Unlike normal interferometers that are designed to detect linear strains, this configuration is an optical cavity designed to detect tilt motions of the end mirror, which has angular amplitude equal to the strain amplitude. In this case, the test mass angular rotation can scatter the laser field into a TEM\(_{01}\) spatial mode with frequency \( \omega_0 - \Omega_{gw} \). Due to the asymmetric mode structure of long optical cavities, the upper sideband is suppressed. We present this as an example of a detector that satisfies the requirement of strongly unbalanced sidebands in which the unbalance is achieved without detuning. In practice the tilt interferometer fails to be a significant detector for the LIGO band because its characteristic length is set by mirror size instead of optical cavity length.

\section*{3.5 Conclusions}

Stimulated by previous work [7] [8], we have been able to obtain a unified understanding of gravitational wave energy absorption by laser interferometers, combining the intrinsic, but tiny, Doppler friction term with an optical damping term which can be tuned by varying the relative amplitude of the signal sidebands. The Doppler friction itself is explained by the Stokes and anti-Stokes sideband photon generation rates having a small unbalance caused by the frequency-dependent optomechanical coupling strength.

Our results show that to an excellent approximation, conventional laser interferometer gravitational wave detectors with balanced sidebands can be treated as lossless parametric transducers in which the energy absorption from gravitational waves is zero. However in a more general case where detuning or some other technique can cause the sidebands to be unbalanced, variation of the relative strength of the Stokes and anti-Stokes sidebands leads to strong optical damping and greatly
enhanced absorption of gravitational wave energy. While our results were derived for monochromatic gravitational waves, they are true in general because every wave can be treated as a superposition of monochromatic waves.

The ability to tune the real and imaginary optical spring stiffness through the relative sideband amplitudes is analogous to the variation in in-phase and anti-phase signal feedback used in electronic operational amplifiers, where variation of the feedback is used to change the gain and the input impedance of the amplifier. The design of gravitational wave detectors can be considered from the same viewpoint. The upper sideband acts as an optomechanical servo to null out the spring due to the lower sideband, canceling the optical spring stiffness. This directly changes the mechanical input impedance of the interferometer. Because of the extremely high gravitational wave impedance of free space $\sim c^3/G$, laser interferometers are always poorly impedance matched to gravitational waves. However by increasing the optical stiffness we increase the detector input impedance, thereby reducing the impedance mismatch with free space and increasing the fraction of absorbed energy.

Whether the results presented here implies any advantage for using gravitational wave detectors with more energy absorption still remains an open question since the benefit of such designs can only be determined by considering the signal to noise ratio. This question is beyond the scope of this Chapter. However, we point out that two designs considered to date, the double optical spring and the detuned resonant sideband extraction interferometer which are equivalent to resonant bar detector design with optical stiffness, achieve enhanced signal to noise ratio that peaks at the frequency where the energy absorption is maximized.

A logical extension of our results is that a detector with an enhanced lower sideband would emit gravitational wave power, acting like a point scattering source with negative cross-section for incoming gravitational waves so that the total gravitational wave power would be slightly enhanced.
3.6 Appendix: Derivation of Manley-Rowe relation

Here we give a formal proof of the Manley-Rowe equations in the multi-mode coupling system such as laser interferometer. We assume that the system consists of nonlinear coupled oscillators (modes): the carrier light, the test masses which are actually pendulums, down-converted sideband light and up-converted sideband light. The Stokes and anti-Stokes interactions are described by three-mode interaction terms. The Hamiltonian of this system can be written as:

\[ H = \frac{1}{2} \sum_{i=1}^{4} \frac{p_i^2}{m_i} + \frac{1}{2} \sum_{i=1}^{4} k_i q_i^2 + \lambda q_1 q_2 q_3 + \lambda q_1 q_2 q_4. \] (3.31)

Here, \( \lambda \) describes the strength of the parametric interaction. The \( q_i \) is the generalized coordinate for the \( i \)th oscillator. Then the equations of motion will be:

\[
\begin{align*}
\ddot{q}_1 + \omega_1^2 q_1 &= -\lambda q_2 q_3 - \lambda q_2 q_4, \\
\ddot{q}_2 + \omega_2^2 q_2 &= -\lambda q_2 q_3 - \lambda q_2 q_4, \\
\ddot{q}_3 + \omega_3^2 q_3 &= -\lambda q_1 q_2, \\
\ddot{q}_4 + \omega_4^2 q_4 &= -\lambda q_1 q_2,
\end{align*}
\] (3.32)
in which \( \omega_i \) here stands for mass-normalized frequency of each oscillator respectively. For our system, \( \omega_1 = \omega_c \) which is the frequency of the main laser; the \( \omega_2 = \Omega \) is the frequency of mechanical oscillation; the \( \omega_{3,4} = \omega_c \pm \Omega \) represent the frequency of two sideband light. By using the slowly-variational amplitude approximation, we have the following results for the evolution of the oscillator amplitudes \( A_i \),

\[
\begin{align*}
\dot{A}_1 &= -\frac{\lambda A_2 A_3}{4m \omega_c} \sin \varphi - \frac{\lambda A_2 A_4}{4m \omega_c} \sin \theta, \\
\dot{A}_2 &= \frac{\lambda A_1 A_3}{4m \Omega} \sin \varphi - \frac{\lambda A_1 A_4}{4m \Omega} \sin \theta, \\
\dot{A}_3 &= \frac{\lambda A_1 A_2}{4m (\omega_c - \Omega)} \sin \varphi, \\
\dot{A}_4 &= \frac{\lambda A_1 A_2}{4m (\omega_c + \Omega)} \sin \theta.
\end{align*}
\] (3.33)

The \( \varphi \) and \( \theta \) terms here are the phase angles of the complex amplitudes. These amplitude evolution equations will lead to:

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\omega_c A_1^2 + (\omega_c - \Omega) A_3^2 + (\omega_c + \Omega) A_4^2) &= 0, \\
\frac{1}{2} \frac{d}{dt} (\omega_c A_2^2 - (\omega_c - \Omega) A_3^2 + (\omega_c + \Omega) A_4^2) &= 0.
\end{align*}
\] (3.34)
Note that above equations relate the time variation of energy in different channels. This is the Manley-Rowe equations. Substituting the oscillator energy $\mathcal{E} = \omega_i^2 A_i^2 / 2$ into the above equation, we can obtain the following results:

$$\frac{W_p}{\omega_c} + \frac{W_+}{\omega_+} + \frac{W_-}{\omega_-} = 0,$$

$$\frac{W_s}{\Omega} - \frac{W_+}{\omega_-} + \frac{W_-}{\omega_+} = 0.$$  \tag{3.35a}\tag{3.35b}

The above equations are the Manley-Rowe equations used in the literature. Here, $W_-$ and $W_+$ represent the power (change rate of $\mathcal{E}_\pm$) in the lower sideband and upper sideband respectively due to the parametric interaction. This point was also mentioned in Manley-Rowe paper[6], in which they use the terminology "power flow". Following the same definition, $W_s$ describe the energy flow into (or out from) the mechanical oscillator in the parametric interaction process, $W_p$ is the energy change of the carrier light. When $\frac{W_-}{\omega_-}$ and $\frac{W_+}{\omega_+}$ have equal value, the $W_s$ is zero. This means that there is no net energy exchange between the light field and the test masses in the system.
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Chapter 4

Noise in laser interferometer gravitational wave detectors due to parametric interaction

4.1 Preface

As discussed in Section 2.7.3 of Chapter 2, triply resonant three mode interactions in long optical cavities have been shown to lead to enhanced scattering of carrier light by the ultrasonic acoustic modes of the test mass mirrors. At high optical power, this can lead to parametric instability (parametric gain $R > 1$) for a few acoustic modes with strong spectral and spatial overlap. Numerous $\sim 10^3$ acoustic modes of the test masses are predicted to have $R > 10^{-2}$. Experimental studies have shown that such modes also strongly scatter the carrier light, enabling very sensitive readout of the acoustic modes. In this Chapter, we are going to explore a possible noise source related to this parametric instability.

This 3-mode scattering from the thermal fluctuation of large population of ultrasonic modes would cause random changes in occupation number of the carrier light and cavity transverse optical modes. Because the thermal fluctuation time scale (set by the acoustic mode relaxation times) is typically a few seconds, the noise spectrum from thermally induced photon number fluctuations is strongly peaked at low frequency. The noise level depends on the acoustic mode structure and acoustic losses of the test masses, the transverse optical mode spectrum of the optical cavities and on the test mass temperature. A theoretical model investigating the possible effect of this noise will be discussed here which shows that in advanced detectors under construction three mode interaction noise is below the standard quantum limit, but
could set limits on future low frequency detectors that aim to exceed the free mass standard quantum limit. This Chapter is a joint research work collaborated with Li Ju, Chunnong Zhao, David Blair, Stefan Danilishin and Slawek Gras. The relevant publication is Class. Quantum Grav. 31 145002.

4.2 Introduction and background

Three mode interactions in laser interferometer gravitational wave detectors were first considered by Braginsky et al in 2001 [1, 2]. It was recognized that the scattering described by the diagrams in Fig. 2.15 was likely to occur in long optical cavities in which the free spectral range was comparable to the internal acoustic mode frequencies of the test masses. Since Braginsky’s prediction, there have been many further detailed studies that confirmed the model [3, 4, 5, 6] and a succession of experimental studies have verified the theory in detail [7, 8, 9, 10]. A complete review of the parametric instability in gravitational wave laser detectors is given by [11].

The parametric gain for three mode interactions (derived by [1], detail form is given in Eq. (4.17)) describes the fraction of ring down power of a test mass internal mode that is returned to the test mass by radiation pressure feedback. Positive gain, characterised by the parametric gain $R$, causes heating of a test mass mode and if $R > 1$ the feedback leads to instability. Negative gain ($R < 0$) corresponds to negative feedback, and causes the cooling of acoustic modes.

The feedback occurs as follows. An acoustic mode (frequency $\omega_m$) of the mirror scatters the carrier light (frequency $\omega_0$). This process would create two sidebands of frequency $\omega_0 + \omega_m$ and $\omega_0 - \omega_m$. However when the optical linewidths are narrow compared with $\omega_m$, the intrinsically asymmetric transverse mode structure of an optical cavity means that only one sideband is likely to be supported. The sideband light beats with the carrier light to create a fluctuating radiation pressure force on the mirror. If the lower sideband is supported, the scattering depletes the carrier, and the beat frequency is in phase with the acoustic mode, giving rise to positive feedback. If the upper sideband frequency is coincident with a transverse mode, the beat frequency is out of phase with the acoustic mode and energy is extracted from
4.3. Theoretical modelling

In this chapter, we are interested in the effect of both types of feedback processes. Our concern is not about parametric instability, which we assume can be always controlled by various techniques [12, 13, 14, 15, 16]. The interaction during the buildup of parametric instability enhances the scattering in the positive feedback domain and suppresses the scattering in the negative feedback domain. For example, for \( R = 0.1 \), the corrections for feedback will be about 10 percent, but if \( R \rightarrow 1 \), the feedback can cause large amplification.

In an advanced gravitational wave detector such as Advanced LIGO, Advanced Virgo [17, 18], the dense spectrum of acoustic modes scatters photons out of the main laser beam through Stokes and anti-Stokes processes associated with each acoustic mode. Both Stokes and anti-Stokes processes contribute to fluctuations of the photon number in the interferometer cavities. Using results from detailed modelling of parametric interactions in typical advanced interferometer cavities [6], it can be seen that out of 5500 acoustic modes in the frequency range 5kHz to 150kHz, there are \( \sim 10 \) acoustic modes with \( R > 1 \) and 70 acoustic modes with \( R > 10^{-1} \). These modes are all independent and will simultaneously contribute random light scattering of the carrier [19] as the thermally excited acoustic mode amplitude fluctuates.

Since in the thermal equilibrium state, all acoustic modes are thermally excited and in equilibrium with the reservoir, and hence have mean energy \( k_B T \). The coupling of each mode to the thermal reservoir is determined by its acoustic quality factor \( Q_i \). This also defines the rate of fluctuation of the mode amplitude. This chapter investigates whether the combined effect of this scattering leads to non-negligible photon number fluctuations.

4.3 Theoretical modelling

4.3.1 Simple estimation

It is possible to make a simple estimate of the magnitude of three mode interaction noise based on previous modeling of parametric instability [15]. We start from the fact that a parametric gain of unity requires the optical power loss from the cavity
to equal to the power dissipation of an acoustic mode. If the mode energy is \(k_B T\), then the energy loss is \(2\pi k_B T/Q\) per cycle, where \(T\) is the test mass temperature and \(Q\) is the acoustic mode quality factor. In general the energy loss from the optical system for a mode with the parametric gain \(R\) is \(2\pi R k_B T/Q\) per cycle. The energy lost from the cavity pump mode per cycle is larger than this by a factor of pumping mode frequency over acoustic mode frequency \(\omega_0/\omega_m\) since the acoustic power is derived from the frequency difference of the two optical modes (We assume that the transverse optical mode is rapidly lost from the optical system and hence does not contribute to the radiation pressure).

For the \(i_{th}\) acoustic mode of gain \(R_i\), the total power loss (energy loss per second) from the cavity would be

\[
\Delta P_{i\text{-total}} = \frac{2\pi R_i k_B T}{Q_i} \frac{\omega_0 \omega_m}{2\pi} = \frac{\omega_0 k_B T R_i}{Q_i}.
\]  

(4.1)

Making the approximation that the total power loss is uniform within the acoustic mode linewidth \(2\gamma_m = \omega_m/Q_i\), then the spectral density of the power loss is,

\[
\Delta P_i(\omega_m) = \frac{\omega_0 k_B TR_i}{\omega_m^2}.
\]  

(4.2)

The sum of all the interaction induced power loss is then:

\[
\Delta P = \omega_0 k_B T \sqrt{\sum_i \left(\frac{R_i}{\omega_m}\right)^2}. 
\]  

(4.3)

This power loss can be equated to the power fluctuation or equivalently to the photon number fluctuation in 1 second, which we use for comparison with quantum power fluctuations \(\Delta P_q\). With parameters \(T \sim 300K\), \(k_B \sim 10^{-23}\) and \(\omega_m \sim 10^6\), and with the fact that there are many acoustic modes with non-negligible parametric gain, the magnitude of photon fluctuation \(\Delta P_q/\hbar\omega_0\) can be seen to be \(\sim 10^8\). However the typical acoustic mode relaxation time is several seconds. In addition, modes with high values of \(|R|\) change the mode temperatures by the parametric amplification process. A spectral noise estimate is necessary to determine whether this noise has any impact on Advanced GW detectors.
4.3.2 Peturbative analysis of the equation of motions

- Hamiltonian and the equations of motions—Now we theoretically analyze the three-mode interaction noise due to the thermal motion of one mechanical vibration mode. Consider a surface-vibration-modulated high-order mode resonant inside the cavity (in the form of the Stokes/anti-Stokes sidebands of pumping in fundamental mode), then a multi-mode Hamiltonian can be given by:

$$\hat{H} = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \hbar \omega_1 \hat{b}^\dagger \hat{b} + \hbar G_0 \hat{x} (\hat{a}^\dagger \hat{b} + h.c) + \hbar G_c (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) \hat{X} + H_m + H_{ext}. \quad (4.4)$$

The \(\hat{a}, \hat{b}, \hat{x}, \hat{X}\) denote the pumping mode with resonant frequency \(\omega_0\), high-order mode with resonant frequency \(\omega_1\), mechanical displacement of the internal mechanical mode and center of mass motion of the test masses, respectively. The optomechanical three-mode interaction strength \(G_0\) is given by \(\sqrt{\Lambda \omega_0 \omega_1 / L^2}\) where \(\Lambda\) is the mode overlap factor and \(L\) is the cavity length. The three mode interaction noise induced by a three-mode parametric coupling between the thermal-fluctuating \(\hat{x}\) and \(\hat{a}, \hat{b}\) fields is given in the third term of Eq. (4.4). Only \(\hat{X}\) couples to gravitational waves. Radiation pressure forces given by the fourth term come from the coupling between the optical fields and the center of mass motion \(\hat{X}\) with strength \(G_c = \omega_0 / L\). Terms \(\hat{H}_m\) and \(\hat{H}_{ext}\) describe the free mechanical degrees of freedom and the coupling between our system and the external thermal/optical bath, respectively.

From the Hamiltonian, we can derive the equations of motion in the rotating frame with the frequency \(\omega_0\):

$$M \ddot{\hat{X}} = -\hbar G_c (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}), \quad (4.5a)$$

$$m (\ddot{\hat{x}} + \gamma_m \dot{\hat{x}} + \omega_m^2 \hat{x}) = -\hbar G_i (\hat{b}^\dagger \hat{a} + h.c) + \xi_{th}, \quad (4.5b)$$

$$\dot{\hat{a}} + \gamma_0 \hat{a} = -i G_c \hat{a} \hat{X} - i G_0 \hat{x} \hat{b} + \sqrt{2 \gamma} \hat{a}_m, \quad (4.5c)$$

$$\dot{\hat{b}} + (-i \Delta_1 + \gamma_1) \hat{b} = -i G_0 \hat{x} \hat{a} - i G_c \hat{b} \hat{X} + \sqrt{2 \gamma} \hat{b}_m, \quad (4.5d)$$

in which \(M\) is the mass of the whole mirror and \(m\) is the effective mass of the acoustic mode; \(\gamma_m, \omega_m\) and \(\xi_{th}\) are the mechanical bandwidth, resonant frequency and the stochastic thermal force which drives the internal modes. The \(\gamma_0\) and \(\gamma_1\) are the bandwidth of pumping mode and the high-order mode. The detuning of the pumping beam with respect to the resonance of the high-order mode is given by \(\Delta_1 = \omega_0 - \omega_1\).
The terms $\hat{a}_{in}$ and $\hat{b}_{in}$ are the vacuum fluctuations of the EM field injected into the fundamental mode and high-order mode channels respectively.

- **Perturbative analysis**—Since the $\hat{a}$ field is pumped by a strong laser, we can perturbatively solve the above problem. Firstly, taking the equilibrium position of the mechanical mode as the zero reference point the steady amplitude of $\hat{a}$ and $\hat{b}$ fields are given by:

\[
\bar{b} = -iG_0 \bar{a} \bar{x}_m = 0, \quad \bar{a} = \sqrt{2/\gamma_0} \bar{a}_{in} = \sqrt{2I_0/\gamma_0 \hbar \omega_0},
\]

with $I_0$ be the power of the pumping beam. Then the optical fields can be expanded as: $\hat{a} = \bar{a} + \hat{a}^{(1)} + \hat{a}^{(2)} + ...$ and $\hat{b} = \bar{b} + \hat{b}^{(1)} + \hat{b}^{(2)} + ....$ From Eq. (4.6), the zeroth-order of $\hat{b}$ does not exist. The radiation pressure force acting on the center of mass degree of freedom can be perturbatively expanded to second order as:

\[
M \ddot{\hat{X}} = -\hbar G_c |\bar{a}|^2 + \bar{a}(\hat{a}^{(1)\dagger} + \hat{a}^{(1)}) + \bar{a}^{(1)\dagger} \hat{a}^{(1)} + \bar{a}(\hat{a}^{(2)\dagger} + \hat{a}^{(2)}) + \hat{b}^{(1)\dagger} \hat{b}^{(1)} + h.c. \]  

The last two terms on the right hand side of Eq. (4.7) contribute to the three-mode-interaction noise. The first one is the stochastic loss of the pumping field $\hat{a}$ due to the thermal motion of the internal mechanical mode $x$. This comes from the second term on the right hand side of Eq. (4.5c).

Since the $\hat{x}$ and $\hat{b}$ are both contributed by thermal fluctuations that can be considered as a perturbation to the mean field value, this term is a 2nd-order perturbative quantity. The last term in Eq. (4.7) is the fluctuation of the high-order optical mode $\hat{b}$ due to the three mode interaction. It comes from the first term on the right hand side of Eq. (4.5d). Because of the strong pumping, $\bar{a}$ is of zeroth order. Hence this term is a 1st-order perturbative quantity. Note that, when $\hat{b}^{(1)}$ contributes to the radiation pressure noise (acting on the center of mass of the mirror) the radiation pressure force is again a 2nd-order perturbative quantity because there is no zeroth order $\hat{b}$-field as we can see from Eq. (4.7).

- **Parametric amplification**—Although the 3-mode interaction noise is a 2nd order effect, it is amplified due to the parametric gain. Therefore, the value of this 2nd order noise need to be carefully studied.
The parametric amplification process happens at the 1st order level, as discussed below. The set of the 1st order linearized equations of motion are as follows:

\[
\dot{\hat{a}}^{(1)} + \gamma_0 \hat{a}^{(1)} = -iG_c \bar{a} \hat{X} + \sqrt{2\gamma_0} \hat{a}_m^{(1)},
\]

\[
\dot{\hat{b}}^{(1)} + (-i\Delta_1 + \gamma_1)\hat{b}^{(1)} = -iG_0 \bar{a} \hat{a} + \sqrt{2\gamma_1} \hat{b}_m^{(1)},
\]

\[
M \ddot{\hat{X}}^{(1)} = -\hbar G_c \bar{a}(\hat{a}^{(1)} + \hat{a}^{(1)*}),
\]

\[
m(\ddot{\hat{x}}^{(1)} + \gamma_m \dot{\hat{x}}^{(1)} + \omega_m^2 \hat{z}^{(1)}) = -\hbar G_i \bar{a}(\hat{j}^{(1)} + h.c) + \xi^{th(1)}.
\]

The \( \hat{b}^{(1)} \) field contains the displacement signal of the internal mode as shown in Eq. (4.8b), which will be fed back to the three mode interaction term in Eq. (4.8d). This feedback process will modify the dynamics of the internal mode, create a shift of mechanical resonant frequency (usually negligible compared to \( \omega_m \)) and an optical damping term \(-m\gamma_{opt} \dot{\hat{x}}^{(1)}\) which can be positive or negative, corresponding to parametric heating or cooling. The parametric gain is defined as \( R = -\gamma_{opt}/\gamma_m \) [20] (for detail form, see Eq. (4.17)).

From the linearized Eq. (4.8b), we can derive the fluctuation field \( \hat{b}^{(1)} \) and \( \hat{a}^{(2)} \) (proportional to \( \hat{b}^{(1)} \hat{x} \)), which is the source of the three-mode interaction noise. From these two terms, we obtain the spectrum of the three-mode interaction noise. In the following section, we will discuss both terms.

### 4.3.3 Noise contributed by the thermally induced fluctuations of high order mode

The thermally induced fluctuations of the high order transverse mode \( \hat{b}^{(1)} \) field is determined by Eq. (4.8b). We choose a rotating frame at \( \omega_m \) to describe \( \bar{b}^{(1)} \) field and neglect the non-interesting quantum fluctuation part \( \hat{b}_m^{(1)} \) which is not relevant to this calculation and is negligible. Then Eq. (4.8b) can be rewritten as:

\[
\dot{\hat{b}}^{(1)} = -((\gamma_1 - i\Delta)\hat{b}^{(1)} - iG_0 \bar{a} \hat{a} e^{i\omega_m t}.
\]

In this case we define \( \Delta = \Delta_1 - \omega_m \), and solve it in the time domain:

\[
\hat{b}(t) = -iG_0 \bar{a} \int_{-\infty}^{t} e^{-i(\Delta + \gamma_1)(t-t')} e^{i\omega_m t'} \hat{x}(t') dt'.
\]
We can use the adiabatic approximation to move $e^{i\omega_m t}x(t)$ out of the integral since it is a slowly varying term. Also note that the frequency band of $e^{i\omega_m t}x(t)$ is of order $\gamma_m \ll \gamma_1$. This means that if $t'$ is deviated from $t$ a little bit, then the contribution is already very small due to its rapid decay. Therefore the main contribution comes from $t' \approx t$, that is:

$$\hat{b}(t) \approx -iG_0\tilde{a}\hat{x}(t)e^{i\omega_m t} \int_{-\infty}^{t} e^{-(i\Delta+\gamma_1)(t-t')}dt'. = \frac{iG_0\tilde{a}\hat{x}(t)e^{i\omega_m t}}{i\Delta+\gamma_1}$$  \hspace{1cm} (4.11)

Then we substitute this time domain solution into the last term of Eq.(4.7) to obtain:

$$F_b(t) \propto (\hat{b}(t)\hat{b}(t) + h.c) = -2\hbar G_0^2 G_c \left[ \frac{\bar{\alpha}^2}{\Delta^2 + \gamma_1^2} \right] \dot{x}(t)^2.$$  \hspace{1cm} (4.12)

The correlation function of this force noise $\text{Cov}_{FF}(t,t')$ is:

$$\text{Cov}_{FF}(t,t') = \frac{1}{2}\langle F_b(t)F_b(t') + F_b(t')F_b(t) \rangle = 4\hbar^2 G_0^2 G_c^2 \left[ \frac{\bar{\alpha}^4}{(\Delta^2 + \gamma_1^2)^2} \right] \langle \dot{x}(t)^2 \dot{x}(t')^2 \rangle.$$  \hspace{1cm} (4.13)

The two-point correlation function for $\dot{x}^2$ has been given in the Appendix of this chapter. What we need to notice here is that the mechanical response function is changed due to the optomechanical feedback discussed in the last section. Now, the optomechanically modified mechanical response function (neglecting the negligible shift of mechanical resonant frequency) is: $\chi_{\text{eff}}(\omega) = m(\omega_m^2 - \omega^2 + i(1 - R)\gamma_m\omega)$. Then the two-point correlation function of $x$ in the time domain becomes:

$$S_{xx}(\tau) = \frac{2k_BT}{m\omega_m^2(1-R)}e^{-(1-R)\gamma_m\tau/2}\cos(\omega_m\tau).$$  \hspace{1cm} (4.14)

Since the $x$ process is well approximated as a Gaussian process, therefore by Wick’s theorem, we have:

$$\langle \dot{x}(t)^2 \dot{x}(t + \tau)^2 \rangle = \frac{4k_BT^2}{m^2\omega_m^4(1-R)^2}(1 + e^{-(1-R)\gamma_m\tau/2}).$$  \hspace{1cm} (4.15)

Taking a cosine transformation to the frequency domain \[21\] and substituting it into Eq. (4.13), the spectrum of the force noise $F_b$ is given by:

$$S_{3\text{Mbh}}^{\text{FF}} = \left( \frac{16G_c}{(\Delta^2 + \gamma_1^2)^2} \right) \left( \frac{k_BT^2}{\omega_m^2(1-R)} \right) \left( \frac{\gamma_1^2\gamma_m^3 R^2}{\Omega^2 + (1-R)^2\gamma_m^2} \right).$$  \hspace{1cm} (4.16)

Now we make use of the fact that \[120\]:

$$R = \frac{\hbar G_0^2 \bar{a}^2}{m\omega_m \gamma_1 \gamma_m} = \frac{2\Lambda P_0 \omega_0}{m\omega_m L^2 \gamma_0 \gamma_1 \gamma_m},$$  \hspace{1cm} (4.17)
4.3. Theoretical modelling

in which the second equality is obtained by substituting the definition of \( G_0 \) with the relationship between external pumping laser power \( P_0 \) and intracavity photon number \( |\bar{a}_0|^2 \). In the case \( \Delta \ll \gamma_1 \) and substituting \( Q = \omega_m/\gamma_m \), the the three mode interaction noise spectral density becomes:

\[
S_{3MIb}^{3MB} = \frac{16\omega_0^2}{\gamma_1^2 L^2} \left( \frac{k_B^2 T^2}{Q^2 (1 - R)} \right) \left( \frac{\gamma_m R^2}{\Omega^2 + (1 - R)^2 \gamma_m^2} \right).
\]

where \( G_c \) has been replaced by \( \omega_0/L \).

4.3.4 Noise contributed by the thermally induced fluctuations of the pumping field

We now repeat a similar analysis for the noise induced through fluctuations in the pumping field. The situation is different because the high order mode force arises from the beating of two first order terms, while the pumping field fluctuation are caused by a beat between the zeroth order pumping field and the second order fluctuating field. However, we will see that this leads to a similar result.

The 2nd-order perturbative equation of motion for the pumping \( \hat{a} \) field is:

\[
\dot{\hat{a}}^{(2)} + \gamma_0 \hat{a}^{(2)} = -iG_c (\bar{a}\dot{X}^{(2)} + \hat{a}^{(1)} \ddot{X}^{(1)}) - iG_0 \hat{b}^{(1)}.
\]

The three-mode interaction contributes to the last term on the right hand side of the above equation. Its corresponding part in \( \hat{a}^{(2)} \) is defined to be \( \hat{a}^{(2)}_{th} \). The \( \hat{b}^{(1)}(t) \) field is given by Eq.(4.11) in the rotating frame with frequency \( \omega_m \). By turning back to the non-rotating frame (by adding a \( e^{i\omega_m t} \) factor), substituting into \(-iG_0 \hat{b}^{(1)}\) in Eq.(4.19), and using the fact that \( \gamma_m \ll \gamma_1 \), we obtain:

\[
\hat{a}^{(2)}_{th}(t) = -\frac{G_0^2 \bar{a}}{i\Delta + \gamma_1} \int_{-\infty}^{t} dt' e^{-\gamma_0 (t-t')} x^2(t') \sim -\frac{G_0^2 \bar{a}}{i\Delta + \gamma_1} \frac{x^2(t)}{\gamma_0}.
\]

The three mode interaction induced thermal radiation pressure force related to this \( \hat{a}^{(2)} \) field is given by:

\[
F_a(t) = \hbar G_c \bar{a}(\hat{a}^{(2)}_{th} + \hat{a}^{(2)}_{th}).
\]

Inserting Eq.(4.20), and using the correlation function of \( x(t)^2 \) shown in Eq.(4.15), we can obtain the three mode interaction noise due to \( \hat{a}^{(2)} \) as:

\[
S_{3MIA}^{3MB} = \left( \frac{16G_c^2}{\gamma_1^2} \right) \left( \frac{k_B^2 T^2}{\omega_m^2 (1 - R)} \right) \left( \frac{\gamma_m^3 \gamma_1^2 R^2}{\gamma_0^2 \gamma_m^2 + (1 - R)^2 \gamma_m^2} \right).
\]

\[(4.22)\]
In the case of $\Delta \ll \gamma_1$, substituting $Q = \omega_m/\gamma_m$, we obtain:

$$S_{3MIa}^{3MLa} = \left( \frac{16G_0^2}{\gamma_0^2} \right) \left( \frac{k_B^2 T^2}{Q^2 (1 - R)} \right) \left( \frac{\gamma_m R^2}{\Omega^2 + (1 - R)^2 \gamma_m^2} \right). \quad (4.23)$$

We see that in spite of the slightly different physics, Eq. 4.23 is identical to Eq. 4.16 except that $\gamma_1$ is replaced by $\gamma_0$.

### 4.3.5 Coherent cancelation

In the above sections, we have derived the spectrum of three mode noise that comes from the $\hat{b}^{(1)}$ and $\hat{a}^{(2)}$ terms that represent the fluctuations of the higher order mode field and the pumping field due to three-mode interaction. In reality, these two noises are not independent of each other but strongly correlated. This point can be easily seen from the fact that in the three-mode parametric interaction process shown in Fig. 2.15, the annihilation (creation) of a pumping field $\hat{a}$ photon will be accompanied with the creation (annihilation) of a high-order mode photon $\hat{b}$. Therefore, an increase of the radiation pressure force contributed by high-order mode will be accompanied with a decrease of the radiation pressure force contributed by the pumping mode. Therefore, the true radiation pressure force noise spectrum should be:

$$S_{3MI}^{3MI} = \langle [F_a(t) - F_b(t)] [F_a(t') - F_b(t')] \rangle = \langle F_a(t) F_a(t') \rangle + \langle F_b(t) F_b(t') \rangle - \langle F_a(t) F_b(t') \rangle - \langle F_b(t) F_a(t') \rangle. \quad (4.24)$$

The first two terms in Eq. 4.24 were calculated in the above two subsections. The calculation of the last two terms is straightforward, following the same method, we obtain:

$$\langle F_a(t) F_b(t') \rangle = \langle F_b(t) F_a(t') \rangle = \left( \frac{16G_0^2}{\gamma_0 \gamma_1} \right) \left( \frac{k_B^2 T^2}{Q^2 (1 - R)} \right) \left( \frac{\gamma_m R^2}{\Omega^2 + (1 - R)^2 \gamma_m^2} \right). \quad (4.25)$$

The final result for the total noise is then:

$$S_{3MI}^{3MI} = \left( \frac{16G_0^2 k_B^2 T^2}{Q^2 (1 - R)} \right) \left( \frac{\gamma_m R^2}{\Omega^2 + (1 - R)^2 \gamma_m^2} \right) \left( \frac{\gamma_0 - \gamma_1}{\gamma_0 \gamma_1} \right)^2. \quad (4.26)$$

Eq. 4.26 shows that if all higher order modes had the same linewidth as the pumping mode, this noise would fall to zero. It is useful to compare this result with the radiation pressure noise due to the quantum vacuum fluctuations of electromagnetic
field in the low frequency region where $\Omega \sim (1 - R)\gamma_m$. It is realistic to assume that $\gamma_1$ is several times larger than $\gamma_0$ since higher order modes usually have higher losses due to their larger spot size. This leads to an upper limit for $S_{\text{FF}}^{\text{3M}}$, given by:

$$
\frac{S_{\text{FF}}^{\text{3M}}}{S_{\text{FF}}^{\text{rad}}} \sim \left( \frac{ck_B^2 T^2 \omega_0}{LhQ^2 P_c \gamma_1 \gamma_m} \right) \left( \frac{R^2}{(1 - R)^3} \right). \quad (4.27)
$$

Here $P_c$ is the intra-cavity pumping power. In the following section, we will use Eq. (4.26) to calculate the noise from a large number of acoustic modes.

Before closing the theoretical modeling of the three-mode noise spectrum, several points need to be clarified:

- **Linearity of the model**—For completeness, we discuss the validity of the perturbation method used here. It is clear from Eq. (4.26), when $R \to 1$, the spectrum goes to infinity, which indicates that the perturbation method is no longer valid. This is due to the fact that the optomechanical interaction heats or cools the mechanical degrees of freedom. Solving this problem fully requires simulation of the non-linear dynamics of the optomechanical system [10]. However, if the root mean square motion is small compared with the cavity’s linear dynamical range, i.e:

$$
\sqrt{x^2} < \lambda/F,
$$

($\sqrt{x^2}$ is the root mean square of mirror displacement due to the thermally excited acoustic mode and $F$ is the cavity finesse), then the system will be within the linear dynamical region and the perturbation method will be valid. For an advLIGO type detector, we have $\lambda/F \sim 10^{-9}$m. Since $\sqrt{x^2} \sim 3 \times 10^{-11}$m, even for $1 - R \sim 10^{-8}$ the system is well within the linear dynamical region. Thus we conclude that the perturbation method is valid in our discussion.

- **The unstable modes**—Last but not least, we want to emphasize that our analysis is only valid for $R < 1$ (the system is under the parametric instability threshold) since it is obvious that Eq. (4.26) will become negative when $R > 1$. In real design, the unstable mode will be controlled and suppressed for stable running of the interferometer. For the case when $R > 1$ and there is no control system to suppress the instability, recent experiment [10] and simulation [10, 22] show that the mechanical internal mode of the mirror will have a saturated motion phase after the initial stage of exponential rising, the analysis of the three-mode noise in this interesting region
will be analyzed in the future using different methods since nonlinear effect need to be accounted [10, 22].

