Geometrical Methods of Inference

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Presented for the degree of Doctor of Philosophy of

The University of Western Australia
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August, 2002
to the memory of my parents
Cuius rei demonstrationem mirabilem sane detexi
hanc marginis exiguitas non caperet.

Pierre de Fermat
Abstract

The central aim of the thesis is to investigate and present results of studies of two geometrical methods of inference:

+ the measure of central tendency in Riemannian manifolds—*the Riemannian mean* and
+ the interpolation of data points in Riemannian manifolds—*the Riemannian variational curves*.

Riemannian manifolds are smooth spaces equipped with a metric allowing to measure geometric quantities like distances and angles. Riemannian geometry—the branch of differential geometry concerning Riemannian manifolds—evolved from Euclid’s plane and solid geometry, and from Gauss’s theory of curved spaces. The thesis develops a geometrical approach to investigations of data in Riemannian manifolds.

In the first part of the thesis two types of statistical measures are of particular interest: the *measure of central tendency* and the *measure of dispersion*. The thesis studies a generalization of mean value in Riemannian manifolds, where the Riemannian distance is regarded as a measure of deviation. The thesis defines the *Riemannian mean* as a set of points in a complete manifold minimizing sum of squares of distances from the sample points. The minimum of the sum is a measure of dispersion—the *Riemannian variance*. The thesis investigates the following properties of the Riemannian mean:

+ the dependence of the mean on the covariance of a sample and the curvature of the space,
+ properties of the measure of dispersion at cut loci of the data point and
+ the convexity property on complete 2-manifolds.

An iterative method of deriving points of the Riemannian mean is proposed and its convergence is investigated. Practical examples demonstrate efficiency of the iterative method in case of Lie groups: $SO(3)$ and $SO(4)$.

In the second part the thesis studies interpolation with variational curves in Riemannian manifolds. Splines in Euclidean spaces have proved to be useful in approximation theory, analysis and statistics. Introduced as piecewise polynomial curves, splines evolved to abstract variational interpolating curves. The thesis explores a variational approach to curves in Riemannian manifolds. This includes the investigations of:

+ the Riemannian cubics in the space of rotations $SO(3)$ endowed with left-invariant metric,
+ the Riemannian cubics in the unit sphere $S^n$,
+ the minimal interval, depending on the initial conditions, where the analytic Riemannian cubics in $SO(3)$ exist,
+ the local form of cubics in complete Riemannian manifolds and
an application of Riemannian cubics to interpolation tangential directions in the plane.

The significance of the research is twofold. Firstly, the research shows that it is possible to: (1) derive statistical quantities like central tendency and dispersion; and (2) interpolate data points with smooth curves in non-Euclidean spaces in geometrically intrinsic way. Secondly, the research gives more insight into Riemannian geometry. We extend studies of geodesics and distances, and variational methods in complete Riemannian manifolds.
Acknowledgements

I would like to express my sincere gratitude to my supervisor, Associate Professor Lyle Noakes for his direct role of guide and mentor, and invaluable help he provided throughout the course of my PhD. His helpful ideas, suggestions and criticism made my research possible. I also wish to acknowledge the assistance of Dr Mike Alder, Dr Luchezar Stoyanov and Professor Melvyn Sargent during my studies.

I gratefully acknowledge the financial support during a two year period of my studies given by The University of Western Australia.

I wish to thank my family and friends, who encouraged and supported me during these years. I am deeply grateful to Catherine McLoughlin for her invaluable encouragement, understanding and proofing of the thesis.
# Contents

Abstract iii

Acknowledgements v

Notation xv

I Introduction 1

1 Introduction 3

1.1 Motivation and Scope 3

1.2 Contribution to the Field 4

1.3 Overview of Thesis 5

2 Literature Review 7

2.1 Riemannian Geometry 8

2.1.1 Background and history of Riemannian geometry 8

2.1.2 Riemannian geometry and statistics 9

2.1.3 Riemannian geometry and splines 11

2.2 Riemannian Means 11

2.2.1 Mean value in non-Euclidean spaces—statistical approach 11

2.2.2 Mean value in non-Euclidean spaces—geometrical approach 15

2.3 Riemannian Variational Curves 17
<table>
<thead>
<tr>
<th>Contents</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3.1 Polynomial splines</td>
<td>17</td>
</tr>
<tr>
<td>2.3.2 Variational splines</td>
<td>18</td>
</tr>
<tr>
<td>2.3.3 Splines in Riemannian spaces</td>
<td>21</td>
</tr>
<tr>
<td>2.3.4 Other curves in Riemannian spaces</td>
<td>22</td>
</tr>
<tr>
<td>2.4 Recapitulation</td>
<td>24</td>
</tr>
<tr>
<td>3  Overview of Riemannian Geometry</td>
<td>27</td>
</tr>
<tr>
<td>3.1 Riemannian Manifold</td>
<td>27</td>
</tr>
<tr>
<td>3.2 Connections</td>
<td>28</td>
</tr>
<tr>
<td>3.3 Riemannian Geodesics</td>
<td>32</td>
</tr>
<tr>
<td>3.4 Curvature</td>
<td>32</td>
</tr>
<tr>
<td>3.5 Calculus of Variations</td>
<td>34</td>
</tr>
<tr>
<td>3.6 Normal Coordinates and the Exponential Map</td>
<td>37</td>
</tr>
<tr>
<td>3.7 Comparison Theorems</td>
<td>38</td>
</tr>
<tr>
<td>3.8 Cut Points and Cut Loci</td>
<td>39</td>
</tr>
<tr>
<td>3.9 Riemannian Geometry in Statistics</td>
<td>41</td>
</tr>
<tr>
<td>II  The Riemannian Mean</td>
<td>43</td>
</tr>
<tr>
<td>4  Definition of the Riemannian Mean and its Properties</td>
<td>45</td>
</tr>
<tr>
<td>4.1 Definition of the Riemannian Mean</td>
<td>47</td>
</tr>
<tr>
<td>4.1.1 Complete manifolds with non-positive sectional curvature</td>
<td>50</td>
</tr>
<tr>
<td>4.2 Properties of the Riemannian Mean</td>
<td>52</td>
</tr>
<tr>
<td>4.3 Critical Points and the Riemannian Mean</td>
<td>53</td>
</tr>
<tr>
<td>4.3.1 Approximating the Riemannian mean</td>
<td>54</td>
</tr>
<tr>
<td>4.3.2 Second derivatives—the Hessian of $\Phi_Q$</td>
<td>66</td>
</tr>
<tr>
<td>4.3.3 The Riemannian mean on spheres and hyperbolic spaces</td>
<td>71</td>
</tr>
</tbody>
</table>
6.3 Preliminaries ................................................. 128
  6.3.1 The real projective plane ................................. 128
  6.3.2 Envelopes .................................................... 129
6.4 Envelopes as a Mapping ......................................... 131
6.5 Interpolation in the Space of Directions ......................... 135
6.6 Conclusion ........................................................ 146

7 Summary ............................................................ 147
  7.1 Introduction .................................................... 147
  7.2 Principal Findings of the Research ............................. 148
  7.3 Limitations and Future Directions ............................. 149

IV Appendices ......................................................... 153

A Riemannian Metric ................................................ 155
  A.1 Taylor Series Derivation ...................................... 155
    A.1.1 Non-symmetric connection ................................. 155
    A.1.2 Symmetric connection ..................................... 156

B Data Sets .......................................................... 157
  B.1 Spherical Data ................................................ 157
  B.2 Normal Population Samples ................................... 159
List of Figures

2.1 Polynomial spline of third degree—cubic—is an example of continuous differentiable spline .......................................................... 18

2.2 One step of the "corner-cutting" algorithm on Riemannian manifold ............ 22

3.1 The tangential and normal components of the ambient connection ............ 31

4.1 Exponential mapping exp$_p$ is a local diffeomorphism .......................... 56

4.2 The variation $\Gamma(s, t)$ ................................................................. 63

4.3 For a set of points on a sphere which are symmetrically distributed at the same longitude the north pole is a critical point of $\Phi_Q$ ............... 65

4.4 Lambert-Schmidt (equal-area) projection of the magnetic remanence measurements ................................................................. 73

4.5 The radial geodesics in the polar-coordinates represented in the Poincaré half-plane ................................................................. 74

4.6 Data points in the Poincaré half-plane .............................................. 75

4.7 Rate of convergence of the iterative algorithm applied to points in $SO(4)$ .... 81

4.8 Geodesic $\gamma_{pq}$ intersects the line $\ell$ ........................................... 83

4.9 Fundamental property of the cut locus: geodesics $\gamma_2 \neq \gamma_1$ are both minimizing . 85

5.1 Smooth curve in a manifold satisfying boundary conditions .................... 99

5.2 $\mathcal{C}^1$-curve in $S^2$ embedded in $\mathbb{R}^3$ .................................... 109

5.3 $\mathcal{C}^1$-curve in $S^2$ represented in spherical coordinates $(\phi, \theta) \in \mathbb{R}^2$ ...... 111
6.1 The contour $C$ and a family of tangent lines ................. 127
6.2 Example of an envelope of line segments in the plane—the astroid .... 131
6.3 Envelope of lines generated by a curve in the projective plane ........ 133
6.4 Duality of $\mathbb{R}^2$ and $\mathbb{R}P^2$: geodesics are mapped into points 134
6.5 Riemannian cubic interpolating four points in $\mathbb{R}P^2$ .............. 136
6.6 The resulting curve (dashed) interpolates a curve (solid gray) given four tangent lines .................................................. 136
6.7 Error of interpolation of tangent directions as a function of density of samples 137
6.8 Interpolation of a plane curve (gray thick line) with one, two and four segments of the envelopes (thin black lines) of the $\mathcal{D}^1$-curves .......... 144
6.9 Error of interpolation as a function of density of samples ............... 145
List of Tables

4.1 Comparison of the Fisher's, embedded and the Riemannian mean in the sphere $S^2$ 73

4.2 Comparison of the Euclidean, embedded and the Riemannian mean in the Poincaré half-plane ........................................ 76

4.3 Convergence of the iterative algorithm applied to points in $SO(4)$ ........... 80

4.4 Link functions corresponding to the Amari's $\alpha$-connections for various distributions ........................................ 88

6.1 Convergence of the interpolation of tangent directions .................. 137

6.2 Convergence of the interpolation .................................................. 143

B.1 Spherical Data ................................................................. 158

B.2 Ten samples of size 10 from a normal population .......................... 159
Notation

Summary of Basic Notations

\( M, N \) \hspace{1cm} \text{manifold}
\( \mathcal{U}, \mathcal{V} \) \hspace{1cm} \text{neighbourhood}
\( \mathcal{W} \) \hspace{1cm} \text{neighbourhood in a tangent (vector) space}
\( B_\varepsilon \) \hspace{1cm} \text{open Riemannian ball}
\( \overline{B}_\varepsilon \) \hspace{1cm} \text{closed Riemannian ball}
\( \partial B_\varepsilon \) \hspace{1cm} \text{boundary of } B_\varepsilon \text{ or } \overline{B}_\varepsilon
\( \mathbb{R}^n \) \hspace{1cm} \text{n-dimensional Euclidean space}
\( S^n \) \hspace{1cm} \text{the } n\text{-dimensional unit sphere}
\( \mathbb{H}^n \) \hspace{1cm} \text{the } n\text{-dimensional hyperbolic space}
\( \mathbb{R}P^n \) \hspace{1cm} \text{real projective } n\text{-space}
\( \mathcal{O}(n) \) \hspace{1cm} \text{orthogonal group (space of all real } n \times n \text{ orthogonal matrices)}
\( \mathcal{S}O(n) \) \hspace{1cm} \text{special orthogonal group (space of all real } n \times n \text{ orthogonal matrices of determinant 1)}
\( \mathcal{C} \) \hspace{1cm} \text{set of complex numbers}
\( \mathbb{Z}_2 \) \hspace{1cm} \text{cyclic group of order 2}
\( \pi \) \hspace{1cm} \text{projection}
\( C \) \hspace{1cm} \text{sectional curvature}
\( \mathfrak{G} \) \hspace{1cm} \text{Lie group}
\( g \) \hspace{1cm} \text{Lie algebra of a Lie group } \mathfrak{G}
\( \mathcal{T}M \) \hspace{1cm} \text{tangent bundle}
\( \mathcal{T}_p M \) \hspace{1cm} \text{tangent space}
\( \mathcal{X}(M) \) \hspace{1cm} \text{space of smooth vector fields on } M
\( \mathcal{X}(\gamma) \) \hspace{1cm} \text{space of smooth vector fields along } \gamma
\( g \) \hspace{1cm} \text{metric (tensor field)}
\( \langle X, Y \rangle \) \hspace{1cm} \text{inner product of vectors } X \text{ and } Y
\( [X, Y] \) \hspace{1cm} \text{Lie bracket}
\begin{tabular}{|l|l|}
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$|X|$</td>
<td>norm of $X$</td>
</tr>
<tr>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$O(x^k)$</td>
<td>terms of order $x^k$ and higher</td>
</tr>
<tr>
<td>$D_t V$</td>
<td>covariant derivative of $V$ along curve $\gamma$</td>
</tr>
<tr>
<td>$\nabla_X Y$</td>
<td>connection, covariant derivative of $Y$ in the direction of $X$</td>
</tr>
<tr>
<td>$R(X,Y)Z$</td>
<td>the (Riemann) curvature endomorphism</td>
</tr>
<tr>
<td>$Rm(X,Y,Z,W)$</td>
<td>the curvature tensor</td>
</tr>
<tr>
<td>$\Pi(X,Y)$</td>
<td>second fundamental form</td>
</tr>
<tr>
<td>$\mathcal{L}_p$</td>
<td>class of integrable functions</td>
</tr>
<tr>
<td>$\mathcal{C}^n$</td>
<td>space of $n$-times continuously differentiable functions</td>
</tr>
<tr>
<td>$1$</td>
<td>identity</td>
</tr>
<tr>
<td>$cm {s_1, \ldots, s_m}$</td>
<td>center of mass</td>
</tr>
<tr>
<td>$I$</td>
<td>compact interval $[a,b] \subset \mathbb{R}$</td>
</tr>
<tr>
<td>$\Pi_m$</td>
<td>set of real polynomials of degree $m$ or less</td>
</tr>
<tr>
<td>$\partial_i$</td>
<td>the partial derivative operators $\partial/\partial x^i$</td>
</tr>
<tr>
<td>$\delta_{ij}$</td>
<td>Kronecker's delta</td>
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Part I

Introduction
Chapter 1

Introduction

This thesis demonstrates how the methods of Riemannian geometry are appropriate tools of inference and have resulted in new and significant findings that extend existing research. The aim of the research presented in this thesis is to investigate quantities derived from data in non-Euclidean spaces. For example measures of central tendency of a sample in the Poincaré half-plane, or a curve interpolating sample points in the projective plane.

1.1 Motivation and Scope

The thesis applies methods of Riemannian geometry to measures of central tendency and methods of interpolation by curves. This is an investigation of major questions related to samples of data:

- central tendency of a sample, and
- interpolation of points of a sample.

The two topics are all familiar to statisticians and computer scientists. What makes this investigation different are the methods of Riemannian geometry. Our mean value (the center of mass) and variational curves\(^1\) have the appeal of being intrinsic and are defined in geometrical terms. They are preserved by isometries, i.e., maps from one space to another that preserve lengths of curves. Also, both quantities

\(^1\)The term "variational curves" used in this thesis refers to the class of curves derived with variational methods.
are naturally defined in terms of Riemannian metric and do not depend on any particular system of coordinates. Since Riemannian geometry—the theory of smooth manifolds equipped with smoothly varying inner products on tangent spaces—has applications in many disciplines, this research may have implications to statistics, physics, robotics and computer science.

1.2 Contribution to the Field

This thesis contributes the following new insights into geometrical methods of inference:

- It derives an approximate formula for the points of local minimum of $\Phi_Q$, defined as a sum of square distances from a given point to every point of a finite set $Q$ (Definition 4.1.1). This formula demonstrates how the points of local minimum, and therefore also points of the Riemannian mean, depend on the curvature of a space and the distribution of the sample points.

- It proves the convergence of an iterative method of deriving points of local minima of $\Phi_Q$ in a complete Riemannian manifold. The iterative method applied to weighted averages on spheres was described in Buss & Fillmore [22].

- It proves that Riemannian mean and cut loci of $Q$ have no points in common other than conjugate points to the points in $Q$. This result extends the property of Riemannian means on spheres in relation with conjugate points obtained by Noakes [74], cf. Section 4.6.

- It provides a geometrical proof of the convexity property of the Riemannian mean in complete 2-manifolds. The proof is significantly different from the one of Corcuera & Kendall [28] who derive geodesic convexity of Riemannian barycenters on 2-manifolds, cf. Section 4.5.

- It derives the necessary and sufficient conditions for the Riemannian cubic in the unit sphere $S^n$. Different set of conditions and invariants of elastic curves on the 2-sphere, suitable for the computations of elastic curves, are given in Brunnett & Crouch [19]. Our result is applicable to higher dimensions.

- It determines the minimal interval, depending on the initial values, on which a unique Riemannian cubic in $SO(3)$ exists. This result extends the theory of Riemannian cubics whose existence and uniqueness conditions where derived in Camarinha [23].
1.3 Overview of Thesis

The thesis is divided into four parts. Part I introduces the subject studied in the thesis. Part II presents results of the research on the Riemannian mean. Part III studies Riemannian variational curves. Appendix A in Part IV, includes calculations of the Taylor series of the Riemannian geodesic and Appendix B includes the data used to illustrate properties of the Riemannian mean.

In Chapter 2 some historical background on Riemannian geometry is presented and a review of areas in current theory relevant to the subject studied in the thesis is examined. The development of ideas that lead to formulation of concepts of mean value and variational curves in terms of Riemannian geometry is outlined. This geometrical approach is founded on Noakes [74] and [77] on relations between information inferred from data and geometrical properties of a space.

Chapter 3 gives a brief overview of Riemannian geometry and establishes notation used in this thesis. The convention and notation closely follows Lee [58]. Riemannian manifolds, covariant derivative and curvature are reviewed and an account on comparison theorems is given. We also mention the geometry of probability measures and the Fisher information as background to the applications of the Riemannian geometry in statistics.

In Chapter 4 the Riemannian mean is defined and investigated. The Riemannian mean extends the spherical mean of a finite discrete set, introduced in Noakes [74]. A number of new results are presented. The dependence of the Riemannian mean on the curvature of a space and the covariance of the data points is established. We prove that under certain conditions the Riemannian mean belongs to any convex subset of the 2-dimensional complete manifold containing the sample. The Riemannian mean in neighbourhoods of cut loci of the data points is also investigated here. Furthermore, we propose and prove convergence of an iterative method of deriving points of the Riemannian mean.

Chapter 5 studies variational curves in complete Riemannian manifolds. Based on

The supervisor's assistance in developing these ideas is acknowledged.
recent developments in [23, 24] the simplest non-trivial curves in symmetric spaces are investigated: the unit sphere and Lie groups. For instance, Riemannian cubics in $\text{SO}(3)$ endowed with left-invariant metrics are examined in order to extend current research to a wider class of manifolds (and metrics) and further applications.

Chapter 6 concentrates on an application of Riemannian cubics to the problem of interpolation of tangential directions in the plane. Such problems arise in approximations of convex shapes from ray-based information. Our approach has a number of advantages. Firstly, interpolation with the Riemannian cubics is isometrically invariant. In other words, it is preserved by any isometric transformation, e.g., reflection or rotation. Secondly, it reflects geometry of a space and does not depend on any particular system of coordinates. Thirdly, it naturally extends to other spaces and allows for generalizations to interpolations with higher order variational curves.

Chapter 7 concludes with some important open problems arising from the research described in this thesis and give directions for further investigations.
Chapter 2

Literature Review

This chapter gives an overview of previous studies of geometrical approaches to measures of central tendency and to interpolation.

These approaches are applicable when data belongs to non-Euclidean space, for example a space of directions (sphere), lines and hyperplanes (projective spaces) transformations (Lie groups) or phase-space in physics. Geometrical approaches reflect natural properties of space (cf. [86, 74, 30]). As a result the central tendency and the methods of interpolation are both intrinsic, i.e., they are preserved by isometries.

Section 2.1 commences with a brief historical background on Euclidean geometry, non-Euclidean geometry and Riemannian geometry. Applications of Riemannian geometry in statistics and interpolation are presented, to provide background to the two topics studied in the thesis. Section 2.2 presents developments in studies of the center of mass. Two approaches are apparent. The first approach studies statistical distributions on smooth compact spaces like circles (cf. [40]) and spheres (cf. [41]). In this approach central tendency is a parameter of a distribution that can be inferred from a sample. The second approach investigates the center of mass in terms of intrinsic distances of space (cf. [78, 49]). The author’s results of research concerning this subject are presented in Part II. Section 2.3 describes recent studies in methods of interpolation in non-Euclidean spaces. In Part III the thesis summarizes research on Riemannian variational curves—a class of smooth curves in Riemannian manifolds. The properties of Riemannian variational curves are investigated and extended by applications to projective spaces.
2.1 Riemannian Geometry

This section describes the genesis of Riemannian geometry (a branch of differential geometry) and its applications to statistics and interpolation. This gives an overview of developments in these two particular areas and shows how the investigations of the thesis relate to contemporary research.

2.1.1 Background and history of Riemannian geometry

What follows is a brief history of developments in geometry that led to Riemannian geometry, named after Bernhardt Riemann who laid the foundations of non-Euclidean geometry in his lecture “On the hypotheses which lie at the foundations of geometry” in 1854.

Euclidean geometry

Euclid (323–285/283 B.C.), who lived in Alexandria at the time of the first Ptolemy, systematized the work of Pythagoras and his followers. About 300 B.C. Euclid wrote a treatise in thirteen books called *Elements*. The first six books deal with triangles, rectangles, circles, polygons, proportion and similarity. The next four concern the theory of numbers: prime numbers, rational and irrational numbers. The last three are on solid geometry: pyramids, cones and cylinders. In *Elements* Euclid used certain primitive propositions, called *postulates or axioms*. His five postulates are (cf. [29]):

A-I A straight line may be drawn from any point to any other point.

A-II A finite straight line may be extended continuously in a straight line.

A-III A circle may be described with any center and any radius.

A-IV All right angles are equal to one another.

A-V If a straight line meets two other straight lines so as to make the two interior angles on one side of it together less than two right angles, the other straight lines, if extended indefinitely, will meet on that side on which the angles are less than two right angles.
Non-Euclidean geometry

Non-Euclidean geometry satisfies all of Euclid’s axioms except either the fifth or the second. In 19th century Johann Carl Friedrich Gauss, Nikolay Lobachevsky and János Bolyai showed the possibility of a consistent geometry without the fifth Euclid’s postulate: two lines in the plane, perpendicular to another line are parallel. In non-Euclidean geometry the lines may become farther apart or closer together. If the lines diverge the geometry is said to be hyperbolic (from the Greek hyperballein, “to throw beyond”). In the latter case, when the rays converge and ultimately intersect, the geometry is said to be elliptic (from elleipein, “to fall short”). In hyperbolic geometry, the fifth axiom is denied. In elliptic geometry, the second axiom is denied, because now the line is closed.

Riemannian geometry

Petersen in [81] gives an interesting insight into the genesis of Riemannian geometry. According to one story, Gauss was on Bernhardt Riemann’s defence committee for his higher doctorate. Riemann submitted three topics on: the Fourier series, integral and foundations of geometry. Riemann hoped that the committee would ask about the first topic that Riemann had already worked on. Gauss, however, decided he wanted to hear whether Riemann had anything to say on the subject of geometry, on which Gauss himself was the expert. Riemann had to ‘invent’ Riemannian geometry to satisfy Gauss’s curiosity. In 1854, Riemann gave a lecture titled “On the hypotheses which lie at the foundations of geometry” presenting a comprehensive view of geometry. With an understanding of the limitations of Euclidean geometry, he formulated a type of geometry without the fifth Euclid’s postulate. Riemann’s work constituted an alternative to Lobachevsky’s (1829) and Bolyai’s (1823) systems of non-Euclidean geometry, of which at that time he was apparently unaware.

Riemann’s geometry played a fundamental role in the mathematical formulation of Einstein’s relativity theory. But it wasn’t until the 1930s (Whitney) that mathematicians developed a clear understanding of abstract Riemannian objects like manifolds. Over the years Riemannian geometry was expanded to modern differential geometry.

2.1.2 Riemannian geometry and statistics

Statistical quantities are closely related to geometry. For example the mean value can be geometrically interpreted as the center of mass. Linear regression is a task of
fitting a straight line into a set of sample points. The exploratory technique known as *multidimensional scaling* represents objects as points in a geometric space (cf. [93]). For example, an investigation of factors in multivariate statistics may be interpreted in terms of finding the nearest neighbour in the $n$-dimensional space ($n$ is a number of factors). Here the distance $d(x, y)$ between two cases $x$ and $y$ is defined as $d(x, y) \overset{\text{def}}{=} \sum (x_i - y_i)^2 / V(i)$, where the sum spans all questions (items) and $V(i)$ denotes the variance for the item $i$. The number of factors corresponds to the minimal number of dimensions of Euclidean space allowing configuration of points with given distance relations.

The need for analyzing directional data (data measured in the form of angles) differently from linear (ordinary) data has been apparent for centuries. The two known early examples are presented below.

Daniel Bernoulli analyzed the uniformity of orbits of the five planets\(^1\) of the Solar system in the following way (cf. [95]). Using the *right hand rule* to each plane of the orbit Bernoulli assigned a point on the sphere. He argued that it would be most unlikely that the similarity of the orbits of the planets would occur by chance. In 1734 Bernoulli won a prize from the Paris Academy of Sciences for his work on orbits of the planets. Furthermore, a written note from 1802 by John Playfair recommends the use of the vector method of averaging directions instead of the arithmetic mean of all the observations [40].

Investigations of directional data in astronomy and navigation ignited interest in analysis of data on curved spaces in 20th century. A range of statistical methods dealing with data on circle and sphere are available, see [95]. These methods offer a wide range of statistical distributions including the most commonly used *Fisher’s distribution* for spherical data (see [41]) and *von Mises distribution* for circular data (see [40]).

Statistical methods concerning estimators of distributions led towards abstract geometry ([5, 4, 70, 9]). This methods investigate *log-likelihood* functions in terms of affine (projective) geometry. Densities are regarded as points of a higher dimensional space and functions are regarded as vectors of translations (cf. [70]). In such a space of log-likelihood functions one defines the *Fisher’s information matrix*. Because the Fisher’s information matrix is positive-definite it is used as a Riemannian metric [50, 66]. Further developments in this direction presented by Amari in [4] introduces an *α-connection* generalizing the Riemannian connection.