4.4 Simulation results and discussion

To estimate the parametric gain for test mass acoustic modes it is necessary to model all the acoustic modes in the frequency range from 5kHz to 150kHz for typical test mass dimensions. We used finite element modeling based on typical Advanced LIGO type test masses to model the acoustic mode structure of many thousands of acoustic modes combined with detailed modeling of the transverse optical mode structure of interferometer cavities [5].

We modeled fused silica test masses similar to those used in Advanced LIGO and Advanced Virgo. The detailed parameters for the fused silica test mass and optical cavity configuration can be found in reference [5]. The key parameters are $P_{\text{cav}} = 830$ kW, acoustic $Q$ factors $(0.5 \sim 1) \times 10^7$, coating loss limited. Based on this data we were able to predict the parametric gain of each test mass mode in an interferometer and determine the dependence of parametric gain on test mass mirror radius of curvature. For comparison, we also included analysis of test masses similar to those proposed for use in KAGRA [23] (although as a cryogenic interferometer, the noise in KAGRA would be reduced). Sapphire test masses have roughly 5 times lower acoustic mode density in the frequency range of interest, so that the number of relevant acoustic modes is $\sim 1000$ in an interferometer with sapphire test masses, compared with $\sim 5500$ for an interferometer with fused silica test masses.

The radius of curvature (RoC) in a long interferometer varies with absorbed power, can be tuned by thermal compensation. Changes in RoC change the optical cavity mode structure and strongly tune the parametric gain of individual modes. However, we find that the statistical distribution of the number of modes with different gain is similar over quite a large range of radii of curvature. An example is the analysis of the gain distribution of modes in sapphire test masses shown in Fig. 4.1. This figure means that a good estimation of three mode interaction noise can be obtained by summing over acoustic modes using a nominal value for radius of curvature. However
4.4. Simulation results and discussion

Figure 4.1 – Histogram of the number of modes of one mirror with different parametric gain at different radius of curvature for typical advanced detector configuration (sapphire test mass). Although the parametric gain depends strongly on mirror radius of curvature for particular acoustic modes, the overall statistical distribution of parametric gain is similar over large range of the Radius of Curvature (RoC).

The large difference in mode density means that the total noise is likely to be higher for fused silica.

Because the total noise depends on $R^2/(1 - R)^3$, it is clear that the noise will be dominated by any mode that has $R \to 1$. However, if $R < 1$, the noise can be roughly estimated from the cumulative value of $R$ summed over all modes. Fig. 4.2 gives model results showing the cumulative sum of $R$ for positive gain modes of a fused silica test mass at RoC of 2191m. This model included one mode with $R \sim 50$ and several modes with $R > 1$. Such modes would normally be controlled using damping or feedback and hence would not contribute to the cumulative gain in an operating interferometer. The modes with gain between 1 and $10^{-2}$ contribute to total gain $\sim 20$, while modes with gain $10^{-2} \sim 10^{-3}$ contribute $R \sim 3$. Modes with $R < 10^{-3}$ make a negligible contribution to the noise.

It is interesting to examine the effect of different $R$ values on the radiation pressure force fluctuation for a single mechanical mode (Eq. (4.26)). Consider a mode of frequency 50kHz with a $Q$-factor of $1 \times 10^7$. The radiation force fluctuation spectral
Figure 4.2 – Cumulative R for positive gain of a fused silica test mass at RoC of 2191m. It can be seen that modes with $1 < R < 0.1$ contribute $\sim 60$ percent to the total $R$ while $R < 1 \times 10^{-3}$ has negligible contribution.

density $S_{\text{FF}}^{\text{omi}}$ is as shown in Fig. 4.3. It can be seen that the fluctuations are at very low frequency due to the long relaxation time of the high mechanical $Q$-factor. The noise level increases strongly as $R$ approaches to unity. Normally $R$ would be expected to be controlled below unity to prevent parametric instability. However, if thermal drifts in the interferometer cause a particular mode to drift towards instability, then we would expect $R$ to increase asymptotically to 1. Fig. 4.3 shows the large increase of noise that would occur as $1 - R$ falls to $10^{-7}$. This would not affect interferometer operation but is an interesting phenomenon in its own right.

It is shown in Fig. 4.3 that the noise for one mode only approaches level of the quantum radiation pressure noise for $1 - R < 10^{-2}$. In the rare situation of parametric gain drifting towards 1, the three-mode interaction noise only exceeds the quantum radiation pressure noise spectrum much below the advanced LIGO frequency band (see Fig. 4.3).

Fig. 4.4 shows the contribution to three mode interaction noise from the combined effects of all acoustic modes used in our model. The curves show the relative noise contribution from a) all modes with $0.9 > R > 0.1$, b) all modes with $0.1 > R > 0.01$,
Figure 4.3 – The radiation pressure noise spectrum due to three-mode interactions for the advanced detector configuration considered in [14]. (a) Dependence of the noise force spectral density on parametric gain for a single mechanical mode. The dashed line is the radiation pressure noise for advanced detectors. (b) Force spectral density at 0.1Hz as a function of $1 - R$. Both models assume a mechanical mode frequency of 50kHz and $Q$-factor of $10^7$. Values of $1 - R$ as small as $10^{-7}$ are considered because it is likely that drift in operating conditions of an interferometer could cause the gain to pass continuously from the stable regime $R < 1$ to the unstable regime $R > 1$. 
and c) all modes with $0.01 > R > 0.001$. It is clear that three mode interaction noise is likely to be dominated by the highest gain modes. The very large population of lower gain modes has a small contribution to the total noise.

For a fused silica test mass in advanced detector configuration, there could be about 10 modes with $R > 1$. We assume that all the unstable modes are controlled to $R < 1$. It is likely that a few modes could have parametric gain be very close to unity. However it can be seen that even with $R \to 1$, the noise level due to three mode interactions is still low compared with radiation pressure noise in the frequency band of interest for advanced detectors.

Changing the test mass acoustic loss has a direct effect on the parametric gain distribution (Fig. 4.1) as well as the spectral distribution. If the $Q$-factor of ultrasonic acoustic modes were reduced as a whole, say by using an acoustic damper scheme (opposed to selective damping a few high parametric gain modes), then the total level of 3 mode interaction noise will drop. Similarly, increasing the power level increases the parametric gain, and hence increases the three mode interaction noise.
4.5 Conclusion

This chapter shows that three mode interactions give rise to a new source of intensity fluctuations in gravitational wave detectors. The noise is due to the effect of light scattering off test mass acoustic modes. Analysis of typical test masses for advanced gravitational wave detectors showed that the acoustic modes contribution to the noise spectrum peaks at frequencies well below the lower frequency limit of ground based gravitational wave detectors. The amplitude of three-mode interaction noise is substantially smaller than the quantum noise within the advanced detector frequency range. Thus the three-mode interaction noise will not limit the current generations of detectors but it should be considered when designing advanced quantum measurement schemes that can surpass the free mass standard quantum limit. It should also be considered when designing future detectors with larger test masses and sensitivity in the 1-5Hz range.

4.6 Appendix: Two-points correlation for $x^2$

In the main text of this chapter, we frequently use the two-points correlation function for $x^2$. In this appendix, we give an derivation of it.

As we know, the $x(t)$ is given by:

$$x(t) = \int_{-\infty}^{t} dt' G(t - t') \xi_{th}(t'),$$

(4.29)

in which the $\xi_{th}(t)$ is the thermal fluctuation force and

$$G(t - t') = \frac{1}{m \omega_m} \sin \omega_m (t - t') e^{-\gamma_m (t-t')/2},$$

(4.30)

is the Green’s function for a damped mechanical oscillator.

Substitute it into $\langle x^2(t) x^2(t + \tau) \rangle$, we have:

$$\langle x^2(t) x^2(t + \tau) \rangle = \frac{D^2}{m^4 \omega_m^4} \int_{-\infty}^{t} \int_{-\infty}^{t+\tau} dt' dt'' dt'_\tau dt''_\tau e^{-\frac{\gamma_m}{2} (t-t')-\frac{\gamma_m}{2} (t-t'')}$$

$$e^{-\frac{\gamma_m}{2} (t+\tau-t'_\tau)-\frac{\gamma_m}{2} (t+\tau-t''_\tau)} [\sin \omega_m (t - t') \sin \omega_m (t - t'')]$$

$$\sin \omega_m (t + \tau - t'_\tau) \sin \omega_m (t + \tau - t''_\tau)] \left\langle \xi(t') \xi(t''_\tau) \xi(t'_\tau) \xi(t''_\tau) \right\rangle$$

(4.31)

in which $D = 4m \gamma_m k_B T$. 

Chapter 4. Noise in laser interferometer gravitational wave detectors due to parametric interaction

Assuming that thermal force is a Gaussian process, therefore it is straightforward to see that:

\[ \langle \xi(t')\xi(t'')\xi(t''\prime)\xi(t''\prime\prime) \rangle = \delta(t'-t'')\delta(t''\prime-t''\prime) + \delta(t'-t''\prime)\delta(t''-t''\prime) + \delta(t'-t'')\delta(t''\prime-t''\prime). \quad (4.32) \]

Substituting Eq. (4.32) into Eq. (4.31) and after a complicated but straightforward calculation, we have:

\[
\begin{align*}
\langle x^2(t)x^2(t + \tau) \rangle &= \frac{D^2}{m^4\omega_m^4} \left[ \frac{4\omega_m^4 Q^2}{(\gamma_m^2 + 4\gamma_m\omega_m^2)^2} + \frac{4\omega_m^4 Q^2}{(\gamma_m^2 + 4\gamma_m\omega_m^2)^2} e^{-\gamma_m\tau} \cos^2 \omega_m\tau \right] \\
&+ 2\frac{D^2}{m^4\omega_m^4} \left[ \frac{2\omega_m^2}{(\gamma_m^2 + 4\omega_m^2)^2} \right] e^{-\gamma_m\tau} \sin^2 \omega_m\tau + \frac{4\omega_m^2 Q}{(\gamma_m^2 + 4\omega_m^2)^2} \right] \left[ 1 + 2e^{-\gamma_m\tau} \cos^2 \omega_m\tau \right].
\end{align*}
\]

(4.33)

Since our \(Q\)-factor is high, therefore we can reexpress the above formula as expansion of \(Q\):

\[
\begin{align*}
\langle x^2(t)x^2(t + \tau) \rangle &= \frac{D^2}{m^4\omega_m^4} \left[ \frac{4\omega_m^4 Q^2}{(\gamma_m^2 + 4\omega_m^2)^2} + \frac{4\omega_m^4 Q^2}{(\gamma_m^2 + 4\omega_m^2)^2} e^{-\gamma_m\tau} \cos^2 \omega_m\tau \right] \\
&+ 2\frac{D^2}{m^4\omega_m^4} \left[ \frac{2\omega_m^2}{(\gamma_m^2 + 4\omega_m^2)^2} \right] e^{-\gamma_m\tau} \sin^2 \omega_m\tau + \frac{4\omega_m^2 Q}{(\gamma_m^2 + 4\omega_m^2)^2} \right] \left[ 1 + 2e^{-\gamma_m\tau} \cos^2 \omega_m\tau \right].
\end{align*}
\]

(4.34)

Apparently, the \(Q^2\)-terms are dominate, thereby we have:

\[
\langle x^2(t)x^2(t + \tau) \rangle = \frac{D^2}{m^4\omega_m^4} \left[ \frac{4\omega_m^4 Q^2}{(\gamma_m^2 + 4\omega_m^2)^2} \right] \left[ 1 + 2e^{-\gamma_m\tau} \cos^2 \omega_m\tau \right].
\]

(4.35)

The further approximation that \(\omega_m \gg \gamma_m\) will lead to:

\[
\langle x^2(t)x^2(t + \tau) \rangle = \frac{D^2}{m^4\omega_m^4} \left[ \frac{4\omega_m^4 Q^2}{16\omega_m^2} \right] \left[ 1 + 2e^{-\gamma_m\tau} \cos^2 \omega_m\tau \right].
\]

(4.36)

Representing \(D\) using \(D = 4m\gamma_m k_B T\), the above formula can take the form of:

\[
\langle x^2(t)x^2(t + \tau) \rangle = \frac{4k_B^2 T^2}{m^2\omega_m^4} \left[ 1 + 2e^{-\gamma_m\tau} \cos^2 \omega_m\tau \right].
\]

(4.37)

The high frequency \(2\omega_m\) part can be neglected, therefore the mechanical displacement four-point correlation function can be expressed as:

\[
\langle x^2(t)x^2(t + \tau) \rangle = \frac{4k_B^2 T^2}{m^2\omega_m^4} \left[ 1 + e^{-\gamma_m\tau} \right].
\]

(4.38)

Here we should notice that this result tells us that the stochastic process \(x(t)\) is a Gaussian process under the approximation we made above. The reason is in this case \(x(t)\) obeys the Wick theorem:

\[
\begin{align*}
\langle x(t)x(t')x(t'')x(0) \rangle &= \langle x(t)x(t') \rangle \langle x(t'')x(0) \rangle + \langle x(t)x(t'') \rangle \langle x(t')x(0) \rangle \\
&+ \langle x(t)x(0) \rangle \langle x(t')x(t'') \rangle.
\end{align*}
\]

(4.39)
Since the two point function of mechanical displacement for a high-Q oscillator in a thermal bath in the time domain can be written as:

\[
\langle x(t)x(t + \tau) \rangle = \frac{2k_BT}{m\omega_m^2} e^{-\frac{\gamma_m}{2}\tau} \cos \omega_m \tau.
\] (4.40)

Then (A.11) can be written as:

\[
\langle x^2(t)x^2(t') \rangle = \langle x^2(t) \rangle \langle x^2(t + \tau) \rangle + 2\langle x(t)x(t + \tau) \rangle^2
\]

\[
= \frac{4k_BT^2}{m^2\omega_m^4} \left[ 1 + e^{-\gamma_m\tau} \cos^2 \omega_m \tau \right] = \frac{4k_BT^2}{m^2\omega_m^4} \left[ 1 + e^{-\gamma_m\tau} (1 + \cos(2\omega_m \tau)) \right].
\] (4.41)

Neglect the high frequency $2\omega_m$ term, we recover the result (A.10).
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due to parametric interaction
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Chapter 5

Narrowing the cavity bandwidth via Optomechanical Interaction

5.1 Preface

In Chapter 2, the basic quantum optomechanical system was discussed. This chapter studies about using optomechanical interaction to narrow the bandwidth of filter cavities for achieving frequency-dependent squeezing in advanced gravitational-wave detectors, inspired by the idea of optomechanically induced transparency (OMIT). This idea can in-principle allow us to achieve a cavity bandwidth on the order of one hundred Hz using small-scale cavities. Additionally, in contrast to a passive Fabry-Pérot cavity, the resulting cavity bandwidth can be dynamically tuned, which is useful for adaptively optimizing the detector sensitivity when switching amongst different operational modes. We will also discuss the experimental requirement for the realization of this proposal. This Chapter is a joint research effort by Haixing Miao, Jiayi Qin, Chunnong Zhao, Stefan Danilishin, Robert Ward, Zach Korth, Yanbei Chen and David Blair. The relevant publication is Phys. Rev. Lett. 113, 151102 (2014) and Phys. Rev. A 89, 041802(R) (2014).

5.2 Introduction and background

In Chapter 2, we have discussed that the advanced interferometric gravitational wave (GW) detectors are expected to be limited by quantum noise over almost the entire detection band. Further enhancement of the detector sensitivity requires manipulation of the optical field and the readout scheme at the quantum level. One approach proposed by Kimble et al. [2] is injecting frequency-dependent squeezed light into the
main interferometer. A series of optical cavities is used to filter the squeezed light and to create proper rotation of the squeezing angle at different frequencies. In order to achieve a broadband reduction of quantum noise, the frequency scale of these filter cavities needs to match that of quantum noise of the main interferometer. For the advanced LIGO, the quantum noise is dominated by quantum radiation pressure noise at low frequencies and shot noise at high frequencies—the transition happens around 100Hz, which determines the required filter cavity bandwidth.

The original proposal in Ref. [2] is using filter cavities of kilometer length. Recently, Evans et al. [3] proposed a more compact (10 meters) filter cavity with $10^5$ finesse to achieve the required bandwidth. With such a high finesse, a small optical losses can degrade the squeezing and become the key limiting factor in the filter cavity performance. Isogai et al. have experimentally demonstrated that the optical losses from current mirror technology are sufficiently small to build such a filter cavity that will be useful for the advanced LIGO [4]. However if we want to further increase the compactness of the filter cavity, then the requirement for optical loss becomes more stringent. In this case, one solution is to go beyond the paradigm of passive cavities. One proposed approach is to actively narrow the cavity bandwidth by using electromagnetically induced transparency (EIT) effect in a pumped atomic system [5]. In principle, the cavity can be made to be on the centimeter-scale while still having a bandwidth comparable to a much longer high-finesse cavity. Additionally, with an active element, the cavity optical properties can be dynamically tuned by changing the power of the control pumping field. This has the advantage of allowing optimization of the filter cavity for different operational modes of the detector, where the quantum noise has different frequency dependencies, e.g., tuned vs. detuned resonant sideband extraction (RSE) in the case of the advanced LIGO.

The active atomic system is generally lossy, which will degrade the squeezing level. Here we propose to narrow the filter cavity bandwidth using the optomechanical analogue of EIT, optomechanically induced transparency (OMIT), which has recently been experimentally demonstrated by Weis et al. [6] and Teufel et al. [7]. In comparison with these OMIT experiments, we consider a different parameter regime and use an overcoupled cavity to attain the desired performance. The scheme integrated
with the main interferometer is illustrated in Fig. 5.1. The filter cavity consists of a mirror-endowed mechanical oscillator with eigenfrequency $\omega_m$ that is much larger than the cavity bandwidth $\gamma$. A control pump laser drives the filter cavity at frequency $\omega_p$, detuned from the cavity resonant frequency $\omega_0$ (also the laser frequency of the main interferometer) by $\omega_m - \delta$ with $\delta$ comparable to the gravitational-wave signal frequency $\Omega$. As we will show, the optomechanical interaction modifies the cavity response and gives rise to the following input-output relation for the sideband at $\omega_0 + \Omega$:

$$\hat{a}_{\text{out}}(\Omega) \approx \frac{\Omega - \delta - i\gamma_{\text{opt}}}{\Omega - \delta + i\gamma_{\text{opt}}} \hat{a}_{\text{in}}(\Omega) + \hat{n}_{\text{th}}(\Omega), \quad (5.1)$$

where $\gamma_{\text{opt}}$ is defined as:

$$\gamma_{\text{opt}} = \frac{4P_c\omega_0}{m\omega_m c^2 T_f}, \quad (5.2)$$

with $P_c$ being the intra-cavity power of the control field, $m$ the mass of the mechanical oscillator and $T_f$ the transmissivity of the front mirror (the end mirror is totally reflective). The first term in Eq. 5.1 gives the input-output relation of a standard optical cavity with the original cavity bandwidth $\gamma$ replaced by $\gamma_{\text{opt}}$, which can be significantly smaller than $\gamma$ as well as dynamically tuned by changing the control beam power.

The second term $\hat{n}_{\text{th}}$ arises from the thermal fluctuation of the mechanical oscillator. It is uncorrelated with the input optical field $\hat{a}_{\text{in}}$ and therefore decohering the squeezed light. In order for its effect to be small, we require:

$$\frac{8k_B T}{Q_m} < \hbar\gamma_{\text{opt}}, \quad (5.3)$$

with $Q_m$ the mechanical quality factor and $T$ the environmental temperature. Given the fact that the desired effective cavity bandwidth is $\gamma_{\text{opt}}/2\pi \approx 100$ Hz, we have

$$\frac{T}{Q_m} < 6.0 \times 10^{-10} \text{ K}. \quad (5.4)$$

This is challenging to achieve even with low-loss materials at cryogenic temperature. A possible solution is to use optical dilution, first proposed by Corbitt et al. [8, 9, 10]. It allows for a significant boost of the mechanical quality factor by using the optical field, to provide most of the restoring force. Later we illustrate its applicability for our purpose.
5.3 Optomechanical induced transparency

This section gives a brief introduction to the optomechanical induced transparency effect. The optomechanical induced transparency effect comes from the mutual interference between different cavity field components. The signal light (with frequency $\omega_s$) and the control light (with frequency $\omega_p$) will beat inside the cavity to create an A-C radiation pressure force on the mechanical oscillator at driving frequency $\omega_c - \omega_p$.

The motion of the oscillator under this pondermotive driving will modulate the control light and create upper and lower sidebands with frequency $\omega_c - \omega_p + \omega_p = \omega_c$ and $\omega_p - (\omega_c - \omega_p) = 2\omega_p - \omega_c$. The lower sideband is far off-resonance and thereby can be safely ignored. However, the upper sideband just have the same frequency as the signal field, which is well within the cavity bandwidth. This upper sideband will destructively interfere with the signal field and decreases the intra-cavity signal field. Therefore equivalently, more signal field are reflected—This is why the reflection of the cavity has a peak in Fig.5.2.

Moreover, the width of the reflection peak is determined by the total damping rate
of the mechanical oscillator. This can be intuitively understood in the following way: When we scan the signal light frequency, it has a tiny detuning so that the frequency of radiation pressure force is detuned by $\Omega$ from the mechanical resonance. Clearly, the mechanical motion will be smaller than the on-resonance case when $\Omega = 0$ and the less upper sideband of the control field will be generated. Consequently, the less signal field will be destructively canceled and the less will be the reflectivity. This picture tells us that the width of the central reflection peak is determined by the amplitude of the mechanical motion, thereby equal to the total mechanical bandwidth. When the optomechanical damping rate $\gamma_{\text{opt}}$ is much larger than the $\gamma_m$. The width of the central reflection peak is just equal to $\gamma_{\text{opt}}$.

Since the signal field with frequency around the cavity resonance almost do not dissipate compare to the situation when there is no optomechanical interaction. This effect is called optomechanical induced transparency (OMIT). As we shall see more clearly, OMIT effect smartly combine the property of the mechanical oscillator $\gamma_{\text{opt}}$ with the property of the optical response: reflectivity together and create an effective cavity whose optical linewidth is the mechanical bandwidth $\gamma_{\text{opt}}$. 

Figure 5.2 – Optomechanical induced transparency: an example. We choose the cavity loss $\eta_c = T_f/(T_f + T_{\text{loss}})$ to be 0.25 and the mechanical detuning $\delta = 0$. This means that the frequency of the radiation pressure force is on-resonance with the oscillator.
5.4 Optomechanical dynamics.

5.4.1 Dynamics of the system and the input-output relation

Here we provide the details behind Eq. (5.1) by analyzing the dynamics of the optomechanical filter cavity, starting from the standard linearized Hamiltonian [12, 13]:

\[
\hat{\mathcal{H}} = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_m^2 \hat{x}^2 + \hbar G_0 \hat{x} (\hat{a}^\dagger + \hat{a}) \\
+ i \hbar \sqrt{2} \gamma (\hat{a}^\dagger \hat{a}_{in} e^{-i\omega_p t} - \hat{a} \hat{a}_{in}^\dagger e^{i\omega_p t}) .
\] (5.5)

In the Hamiltonian, \( \hat{a} \) is the annihilation operator of the cavity mode and \( \hat{a}_{in} \) is the annihilation operator for the input optical field (the squeezed light in our case); \( \hat{x} (\hat{p}) \) is the oscillator position (momentum); \( \bar{G}_0 = [2P_c \omega_0 / (\hbar c L)]^{1/2} \) with \( L \) being the cavity length. In the rotating frame at frequency \( \omega_p \) of the control laser, the Heisenberg equation of motion reads:

\[
m \ddot{\hat{x}} + \gamma m \dot{\hat{x}} + \omega_m^2 \hat{x} = -\hbar \bar{G}_0 (\hat{a} + \hat{a}^\dagger) + \hat{F}_{th},
\] (5.6)

\[
\dot{\hat{a}} + (\gamma + i\Delta) \hat{a} = -i \bar{G}_0 \dot{\hat{x}} + \sqrt{2} \gamma \hat{a}_{in},
\] (5.7)

where \( \Delta \equiv \omega_0 - \omega_p \) is the detuning frequency and we have included the mechanical damping and associated Langevin force \( \hat{F}_{th} \). Solving these equations of motion in the frequency domain yields

\[
\hat{x}(\omega) = \chi_m(\omega) \left\{ \hbar \bar{G}_0 [\hat{a}(\omega) + \hat{a}^\dagger(-\omega)] + \hat{F}_{th}(\omega) \right\},
\] (5.8)

\[
\hat{a}(\omega) = \chi_c(\omega) [-i \bar{G}_0 \hat{x}(\omega) + \sqrt{2} \gamma \hat{a}_{in}(\omega)].
\] (5.9)

We have defined the susceptibilities \( \chi_m \equiv -[m(\omega^2 - \omega_m^2 + i\gamma_m \omega)]^{-1} \) and \( \chi_c \equiv [\gamma - i(\omega - \Delta)]^{-1} \).

We consider the parameter regime leading to Eq. (5.1). This requires \( \Delta = \omega_m - \delta \) with \( \omega_m \gg \delta \), and the so-called resolved-sideband regime \( \omega_m \gg \gamma \). Correspondingly, the lower sideband of the cavity mode \( \hat{a}(-\omega) \) in Eq. (5.8) is negligibly small and can be ignored (we will analyze the effect of this approximation later). We therefore obtain [cf. Eqs. (5.8) and (5.9)]:

\[
\hat{a}(\omega) \approx \frac{\sqrt{2} \gamma \hat{a}_{in}(\omega) - i \bar{G}_0 \chi_m(\omega) \hat{F}_{th}(\omega)}{\chi_c^{-1}(\omega) + i \hbar G_0^2 \chi_m(\omega)} .
\] (5.10)
Since we are interested in the signal sidebands around $\omega_0$, we rewrite the above expression in terms of $\Omega$ by using the equality $\omega = \Delta + \Omega$ [cf. the inset of Fig. 5.1]. Given $\Omega \approx \delta \ll \omega_m$, we have $\chi_m \approx -[2m\omega_m(\Omega - \delta + i\gamma_m)]^{-1}$ and $\chi_c \approx \gamma^{-1}$. Together with $\hat{a}_{out} = -\hat{a}_{in} + \sqrt{2\gamma} \hat{a}$, we obtain

$$\hat{a}_{out}(\Omega) \approx \frac{\Omega - \delta + i\gamma_m - i\gamma_{opt}}{\Omega - \delta + i\gamma_m + i\gamma_{opt}} \hat{a}_{in}(\Omega) + \hat{n}_{th}(\Omega)$$

(5.11)

with the additional noise term $\hat{n}_{th}$ defined as

$$\hat{n}_{th}(\Omega) = \frac{i\sqrt{2\gamma\gamma_{opt}} \hat{F}_{th}(\Omega)}{\hbar G_0(\Omega - \delta + i\gamma_m + i\gamma_{opt})}.$$  

(5.12)

For a high quality factor oscillator $\gamma_m \ll \gamma_{opt}$, we can ignore $\gamma_m$ and recover the input-output relation shown in Eq. (5.1).

- **Effective Hamiltonian**—To intuitively understanding this new input-output relation, let us rewrite it in the following way: Since $\hat{F}_{th}$ satisfies $\langle \hat{F}_{th}(\Omega) F_{th}(\Omega') \rangle = 4m\gamma_m k_B T$ and $k_B T = \hbar \omega_m (\hat{b}^\dagger \hat{b})$ where $\hat{b}^{(i)}$ is the annihilation(creation) operator of the thermal bath excitation quanta, then we can establish the relationship between the $\hat{b}_{th}$ and $\hat{F}_{th}$ roughly as: $F_{th}(\Omega) \sim \sqrt{4m\gamma_m \hbar \omega_m \hat{b}_{th}(\Omega)}$. Then in terms of $\hat{b}$, the above input-output relation (in the limit of $\gamma_m \ll \gamma_{opt}$) can be written as:

$$\hat{a}_{out}(\Omega) \approx \left[1 - \frac{2\gamma_{opt}}{\gamma_{opt} - i(\Omega - \delta)}\right] \hat{a}_{in}(\Omega) + \sqrt{2\gamma_{opt} / \gamma_{opt} - i(\Omega - \delta)} \hat{b}_{th}(\Omega).$$

(5.13)

The first term is easy to be understood, which is just the reflection of the optical injection by this optomechanical cavity. The second term here represents the “transmission” of the thermal-excited mechanical mode $\hat{b}_{th}$. Interestingly, in this problem, the susceptibility function of the optomechanical cavity is the same as the susceptibility of the mechanical degrees of freedom. The reason is the mechanical response
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toward the thermal bath is given by the equation of motion of the creation and annihilation operator of the mechanical oscillator:

\[ \dot{\hat{b}} = (-i\delta + \gamma_{\text{opt}})\hat{b} + \sqrt{2\gamma_m}\hat{b}_{\text{th}} \]  
(5.14)

where \( \delta \), as we can see from Fig.5.1, is the detuning of the center of the driving frequency with respect to the mechanical resonance \( \omega_m \). The Fourier transformation of the above formula will lead to:

\[ \hat{b}(\Omega) = \sqrt{2\gamma_m} \frac{\sqrt{\gamma_{\text{opt}}}}{\gamma_{\text{opt}} - i(\Omega - \delta)} \hat{b}_{\text{in}}(\Omega) \]  
(5.15)

and we also have:

\[ \hat{a}(\Omega) = \sqrt{2\gamma_{\text{opt}}} \frac{\sqrt{\gamma_{\text{opt}}}}{\gamma_{\text{opt}} - i(\Omega - \delta)} \hat{a}_{\text{in}}(\Omega) \]  
(5.16)

Then the input-output relation, in a more compact way, can be written as:

\[ \hat{a}_{\text{out}}(\Omega) = -\hat{a}_{\text{in}}(\Omega) + \sqrt{2\gamma_{\text{opt}}} \left[ \hat{a}(\Omega) + \hat{b}(\Omega) \right] \]  
(5.17)

This effective description is given in Fig.5.3. Then the effective Hamiltonian of this optomechanical cavity can be written simply as:

\[
\frac{H_{\text{eff}}}{\hbar} = \frac{i}{2} \int_{-\infty}^{\infty} dx \left[ \frac{\partial a_x}{\partial x} \hat{a}^\dagger_x - \frac{\partial a_x^\dagger}{\partial x} a_x \right] + \delta(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}) \\
+ i\sqrt{2\gamma_{\text{opt}}} \hat{a}_0 (\hat{a}^\dagger + \hat{b}^\dagger) + i\sqrt{2\gamma_m} \hat{b}_{\text{th}} \hat{b}^\dagger + h.c
\]  
(5.18)

where we have already folded the outgoing field from \([-\infty, 0]\) to \([0, \infty]\). This effective Hamiltonian shows the duality between the \( \hat{a} \) and \( \hat{b} \), which is the key property of our optomechanical cavity.

5.5 Noises associated with the optomechanical filter cavity

• Thermal Noise—To maintain coherence of the squeezed light, the fluctuations due to the thermal noise term \( \hat{n}_{\text{th}} \) need to be much smaller than those due to the input field; equivalently, the quantum radiation pressure noise on the mechanical oscillator from the squeezed light needs to dominate over thermal noise of the oscillator. Given
the fact that \( \langle \hat{F}_{\text{th}}(\Omega)\hat{F}_{\text{th}}(\Omega') \rangle = 4m\gamma_m k_B T \delta(\Omega - \Omega') \), the requirement on the noise spectrum for \( \hat{n}_{\text{th}} \) reads

\[
S_{\text{th}}(\Omega) = \left( \frac{8k_B T}{\hbar \gamma_{\text{opt}} Q_m} \right) \frac{\gamma^2_{\text{opt}}}{(\Omega - \delta)^2 + \gamma^2_{\text{opt}}} < 1. \tag{5.19}
\]

The thermal noise effect is maximal around \( \Omega \sim \delta \), from which we obtain the condition shown in Eq. (5.3).

- **Noise associated with optical loss and finite cavity bandwidth**— Apart from the above-mentioned thermal noise, there are other decoherence effects: (i) the additional radiation pressure noise introduced by the optical loss, and also (ii) the effect of the lower sideband due to the finite cavity bandwidth, ignored in the resolved-sideband limit. Their effects are similar to the above thermal force noise; therefore we can quantify their magnitude using the noise spectrum referred to the output. For the optical loss,

\[
S_{\epsilon}(\Omega) = \left( \frac{c \epsilon}{\gamma L} \right) \frac{\gamma^2_{\text{opt}}}{(\Omega - \delta)^2 + \gamma^2_{\text{opt}}}, \tag{5.20}
\]

where \( \epsilon \) is the magnitude of the optical loss (e.g., \( \epsilon = 10^{-5} \) for 10ppm loss). Similarly, for the contribution from the lower sideband, we have

\[
S_{-\omega_m}(\Omega) = \left( \frac{\gamma}{\omega_m} \right)^2 \frac{\gamma^2_{\text{opt}}}{(\Omega - \delta)^2 + \gamma^2_{\text{opt}}}. \tag{5.21}
\]

These two need to be taken into account when estimating the performance of this optomechanical filter cavity.
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5.6 Optical dilution

5.6.1 General concept of optical dilution

We have shown in Eq. (5.4) that the most significant issue is the thermal noise, which puts a stringent requirement on the mechanical system and the environmental temperature. As we mentioned earlier, one possible way to mitigate this is using optical dilution explored by Corbitt et al. [8], in which the optical restoring force is due to the linear dependence of radiation pressure force on the oscillator position.

The general idea of the optical dilution is to use the opto-mechanical interaction to boost up the rigidity of the mechanical mode so that the quality factor of the mechanical mode after the optical dilution is given by:

\[ Q = \sqrt{\frac{\omega_m^2 + \omega_{opt}^2}{\gamma_m}} \]  (5.22)

where \( \omega_{opt}^2 = \frac{K_{opt}}{m} \) and \( K_{opt} \) is the pondermotive rigidity.

The mean thermal occupation number for the mechanical oscillator now becomes (if \( \omega_{opt} \gg m\omega_m \)):

\[ \bar{n}_{th} \sim \frac{k_B T}{\hbar \omega_m Q} = \bar{n}_{0th} \frac{\omega_m}{\omega_{opt}} \]  (5.23)

where \( Q_0 = \omega_m \gamma_m \) and \( \bar{n}_{0th} = \frac{k_B T}{\hbar \omega_m Q_0} \) are the Q-factor and the mean thermal occupation number of the mechanical mode before the optomechanical interaction. From the above formula, it is clear that after the optical dilution, ideally the mean thermal occupation number decreased by a factor of \( \frac{\omega_{opt}}{\omega_m} \).

The above dilution factor is “ideally” because there exists some issues in the optical dilution.

- **Possible instabilities**—Some optical dilution schemes (such as the one proposed by Corbitt et al.) actually make use of the linear “optical spring” effect we discussed above. As we have discussed before, optical spring usually accompanies with the optical anti-damping \( \gamma_{opt} < 0 \). If \(|\gamma_{opt}| > \gamma_m \), the mechanical oscillator will suffer instability. One way to solve this problem is by using double (multiple) optical spring. This scheme has been proposed in [14].