\(^1\)Until the beginning of the 17th century the known universe consisted of only 8 bodies: Earth, Sun, Moon, Mercury, Venus, Mars, Jupiter and Saturn. Uranus was discovered in 1781 by William Herschel.
In brief, since the 1970s a number of statistical methods have been developed for analyzing non-linear data. Riemannian geometry is particularly well suited to work with spheres and more complicated spaces. A geometrical approach offers more intuitive interpretations of statistical quantities characterizing data and populations in abstract spaces. However, many areas need to be further investigated.

2.1.3 Riemannian geometry and splines

Mathematical splines take their name from the flexible strip of wood used in drawing curves. By attaching weights to a strip of wood one can obtain different shapes of curves. In mathematics, splines are smooth curves connecting given points—nodes. For example, the simplest polynomial splines that are differentiable at the nodes are cubics, the curves of third order (see Figure 2.1 on page 18). Another approach of studying splines investigates their minimizing properties (cf. [1, 33]). In the variational approach splines are defined as optimal interpolating functions, [26]. Depending on the minimized quantity, conditions at the nodes and space there is whole range of splines with various properties and applications. Multivariate splines are the interpolating functions of several variables, [31]. Atteia studies abstract splines in Hilbert spaces, see [26]. Noakes et al. ([77]) and Camarinha et al. ([24]) investigate "geometrical splines" in non-Euclidean spaces. This thesis further investigates the above class of curves (named $\mathcal{D}^k$-curves in this thesis) in Chapter 5.

2.2 Riemannian Means

The next two sections present recent developments in statistics and geometry on mean value in non-Euclidean spaces. The intention is to present a variety of definitions of the central tendency in non-Euclidean (curved) spaces.

2.2.1 Mean value in non-Euclidean spaces—statistical approach

A number of distributions are available for statistical models on simple spaces like the circle and sphere. The most important ones are briefly described below. For circular data, samples of directions (angles), the most commonly used distributions are listed below.

Uniform distribution This is the simplest model with the probability density func-


12 literature review

\[ f(\theta) = \frac{1}{2\pi}. \]

**Cosine distribution** The example of symmetric unimodular distribution where the probability density function is defined as:

\[ f(\theta) = \frac{1}{2\pi} [1 + 2\rho \cos(\theta - \mu)], \quad \text{where} \quad 0 \leq \rho \leq \frac{1}{2}. \]

**Wrapped distribution** Let \( g \) be a probability density function defined on the real line. Then \( f \) can be defined as

\[ f(\theta) = \sum_{k=-\infty}^{\infty} g(\theta + 2k\pi). \]

**Von Mises distribution** The von Mises distribution (cf. [40]) has the probability density function:

\[ f(\theta) = \frac{1}{2\pi I_0(\kappa)} \exp[\kappa \cos(\theta - \mu)], \quad \text{for} \quad 0 < \kappa < \infty, \quad (2.1) \]

where

\[ I_0 = \frac{1}{2\pi} \int_{0}^{2\pi} \exp[\kappa \cos(\phi - \mu)]d\phi. \]

The von Mises distribution (2.1) for circular data is as important as the normal distribution for linear data (cf. [40]). The mean value \( \mu \) for the von Mises distribution is found by adding unit vectors. Let

\[ C = \sum_{i=1}^{n} \cos \theta_i, \quad S = \sum_{i=1}^{n} \sin \theta_i \quad \text{and} \quad R^2 = C^2 + S^2. \]

The direction \( \bar{\theta} \) of the resultant vector is given by

\[ \cos \bar{\theta} = \frac{C}{R} \quad \text{and} \quad \sin \bar{\theta} = \frac{S}{R} \quad (2.2) \]

is known as the mean direction (cf. [40]). The mean resultant length \( \bar{R} \) associated with with the mean direction \( \bar{\theta} \) is defined by

\[ \bar{R} = \frac{R}{n}. \quad (2.3) \]

The quantity \( V = 1 - \bar{R} \) is called sample circular variance. The sample circular standard deviation is defined as

\[ v = [\ln(1 - V)]^{1/2}. \]

The other two measures of reference direction for circular data are the sample median direction and sample modal direction. The sample median direction \( \bar{\theta} \) is found by
choosing an axis which divide the data into two equal groups and then selecting the 
end giving the smaller value of $\bar{\theta}$, where $\bar{\theta}$ can be found by minimizing the function
\[
d(\theta) = \pi - \sum_{i=1}^{n} |\pi - |\theta - \theta_i||.
\]
The sample modal direction $\hat{\theta}$ is the direction corresponding to the maximum con­
centration of the data. The sample modal direction is not well defined for a sample 
of discrete measurements. It can be defined as the value of $\theta$ maximizing a density 
estimate $\hat{f}(\theta)$.

The distributions for spherical data are:

**Uniform distribution** In the *uniform distribution* all directions are spread uni­
formly, cf. [41]. The probability density function is
\[
f(\theta, \phi) = \frac{\sin \theta}{4\pi},
\]
where $(\theta, \phi)$ are the spherical polar coordinates: $0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$.

**Fisher's distribution** This is the model for directions distributed unimodally with 
rotational symmetry (cf. [41]). It serves as an all-purpose probability model 
for directions in space and directional measurement errors (cf. [40]). This is 
much like the *von Mises* distribution for directions in the plane and the normal 
distribution for observations on the line.
The probability density function is
\[
f(\theta, \phi) = C_F \exp \left( \kappa (\sin \theta \sin \alpha \cos(\phi - \beta) + \cos \theta \cos \alpha) \right) \sin \theta,
\]
for $0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$, where $C_F = \kappa/(4\pi \sinh \kappa)$. There are three 
parameters: $\kappa$ is a shape parameter called the *concentration parameter*, $\alpha$ and $\beta$ 
are location parameters. Fisher’s distribution has rotational symmetry around 
the direction $(\alpha, \beta)$. For $\alpha = 0$, the probability density function for the Fisher’s 
distribution (2.4) becomes
\[
f(\theta, \phi) = C_F \exp (\kappa \cos \theta) \sin \theta,
\]
does not depend on $\phi$.

**The Watson distribution** This is the basic model for undirected lines distributed 
with rotational symmetry in either bipolar or equatorial form (cf. [41]). The 
probability density function is
\[
f(\theta, \phi) = C_W \exp \left( \kappa (\sin \theta \sin \alpha \cos(\phi - \beta) + \cos \theta \cos \alpha)^2 \right) \sin \theta,
\]
\[
(2.5)
\]
for $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, where

$$C_W = \left( 4\pi \int_0^1 \exp(\kappa u^2) \, du \right)^{-1}.$$

There are three parameters: $\kappa$ is a concentration parameter, $\alpha$ and $\beta$ are location parameters. Depending on the sign of the concentration parameter $\kappa$ the Watson distribution (2.5) is

- bipolar for $\kappa > 0$ and
- equatorial for $\kappa < 0$.

The Watson distribution has rotational symmetry around the direction $(\alpha, \beta)$. For $\alpha = 0$, the probability density function for the Watson distribution (2.5) becomes

$$f(\theta, \phi) = C_W \exp(\kappa \cos^2 \theta),$$

which does not depend on $\phi$.

In the case of symmetric unimodular distribution, the sample estimate of the mean direction $(\hat{\alpha}, \hat{\beta})$ is simply the direction of the resultant vector of the $n$ unit vectors in the similar way as (2.2). For anisotropic data the spherical median $P$ is more suited than the mean direction (cf. [41]). The spherical median is defined as the direction for which the sum of arc lengths from $P$ to each $P_i$ is minimized.

**Remark 2.2.1** The sample spherical median is an estimate of the population spherical median. Let $X$ be a random vector from the population under study, and $\alpha$ an arbitrary direction, and let $\theta_\alpha = \arccos(X \cdot \alpha)$, the angular deviation of $X$ from $\alpha$. The spherical median is the direction $\alpha$ for which the average value of $\theta_\alpha$ is minimum. The mean direction is the direction for which the average value of $\cos \theta_\alpha$ is minimum.

The most common statistical models are:

- normal distribution for linear data;
- von Mises distribution for circular data, and
- Fisher's distribution for spherical data.

Sometimes, the mean direction for circular and spherical data is taken as an average of $n$ unit vectors (2.2). Section 2.2.2 discusses disadvantages of this approach and presents alternative geometrical definitions of the mean value.
2.2 riemannian means

2.2.2 Mean value in non-Euclidean spaces—geometrical approach

The mean direction $\bar{\theta}$ defined by (2.2) and the resultant length $\bar{R}$ defined by (2.3) are not reliable indicators if the data points are scattered. For instance, a sample consisting of pairs of antipodal points produce the resultant length $\bar{R} = 0$. A number of authors define mean value or center of mass in terms of geometric (Riemannian) distance (e.g., Kobayashi [54], Karcher [49], Buss & Fillmore [22], Noakes [74] and Oller & Corcuera [78]). The distance is a measure of deviation from the central tendency. Depending on the space and applications different definitions can be found. Most common are presented below.

**Spherical mean and spherical averages** Noakes [74] investigates two measures of dispersion on spheres. For a finite set of points $S \subset S^n$ the two functions $\Phi_S, \Psi_S: S^n \to \mathbb{R}$ measure average spreads of $S$, where

$$
\Phi_S(p) = \frac{1}{\#S} \sum_{s \in S} d(p,s)^2 \quad \text{and} \quad \Psi_S(p) = \frac{1}{\#S} \sum_{s \in S} d(p,s),
$$

and where $d(p,q)$ is the spherical distance—the length of the shortest great-circle arc joining $p, q$, (see also Remark 2.2.1).

The spherical mean is a point or a set of points minimizing $\Phi_S$. The dispersion of $S$ is measured by the spherical variance—the minimum of $\Phi_S$.

**Example 2.2.2** Let $S$ contain two antipodal points $s$ and $-s$ on sphere $S^n$. By symmetry, the spherical mean is infinite. It is a great circle equidistant from the two points. The spherical variance is $\pi^2 / 4$.

The weighted spherical averages are defined by Buss & Fillmore [22] as points in the unit sphere minimising

$$
\Phi(q) \overset{\text{def}}{=} \frac{1}{2} \sum_i w_i \cdot \|q - p_i\|^2,
$$

where $\{w_i\}$ are the weight values such that $w_i \geq 0$ and $\sum w_i = 1$.

**Center of mass** Let $(M, g)$ be a complete simply connected Riemannian manifold whose sectional curvature is non-positive everywhere. Consider a modified distance function $f_{0,s}: M \to \mathbb{R}$ defined as

$$
f_{0,s}(x) \overset{\text{def}}{=} \frac{1}{2} d(x,s)^2.
$$

The modified distance function is strictly convex when restricted to a geodesic. For a finite set of points $S = \{s_1, s_2, \ldots, s_m\} \subset M$ define a function $\Phi: M \to \mathbb{R}$ as follows (cf. [81])

$$
\Phi(x) = \max_{1 \leq i \leq m} f_{0,s_i}(x). \quad (2.6)
$$
The unique minimum of $\Phi$ is called the center of mass of $S$ and is denoted by $cm\{s_1, s_2, \ldots, s_m\}$. In the Euclidean space we have the well known formula
\[
cm\{s_1, s_2, \ldots, s_m\} = \frac{1}{m} \sum_{i=1}^{m} s_i. \tag{2.7}
\]

**Center of gravity** Let $\mu$ be a positive measure on a compact topological space $A$, $f: A \to M$ be a continuous mapping from $A$ into a complete, simply connected Riemannian manifold $M$ with non-positive sectional curvature, and $J: M \to \mathbb{R}$
\[
J(x) \overset{\text{def}}{=} \int_A d(x, f(a))^2 \, d\mu(a), \tag{2.8}
\]
where $d(x, f(a))$ is the distance between $x$ and $f(a)$. $J$ attains its minimum at precisely one point called center of gravity of $f(A)$ with respect to $\mu$ (see [54]). In particular it is shown that if $\exp_p X = x \neq p$ then
\[
J(p) = J'(0) < J'(X) \leq J(x).
\]

A special case of the center of gravity, but not restricted to non-positive sectional curvature manifolds, is considered by Karcher [49]. Let $A$ be a measure space of volume 1 (compact Riemannian manifold or a finite set of points), $M$ be a complete Riemannian manifold and $B_\rho(m)$ a convex open ball of radius $\rho$ centered at $m \in M$, $f: A \to B_\rho$ a measurable map called a mass distribution on $B_\rho$, define $P_f: \overline{B}_\rho \to \mathbb{R}$ as
\[
P_f(m) \overset{\text{def}}{=} \frac{1}{2} \int_A d(m, f(a))^2 \, da. \tag{2.9}
\]
It is shown in [49] that there is such a $\rho > 0$ that $P_f$ is convex and has a unique point of minimum in $B_\rho(m)$ called a center of mass. The convexity of $P_f$ and uniqueness of the center of mass is further studied in Kendall [51] who improves Karcher’s result, see Example 4.3.12.

**Riemannian barycenters** Corcuera & Kendall in [28] define barycenters as follows. Let $M$ be a smooth complete Riemannian manifold, $\mu$ a probability measure defined on $M$. A point $w \in M$ is a (Riemannian) barycenter of $\mu$ if there is a probability measure $\tilde{\mu}$ on $\mathcal{T}_w M$ which is mapped onto $\mu$ by the exponential map (see Section 3.6 for its definition) $\tilde{\mu}(\exp^{-1}_w A) = \mu(A)$, for all Borel $A \subseteq M$, and satisfies the critical condition
\[
\int_{\mathcal{T}_w M} \nu \tilde{\mu}(dv) = 0.
\]
The above definition of Riemannian barycenters allows for non-uniqueness and non-minimality of geodesics, cf. Remark 4.1.3. Corcuera & Kendall prove the
following result: the Riemannian barycenter of a probability measure supported by a subset $\mathcal{U} \subset M$, where $M$ is a complete 2-manifold, must also lie in $\mathcal{U}$, see also Remark 4.5.2.

Schattner [86] provides an example of a special application of the concept of center of mass in relativity theory. Here the central role is played by the stress-energy tensor $T^{ab}$ describing the density, momentum and stress of the matter and energy present at each point in space-time. Let $T$ denote a geodesic convex hull of the support of the stress-energy tensor (supp $(T^{ab})$). A center of mass line is smooth time-like curve contained in a convex hull of the world tube of the body formally defined as follows. Let $m^i$ be a mass dipole moment $m^i \overset{\text{def}}{=} n_k S^{ik}$, where $S^{ik}$ is the angular momentum. A $C^1$ curve $z(s)$ is said to be a center of mass line if $z(s) \in T$ and $m^k(z(s)) = 0$, for all $s$. Schattner proves the existence of a center of mass line.

Oiler & Corcuera [78] have developed a statistical inference procedure invariant under transformation of the parameters in a parametric model for the probability distribution. The parametric statistical models have a Riemannian manifold structure given by the information metric. Now the estimators of the parameters are random points in the manifold of the statistical model. The new approach named intrinsic analysis is based on the geometrical structure of the statistical model. Oiler & Corcuera extend notions of mean value and moments of random objects in connected Hausdorff manifolds. The mean value defined in this framework is closely related to the Riemannian barycenter and called an exponential barycenter. An intrinsic estimator (function on an complete manifold) is an uniformly minimum Rao distance (cf. [70, 66]) estimator for a fixed bias vector. Oiler & Corcuera also prove that the maximum likelihood estimators are asymptotically intrinsically unbiased.

2.3 Riemannian Variational Curves

Splines in Euclidean spaces are smooth curves connecting given points (nodes), see Figure 2.1 on the next page. Splines play a major part in today's applied (cf. [47, 44]) and theoretical (cf. [90]) mathematics.

2.3.1 Polynomial splines

Let $\Pi_m$ be the set of real polynomials of degree $m$ or less. Given an interval $I = [a, b] \subset \mathbb{R}$ partitioned by the interior points (knots) $\{t_1, t_2, \ldots , t_n\}$, such as $a = t_0 < t_1 < \cdots < t_n < t_{n+1} = b$ curve $\sigma : I \to \mathbb{R}$ is a polynomial spline of degree $m$ (or less) if $\sigma$
Figure 2.1: Polynomial spline of third degree—cubic—is an example of continuous differentiable spline

is $C^{m-1}$-smooth ($\sigma \in C^{m-1}(I)$), and for each $i \in \{0, 1, \ldots, n\}$, $\sigma$ is piecewise polynomial of degree $m$ or less, to be precise $\sigma|_{[t_i,t_{i+1}]} \in \Pi_m$ (cf. [1, 34, 33]).

2.3.2 Variational splines

The variational approach to splines originated with Holladay's theorem (1957) and gave rise to large variety of splines obtained as families of curves minimizing certain functionals. Holladay's theorem (cf. [1]) states that the cubic spline (piecewise polynomial curve or third order) is optimal in a sense that it minimizes $\int_I \gamma(t)^2 \, dt$ over all functions of class $C^2$ (twice differentiable). In other words it is the smoothest curve, among all curves in $C^2$, that coincides with a given function at the given nodes (the optimal interpolating function). De Boor [33] introduced $D^m$-splines satisfying the interpolation conditions and minimizing $\int_I (D^m \gamma(t))^2 \, dt$ among all functions of class $C^{2m-1}$, for $m \geq 2$. De Boor proves the existence of $D^m$-splines and shows that if $\sigma: I \to \mathbb{R}$ is an interpolating function satisfying the minimizing condition, then

- $\sigma \in C^{2m-2}(I)$;
- $\sigma|_{[t_i,t_{i+1}]} \in \Pi_{2m-1}$, for $i = 1, 2, \ldots, n - 1$ and
- $\sigma|_{[a,t_1]} \in \Pi_{m-1}$ and $\sigma|_{[t_n,b]} \in \Pi_{m-1}$.

This generalizes Holladay's result, which is the case of $m = 2$.

On the other side of the spectrum of variational splines we have the following result of Atteia (cf. [26]). Let us introduce the following notation. For any natural number $m$, $H^m(I)$ will denote the set of functions $x: I \to \mathbb{R}$ such that $x^{(m-1)}$ is absolutely
continuous on $I$ and $x^{(m)} \in \mathcal{L}^2(I)$. If we define an inner product on $H^m(I)$ by

$$\langle x_1, x_2 \rangle = \int_I \sum_{j=0}^{m} x^{(j)}_1(t) x^{(j)}_2(t) \, dt$$

then $H^m(I)$ becomes a Hilbert space.

Consider the following diagram

$$\begin{array}{ccc}
    X & \xrightarrow{T} & Y \\
    \downarrow A & & \downarrow & \\
    Z & &
\end{array}$$

where $X, Y, Z$ are Hilbert spaces, $T, A$ are continuous, linear surjections, $\ker T + \ker A$ is closed in $X$, $\ker T \cap \ker A = \{0_X\}$ (zero element of the vector space $X$) and for any $z \in Z$ let $I(z) = \{x \in X \mid Ax = z\}$, then there exists one and only one $\sigma \in I(z)$ such as

$$\|T\sigma\|_Y = \min \{\|Tx\|_Y \mid x \in I(z)\}.$$ 

The optimal $\sigma$ is known as a variational interpolating spline. The role of the condition $\ker T + \ker A$ is to ensure the existence of $\sigma$ whereas the role of the condition $\ker T \cap \ker A$ is to ensure the uniqueness of $\sigma$. Putting $X = H^2(I)$, $Y = \mathcal{L}^2(I)$, $Z = \mathbb{R}^n$, $T(x) \overset{\text{def}}{=} x(2)$ and $A(x) \overset{\text{def}}{=} (x(t_1), x(t_2), \ldots, x(t_n))$ we obtain the Holladay's theorem.

Schoenberg in [88] introduced $g$-splines. This is an extension of Hermite's idea of constructing a polynomial of minimal degree such that

- the polynomial assumes prescribed values at given nodes and
- the derivatives of certain orders of the polynomial also assumed prescribed values at the nodes.

Another example of variational splines are $L$-splines studied by Schultz & Varga [89]. The $L$-splines extend the notion of $D^m$-splines, where the differential operator $D^m$ is replaced by a more general operator $L$ defined as follows. Let $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{R}$ and $X = H^2(I)$. Denote $I_n = \{x \in X \mid x(t_i) = z_i, 1 \leq i \leq n\}$. Then

$$L: X \rightarrow \mathcal{L}^2(I) \text{ so that } L[x](t) = \sum_{j=0}^{m} a_j(t) D^j x(t),$$
where $X = H^m(I)$, $a_j \in C^j(I)$ and $a_m$ is positive bounded $a_m(t) \geq w > 0$ on $I$, and $L$ has Pólya’s property\footnote{L has Pólya’s property $W$ on $I$ if $L[x] = 0$ has $m$ solutions $x_1, x_2, \ldots, x_m$ such that, for all $t \in I$ and for all $k \in \{1, 2, \ldots, m\}$}

$W$ on $I$ then there exists such a $\sigma \in I_n$ $(n \geq m)$ that

$$\int_I L[\sigma](t)^2 \, dt = \min \left\{ \int_I L[x](t)^2 \, dt \mid x \in I_n \right\}.$$ 

The consequence of the Pólya’s property $W$ is that if

- $L[x] = 0$, and
- $x$ has $m$ or more zeros on $I$,

then $x \equiv 0$.

Duchon [38] proved a result regarded as a significant step towards developing a variational approach to interpolating functions of several variables. For an arbitrary element of $\mathbb{R}^2$ denoted by $t = (\xi_1, \xi_2)$ let us define a norm by $\|t\| \overset{\text{def}}{=} \xi_1^2 + \xi_2^2$, and linear polynomials by

$$\Pi_1 \overset{\text{def}}{=} \{p_1(t) = a_0 + a_1 \xi_1 + a_2 \xi_2 \mid a_1, a_2, a_3 \in \mathbb{R} \}.$$ 

Let $X = H^2(\mathbb{R}^2)$ and the knots $\{t_1, t_2, \ldots, t_n\} \subset \mathbb{R}^2$. If $p \in \Pi_1$ and $p_1(t_1) = \cdots = p_1(t_n) = 0$ then $p_1 \equiv 0$. Finally define the functional $J: X \to \mathbb{R}$ such as

$$J(x) = \int \int_{\mathbb{R}^2} \left( \left( \frac{\partial^2 x}{\partial \xi_1^2} \right)^2 + \left( \frac{\partial^2 x}{\partial \xi_1 \partial \xi_2} \right)^2 + \left( \frac{\partial^2 x}{\partial \xi_2^2} \right)^2 \right) d\xi_1 d\xi_2,$$

then there is $\sigma \in I_n$ such that

$$J(\sigma) = \min \{ J(x) \mid x \in I_n \}.$$ 

The optimal function $\sigma$, named a \textit{thin plate spline}, has the following remarkable, unlike piecewise polynomial function, form

$$\sigma(t) = \sum_{i=1}^{n} \lambda_i \| t - t_i \|^2 \ln \| t - t_i \| + p_1(t),$$

where $p_1 \in \Pi_1$ and for all $q \in \Pi_1$ there is $\sum_{i=1}^{n} \lambda_i q(t_i) = 0$. 

$$\begin{vmatrix}
    x_1(t) & x_2(t) & \cdots & x_k(t) \\
    Dx_1(t) & Dx_2(t) & \cdots & Dx_k(t) \\
    \vdots & \vdots & \ddots & \vdots \\
    D^{k-1}x_1(t) & D^{k-1}x_2(t) & \cdots & D^{k-1}x_k(t)
\end{vmatrix} \neq 0.$$ 


2.3 Riemannian variational curves

We have seen how the variational approach to splines naturally generalizes to other spaces. As is the case with interpolating functions in several variables, such generalizations produce functions far more complex than polynomial. In the next section we will introduce another generalization of variational curves applicable to Riemannian spaces.

2.3.3 Splines in Riemannian spaces

This section provides a short survey on recent developments in Riemannian geometry concerning variational curves.

Noakes et al. [77] consider variational curves in a Riemannian manifold M. The Riemannian cubic is a smooth curve minimizing

$$\Phi(x) \overset{\text{def}}{=} \int_0^1 \langle D_t \dot{x}(t), D_t \dot{x}(t) \rangle dt,$$

where $D_t$ is a covariant derivative along $x$. Putting $M = \mathbb{R}^n$ results in the variational property of the cubic spline. Noakes et al. present necessary conditions (Euler-Lagrange equations) for a curve $x: [0, 1] \rightarrow M$ to be an extreme of the functional $\Phi$. For the group of rotations $M = \text{SO}(3)$ the Euler-Lagrange equations have particularly simple form

$$\dot{V}(t) = V(t) \times V(t) + \text{constant},$$

where $V: I \rightarrow \mathbb{R}^3$ is a curve in $\mathbb{R}^3$, by identification $\mathfrak{so}(3) \cong \mathbb{R}^3$, $V(t) = \dot{x}(t)$, and ‘×’ denotes vector product in $\mathbb{R}^3$.

There is more research done on interpolating curves on non-Euclidean spaces. Brunnett & Crouch [19] investigate elastic curves on the sphere, together with a differential equation for the geodesic curvature of spherical elastica and a formula which expresses the squared torsion of a spherical elastica as a rational function of its curvature. Brunnett & Crouch present a classification of the fundamental forms of these curves. Crouch & Silva Leite [30] and Camarinha et al. [24] investigate higher order splines on complete Riemannian manifolds. Camarinha et al. in [24] consider curves $x: [0, T] \rightarrow M$ of class $C^{2m-3}$ satisfying the interpolating conditions and being an extreme of the functional

$$J(x) = \frac{1}{2} \int_0^T \langle D_t^{m-1}x(t), D_t^{m-1}x(t) \rangle dt.$$  \hspace{1cm} (2.10)

Their investigations lead to the Euler-Lagrange equations of a higher-order variational problem (2.10). Camarinha et al. obtain a necessary conditions for $x$ to be an
Figure 2.2: One step of the “corner-cutting” algorithm on Riemannian manifold

The extreme of the functional $J: x$ is of class $C^{2m-2}$ and, for $t \in [t_{i-1}, t_i]$, and $i = 1, 2, \ldots, N$

$$D_t^{2m-1}V(t) + \sum_{j=2}^{m} (-1)^j R(D_t^{2m-j-1}V(t), D_t^{j-2}V(t))V(t) \equiv 0,$$  \hspace{1cm} (2.11)

where $V(t) = \dot{x}(t)$ and $R$ is the Riemannian curvature tensor of the Levi-Civita connection on $M$. Camarinha et al. define a $C^{2m-2}$ geometric spline on $M$ as a curve $x$ whose velocity vector field $V(t)$ along $x$ satisfies condition (2.11). It is proved in [24] that if $\{X_1, X_2, \ldots, X_n\}$ is a frame field of left (or right) invariant vector fields on a connected compact abelian Lie group $G$, then the geometric spline of class $C^{2m-2}$ in $G$ is given by

$$x(t) = \exp (f_1(t)X_1) \cdot \exp (f_2(t)X_2) \cdot \ldots \cdot \exp (f_n(t)X_n),$$

where $f_i(t)$ is a polynomial spline of degree $2m - 1$, for $i = 1, 2 \ldots, n$.