- **Radiation pressure noises**—The schemes based on linear optical spring also has a limitation from quantum back action noise associated with a linear position
response. Korth et al. [14], have showed how measurement-based feedback can cancel the quantum back action. Such a cancelation is, however, limited by quantum efficiency of the photodiode for measurement. The instability and the radiation pressure noises can be in-principle avoided if we trap the mechanical oscillator is a quadratic way. In the next section, we will propose an new optical dilution scheme based on dissipative coupling, of which the spring comes from the linear optomechanics, but the mechanical oscillator can be equivalently treated as being set in a quadratic way (for this equivalency, see [18]).

5.6.2  A noise-free optical dilution scheme with internal feedback

In this section, we propose an new optical dilution scheme with very small radiation pressure noise (in-principle the radiation pressure noise can be zero) based on the configuration shown in Fig 5.4, with a mirror-endowed oscillator placed in the middle of a Fabry-Pérot cavity, first implemented by Thompson et al. [15, 16]. Interestingly, this scheme allows for an internal cancellation of the quantum back action associated with a linear optical spring, and thus it avoids the limitation of the scheme in Ref. [14]. A detailed analysis is given below. An intuitive picture behind this back-action evasion effect can be described as follows (also see Fig.5.6). The optical field on the left-hand side of the middle oscillator consists of two parts: (i) the immediate reflection from the oscillator and (ii) the transmitted field from the right-hand side, both containing the position information of the oscillator. The coupled cavity has a doublet resonance. It turns out that, when the trapping field is resonantly tuned to one of the doublet and the end mirror is perfectly reflective, the position information from these two parts destructively interfere, resulting in a cancelation of the back action.

Strong trapping beam can induce an optical spring frequency $\omega_{opt} \gg \omega_{m0}$ with:

$$\omega_{opt}^2 = \frac{2P_{\text{trap}}\omega'_0}{mc^2 \sqrt{T_s T_f}},$$  \hspace{1cm} (5.24)

where $P_{\text{trap}}$ and $\omega'_0$ are the input power and optical frequency of the trapping beam, the $T_s$ and $T_f$ are the transmissivity of the mirror-endowed oscillator and the front mirror, respectively. The modified quality factor can be greatly boosted since the
mechanical dissipation rate $\gamma_m$ is unchanged.

This optical dilution scheme also has its own limitations. Firstly, in reality there is no perfectly reflective mirror and always some optical loss, and so the above-mentioned cancelation cannot be perfect. The residual radiation pressure noise, referred to the output, is given by:

$$ S_{\text{opt}}^{\epsilon} (\Omega) = \frac{4\omega_0^2 P_{\text{trap}}^{\epsilon}}{m\gamma_m \omega_m c^2 T_s T_j} \frac{\gamma_{\text{opt}}^2}{(\Omega - \delta)^2 + \gamma_{\text{opt}}^2}. $$

(5.25)

Secondly, the optical spring effect is frequency dependent: $K_{\text{opt}}(\omega) \approx m\omega_m^2 - im\gamma_m\omega - m_{\text{opt}}\omega^2$. This tells us that the optical spring can modify not only the resonance frequency, but also the mechanical damping and the effective inertia (mass), which could induce instability. Lastly, finite absorption of the laser power in the oscillator will increase its temperature and may increase the thermal noise. The size of this effect, however, depends on the mechanical structure and the detailed loss mechanism.

**Hamiltonian and equation of motion**— The Hamiltonian of the system can be written as:

$$ H = \hbar \omega_0 (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) + \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_m^2 \hat{x}^2 + \hbar \omega_s (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) $$

$$ + \hbar G_0 (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) + H_{\text{opt}}^{\text{ext}} + H_{\text{m}}^{\text{ext}}. $$

(5.26)

Here, $\hat{a}, \hat{b}$ are annihilation operators for cavity modes in left and right sub-cavity (with resonant frequency $\omega_c$) respectively. $\hat{x}, \hat{p}$ are the position and momentum operators of the vibrating mirror. $\omega_s$ is the coupling constant for $\hat{a}$ and $\hat{b}$ and $G_0$ is defined to be $\omega_0/L$. $H_{\text{ext}}^{\text{opt}} = i\hbar \sqrt{2\gamma_f} (\hat{a}^\dagger \hat{a}_{\text{in}} - \text{h.c}) + i\hbar \sqrt{2\gamma_f} (\hat{b}^\dagger \hat{b}_{\text{in}} - \text{h.c})$ and $H_{\text{ext}}^{\text{m}}$ correspond to the coupling of the system to the environment.

The Heisenberg equations of motion in the rotating frame of the trapping beam

![Figure 5.5](image)

**Figure 5.5** – Basic configuration of the proposed scheme: a vibrating mirror trapped in a Fabry-Pérot cavity. $\hat{a}$ and $\hat{b}$ are the light field operators in the left and right subcavities, respectively.
at frequency $\omega'_0$ can be derived as:

\[
\dot{a} = i\Delta_t\hat{a} - \gamma_f\hat{a} - i\omega_s\hat{b} - iG_0\hat{x}_a + \sqrt{2\gamma_f}\hat{a}_{in}, \tag{5.27a}
\]

\[
\dot{\hat{b}} = i\Delta_t\hat{b} - \gamma_f\hat{b} - i\omega_s\hat{a} + iG_0\hat{x}_b + \sqrt{2\gamma_f}\hat{b}_{in}, \tag{5.27b}
\]

\[
\dot{\hat{p}} = -m\omega_m^2\hat{x} - \gamma_m\hat{p} - \hbar G_0(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) + F_{th}, \tag{5.27c}
\]

\[
\dot{\hat{x}} = \hat{p}/m. \tag{5.27d}
\]

Here, $\gamma_f = cT_f/4L$ and $\gamma_e = c\epsilon/4L$, $T_f$ and $\epsilon$ are the transmissivity of the front mirror and the loss of the system through the end mirror, $\Delta_t = \omega'_0 - \omega_0$ is the detuning of the pumping laser field with respect to the half-cavity resonance. Suppose we pump the cavity by injecting a laser field through the front mirror (single-side pumping), then $\hat{b}_{in} = 0$. These equations can be solved perturbatively. The zeroth order terms give us the classical amplitude of the intra-cavity mode in both sub-cavities and the first order terms carry information about the mirror vibration along with quantum noise due to the non-zero transmissivity of the cavity end mirror.

From the above Heisenberg equations of motion, we have the steady state fields in the two sub-cavities:

\[
\hat{a} = \frac{(i\Delta_t - \gamma_e)\sqrt{2\gamma_f}\hat{a}_{in}}{\Delta_t^2 - \omega_s^2 - \gamma_f\gamma_e + i\Delta_t(\gamma_f + \gamma_e)}, \tag{5.28a}
\]

\[
\hat{b} = \frac{i\sqrt{2\gamma_f}\omega_s\hat{a}_{in}}{\Delta_t^2 - \omega_s^2 - \gamma_f\gamma_e + i\Delta_t(\gamma_f + \gamma_e)}. \tag{5.28b}
\]

As we can see from above equations, when we set the detuning of the trapping beam to be $\Delta_t = \omega_s$ and set $\gamma_e/\gamma_f \ll 1$, the intracavity field amplitude is strong: $\hat{a} = \hat{b} = \sqrt{2/\gamma_f}\hat{a}_{in}$ with $\hat{a}_{in} = \sqrt{P_{trap}/\hbar\omega'_0}$. The fluctuating field consists of mechanical modulation and quantum fluctuations as:

\[
\hat{a}(\omega) = \frac{-G_0\omega_s\hat{x} + i\omega_s\sqrt{2\gamma_f}\hat{b}_{in} + [i(\omega + \Delta_t) - \gamma_e][-iG_a\hat{x} + \sqrt{2\gamma_f}\hat{a}_{in}]}{(\omega + \Delta_t)^2 - \omega_s^2 - \gamma_e\gamma_f + i(\omega + \Delta_t)(\gamma_f + \gamma_e)}, \tag{5.29a}
\]

\[
\hat{b}(\omega) = \frac{G_a\omega_s\hat{x} + i\sqrt{2\gamma_f}\omega_s\hat{a}_{in} + [i(\omega + \Delta_t) - \gamma_f][iG_b\hat{x} + \sqrt{2\gamma_f}\hat{b}_{in}]}{(\omega + \Delta_t)^2 - \omega_s^2 - \gamma_e\gamma_f + i(\omega + \Delta_t)(\gamma_f + \gamma_e)}, \tag{5.29b}
\]

with $G_a \equiv G_0\hat{a}$ and $G_b \equiv G_0\hat{b}$ (notice that in case of $\Delta_t = \omega_s$, we have $G_a = G_b$). The radiation pressure force acting on the trapped mirror is given by

\[
\hat{F}_{rad}(\omega) = \hbar[G^*_a\hat{a}(\omega) + G_a\hat{a}^\dagger(-\omega) - G^*_b\hat{b}(\omega) - G_b\hat{b}^\dagger(-\omega)], \tag{5.30}
\]
which can be split into two parts:

\[
\hat{F}_{\text{rad}}(\omega) = -K_{\text{opt}}(\omega)\hat{x}(\omega) + \hat{F}_{\text{BA}}(\omega).
\]  

(5.31)

The first and second term represent the pondermotive modification of the mechanical dynamics and the back-action quantum radiation pressure noise respectively. The \(K_{\text{opt}}(\omega)\) here is the optomechanical rigidity which can be expanded in terms of \(\omega\) if the typical frequency of mechanical motion is smaller than the other frequency scale in the trapping system:

\[
K_{\text{opt}}(\omega) \approx K_{\text{opt}}(0) + \frac{\partial K_{\text{opt}}}{\partial \omega} \omega^2 + \frac{1}{2} \frac{\partial^2 K_{\text{opt}}}{\partial \omega^2} \omega^2 \equiv m\omega^2_{\text{opt}} - im\gamma{\omega} - m_{\text{opt}}\omega^2.
\]  

(5.32)

The first term in (5.32) gives the trapping frequency and the second and third terms give the velocity and acceleration response of the trapped mirror which are optical (anti-)damping \(\gamma\) and optomechanical inertia \(m_{\text{opt}}\), respectively.

Substituting (5.28) and (5.29) into (5.30) and taking the expansion with respect to detection frequency \(\omega\), we can get analytical expressions of the optical rigidity and radiation pressure noise. However, they are too cumbersome to show. In the following, we show approximate results in the interesting parameter region of \(\Delta_t \sim \omega_s\) and \(\gamma_\epsilon \ll \gamma_f\) in which the back-action noise can be coherently canceled.

- **Dynamics and back-action**— The optical spring frequency is given by:

\[
\omega^2_{\text{opt}} = \frac{\hbar G^2 a}{m\omega_s} + \mathcal{O}(\eta).
\]  

(5.33)

Substitute \(\bar{a}_{\text{in}}, G, \omega_s\) and \(\gamma_f\) in, and we have Eq. (5.24). The \(\mathcal{O}(\eta)\) here describes all the high order terms with \(\eta \sim (\Delta_t - \omega_s)/\omega_s, \gamma_\epsilon/\gamma_f\). Notice that this optical spring can be treated effectively as a quadratic trap of the vibrating mirror on the anti-node of our trapping beam as shown in [18].

The optical (anti)-damping factor \(\gamma\) is given by (to 1st order of \(\omega\)):

\[
\gamma = \frac{16hG^2 a}{m\gamma_f\omega_s} \left(\frac{\Delta_t - \omega_s}{\omega_s}\right) - \frac{8hG^2 a\gamma_f}{m\omega_s^3} \left(\frac{\gamma_\epsilon}{\gamma_f}\right) + \mathcal{O}(\eta^2)
\]  

(5.34)

It is clear from this formula that in the ideal case when \(\Delta_t = \omega_s\) and \(\gamma_\epsilon = 0\), the optical damping is completely cancelled. Therefore by carefully choosing the system parameters, we can achieve a small positive damping when the end mirror is not perfectly reflective.
5.6. Optical dilution

The main contribution to the optomechanical inertia is at zeroth order of $\epsilon$:

$$m_{\text{opt}} = -\frac{\hbar G_a^2}{\omega_s^3} + O(\eta), \quad (5.35)$$

which is extremely small as we have shown in the main text.

Finally, the back-action radiation pressure force noise spectrum is given by:

$$S_{FF}^{\text{rad}} = \frac{2\hbar^2 G_a^2 \gamma_f}{\omega_s^2} \left( \frac{\gamma_s}{\gamma_f} \right) + O(\eta^2). \quad (5.36)$$

Substitute $\bar{a}_{\text{in}}, G_a, \omega_s$ and $\gamma_f$ in, and we have Eq. (5.25). Notice that the back-action force spectrum is zero when the end mirror is perfectly reflective ($\gamma_s = 0$).

The physical explanation of this back-action evasion phenomenon is shown in Fig. 5.6. The part of the outgoing fields which contains the displacement signal can be written as (suppose the end mirror is perfectly reflective):

$$\hat{a}_{\text{out}}^m = -2iG_a \hat{x} + 2iG_a \frac{\omega_s^2}{\Delta t} \hat{x}. \quad (5.37)$$

The first term on the right hand-side is the field directly reflected from the trapped mirror while the second term is the field transmitted out of the cavity. We can see that they cancel when $\Delta t = \omega_s$. Therefore in this case the output field does not contain the $x$-information.

Given the parameters listed in Tab.I of the main text, we use (5.33)-(5.36) to calculate the modification of the mechanical dynamics by the trapping beam, and list the effective parameters in Tab.II of the main text. We can see that velocity response is a mechanical damping factor $\gamma$ which will not cause instability and is too small to affect the OMIT effective cavity bandwidth $\gamma_{\text{opt}}$. The negative inertia $m_{\text{opt}}$ is also too small to be comparable to the mass of the mechanical oscillator.
Finally, it is interesting to notice that our proposed scheme, although based on the linear optical spring effect, can be effectively mapped to the quadratic spring effect. This effective mapping has been discussed in [18].

### 5.7 An example

We illustrate the requirements for experimentally realizing the optomechanical filter cavity using optical dilution shown in Fig. 5.4 with some example parameters in Table 5.1. These values are chosen after considering the above mentioned effects, which can cause decoherence to the squeezed light, such that

\[
S_{\text{tot}}^{\max} = S_{\text{th}}(\delta) + S_{\epsilon}(\delta) + S_{-\omega m}(\delta) + S_{\text{opt}}^{\epsilon}(\delta) < 1. \tag{5.38}
\]

In addition, once we fix the oscillator mass \( m \) and transmissivity \( T_s \), we can minimize \( S_{\text{tot}}^{\max} \) by looking into the scaling of different parameters, which determines the trapping beam power \( P_{\text{trap}} \), the front mirror transmissivity \( T_f \), and the environmental temperature \( T \). We end up with the following scaling of \( S_{\text{tot}}^{\max} \) in terms of optical loss and cavity length:

\[
S_{\text{tot}}^{\max} \approx 3 \times 10^4 \epsilon^{4/5} / L^{2/5}. \tag{5.39}
\]

The resulting degradation to the squeezing factor due to optical loss is shown in Fig. 5.7 for a cavity length of 50cm. In comparison to a passive filter cavity for which the performance degrades as \( \epsilon / L \)[4], the optomechanical filter cavity using the optical-dilution scheme has a milder dependence on \( L \), which yields the possibility of being small scale.

The mechanical dynamics are modified by the opto-mechanical interaction, and the new effective parameters of the oscillator are summarised in Table 5.2. The optical spring shifts the mechanical resonant frequency from its bare value of 100Hz to 20kHz, which results in a two hundred fold increase in the quality factor. Comparing Table 5.1 and Table 5.2, we can see that the negative optical damping and inertia do not pose an important problem.

We would like to point out that this scheme might not function as expected due to heating from finite absorption of the laser power. The intra-cavity power of the
Table 5.1 – Example parameter values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>filter cavity length</td>
<td>50cm</td>
</tr>
<tr>
<td>$T_f$</td>
<td>front mirror transmissivity</td>
<td>250ppm</td>
</tr>
<tr>
<td>$T_s$</td>
<td>transmissivity of oscillator</td>
<td>3000ppm</td>
</tr>
<tr>
<td>$P_{\text{trap}}$</td>
<td>trapping beam input power</td>
<td>1.6mW</td>
</tr>
<tr>
<td>$\lambda'_0$</td>
<td>trapping beam wavelength</td>
<td>532nm</td>
</tr>
<tr>
<td>$m$</td>
<td>oscillator mass</td>
<td>500ng</td>
</tr>
<tr>
<td>$\omega_{\text{mo}}/(2\pi)$</td>
<td>bare mechanical frequency</td>
<td>200Hz</td>
</tr>
<tr>
<td>$Q_{\text{mo}}$</td>
<td>bare mechanical quality factor</td>
<td>$10^{12}$</td>
</tr>
<tr>
<td>$T$</td>
<td>environmental temperature</td>
<td>1K</td>
</tr>
<tr>
<td>$P_c$</td>
<td>control beam intra-cavity power</td>
<td>0.1mW</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>control beam wavelength</td>
<td>1064nm</td>
</tr>
<tr>
<td>$\gamma_{\text{opt}}/(2\pi)$</td>
<td>effective cavity bandwidth</td>
<td>100Hz</td>
</tr>
</tbody>
</table>

Table 5.2 – Effective oscillator parameters

| $\omega_{\text{opt}}/(2\pi)$ | optical spring frequency [Eq.(5.24)] | 20kHz     |
| $Q_m$     | final mechanical quality factor   | $2 \times 10^{10}$ |
| $\gamma/(2\pi)$ | optical (anti-)damping rate [Eq.(A.9)] | $-8$mHz  |
| $m_{\text{opt}}$ | negative optical inertia [Eq.(A.10)] | $-8.5$pg |

trapping beam, given the listed parameter values, is around 10W. For 10 ppm absorption, this amounts to 0.1mW of heat deposited into the nano-mechanical oscillator. We make an order-of-magnitude estimate in the following paragraph and find this can create a nonuniform temperature distribution with a maximum around 10K near the beam spot. Further detailed study is required to estimate how this nonuniform temperature distribution on the oscillator affects the total thermal noise. Specifically in this case, the dissipation mainly comes from the clamping point where the tem-
temperature is still low. If this nonuniform temperature distribution indeed introduces significant thermal noise, then alternative materials with higher thermal conductivity at low temperature would need to be manufactured.

- **Effective temperature of the mirror-endowed oscillator**—Here we estimate the temperature of the mirror-endowed oscillator due to the additional heating caused by optical absorption. We assume the oscillator to be a silicon cantilever mirror with thickness $h$, Young’s modulus $Y$ and density $\rho$. Under the assumption that $l > b, h$, the fundamental frequency of the cantilever is given by \[ \omega_{m0} = 1.875^{2} \sqrt{\frac{YI}{\rho Sl^{2}}}, \] \[(5.40)\]

where the $S = bh$ and $l$ are the cross-sectional area and length of the cantilever. Then $I = bh^{3}/12$ is the moment of inertia of the beam cross-section. Using the parameters given in Tab. 5.3, the resonant frequency has the value about 180Hz.

We also assume that the suspended mirror inside the cavity has thermal conductivity $\kappa(T) = \kappa_0 T^n$. The cantilever is illuminated by the trapping field with intra-cavity power $P_{\text{trap}}^c$. As a simple 1-D heat transport problem, Fourier’s law says that the heat power passing through the cross-section $S = bh$ of the mirror material at distance $z$ from its center equals to $P_{\text{cond}} = -S\kappa T'(z)$. Integrating the heat transport equation from the illuminated spot center with temperature $T_0$ to the boundary with temperature $T$, we have the relation between the $T_0$ and the absorbed power for a rectangular

\[ P_{\text{trap}}^c = \int_{0}^{L} -S\kappa T'(z) dz. \]

\[ \epsilon = 30\text{ppm} \]

\[ \epsilon = 10\text{ppm} \]

\[ \epsilon = 2\text{ppm} \]

\[ L = 50\text{cm} \]

**Figure 5.7** – Resultant squeezing level from injecting 10 dB of input squeezing into an optomechanical filter cavity using the optical-dilution scheme in Fig. 5.4, with parameters in Table 5.1, for several values of the optical loss $\epsilon$. 
shape mirror $P_{\text{abs}}$ as:

$$P_{\text{abs}} = \frac{2S\kappa_0}{l(n+1)} (T_{0}^{n+1} - T^{n+1}). \quad (5.41)$$

Typically we have $n \sim 2$ at cryogenic temperature. Using the sample parameters 10ppm, $P_c \sim 15$W and the conductivity of the material $[20] \kappa_0 = 10$W/(m.K)$^n$, we have $T_0 = 6$K from Eq. (5.41).

How this absorption-induced 6K temperature around the hot spot and its nonuniform distribution across the beam cantilever influences the thermal noise is not entirely clear and needs further study. The loss of the cantilever motion can be classified as surface loss and body loss. The body loss is mainly through the clamping point where the temperature is around the cryogenic environment temperature. The surface loss, on the other hand, influences the cantilever motion through the coupling of the material surface motion with the local thermal bath, which has a raised temperature from the trapping beam heating. Whether the thermal noise due to surface loss degrades the squeezed light or not depends on detailed design of the experiment and needs a more sophisticated study. Moreover, the trapping beam does not illuminate the cantilever beam uniformly thereby a heat flux will be built up across the cantilever beam with temperature gradient about $\nabla T \sim 3 \times 10^3$K/m. The non-equilibrium thermal noise associated with this heat flux is also unclear and needs to be addressed in future research.
Chapter 5. Narrowing the cavity bandwidth via Optomechanical Interaction

5.8 Experimental demonstration

5.8.1 General description

In UWA optics basement laboratory, the above discussed optomechanical filtering effect was demonstrated by using a noise-added signal light to mimic the squeezed light in a room temperature system. In this experiment, a control light is injected into the same port for generating OMIT effect (See Fig 5.8). The frequency dependent noise ellipse rotation was demonstrated in a tunable OMIT cavity of which the linewidth can be tuned from 3Hz to several hundred Hz. The beating of the signal field and the control field at the transmission port of the coupled system was detected in order to measure the noise ellipse rotation of the single-mode signal field by the lock-in technique [21]. The result proves that the OMIT cavity has the same amplitude and phase response as a simple filter cavity, which can rotate the noise ellipse of a classical signal light with squeezed added noise in close agreement with the theoretical phase response. This shows the potential of FD squeezed vacuum generation in a small scale compact system with future implementations of low temperature environment and proper optical dilution.

The core elements of our OMIT apparatus consists of an 85 mm high-finesse optical cavity with a high stress silicon nitride membrane, which has a quality factor of $\sim 1.5 \times 10^6$ at the mechanical resonance. By changing the frequency separation between signal field and control field, the angle rotation of noise ellipses of the signal light was observed, which is shown in Fig. 5.14.

Following the same derivation as in Section IV.A, the effective transmissivity of the OMIT system can be written as:

$$t(\Omega) = 2\sqrt{\eta_c (1 - \eta_c)} \frac{\Omega - \omega_m + i\gamma_m}{\Omega - \omega_m + i\gamma_m + i\gamma_{opt}}, \quad (5.42)$$

where the cavity coupling parameter is $\eta_c = \gamma_1 / (\gamma_1 + \gamma_2)$.

The phase $\phi(\Omega)$ of the system transmissivity can be written as:

$$\phi(\Omega) = -\arctan \left( \frac{\gamma_{opt}}{\Omega - \omega_m} \right), \quad (5.43)$$

which is equivalent to the transmissivity phase response of a simple Fabry-Pérot cavity with the resonant point at $\Omega = \omega_m$. The linewidth $\gamma_{opt}$ is much smaller than
5.8. Experimental demonstration

**Figure 5.8** – Configuration schematics of the experiment. The signal light with squeezed added noise having frequency $\omega$ respect to cavity resonance $\omega_c$ is injected into an optical cavity with a high Q−factor membrane in the middle which acts as an oscillator at the resonant frequency $\omega_m$. The position of the membrane is chosen to introduce a linear optomechanical coupling.

**Figure 5.9** – (Color online) Experimental setup. The 85 mm long cavity sits in a vacuum chamber with a central silicon nitride membrane oscillator (1mm×1mm×50nm, effective mass 40 ng). The green line represents the locking light for stabilizing the laser frequency to the cavity resonance using Pound-Drever-Hall (PDH) method [22]. The blue line represents the control light, with polarization orthogonal to the locking light. The broadband electro-optic modulator (BEOM) generates an upper-sideband from the control light, which is our signal light (red line). The $\Delta \sim 400$ kHz for the control light was created using a pair of 80 MHz AOMs in the locking path.

the original cavity bandwidth $\gamma$, and is tunable, principally through $\tilde{G}_0$ which depends on the control light input power [6].

When we inject the signal light into the system at different frequencies, we will
observe a rotation of the noise ellipse at the cavity output. For the single mode signal field with classical squeezed added noise, the rotation angle $\theta(\Omega)$ is determined by the phase response $\phi(\Omega)$ of the cavity transmissivity $t(\Omega)$, which is given by:

$$\theta(\Omega) = -\arctan\frac{\gamma_{\text{opt}}(\Omega - \omega_m)}{(\Omega - \omega_m)^2 + \gamma_m \gamma_{\text{opt}}}, \quad (5.44)$$

which is shown as theoretical curves in Fig. 5.13 (b)(c).

### 5.8.2 Experiment

In our experimental setup shown in Fig. 5.9, the weak signal light is generated by passing the carrier control light through a broadband electro-optic modulator (BEOM).

![Graph](image)

**Figure 5.10** – Detected OMIT transmissivity. (a) Normalized transmissivity amplitude $|t_n(\Omega)|$ vs. frequency offset $\Delta f$ (Hz), where $\Delta f \equiv \Omega/2\pi - 402.5$kHz. The control light powers were 0.5, 1.3, 2.7 and 4.0 mW respectively. (b) Transmissivity amplitude vs. frequency difference $\Omega/2\pi$ (kHz) in a span of 200kHz. The coupled cavity full linewidth was 170 kHz in this measurement. (c) Normalized transmissivity amplitude peak value $|t_n(\Omega - \omega_m = 0)|$ vs. the control light input power. (d) OMIT cavity full linewidth vs. the control light input power. The full linewidth data correspond to the Lorentzian transmissivity of the OMIT cavity. In this measurement, the mechanical resonance frequency was $\omega_m = 2\pi \times 402.7$ kHz.
The BEOM modulates the control light and generates an upper-sideband, which is our signal field. The lower sideband \((\omega_p - \Omega)\) generated by the BEOM is far detuned from the cavity resonance, so it is totally reflected and can only be neglected at the transmission port. This method guarantees a common optical path for the signal light and the control light so as to avoid the fluctuating phase difference from an unlocked optical path. The voltage \(V = A \cos \Omega t\) from function generator FG1, acting on the BEOM with modulation index \(\beta = 15 \text{ mrad/V}\), determines the amplitude \(|\hat{a}^s_{\text{in}}| = \beta A |\hat{a}^c_{\text{in}}|\) and the frequency \(\omega_s = \omega_p + \Omega\) of the signal light. By adding random noise \(\delta A\) to the voltage amplitude \(A\), we increase the amplitude uncertainty of the signal light to simulate the “phase squeezed light”.

- **Optical cavity**—Our optical cavity was mounted on an invar spacer machined by electrical discharge machining with accuracy of 0.1 \(\mu\)m and fixed in a vibration isolated vacuum tank. The M1 and M2 were clamped at the ends of the spacer. In order to optimize the optical coupling, we built an over-coupled cavity. The transmissivity \(T_1\) of M1 was chosen to be much larger than that of M2 \((T_1 = 245.1 \pm 2.8 \text{ ppm}, T_2 = 16.93 \pm 0.20 \text{ ppm})\). This experiment was conducted at room temperature using a 1064nm Nd:YAG laser.

- **Mechanical membrane**—The mechanical oscillator in this study was a high \(Q\)-factor stoichiometric silicon nitride membrane window. In order to adjust the position and alignment of the membrane in the vacuum, it was attached to a piezoelectric actuator which was glued to a motorized optical mounts attached to the invar cavity spacer. To reduce the bonding loss, the membrane frame was bonded onto the actuator with Yacca gum, a natural resin with low intrinsic loss [24]. After gluing, we measured the quality factor of the membrane with a He-Ne laser to characterize the extra mechanical loss \(\gamma_{\text{gas}}\) introduced by the background gas. When the background gas pressure \(P_{\text{gas}}\) is smaller than \(3 \times 10^{-5}\) mbar, the gas damping was negligibly small and the membrane quality factor was \(\sim 1.5 \times 10^6\) at its mechanical resonance \(\sim 400\) kHz.

As a more concrete description of the measurement of the mechanical \(Q\)-factor in our experiment, the details are given as follows: When the background gas pressure \(P_{\text{gas}} > 10^{-2}\) mbar, the size of the membrane is ignorable compared with the mean free
path of the background gas. This satisfies the requirements for the non-interacting
gas molecule model [26]. The background gas leads to a mean damping force \( dp/dt = -\gamma_{\text{gas}}p/2 \). The background gas damping rate is \( \gamma_{\text{gas}} = 16P_{\text{gas}}/(\pi \rho_m d \bar{v}_{\text{gas}}) \), where \( d = 50 \text{ nm} \) is the membrane thickness and \( \bar{v}_{\text{gas}} = \sqrt{3RT/M_{\text{gas}}} \) is the gas mean speed [25]. The quality factor is given by \( Q = 2\pi \omega_m/(\gamma_m + \gamma_{\text{gas}}) \).

We used He-Ne laser to measure the ringdown time \( \tau \) of the mechanical oscillation amplitude in different gas pressures. The quality factor is given by \( Q = \pi f \tau \). As shown in Fig. 5.11 when the background gas pressure \( P_{\text{gas}} \) reaches \( \sim 10^{-5} \text{ mbar} \), the \( \gamma_{\text{gas}} \) is negligible small and the quality factor of the membrane is \( \sim 1.5 \times 10^6 \) at its resonant frequency of \( \sim 400 \text{ kHz} \).

![Figure 5.11 – Measured membrane quality factor vs background gas pressure.](image)

- **Effect of membrane to the cavity field: cavity finesse and Optomechanical coupling** \( G_0 \)— In the coupled cavity system, optical absorption by the membrane changes the coupled cavity \( \mathcal{F}(z) \) in different positions along the cavity axis [16, 23]. The coupled cavity linewidth \( \gamma = \pi f_{\text{FSR}}/\mathcal{F} \), where \( f_{\text{FSR}} = c/2L \) is the free spectral range of the cavity. According to previous theoretical work [23], the coupled cavity finesse \( \mathcal{F}(z) \) is a periodic function of the membrane position \( z \) with period equal to half wavelength of the optical mode. In Fig. 5.12 we show the experimental results of the coupled cavity finesse \( \mathcal{F}(z) \) as a function of \( z \). And the optical loss of the system also depends on the alignment of the membrane in the transverse plane orthogonal to the cavity axis.

The linear optomechanical coupling constant \( G_0 \) is the change of cavity resonance
5.8. Experimental demonstration

Figure 5.12 – Coupled cavity finesse $F(z)$ vs. membrane position $z$. Optical wavelength is $\lambda = 1064$ nm. The empty cavity finesse is $F_0 = 22400$. The membrane index of refraction $n = 2 + 2.5 \times 10^{-5} i$.

5.8.3 Optomechanically induced transparency

In our system, the linewidth of the OMIT cavity can be changed in two ways: (a) The input power of the control light can be adjusted by a half-wave plate before a polarized beam splitter (PBS3 in Fig. 5.9). (b) The coupled cavity linewidth $\gamma$ and the optomechanical coupling strength $G_0$ can be tuned by changing the position and alignment of the membrane in the cavity [16, 23]. We achieved a widely tunable linewidth of the OMIT cavity changing from 3 Hz to several hundred Hz.

In order to achieve an extremely narrow linewidth, we tuned the position and alignment of membrane and reduced the control light input power until the characteristic frequency $\gamma_{\text{opt}}$ was close to the mechanical linewidth $\gamma_m$. In Fig. 5.10 we show the experimental results for the lowest linewidth data 3~15 Hz. Here, we define an
normalized transmissivity amplitude as $t_n(\Omega) \equiv t(\Omega)/t_0$, where $t_0$ is the transmissivity amplitude of the signal light in the absence of the control field. The measurement data points in Fig. 5.10 (c) and (d) are well matched with the theoretical model shown as the black solid lines.

5.8.4 Frequency dependent noise ellipse rotation

The above results show that OMIT effect can be used to create cavities with tunable linewidth down to a few Hz. We now demonstrate that such cavities have the appropriate phase response, and that they rotate the angle of the noise ellipse of the signal light as required for one simple filter cavity.

In order to characterize the noise ellipse rotation of the signal light in phasor diagram, we tuned the OMIT cavity linewidth to several hundred Hz and demonstrate the noise ellipse rotation in phasor diagram. The phase response and the rotation angles of the noise ellipses are detected by lock-in technique (See Fig. 5.9). We take results of the coupled cavity with linewidth of $\sim 600$ Hz as an example.

![Diagram](image)

**Figure 5.13** – OMIT cavity transmissivity amplitude $|t(\Omega)|$ (a), phase $\phi(\Omega)$ (b) and rotation angle $\theta(\Omega)$ (c) of the noise ellipse as a function of $\Delta f$ (kHz). The frequency offset is defined as $\Delta f \equiv \Omega/2\pi - 390kHz$. In this measurement, the mechanical resonance frequency was $\omega_m = 2\pi \times 394.6$ kHz.
Figure 5.14 – Contour plots of the normalized noise ellipse in phasor diagram at different frequency offsets $\Delta f$ corresponding to (c) in Fig. 5.13. In each phasor diagram, the horizontal axis is the amplitude quadrature and the vertical axis is the phase quadrature. The frequency offset of the OMIT cavity resonance is 4.6 kHz. As the frequency offset increases, the rotation pattern is changed from $0^\circ$ (off resonance) to $90^\circ$ anticlockwise. Near the resonance, it flips by $-180^\circ$. Above resonance, the rotation pattern is changed from $-90^\circ$ to $0^\circ$ anticlockwise.

In Fig. 5.13 (a) and (b), we show the experimental results of the amplitude $|t(\Omega)|$ and the phase $\phi(\Omega)$ of the OMIT cavity transmission. The phase drop in the vicinity of the OMIT cavity ($|\Omega - \omega_m| < \sqrt{\gamma_m/\gamma_{opt}}$) resonance was measured and shown in Fig. 5.15. This experimental measurements of the phase drop near the OMIT cavity resonance clearly demonstrates that the phase response of the OMIT cavity behaves just like an ordinary optical cavity, but with much narrower linewidth.