2.3.4 Other curves in Riemannian spaces

Buss & Fillmore [22] study a method for calculating weighted averages on spheres based on the spherical distance. Based on the weighted spherical averages Buss & Fillmore define Bézier and spline curves on spheres with applications to smooth averaging of quaternions. Given a finite number of points $\{p_i\}$ in the unit sphere $S^n \subset \mathbb{R}^{n+1}$ and weight values $\{w_i\}$, where $w_i \geq 0$ and $\sum w_i = 1$, let

$$\Phi(q) \overset{\text{def}}{=} \frac{1}{2} \sum_i w_i \cdot \|q - p_i\|^2.$$
Then $\text{Avg}(w_1, p_1; \ldots; w_n, p_n)$ is the point of a sphere that minimizes $\Phi$. There are a number of situations when the point of minimum of $\Phi$ is unique, for example if the points $\{p_i\}$ all lie in a hemisphere and at least one point lie in the interior of the hemisphere. Buss & Fillmore define spline curves on spheres with weighted spherical averages, where the role of weights $\{w_i\}$ is taken by blending functions: $w_i: [a, b] \rightarrow \mathbb{R}$. Since (under certain conditions) the minimum of $\Phi$ smoothly depends on the weights $\{w_i\}$, smooth blending functions generate smooth spline curves on spheres.

Silva Leite et al. [90] investigate elastic curves, a generalization of Riemannian cubics. Silva Leite et al. consider a non-linear interpolation problem on Riemannian manifolds. The solution curve $x: [0, T] \rightarrow M$ passes through specific points in the configuration space and minimizes the following functional

$$J(x) \overset{\text{def}}{=} \frac{1}{2} \int_0^T \left( \langle D_t \dot{x}(t), D_t \dot{x}(t) \rangle + \tau^2 \langle \dot{x}(t), \dot{x}(t) \rangle \right) dt.$$ 

Silva Leite et al. further investigate more complex control problems, where the velocity vector $\dot{x}(t)$ is additionally constrained to lie in some subspace of the tangent bundle $TM$. The necessary conditions for elastic curves to satisfy additional constraints are presented for the case of compact and connected Lie groups.

Noakes in [73, 76] investigates curves derived through the corner-cutting algorithm in Riemannian spaces. It is known that a corner-cutting algorithm in $\mathbb{R}^n$ produces splines [36, 35, 67]. The algorithm in a Riemannian manifold $M$ generalizing construction for quadratics can be described as follows. Define a midpoint map $M: U \times U \rightarrow U$ as $M(x_a, x_b) \overset{\text{def}}{=} \gamma((a + b)/2)$, where $U \subset M$ is convex and compact, and $\gamma: [a, b] \rightarrow U$, is a unique geodesic from $x_a$ to $x_b$. A Riemannian scaled triple is $Y = (y_0, y_1, y_2, h) \in U^3 \times \mathbb{R}^+$, where $y_0, y_1, y_2$ are vertices and $h$ is the scale. The fundamental polygon $p: [0, 2h] \rightarrow U$ defined as a piecewise geodesic consisting of two segments: from $y_0$ to $y_1$ and from $y_1$ to $y_2$. The midpoint map $M$ applied to the vertices of a fundamental polygon $Y$ splits $Y$ into left triple $Y^L = (y_0, y_3, y_5, h/2)$ and right triple $Y^R = (y_5, y_4, y_2, h/2)$, where $y_3 = M(y_0, y_1)$, $y_4 = M(y_1, y_2)$, and $y_5 = M(y_3, y_4)$ (see Figure 2.2 on the facing page). The same procedure is applied to the descendant triples $Y^L$ and $Y^R$ producing polygon $p_2: [0, 2h] \rightarrow U$ consisting of the track sum of fundamental polygons of triples $Y^{LL}$, $Y^{LR}$, $Y^{RL}$, and $Y^{RR}$. The sequence $\{p_m | m \geq 1\}$ converges uniformly in $\mathcal{C}([0, 2h])$ to a curve $p_\infty \in \mathcal{C}([0, 2h])$, and $p_\infty$ is differentiable on $(0, 2h)$. Noakes proves that:

- the velocity $\dot{p}_\infty$ is left-differentiable on $(0, 2h]$ and right-differentiable on $[0, 2h)$;
- the left-acceleration $\ddot{p}_\infty$ is left-continuous and the right-acceleration $\ddot{p}_\infty$ is right-continuous, and
\( \phi_\infty \) is differentiable on the complement of multiples of \( h \) by dyadic rationales in 
\([0, 2h]\) (countable dense subset of the domain).

The two above examples demonstrate that well known properties of curves in Euclidean properties are becoming quite complex to investigate, when considered in a more general setup in Riemannian geometry.

### 2.4 Recapitulation

Developments in Riemannian geometry have made it a valuable tool in research of inferential methods in non-Euclidean spaces. The thesis presents new results of investigations in two areas: a measure of central tendency and Riemannian variational curves. In these areas a geometrical approach help us to understand relations between information inferred from data and geometrical properties of a space, e.g., curvature.

#### Developments in Riemannian means

The most commonly used method of deriving the mean value in non-Euclidean spaces is embedded (or projected) mean. This approach is taken for circular and spherical data, where the mean value is evaluated as a normalized sum of unit vectors. It is shown that if data is concentrated, the derived mean value is a good estimate of the central tendency (cf. [41]). For example, for circular data Fisher's distribution (2.4) is commonly used. While the Normal distribution is a satisfactory model irrespective of the dispersion parameter \( \sigma^2 \), Fisher's distribution may not be acceptable for large values of the concentration parameter \( \kappa > 2 \). This highlights a deficiency of statistical models on spheres in the case of significant densities at the antipodes. A geometrical approach seems to be more appropriate. Here the geometrical distance, an intrinsic quantity of space, is a measure of deviation from the central tendency. On the sphere, for example, the measure of deviation is the spherical distance, the length of the shortest great-arc joining two points (cf. [74]). Thus, geometrically the mean value becomes the center of mass. In different contexts, several authors define the center of mass in various ways (cf. Section 2.2.2). Karcher [49], for example provides a definition of the center of mass for a mass distribution on a measurable space. If the space is taken to be a finite set of points in a sphere, the center of mass coincides with the spherical mean of [74], if the spherical mean is a single point. The thesis investigates the (Riemannian) mean for a finite set of points on compact Rie-
mannian manifolds. Methods of deriving the Riemannian mean and its relationship with Riemannian curvature and covariance of a sample are presented. In particular, Section 4.4 demonstrates an iterative method of finding the Riemannian mean for classical Lie groups. The approximation formula in Section 4.3.1 shows the dependence of the mean on the curvature of the space. The difference between the Riemannian mean and embedded mean for spaces of constant curvature are investigated.

**Developments in Riemannian variational curves**

Splines are important tools in the theory of interpolation and approximation. Recent developments concern methods of multivariate splines (cf. [31]), interpolation by surface splines (cf. [47]) and variational curves in non-Euclidean spaces (cf. [24]). There is also growing interest in interpolating curves in Riemannian manifolds, e.g., Riemannian quadratics (cf. [76]) and elastic curves in the sphere and Lie groups (cf. [90]). The thesis investigates (Riemannian) variational curves in Riemannian complete manifolds. In Section 5.3.2 the differential equations in local coordinates of a cubic are derived and discussed. The minimal interval, where Riemannian cubics in SO(3) exists, is investigated in Section 5.3.4. Section 5.3.4 derives a system of differential equations satisfied by cubics in SO(3) endowed with left-invariant metric. Section 6.5 investigates properties of cubics in projective spaces with applications to interpolation of tangent directions in the plane. It derives necessary and sufficient conditions for a plane curve to be the envelope of the Riemannian cubic in the projective plane. Example 6.5.1 and Example 6.5.6 demonstrate applications of variational curves to approximation of a shape of convex body in the plane.
Chapter 3

Overview of Riemannian Geometry

In this chapter some standard results of Riemannian geometry used throughout the thesis are summarized. The purpose is to collect known results, that are useful in the subsequent chapters of the thesis, in this short overview. It was also an intention of this chapter to introduce the notation, adopted by the author, that is used throughout the thesis.

3.1 Riemannian Manifold

One may regard a manifold as a topological space which is locally Euclidean. Riemannian geometry concerns manifolds that are smooth, i.e., there is an open cover \{U_\alpha\} of the manifold M and homeomorphisms \(x_\alpha: U_\alpha \to x_\alpha(U_\alpha)\) onto open subsets of \(\mathbb{R}^n\), such that \(x_\alpha \circ x_\beta^{-1}\) is smooth. We write local coordinates on any open subset \(U_\alpha \subset M\) as \(x_\alpha = (x^1, x^2, \ldots, x^n)\). Each pair \((U, (x^i))\) is called a chart.

For any \(p \in M\), the tangent space \(T_pM\) is the set of equivalence classes of curves through \(p\) under a suitable equivalence relation. Local coordinates \((x^i)\) give a basis for \(T_pM\) consisting of the partial derivative operators \(\partial/\partial x^i\), denoted in this thesis as \(\partial_i\).

On a finite-dimensional vector space \(E\) with its standard smooth manifold structure a vector \(X \in E\) with its directional derivative is identified as follows

\[ Xf \overset{\text{def}}{=} \left. \frac{d}{dt} \right|_{t=0} f(p + tX). \]
This corresponds to the usual identification \((x^i) \leftrightarrow x^i \partial_i\). Here and throughout the thesis we use the *Einstein summation convention*, i.e., if in any term the same index name appears twice, as both an upper and a lower index, that term is assumed to be summed over all possible values of that index. The disjoint union of the tangent spaces \(T_pM\), for all \(p \in M\), is called the *tangent bundle*.

Let \(\mathcal{T}(M)\) be the space of smooth vector fields on \(M\). A *Riemannian metric* on a smooth manifold \(M\) is a 2-tensor field \(g: \mathcal{T}(M) \times \mathcal{T}(M) \to \mathbb{R}\) that is *symmetric* and positive definite. A Riemannian metric determines an inner product on each tangent space \(T_pM\). We denote \((X, Y) \overset{\text{def}}{=} g_p(X, Y)\), for \(X, Y \in T_pM\). A manifold together with a given Riemannian metric is called a *Riemannian manifold* \((M, g)\).

### 3.2 Connections

Let \(\mathcal{T}(M)\) be the space of smooth vector fields on \(M\). A *linear connection* on a smooth manifold \(M\) is a map \(\nabla: \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{V}(M)\), which assigns to each two vector fields \(X, Y\) in \(M\), a new vector field \(\nabla_X Y\) satisfying the following properties:

1. \(\nabla_X Y\) is linear over smooth functions in \(X\):
   \[
   \nabla_{fX_1 + gX_2} Y = f \nabla_{X_1} Y + g \nabla_{X_2} Y, \quad \text{for } f, g \in \mathcal{C}^\infty(M); 
   \]
2. \(\nabla_X Y\) is linear over \(\mathbb{R}\) in \(Y\):
   \[
   \nabla_{X(aY_1 + bY_2)} = a \nabla_{X} Y_1 + b \nabla_{X} Y_2, \quad \text{for } a, b \in \mathbb{R}; 
   \]
3. \(\nabla_X Y\) satisfies the following product rule:
   \[
   \nabla_{X} fY = f \nabla_{X} Y + (Xf) Y, \quad \text{for } f \in \mathcal{C}^\infty(M). 
   \]

In any local frame \(\{E_i\}\) for \(\mathcal{T}M\), the connection \(\nabla\) can be written in terms of the same frame as
\[
\nabla_{E_i} E_j = \Gamma_{ij}^k E_k,
\]
where functions \(\Gamma_{ij}^k\) are called the *Christoffel symbols* of \(\nabla\) with respect to this frame. By the above formula Christoffel symbols determine the connection completely and conversely, given any smooth real functions \(\Gamma_{ij}^k\) on \(U \subset M\), one can define \(\nabla\) on \(U\) by the formula (3.1).

A connection \(\nabla\) satisfying
\[
\nabla_X (Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z), \quad \text{for all vector fields } X, Y, Z \text{ in } M
\]
is said to be *compatible* with metric $g$. A connection $\nabla$ satisfying

$$\nabla_X Y - \nabla_Y X = [ X , Y ],$$

for all vector fields $X, Y$ in $M$, is said to be *symmetric*, where $[ \cdot , \cdot ]$ is the *Lie bracket*, $[ X , Y ] f = (XY - YX) f$. A connection compatible with metric $g$ and symmetric is called the *Riemannian connection* (or the Levi-Civita connection). In this thesis the Riemannian connection is always assumed. We have the following important lemma:

**Lemma 3.2.1 (Fundamental Lemma of Riemannian Geometry, cf. [58, 69, 81])**

A Riemannian manifold possesses one and only one symmetric connection which is compatible with its metric.

The Riemannian connection can be determined from the metric by the following *second Christoffel identity* (cf. [69])

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} ( \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} ), \quad (3.2)$$

where the metric coefficients are defined by $g_{ij} \overset{\text{def}}{=} \langle \partial_i , \partial_j \rangle$ and $g^{kl}$ are the coefficients of the inverse matrix of $(g_{ij})$.