In Fig. 5.13 (c) and Fig. 5.14, we show the measured rotation angles $\theta(\Omega)$ and the corresponding normalized noise ellipses in phasor diagram. As shown in Fig. 5.13 (c), the measured results for angle rotation of the noise ellipses well match both the theoretical model and the previous measurement of the phase $\phi(\Omega)$. 
In summary we have shown an extremely narrow cavity linewidth created by optomechanical interaction in an optical cavity with a silicon nitride membrane in the middle. Classical light with a noise ellipse simulating quantum squeezed light was injected into the cavity. It demonstrates the frequency dependent noise ellipse rotation. The rotation angle follows the theoretical prediction in the detection band of advanced gravitational wave detectors. To use the current setup to develop a system for realizing frequency dependent squeezed vacuum in GW detectors in the future, it will be necessary to cool the resonator to the mK temperature range and dilute the mechanical losses by optical springs as discussed in the previous sections and [14].

![Figure 5.15](image)

Figure 5.15 – The phase $\phi(\Omega)$ of the OMIT cavity transmission. The minimum sweep step was 0.25 Hz and the intensity of the signal on the resonance was very small.

### 5.9 Conclusion

In this chapter, we have considered the use of optomechanical interaction to narrow the bandwidth of a filter cavity for frequency-dependent squeezing in future advanced gravitational-wave detectors. However, due to susceptibility to thermal decoherence, its feasibility is conditional on advancements in low-loss mechanics and optics. As a trial for solving the thermal noise issue, we discussed an interesting optical dilution scheme with very low quantum radiation pressure noise. As an experimental demonstration of the theoretical discussion of this chapter, we have shown an extremely narrow cavity linewidth created by optomechanical interaction in an optical cavity with a silicon nitride membrane in the middle. Classical light with a noise ellipse
simulating quantum squeezed light was injected into the cavity. It demonstrates the frequency dependent noise ellipse rotation.


CHAPTER 6

Broadening the detector’s bandwidth using acasual optomechanical cavity

6.1 Preface

The Chapter 5 discussed the implement of the optomechanical cavity on filtering the squeezed light in gravitational wave detectors. In that case, the optomechanical interaction cools down the mechanical degree of freedom, broaden its bandwidth to be $\gamma_{\text{opt}}$. Quite different from the optomechanical cavity we have discussed in Chapter 5 where the mechanical degree of freedom is cooled down by the optomechanical interaction, this chapter is going to discuss the case when the optomechanical interaction heats up the mechanical degree of freedom. The system working in this region can in principle be used to broaden the detection bandwidth of an interferometric gravitational wave detector with a feedback control loop. Basically, we propose to embed an active unstable filter, a cavity-assisted optomechanical device operating in the instability regime, inside the signal-recycling cavity to cancel such a linear phase. Therefore the sideband field which carries the test mass displacement signal can resonant over a broad frequency range. To make this scheme function, a feedback control is needed for stabilizing the entire system, and cryogenic temperature is necessary to minimize thermal fluctuation of the mechanical oscillator in the optomechanical device. This scheme follows the white-light-cavity idea first proposed by Wicht et al. [1] but with a different realization.

In this Chapter, Section 6.2 gives an introduction of the research background then Section 6.3 discusses the optomechanical dynamics of the system. Section 6.4 analyzes
the feed-back control of the optomechanical instability.

6.2 Introduction and background

As discussed in Chapter 2, the advanced interferometric gravitational wave detector with the dual-recycling structure with PRM and SRM can be mapped to a single cavity with tunable “front mirror” (which is actually the signal-recycling cavity). In the presence of SRM, the sideband fields which carry the gravitational wave signals are coherently reflected back to the interferometer. The detection bandwidth is no longer proportional to the transmissivity of the ITM, but rather is determined by the effective transmissivity of the signal-recycling cavity $T_{\text{SRC}}$, as a compound mirror, formed by ITM and SRM. This compound mirror increase the tunability of the detector. However, such tunability can not surpass a trade-off between the signal gain and the bandwidth is illustrated in Fig[6.3]. The white light cavity scheme based on anomalous dispersive systems was proposed to surpass this trade-off. This trade-off leads to a limitation of the integrated sensitivity which is firstly discussed by Mizuno [2]:

$$
\int_0^\infty \frac{1}{S_{hh}(\Omega)} d\Omega = \frac{4\pi\omega_0 I_c L_{\text{arm}}}{\hbar c},
$$

(6.1)

where $S_{hh}^h$ is the shot-noise limited sensitivity of the detector. In Chapter 8, a proof of the Mizuno hypothesis will be given.

The physical picture of the Mizuno hypothesis is the following: The decreasing of the detector sensitivity at high frequency region is due to the decreasing of the signal amplitude. the sideband field which carry the GW signal will obtain a phase $\phi(\Omega) = 2\Omega L_{\text{arm}}/c$ after a round-trip propagation, where $L_{\text{arm}}$ is the (arm-)cavity length. Suppose the $\Omega$ takes the value of $\pi c/2L$, then the phase factor will be $e^{-i\phi} = -1$. The field with this negative phase factor will destructively interfere with the incoming field transmitted through the front mirror into the cavity —this is why the signal field amplitude decreases with the increasing of the frequency.

For broadening the detection bandwidth without sacrificing the peak sensitivity, various literatures proposed to add a filter cavity inside the signal recycling mirror so that the round-trip phase of the sideband light $\phi(\Omega) = 2\Omega L_{\text{arm}}/c + \phi_f(\Omega)$ is always
6.2. Introduction and background

Figure 6.1 – Mizuno hypothesis: Trade-off between the peak sensitivity and the detection bandwidth. With the increase of the reflectivity of the signal recycling mirror (SRM), the bandwidth becomes smaller and the noise spectrum peak becomes lower.

Figure 6.2 – Round trip phase of the intra-cavity sideband fields. On the left panel, the round-trip phase obtained by the sideband field is varying with the changing of the sideband frequency.
equal to $2n\pi$, that is, ideally the sideband field with any $\Omega$ can resonate inside the cavity.

In this case, the $\phi_f(\Omega)$ have to satisfy the following equation:

$$\frac{d\phi_f(\Omega)}{d\Omega} = -\frac{2L_{\text{arm}}}{c}.$$  

(6.2)

which means the filter has to be a negative-dispersive medium.

In Chapter 7, we will discuss the previously proposed scheme of using negative dispersive atomic medium to surpass the Mizuno limit. In this Chapter, we propose an approach to achieve such a broadband enhancement by placing an optomechanical device operating in the instability regime inside the signal-recycling cavity, as illustrated schematically in Fig. 6.3, and the entire setup is stabilized via feedback control. Inside the loop, the unstable filter acts as a phase compensator which negates the linear propagation phase of the sideband signal, and there is no violation of causality,
as the entire scheme is stable and always lags behind incoming GW signals. This idea is inspired by recent experimental observations of the optomechanical analogue of the electromagnetically induced transparency by S. Weis et al. [3], Teufel et al. [4] and Safavi-Naeini et al. [5], and also a more recent theoretical proposal by Ma et al. to use such a phenomenon for realizing a short optical filter cavity but with a narrow bandwidth [6]. In these experiments, a red-detuned control laser with frequency lower than the cavity resonant frequency \( \omega_0 \) is used to demonstrate the optomechanically induced transparency. In contrast, the control laser in our proposed scheme is blue detuned with a frequency equal to \( \omega_0 + \omega_m \) with \( \omega_m \) being the mechanical resonant frequency. Safavi-Naeini et al. [5] also experimentally investigated such a case and was focusing on the regime where the mechanical oscillator remains stable. In our case, however, the mechanical oscillator is highly unstable without the feedback. Notice that the feedback only uses the signal after the SRM, as indicated by the flow chart in Fig. 6.3(b), which contains the GW signal together with noise, and therefore it does not influence the signal-to-noise ratio (ignoring noise in the sensor and actuator), as also proven in Refs. [7, 8]. We start with the dynamics of the optomechanical filter, as shown schematically in Fig. 6.4(a), derive its input-output relation for the sideband signal, and later combine it with the main interferometer. Such an optomechanical device has been studied extensively in the literature (see recent reviews: Refs. [9, 10]).

### 6.3 Optomechanical dynamics

Qualitatively, this optomechanical dynamics can be described as follows: the lower sideband of the red-detuned pumping field inside the cavity will be generated by the motion of the test mass. This motion is under the driving of radiation pressure force, which is contributed by the beating of the pumping fields and the probe field. Different from optomechanical induced transparency effect: 1) this lower sideband will constructively interfere with the probe field, therefore the response of the system toward the probe fields behaves in an “optomechanical induced absorption” way. 2) The mechanical oscillator will not be stable. 3) More interestingly, the probe fields will feel an apparent anti-casual phase shift when it propagates through the cavity.
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Figure 6.4 – (a) schematics for the optomechanical filter; (b) frequency of interests: cavity resonance $\omega_0$ (red) and control laser frequency $\omega_0 + \omega_m$ (blue) and the common sideband of these two (black). GW signals are at frequency $\Omega$. (round-trip). The details are presented in the following discussion.

Specifically, the total Hamiltonian of the system is given by $\hat{H}_{\text{tot}} = \hat{H}_0 + \hat{H}_{\text{int}} + \hat{H}_{\gamma_f} + \hat{H}_{\gamma_m}$. Here the free Hamiltonian $\hat{H}_0$ is

$$\hat{H}_0 = \hbar \omega_0 \hat{a} \hat{a}^\dagger + \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_m^2 \hat{x}^2;$$

where the first term is the free Hamiltonian of the cavity field while the second term is the free Hamiltonian of the mechanical oscillator. The linearized interaction Hamiltonian $\hat{H}_{\text{int}}$ is

$$\hat{H}_{\text{int}} = -\hbar g_0 \left[ \hat{a} e^{i(\omega_0 + \omega_m)t} + \hat{a}^\dagger e^{-i(\omega_0 + \omega_m)t} \right] \hat{x}$$

with $g_0 \equiv \omega_0 \hat{a}/L_f$, in which $\hat{a} = (2P_c L_f / (\hbar \omega_0 c))^{1/2}$ is the mean amplitude of the cavity mode; $\hat{H}_{\gamma_f}$ describes how the cavity mode $\hat{a}$ interacting with ingoing field $\hat{a}_{\text{in}}$ and outgoing field $\hat{a}_{\text{out}}$, and $\hat{H}_{\gamma_m}$ describes the coupling between the mechanical oscillator with the environmental heat bath. Notice that in this interaction Hamiltonian, we have already wrote the operators in the rotation frame of $\omega_0 + \omega_m$, which is the frequency of the pumping field.

The parameter regime to obtain the desire performance is $\omega_m \gg \gamma_f \gg \Omega$. In this regime, the upper sideband of the mechanical oscillator is far outside of the filter cavity linewidth—the so-called resolved-sideband regime, as shown schematically in Fig. 6.4(b). This allows us to gain a qualitative picture of the optomechanical processes in terms of phonon-photon interaction as follows. By moving into the interaction picture with respect to $\hat{H}_0$ and using the rotating-wave approximation
(RWA), which leads to
\[ \hat{H}_{\text{int}}^{\text{RWA}} = -\hbar g (\hat{a} \hat{b} + \hat{a}^\dagger \hat{b}^\dagger), \]
where we have introduced annihilation operator \( \hat{b} \) for the mechanical oscillator through \( \hat{x}(t) \equiv x_q (\hat{b} e^{-i\omega_m t} + \hat{b}^\dagger e^{i\omega_m t}) \) and \( g \equiv g_0 x_q \) with \( x_q \) being the zero-point motion. From this Hamiltonian, it is clear that the optomechanical interaction converts a pumping field photon to be a pair of lower sideband photon and a phonon, vice versa.

Solving the Heisenberg equations of motion in the frequency domain leads to the following input-output relation for the sideband at frequency \( \omega_0 + \Omega \):
\[ \hat{a}_{\text{out}}(\Omega) \approx \frac{\Omega + i(\gamma_m + \gamma_{\text{opt}})}{\Omega + i(\gamma_m - \gamma_{\text{opt}})} \hat{a}_{\text{in}}(\Omega) - \frac{\sqrt{2\gamma_{\text{opt}}}}{\Omega + i(\gamma_m - \gamma_{\text{opt}})} \sqrt{2\gamma_m} \hat{b}_{\text{th}}^\dagger(-\Omega), \]
where in deriving the above relation, we have used the fact \( \gamma_f \gg \Omega \) and \( \gamma_{\text{opt}} \gg \gamma_m \), and we have defined \( \gamma_{\text{opt}} \) as follows:
\[ \gamma_{\text{opt}} \equiv \frac{g^2}{\gamma_f} \approx \frac{P_c \mathcal{F} \omega_0}{(m \omega_m c^2)}. \]

This input-output relation can be qualitatively understood in the same way as the input-output relation for the optomechanical cavity for filtering the squeezed light. The \( \hat{a}_{\text{in}} \) and \( \sqrt{2\gamma_m} \hat{b}_{\text{th}}^\dagger \) can be treated as optical and mechanical “input” field, interact with the optomechanical cavity with coupling strength \( \sqrt{2\gamma_{\text{opt}}} \). The transfer function of the system is given by \( 1/(\Omega + i\gamma_m - i\gamma_{\text{opt}}) \).

The first term in the input-output relation, in the limit of \( \gamma_{\text{opt}} \gg \gamma_m \), shows the ideal frequency-dependent phase: \( \phi \approx -2 \arctan (\Omega/\gamma_{\text{opt}}) \approx -2\Omega/\gamma_{\text{opt}} \):
\[ \hat{a}_{\text{out}}(\Omega) = \frac{\Omega + i\gamma_{\text{opt}}}{\Omega - i\gamma_{\text{opt}}} \hat{a}_{\text{in}}(\Omega) \approx -\exp \left( -\frac{2\Omega}{\gamma_{\text{opt}}} \right) \hat{a}_{\text{in}}(\Omega). \]

- This relation exhibits phase lead rather a phase lag in the case of a passive optical cavity, which can be used for compensate the round-trip phase \( \phi_{\text{arm}} = 2\Omega L_{\text{arm}}/c \) of the sideband field in the interferometer. The requirement for the \( \gamma_{\text{opt}} \) is approximately:
\[ \gamma_{\text{opt}} = c/L_{\text{arm}}. \]

- This formula is only valid under (i) the rotating-wave approximation by ignoring the upper sideband \( \omega_0 + 2\omega_m + \Omega \) of the control laser, and (ii) the assumption that frequency-dependent response of the cavity is small around \( \Omega \) with \( \gamma_f \gg \Omega \). In order
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Figure 6.5 – The frequency-dependent reflection phase $\phi_f$ of the optomechanical filter given different parameters in comparison with the ideal case from using the rotating-wave approximation and assuming the large bandwidth limit.

to get close to the ideal performance, we require $\omega_m \gg \gamma_f \gg \gamma_{opt}$. The physical reason is clear: the smaller the $\gamma_f/\omega_m$ is, the less effect the upper-sideband will have on the input-output relation (or in other words, the better the resolved sideband approximation is); the smaller the $\gamma_{opt}/\gamma_f$ is, the original optical cavity resonance structure has less effect on the above input-output relation of the effective optomechanical cavity. In Fig. 6.5, we compare the ideal case with the actual frequency-dependence of the phase without the above-mentioned approximations.

- For $\gamma_{opt}$ reaching $c/L_{arm}$, the resulting control power scales as:

$$P_c = 50W \left( \frac{4km}{L_{arm}} \right) \left( \frac{m}{1mg} \right) \left( \frac{\omega_m/2\pi}{10MHz} \right) \left( \frac{10^5}{F} \right). \quad (6.10)$$

For the system with $m = 1mg$, $L_{arm} = 4km$, $\omega_m/2\pi = 10MHz$ and $F = 10^5$, the intra-cavity power is around 50W. This is achievable in the real experiment.

The second term in Eq. (6.6) arises from the environmental thermal noise of the mechanical oscillator and $\hat{b}_{th}$ is the associated annihilation operator of the heat bath, with a white noise spectrum $S_{b_{th}}(\Omega) \approx k_B T_{envir}/(\hbar \omega_m)$ in the high temperature limit. It has the same effect as the optical loss, and introduces decoherence to the input field $\hat{a}_{in}$. As shown in Fig. 6.6, its influence is most prominent at low frequencies. As a rule of thumb, in order for the thermal noise to be insignificant compare to the
quantum shot noise level, we require:

\[ \frac{8k_B T_{\text{envir}}}{Q_m} < \hbar \gamma_{\text{SRM}}, \]  

(6.11)

where \( \gamma_{\text{SRM}} = c T_{\text{SRM}}/(4L_{\text{arm}}) \) and \( T_{\text{SRM}} \) being the power transmissivity of the SRM. Note that this is quite different from the condition we asked for the OMIT filter cavity \( 8k_B T_{\text{envir}}/Q_m < \hbar \gamma_{\text{opt}} \). This difference is due to the fact that the filter cavity here is put inside the signal recycling cavity so that the life time of the sideband photon is affected by the interaction between the main interferometer field and the filter cavity field, we will discuss these details in the next section. The condition here is more stringent since \( \gamma_{\text{SRM}} < \gamma_{\text{opt}} \) if \( \gamma_{\text{opt}} = c/L_{\text{arm}} \). As an order of magnitude estimation, the required environmental temperature scales as:

\[ T_{\text{envir}} \leq 6 \text{mK} \left( \frac{Q_m}{10^7} \right) \left( \frac{\gamma_{\text{SRM}}}{2 \pi \times 100 \text{Hz}} \right). \]  

(6.12)

Note that the \( \gamma_{\text{SRM}} \) is the original bandwidth of the interferometer, as ITM and iSRM are impedance matched and the bandwidth is solely determined by the SRM transmissivity. Therefore the lower the bandwidth we start off, the higher requirement will be imposed on the temperature. As discussed in Chapter 2 and Chapter 5, one way to mitigate the thermal noise is to optically dilute the mechanical oscillator, which allows for the enhancement of \( Q_m \) by a factor of hundred or even more.

Notice that this thermal noise term appears as an creation operator in the above input-output relation. This reflects the fact that our system is a parametric amplifier. Even when the environmental thermal temperature is zero, this term will still contribute an additional quantum noise effect to the sensitivity of the system.

However, there is one additional issue arising from the instability of the mechanical motion due to the optomechanical interaction. This is readily seen from the resulting susceptibility of the mechanical sideband \( \hat{b}(\Omega) \) at \( \omega_m + \Omega \):

\[ \chi_m(\Omega) = -(i\Omega + \gamma_m - \gamma_{\text{opt}})^{-1}. \]  

(6.13)

Since the intrinsic mechanical damping \( \gamma_m \) is orders of magnitude smaller than \( \gamma_{\text{opt}} \) in the parameter regime of interest, the above susceptibility implies that the mechanical motion is unstable with a timescale of \( \gamma_{\text{opt}}^{-1} \). A feedback control is therefore needed to stabilize it with a control bandwidth up to \( \gamma_{\text{opt}} \). However, as we have experienced
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for the thermal noise condition Eq. (6.11), this native treatment regardless of the effect of the mutual interaction between the interferometer optical mode and the filter cavity mode is not correct. Also notice that we cannot simply apply feedback control by measuring the output of the filter cavity, as this will also destroy the quantum coherence of the input field which contains the GW signal. Instead, we have to use the measurement output after the signal-recycling mirror so that the coherence is maintained, as shown schematically in Fig. 6.3(a). We shall comment on these points with more details after considering the filter cavity together with the main interferometer.

6.4 Surpassing the Mizuno limit—intuitive argument

The dynamics including both the optomechanical filter and the main interferometer is rather involved. This does not pose any issue for evaluating the sensitivity of the full scheme, but indeed for gaining an intuitive understanding. To gain some insights, we therefore continue using the above-mentioned rotating-wave approximation and

![Figure 6.6](image)

**Figure 6.6** – Effect of the thermal fluctuation of the mechanical oscillator on the low-frequency sensitivity. The mechanical quality factor is assumed to be fixed at $5 \times 10^7$ for different temperature
focus on the high frequency where the dynamics of the test masses in the main interferometer can be ignored. Later, we will outline the procedure for obtaining the rigorous results which are used for computing the sensitivities.

We will only consider the differential mode of the interferometer so that the two arms can be folded into a single cavity, following the treatment as applied in Ref. [7]. Our system works in the resolved sideband extraction mode (see Chapter 2) so that the sideband field sees a transparent signal recycling cavity. At high frequency where the shot noise dominates, the input-output relation of the arm cavity (in terms of sideband at $\omega_0 + \Omega$) is given by [cf. Eq. (16) in Ref. [12] by ignoring the radiation pressure noise term]:

$$\hat{A}_{\text{out}}(\Omega) = e^{2i\phi_{\text{arm}}} \hat{A}_{\text{in}}(\Omega) + ie^{i\phi_{\text{arm}}} \sqrt{K} h(\Omega)/h_{\text{SQL}}(\Omega),$$

(6.14)

where $K \equiv 8\omega_0 P_c/[ML_{\text{arm}}^2 \Omega^2]$ with $P_c$ being the intra-cavity optical power and $h_{\text{SQL}}(\Omega) \equiv \sqrt{8\hbar/(M\Omega^2L_{\text{arm}}^2)}$ is the Standard Quantum Limit [11] for strain sensitivity when the signal-recycling cavity is tuned.

By combining the above input-output relation with that of the optomechanical filter in the rotating-wave approximation and the large-bandwidth limit, shown in Eq. (6.6), the input-output relation of the entire scheme reads (at the dark port):

$$\hat{d}_{\text{out}} \approx \hat{d}_{\text{in}} + 2ie^{i\phi_{f}} \frac{\sqrt{K}}{T_{\text{SRM}}} \frac{h(\Omega)}{h_{\text{SQL}}} - \frac{4\sqrt{\gamma_{\text{opt}} \gamma_{m}}}{\sqrt{T_{\text{SRM}}}} \frac{\hat{b}_{\text{th}}^\dagger(-\Omega)}{\Omega - i\gamma_{\text{opt}}},$$

(6.15)

where we have used the fact that $1 - r_{\text{SRM}} \approx T_{\text{SRM}}/2$. It is important to note that in deriving the above formula, the perfect phase-cancelation condition $\phi_{\text{arm}} + \phi_{f} = 0$ was assumed. This input-output relation implies that the GW signal is enhanced by a factor of $2/\sqrt{T_{\text{SRM}}}$ without decreasing the detection bandwidth, but with additional thermal noise from the mechanical oscillator, which is the main result of this Chapter. Since $\phi_{f}$ is negative, this gives us the impression that the outgoing field comes out before the signal arrives, which is, however, only true when the feedback is engaged.

From Eq. (6.15), we can proof the surpassing of Mizuno theorem.

$$\int_{0}^{\omega_{\text{far}}} \frac{1}{S_{\text{hh}}(\Omega)} d\Omega = \frac{4\pi\omega_0 L_{\text{arm}}}{hcT_{\text{SRM}}}$$

(6.16)
The above intuitive argument assumes the perfect cancelation of the propagation phase by the optomechanical filter, which is not correct over the full frequency band, as we can see from Fig. 6.8.

As mentioned earlier, we need to use feedback to stabilize the mechanical oscillator by using the final measurement readout. From the third term in Eq. (6.15), we learn that the upper optical sideband $\omega_0 + \Omega$ at the output is proportional to the lower mechanical sideband $\omega_m - \Omega$. This means that if we use homodyne detection to detect the output phase quadrature $\hat{y}_2 \equiv [\hat{d}_{\text{out}}(\Omega) - \hat{d}_{\text{out}}^\dagger(-\Omega)]/(i\sqrt{2})$ which contains the GW signal, only one of the two mechanical quadratures: $\hat{x}_1 \equiv [\hat{b}(\Omega) + \hat{b}^\dagger(-\Omega)]/\sqrt{2}$

---

**Figure 6.7** – (a) A more elaborated scheme. Compared with Fig. 6.3(a), heterodyne detection is used in order to measure both the amplitude and phase quadratures (at a price of a higher shot noise than homodyne detection [13]), and a small signal-recycled interferometer is served as the optomechanical filter to have a clean separation of the filtered signal and the control laser. The signal-recycling cavity is folded to avoid using a three-way circulator (a Faraday isolator) which is usually quite lossy; (b) A flow chart for showing the propagation of signal; (c) Sidebands involved in the rigorous calculation. Sideband pair 1&4, pair 2&3 are mixed due to the optomechanical interaction in the filter, while 1&2 are mixed because of the optomechanical interaction with the test masses in the main interferometer. Even lower/higher sidebands are ignored, which contribute little to the dynamics.
will be measured, while leaving the conjugated quadrature $\hat{x}_2 \equiv [\hat{b}(\Omega) - \hat{b}^\dagger(-\Omega)]/(i\sqrt{2})$ unmeasured. Since both mechanical quadratures are unstable [cf. Eq. (6.13)], only feeding back the measurement output of the phase quadrature will not be able to stabilize the mechanical motion. Therefore, we have to measure both the amplitude $\hat{y}_1$ and phase quadrature $\hat{y}_2$ of the output by using either an additional beamsplitter or using heterodyne detection—the later is preferable because we can use the control laser at frequency $\omega_0 + \omega_m$ serving as the local oscillator, namely $L(t) = \cos(\omega_0 + \omega_m)t$, to demodulate the output: $\hat{y}_1 \cos \omega_0 t + \hat{y}_2 \sin \omega_0 t$. Such a consideration leads to a more elaborated setup shown schematically in Fig.6.7(a). Moreover, when the detection frequency $\Omega$ is comparable to $\pi c/L_{\text{arm}}$, the $\phi_{\text{arm}} + \phi_f = 0$ condition is not correct (see Fig.6.8), which means we have to take into account of this phase-delay in choosing the feedback control scheme. In the next section, we will first give a simple model to study the possible feedback control scheme, then extend the discussion to a more accurate case.

6.5 Stability analysis

6.5.1 Single-mode effective model

We now consider the dynamics of the main interferometer and its coupling with the optomechanical filter described by the total Hamiltonian: $\hat{H}_{\text{tot}} = \hat{H}_{\text{ifo}} + \hat{H}_f + \hat{H}_{\text{if}}$

![Figure 6.8](image-url) - Imperfect Phase cancelation at higher frequency. The nonzero $\phi_{\text{sum}} = \phi_{\text{arm}} + \phi_{\text{filter}}$ is due to the effect of sideband with frequency $2\omega_m \pm \Omega$. The imperfect phase cancelation is due to the higher-order terms of $\Omega$ in the $e^{i\phi_{\text{arm}}(\Omega)}$. 
where the $\hat{H}_{ifo}$ is the free Hamiltonian for the main interferometer given by:

$$\hat{H}_{ifo} = \hbar \omega_0 \hat{A}^\dagger \hat{A} + \frac{\hat{P}^2}{2M} + \hbar G_0 (\hat{A} + \hat{A}^\dagger) \hat{X} + \hat{X} F_{GW} + H_{\gamma_{ifo}}. \quad (6.17)$$

Since in this Chapter, only the shot-noise limited sensitivity is considered, therefore we can treat the interferometer in the infinite mass limit, therefore the interferometer Hamiltonian can be simply written as $\hat{H}_{ifo} = \hbar \omega_0 \hat{A}^\dagger \hat{A} + \hat{H}_{\gamma_{ifo}}$ where the $\hat{H}_{\gamma_{ifo}}$ describes the interaction between the main cavity field and the external electromagnetic bathes.

The interaction between the filter cavity and the interferometer is given by:

$$\hat{H}_{if} = \hbar \omega_s (\hat{A}^\dagger \hat{a} + \hat{A} \hat{a}^\dagger), \quad (6.18)$$

where $\omega_s = \sqrt{c \gamma_f / L_{arm}}$ is the “sloshing frequency” between the main interferometer and the filter cavity. The corresponding equations of motion is given by:

$$\dot{\hat{b}} = ig\hat{a}^\dagger - \gamma_m \hat{b} + \sqrt{2 \gamma_m} \hat{b}_{th}, \quad (6.19)$$

$$\dot{\hat{a}} = -i \omega_s \hat{A} + ig \hat{b}^\dagger - \gamma_f \hat{a}, \quad (6.20)$$

$$\dot{\hat{A}} = -i \omega_s \hat{a} - \gamma_{SRM} \hat{A} + \sqrt{2 \gamma_{SRM}} \hat{A}_{in}. \quad (6.21)$$

Under the approximation $\gamma_m \ll \gamma_{opt}$, the field $\hat{a}(t)$ can be solved out from the equation of motion of $\hat{b}(t)$ as: $\hat{a}^\dagger(t) = \hat{b}(t)/ig$. In the large $\omega_s$ limit where $\omega_s$ is much larger than $\gamma_f$ and the typical variation frequency of $\hat{a}$, the equation of motion of $\hat{a}$ will lead to:

$$\hat{a}^\dagger(t) = \hat{b}(t)/ig = i \omega_s \hat{A}^\dagger / g^2. \quad (6.22)$$

Adiabatically eliminating $\hat{a}$, we have the equations of motion for the main cavity field $\hat{A}$ as:

$$\left(1 - \frac{\omega_s^2}{g^2}\right) \dot{\hat{A}} + \gamma_{SRM} \hat{A} = \sqrt{2 \gamma_{SRM}} \hat{A}_{in}. \quad (6.23)$$

This formula demonstrates that for maximumly broadening the bandwidth of the main cavity (or maximumly canceling the propagation phase), the $\omega_s$ can be chosen as equal to $g$. The Eq. (6.23) is derived under several approximations, therefore it describes an over-optimistic situation where the propagation can be completely canceled out and the bandwidth is infinity. In reality, as shown in Fig. 6.5, There will only be a finite frequency region where the phase cancelation can be well realized Fig. 6.9 in
Figure 6.9 – The resonance structure of the main interferometer cavity when the interaction with the filter cavity is switched off (blue) and on (red) using the exact formula in the single-mode model. The parameter chosen here is \( g = \omega_s = 10\gamma_{SRM} \). The resonance structure is extended from \( \sim [-0.1\gamma_{SRM}, 0.1\gamma_{SRM}] \) to \( \sim [-5\gamma_{SRM}, 5\gamma_{SRM}] \), which is very significant.

plotting which we used the exact formula of the dynamics of this single-mode model:

\[
\dot{\hat{A}}(\Omega) = \frac{\sqrt{2\gamma_{SRM}}}{\gamma_{SRM} - i\Omega + i\Omega\omega_s^2/(g^2 + i\Omega\gamma_f + \Omega^2)}\hat{A}_{in}(\Omega).
\]

(6.24)

The instability for the mechanical oscillator actually comes from the optomechanical interaction contributed by the \( ig\hat{a}^\dagger(t) \) term. If we applied a feedback to cancel this optomechanical interaction term, there will be no instability. This feedback force can be added in the following way:

\[
\hat{F}_{fb}(t) = C[\hat{A}_{out}(t) + \gamma_{SRM}\hat{A}_{out}(t)]e^{i\omega_m t} + h.c.
\]

(6.25)

This kind of proportional-differential controller with \( C = -1/\sqrt{2\gamma_{SRM}} \), when added on the mechanical oscillator, will modify the equation of motion to be:

\[
\dot{\hat{b}} = -\gamma_m \hat{b} + \sqrt{2\gamma_m} \hat{b}_h(t) - \frac{1}{\sqrt{2\gamma_{SRM}}} (\hat{A}_{in} - \gamma_{SRM}\hat{A}_{in}).
\]

(6.26)

This demonstrated that applying the above feedback control scheme can indeed stabilize the system. However, this scheme is worked out under the single-mode assumption. The single mode assumption require us to have \( \Omega \ll \pi c/L_{arm} \) and \( \Omega \ll \pi c/L_f \). However, the detection frequency of the detector can be comparable to the \( \pi c/L_{arm} \).
which is the free spectral range of the main interferometer. Therefore in this case, the single mode approximation is no longer valid.

In the next section, we will provide an more exact analysis on the stability issue.

6.5.2 More exact analysis on the stability issue

Since the free spectral range of the main interferometer is much larger than the small filter cavity \( \text{FSR}_{\text{ifo}}/\text{FSR}_f = L_f/L_{\text{arm}} \), typically \( \sim 10^{-5} - 10^{-6} \), therefore it is still safe to treat the filter cavity using single-mode approximation.

The optomechanical dynamics of the full scheme can be described by the flow chart shown in Fig.6.7(b). The stability of this system can be analyzed using Nyquist theorem (See the appendix of this chapter), which basically means that the stability issue can be addressed through analyzing analytical behavior of the open-loop transfer function of the system. In order to obtain the open-loop transfer function, we truncate the closed loop as shown in the Fig.6.10(a) (In this simplified version, we only consider the shot-noise-limited sensitivity). For writing down the expression of the open loop gain between 1 and 2 in Fig.6.10(a), we have to first calculate the transfer function of the closed loop related to the feedback control process inside the open loop between 1 and 2, named as “inner loop”.

- **Inner loop transfer function**— First, we truncate the inner loop as shown in Fig.6.10. Suppose a field \( s \) was injected into 5 in Fig.6.10(b), After a round-trip
propagation to 6, the gain $T_{56} (\Omega)$ is given by:

$$T_{56} (\Omega) = - \frac{i t_{SRM} \sqrt{2 \gamma_f g}}{g^2 + i \gamma_f \Omega + \Omega^2} K (\Omega), \quad (6.27)$$

where $K (\Omega)$ is the feedback control kernel. Then the transfer function between 3 and 2 can be derived as:

$$\hat{a}_3 (\Omega) = C (\Omega) \left[ \frac{- g^2 + i \Omega \gamma_f - \Omega^2}{g^2 + i \Omega \gamma_f + \Omega^2} \hat{a}_2 (\Omega) - T_{56} (\Omega) \frac{r_{SRM}}{t_{SRM}} \hat{d}_{in} (\Omega) \right], \quad (6.28)$$

where $C (\Omega)$ is given by:

$$C (\Omega) = \frac{1}{1 + i t_{sr} \sqrt{2 \gamma_f g K (\Omega) / (g^2 + \Omega^2 + i \Omega \gamma_f)}}, \quad (6.29)$$

in which the $\hat{a}_{2,3}$ represents the field at 2,3 in Fig. 6.10.

- **Open/Closed loop transfer function between 1 and 2** — The open loop transfer function between 1 and 2 can be easily seen as:

$$T_{12} (\Omega) = r_{SRM} C (\Omega) \frac{- g^2 + i \Omega \gamma_f - \Omega^2}{g^2 + i \Omega \gamma_f + \Omega^2} e^{2i \Delta \tau}. \quad (6.30)$$

For determine the analytical behavior of this open loop transfer function, we have to specify the concrete form of the filter function $K (\Omega)$. It is implied in the single mode situation that the kernel is a differential kernel, therefore we assume $K (\Omega)$ to be:

$$K (\Omega) = \frac{1}{\sqrt{\gamma_c}} (\beta_1 \gamma_c - i \beta_2 \Omega). \quad (6.31)$$

In this case, the denominator $T_{12} (\Omega)$ is equal to zero has the form of:

$$g^2 + i \Omega \gamma_f + \Omega^2 + i t_{sr} g \sqrt{2 \gamma_f / \gamma_c} (\beta_1 \gamma_c - i \beta_2 \Omega) = 0. \quad (6.32)$$

The number of solutions of this equation with positive imaginary part will be the P-number in Nyquist analysis.