The importance of the Riemannian connection comes from the fact that symmetry and compatibility are invariantly-defined properties that force the connection to coincide with the tangential connection whenever $M$ is realized as a submanifold of $\mathbb{R}^n$ with the induced metric. The symmetry and compatibility conditions of the Riemannian connections implies the following important fact:

**Lemma 3.2.2 (cf. [58, 69])** Let $(M, g)$ be a Riemannian manifold and $\nabla$ be a Riemannian connection (symmetric and compatible with $g$). Then for any vector fields $X, Y, Z$ the following holds

$$2(\nabla_X Y , Z) = X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle .$$

Let $\mathfrak{G}$ be a Lie group with Lie algebra $\mathfrak{g}$. We have for each $p \in \mathfrak{G}$ the diffeomorphisms: *left translation* $L_p : g \mapsto pg$ and *right translation* $R_p : g \mapsto gp$. A Riemannian metric $g$ on $\mathfrak{G}$ is said to be *left invariant* if it is invariant under all left translations $L_p^* g = g$, for all $p \in \mathfrak{G}$. Similarly, $g$ is *right invariant* if it is invariant under all right translations, and is *bi-invariant* if it is both left and right invariant.
Recall that the adjoint $L^*$ of a linear transformation $L$ between metric vector spaces is defined by the formula $(Lx, y) = (x, L^*y)$. If $L^* = -L$ then the linear transformation $L$ is called skew adjoint. If $G$ is a connected group, then a left invariant metric is bi-invariant if and only if the linear transformation $y \mapsto [x, y]$, from a Lie algebra of $G$ to itself, is skew adjoint for every $x$ in the Lie algebra of $G$, cf. [68]. If a Lie group is compact and simple then the bi-invariant metric is unique (up to multiplication by a positive constant). For example, both $SU(2)$ — group of $2 \times 2$ unitary matrices of determinant 1, and $SO(3)$ — rotation group of 3-space are compact and simple. A specific Riemannian bi-invariant metric $(x, y)$ is given by the real part of trace $(x \cdot y^i)$, for any $x, y$ in the Lie algebra, where $y^i$ is the conjugate transpose of $y$.

One has the following property of the Riemannian connection of a bi-invariant metric.

**Theorem 3.2.3 (cf. [58, 69])** Let $G$ be a Lie group and $g$ its Lie algebra. Let $g$ be a bi-invariant metric on $G$. For any two left invariant vector fields $X, Y$ on $G$

$$\nabla_X Y = \frac{1}{2} [X, Y].$$

Let $\gamma: I \to M$ be a smooth curve, where $I \subset \mathbb{R}$ is some interval. A **vector field along a curve** $\gamma: I \to M$ is a smooth map $V: I \to TM$ such that $V(t) \in T_{\gamma(t)}M$, for all $t \in I$. A vector field along $\gamma$ is called extendible if there exists a vector field $\tilde{V}$ on a neighbourhood of the image of $\gamma$, such that $V(t) = \tilde{V}_{\gamma(t)}$. Not every vector field along a curve is extendible. For example, if $\gamma(t_1) = \gamma(t_2)$ and $\dot{\gamma}(t_1) \neq \dot{\gamma}(t_2)$ then the vector field $\dot{\gamma}(t)$ is not extendible. Let $\mathcal{F}(\gamma)$ be the space of smooth vector fields along $\gamma$. We define a **covariant derivative** $D_t: \mathcal{F}(\gamma) \to \mathcal{F}(\gamma)$ assigning to a vector field $V$ along $\gamma$ a new vector field $D_tV$ along $\gamma$ satisfying the following properties:

i) $D_t$ is linear over $\mathbb{R}$:

$$D_t (aV + bW) = aD_tV + bD_tW, \quad \text{for } a, b \in \mathbb{R};$$

ii) $D_t$ satisfies the following product rule:

$$D_t (fY) = fV + fD_tV, \quad \text{for } f \in C^\infty(M);$$

iii) if $V$ is extendible then

$$D_tV(t) = \nabla_{\dot{\gamma}(t)}\tilde{V}, \quad \text{for any extension } \tilde{V} \text{ of } V.$$

To conclude this section we note on connections on Riemannian submanifolds. The **Gauss Formula Along a Curve** (3.3), is later used in Section 5.3.3 of the thesis.
Let \((\mathcal{M}, \tilde{g})\) be a Riemannian manifold, \(M\) a manifold and \(\iota: M \to \mathcal{M}\) be an immersion. If \(\iota\) is injective and \(M\) is given the induced Riemannian metric \(g \overset{\text{def}}{=} \iota^* \tilde{g}\), then \(M\) is said to be a Riemannian submanifold of \(\mathcal{M}\) and \(\mathcal{M}\) the ambient manifold. At each point \(p \in M\), the ambient tangent space \(\mathcal{T}_p \mathcal{M}\) splits into an orthogonal direct sum \(\mathcal{T}_p \mathcal{M} = \mathcal{T}_p M \oplus N_p M\), where \(N_p M \overset{\text{def}}{=} (\mathcal{T}_p M)^\perp\) is the normal space at \(p\) with respect to \(\tilde{g}\). The disjoint union

\[
NM \overset{\text{def}}{=} \bigsqcup_{p \in M} N_p M
\]

is a smooth vector bundle over \(M\) called the normal bundle of \(M\). One defines the tangential \(\pi^T: \mathcal{T}\mathcal{M}|_M \to \mathcal{T}M\) and normal \(\pi^\perp: \mathcal{T}\mathcal{M}|_M \to NM\) projections as orthogonal projections, at each point \(p \in M\), into the subspaces \(\mathcal{T}_p M\) and \(N_p M\), respectively.

Any two vector fields \(X, Y \in \mathcal{X}(M)\) can be extended to vector fields on \(\mathcal{M}\). Given \(X\) and \(Y\) vector fields on \(M\) there is a natural decomposition of the ambient connection \(\tilde{\nabla}_X Y = (\tilde{\nabla}_X Y)^T + (\tilde{\nabla}_X Y)^\perp\), where \((\tilde{\nabla}_X Y)^T\) denotes tangential component \(\pi^T(\tilde{\nabla}_X Y)\) and \((\tilde{\nabla}_X Y)^\perp\) denotes normal component \(\pi^\perp(\tilde{\nabla}_X Y)\) of the ambient connection, cf. Figure 3.1. The normal component of the ambient connection is called the second fundamental form \(\text{II}(X, Y)\), i.e., \(\text{II}(X, Y) \overset{\text{def}}{=} (\tilde{\nabla}_X Y)^\perp\), for two vector fields \(X, Y\) extended arbitrarily to \(\mathcal{M}\). The following theorem identifies the tangent component of the ambient connection as the intrinsic connection.

**Theorem 3.2.4 (The Gauss Formula Along a Curve, cf. [58])** Let \(M \hookrightarrow \mathcal{M}\) be a
Riemannian submanifold of $\tilde{M}$, and $\gamma$ a curve in $M$. For any vector field $V$ tangent to $M$ along $\gamma$

$$\tilde{D}_t V = D_t V + H(\dot{\gamma}, V). \tag{3.3}$$

### 3.3 Riemannian Geodesics

A parametrized curve $\gamma: I \to M$, where $I \subset \mathbb{R}$ denotes any interval of real numbers, is called a geodesic if the acceleration vector field is identically zero, i.e., $D_t \gamma(t) \equiv 0$. In local coordinates

$$\dot{\gamma}^k(t) = -\Gamma^k_{ij} \gamma^i(t) \dot{\gamma}^j(t), \tag{3.4}$$

for all $t \in I$, where $\Gamma$ are the Christoffel symbols. Since (3.4) is a second order system of ordinary differential equations for the functions $\gamma^i(t)$, by the existence and uniqueness of a solution for ODEs one proves the following theorem.

**Theorem 3.3.1 (Existence and uniqueness of geodesics, cf. [58, 69])** Let $(M, g)$ be a Riemannian manifold. For any $p \in M$, any $V \in T_pM$, and any $t_0 \in \mathbb{R}$, there exists an open interval $I \subset \mathbb{R}$, such that $t_0 \in I$, and a unique geodesic $\gamma: I \to M$ satisfying: $\gamma(t_0) = p$ and $\dot{\gamma}(t_0) = V$.

### 3.4 Curvature

An important question of Riemannian geometry is whether all Riemannian manifolds are locally isometric. To answer this question one defines the Riemannian curvature as follows.

**Definition 3.4.1 (Curvature endomorphism, cf. [58])** Let $M$ be a Riemannian manifold and $\mathcal{T}(M)$ be the space of smooth vector fields on $M$, the (Riemann) curvature endomorphism is the map $R: \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ defined by

$$R(X,Y)Z \overset{\text{def}}{=} \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$ 

It is appropriate to note that some authors, notably Milnor [69, 68], Gallot et al. [42] and Gray [43], adopt definition of the Riemannian curvature endomorphism that is the negative of ours.
One proves that the curvature endomorphism is a \((3,1)\)-tensor field. We define the Riemannian curvature tensor \(Rm\) by \(Rm(X,Y,Z,W) \overset{\text{def}}{=} (R(X,Y)Z, W)\). A Riemannian manifold is locally isometric to Euclidean space if and only if its curvature tensor vanishes identically.

The curvature tensor on a Riemannian manifold has a number of symmetries.

**Theorem 3.4.2 (Symmetries of the Riemannian curvature tensor, cf. \([58, 81]\))**

The Riemannian curvature tensor satisfies the following properties:

i) **Antisymmetry in the first and last pair of entries:**
\[
Rm(W, X, Y, Z) = -Rm(X, W, Y, Z) \quad \text{and} \quad Rm(W, X, Y, Z) = -Rm(X, W, Z, Y).
\]

ii) **Symmetry between the first and last pair of entries:**
\[
Rm(W, X, Y, Z) = Rm(Y, Z, X, W).
\]

iii) **Algebraic Bianchi's identity (or first Bianchi's identity)**
\[
Rm(W, X, Y, Z) + Rm(X, Y, W, Z) + Rm(Y, W, X, Z) = 0.
\]

**Lemma 3.4.3 (cf. \([58]\))** Let \((M, g)\) be a Riemannian \(n\)-manifold with constant sectional curvature \(C\) given by \(C = Rm(\partial_i, \partial_j, \partial_j, \partial_i)\), where \(\partial_i, \partial_j\) are any two orthogonal unit vectors at \(p \in M\). Then the curvature endomorphism and curvature tensor of \(g\) are given by:
\[
R(X,Y)Z = C((Y,Z)X - (X,Z)Y);
\]
\[
Rm(X,Y,Z,W) = C((X,W)(Y,Z) - (X,Z)(Y,W)).
\]

In the specific case of Lie groups one has the following theorem.

**Theorem 3.4.4 (cf. \([58, 69]\))** Let \(\mathbb{G}\) be a Lie group with a bi-invariant metric \(g\). For any left invariant vector fields \(X, Y, Z\) on \(\mathbb{G}\) the Riemannian curvature endomorphism of \(g\) is
\[
R(X,Y)Z = \frac{1}{4} [Z, [X, Y]].
\]
3.5 Calculus of Variations

In studies of minimizing geodesics in a Riemannian manifold one considers the first and the second derivative of the length functional. Let \( \gamma : [a, b] \to M \) be a curve in a Riemannian manifold \( M \) smooth on \([a, b]\) except for a final set of points \( \{a_i\} \subset [a, b] \). Such a curve is said to be an admissible curve. The length of \( \gamma \) is defined by the following integral

\[
L(\gamma) \overset{\text{def}}{=} \int_a^b \|\dot{\gamma}(t)\| \, dt.
\]

It is easy to show that the length of a curve is independent of parametrization, i.e.,

\[
L(\gamma) = L(\gamma \circ \varphi), \quad \text{where } \varphi : [c, d] \to [a, b] \text{ is a smooth map with smooth inverse.}
\]

Suppose \( M \) is a connected Riemannian manifold. For any pair of points \( p, q \in M \), we define the Riemannian distance \( d(p, q) \) to be the infimum of the lengths of all admissible curves from \( p \) to \( q \). An admissible curve \( \gamma \) from \( p \) to \( q \) is said to be minimizing if its length is equal to \( d(p, q) \).

For any connected Riemannian manifold the Riemannian distance function \( d \), defined above, is a metric and the Riemannian manifold is a metric space whose induced topology is the same as the given manifold topology.

A Riemannian manifold \( M \) is said to be geodesically complete if every geodesic segment can be extended indefinitely, cf. [69, 58]. The following result states that all concepts of completeness are equivalent ([81]).

**Theorem 3.5.1 (Hopf and Rinow)** A Riemannian manifold \( M \) is geodesically complete if and only if it is complete as a metric space.

An important implication of the Hopf-Rinow theorem is that \( M \) is complete if and only if any two points in \( M \) can be joined by a minimizing (but not necessarily unique) geodesic.

The following theorem gives a global characterization of complete manifolds of constant sectional curvature.

**Theorem 3.5.2 (Uniqueness of constant curvature metrics, cf. [58])** Let \( M \) be a complete, simply-connected Riemannian \( n \)-manifold with constant sectional curvature \( C \). Then \( M \) is isometric to one of the spaces: Euclidean \( \mathbb{R}^n \), the \( n \)-sphere \( S^n_R \) of radius \( R \) or the hyperbolic space \( \mathbb{H}^n_R \) of radius \( R \).
If $\gamma$ is a curve in a Riemannian manifold, the speed of $\gamma$ at any time $t$ is the length of its velocity vector $||\dot{\gamma}(t)||$. If $||\dot{\gamma}(t)||$ is independent of $t$ then $\gamma$ is said to be constant speed. As an immediate consequence of the definition of geodesic one has the following lemma.