For determine the value of these coefficients, we have to consider (1): $\gamma_{opt} / \gamma_f$ as small as possible, (2) the $\gamma_f$ can not be too large since the optomechanical dynamics of the filter cavity should obey resolved sideband approximation, (3) the broadening of the bandwidth is as large as possible, which means $g = \omega_s = \sqrt{\epsilon \gamma_f / L_{arm}}$. Suppose we chose $g = \alpha \gamma_{sr}$, this leads to $T_f = 4 \alpha^2 T_{SRM}^2 L_f / L_{arm}$. If we set $T_{SRM} = 0.2$, $L_f = 5 \text{mm}$ and $L_{arm} = 4 \text{km}$, the $T_f$ is given by $\alpha^2 \times 2 \times 10^{-7}$. If the finesse of the filter cavity
Figure 6.11 – The Nyquist plot of $T_{12}(\Omega)$ when the feedback control is not added in. The black, red and blue curves correspond to the situation when the signal recycling mirror power transmissivity is $T_{\text{SRM}} = 0.8, 0.5, 0.2$ respectively. It is clear from this figure that all of them are unstable since their winding number around $(-1, 0)$ is zero.

is around $10^5$, then we have the value of $\alpha$ is around 10. The $\gamma_{\text{opt}}/\gamma_{\ell} \sim 0.2$, which is acceptable. Choosing $g = \alpha \gamma_{\text{SRM}}$ and normalizing the open loop function by $\gamma_{\text{SRM}}$, we obtain:

$$T_{12}(x) = \frac{r_{\text{SRM}} e^{2ixT_{\text{SRM}}/4}(-\tilde{g}^2 - x^2 + ix\alpha \tilde{g}T_{\text{SRM}}/4)}{\tilde{g}^2 + x^2 + ix\alpha \tilde{g}T_{\text{SRM}}/4 + i\tilde{t}_{\text{SRM}} \sqrt{2\tilde{\gamma}_f/\tilde{\gamma}_c} \tilde{g}[\beta_1 \tilde{\gamma}_c - i\beta_2 x]} \quad (6.33)$$

where all the quantities with tilde are the quantities normalized by $\gamma_{\text{SRM}}$. In plotting the Nyquist plot of the $T_{12}(x)$, we assume that $\tilde{g} = 10$ (equivalently, $\alpha = 10$) and $\tilde{\gamma}_c = 1$. The result is given in Figs. 6.12, 6.11 and 6.13.

The stability of the system is based on the Nyquist analysis of the external transfer function $T_{12}(\Omega)$. Nyquist theorem requires that the number of poles $P$ of $T_{\text{ext}}(\Omega)$ enclosed by the Nyquist contour, the number of roots of the closed loop transfer function $Z$ and the winding number of the $T_{\text{ext}}(\Omega)$ around the $-1$ should satisfy:

$$Z = N + P. \quad (6.34)$$
Figure 6.12 – The Nyquist plot of $T_{12}(\Omega)$ when the feedback control is chosen to be the one proposed in the single-mode example where we have $\beta_1 = \beta_2 = i/\sqrt{2}$. The black, red and blue curves corresponds to the situation when the signal recycling mirror power transmissivity is $T_{\text{SRM}} = 0.8, 0.5, 0.2$ respectively. It is clear from this figure that all of them are unstable since their winding number around $(-1,0)$ is zero.
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Figure 6.13 – The Nyquist plot of $T_{12}(\Omega)$ when the feedback control is slightly enhanced by setting $\beta_1 = \beta_2 = i$. The black, red and blue curves corresponds to the situation when the signal recycling mirror power transmissivity is $T_{\text{SRM}} = 0.8, 0.5, 0.2$ respectively. It is clear from this figure that all of them are stable since their winding number around $(-1,0)$ is 1.
This means that if the system is stable \((Z = 0)\), the \(N\) and \(P\) must satisfy

\[
N = -P. \tag{6.35}
\]

It is straightforward to show that the one (and only one) of the two zeros of the denominator of \(T_{12}(\Omega)\) is located on both the upper and lower half-planes. This means in above criteria, \(P = 1\) inside the Nyquist contour. In order for the system to be stable, The \(N = -1\), which means the Nyquist plot of \(T_{12}(\Omega)\) should wind around \(-1\) counter-clock-wisely once and only once.

As shown in these Nyquist diagrams, the system is still unstable if we use the feedback control as in the single-mode case, namely \(\beta_1 = \beta = i/\sqrt{2}\) (Fig. 6.13). Strengthen the feedback strength by \(\sqrt{2}\) (namely, \(\beta_1 = \beta_2 = i\)) will change this situation, in Fig. 6.13, the Nyquist diagram encircles \(-1\) counter-clock-wisely only once. This subtle difference between our more exact analysis and the single-mode analysis implies that we need to be careful in choosing the model when we study the stability of these systems.

## 6.6 Conclusions

This Chapter discussed another possible application of optomechanical filter cavity for surpass the Mizuno hypothesis in laser interferometer gravitational wave detectors. For the filter having an advanced phase, the system works in a unstable region where the optomechanical filter cavity is pumped in a blue-detuned way. To properly control the unstable dynamics of the system, we designed a differential feedback control scheme by using Nyquist analysis. The requirement for the feasibility of this scheme is quite strict, since it is very sensitive to the thermal noise.

## 6.7 Appendix: Nyquist criteria

The behavior of control systems is usually described by gain functions. For a control system with feedback process, the open-loop gain function \(G_o(\Omega)\) is used to describe the information transfer ignoring the feedback process, while the closed loop gain function \(G_c(\Omega)\) includes the effect of the feedback process. The relationship between
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\[ G_o(\Omega) \text{ and } G_c(\Omega) \text{ can be written as:} \]

\[ G_c(\Omega) = \frac{G_o(\Omega)}{1 + H(\Omega)G_o(\Omega)}. \]  

(6.36)

The \( H(\Omega) \) is the gain function for the feedback process itself, it is clear from Fig 7.7 that in our system, it is just the reflection of the SRM: \(-r_s\).

The stability of the system depends critically on the poles of the close-loop transfer function, that is, it depends on the poles of \( G_o(\Omega) \) and also the zeros of \( 1 - r_sG_o(\Omega) \). However, computing the poles and zeros of these gain functions is generally a difficult task when these functions are non-rational. The Nyquist stability criterion is a graphical technique for determining the stability of a control system, which is based on the following \textit{Lemma}: Cauchy argument principle.

The Cauchy argument principle starts from the Nyquist mapping, which maps the complex argument \( \Omega \)-plane to the complex \( F(\Omega) \)-plane. If we have a clockwise contour in \( \Omega \)-plane encircling a \textit{zero} of \( F(\Omega) \), correspondingly, the contour also encircles the \textit{origin clock-wisely} in the \( F(\Omega) \)-plane. However, if we have a clockwise contour in \( \Omega \)-plane encircling a \textit{pole}, then the corresponding contour will encircles the \textit{infinity clock-wisely} in the \( F(\Omega) \)-plane thereby encircling the \textit{origin} in an \textit{anti-clockwise} way.

In general, if we have a contour in the \( F(\Omega) \)-plane encircling the origin \( N \) times clockwise, that means in the \( \Omega \)-plane, the number of zeros (\( Z \)) and the number of poles (\( P \)) satisfy:

\[ Z = N + P. \]  

(6.37)

This equality is the Cauchy argument principle.

The transformation for quantity \( A(t) \) between the frequency domain and the time domain is defined as: \( A(t) = \int_{-\infty}^{\infty} A(\Omega)e^{-i\Omega t}. \) Therefore if \( A(\Omega) \) has poles in the upper-half plane, we will have instabilities for causal system (\( t > 0 \)). Now we choose the contour encircling the upper-half \( \Omega \)-plane as “Nyquist contour”. If the system is stable, then the \( Z \) of \( 1 - r_sG_o(\Omega) \) (the denominator of close-loop gain function) inside the Nyquist contour should be zero. As a result, the Cauchy argument principle becomes \( N = -P \), which is the Nyquist criteria.


Chapter 7

Quantum noise of white cavity based on double-pumped gain medium

7.1 Preface

In the Chapter 1, we have discussed that the sensitivity of a dual recycling interferometer has a trade-off between the peak sensitivity and the detection bandwidth. This trade-off is called Mizuno hypothesis. Several proposals are discussed for surpassing this Mizuno hypothesis by modifying the dispersion behavior of the interferometer. In this chapter, we will discuss one of these proposals which based on the double-pumped gain medium. Our discussion will focus on the stability and additional noise of these schemes. It will be revealed that such schemes actually can not surpass the Mizuno hypothesis. This chapter based on the collaborative theoretical works done with Haixing Miao, Chunmeng Zhao and Yanbei Chen, and the collaborative experimental works done with Jiayi Qin, Chunmeng Zhao, Li Ju and David. G. Blair. The relevant arxiv papers can be found in \url{http://arxiv.org/abs/1501.01349} (submitted to Phys. Rev. A) and \url{http://arxiv.org/abs/1502.06083} (submitted to Optics Letter).

7.2 Background and introduction

Second-generation large-scale interferometric gravitational wave (GW) detectors, such as advanced LIGO\cite{1}, advanced VIRGO\cite{2} and KAGRA\cite{3}, are designed to operate at better sensitivity than the first generation detectors. This improvement in sensi-
tivity comes from increase in the optical power and introduction of a signal recycling mirror (SRM) to the initial configuration \[4\]. The SRM on the dark port forms a signal recycling cavity (SRC) with the input test mass mirror (ITM). The position of SRM determines the propagation phase of the signal light inside the SRC, and control of the SRM parameters allows for adjustments to the frequency response of the interferometer \[5, 6\]. Two typical operational modes are the signal recycling mode and resonant sideband extraction mode (RSE). The signal recycling mode enhances the sideband carrier of the GW signal inside the cavity, while the RSE mode increases the detection bandwidth which is the effective bandwidth of the combined SRC and arm cavity \[7\].

However, broadening the detection bandwidth in the RSE mode comes at the loss of the peak sensitivity; while enhancing the peak sensitivity in SRC results in a narrower detection bandwidth. This trade-off is represented by the integrated sensitivity:

$$\rho = \int_{0}^{\omega_{\text{FSR}}} \frac{1}{S_{\text{hh}}(\Omega)} d\Omega = \frac{4\pi L_{\text{arm}} P_{c}\omega_{0}}{\hbar c},$$  \hspace{1cm} (7.1)$$

which only depends on the intra-cavity power $P_{c}$ and cavity length $L_{\text{arm}}$, and is independent of the property of SRC. The $\omega_{0}, \omega_{\text{FSR}}, c$ here are the laser frequency, free-spectral range, and the speed of light, respectively. Here, we only consider the shot noise limited strain sensitivity $S_{\text{hh}}(\Omega)$ since radiation pressure noise can in principle be evaded using frequency dependent readout or sufficiently heavy test masses. Such a trade-off between bandwidth and peak sensitivity is due to the accumulated phase of the sideband field propagating inside the arm cavity. There are several proposals in the literature that try to achieve the resonant amplification of the signal without decreasing the bandwidth, using the idea of white light cavity. Among those, Wicht et al. were the first to suggest placing an atomic gain medium with anomalous dispersion inside the SRC to cancel the propagation phase \[8, 9\]. This idea was then followed by Pati and Yum et al. with different types of active mediums \[10, 11\].

The anomalous dispersion phenomenon and the interesting “superluminal” physics of the propagation of light pulse in these active mediums have been theoretically discussed \[12\] and experimentally demonstrated \[13, 14\]. In these experiments, the anomalous dispersion is usually realized by using a double-pumped gain medium in
7.2. Background and introduction

Figure 7.1 – (a) the typical dual recycled interferometer configuration for an advanced gravitational wave detector, with an atomic gain medium (blue block) embedded inside the SRC to compensate the phase delay of the arm cavity. An internal SRM (iSRM) with the same transmissivity as ITM is introduced to make impedance matching so that effectively we can view the compound mirror (consists of ITM and iSRM) as transparent to the sideband field [17]; (b) the energy levels of the gain medium atoms. Two far-detuned strong control laser with frequency $\omega_a$ and $\omega_b$ couple the energy levels $|3\rangle$ and $|1\rangle$. The signal field interacts with $|2\rangle$ and $|3\rangle$.

which the anomalous dispersion lies in between the two gain peaks. As discussed by Kuzmich et.al [15], the gain medium is subject to quantum noise that accompanies the amplification process. In addition, the gain medium could cause lasing when placed inside a resonant cavity. To investigate how the quantum noise and associated gain influence the detector sensitivity and dynamics, we develop an input-output formalism for the optical field propagating through the gain medium. Using this formalism, we make a detailed analysis of the quantum shot noise limited sensitivity for a typical gravitational wave detector configuration implementing the white light cavity idea, as shown in Fig.7.1. Specifically, we consider: (i) the requirement for canceling the propagation phase shift; (ii) the optical stability of the interferometer system with the gain medium; (iii) the noise associated with the amplification process. Taking these factors into account, we find that the integrated shot noise limited sensitivity is still limited by Eq.(7.1) when the gain medium itself is stable (not lasing) and stationary.
7.3 Basic theory of opto-atomic dynamics

It is well known that in the energy representation, the free-Hamiltonian for atoms can be written as:

$$\hat{H}_a = \sum_i \hbar \omega_i \hat{\sigma}_{ii}$$  (7.2)

where \(\hat{\sigma}_{ii}(t) = |i(t)\rangle \langle i(t)|\) is the population operator of the atomic energy level \(a\). It is easy to see that \(\hat{H}_a |i(t)\rangle = \hbar \omega_i |i(t)\rangle\) where \(|i(t)\rangle = |i\rangle e^{-i\omega_i t}\). The free-Hamiltonian for the electromagnetic field, as discussed in Chapter 2, is given by:

$$H_f = \int d\mathbf{k} \hbar c |\mathbf{k}| \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}.$$  (7.3)

The interaction between the light field and the atoms are described by the following Hamiltonian:

$$\hat{H}_{\text{int}} = -e \hat{\mathbf{E}} \cdot \hat{\mathbf{x}},$$  (7.4)

where \(\mathbf{E}\) is the electric field of the light and \(e \mathbf{x}\) is the dipole moment of the atom. In the energy representation, this Hamiltonian can be written as

$$\hat{H}_{\text{int}} = -e \mathbf{E} \cdot \sum_{ij} |i\rangle \langle j| \langle i| \mathbf{x} |j\rangle \equiv -\frac{\hbar}{2} \sum_{ij} \mathbf{\mu}_{ij} \cdot \mathbf{E} \hat{\sigma}_{ij}$$  (7.5)

The \(|i| \mathbf{x} |j\rangle\) only has non-diagonal components since the dipole transition must happen between different energy levels. This is the Hamiltonian we are going to use in the following discussion.

The electric field can be written as:

$$\mathbf{E} = \sum_k \sqrt{\frac{2\pi \hbar c |\mathbf{k}|}{V}} \epsilon_k(t) e^{ik \cdot \mathbf{x}} + h.c,$$  (7.6)

where \(\epsilon\) is the polarization vector of the electric field. Substituting the above electric field equation into the interaction Hamiltonian gives:

$$\hat{H}_{\text{int}} = -\frac{\hbar}{2} \sum_k \sum_{ij} (\mathbf{\mu}_{ij} \cdot \epsilon) \sqrt{\frac{2\pi \hbar c |\mathbf{k}|}{V}} \hat{a}_k(t) e^{ik \cdot \mathbf{x}} \hat{\sigma}_{ij} + h.c$$  (7.7)

where we have redefine the \(\mu_{ij}\) to absorb the \(\sqrt{2\pi \hbar c |\mathbf{k}|/V}\). Assuming that the dipole moments of atoms are all aligned to one direction (say \(z\)–direction), then we have \(\mu_{ij} \cdot \epsilon = \mu_{ij}^z\).
7.3.1 A simple example: two-level-system (TLS)

For demonstrating the opto-atomic dynamics, this section considers a simple example so that the atom is a two-level system and we have totally $N$ TLS atoms. The equations of motion are then given by:

$$
\dot{\sigma}_{21}^m = -i\omega_a \sigma_{21}^m + \frac{i}{2} \sum_k (\mu_{21}^m \cdot \epsilon) \hat{a}_k(t) e^{-i k x_m} (\sigma_{22}^m - \sigma_{11}^m), \tag{7.8a}
$$

$$
\dot{a}_k = -i\omega_k \hat{a}_k + \frac{i}{2} \sum_m (\mu_{21}^m \cdot \epsilon) e^{-i k x_m} \sigma_{12}^m. \tag{7.8b}
$$

The equation of motion for $\hat{a}_k$ can be solved:

$$
\hat{a}_k(t) = \hat{a}_k(0) e^{-i\omega_k t} + \frac{i}{2} \sum_m (\mu_{21}^m \cdot \epsilon) \int_{-\infty}^{t} e^{-i k x_m + i\omega_k (t' - t)} \sigma_{12}^m(t') dt'. \tag{7.9}
$$

The first term corresponding to the free evolution of the electromagnetic fields. The second term is the contribution from the interaction between the electromagnetic field and the atoms. Specifically, the term where $m' = m$ corresponds to the radiative reaction. Substituting the $\hat{a}_k$ back into the equation of motion of $\sigma_{21}^m$ leads to:

$$
\dot{\sigma}_{21}^m = -i\omega_a \sigma_{21}^m + \frac{i}{2} \sum_k (\mu_{21}^m \cdot \epsilon) \hat{a}_k(0) e^{i\omega_k t - i k x_m} (\sigma_{22}^m - \sigma_{11}^m)
+ \frac{1}{4} \sum_{k,m'} (\mu_{21}^m \cdot \epsilon) \int_{-\infty}^{t} (\mu_{21}^{m'} \cdot \epsilon) e^{i k (x_{m'} - x_m) + i\omega_k (t' - t)} \sigma_{21}^{m'}(t') (\sigma_{22}^m(t) - \sigma_{11}^m(t)) dt'.
$$

(7.10)

When $m = m'$, the last term reduces to:

$$
-\frac{1}{4} \sum_k |\mu_{21}^m \cdot \epsilon|^2 \int_{-\infty}^{t} e^{-i\omega_k (t' - t)} \sigma_{21}^m(t') \sigma_{11}^m(t) dt' \tag{7.11}
$$

$$
= -\frac{1}{4} \sum_k |\mu_{21}^m \cdot \epsilon|^2 \int_{-\infty}^{t} e^{-i\omega_k (t' - t)} \sigma_{21}^m(t) \delta(t - t') dt',
$$

where the $\sigma_{22}$ term is ignored since its right operator product with $\sigma_{21}$ is apparently zero. By using Wigner-Weisskopf approximation \[19\], we have

$$
\dot{\sigma}_{21}^m = -i\omega_a \sigma_{21}^m - \gamma \sigma_{21}^m + \frac{i}{2} \sum_k (\mu_{21}^m \cdot \epsilon) \hat{a}_k(0) e^{i\omega_k t - i k x_m} (\sigma_{22}^m - \sigma_{11}^m)
+ \frac{1}{4} \sum_{k,m' \neq m} (\mu_{21}^m \cdot \epsilon) \int_{-\infty}^{t} (\mu_{21}^{m'} \cdot \epsilon) e^{i k (x_{m'} - x_m) - i\omega_k (t' - t)} \sigma_{21}^{m'}(t') (\sigma_{22}^m(t) - \sigma_{11}^m(t)) dt'.
$$

(7.12)
where the last term corresponds to the “polarization” of the \( m_{th} \) atom induced by the electromagnetic field from the transition of all the other atoms, we call this effect “collectively induced polarization” [20].

In general, the atomic dynamics consists of three parts: 1) original EM field driving, 2) radiation reaction, 3) collectively induced polarization.

7.3.2 Maxwell-Bloch equation

Based on the single-atom optoatomic dynamics equations discussed in the previous subsection, this subsection formulates the effect of the opto-atomic interaction to the propagation of optical fields. We first define the collective atomic operators as

\[
\hat{O}_{x}(t) = \frac{1}{N_d} \sum_{m \in m_{x}} \hat{O}_{m}(t),
\]

in a small cell with length \( d \), which is much larger than the mean distance between atoms, but still smaller than the optical wavelength. The summation is over the coordinate \([x - d/2, x + d/2]\). These collective operators satisfy the following equation of motion:

\[
\dot{\sigma}_{21}^{x} = -(i \omega_{a} + \gamma)\sigma_{21}^{x} + \frac{1}{4 N_d} \sum_{k} |\mu_{12} \cdot \epsilon|^2 \int_{-\infty}^{t} dt' e^{-i\omega_{k}(t'-t)} \left[ \sum_{m' \neq m} e^{i k \cdot x_{m}} [\hat{\sigma}_{m'}^{m_{21}}(t) - \hat{\sigma}_{m'}^{m_{11}}(t)] \right] + \text{Langevin terms}.
\]

Note that the summation over \( m \) is within the small cell while the summation over \( m' \) is over all atoms except the \( m_{th} \) one. For express the summation over \( \hat{\sigma}_{m'}^{m_{21}} \) in terms of the collective atomic operators, we need to average over \( x_{m}' \) within the small cell, that is, integrate over the above formula as:

\[
\frac{1}{Sd} \int_{\text{cell}} e^{i k x_{m}'} = \frac{1}{d} \int_{x-d/2}^{x+d/2} e^{i k x_{m}'} \frac{1}{S} \int_{S} dydz e^{ik_{y} y + ik_{z} z} (7.15)
\]

Using the fact that \( k \cdot d \ll 1 \), and \( |k_{x}| \approx k \) the above integral can be simplified to be: \( e^{i k x} \). In another words, we can integrate out the \( y, z \) direction by noticing that the diffraction function:

\[
F(k_{y}, k_{z}) = \frac{1}{S} \int_{S} dydz e^{ik_{y} y + ik_{z} z} \approx 1 \quad (7.16)
\]
where $S$ is the transversal cross-section of the small cell, then all the $k \cdot x(x')$ in the above formula reduce to a scalar product: $k_2 x(x')$. Then the dynamics of collective $\hat{\sigma}_{21}$ can be written as:

$$
\dot{\hat{\sigma}}_{21}^x = -(i\omega_a + \gamma)\hat{\sigma}_{21}^x + \frac{1}{4}N_d|\mu_{21}|^2 \sum_{k,x'} \int_{-\infty}^t e^{ik_2(x'-x) - i\omega_a(t'-t)}
$$

$$
\sigma_{21}(t')[\hat{\sigma}_{22}(t) - \hat{\sigma}_{11}(t)]dt' + \text{Langevin terms.} \tag{7.17}
$$

Further treatment of the above equation is to separate the fast oscillation part of the $\hat{\sigma}_{21}$ by noticing that:

$$
\dot{\hat{O}}_x(t) = O_L(x,t)e^{-ik_2x - i\omega_a t} + O_R(x,t)e^{ik_2x - i\omega_a t} \tag{7.18}
$$

where the $O_{L/R}(x,t)$ is the left/right propagating slowly-varying amplitude operator. Substitute this definition into the above equation of motion for collective operators, we can have:

$$
\dot{\hat{\sigma}}_{21}^L = -\gamma\hat{\sigma}_{21}^L + \frac{1}{4}|\mu_{21}|^2 N_d \sum_{x',k} \int_{-\infty}^t e^{i(k-k_0)(x-x') + c(t'-t)}
$$

$$
\sigma_{21}(t')[\hat{\sigma}_{22}(t) - \hat{\sigma}_{11}(t)]dt' + \text{Langevin terms.} \tag{7.19}
$$

The summation over $k$ can be worked out in the continuous limit:

$$
\sum_k e^{i(k-k_0)(x-x') + c(t'-t)} \rightarrow L \int_0^\infty dk e^{i(k-k_0)(x-x') + c(t'-t)} = L \int_{-k_0}^\infty d\delta k e^{i\delta k[(x-x')+c(t'-t)]}.
$$

The narrowband approximation we have discussed in Chapter 2 tells us that the lower limit of the above integration can be extended to $-\infty$ as an approximation. Then the integral in the dynamics of $\hat{\sigma}_{21}^L$ can be written as a $\delta-$function.

$$
\dot{\hat{\sigma}}_{21}^L = -\gamma\hat{\sigma}_{21}^L + \frac{1}{4}|\mu_{21}|^2 N_dL_d \sum_{x'} \int_{-\infty}^t \delta [(t' - t) + \frac{x' - x}{c}]
$$

$$
\sigma_{21}(t')[\hat{\sigma}_{22}(t) - \hat{\sigma}_{11}(t)]dt' + \text{Langevin terms.} \tag{7.21}
$$

If $t > t' - (x' - x)/c$, we have:

$$
\dot{\hat{\sigma}}_{21}^L = -\gamma\hat{\sigma}_{21}^L + |\mu_{21}|^2 N_dL_d \sum_{x'>x} \int_{x'}^t \delta [(x' - t) - x/c] \dot{\hat{\sigma}}_{21}^L(x', t - \frac{x' - x}{c})dt' + \text{Langevin terms.} \tag{7.22}
$$

\footnote{The equation for the right propagating operators can be obtained through the same way.}
Chapter 7. Quantum noise of white cavity based on double-pumped gain medium

It is easy to see that this formula can be viewed as collective Rabi-oscillation of \( \hat{\sigma}_{21}^L \) with the matter-induced EM field:

\[
\hat{A}_L(x, t) = \frac{N_d L_d}{4c} |\mu_{21}| \epsilon |^2 \sum_{x' < x} \hat{\sigma}_{21}^L (x', t - \frac{x - x'}{c}).
\] (7.23)

This field operator apparently satisfies the wave equation:

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \hat{A}_L(x, t) = \text{source term}.
\] (7.24)

This is the Maxwell-Bloch equation of the matter-induced electromagnetic fields propagate inside the atomic gas.

As a summary, the above derivation tells us that if the typical wavelength of the slowly-varying amplitude is much larger than the size of the source, then the interaction between the slowly-varying amplitude and the source can be treated as a point interaction, all the atoms of the source collectively interacts with the field.

Notice that this process is somewhat similar to the so-called superradiance effect first discussed by Robert Dicke [21]. However, the difference between Dicke superradiance and the collective interaction here is that:

1) In Dicke’s model, the typical size of the medium is actually smaller than even one optical wavelength. Therefore it is the collective interaction between the medium and the fast-oscillating optical field, instead of its slowly varying amplitude.

2) Dicke’s superradiance only happens for the atoms prepared in the special Dicke states.

The above discussion about how the optical field interact with the atoms follows the traditional way given by D. Polder [20]. In the next subsection, we are going to discuss this problem in a more simpler way starting by coarse-graining directly on the Hamiltonian. The analyze of the quantum noise of the double-gain medium will be based on this coarse-grained Hamiltonian.
7.3. Basic theory of opto-atomic dynamics

7.3.3 Hamiltonian coarse-graining

For a continuous EM field mode (uni-directional) and atoms as an 1-dimensional model, we have the following Hamiltonian:

\[
\hat{H} = \hbar c \int_0^\infty dk k \hat{a}_k \hat{a}_k^\dagger + \frac{1}{2} h \mu_{21} \int dk \sum_{j=1}^N \sqrt{\frac{2 \pi \hbar c k}{L}} \sigma_{21}^j \hat{a}_k(t) e^{ikx_j} + \text{h.c}
\]

+ interaction of other energy levels with EM field.

(7.25)

Since the probe field only interacts with the \(\hat{\sigma}_{21}\), therefore in the above Hamiltonian, we temporarily hide the EM interaction of other energy levels. Since the interested frequency scale is around \(k_0\), therefore the \(\sqrt{2 \pi \hbar c k / L}\) can be approximated to be \(\sqrt{2 \pi \hbar c k_0 / L}\). Then the above interaction Hamiltonian can be written as

\[
\hat{H}_{\text{int}} = -\frac{1}{2} h \mu_{21} \sqrt{\frac{2 \pi \hbar c k_0}{L}} \sum_{j=1}^N \hat{\sigma}_{21}^j \int_{-k_0}^{k_0} dk' \hat{a}_{k_0+k'}(t) e^{ik'x_j} e^{ik_0x_j},
\]

(7.26)

in which the integration can be treated as Fourier transformation of \(\hat{a}_{k_0+k'}\) in the narrow band approximation:

\[
\hat{H}_{\text{int}} = -\frac{1}{2} h \mu_{21} \sqrt{\frac{2 \pi \hbar c k_0}{L}} \sum_{j=1}^N \hat{\sigma}_{21}^j \hat{a}(x_j) e^{ik_0x_j},
\]

(7.27)

with \(a(x_j)\) is the slowly varying amplitude of the EM field in the \(x-\) domain. Suppose the coarse-graining cell has the length \(d\) ([\(x - d/2, x + d/2\)]) which is much smaller than \(\lambda\) but contains enough atoms, then we can average the above formula over these cells:

\[
\hat{H}_{\text{int}} = -\frac{1}{2} h \mu_{21} \sqrt{\frac{2 \pi \hbar c k_0}{L}} \sum_x \hat{\sigma}_{21}(x) \hat{a}(x).
\]

(7.28)

Notice that \(\hat{\sigma}_{21}(x)\) also has a spatially slowly varying part which is \(e^{-ik_0x}\), which can cancel \(e^{ik_0x}\) associated with \(\hat{a}(x)\). In the above equation both the \(\hat{\sigma}_{21}\) and \(\hat{a}\) are actually their spatial slowly varying parts.
Chapter 7. Quantum noise of white cavity based on double-pumped gain medium

Since the $\hat{a}$ we are interested in has spatial frequency $\sim \Omega$, where $\Omega$ is about the gravitational wave frequency. Therefore, the spatial resolution scale is $\sim 2\pi c/\Omega$ which is much much longer than the scale of atomic cloud (See Fig. 2). In this case, the $\hat{a}(x)$ is almost a constant across the atomic cloud. This allows us to further approximate the above coarse-graining Hamiltonian to be

$$\hat{H}_{\text{int}} = -\frac{1}{2} \hbar \mu_{12} \sqrt{\frac{2\pi \hbar c k_0}{L}} \left[ \sum_x \hat{\sigma}_{21}(x) \right] \hat{a}(x) + h.c, \quad \text{with} \quad \sum_x \hat{\sigma}_{21}(x) = \sum_j^{N} \hat{\sigma}_{j1}^{\dagger}.$$  \hfill (7.29)

We can define $\hat{\sigma}_{21} = \sum_x \hat{\sigma}_{21}(x) = \sum_j^{N} \hat{\sigma}_{j1}^{\dagger}$ as collective atomic transition operator. Notice that this definition is different from the definition of collective spin operators in the previous sections. The benefit of this definition is that the newly-defined operator still satisfies the same commutation relation as the single-atom operator. Using this collective atom operator, we have coarse-grained Hamiltonian:

$$\hat{H}_{\text{int}} = -\frac{1}{2} \hbar \mu_{12} \sqrt{\frac{2\pi \hbar c k_0}{L}} \hat{\sigma}_{12} \hat{a}(x) + h.c$$  \hfill (7.30)

The following discussion will use this coarse-grained Hamiltonian to derive the input-output relation of the gain medium. But firstly, let us give a brief description of our main result.

### 7.4 A brief summary of the main results

Before presenting the detailed analysis, we briefly summarize our main results in this section. The susceptibility of the double-pumped gain medium $\chi(\Omega)$ that we derive is given by (the same as in Refs. [14] [12] but with slightly different notations):

$$\chi(\Omega) = \frac{2i\gamma_{\text{opt}}}{i(\Delta_0 + \Omega) - \gamma_{12} + \gamma_{\text{opt}}} + \frac{2i\gamma_{\text{opt}}}{i(-\Delta_0 + \Omega) - \gamma_{12} + \gamma_{\text{opt}}},$$  \hfill (7.31)

where $\Delta_0$ is one half of the frequency difference between two control fields, and $\Omega$ is the sideband frequency of the probe field with the carrier frequency $\omega_p$. The damping rate $\gamma_{12}$ is the effective atomic transition rate from state $|2\rangle$ to $|1\rangle$, while $\gamma_{\text{opt}}$, which depends on pumping power of the control fields, is the transition rate between $|1\rangle$ to $|2\rangle$ mediated by a virtual excitation of $|3\rangle$. In terms of $\chi$, the ingoing and outgoing field $\hat{a}_{\text{in}}, \hat{a}_{\text{out}}$ are related by (temporarily ignoring additional noise term that will be
7.4. A brief summary of the main results

\[ \hat{a}_{\text{out}}(\Omega) = [1 + i\chi(\Omega)/2]\hat{a}_{\text{in}}(\Omega). \] (7.32)

In deriving the effective phase advance, here we use weak-coupling approximation which allows us to express the input-output relation Eq. (7.32) for an unidirectional sideband field passing through the gain medium as:

\[ \hat{a}_{\text{out}}(\Omega) \approx e^{i\chi_r(\Omega)/2}e^{-\chi_i(\Omega)/2}\hat{a}_{\text{in}}(\Omega). \] (7.33)

Here \( \chi_r(\Omega) \) and \( -\chi_i(\Omega) \) are the real and imaginary part of the susceptibility \( \chi(\Omega) \) of the medium, which describe, respectively, the phase accumulation and the amplitude change of the sideband field after passing through the medium as shown in Fig. 7.3. The weak-coupling approximation here is:

\[ \Delta_0^2 + (\gamma_{12} - \gamma_{\text{opt}})^2 \gg \gamma_{\text{opt}}^2. \] (7.34)

In order to compensate the round-trip propagation phase inside the arm cavity thereby broadening the bandwidth of the optical cavity, the susceptibility should satisfy \( d\chi(0)/d\Omega \approx -2L_{\text{arm}}/c \) (negative dispersion), which leads to:

\[ \gamma_{\text{opt}}[(\gamma_{12} - \gamma_{\text{opt}})^2 - \Delta_0^2] \leq \frac{L_{\text{arm}}}{c}. \] (7.35)

The above phase-cancelation condition will reduce to \( \gamma_{\text{opt}} = \Delta_0^2L_{\text{arm}}/c \) when \( |\gamma_{12} - \gamma_{\text{opt}}| \ll \Delta_0^2 \). In our calculation, we have used the exact formula Eq. (7.35).

If we fixed the value of \( \gamma_{12} \) and \( \gamma_{\text{opt}} \), then the phase cancelation condition becomes a second order algebraic equation for \( \Delta_0^2 \). Suppose this equation has two roots \( x_1, x_2 \), then we have:

\[ x_1x_2 = (\gamma_{12} - \gamma_{\text{opt}})^2[(\gamma_{12} - \gamma_{\text{opt}})^2 + \gamma_{\text{opt}}c/(2L_{\text{arm}})], \]

\[ x_1 + x_2 = \gamma_{\text{opt}}c/(2L_{\text{arm}}) - 2(\gamma_{12} - \gamma_{\text{opt}})^2. \] (7.36)

Notice that in the above equations, \( x_1x_2 \) is always positive, thereby \( x_1 + x_2 \) can only be positive:

\[ (\gamma_{12} - \gamma_{\text{opt}})^2 < \gamma_{\text{opt}}c/(4L_{\text{arm}}). \] (7.37)

On the other hand, Eq. (7.35) must have real roots, which gives:

\[ (\gamma_{12} - \gamma_{\text{opt}})^2 < \gamma_{\text{opt}}c/(8L_{\text{arm}}), \] (7.38)
Figure 7.3 – Phase angle and amplitude gain of the sideband field propagating through the atomic gain medium as functions of the normalized (by $\Delta_0$) sideband frequency. The top figure shows the negative dispersion of the atomic gain medium. The white light cavity bandwidth is the linear region between $-\Delta_0$ and $\Delta_0$. The bottom figure shows that the gain is negligibly small, except when $\Omega \sim \Delta_0$. In these frequency regions, the gain is high and need to be considered in the design for preventing the possible instability (See Section III for detailed analysis).
which is a more stringent condition than Eq. (7.37).