**Lemma 3.5.3 (cf. [58])** All Riemannian geodesics are constant speed.

**Remark 3.5.4** The important consequence of the Lemma 3.5.3 is that if $\gamma:[0,1] \to M$ is a minimizing geodesic (parameterized by arc-length) then the length of $\gamma$ (the distance between $\gamma(0)$ and $\gamma(1)$) is equal to the speed $||\dot{\gamma}||$, because

$$L(\gamma) = \int_0^1 ||\dot{\gamma}(t)|| \, dt = \int_0^1 ||\dot{\gamma}|| \, dt = ||\dot{\gamma}||.$$

Let be given a Riemannian manifold $(M, g)$ and let $\nabla$ denote the Riemannian connection. Furthermore, let a continuous map $\Gamma: (-\epsilon, \epsilon) \times [a, b] \to M$ be a family of curves. A vector field along $\Gamma$ is a continuous map $V: (-\epsilon, \epsilon) \times [a, b] \to TM$ such that $V(s, t) \in T_{\Gamma(s, t)}M$, for each $(s, t)$. The family $\Gamma$ defines two collections of curves: the main curves $\Gamma_s(t) \overset{\text{def}}{=} \Gamma(s, t)$ defined on $[a, b]$ by setting $s =$ constant and the transverse curves $\Gamma_t(s) \overset{\text{def}}{=} \Gamma(s, t)$ defined on $(-\epsilon, \epsilon)$ by setting $t =$ constant. Wherever $\Gamma$ is smooth, the tangent vectors to these two families of curves form vector fields along $\Gamma$:

$$\partial_t \Gamma(s, t) \overset{\text{def}}{=} \frac{d}{dt} \Gamma_s(t) \quad \text{and} \quad \partial_s \Gamma(s, t) \overset{\text{def}}{=} \frac{d}{ds} \Gamma_t(s).$$

**Lemma 3.5.5 (Symmetry Lemma, cf. [58])** Let $\Gamma: (-\epsilon, \epsilon) \times [a, b] \to M$ be a family of curves in a Riemannian manifold $M$, on any rectangle where $\Gamma$ is smooth

$$D_s \partial_t \Gamma = D_t \partial_s \Gamma.$$

Let us denote $T(s, t) \overset{\text{def}}{=} \partial_t \Gamma(s, t)$ and $S(s, t) \overset{\text{def}}{=} \partial_s \Gamma(s, t)$.

**Lemma 3.5.6 (cf. [58])** If $\Gamma$ is any smooth family of curves, and $V$ is a smooth vector field along $\Gamma$, then

$$D_s D_t V - D_t D_s V = R(S, T)V,$$

where $R$ is the Riemannian curvature endomorphism (Definition 3.4.1).

For a curve $\gamma: [a, b] \to M$, a variation of $\gamma$ is a family of curves $\Gamma$ such that $\Gamma_0(t) = \gamma(t)$. The variation is proper if $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$, for all $s$. The variation field of $\Gamma$ is the vector field $V(t) = \partial_s \Gamma(0, t)$ along $\gamma$. A vector field $V$ along $\gamma$ is proper if $V(a) = V(b) = 0$. From this it is obvious that the variation field of a proper variation is proper.
Lemma 3.5.7 (cf. [58]) If $\gamma$ is an admissible curve and $V$ is a vector field along $\gamma$, then $V$ is the variation field of some variation of $\gamma$. If $V$ is proper, the variation can be taken proper as well.

Let $p, q \in M$ be two points on the geodesic $\gamma$. Then $q$ is said to be conjugate to $p$ along $\gamma$ if there exists a non-zero Jacobi field along $\gamma$ which vanishes at $p$ and $q$. If $p$ and $q$ are not conjugate along $\gamma$, then a Jacobi field along $\gamma$ is determined by its values at $p$ and $q$, cf. [27].

Let $\Gamma: (-\epsilon, \epsilon) \times [a, b] \to M$ be a proper variation of $\gamma$. Let $V: (-\epsilon, \epsilon) \times [a, b] \to TM$ be a variation field of $\Gamma$ such that $V(a) = V(b) = 0$. Finally, let $L$ be the length functional on the set of admissible curves. We have the two following important facts of Riemannian geometry.

Theorem 3.5.8 (First Variational Formula, cf. [58]) Let $\gamma: [a, b] \to M$ be any unit speed admissible curve, $\Gamma$ a proper variation of $\gamma$ and $V$ variational field of $\Gamma$. Then

$$\frac{d}{ds} |_{s=0} L(\Gamma_s) = -\int_a^b \langle V, D_t \gamma \rangle dt - \sum \langle V(a_i), \dot{\gamma}(a_i^+) - \dot{\gamma}(a_i^-) \rangle,$$

where $\dot{\gamma}(a_i^+) - \dot{\gamma}(a_i^-)$ is the jump in the tangent vector field $\gamma$ at $a_i$.

A unit speed admissible curve $\gamma$ is a critical point for the length functional $L$ if and only if it is a geodesic. Thus the geodesic equation $D_t \dot{\gamma} = 0$ characterizes the critical points of the length functional. To answer the question of which geodesics minimize $L$ one needs to study the second variation of the length functional.

If $V$ is a vector field along $\Gamma$ the covariant derivatives of $V$ either along main curves or along transverse curves produce vector fields along $\Gamma$: $D_t V$ and $D_s V$, respectively.

Theorem 3.5.9 (The Second Variational Formula, cf. [58]) Let $\gamma: [a, b] \to M$ be any unit speed geodesic, $\Gamma$ a proper variation of $\gamma$ and $V$ variational field of $\Gamma$. Then

$$\frac{d^2}{ds^2} |_{s=0} L(\Gamma_s) = \int_a^b \left( \langle D_t V^\perp, D_t V^\perp \rangle - Rm(V^\perp, \dot{\gamma}, \dot{\gamma}, V^\perp) \right) dt,$$

where $V^\perp$ is the normal (orthogonal to $\gamma$) component of $V$ and $Rm$ is the Riemannian curvature tensor (see its definition on page 33).

Define a symmetric bilinear form $I$ called the index form on the space of proper normal vector fields along $\gamma$ by

$$I(V, W) \overset{\text{def}}{=} \int_a^b \langle (D_t V, D_t W) - Rm(V, \dot{\gamma}, \dot{\gamma}, W) \rangle dt. \quad (3.5)$$
If $\Gamma$ is a proper variation of a unit speed geodesic $\gamma$ whose variational field is a proper normal vector field $V$ then the second variation of $L(\Gamma)$ is $I(V, V)$. In particular if $\gamma$ is minimizing then $I(V, V) \geq 0$, for any proper normal vector field along $\gamma$. Conversely, if $\gamma$ is a geodesic segment from $p$ to $q$ that has an interior conjugate point to $p$, then there exists a proper normal vector field $V$ along $\gamma$ such that $I(V, V) < 0$. In particular, $\gamma$ is not minimizing.

An immediate consequence of the above statement is that if $Rm(V, \dot{\gamma}, \dot{\gamma}, V)$ is non-positive for every vector field $V$ along $\gamma$, then for any proper normal vector field $V$ along $\gamma$ there is $I(V, V) \geq 0$, where the equality holds only when $V = 0$. Hence no points are conjugate.

If $\Gamma$ is a variation through geodesic, i.e., the main curves $\Gamma_s(t)$ are also geodesics, then one has the following theorem.

**Theorem 3.5.10 (The Jacobi Equation, cf. [58])** Let $\gamma$ be a geodesic and $V$ a vector field along $\gamma$. If $V$ is the variation field of a variation through geodesics, then $V$ satisfies

$$D_t^2 V + R(V, \dot{\gamma}) \dot{\gamma} = 0. \tag{3.6}$$

The (3.6) is called the Jacobi equation and any vector field along a geodesic satisfying the Jacobi equation is called a Jacobi field. Furthermore, every Jacobi field along a geodesic $\gamma$ is the variation field of some variation of $\gamma$ through geodesics, cf. [58].

### 3.6 Normal Coordinates and the Exponential Map

Let $(M, g)$ be a complete Riemannian manifold. Then there exists a map $\exp: TM \to M$ called the exponential map defined by $\exp(V) \overset{\text{def}}{=} \gamma_V(1)$ (see [69] for motivation of this terminology). We have the following important fact.

**Lemma 3.6.1 (Normal Neighbourhood Lemma, cf. [58])** For any $p \in M$, there is a neighbourhood $W$ of the origin in $T_pM$ and a neighbourhood $U$ of $p \in M$ such that $\exp_p: W \to U$ is a diffeomorphism.

Any open neighbourhood $U$ of $p \in M$ that is the diffeomorphic image under $\exp_p$ of an open neighbourhood of $0 \in T_pM$ is called a normal neighbourhood of $p$. 
Lemma 3.6.2 (cf. [58]) Suppose \( p \in M, \ V \in T_pM \) and \( q = \exp_p V \). Then \( \exp_p \) is a local diffeomorphism in a neighbourhood of \( V \) if and only if \( q \) is not conjugate to \( p \) along the geodesic \( \gamma(t) = \exp_p tV, \ t \in [0,1] \).

An orthonormal basis \( \{E_i\} \) for \( T_pM \) gives an isomorphism \( E: \mathbb{R}^n \to T_pM \) defined by \( E(x^1,\ldots,x^n)^i = x^i E_i \). If \( U \) is a normal neighbourhood of \( p \), we can combine this isomorphism with the exponential map to get a coordinate chart

\[
\varphi \overset{\mathrm{def}}{=} E^{-1} \circ \exp^{-1}: U \to \mathbb{R}^n.
\]

Any such coordinates are called \((\text{Riemannian}) \) normal coordinates centered at \( p \).

Given \( p \in M \) and a normal neighbourhood \( U \) of \( p \), there is a one-to-one correspondence between normal coordinate charts and orthonormal bases at \( p \).

In any normal coordinate chart centered at \( p \), define the radial distance function \( r \) by

\[
r(x) \overset{\mathrm{def}}{=} \left( \sum_{i=1}^{n} (x^i)^2 \right)^{1/2},
\]

and the unit radial vector field \( \partial/\partial r \) by

\[
\frac{\partial}{\partial r} \overset{\mathrm{def}}{=} \frac{x^i}{r} \frac{\partial}{\partial x^i}.
\]

In Euclidean space, \( r(x) \) is the distance to the origin and \( \partial/\partial r \) is the unit vector field tangent to straight lines through the origin.

Since the velocity of a radial geodesic is equal to \( \partial/\partial r \), which is a unit vector in both, the \( g \) norm and the Euclidean norm in normal coordinates, the \( g \)-length of the geodesic is equal to its Euclidean length, which is \( r(x) \).

### 3.7 Comparison Theorems

This section, based on Karcher [49], is later used in Section 4.3.2 of the thesis. More on comparison theorems can be found in [27, 49, 87, 81].

Let \( M \) be a Riemannian manifold with a linear connection \( \nabla \). A vector field \( V \in TM \) along a curve \( \gamma \) is said to be \textit{parallel along} \( \gamma \) with respect to \( \nabla \) if \( D_V V = 0 \). For example a geodesic can be characterized as a curve whose velocity vector field is parallel along the curve. The fundamental fact about parallel vector fields is that any tangent vector \( V_0 \) at any point on a curve can be uniquely extended to a parallel vector field along the entire curve. Such a vector field is called the \textit{parallel translate} of \( V_0 \) along \( \gamma \). If
3.8 Cut Points and Cut Loci

Let $\gamma: I \rightarrow M$ be a curve and $t_0, t_1 \in I$, parallel translation defines a linear isomorphism $P_{t_0 t_1}$ between $\mathcal{T}_{\gamma(t_0)} M$ and $\mathcal{T}_{\gamma(t_1)} M$

$$P_{t_0 t_1}: V_0 \mapsto V(t_1),$$

where $V$ is the parallel translate of $V_0$ along $\gamma$.

The following comparison theorems are described in terms of the matrix version of the Jacobi equation (3.6), as in [49]. Define the following $R_t: \mathcal{T}_{\gamma(t_0)} M \rightarrow \mathcal{T}_{\gamma(t_0)} M$ to be

$$R_t \cdot X = P_{t_0 t}^{-1} \cdot R(P_{t_0 t} \cdot X, \dot{\gamma}(t)) \dot{\gamma}(t),$$

then the matrix version of the Jacobi equation becomes

$$T'' + R_t \cdot T = 0,$$ (3.7)

where $J(t) = P_{t_0 t} \cdot T(t) \cdot V_0$, for any $V_0 \in \mathcal{T}_{\gamma(t_0)} M$. Since $P_{t_0 t_0}$ is the identity, we immediately get

$$J(0) = T(0) \cdot V_0 \quad \text{and} \quad D_t J(0) = T'(0) \cdot V_0.$$

Let $\delta$ and $\Delta$ be lower and upper bounds of the sectional curvature $C$ along geodesic $\gamma$, i.e., $\delta \leq C \leq \Delta$. Furthermore, let $\kappa: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and $S_\kappa$ the solution of the differential equation

$$S''_\kappa + \kappa \cdot S_\kappa = 0, \quad \text{where} \ S_\kappa(0) = 0 \text{ and } S'_\kappa(0) = 1.$$

**Theorem 3.7.1 (Upper and lower bounds estimate, cf. [49])** Let $T$ be a solution of the matrix version of the Jacobi equation (3.7) with the boundary conditions $T(0) = 0$ and $T'(0) = 1$. Let $\gamma$ and $u$ be orthonormal and $S_\Delta$ be positive on $(0, s_0)$, then

$$S_\delta(s) \langle T'(s) \cdot u, T(s) \cdot u \rangle \leq S'_\delta(s) \langle T(s) \cdot u, T(s) \cdot u \rangle$$ (3.8)

and

$$S_\Delta(s) \langle T'(s) \cdot u, T(s) \cdot u \rangle \geq S'_\Delta(s) \langle T(s) \cdot u, T(s) \cdot u \rangle$$ (3.9)

in $[0, s_0]$.