In summary, considering weak coupling approximation and the requirement of the phase-cancelation condition, our parameters must satisfy Eq. (7.38). It is important to notice that there are always two $\Delta_0^2$ corresponding to a fixed set of $(\gamma_{12}, \gamma_{\text{opt}})$. In plotting the Fig. 7.5 and Fig. 7.9 we should take into account both roots.

Under the condition in Eq. (7.34) and Eq. (7.38), we explore the relevant parameter regime for studying the dynamical behavior of the gain medium. Firstly, as we analyze in detail in Section III and IV(A), the system has two different types of instability (lasing) 1) if $\gamma_{12} < \gamma_{\text{opt}}$, there will be a population inversion between level $|1\rangle$ and $|2\rangle$, the gain medium starts lasing by itself, which we name as “atomic instability”; 2) if the photon loss rate for each round trip inside the cavity is less than the photon increasing rate through the amplification by the gain medium, the cavity-medium system starts lasing, which we name as “optical instability”. In Fig. 7.4, we plot the phase diagram for the stability of the system. This figure gives a constraint on the possible parameter region for $\gamma_{12}$ and $\gamma_{\text{opt}}$ of the atomic gain medium (with fixed SRM reflectivity $r_s$), if lasing were to be avoided. Notice that we choose the re-scaled parameter $(\eta = \gamma_{\text{opt}}/\gamma_{12}, \xi = 8(\gamma_{12} - \gamma_{\text{opt}})^2L/c\gamma_{12})$ instead of $(\gamma_{12}, \gamma_{\text{opt}})$ and survey them within $0 < \eta, \xi < 1$. These new parameters help us exclude the atomic instability region $(\eta > 1)$ and the region where the phase-cancelation condition is unsatisfied $(\xi > 1)$.

Secondly, as implied by the above input-output relation, the gain medium is a parametric amplifier. Therefore, as first discussed by Caves [18], there must be an additional noise term on the right hand side of Eq. (7.33) for keeping the commutation relation for $\hat{a}_{\text{out}}$ to be $[\hat{a}_{\text{out}}(t), \hat{a}_{\text{out}}^\dagger(t')] = \delta(t - t')$. This additional noise is due to the quantum fluctuation that causes spontaneous transition between $|1\rangle$ and $|2\rangle$. and degrades the signal to noise ratio. From the Hamiltonian, we can derive the noise terms from Heisenberg equations of motion. Their effect on the integrated shot noise limited sensitivity improvement factor (defined in Eq. (7.84)) is given in Fig. 7.5 (with tunable parameters of atomic system and fixed SRM reflectivity).

From these two figures, it is clear that 1) the stability condition and the phase cancelation condition put a strong constraint on the possible parameter region; 2)
Figure 7.4 – Stability region of the full interferometer scheme with double-pumped gain medium (optical stability only). The SRM power reflectivity $r_s^2 = 0.5, 0.8, 0.9$ are chosen from the top panel to the bottom, while we survey the parameter region for $\gamma_{\text{opt}}, \gamma_{12}$. The horizontal and vertical axis are $\eta = \gamma_{\text{opt}}/\gamma_{12}$ and $\xi = 8(\gamma_{12} - \gamma_{\text{opt}})^2L/c\gamma_{12}$. We survey $\eta, \xi$ between 0 and 1 so that atomic instability is excluded and the phase cancellation condition can be satisfied. For each $r_s$, the left panel and right panel correspond to two roots of $\Delta_0^2$ in Eq. (7.35), respectively. The purple region is the only stable region. In the blue region (“optical instability region”), the atomic medium is stable by itself but the dynamics of the full interferometer system is unstable (see Section IV for details). The red region corresponds to the situation when the system becomes non-stationary, i.e. breaking down of condition in Eq. (7.34). With the increasing of the SRM reflectivity, the stable region shrinks due to the enhancement of the optical instability effect.
7.4. A brief summary of the main results

The double-pumped gain medium itself is stable and stationary. The integrated shot noise limited sensitivity improvement factor is larger than 1, when shown in Fig. 7.4. It is clear from this figure that there is no parameter region where to the two roots of Eq. (7.35). The dashed line is the boundary of the stable region identical to the one for producing Fig. 7.4. The left panel and right panel correspond into account of the effect of additional noise. The specification for the parameters is Eq. (7.84)) of the full interferometer scheme with double-pumped gain medium, taking Figure 7.5

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**Figure 7.5** – Integrated shot noise limited sensitivity improvement factor (defined in Eq. (7.84)) of the full interferometer scheme with double-pumped gain medium, taking into account of the effect of additional noise. The specification for the parameters is identical to the one for producing Fig. 7.4. The left panel and right panel correspond to the two roots of Eq. (7.35). The dashed line is the boundary of the stable region shown in Fig. 7.4. It is clear from this figure that there is no parameter region where the integrated shot noise limited sensitivity improvement factor is larger than 1, when the double-pumped gain medium itself is stable and stationary.
There is no parameter region where the shot-noise limited sensitivity is improved. This indicates that placing a stable double-pumped gain medium with anomalous dispersion inside the SRC can not broaden the detection bandwidth while increasing the shot-noise limited sensitivity. Therefore, one shall explore other types of gain mediums or different parameter regimes for realizing the white light cavity.

### 7.5 Input-output relation of double gain atomic medium

After summarizing the main results, we now start a detailed discussion by first developing an input-output formalism for light propagating through the double-pumped gain medium in the Heisenberg picture. As we have briefly mentioned in the *Introduction*, our gain medium consists of three-level atoms schematically shown in Fig. 7.1 with two red (blue)-detuned (with respect to frequency difference between $|3\rangle$ and $|1\rangle$) control lasers. The polarizations of the control and probe fields are orthogonal to each other and only sensitive to the atomic transitions between $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$, respectively. In modeling the gain medium, we treat the atoms as non-interacting distinguishable particles. Nevertheless, all the atoms have the same energy level structures. In this section, we first derive the atomic dynamics for a single three-level atom, then extend the result to many-atoms case under the approximation that the length of the gain medium is much smaller than the spatial scale of the optical sideband field $2\pi c/\Omega$ where $\Omega$ is the gravitational wave frequency.

#### 7.5.1 Slowly-varying amplitude Hamiltonian of the electromagnetic field

In the main text, the Hamiltonian of the electromagnetic field is given in Eq.(7.47). Unlike the usual free field Hamiltonian written in the $k$—space, this Hamiltonian is written in the $x$—space and the $\hat{a}_x$ is the slowly varying amplitude of the optical field.

In the $k$—space, the free-field Hamiltonian for an unidirectional propagating field
7.5. Input-output relation of double gain atomic medium

The input-output relation of double gain atomic medium can be written as:

\[ \hat{H}_f = \hbar c \int_0^\infty dk k \hat{a}_k^\dagger \hat{a}_{-k} = \hbar c \int_{-k_0}^{\infty} dk'(k_0 + k') \hat{a}_{k_0-k'}^\dagger \hat{a}_{-k_0-k'}, \tag{7.39} \]

where we have shifted the integral to the deviation of the wave vector to the center wave-vector \( k_0 \) in the second equality. Since we are interested in the \( k' \ll k_0 \), we can approximate the above formula to be:

\[ \hat{H}_f \approx \hbar c \int_{-k_0}^{\infty} dk'(k_0 + k') \hat{a}_{k_0-k'}^\dagger \hat{a}_{-k_0-k'} \tag{7.40} \]

This approximation is called narrow band approximation.

Then we can define the optical field operator in the \( x \)-space by Fourier transformation:

\[ \hat{a}_x = \int_{-\infty}^{\infty} \hat{a}_{k_0-k'} e^{-ik'x} dk' \tag{7.41} \]

Substituting the above definition into the Eq.(7.40), we obtain:

\[ \hat{H}_f = \hbar c k_0 \int_{-\infty}^{\infty} dx \hat{a}_x^\dagger \hat{a}_x + \frac{i\hbar c}{2} \int_{-\infty}^{\infty} dx \left[ \frac{\partial \hat{a}_x^\dagger}{\partial x} \hat{a}_x - \hat{a}_x^\dagger \frac{\partial \hat{a}_x}{\partial x} \right] \tag{7.42} \]

Further more, if we work in the rotating frame of \( ck_0 \): \( \hat{a}_x = \hat{a}_x e^{-ick_0 t} \) then the above Hamiltonian will be:

\[ \hat{H}_f = \frac{i\hbar c}{2} \int_{-\infty}^{\infty} dx \left[ \frac{\partial \hat{a}_x^\dagger}{\partial x} \hat{a}_x - \hat{a}_x^\dagger \frac{\partial \hat{a}_x}{\partial x} \right] \tag{7.43} \]

Notice that the \( \hat{a}_x \) satisfies the commutation relation: \([\hat{a}_x, \hat{a}_x^\dagger] = \delta(x-x')\). The \( \hat{a}_x \) here is the spatially and temporary slowly varying amplitude of the electromagnetic field. This fact can be seen from the definition of the electric field under the narrow-band approximation:

\[ \hat{E}^{(+)}(x,t) = \hat{E}^{(+)}(x,t) e^{i\omega_0 t+ik_0 x} \approx \sqrt{2\pi \hbar c k_0} \int_{-\infty}^{\infty} dk' \hat{a}_{k_0-k'} e^{-ik'x-ick_0 t} = \sqrt{2\pi \hbar c k_0} \hat{a}_x \tag{7.44} \]

7.5.2 Single-atom dynamics

The above physical modeling leads to the following system Hamiltonian for a single atom

\[ \hat{H} = \hat{H}_{\text{atom}} + \hat{H}_f + \hat{H}_{\text{int}} + \hat{H}_\gamma. \tag{7.45} \]
The $\hat{H}_{\text{atom}}$ is the free Hamiltonian for a three level atoms in the form of:

$$\hat{H}_{\text{atom}} = \sum_{a=1,2,3} \hbar \omega_a \hat{\sigma}_{aa},$$

(7.46)

where $\omega_a$ is the Bohr frequency of energy level $a$ and $\hat{\sigma}_{aa}$ is the atomic population operator. The $\hat{H}_f$ is the free Hamiltonian for the sideband probe fields propagating in the main GW detector, given by [24, 23]:

$$\hat{H}_f = i\frac{\hbar c}{2} \int_{-\infty}^{\infty} \left[ \frac{\partial \hat{a}_x^{\dagger}}{\partial x} \hat{a}_x - \hat{a}_x^{\dagger} \frac{\partial \hat{a}_x}{\partial x} \right] dx.$$  

(7.47)

Notice that the $\hat{a}_x$ is the slowly varying amplitude operator (both spatially and temporarily) with respect to $e^{-i\omega_p x/c - i\omega_p t}$, defined as: $\hat{E}_p(t,x) = \hat{a}_x(t) e^{-i\omega_p x/c - i\omega_p t} + h.c.$ The probe field propagates unidirectionally (right to left), encounters and interacts with the atom at position $x_0$. This interaction is given by a Jaynes-Cumming type of Hamiltonian under the rotating wave approximation [19]:

$$\hat{H}_{\text{int}} = -\frac{\hbar}{2} \mu_{23} \hat{a}_x^{\dagger} e^{i\omega_p t} \hat{\sigma}_{23} - \frac{\hbar}{2} \mu_{13} E_c e^{i\omega_0 t} \hat{\sigma}_{13} + h.c,$$

(7.48)

where the first term describes the atomic transition between $|2\rangle$ and $|3\rangle$ under the driving of probe fields with transition operator $\hat{\sigma}_{23}$ and the second term describes the atomic transition between $|1\rangle$ and $|3\rangle$ under the pumping of control fields with the transition operator $\hat{\sigma}_{13}$. The $E_c = E_a e^{i\omega_a t} + E_b e^{i\omega_b t}$ describes the two classical amplitude of control fields with frequency $\omega_{a,b}$ and the $\mu_{mn}(m,n = 1,2,3)$ are the dipole moments of the atom. The atom transition operators satisfy the algebra:

$$\hat{\sigma}_{mn} \hat{\sigma}_{kl} = \hat{\sigma}_{ml} \delta_{nk}; (\hat{\sigma}_{mn})^{\dagger} = \hat{\sigma}_{nm}.$$  

(7.49)

The coupling between an atom with other bath sources at position $x_0$ is introduced phenomenologically by:

$$\hat{H}_{\gamma} = i\hbar \sqrt{2\gamma_{12}} \hat{n}_{12} e^{i\omega_{12} t} \hat{\sigma}_{12} - \frac{\hbar}{2} \mu_{13} \hat{a}_c^{\dagger} e^{i\omega_0 t} \hat{\sigma}_{13} + h.c$$

(7.50)

where $\hat{n}_{12}$ is the noise operator which couples to the atomic transition operator between $|1\rangle$ and $|2\rangle$. The $\hat{a}_c$ is the quantum fluctuation associated with the control field. In our 1-D model, the quantum fluctuation associated with the probe field has been included in the $\hat{a}_x$ field of Eq.(7.48). The noise bathes model can be attributed
to mutual collision of atoms or the stimulation of the external electromagnetic vacuum bath. Here to study the minimal additional noise, we consider an effectively zero-temperature external bath.

With the Hamiltonian, we can now analyze the dynamics of the gain medium. The Heisenberg equation of motion for the probe field derived from Eq. (7.47) can be written as:

\[
\frac{\partial \hat{a}_x}{\partial t} - c \frac{\partial \hat{a}_x}{\partial x} = \frac{i}{2} \mu_{23} \hat{\sigma}_{23} e^{-i\omega_p t} \delta(x - x_0),
\]

(7.51)

which reflects the fact that the probe field propagates from right to left (unidirectional).

The Heisenberg equations of motion for the atomic transition operators of a single atom in this gain medium are given by:

\[
\dot{\hat{\sigma}}_{13} + (i\omega_{31} + \gamma_{13})\hat{\sigma}_{13} = i\sqrt{2\gamma_{13}} (\hat{\sigma}_{11} - \hat{\sigma}_{33}) \hat{a}_{\text{cin}} e^{-i\omega_0 t} + \frac{i}{2} \mu_{13} (\hat{\sigma}_{11} - \hat{\sigma}_{33}) E_c + \frac{i}{2} \mu_{23} \hat{\sigma}_{12} \hat{a}_{x_0} e^{-i\omega_p t} \quad (7.52a)
\]

\[
\dot{\hat{\sigma}}_{23} + (i\omega_{32} + \gamma_{23})\hat{\sigma}_{23} = \frac{i}{2} \mu_{13} \hat{\sigma}_{21} (E_c + \hat{a}_c e^{-i\omega_0 t}) + \frac{i}{2} \sqrt{2\gamma_{23}} (\hat{\sigma}_{22} - \hat{\sigma}_{33}) \hat{a}_{\text{in}} e^{-i\omega_p t} \quad (7.52b)
\]

\[
\dot{\hat{\sigma}}_{12} + (i\omega_{21} + \gamma_{12})\hat{\sigma}_{12} = -\sqrt{2\gamma_{12}} (\hat{\sigma}_{11} - \hat{\sigma}_{22}) \hat{n}_{12\text{in}} e^{-i\omega_{21} t} + \frac{i}{2} \mu_{23} \hat{\sigma}_{13} \hat{a}^\dagger_{x_0} e^{i\omega_p t} - \frac{i}{2} \mu_{13} \hat{\sigma}_{32} (E_c + \hat{a}_c e^{-i\omega_0 t}) \quad (7.52c)
\]

Notice that the \( \hat{n}_{12\text{in}}, \hat{a}_{\text{in}} \) and \( \hat{a}_{\text{cin}} \) are the incoming noise fields, whose relations with the \( \hat{n}_{12}, \hat{a}, \hat{a}_c \) of Eq. (7.50) are given in the way of Eq. (7.56)(7.57). The \( \gamma_{13} = \mu_{13}^2/8, \gamma_{23} = \mu_{23}^2/8 \) can be derived from Eq. (7.48). Besides, the condition that the majority of atoms are initially prepared at \( |1\rangle \) is set as an assumption. In a real experiment, this population preparation can be achieved through various methods such as introducing an additional optical pumping field [14].

It is clear that the above equations of motion are generally nonlinear. However the system dynamics can be simplified by exploring the linear regime where the scheme is proposed to operate. The simplification can be done using perturbative method by noticing that 1) the control fields have large detuning with respect to \( \omega_{31} \) and therefore the population of atoms on \( |3\rangle \) is still small compared to that on \( |1\rangle \); 2) the transition between \( |1\rangle - |3\rangle \) is much stronger than the transition between \( |1\rangle - |2\rangle \).
and $|2\rangle - |3\rangle$ since it is induced by strong control beams; 3) The probe field is very weak compared to the control field since it is around the quantum level.

There are three dimensionless expansion parameters in this system of equations of motion: $\epsilon \sim \mu_{mn} E_c / \Delta_0$, $\alpha \sim \mu_{mn} \hat{a} / \Delta_0$ and $\alpha \ll \epsilon \ll 1$ (notice that the denominator could also be other frequency scales such as $\omega_{31} - \omega_{a,b}$, we choose the smallest one here for briefness). Writing the $\hat{\sigma}_{13}$, $\hat{\sigma}_{23}$, $\hat{\sigma}_{12}$ in the rotating frame of $\omega_0 = (\omega_a + \omega_b) / 2$, $\omega_p$ and $\omega_0 - \omega_p$ respectively, the leading order ($\sim \epsilon$) of $\hat{\sigma}_{13}$ dynamics can be derived as:

$$\dot{\hat{\sigma}}_{13} - i(\omega_0 - \omega_{31} + i\gamma_{13})\hat{\sigma}_{13} = \frac{i}{2}\mu_{13}\hat{\sigma}_{11}(E_a e^{-i\Delta_0 t} + E_b e^{i\Delta_0 t}),$$  \hspace{1cm} (7.53)

in which we can approximate the collective population operator on $|1\rangle$ as $\hat{\sigma}_{11} \approx 1$ and $\hat{\sigma}_{mn}$ is the mean value of $\sigma_{mn}^i$ averaged over all the atoms. The solution of Eq.(7.53) is given by:

$$\hat{\sigma}_{13} \approx \frac{1}{2} \frac{\mu_{13} E_a e^{-i\Delta_0 t}}{\omega_{31} - \omega_a} + \frac{1}{2} \frac{\mu_{13} E_b e^{i\Delta_0 t}}{\omega_{31} - \omega_b}.$$ \hspace{1cm} (7.54)

Here, we have neglected the $\gamma_{13}$ which is assumed to be much smaller than the detuning: $\gamma_{13} / (\omega_{a,b} - \omega_{31}) \ll \epsilon$. In the same rotation frame, the leading order of the $\hat{\sigma}_{23}$ ($\sim \epsilon^2 \alpha$) and $\hat{\sigma}_{12}$ ($\sim \epsilon \alpha$) dynamics can be written as:

$$\dot{\hat{\sigma}}_{23} - i(\omega_p - \omega_{32} + i\gamma_{23})\hat{\sigma}_{23} = \frac{i}{2}\mu_{13}\hat{\sigma}_{21}\hat{E}_c,$$  \hspace{1cm} (7.55a)

$$\dot{\hat{\sigma}}_{12} + \gamma_{12}\hat{\sigma}_{12} = \frac{i}{2}\mu_{23}\hat{a}_{x_0}\hat{\sigma}_{13} - \frac{i}{2}\mu_{13}\hat{\sigma}_{32}\hat{E}_c - \sqrt{2}\gamma_{12}\hat{n}_{12\text{in}},$$ \hspace{1cm} (7.55b)

where $\hat{E}_c = E_a e^{-i\Delta_0 t} + E_b e^{i\Delta_0 t}$ is the pumping field strength in the rotating frame of $\omega_0$ and we have used the fact that $\omega_0 = \omega_p + \omega_{21}$ (See Fig.7.1). We also make use of the fact that $\hat{\sigma}_{11} \approx 1$. In deriving Eq.(7.55), we also assume that system parameters satisfy: $\gamma_{23} / (\omega_p - \omega_{32}) \ll \epsilon^2 \alpha$.

For the probe field, we can integrate Eq.(7.51) around $x_0$ and obtain:

$$-\hat{a}_{x_0+} + \hat{a}_{x_0-} = \frac{i}{2}\mu_{23}\hat{\sigma}_{23},$$ \hspace{1cm} (7.56)

in which the $\hat{a}_{x_0+}$ and $\hat{a}_{x_0-}$ are the incoming and outgoing sidebands fields (respectively) defined in the vicinity of the interaction point $x_0$ (in the following, we will use $\hat{a}_{\text{in}}$ and $\hat{a}_{\text{out}}$ to represent them, respectively). The probe field at $x_0$ is connected with
these vicinity fields through the junction condition:

$$\dot{a}_x = \frac{1}{2} (\dot{a}_{in} + \dot{a}_{out}).$$ (7.57)

The dynamics of $\hat{\sigma}_{32}$ can be obtained by solving Eq. (7.55) (a), we have:

$$\dot{\hat{\sigma}}_{32} = \frac{\mu_{13}}{2} \frac{\hat{E}_{c}^*}{\omega_{32} - \omega_p} \hat{\sigma}_{12},$$ (7.58)

In deriving the above equation, we assume that $\gamma_{23} \ll \omega_{32} - \omega_p$ and make use of the fact that $\sigma_{12}$ is a slowly varying amplitude thereby $\dot{\hat{\sigma}}_{32} \approx 0$ in Eq. (7.55). Substituting the solution Eq. (7.58) and Eq. (7.57) into Eq. (7.55) (b) and Eq. (7.56), we can adiabatically eliminate the $\hat{\sigma}_{23}$ so that the Eq. (7.55) (b) and Eq. (7.56) form a closed equation set:

$$\dot{\hat{a}}_{\text{out}} = \hat{a}_{\text{in}} - i (\sqrt{2 \gamma_{\text{opta}}} e^{i \Delta_{0a} t} + \sqrt{2 \gamma_{\text{optb}}} e^{-i \Delta_{0a} t}) \hat{\sigma}_{12}$$ (7.59a)

$$\dot{\hat{\sigma}}_{12} = i \left(\sqrt{2 \gamma_{\text{opta}}} e^{-i \Delta_{0a} t} + \sqrt{2 \gamma_{\text{optb}}} e^{i \Delta_{0a} t}\right) \hat{a}_{\text{in}} + (\gamma_{\text{opta}} + 2 \Delta_{0a}) \hat{\sigma}_{12} - i \omega_{\text{opt}} \hat{\sigma}_{12} - \sqrt{2 \gamma_{\text{optb}}} \hat{n}_{12in}.$$ (7.59b)

This equation set describes the coupling between the composite system (consisting of the atom and the pumping fields) to the probe field.

The second term on the right-hand-side of Eq. (7.59) is the sum of an anti-damping term $\gamma_{\text{opt}}^{\text{a}} \hat{\sigma}_{12}$:

$$\gamma_{\text{opt}}^{\text{a}} = \gamma_{\text{opta}} + \gamma_{\text{optb}} = \frac{\mu_{23}^2 \mu_{13}^2}{32} \left[ \left| E_a \right|^2 \Delta_a^2 + \left| E_b \right|^2 \Delta_a^2 \right],$$ (7.60)

and a high-oscillating term $\gamma_{2 \Delta_{0a}} \hat{\sigma}_{12}$:

$$\gamma_{2 \Delta_{0a}} = \frac{\mu_{23}^2 \mu_{13}^2}{32} \left[ \frac{E_a E_b e^{2i \Delta_{0a} t}}{\Delta_a^2} + \frac{E_b E_a e^{-2i \Delta_{0a} t}}{\Delta_a^2} \right].$$ (7.61)

These formulae are derived under the approximation $\Delta_{0} \ll \Delta_{p,a,b}$ (thereby $\Delta_{p} \approx \Delta_{a} \approx \Delta_{b}$). These are good approximations to the situation in the proposed experiments $[12, 14, 10]$.

In the symmetric pumping case when $E_a = E_b = E_0$:

$$\gamma_{\text{opt}}^{\text{a}} \approx \frac{\mu_{23}^2 \mu_{13}^2 E_0^2}{(16 \Delta_a^2)}.$$ (7.62)

When $\gamma_{\text{opt}}$ is larger than $\gamma_{12}$, we have the population inversion between energy level $|1\rangle$ and $|2\rangle$, i.e., the atomic instability, mentioned earlier.
Solving the Eq. (7.59)(a)(b) in the frequency domain, we can obtain the input-output relation for the probe field:

\[ \hat{a}_{\text{out}}(\Omega) = M(\Omega)\hat{a}_{\text{in}}(\Omega) + N_{+}(\Omega)\hat{n}_{12\text{in}}^{\dagger}(\Delta_{0} - \Omega) + N_{-}(\Omega)\hat{n}_{12\text{in}}^{\dagger}(-\Delta_{0} - \Omega), \]

with \( M(\Omega) \) and \( N(\Omega) \) given by:

\[ M(\Omega) = 1 - \frac{\gamma_{\text{opt}}^{\ast}}{i(\Omega + \Delta_{0}) - \gamma_{12} + \gamma_{\text{opt}}} - \frac{\gamma_{\text{opt}}^\ast}{i(\Omega - \Delta_{0}) - \gamma_{12} + \gamma_{\text{opt}}}, \]

\[ N_{\pm}(\Omega) = \pm \frac{\sqrt{2\gamma_{12}\gamma_{\text{opt}}}}{i\Delta_{0} - i\Omega + \gamma_{12} - \gamma_{\text{opt}}}. \]

Notice that 1) \( N_{\pm}^{\ast}(-\Omega) = N_{\mp}(\Omega) \). 2) Here and after, for simplicity, we will only consider the symmetric pumping case where \( E_{a} = E_{b} \) because the non-symmetric pumping will only induce an additional rotation int he quadrature plane, which does not affect our main results. 3) In obtaining the required susceptibility Eq. (7.31) and the corresponding input-output relation, we neglect the above oscillating terms at \( 2\Delta_{0} \) which is contributed by the driving from the beating of two control fields. This approximation leads to the condition in Eq. (7.35); 2) The tiny Stark frequency shift \( \omega_{\text{opt}} = \mu_{13}^{2}|E_{c}(t)|^2/(4\Delta_{p}) \ll \Delta_{0} \) on the right hand side of Eq. (7.59) has been neglect here.

The above input-output relation describes a phase-insensitive parametric amplification process. Therefore, there is an additional noise given by the \( \hat{n}_{12\text{in}}^{\dagger} \) terms in Eq. (7.63). This noise comes from the stochastic dynamics of \( \hat{\sigma}_{12} \) driven by the last term on the right hand side of Eq. (7.59). The usual method for introducing the additional noise for parametric amplifier is using the argument given by Caves [18], which base on the principle that \( \hat{a}_{\text{out}} \) field should satisfy Bosonic commutation relation. However, Caves’s method can not be directly applied to our system since the additional noise has two different frequency channels. Solving the dynamics from the full Hamiltonian of the system can allows us to pin down the source of the additional noise and give the correct formula for the noise contribution.

### 7.5.3 Extension to many-atoms case

In the above subsection, we discussed the input-output relation for the probe field interacting with a single atom. In this subsection, we extend the above results to the
7.5. Input-output relation of double gain atomic medium

For many-atom case, we have to use the coarse-grained Hamiltonian to do analysis. As we have discussed before, since the size of the gain medium (centimeter scale) is much smaller than the spatial scale of the slowly varying amplitude of the probe field (kilometer scale), therefore the slowly changing amplitude of the probe field interacts with all the atoms together. In this case, the equation of motion for the probe field should be written as:

$$\frac{\partial \hat{a}_x}{\partial t} - c \frac{\partial \hat{a}_x}{\partial t} = \frac{i}{2} \mu_{23} \sum_{i=0}^{N} \hat{\sigma}_{23}^i e^{-i\omega_p t} \delta(x - x_i) = \frac{i}{2} \mu_{23} N \hat{\sigma}_{23} e^{-i\omega_p t} \delta(x - x_0)$$

(7.65)

where we have used the fact that the size of the atomic medium is much smaller than the typical length scale of the slowly-varying amplitude so that we can treat the interaction as a point-interaction happens on $x_0$. This equation is actually the Maxwell-Bloch equation we have derived before. Integrate this equation around $x_0$, we obtain:

$$-\hat{a}_{x_0} + \hat{a}_{x_0} = \frac{i}{2} N \mu_{23} \hat{\sigma}_{23}.$$  

(7.66)

Beside the probe field, the atoms are also interacting with the bath field which is associated with noise, the modeling of the noise field is a bit subtle. Notice that in real experiment, the noise field, can interact with the atoms in both collective way and local way. For example, in some experiment, the $\gamma_{12}$ above is mainly contributed by an optical pumping laser propagate in the same direction of the probe field rather than the interaction between the atoms and the vacuum fluctuation. In this case, the transition between two energy levels $|1\rangle - |2\rangle$ is actually a collective phenomenon as we shall see later. In this case, the fluctuation noise field associated with the optical pumping laser also interacts with the system collectively. On the other hand, in some cases, the $\gamma_{12}$ is contributed by the local noise bath associated with each atom. Fig.7.6 is the simplest case when the atoms are perfectly aligned dipole, but interact non-locally with the environment bath.

In this many-body case, similar calculation using the many-atoms Hamiltonian will show that the anti-damping rate will be enhanced by a factor of $N$ where $N$ is the number of the atoms. Therefore the $M$ coefficients in the above input-output
Noise couple to a polarized atom/polarized atom clouds

\[ M(\Omega) = 1 - \frac{\gamma_{opt}}{i(\Omega + \Delta_0) - \gamma_{12} + \gamma_{opt}} - \frac{\gamma_{opt}}{i(\Omega - \Delta_0) - \gamma_{12} + \gamma_{opt}} \] (7.67a)

where \( \gamma_{opt} = N\gamma_{opt}^s \).

The concrete formulation of noise field in the input-output relation for many-atoms depends on the specific modeling of the interaction between the noise field and the atoms.

- **Noise interacts with atoms locally**—In this case, each atom is associated with its own noise bath. The noise term will be represented by:

  \[ \text{Noise term} = \sum_{\pm} \sum_{j=1}^{N} \mathcal{N}_{\pm}(\Omega) \hat{n}_{12\text{in}}(\pm\Delta_0 - \Omega), \] (7.68)

  where

  \[ \mathcal{N}_{\pm}(\Omega) = \sqrt{\frac{2\gamma_{12}\gamma_{opt}}{i\Delta_0 - i\Omega + \gamma_{12} - \gamma_{opt}}}. \] (7.69)

  Using the fact that the noise operators \( \hat{n}_j \) for different atoms are independent of each other, it is straightforward to check that this extension of the input-output relation satisfies the commutation relation \( [\hat{a}_{\text{out}}(\Omega), \hat{a}_{\text{out}}^\dagger(\Omega')] = \delta(\Omega - \Omega') \).

- **Noise interacts with atoms collectively**—In some cases, the noise is introduced through processes where the electromagnetic field amplitude interacts with all the atoms collectively as what the slowly-varying amplitude of the probe field does.

\[ \text{Figure 7.6} \] – For polarized atoms, the EM vacuum fluctuation whose polarization is orthogonal to \( \mu \) can not stimulate the atom, but still, the EM vacuum fluctuation can excites the atom from other directions. Therefore local noise operator is more realistic relation can be written as:
For example, the $\gamma_{12}$ is induced by applying an additional pumping laser such as the experiment done in [10]. In this situation, the noise term will be represented by:

$$\text{Noise term} = \sum \pm \mathcal{N}_\pm^\gamma(\Omega) \hat{n}_{12\text{in}}^\dagger(\omega_{21} \pm \Delta_0 - \Omega),$$

with

$$\mathcal{N}_\pm^\gamma(\Omega) = \frac{\sqrt{2\gamma_{12}\gamma_{\text{opt}}}}{\pm i\Delta_0 - i\Omega + \gamma_{12} - \gamma_{\text{opt}}}. \quad (7.71)$$

Notice that the $\gamma_{12}$ here (also accordingly in $\mathcal{M}(\Omega)$) should be understood as $\gamma_{12} = N\gamma_\text{s12}$. The $\gamma_\text{s12}$ is the transition rate from $|2\rangle$ to $|1\rangle$ for one single atom which is proportional to the intensity of the additional pumping laser the experiment in [10].

The input-output relation based on this noise model also satisfies the Bosonic commutation relation of $\hat{a}_{\text{out}}(\Omega)$.

As we shall see later, the subtleties of the noise model do not affect the sensitivity of the gravitational wave detector.

### 7.5.4 Some Physical Discussion

After deriving the system dynamics and input-output relation, we are going to give some intuitive discussion of the system dynamics and the additional noise.

Firstly, the “anti-damping” dynamics of $\hat{\sigma}_{12}$ can be understood in the following way: a small amount of atoms initially populated on $|1\rangle$ can be pumped to $|3\rangle$ by the detuned control fields, then jump to $|2\rangle$ due to their interactions with the probe field. During this indirect transition between $|1\rangle$ and $|2\rangle$ mediated by $|3\rangle$, the population of atoms on $|2\rangle$ will increase indefinitely if the decay rate from $|2\rangle$ to $|1\rangle$ is not sufficiently large—a “population inversion process”. Physically, this process could cause lasing (“atomic instability” in Section II) and our approximation will fail as the population on $|2\rangle$ becomes larger than the population on $|1\rangle$. One may argue that this instability will not happen in real experiments with the atom population being prepared using additional pumping fields. However, the thermal collision relaxation rate can be tuned to be small if we decrease the gas temperature, increase the pumping beam waist and fill in the “buffer” gas [22, 12]. In this case a small transition rate contributed by the optical pumping beam could be sufficient for the population...
preparation. Therefore, in principle the lasing can still happen as long the control fields are strong enough and $\gamma_{\text{opt}} > \gamma_{12}$.

Secondly, for the additional noise, the stochastic dynamics driven by the $\hat{n}_{12 \text{in}}$ can be attributable to 1) the collision of atoms due to the Van-der-Waals mutual interaction or the thermal collision [12], 2) the transition between $|2\rangle - |1\rangle$ induced by environmental black-body radiation, 3) the contribution of the quantum noise associated with the additional optical pumping process as in [10]. In Eq. (7.63) the $\hat{a}_{\text{out}}$ field contains terms related to the additional noise $\hat{n}$ in the way that the stochastic fluctuations of the population on $|1\rangle$ and $|2\rangle$ will cause fluctuations of the transition between $|2\rangle$ and $|3\rangle$, since $\hat{\sigma}_{23}$ is slaved by $\hat{\sigma}_{12}$.

In this section, we have derived the input-output relation for the sideband probe field propagating through the double gain medium from the full Hamiltonian. We also discussed the opto-atom dynamics and origin of the additional noise. In the next section, we will apply these results to the interferometer configuration shown in Fig. 7.1 and analyze its strain sensitivity.

### 7.6 Interferometer with gain medium

The propagation of sideband fields inside the interferometer shown in Fig. 7.1 can be schematically described by the flow chart shown in Fig. 7.7. Here, we only study the differential mode of this interferometer which carries the gravitational wave signal and can be mapped into a signal cavity containing a gain medium [25]. In this scheme, an internal signal recycling mirror is used to effectively remove the frequency response of the arm cavities so that the sideband fields see an input-output relation for the amplitude and phase quadrature of sideband field in the following form (we ignore the optomechanical back-action term by assuming the infinitely heavy test masses):

$$e(\Omega) = M_0(\Omega)d(\Omega) + D(\Omega)h(\Omega), \quad (7.72)$$

or more explicitly, we can expand the vectors $d, e, D$ and matrix $M_0$

$$\begin{bmatrix} \hat{e}_1(\Omega) \\ \hat{e}_2(\Omega) \end{bmatrix} = e^{2i\Omega \tau} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{d}_1(\Omega) \\ \hat{d}_2(\Omega) \end{bmatrix} + e^{i\Omega \tau} \begin{bmatrix} 0 \\ \sqrt{2K} \end{bmatrix} h(\Omega) \quad (7.73)$$
Figure 7.7 – A flow chart showing the propagation of fields in the full system. The optical stability of the system is determined by the part inside the blue dashed box, whose gain function is given by Eq. (7.72) and (7.82). The test mass (end mirror) \( m \) is driven by gravitational waves while the double gain medium is affected by the additional noise \( \hat{n} \).

with \( K = P_c \omega_0 L^2 / (hc^2) \).

(A) In case of local noise model, the input-output relation for phase and amplitude quadrature of light field propagates through the gain medium is (Ref. Eq.(7.67a)–(7.71)):

\[
d(\Omega) = M(\Omega) c(\Omega) + N_+(\Omega) + N_-(\Omega). \tag{7.74}
\]

where \( N_\pm \) represent the additional noise terms, the forms of which depends on the specific noise modeling as we will show later.

The combined effect of the gain medium and the main interferometer is described by \( M_{\text{tot}}(\Omega) = M(\Omega) M_0(\Omega) \) with the noise terms \( N_\pm(\Omega) M_0(\Omega) \). Then the final input-output relation for the sideband field is given by:

\[
j(\Omega) = M_k(\Omega) k(\Omega) + t_s e^{i\Omega \tau} M_c(\Omega) D(\Omega) h(\Omega)
+ t_s e^{2i\Omega \tau} M_c(\Omega) (N_+(\Omega) + N_-(\Omega)), \tag{7.75}
\]

in which \( M_k(\Omega) = -r_s I + t_s^2 M_c(\Omega) M_{\text{tot}}(\Omega) \) and \( M_c(\Omega) = [I - r_s M_{\text{tot}}(\Omega)]^{-1} \) with \( r_s, t_s \) the amplitude reflectivity and transmissivity of the signal recycling mirror. The \( k(\Omega) \) and \( j(\Omega) \) are the input and output field of the entire configuration as shown in Fig.7.1. The first term in Eq.(7.75) is the quantum noise contributed by the vacuum injection outside of the SRM, the second term is the contribution of additional noise introduced by the gain medium and the last term is the signal term. The SRM feeds the optical field back into the gain medium and the main interferometer, which is
described by dashed blue box in the flow chart Fig. 7.3. This feedback process will bring in another potential lasing (the “optical instability” mentioned in Section II) even if $\gamma_{12} > \gamma_{\text{opt}}$, discussed in the following subsection A.

For the additional noise terms $N_{\pm}(\Omega)$ in the above relations, suppose the additional noise interacts with atoms locally, they are given by

$$N_+(\Omega) + N_-(\Omega) = \sum_{j=1}^{N} [N_{\pm}(\Omega) \cdot n_{\pm}(\Omega)]. \quad (7.76)$$

The $n_{\pm}$ are the vectors ($\hat{n}_{1}^{\pm}$, $-\hat{n}_{2}^{\pm}$), with $\hat{n}_{1(2)}^{\pm}$ is the amplitude (phase) quadrature of the noise field $\hat{n}_{12\text{in}}$ with respect to the central frequency $\pm \Delta_0$. The noise matrix $N_{\pm}$ can be derived using the sideband-quadrature transfer matrix $M_{qs}$ defined as:

$$M_{qs} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad (7.77)$$

which leads to:

$$N_{\pm}(\Omega) = M_{qs} \cdot \begin{pmatrix} N_{\pm}(\Omega) & 0 \\ 0 & N_{\mp}(\Omega) \end{pmatrix} \cdot M_{qs}^{-1} \quad (7.78)$$

in obtaining the above formula, we have made use of the relation: $N_{\mp}(\Omega) = N_{\pm}(\Omega)$.

If the noise interacts with the atoms collectively, the noise term is given by:

$$N_+(\Omega) + N_-(\Omega) = \sum_{\pm} M_{qs} \cdot \begin{pmatrix} N_{\pm}(\Omega) & 0 \\ 0 & N_{\mp}(\Omega) \end{pmatrix} \cdot M_{qs}^{-1} n_{\pm}(\Omega) \quad (7.79)$$

where formally we have:

$$N_{\pm}(\Omega) = \sqrt{\gamma_{12}} - \sqrt{2\gamma_{12} \gamma_{\text{opt}}} \quad (7.80)$$

where we have used the fact that $\Gamma_{\text{opt}} = N\gamma_{\text{opt}}$ while as before, the $\gamma_{12}$ here should be understood as $N$ times of the $|2\rangle \rightarrow |1\rangle$ transition rate for a single atom. When we calculate the sensitivity, the $\sqrt{N}$ in the above equation will give us the same $N$ factor in the following Eq. (7.81) while the rest part of the above equation has the same form as $N_{\pm, j}$. Therefore if the the noise is modeled as interacting with the atoms collectively, the strain sensitivity formula will have the same form except that the $\gamma_{12}$ are different. However, since we will scan over all possible values of $\gamma_{12}$ and $\Gamma_{\text{opt}}$ in the sensitivity calculation, this difference of $\gamma_{12}$ will not affect our final result.
Finally, the shot-noise limited strain sensitivity of the interferometer derived from Eq. (7.81), which is given by:

\[
S_{hh}^a(\Omega) = \frac{v_h.\mathbf{M}_k(\Omega).\mathbf{M}_k^\dagger(\Omega)v_h^T}{|t_s v_h.\mathbf{M}_c(\Omega).\mathbf{D}(\Omega)|^2} + N \sum_{\pm} \frac{v_h \mathbf{M}_c(\Omega).\mathbf{N}_{j\pm}(\Omega).\mathbf{N}_{j\pm}^\dagger(\Omega)\mathbf{M}_c^\dagger(\Omega)v_h^T}{|v_h.\mathbf{M}_c(\Omega).\mathbf{D}(\Omega)|^2},
\]

in which the \( v_h = (\sin \xi, \cos \xi) \) is the homodyne readout vector with \( \xi \) is the homodyne angle. In our calculation, we will choose \( \xi = 0 \) which means we only measure the phase quadrature \( j_2(\Omega) \) of the output field.

We will discuss the numerical result of this shot-noise limited strain sensitivity in the following subsection B.

### 7.6.1 Stability criteria

Besides the atomic instability due to “population-inversion process” when \( \gamma_{12} < \gamma_{opt} \), it is important to notice that the dynamics of the interferometer with the gain medium may still be unstable (start lasing) even if \( \gamma_{12} > \gamma_{opt} \).

This instability is related to the feedback process discussed below Eq. (7.75) due to the reflection of the SRM. Intuitively, when the reflectivity of SRM \( r_s \) becomes high (or equivalently, \( t_s \) decreases), the photon loss rate through the transmission for each round trip can be less than the photon increasing rate through amplification by the gain medium, corresponding to the “optical instability”. The criterion of this instability is determined by the analytical behavior of the close-loop transfer matrix \( \mathbf{M}_c \). In our configuration, this close-loop transfer matrix is diagonal so that it can be simplified as a close-loop transfer function \( G_c(\Omega) \) (\( \mathbf{M}_c = G_c(\Omega)\mathbf{I} \)):

\[
G_c(\Omega) = \frac{1}{1 - r_s G_o(\Omega)},
\]

with \( G_o(\Omega) = e^{2\Omega \tau} M(\Omega) \). The stability of the full system is determined by the poles of the denominator, which can be obtained by solving the equation \( 1 - r_s G_o(\Omega) = 0 \).

However, the time-elapsed factor \( e^{2\Omega \tau} \) in the gain function makes it difficult to find the root of the above mentioned equation. The Nyquist criteria provides us another way to understand the stability through the analytical behavior of \( G_o(\Omega) \) instead of \( G_c(\Omega) \):
In our system with
\[ G_o(\Omega) = \chi(\Omega)e^{2i\omega r}, \] (7.83)
we have the poles of \( r_s G_o(\Omega) \) which are \( \Omega_{1,2} = \pm \Delta_0 - i(\gamma_{12} - \gamma_{opt}) \). Both of them fall outside of Nyquist contour because \( \gamma_{12} - \gamma_{opt} > 0 \) for the requirement of the stability of the atomic gain medium itself. Then we can conclude that \( P = 0 \) inside the Nyquist contour. In this case, the Nyquist criteria requires \( N = 0 \) to keep the stability for the full system, that is, in the Nyquist diagram, the contour of \( 1 - r_s G_o(\Omega) \) should not encircle the origin at all. In other words, the contour of \( r_s G_o(\Omega) \) should not encircle the point \((1,0)\) in the \((\text{Re}[r_s G_o(\Omega)], \text{Im}[r_s G_o(\Omega)])\) plane.

Specifically in our system, the Nyquist criteria can be stated in a way that the Nyquist contour of \( r_s G_o(\Omega) \) should not encircle the point \((1,0)\) in the \((\text{Re}[r_s G_o(\Omega)], \text{Im}[r_s G_o(\Omega)])\) plane at all. This criteria is equivalent to the lasing condition that the round-trip gain is smaller than one when the phase is the integer number of \( 2\pi \). For illustration purpose, several examples of the Nyquist contour of \( r_s G_o(\Omega) \) are shown in Fig.\[7.8\] given typical parameters of \( \gamma_{12} \) and \( \Delta_0 \). This plot demonstrates that increasing the SRM reflectivity can lead to system instability. We further search the parameter region \( 0 < \eta, \xi < 1 \) and give the plot on Fig.\[7.4\], from which we can see that the stability criteria imposes a very strong constrain on the possible parameter region. Only the interferometer system with parameters in the stable region is useful.

### 7.6.2 Integrated shot noise limited sensitivity improvement factor

For quantitatively describing the improvement of the integrated shot noise limited sensitivity, we define a quantity: the integrated shot noise limited sensitivity improvement factor (iSNS improvement factor) \( \rho_r \) in the following way:

\[ \rho_r = \frac{\int_{\omega_{\text{FSR}}}^{\omega_{\text{FSR}}} \frac{1}{S_{hh}(\Omega)} d\Omega}{\int_{\omega_{\text{FSR}}}^{\omega_{\text{FSR}}} \frac{1}{S_{hh}(\Omega)} d\Omega}, \] (7.84)

where \( S_{hh}^n \) is the shot noise limited gravitational wave strain sensitivity of the laser interferometer with double-pumped gain medium given by Eq.(7.81), while the denominator is that of the conventional interferometers given in Eq.(7.1). In case of \( \rho_r > 1 \), the system with double gain media will improves the signal to noise ratio by
Figure 7.8 – Nyquist contours of the $r_S G_o(\Omega)$ for the full system with fixed parameters of the gain medium $\eta = 0.1, \xi = 0.4$ while varying the SRM amplitude reflectivity $r_s$. The dashed (magenta), solid (red), dotdashed (black), dotted (blue) curves are the Nyquist contours when $r_s^2 = 0.9, 0.8, 0.7$ (unstable cases), and 0.5 (stable cases), respectively. The upper-part zooms in the full contour (lower-part) near (1,0) (the red spot). It is clear here that when the SRM reflectivity increases, the instability develops and the stable region in Fig. 7.4 shrinks.
breaking the trade-off between the detection bandwidth and peak sensitivity. However, we need to be cautious about the stability of the system at the same time.

We can calculate the strain sensitivity and hence the iSNS improvement factor. By fixing the SRM power reflectivity to be $r_s^2 = 0.8$, we calculate the iSNS improvement factor by searching the parameter region for $(\eta, \xi)$ within the range $[0, 1]$ constrained by the phase cancelation condition. For illustration purpose, we first calculate the $\rho_r$ by ignoring the effect of additional noise introduced by the atom system and give the plot in Fig.7.9. This figure clearly shows that there could be some parameter regions where the system is stable and $\rho > 1$.

However, taking into account of the additional noise, the results dramatically changed as we can see from Fig.7.5. It turns out that there is no region where $\rho > 1$ and the $\rho-$ contours are significantly distorted due to the additional noise. According to our numerical test, this conclusion does not change with the variation of the SRM reflectivity.

Until now, we applied the input-output formalism developed from the Hamiltonian of light-atom interaction to study the quantum noise of white light cavity using double gain medium. We find that not only does the additional noise associated with the parametric amplification process affects the system, but the requirement for the system stability also introduces an additional issue to take into account for its implementation. We conclude that the net sensitivity can not be enhanced by using the anomalous dispersive behavior of the stable double gain medium when the system is stable. The next section shows that the above analysis can also be used to analyze the optomechanical double-pumped gain system.

### 7.7 Optomechanical double-pumped gain system

Recently, in UWA lab, the negative dispersive phenomenon was already observed in a membrane-in-the middle optomechanical system. Using an 85-mm optical cavity coupled with a silicon nitride membrane, the optically tunable negative dispersion was demonstrated, which has a phase derivative $d\phi/d\Omega$ from $-0.14$ deg. Hz$^{-1}$ to
7.7. Optomechanical double-pumped gain system

Figure 7.9 – Integrated shot noise limited sensitivity improvement factor (defined in Eq. (7.84)) of the full interferometer scheme with double gain medium, without the effect of additional noise. The specification for the parameters is identical to the one for producing Fig. 7.4 and Fig. 7.5. The left panel and right panel correspond to the larger and smaller roots of Eq. (7.35), respectively. The dashed line is the boundary of the stable region shown in Fig 7.4. In this figure, when the detuning takes the larger solution, there are some regions where $\rho > 1$ and the system is stable at the same time. However, as we can see from Fig. 7.5, these regions will disappear when we take into account of the effect of the additional noise.
Chapter 7. Quantum noise of white cavity based on double-pumped gain medium

Figure 7.10 – Basic scheme of double-pumped gain optomechanical system

Figure 7.11 – Experimental measurement of the transmissivity in the optomechanical double-pumped gain scheme

\(-4.2 \times 10^{-3} \text{ deg.Hz}^{-1}\). In this experiment, the quality factor of the membrane is \(1.5 \times 10^6\) at its fundamental mechanical resonance \(\sim 378.5\)kHz, measured at the vacuum level of \(10^{-5}\)mbar. Two blue detuned control fields \(\hat{a}_{\pm}\) are injected into the coupled cavity with frequencies \(\omega_c \pm \Delta_0\) where \(\omega_c\) is the resonance frequency.

The field \(\hat{a}_-\) passes through a broadband electric-optic modulator to generate the weak signal light \(\hat{a}_n^s\). Using a lock-in amplifier, the transmission of the cavity is measured by detecting the beat signals between the signal field and the control fields at the transmission port. The result is shown in the Fig. 7.11.

Theoretically, the dynamics of the atomic system can be mapped to the optomechanical system. This mapping is given in the following general formalism: The Hamiltonian of the free-propagating sideband field is the same as that in the atomic gain medium case. However, the atomic/mechanical system is modeled as three os-
cillators in a general sense. Specifically, for atomic system, the three energy levels are modeled as three oscillators (Note that the traditional way defines the normalized transition operators in the linear approximation to be the harmonic oscillator quadrature operators [28], here the \( \hat{b}_l \) describes the annihilation of the atomic state at energy level \( l \) to the vacuum instead of the lower atomic state.) while for optomechanical system, these three oscillators corresponds to the mechanical resonator, intra-cavity pumping mode and signal models. This three-oscillators Hamiltonian can be written as:

\[
\hat{H} = \sum_l \hbar \omega_l \hat{b}_l^\dagger \hat{b}_l,
\]  

(7.85)

and the interaction is given by:

\[
\hat{H}_{\text{int}} = -\hbar g_1 E_c(y,t) \hat{x}_1(t) \hat{x}_3(t) + \hbar g_2 \hat{E}_p(y,t) \hat{x}_2(t) \hat{x}_3(t),
\]  

(7.86)

where \( E_c \) and \( E_p \) are two external fields: pumping field and the probe field. The \( x_l(t) = \sqrt{\hbar/m_l \omega_l} [\hat{b}_l(t) e^{-i \omega_l t} + \hat{b}_l^\dagger(t) e^{i \omega_l t}] \) is the oscillators’ displacement operator. The \( g \) measures the coupling strength between the harmonic oscillators and the external fields.

Working in the rotating frame \( \hat{b}_3 \rightarrow \hat{b}_3 e^{-i \omega_3 t} \) (\( \omega_0 = (\omega_a + \omega_b)/2 \)) and \( \hat{b}_1 \rightarrow \hat{b}_1 e^{-i \omega_1 t} \), we obtain:

\[
\bar{b}_3 \approx \frac{2g_1 E_c \cos(\Delta_3 t)}{\Delta_3} \hat{x}_1
\]  

(7.87)

where \( \Delta_c = \omega_3 - \omega_c \gg \gamma_3 \). After adiabatic eliminating of \( \hat{b}_3 \) as what we have done in the atomic medium example and neglecting the non-rotating wave terms, we have the effective interaction Hamiltonian between oscillator \( x_2 \) and \( x_3 \) as:

\[
\hat{H}_{23}^{\text{int}} \approx -\hbar g \hat{b}_2\hat{a}_p^\dagger \cos(\Delta_0 t) e^{-i(\omega_0 - \omega_2 - \omega_p)t} + h.c
\]  

(7.88)

in which \( g = 2g_1 g_2 E_c \bar{x}_1 / \Delta_3 \). Clearly, this is a typical Hamiltonian for a parametric amplification process and the corresponding equations of motion can be written as:

\[
\dot{\hat{b}}_2 = -\gamma_2 \hat{b}_2 + i g \hat{a}^\dagger(y) \cos \Delta_0 t + \sqrt{2\gamma_2} \hat{n}
\]

\[
\frac{\partial \hat{a}_x}{\partial t} - c \frac{\partial \hat{a}_x}{\partial x} = i g \hat{b}_2^\dagger \cos(\Delta_0 t) \delta(x-y)
\]  

(7.89)

This equation directly maps to the Eq. (7.59). Therefore it is clear that for optomechanical system, the input-output relation for the probe light has the almost the same
form as that of the light field interacting with a single-atom. In this case, the $\gamma_{\text{opt}}$ is the rate of optical heating which will be balanced by the damping rate $\gamma_m$, describing the mechanical damping through interaction with the thermal bath. Then the additional noise is just the thermal noise. In the extreme case when the environmental temperature is zero, the additional noise is contributed by the quantum zero-point fluctuation of the thermal bath degrees of freedom.

In conclusion, the mapping between the optomechanical system and the optoatomic system shows that the Mizuno theorem also can not be surpassed if we use double-pumped gain optomechanical filter cavity to cancel the propagation phase of the sideband field.

7.8 Conclusions

In this chapter, we applied the input-output formalism developed from the Hamiltonian of light-atom interaction to study the quantum noise of white light cavity using double gain medium. We found that not only does the additional noise associated with the parametric amplification process affects the system, but the requirement for the system stability also introduces an additional issue to take into account for its implementation. We concluded that the net sensitivity can not be enhanced by using the anomalous dispersive behavior of the stable double gain medium when the system is stable. We also discussed the similar double gain system using optomechanical interaction and experimentally demonstrated the negative dispersive phenomenon. For using such double gain system to improve the sensitivity, one might need to consider other interferometer configurations, for example, broadening the narrowband sensitivity within the optical resonance of a detuned double-recycling interferometer [29].
Bibliography


In Chapter 6 and Chapter 7, we have discussed the two proposed methods for sur-
passing the Mizuno sensitivity limit. The key for surpassing the Mizuno sensitivity
limit is to cancel the round-trip propagation phase of the sideband field through the
main interferometer, which requires a negative dispersive filter. The optical dynamics
of the dispersive filter shows that it is a parametric amplifier. In this Chapter, we
are going to give a more general discussion of the sensitivity limitation for a large
class of optical parametric measurement devices. These devices satisfy the following
descriptions:

1) A low frequency signal is coupled through the length of a high-finesse cavity,
with signal band much lower than the free spectral range of the cavity— which means
these systems contains optomechanical interaction and can be described using single-
mode approximation.

2) The cavity is coherently pumped and the entire optical system is driven by the
vacuum fluctuations from open ports—which means the sensitivity we are interested
in is the quantum noise limited sensitivity.

3) The system contain parametric amplifiers, which includes those “white light
cavity” interferometer designs.

The outline of this chapter is as follows. In Section 8.2, the background is intro-
duced. Specifically, we briefly summarize the Mizuno sensitivity limit and the basic
theory of a parametric amplifier. In Section 8.3, we give a proof of the Mizuno theo-
rem. In Section 8.4, the sensitivity bound for the devices with a parametric amplifier
is discussed. In the end, we give a summary of this Chapter

This is an ongoing project collaborated with Yanbei Chen and Haixing Miao.
8.1 Research Background

8.1.1 A brief summary of Mizuno sensitivity limit

As signal-recycling [1] and resonant side-band extraction [2] techniques were developed in the gravitational-wave community, J. Mizuno noticed that given optical power circulating in the arm cavity of a gravitational-wave detector, the modification of the optical resonant structure of the interferometer only provides a trade-off between the peak sensitivity of the detector and its detection bandwidth [2] — these two important characteristics of the detector cannot be simultaneously improved. In fact, if the minimum noise spectral density is $S_{x}^{\text{min}}$, and the bandwidth is $\Gamma$, then we should have

$$\frac{\Gamma}{S_{x}^{\text{min}}} \propto P_{c}. \quad (8.1)$$

where $P_{c}$ is the optical power inside the cavity.

More precisely, let us first assume that energy $E_{c}$ is stored in the cavity of length $L$. Let us also presume that the cavity has a high finesse, and that the carrier frequency is close to one of the resonant frequencies of the cavity. In this case, it will be reasonable to assume (and certainly straightforward to verify) that only signal from the nearly resonant mode of the cavity is relevant for detection sensitivity at low frequencies (compare to the free-spectral range). In this Chapter, we shall first prove that, if the interferometer is driven by usual vacuum, and that the steady state of the cavity mode is a coherent state, then the sensitivity of the interferometer will be constrained in the following way:

$$\int_{0}^{\text{FSR}} \frac{1}{S_{x}(\Omega)} d\Omega \leq \frac{4}{\tau^{2}} \left( \frac{E_{c}}{\hbar \omega_{0}} \right) \left( \frac{2 \pi}{\lambda} \right)^{2} = \frac{2}{\tau} \left( \frac{P_{c}}{\hbar \omega_{0}} \right) \left( \frac{2 \pi}{\lambda} \right)^{2}. \quad (8.2)$$

Here $E_{c}$ is energy stored in the cavity, $\lambda$ is the optical wavelength, and

$$\tau = \frac{L}{c} \quad (8.3)$$

is the time it takes light to travel across the cavity. This includes signal-recycling interferometers (for reaching the above limit, the infinite mass limit for the detector is taken, or in other words, the radiation pressure noise is removed) [1 2 4 3 5 6], signal recycling with double sidebands [7], the so-called speed meters [8 9 10 11]. For
surpassing this Mizuno limit, the negative dispersive medium need to be introduced, which is usually a parametric amplifier.

### 8.1.2 Parametric amplifier

It was discussed in Chapter 7 that the effective Hamiltonian describe the gain medium under the external pumping can be written as:

$$\hat{H}_{int}^{\text{int}} = -\hbar g \hat{b}^\dagger \hat{a}_p \cos(\Delta_0 t) + h.c. \quad (8.4)$$

A general parametric amplifier (or a parametric down-converter, or a second harmonic generator (See Fig.8.1)) consists of the signal \( \hat{a} \) and idler modes \( \hat{b} \) at different frequencies, coupled through nonlinear process \( \chi^{(2)} \) with the pumping field. Take optomechanical system as an example, this nonlinearity is introduced through the optomechanical interaction. For example: \( \hat{H}_{int} = \hbar G_0 \hat{a}^\dagger \hat{a} \hat{x} \), where the \( \hat{x} \) is a combination of \( \hat{b} \) and \( \hat{b}^\dagger \). The linearized form of the optomechanical interaction \( \propto \bar{a} (\hat{a} + \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger) \) contains the \( \chi^{(2)} \) interaction between the signal field \( \hat{a} \), the idler field \( \hat{b} \) and pumping field \( \bar{a} \).

Besides the white light cavity, parametric amplification process has lots of other application in quantum optics physics and the precision measurement, for example the generation of squeezed light is through the parametric amplification process. In particular, the generation of squeezed light is associated with a specific kind of parametric amplification process called “degenerate parametric amplification” in which the idler mode has the same frequency as the signal mode. This amplification process can happen for a pumped OPO (optical parametric oscillator) crystal. The Hamiltonian for the OPO can be written as: \( \hat{H}_{int} = i(g/2) \hat{a}^\dagger \hat{a}^\dagger + h.c \), which leads to the equations of motion\(^1\) as:

$$\begin{align*}
\dot{\hat{a}} &= -\gamma_a \hat{a} + \sqrt{2\gamma_a} \hat{a}_\text{in} + g \hat{a}^\dagger, \\
\dot{\hat{a}}^\dagger &= -\gamma_a \hat{a}^\dagger + \sqrt{2\gamma_a} \hat{a}_\text{in}^\dagger + g \hat{a}, \\
\hat{a}_\text{out} &= -\hat{a}_\text{in} + \sqrt{2\gamma_a} \hat{a}.
\end{align*} \quad (8.5)$$

\(^1\)Here we study an oversimplified model with lossless OPO crystal and a cavity with bandwidth \( \gamma_a \).
The input-output relation can be written as:

\[
\hat{a}_{\text{out}}(\Omega) = \left[-1 + \frac{2\gamma_a(\gamma_a - i\Omega)}{(\gamma_a - i\Omega)^2 - g^2}\right] \hat{a}_{\text{in}}(\Omega) + \frac{2g\gamma_a}{(\gamma_a - i\Omega)^2 - g^2} \hat{a}_{\text{in}}^\dagger(\Omega). \tag{8.6}
\]

The corresponding quadratures input-output relation can be written as:

\[
\begin{align*}
\hat{a}_{\text{out}2}(\Omega) &= \frac{\gamma_a + i\Omega - g}{\gamma_a - i\Omega + g} \hat{a}_{\text{in}2}(\Omega) \Rightarrow V_{22} = \frac{(\gamma_a - g)^2 + \Omega^2}{(\gamma_a + g)^2 + \Omega^2}, \\
\hat{a}_{\text{out}1}(\Omega) &= \frac{\gamma_a + i\Omega + g}{\gamma_a - i\Omega - g} \hat{a}_{\text{in}1}(\Omega) \Rightarrow V_{11} = \frac{(\gamma_a + g)^2 + \Omega^2}{(\gamma_a - g)^2 + \Omega^2}. \tag{8.7}
\end{align*}
\]

Therefore it is clear that the phase quadrature and amplitude quadrature for the output light is not equal to each other (and note that \(V_{22}V_{11} = 1\)) thereby create squeezing\(^2\).

Those parametric amplification process which creates squeezing effect is called “Phase sensitivity parametric process” such as the above example, while those parametric amplification process which do not create squeezing effect is called phase-insensitive parametric process, the general feature of their input-output relation is described as follows \([12]\).

For phase sensitive parametric amplifier, the input(\(\hat{a}\))-output \(\hat{b}\) relation normally have the form of

\[
\hat{b} = \mu \hat{a} + \nu \hat{a}^\dagger + \eta \hat{n}^\dagger + \xi \hat{n}, \tag{8.8}
\]

\(2\) the intracavity squeezing has a limit called 3dB limit, means the squeezing of phase quadrature has a limitation. However, with the cavity structure and explore the outgoing field, this limit does not exist. In principle, one can reach any squeezing degree.
in quadrature picture:

\[
\begin{aligned}
\hat{b}_1 &= \frac{1}{2} (\mu + \mu^* + \nu + \nu^*) \hat{a}_1 + \frac{i}{2} (\mu - \mu^* + \nu - \nu^*) \hat{a}_2 \\
&\quad + \frac{1}{2} (\eta + \eta^* + \xi + \xi^*) \hat{n}_1 + \frac{i}{2} (\eta^* - \eta + \xi - \xi^*) \hat{n}_2 \\
\hat{b}_2 &= \frac{1}{2i} (\mu - \mu^* + \nu - \nu^*) \hat{a}_1 + \frac{1}{2} (\mu + \mu^* - \nu - \nu^*) \hat{a}_2 \\
&\quad + \frac{1}{2i} (\eta - \eta^* + \xi - \xi^*) \hat{n}_1 - \frac{1}{2} (\eta^* - \eta - \xi - \xi^*) \hat{n}_2.
\end{aligned}
\] (8.9)

In general, there will be squeezing since \((\Re[\mu] + \Re[\nu])^2 \neq (\Re[\mu] - \Re[\nu])^2\).

If \(\nu = 0, \xi = 0\) or \(\mu = 0, \eta = 0\), in this case, there will be no squeezing and the input-output will be:

\[
\hat{b} = \mu \hat{a} + \eta \hat{n}_1, \quad \text{or} \quad \hat{b} = \nu \hat{a}^\dagger + \xi \hat{n},
\] (8.10)

respectively. The first relation corresponds to the “Phase-preserving parametric amplifiers” while the second relation corresponds to the “Phase-conjugate parametric amplifiers”. In the quadrature picture, the above relation can be written as:

\[
\hat{b}_1 = \mu \hat{a}_1 + i\eta^* \hat{n}_2, \quad \hat{b}_2 = \mu \hat{a}_2 + i\eta^* \hat{n}_1,
\] (8.11)

which shows that there is no correlation between \(\hat{b}_1\) and \(\hat{b}_2\).

After introducing these two background concept of this Chapter, the next section will give a prove of Mizuno theorem when there is no parametric amplifier, which is a base for the extended proof to the case with parametric amplifier.

### 8.2 Proof of the Mizuno theorem

#### 8.2.1 The Hamiltonian and the dynamics

For proving the Mizuno theorem, we consider a single optomechanical cavity using the following Hamiltonian:

\[
\hat{H}_{\text{cav}} = \hbar \Delta \hat{a}^\dagger \hat{a} + \sqrt{2\gamma} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left[ \hat{c}_{\omega}^\dagger \hat{a} + \hat{c}_\omega \hat{a}^\dagger \right] \\
+ \hbar G_0 \left( A\hat{a}^\dagger + A^* \hat{a} \right) \hat{x}.
\] (8.12)

The first term of Eq. (8.12) is the free Hamiltonian of the caivity field. Here we have removed an \(\omega_0\) oscillation (pumping laser frequency) via conversion to the interaction
picture, which means the cavity resonant frequency is in fact $\omega_0 + \Delta$. The second term describes the interaction between the cavity field and the external continuum field with strength $\sqrt{2}\gamma$ in which the $\gamma$ is the decay rate of the cavity. The third term describes the pondermotive interaction between the optical field and the mechanical displacement, with coupling strength $G_0 = \omega_0/L$ while the $A = \sqrt{E_c/\hbar \omega_0}$ is the square root of the mean excitation number of the cavity mode and $\hat{a}$ is the deviation from $A$. We can choose our phase reference so that $A$ is real valued.

The Heisenberg equations of motion can be written as:

$$\dot{\hat{a}} + (\gamma + i\Delta)\hat{a} = -iG_0 A \dot{x} + \sqrt{2\gamma} \hat{c}_{\text{in}}. \quad (8.13)$$

$$\hat{c}_{\text{out}} = -\hat{c}_{\text{in}} + \sqrt{2\gamma} \hat{a}. \quad (8.14)$$

The signal to noise ratio is defined as

$$\rho = \int d\Omega |h_n(\Omega)|^2 \frac{S_{hh}(\Omega)}{S_{hh}(\Omega)} \quad (8.15)$$

For proving the Mizuno theorem, we assume the signal is a delta-function $x(t) = L h(t) \propto \delta(t)$ so that its Fourier transform is a constant, which leads to:

$$\rho = \int d\Omega \frac{1}{S_{xx}(\Omega)}, \quad (8.16)$$

and this is exactly the formula that Mizuno theorem constraints.

Solving the Heisenberg equations of motion Eq. (8.13) can give us the input-output relation as (for $t > 0$, after the impulse injection):

$$\hat{c}_{\text{out}}(t) = -\hat{c}_{\text{in}}(t) + 2\gamma \int_{t_0}^t dt'e^{-(\gamma + i\Delta)(t-t')} \hat{c}_{\text{in}}(t') + \sqrt{2\gamma} e^{-\gamma t - i\Delta t} a(0-) - i \sqrt{2\gamma} e^{-\gamma t - i\Delta t} AG_0, \quad (8.17)$$

where we define $a(0-) = \int_{-\infty}^0 dt'e^{(\gamma + i\Delta)t'} \hat{c}_{\text{in}}(t')$ which encodes the contribution of the vacuum injection during $t \in [-\infty, 0]$ to the intra-cavity field at $t = 0$. It is clear that $\hat{a}(0-) = 0$ is independent from the rest of the noise in $\hat{c}_{\text{out}}(t)$, because they arrive at $t = 0$ or after. The first two terms on the right hand side are the contribution of vacuum injection to $\hat{c}_{\text{out}}(t)$ after the impulse. Finally, the term $-iAG_0$ is actually the signal. Also if $t < 0$, we have:

$$\hat{c}_{\text{out}}(t) = -\hat{c}_{\text{in}}(t) + 2\gamma \int_{-\infty}^t dt'e^{-(\gamma + i\Delta)(t-t')} \hat{c}_{\text{in}}(t'). \quad (8.18)$$
8.2.2 Filtering of the outgoing field

- **Optimal sensitivity**—Suppose we measure the outgoing light using filter $f(t)$ and the measurement result is given by:

$$
\hat{X} = \hat{y} + \hat{y}^\dagger = \int_{0}^{+\infty} \left[ f(t)\hat{c}_{\text{out}}(t) + f^*(t)\hat{c}_{\text{out}}^\dagger(t) \right] dt + \int_{-\infty}^{0} \left[ f(t)\hat{c}_{\text{out}}(t) + f^*(t)\hat{c}_{\text{out}}^\dagger(t) \right] dt,
$$

(8.19)

The first term corresponds to the measurement after the impulse signal injection while the second term corresponds to the measurement before the signal injection. Note that the integration range is extend to infinity since the formulation of Mizuno theorem requires to integrate over the full bandwidth. Let us focus on the first term. We can proof that the vacuum noise injected after $t = 0$ (first two terms on the right hand side of Eq.(8.17)) can be completely removed.

If we want to remove the vacuum noise injected after $t = 0$, the first condition the filter function $f(t)$ should satisfy is:

$$
\hat{y} = \int_{0}^{+\infty} f(t) \left[ -\hat{c}_{\text{in}}(t) + 2\gamma \int_{0}^{t} dt' e^{-(\gamma+i\Delta)(t-t')} \hat{c}_{\text{in}}(t') \right] dt = 0,
$$

(8.20)

which can be transformed to:

$$
\hat{y} = -\int_{0}^{+\infty} f(t) \hat{c}_{\text{in}}(t) dt + 2\gamma \int_{0}^{+\infty} f(t') dt' \int_{0}^{+\infty} dt e^{(\gamma+i\Delta)(t-t')} \hat{c}_{\text{in}}(t)
$$

$$
\Rightarrow -f(t) + 2\gamma \int_{t}^{+\infty} e^{(\gamma+i\Delta)(t-t')} f(t') dt' = 0.
$$

(8.21)

If this condition is satisfied, then what we have is:

$$
\hat{X} = \sqrt{2\gamma} \int_{0}^{+\infty} \left\{ \sqrt{2} \Re[f(t)e^{-(\gamma+i\Delta)t}][\hat{a}_1 - \sqrt{2} \Im[f(t)e^{-(\gamma+i\Delta)t}](\hat{a}_2 - \sqrt{2} A_G)] \right\},
$$

(8.22)

where $\hat{a}_1, \hat{a}_2$ are the amplitude and phase quadrature of the intra-cavity field at time $t = 0$.

For removing the $\hat{a}_1$ noise, the filter function $f(t)$ should satisfy another relation that is:

$$
\Re[f(t)e^{-(\gamma+i\Delta)t}] = 0,
$$

(8.23)

and finally, the optimized measured quantity is:

$$
\hat{X}_{\text{opt}} = F(t)[\hat{a}_2 - \sqrt{2} A_G],
$$

(8.24)

where $F(t) = -\sqrt{2\gamma} \int_{0}^{+\infty} \sqrt{2} \Im[f(t)e^{-(\gamma+i\Delta)t}]$. 


Since the $\hat{a}_2$ is the intra-cavity field, its variance $V_{\hat{a}_2}$ is equal to one if the measurement result within $[−\infty, 0]$ does not decrease the variance of $\hat{a}_2$ below the vacuum limit (This fact will be proved later). Then we can conclude that the signal to noise ratio is:

$$\text{SNR}_{\text{max}} = 2A^2G_0^2 = \frac{2E_c}{\hbar\omega_0} \left( \frac{2\pi}{\lambda} \right)^2 \frac{1}{\tau^2}.$$  

(8.25)

This is the Mizuno limit.

- **Realization of the optimal filtering**—As shown in the last subsection, for achieving the Mizuno limit, the filter function should satisfy two conditions Eq.(8.21) (8.23). These two conditions are actually compatible.

The first condition, by redefining $t' - t \to t'$, has the form of:

$$-f(t) + 2\gamma \int_{-\infty}^{\infty} dt' f(t' + t) K(t') = 0,$$

(8.26)

in which $K(t) = e^{-(\gamma+i\Delta)t} \Theta(t)$. Reexpress the above formula as the Fourier transformation of its counterpart in frequency domain will leads to:

$$\mathcal{F}\left[-f(\Omega) + 2\gamma \int_{-\infty}^{\infty} f(\Omega) K(\Omega) d\Omega\right] = 0,$$

(8.27)

in which the Fourier transformation of $K(\Omega)$ is:

$$K(\Omega) = \frac{1}{\gamma + i(\Omega + \Delta)} \text{ when } t > 0.$$  

(8.28)

Therefore, we have the final result as:

$$\mathcal{F}\left[\frac{\gamma - i(\Omega + \Delta)}{\gamma + i(\Omega + \Delta)} f(\Omega)\right] = 0 \text{ when } t > 0.$$  

(8.29)

The solution of the above equation is:

$$f(\Omega) = \frac{C}{\gamma - i(\Omega + \Delta)}$$

(8.30)

with $C$ as an integral constant (complex). The form of this filter function solution in the time domain would be:

$$f(t) = Ce^{-(\gamma-i\Delta)t}$$

(8.31)

Substitute it back to the other Maximum SNR condition, we have the equation for $C$:

$$\text{Re}[C \int_0^{\infty} e^{-2\gamma t} dt] = \text{Re}[C \int_0^{\infty} e^{-2\gamma t} dt] = 0 \Rightarrow \frac{1}{2\gamma} \text{Re}[C] = 0$$

(8.32)

As long as $C$ is a pure imaginary number, the second condition will be satisfied.
8.2.3 Relevance of measurement before the impulse injection

This subsection discusses the effect of the measurement before the impulse injection to the variance of the \( \hat{a}_2(0) \). The measurement result when \( t < 0 \) can be rewritten as:

\[
\hat{X} = \int_{-\infty}^{0} dt |f(t)| \left[ \hat{c}_{\text{out}}(t)e^{i\theta_t} + \hat{c}_{\text{out}}(t)e^{-i\theta_t} \right],
\]

while the general Hermitian operators \( \hat{Q}_\theta \) for the measurement result of \( a(0) \) can be written as:

\[
\hat{Q}_\theta = \sqrt{2\gamma}|f(0)| [\hat{a}(0)e^{i\theta_0} + \hat{a}^\dagger(0)e^{-i\theta_0}].
\]

In these formulae, the \( \theta_t \) is the phase of the filter function \( f(t) \) and \( |f(t)| \) is the absolute value of the filter function. Note that \( \hat{a} \) and \( \hat{c}_{\text{out}} \) are two different degrees of freedom, therefore we have \([\hat{a}^\dagger(0), \hat{c}_{\text{out}}(t)] = 0 \) when \( t < 0 \) (it is also straightforward to proof it mathematically). Then similarly, we also have:

\[
[\hat{Q}_\theta, \hat{X}] = 0.
\]

Moreover, using the fact that the cavity mode and the output fields are at their respective vacuum states \( |0\rangle_a \) and \( |0\rangle_{\hat{c}_{\text{out}}} \) before the impulse, the causality condition also leads to:

\[
\langle \hat{Q}_\theta \hat{X} \rangle = \langle \hat{Q}_\theta \rangle \langle \hat{X} \rangle = 0
\]

because \( \langle \hat{a}(0)\hat{c}_{\text{out}}^\dagger(0) \rangle = \langle [\hat{a}(0), \hat{c}_{\text{out}}^\dagger(0)] \rangle + \langle \hat{c}_{\text{out}}^\dagger(0)\hat{a}(0) \rangle = 0 \). These two results Eq.(8.35)(8.36) tells us that even though \( \hat{Q}_\theta \) and \( \hat{X}(t) \) are simultaneously measurable, measuring \( \hat{X}(t) \) will not provide any information about \( \hat{Q}_\theta \) since they have no correlation at all. In another words, the conditional variance of \( \hat{a}_2 \) under the measurement of \( \hat{X} \) is the same as the unconditional one.

As a summary, in this section, we have proved Mizuno theorem by studying the optimization of signal to noise ratio of the system given in Eq.(8.12). The key of this prove can be summarized as the following two points: 1) Mizuno theorem can be formulated with a model describing an optomechanical system triggered by a delta-function signal. 2) the measurement of the outgoing field before the signal injection does not provide information about the intra cavity field at time \( t = 0 \), thereby can not decrease the variance of the intra-cavity field quadrature below the quantum vacuum level.
8.3 Sensitivity bound for the devices with a parametric amplifier

Discussion on the sensitivity limit for the devices with a parametric amplifier follows the same logic as the proof of Mizuno theorem above. We first give a brief discussion about the dynamics and input-output relation using a simple example with delta-function signal injected at \( t = 0 \) and then generalize the result. Once the input-output relation is obtained, we will discuss the filtering of the measurement result and the optimization of the sensitivity. The relevance of the measurement result before the signal injected will also be discussed. Finally, we will give a physical discussion of the meaning of our results.

8.3.1 An example system and input-output relation

Let us consider a single optomechanical system whose intra-cavity field interacts with a parametric amplifier with an internal degree of freedom \( \hat{b} \), whose Hamiltonian is written as:

\[
\hat{H} = \hbar \Delta \hat{a}^\dagger \hat{a} + \sqrt{2 \gamma} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left[ \hat{c}_\omega^\dagger \hat{a} + \hat{c}_\omega \hat{a}^\dagger \right] + \hbar G_0 \left( A \hat{a}^\dagger + A \hat{a} \right) \hat{x} + \hat{a} + \hat{V}[\hat{a}, \hat{b}, \hat{n}] \tag{8.37}
\]

where

\[
\hat{V}[\hat{a}, \hat{b}, \hat{n}] = f \hat{b}^\dagger \hat{a}^\dagger + f^* \hat{b} \hat{a} + i \sqrt{2 \gamma_0} (\hat{b}^\dagger \hat{n} - \hat{b} \hat{n}^\dagger) + \hbar \omega_0 \hat{b}^\dagger \hat{b}. \tag{8.38}
\]

In this interaction Hamiltonian \( \hat{V}[\hat{a}, \hat{b}, \hat{n}] \), the third terms describes the coupling between the amplifier’s internal degree of freedom and the external continuum and the last term is the free Hamiltonian of the \( \hat{b} \)-field. For simplicity, we assume the parametric coupling strength is a constant.

The equations of motion can be written as:

\[
\begin{align*}
\dot{\hat{a}} &= - (\gamma + i \Delta) \hat{a} - i G_0 \hat{A} \hat{x} + \sqrt{2 \gamma} \hat{c}_\text{in} - i f \hat{b}^\dagger \\
\dot{\hat{b}} &= - i f \hat{a}^\dagger + \sqrt{2 \gamma_0} \hat{n}_\text{in} - \gamma_0 \hat{b} \\
\hat{c}_\text{out} &= - \hat{c}_\text{in} + \sqrt{2 \gamma} \hat{a} \\
\hat{n}_\text{out} &= \hat{n}_\text{in} + \sqrt{2 \gamma} \hat{b}.
\end{align*} \tag{8.39}
\]
Solving these equations using Laplace transformation, we can separate the field before and after $t = 0$:

$$
\begin{align*}
&s\hat{b}(s) - \hat{b}(0) + \gamma_b \hat{b}(s) = \sqrt{2}\gamma_0 \hat{n}_{\text{in}}(s) - if\hat{a}^\dagger(s) \\
&s\hat{a}(s) - \hat{a}(0) + (\gamma + i\Delta)\hat{a}(s) = \sqrt{2}\gamma \hat{c}_{\text{in}}(s) - if\hat{b}^\dagger(s).
\end{align*}
$$

(8.40)

Then with signal $x_0\delta(t)$ injected in, the cavity field can be written as:

$$
\left[s + \gamma + i\Delta - \frac{f^2}{s + \gamma_b}\right] \hat{a}(s) = \sqrt{2}\gamma \hat{c}_{\text{in}}(s) + [\hat{a}(0) - iG_0\tilde{A}x_0] - \frac{if}{s + \gamma_b} \hat{b}^\dagger(0) - \frac{if\sqrt{2}\gamma_0}{s + \gamma_b} \hat{n}_{\text{in}}^\dagger(s).
$$

(8.41)

It is clear from the above equation that the parametric interaction between the cavity field and $\hat{b}$ affect the bandwidth and resonant frequency of the cavity field. Besides, the the enhancing of the amplifier’s mode $\hat{b}$ due to the parametric interaction as what we have seen in the last two chapters also shows here. Note that substituting the equation of motion for $\hat{a}$ into the parametric interaction term $-if\hat{a}^\dagger(s)$ in the equation of $\hat{b}$, in the adiabatic region where $\Omega$ is smaller than $\gamma_b$, we obtain:

$$
[-if\hat{a}^\dagger(s)]_b \approx \frac{f^2}{\gamma}
$$

(8.42)

where $[\ldots]_b$ means the term associated with the $\hat{b}$–field. This anti-damping factor is the same as our $\gamma_{\text{opt}}$ in the last section.

From the Eq(8.41), one can derive the input-output relation of $\hat{c}$–field:

$$
\begin{align*}
\hat{c}_{\text{out}}(t) &= -\hat{c}_{\text{in}}(t) + 2\gamma \int_0^t \chi_{11}(t - t')\hat{c}_{\text{in}}(t')dt' + \int_0^t \chi_{12}(t - t')\hat{n}_{\text{in}}^\dagger(t')dt' \\
&\quad + g(t)[\hat{a}(0) - iG\tilde{A}x] + h(t)\hat{b}^\dagger(0).
\end{align*}
$$

(8.43)
where $\chi_{11}(\Omega), \chi_{12}(\Omega), g(t)$ and $h(t)$ is the inverse transformation of the corresponding coefficients in Eq. [8.41]. The $\hat{a}(0)$ and $\hat{b}^\dagger(0)$ is the intra-cavity field and the internal amplifier mode at time $t = 0$. The form of this input-output relation tells us that this system works as a parametric phase-insensitive amplifier. Similar relation can also be obtained for the $\hat{n}$-mode.

In general, for a device with phase-insensitive amplifiers, the input-output relation can be written as (for $t > 0$):

$$
\hat{c}_{\text{out}2}(t) = \alpha_{11} \circ \hat{c}_{\text{in}2} + \alpha_{12} \circ \hat{n}_{\text{in}1} + g_1(\hat{a}_2(0) - iGAx) + h_1(t)\hat{b}_1(0),
$$

$$
\hat{n}_{\text{out}1}(t) = \alpha_{21} \circ \hat{c}_{\text{in}2} + \alpha_{22} \circ \hat{n}_{\text{in}1} + g_2(\hat{a}_2(0) - iGAx) + h_2(t)\hat{b}_1(0),
$$

where the “$\circ$” operation actually means an integral:

$$
A \circ B = \int_0^t A(t - t')B(t')dt'.
$$

This expression of input-output relation in the quadrature picture is also quite general since for parametric phase-insensitive amplifier, we can always align the input-output relation in this form.

For $t < 0$, the input-output relation can be written as

$$
\hat{c}_{\text{out}2}(t) = \alpha_{11} \circ \hat{c}_{\text{in}2} + \alpha_{12} \circ \hat{n}_{\text{in}1},
$$

$$
\hat{n}_{\text{out}1}(t) = \alpha_{21} \circ \hat{c}_{\text{in}2} + \alpha_{22} \circ \hat{n}_{\text{in}1}.
$$

Here where the “$\circ$” operation means a similar integral with a different integration range:

$$
A \circ B = \int_{-\infty}^t A(t - t')B(t')dt'.
$$

### 8.3.2 Filtering of the outgoing field

From the above input-output relation, it is clear that the displacement signal not only be carried by the phase quadrature of the cavity output channel $\hat{c}_{\text{out}2}$, but also be carried by the amplitude quadrature of the amplifier output channel $\hat{n}_{\text{out}1}$ in the devices with the parametric amplifier. This is due to the interaction between the cavity mode and the amplifier mode. For reaching the maximum sensitivity of this device, we have to extract the full information of the displacement signal, that is we have to measure both of the quadratures.
8.3. Sensitivity bound for the devices with a parametric amplifier

Suppose we have two filter functions \( f_1(t) \) and \( f_2(t) \), the measurement result can be written as

\[
\hat{X}_{\text{out}} = \hat{X}_{\text{cout}} + \hat{X}_{\text{nout}}
\]

\[
\hat{X}_{\text{cout}} = (f_1|\alpha_1 \circ \hat{c}_{2\text{in}} + \alpha_2 \circ \hat{n}_{1\text{in}}) + (f_1|g_1(t)|\hat{a}_2(0) - \sqrt{2}GAx) + (f_1|h_1(t)|\hat{b}_1(0)),
\]

\[
\hat{X}_{\text{nout}} = (f_2|\alpha_2 \circ \hat{c}_{2\text{in}} + \alpha_2 \circ \hat{n}_{1\text{in}}) + (f_2|g_2(t)|\hat{a}_2(0) - \sqrt{2}GAx) + (f_2|h_2(t)|\hat{b}_1(0)),
\]

(8.48)

in which the \( \langle A|B \rangle \) is defined as

\[
\langle A|B \rangle = \int_{-\infty}^{\infty} A(t)B(t)dt.
\]

(8.49)

Following the procedure for proving the Mizuno theorem, we also separate the integral into two parts: the integral over \([-\infty, 0]\) and the integral over \([0, +\infty]\). Where the latter one carries the displacement signal. Let us focus first on the measurement date collected over \([0, +\infty]\).

Similar to the case of Sec. III B, we can remove the vacuum injection after \( t = 0 \) by choosing the filter functions as:

\[
\alpha_1|f_1 \rangle + \alpha_2|f_2 \rangle = 0,
\]

\[
\alpha_2|f_1 \rangle + \alpha_2|f_2 \rangle = 0.
\]

(8.50)

Note that these two filter function condition equations can be viewed as matrix equation if the time is discretized\(^3\) Applying the filter satisfying the Eq.(8.50) to the measurement output \( \hat{X}_{\text{out}} \), we have:

\[
\hat{X}_{\text{out}} \rightarrow [(f_1(t)|g_1(t)) + (f_2(t)|g_2(t))[\hat{a}_2(0) - \sqrt{2}GAx] + [(f_1(t)|h_1(t)) + (f_2(t)|h_2(t))\hat{b}_1(0)]
\]

(8.51)

The noise variance refereed to the signal can be written as

\[
G^2A^2\langle XX \rangle_n = V_{a_{2a_2}} + \alpha^2V_{b_{1b_1}} + 2\alpha V_{a_{2b_1}} = V_{b_{1b_1}} \left[ \alpha + \frac{V_{a_{2b_1}}}{V_{b_{1b_1}}} \right]^2 + V_{a_{2a_2}} - \frac{V_{a_{2b_1}}^2}{V_{b_{1b_1}}},
\]

(8.52)

\(^3\)For the time discretized into \( N \) portions, \( |f_{1,2}(t) \rangle \) is now a column vector with elements \( f_{1,2}(t_i) \) and \( \alpha_{11} \) is now a \( N \times N \) matrix with elements \( \alpha_{11}(t_i, t_j) \). In the continuous case, the matrix equation becomes a integral equation.
where the coefficient $\alpha$ is defined to be:

$$
\alpha \equiv \frac{\left< f_1(t)|h_1(t)\right> + \left< f_2(t)|h_2(t)\right>}{\left< f_1(t)|g_1(t)\right> + \left< f_2(t)|g_2(t)\right>}. \tag{8.53}
$$

If $\alpha$ can take the value:

$$
\alpha = -V_{a_2b_1}/V_{b_1b_1}, \tag{8.54}
$$

then the noise variance can take the minimum value and the signal to noise ratio takes the maximum value:

$$
\text{Min}[\langle XX \rangle_n] = \frac{1}{2G^2A^2} \left[ V_{a_2a_2} - \frac{V_{a_2b_1}^2}{V_{b_1b_1}} \right], \quad \left[ \frac{S}{N} \right]_{\text{max}} = \frac{2G^2A^2}{V_{a_2a_2} - \frac{V_{a_2b_1}^2}{V_{b_1b_1}}}. \tag{8.55}
$$

However, the two conditions Eqs. (8.50)(8.54) are not always compatible with each other since Eqs. (8.50) already gives a complete solution of filter function $f_1(t)$ and $f_2(t)$. Therefore, the maximum sensitivity can be only a bound which may not be achievable. Therefore, we have the following inequality:

$$
\frac{S}{N} < \frac{2G^2A^2}{V_{a_2a_2} - \frac{V_{a_2b_1}^2}{V_{b_1b_1}}}. \tag{8.56}
$$

- (a) It is important to notice that the variance $V_{a_2a_2}$, $V_{a_2b_1}$ and $V_{b_1b_1}$ are all variance conditioned on the measurement result when $t < 0$. Therefore it is important to study the effect of the measurement before $t = 0$ to the $\hat{a}$ and $\hat{b}$ fields at $t = 0$.

- (b) The interaction between the cavity field $\hat{a}$ and $\hat{b}$ entangles these two fields. Therefore the $\hat{a}(\hat{b})$ itself is in a mixed state. However, the denominator of Eq. (8.56) can be viewed as the variance of $\hat{a}_2$ conditioned on the measurement of $\hat{b}_1$. This conditional variance of the $\hat{a}_2$ field must be smaller than its unconditional variance. In another words, Since the $\hat{b}_1$ quadrature contains the information of $\hat{a}_2$, therefore we will obtain information on $\hat{a}_2$ if $\hat{b}_1$ was measured. This leads to the decrease of the uncertainty of $\hat{a}_2$. However, as we shall see later, the uncertainty of $\hat{a}_2$ can not be decreased without any limit. This limit can give a looser bound of signal to noise ratio than Eq. (8.56).

- (c) It is clear that if there is no parametric interaction between $\hat{a}$ and $\hat{b}$, there will be no correlation between $\hat{a}_2$, $\hat{b}_1$: $V_{a_2b_1} = 0$. At the same time, the $\hat{a}$ is in a pure state with variance $V_{a_2a_2} = 1$. In this case, the signal to noise ratio goes back to the Mizuno limit $2G^2A^2$. 
8.3. Sensitivity bound for the devices with a parametric amplifier

Figure 8.3 – Noise error ellipse of the $\hat{a}_1$ and $\hat{a}_2$ quadratures. The red curve corresponds to the unconditional noise ellipse, while the blue curve corresponds to the noise ellipse of $(\hat{a}_1, \hat{a}_2)$ conditioned on the measurement of $\hat{b}_1$. The unconditional variance of $\hat{a}_1$ and $\hat{a}_2$ are conditional on the measurement result when $t < 0$. The $V(\hat{a}_1 | \hat{b}_1)$ is equal to unconditional variance of $\hat{a}_1$ because of there is no correlation between $\hat{a}_1$ and $\hat{b}_1$ for the phase-insensitive parametric amplifier. The “squeeze” of the conditional variance of $\hat{a}_2$ can not be too large because of the limitation of Heisenberg uncertainty principle. This argument can also be used for the variance conditioned on the measurement result before the signal impulse injection.

8.3.3 A looser sensitivity bound

In the last subsection, we have mentioned that the conditional variance of $\hat{a}_2$ can not be decreased without limit. This is actually due to the Heisenberg uncertainty principle:

$$V(\hat{a}_2 | \hat{b}_1)V(\hat{a}_1 | \hat{b}_1) > 1 \Rightarrow V(\hat{a}_2 | \hat{b}_1) > \frac{1}{V(\hat{a}_1 | \hat{b}_1)},$$

where the variance $V(\hat{A} | \hat{B}) = V_{AA} - V_{AB}^2/V_{BB}$ is the conditional variance of $\hat{a}_2$ under the measurement of $\hat{b}_1$ (See Fig. 8.3). Therefore, we can write Eq.(8.56) as:

$$\frac{S}{N} < \frac{2G^2A^2}{V(\hat{a}_2 | \hat{b}_1)} < 2G^2A^2V(\hat{a}_1 | \hat{b}_1) = 2G^2A^2V_{a_1a_1},$$

in which the last equality is due to the fact that $\hat{a}_1$ and $\hat{b}_1$ are uncorrelated and $V_{a_1a_1}$ is the unconditional variance of the $\hat{a}_1$ quadrature.

Note that the validity of this inequality depends on the measured quadrature. For example, if we measured $\hat{b}_1$ for predict $\hat{a}_2$ and measured $\hat{b}_2$ for predict $\hat{a}_1$, then $V(\hat{a}_2 | \hat{b}_1)V(\hat{a}_1 | \hat{b}_2)$ could be smaller than 1 [13], which is a essential feature of the EPR paradox [14].
The $V_{\hat{a}_1\hat{a}_1} > 1$ for mixed state, therefore the Mizuno limit could be surpassed for the devices with parametric amplifier and the maximum sensitivity is depended on the details of the devices.

### 8.3.4 Relevance of measurement before $t = 0$

Now let us study the point (a) listed at the end of Subsection(B), namely, how the measurement result before $t = 0$ affects the $\hat{a}$ and $\hat{b}$ field at time $t = 0$. For simplicity, we ignore the entanglement between the $\hat{a}$ and $\hat{b}$ at $t = 0$, only focus on the entanglement between the outgoing field when $t < 0$ and the intra-cavity field at $t = 0$.

Intuitively, if we measured the $\hat{c}_{2\text{out}}$ and $\hat{n}_{1\text{out}}$ before $t = 0$, there should be no correlation between the measurement result and the $\hat{a}_1$ quadrature at $t = 0$, since what is discussed here is the phase-insensitive parametric amplifier. However, there will be effect on the $\hat{a}_2$ quadrature. For example, $\hat{a}_2$ quadrature can be squeezed by conditioning on the measurement result. However, its squeezing will have a limited bound due to Heisenberg uncertainty principle as we have discussed in the last section.

The relevant quantities are the quadratures of the intra-cavity field $\hat{Q}_0 = (\hat{a} + \hat{a}^\dagger)$ or $\hat{Q}_{\pi/2} = -i(\hat{a} - \hat{a}^\dagger)$ and $\hat{P}_0 = \hat{b} + \hat{b}^\dagger$, $\hat{P}_{\pi/2} = -i(\hat{b} - \hat{b}^\dagger)$ and the quantities we measured before $t = 0$ is:

$$\hat{X}_{\text{out}} = \hat{X}_{\text{c}} + \hat{X}_{\text{n}}.$$

(8.59)

where $\hat{X}_{\text{c}}$ and $\hat{X}_{\text{n}}$ are the cavity output channel and the amplifier output channel. They can be represented as:

$$\hat{X}_{\text{c}} = \langle f_1 | \alpha_{11} \hat{c}_{\text{in}}^2 \rangle + \langle f_1 | \alpha_{12} \hat{n}_{\text{in}}^1 \rangle,$$

$$\hat{X}_{\text{n}} = \langle f_2 | \alpha_{21} \hat{c}_{\text{in}}^2 \rangle + \langle f_1 | \alpha_{22} \hat{n}_{\text{in}}^1 \rangle.$$

(8.60)

Using the fact that:

$$\int_{-\infty}^{0} \int_{-\infty}^{t} f_1(t)\alpha(t - t')\hat{m}_{\text{in}}(t') = \int_{-\infty}^{0} dt' \int_{t'}^{0} dt\alpha(t - t')f_1(t)\hat{m}_{\text{in}}(t').$$

(8.61)

in which the $\alpha(t)$, $f(t)$ and $m_{\text{in}}(t)$ are arbitrary functions and operators. The underlined part is defined to be a function $\lambda(t')$. Then the measurement operator can be
written as:

\[
\hat{X}_{\text{out}} = \lambda_{11} \hat{c}_{2\text{in}} + \lambda_{12} \hat{n}_{1\text{in}} \\
\hat{X}_{\text{out}} = \lambda_{21} \hat{c}_{2\text{in}} + \lambda_{22} \hat{n}_{1\text{in}}.
\] (8.62)

The \( \hat{a}(0) \) field can be written as:

\[
\hat{a}_2(0) = (\beta_{21}|\hat{c}_{2\text{in}}) + (\beta_{22}(t)|\hat{n}_{1\text{in}}), \quad \hat{a}_1(0) = (\beta_{11}|\hat{c}_{1\text{in}}) + (\beta_{12}|\hat{n}_{2\text{in}})
\] (8.63)

\[
\hat{b}_1(0) = (\epsilon_{11}|\hat{c}_{2\text{in}}) + (\epsilon_{12}(t)|\hat{n}_{1\text{in}}), \quad \hat{b}_2(0) = (\epsilon_{21}|\hat{c}_{1\text{in}}) + (\epsilon_{22}|\hat{n}_{2\text{in}}),
\]

where

\[
(A|B) = \int_{-\infty}^{0} A(t)B(t)dt.
\] (8.64)

The conditional variance for any \( \hat{O} \) field at \( t = 0 \) can be written as:

\[
\langle \hat{O}^2 \rangle_c = \langle \hat{O}^2 \rangle - \frac{\langle \hat{O}\hat{X}_{\text{out}} \rangle^2_{\text{sym}}}{\langle \hat{X}_{\text{out}}^2 \rangle},
\] (8.65)

where the first term is unconditional variance of \( \hat{O} \). Therefore the effect of the measurement before \( t = 0 \) on the operator at \( t = 0 \) is encoded in the correlation function

\[
\langle \hat{O}\hat{X}_{\text{out}} \rangle_{\text{sym}} = \frac{1}{2} \langle \hat{O}\hat{X}_{\text{out}} + \hat{X}_{\text{out}}\hat{O} \rangle
\] (8.66)

Then turn to the looser sensitivity bound discussed in Section IV.C, which is given by \((S/N) < 2G^2A^2V_{a1a1}\), where the \( V_{a1a1} \) is the variance of \( \hat{a}_1 \) conditioned on the measurement data before \( t = 0 \), but unconditioned on \( \hat{b}_1 \) at \( t = 0 \). It is straightforward to calculate this correlation function, as an example, for \( \hat{O} = \hat{Q}_0, \hat{P}_{\pi/2} \), we obtain:

\[
\langle \hat{Q}_0\hat{X}_{\text{out}} \rangle_{\text{sym}} = 0, \quad \text{and} \quad \langle \hat{P}_{\pi/2}\hat{X}_{\text{out}} \rangle_{\text{sym}} = 0.
\] (8.67)

However for \( \hat{O} = \hat{Q}_{\pi/2}, \hat{P}_0 \), we have:

\[
\langle \hat{Q}_{\pi/2}\hat{X}_{\text{out}} \rangle_{\text{sym}} = (\lambda_{11}|\beta_{21}) + (\lambda_{21}|\beta_{22}) \quad \text{and} \quad \langle \hat{P}_{\pi/2}\hat{X}_{\text{out}} \rangle_{\text{sym}} = (\lambda_{11}|\epsilon_{21}) + (\lambda_{21}|\epsilon_{22}).
\] (8.68)

They are in general not equal to zero.

Therefore, generally speaking, the value of the conditional variance in Eq. (8.56) will be affected by the measurement before \( t = 0 \). However, following the argument in Section 8.4.3, the looser sensitivity bound is unchanged since \( V_{a1a1} \) is unaffected by the measurement. In another words, even if the measurement result before \( t = 0 \) can
somehow squeeze the variance of the $\hat{a}_2$ field at $t = 0$ to be $V(\hat{a}_2|\hat{c}_{\text{out}2}, \hat{b}_{\text{out}1} t < 0) < V_{\hat{a}_2}$, this variance can not be decreased without any limit because of the Heisenberg uncertainty principle (See. Fig.8.3):

$$V(\hat{a}_2|\hat{c}_{\text{out}2}, \hat{b}_{\text{out}1} t < 0) V(\hat{a}_1|\hat{c}_{\text{out}2}, \hat{b}_{\text{out}1} t < 0) > 1 \Rightarrow$$

$$V(\hat{a}_2|\hat{c}_{\text{out}2}, \hat{b}_{\text{out}1} t < 0) > \frac{1}{V(\hat{a}_1|\hat{c}_{\text{out}2}, \hat{b}_{\text{out}1} t < 0)} = \frac{1}{V_{\hat{a}_1\hat{a}_1}}. \quad (8.69)$$

### 8.4 Conclusion and summary

In this chapter, firstly, we proved the Mizuno theorem in the case of an ordinary optomechanical system with out the amplifier. Secondly, we extended the method developed in proving the Mizuno theorem and gives two sensitivity bounds for the devices with parametric amplifier. The first bound Eq.\ref{eq:8.56} is a more tight bound the value of which depends on the measurement of the data before the injection of the signal impulse, while the second bound Eq.\ref{eq:8.58} is a looser bound, but the result does not depend on the measurement of the data before the signal injection. This research gives an estimation of the upper-limit of the shot-noise limited sensitivity of such kind of devices.
Bibliography


A number of closely related subjects in were studied in this thesis. These subjects explored the optomechanical physics in the design of laser interferometer gravitational wave detectors, which could be the basis of future improvement work for gravitational wave detectors.

We explored the effect of optomechanical modification of mechanical dynamics on the energy interaction between gravitational waves and the interferometers in Chapter 3. We found that the optical damping due to the unbalanced sideband field causes the laser interferometer to absorb energy from gravitational waves. A further question about the relationship between the energy absorption and the sensitivity was inspired: does extraction more energy from gravitational waves allow substantial increases in detector sensitivity? Future work on answering this question may not only help us understand the working mechanism of interferometer thereby improving its design, but also motivate the study a more general relation between energy absorptions and sensitivities for general measurement devices.

Moreover the effect of optomechanical modification of mechanical dynamics on thermal noise was discussed using the example of three mode interaction noises in Chapter 4. Although our results show that this noise has negligible effect on current detectors in the frequency band of interest, it may have a large effect in the noise budget of other low-frequency gravitational wave detectors. Moreover, the physics of this noise itself is interesting since the noise is amplified due to optomechanically modified mechanical dynamics. This indicates that noises generated from optomechanical interaction process may not be negligible in certain conditions.

The application of optomechanical devices for improving the detector quantum sensitivity was discussed in Chapter 5 and 6. We found that the compactness and tunability of optomechanical devices in principle allow us to build a filter for rotating
the squeezed light before entering the dark port of the interferometer. We also in-
vvented an optomechanical approach for realising a white light cavity design for laser
interferometer gravitational wave detectors, which can surpass the Mizuno theorem.
We concluded that the main challenge for realising these designs is the serious effect
of thermal noise.

To dilute the effect of thermal noise, we discovered a novel optical dilution scheme
based on dissipative optomechanics in Chapter 5. This scheme in principle does
not have additional quantum noise and optomechanical instabilities caused by the
coupling between the mechanical motion and the dilution laser beam. However, the
effect of trapping beam power absorption on the thermal noise has not been entirely
understood. Especially, the temperature gradient developed across the oscillator can
induce a non-equilibrium thermal noise, which also exists in cryogenic large-scale
gravitational wave detectors such as KAGRA. Future work on understanding the
physics of this non-equilibrium noise and calculating its spectrum is important not
only for the optical dilution experiment, but also for evaluating the sensitivity of
cryogenic gravitational wave detectors.

Additionally, the optical dilution scheme that was proposed in Chapter 5 can
also be used to experimentally observe the quantum jump of mechanical membrane
among different energy levels. S. Danilishin and I developed a program to simulate
this experiment aiming for searching of optimal experimental parameters. Currently
this simulation is constrained by its long running time, but in the future, a more
optimised code can be developed for studying the quantum jump phenomenon.

The similarity between linear optoatomic systems and optomechanical systems
allows us to develop a systematic formalism to study the previous proposals of white
light cavity design based on the stable double-pumped atomic gain media in Chapter
7. We found that the sensitivity of these proposals are still limited by the Mizuno
limit due to the stability requirement and effect of additional quantum noise brought
by atomic media. In the future, it will be interesting to investigate if we can use
atomic system to design a unstable white light cavity interferometer with feedback
control similar to the one shown in Chapter 6.

More importantly, the study of these topics on the use of optomechanical/opto-
atomic devices for surpassing quantum limits inspires a number of questions: what is the ultimate quantum noise limited sensitivity of a gravitational wave detector, or more generally any linear measurement devices? If this ultimate limit exists, what parameters will it depend on? What are the requirements to reach this ultimate sensitivity? In Chapter 8 we reported that the ultimate quantum limit of general optomechanical devices with parametric amplifier depends on the amplitude quadrature fluctuation of cavity fields. However this discovery is not general enough and the time-domain formalism is insufficient to provide us an efficient way of computing. In the future, more work need to be done in analysing the ultimate quantum limited sensitivity. Answering these questions will not only help us to understand more deeply about the working mechanism of gravitational wave detectors, thus improve their future designs.