### 3.8 Cut Points and Cut Loci

Let $M$ be a complete Riemannian manifold. Suppose $\gamma$ is a geodesic starting at $p$. Let

$$B \overset{\text{def}}{=} \sup \{ b > 0 | \gamma|_{[0,b]} \text{ is minimizing} \}.$$
If $B < \infty$, we call $q = \gamma(B)$ the cut point of $p$ along $\gamma$. The cut locus of $p$, denoted as $\text{Cut}(p)$, is the set of all points $q \in M$ such that $q$ is the cut point of $p$ along some geodesic. It follows directly from the definition that if $q$ is a cut point of $p$ along $\gamma$, then $p$ is the cut point of $q$ along $\gamma$ in the reversed direction. In particular $q \in \text{Cut}(p)$ if and only if $p \in \text{Cut}(q)$.

Note that if $M$ is compact, its diameter is finite, hence there exists a cut point for any $p \in M$ along any geodesic starting at $p$. The converse is also true. If $M$ is complete and there exists $p \in M$ which has a cut point for every geodesic starting from $p$ then $M$ is compact.

**Theorem 3.8.1 (cf. [27, 37])** Let $q$ be the cut point of $p$ along $\gamma_1$. Then either $q$ is conjugate to $p$ along $\gamma_1$, or there is a geodesic $\gamma_2 \neq \gamma_1$ from $p$ to $q$ such that length of $\gamma_2$ is equal to the length of $\gamma_1$.

It is known (cf. [27]) that the distance of a point $p$ to its cut locus is a continuous function of $p$ and $\text{Cut}(p)$ is a closed set, for any $p \in M$. The complement $M \setminus \text{Cut}(p)$ of $\text{Cut}(p)$ is the maximal open set in $M$ such that each of its points can be joined to $p$ by exactly one minimizing geodesic, see [53] for instance.

Let us give a definition of a (geodesically) convex set since there are various definitions and names adopted by different authors. Notably, in [27, 53] the convex sets are named strongly convex.

**Definition 3.8.2 (Convex set, cf. [58])** A subset $\Omega$ of a Riemannian manifold $M$ is said to be convex if for any $p, q \in \Omega$, there is a unique minimizing geodesic in $M$ from $p$ to $q$ such that this geodesic lies entirely in $\Omega$.

Let $\varrho: M \to \mathbb{R}$ be the distance of a point $p$ to its cut locus. Let $B_\varrho(p)$ be contained in some compact set $C$. Let $\mathcal{I}$ denote the infimum of $\varrho$ over $C$. Let $K$ denote the supremum of sectional curvatures at points of $C$. In what follows interpret $\pi/\sqrt{K}$ as infinity if $K < 0$.

**Theorem 3.8.3 (Whitehead, cf. [27])** If $r < 1/2 \min \left\{ \pi/\sqrt{K}, \mathcal{I} \right\}$, then $B_\varrho(p)$ is convex. In particular, there exists a positive continuous function $r(p)$, the convexity radius, such that $r < r(p)$ implies $B_\varrho(p)$ is convex.
3.9 Riemannian Geometry in Statistics

This section presents some definitions, facts and theorems that are relevant to Section 4.3.4 and Section 4.7 of the thesis. Families of probability distributions have a natural geometric structure where parameters of a distribution play a role of coordinate system in a differentiable manifold equipped with a metric. As it turns out, the metric defined by the Fisher information matrix, is a Riemannian metric that can be used to measure distances between distributions belonging to the same parametric family. This measure is invariant in the sense that it is unaffected by a reparametrization of the distributions.

We will describe statistical manifold following Murray & Rice [70]. However, other definition of a statistical manifold are also possible. Let $\Omega$ be a measure space. We consider the equivalence class $\mathcal{M}$ of a family of non-negative measures on $\Omega$ up to scale, where two measures $\mu_1, \mu_2$ are equivalent if there is a constant $c$ such that $\mu_1 = c \mu_2$. $\mathcal{M}$ forms an affine space of measures. Let $\mathcal{P}$ denotes the space of all finite positive probability measures in $\mathcal{M}$, $\mathcal{P} \subset \mathcal{M}$.

Let $P = \{p(\theta)\}$ be a parameterized family of probability measures thought as a submanifold in $\mathcal{P}$, where $\theta = (\theta^1, \theta^2, \ldots, \theta^n)$ are local coordinates. Let $\mathcal{R}_\Omega$ denotes a vector space of measurable functions on $\Omega$. We define a function $f: P \to \mathcal{R}_\Omega$ to be smooth, if for every $x$ the real valued function $p \mapsto f(p)(x)$ on $P$ is smooth. Let us assume that:

i) the inclusion $P \subset \mathcal{P}$ is smooth, and

ii) the inclusion $P \subset \mathcal{P}$ is an immersion.

**Definition 3.9.1 (cf. [70])** Define a map $\ell: P \to \mathcal{R}_\Omega$ such that $\ell(p \, d\mu) = \log(p)$. When $p$ is a probability density $\ell(p \, d\mu)$ is called the log-likelihood.

The use of log-likelihoods allows for geometrical approach to statistical parametric models, [5, 70]. Function $f$ acts on densities through multiplication by $e^\theta$ with the property $e^\theta e^{\theta'} p = e^{(\theta + \theta')} p$. Thus we may regard functions as vectors of translations in $\mathcal{M}$.

Since the inclusion $P \subset \mathcal{P}$ is smooth (condition i), the log-likelihood function $\ell: p = e^\theta \mu \mapsto f$ is smooth, i.e., $p(x) \mapsto \ell(p)(x)$ is smooth, for all $x \in \mathcal{R}_\Omega$. Because the inclusion $P \subset \mathcal{P}$ is an embedding (condition ii), then for $p \in P$ and any coordinates $\theta$ in a neighbourhood of $p$, the random variables $\frac{\partial \ell}{\partial \theta}(p)$ are linearly independent.
Let $E_p: \mathcal{R}_\Omega \to \mathbb{R}$ denote the expectation with respect to the measure $p$:

$$E_p(f) \overset{\text{def}}{=} \int_{\Omega} f \, p \, \mu.$$  

Define an inner product by the formula

$$\langle f , g \rangle_p = E_p(fg)$$

on the space of $p$ square integrable random variables $f$, i.e., $E_p(f^2) < \infty$. Let $\ell_i$ be $i$-th component of the derivative of log-likelihood $\ell$, then

$$g_{ij}(p) = E_p(\ell_i \ell_j) = E_p\left( \frac{\partial \ell}{\partial \theta^i} \frac{\partial \ell}{\partial \theta^j} \right) \quad (3.10)$$

is called the Fisher information matrix.

Let $\theta: P \to \mathbb{R}^n$ are global coordinates and $u: \Omega \to P$ is an unknown unbiased estimator, $E_p(\theta \circ u) = \theta(p)$. It can be shown (cf. [5, 70]) that the covariance matrix for the random variable $\theta^i \circ u$ is greater than or equal to the inverse of the Fisher information matrix, namely that

$$\text{Cov}(\theta^i \circ u, \theta^j \circ u) \geq g^{ij} \quad (3.11)$$

is positive semi-definite. The property (3.11) is called the Cramér-Rao inequality. From the definition of the Fisher information matrix (3.10) it follows that $g_{ij}$ is linear and symmetric, and the Cramér-Rao inequality implies that Fisher information matrix is positive definite. Hence, $g_{ij}$ is a metric, often called the information metric.
Part II

The Riemannian Mean
Chapter 4

Definition of the Riemannian Mean and its Properties

In this chapter the dependence of the Riemannian mean on the curvature of a space and the covariance of a sample is investigated. A simple approximation formulae for the Riemannian mean in spaces of constant sectional curvature is derived. Further properties of the Riemannian mean with numerous examples are also presented here.

This chapter sets out to achieve the following objectives:

✦ to define the Riemannian mean,
✦ to derive a simple approximate formula for the Riemannian mean in the $n$-sphere $S^n_R$ and the hyperbolic space $H^n_R$,
✦ to present an iterative method for computing the Riemannian mean,
✦ to prove the convexity property of the Riemannian mean in a complete Riemannian 2-manifold,
✦ to investigate the Riemannian mean at points where the Riemannian geodesics are not minimizing,
✦ to derive the Riemannian mean in the space of Poisson distributions.

The Riemannian center of mass and the Riemannian barycenter have been investigated by many authors in a number of contexts. The barycenter of a probability measure on a Riemannian manifold is investigated in Karcher [49] and Corcuera & Kendall [28]. Given the Fisher information matrix of a model, the parametric statistical model has a natural Riemannian structure. In this context Oller & Corcuera [78]
investigate the mean value and moments of random objects defined as measurable maps from a sample space to subsets of a manifold. They derive *intrinsic* versions of the Rao-Blackwell and Lehmann-Scheffé theorems. The center of mass in Riemannian geometry was already studied by Cartan (cf. [81, 54]). The case of a discrete support measure on spheres is investigated in [74], where the *spherical mean* is defined in *global* terms.

This chapter extends the definition of the spherical mean (cf. Section 2.2.2) to any complete Riemannian manifold. We call it the *Riemannian mean* to emphasize its relationship with Riemannian geometry. The Riemannian mean is a set, which is non-empty for a complete manifold but not always a singleton, for instance, when sample points are symmetrically distributed in a symmetric space. On the other hand, for a complete manifold with non-positive sectional curvature the mean is a singleton. If the sample points are concentrated in a small region of a complete manifold then the mean contains a unique point in this region.

The chapter is organised as follows. Section 4.1 defines the Riemannian mean to be the set of points where the energy\(^1\) function \(\Phi_Q: M \rightarrow \mathbb{R}\) attains its minimum. A definition applicable to metric spaces is also presented. This section is concluded by an example deriving critical points of \(\Phi_Q\) defined on the group of rotations \(\text{SO}(n)\). Section 4.1.1 describes the Riemannian mean in the case of a complete manifold with non-positive sectional curvature. Section 4.2 presents the fundamental properties of the Riemannian mean following directly from its definition. Section 4.3 investigates the critical points of \(\Phi_Q\). Section 4.3.1 investigates the difference between the Riemannian mean and embedded (or projected) mean, when the sample points are concentrated in a small region. It is shown that the difference is related to the Riemannian curvature tensor and the covariance of a sample. Section 4.4 establishes a general iterative method of finding the local minima of \(\Phi_Q\). This method is illustrated by examples of classical Lie groups in Section 4.4.1. Section 4.5 proves that for a complete Riemannian 2-manifold (surface), under certain conditions, any convex set containing \(Q\) also contains a critical point of \(\Phi_Q\). The discussion on conjugate and cut points in relation to the Riemannian mean is given in Section 4.6. It is shown that no cut point to a point in \(Q\), not conjugate to any point of \(Q\), can be a point of local minimum of \(\Phi_Q\) and therefore does not belong to the Riemannian mean. Finally,

\(^1\)The term “energy” used by some authors comes from the interpretation of the integral

\[
E = \int_0^1 \|\dot{\gamma}(t)\|^2 \, dt.
\]

This quantity is also often called the “action” integral. See the introduction in Milnor [69] where this matter is discussed. In this thesis there are no references to any physical interpretation of this integral and the word “energy” is seldom used.
applications of the Riemannian mean to parametric statistical models are given in Section 4.7.

4.1 Definition of the Riemannian Mean

Noakes [74] (unpublished) investigates both spherical mean and spherical variance. These two statistical measures are defined in terms of the spherical distance and therefore have properties related to spherical geometry. We will introduce the notion of Riemannian mean by extending the definition of the spherical mean to a complete Riemannian manifold. The Riemannian mean is a geometrical interpretation of the center of mass of a finite number of points and is closely related to measures of centrality considered by Oller & Corcuera [78] and Karcher [49], cf. Section 2.2.2. However, these measures of centrality are defined locally as the critical points of certain functionals. The Riemannian mean, instead, is a subset of the manifold, on which the minimum of $\Phi_Q$ is achieved—it may consist of a number of points, e.g., it is a great circle for any two antipodal points in the sphere, cf. Example 2.2.2. We define the Riemannian mean as follows.

Let $(M, g)$ be a complete Riemannian manifold ($g$ is a Riemannian metric) and let $d(p, q)$ be a Riemannian distance, for $p, q \in M$. Let $Q$ be a finite set of points in $M$.

**Definition 4.1.1** Let $\Phi_Q : M \to \mathbb{R}$ be a function defined as

$$\Phi_Q(x) \overset{\text{def}}{=} \frac{1}{|Q|} \sum_{q \in Q} d(x, q)^2. \quad (4.1)$$

The Riemannian variance $\sigma_Q^2$ is the global minimum value of $\Phi_Q$. The Riemannian mean $\bar{Q} \in M$ is the set of points at which $\Phi_Q = \sigma_Q^2$.

First we point out that the Riemannian mean is non-empty by demonstrating that $\Phi_Q$ takes its minimum at points in $M$.

Choose any point $x_0 \in M$ and a positive number $\delta$ such that $d(x_0, q) \leq \delta$, for all $q \in Q$. Let $K \overset{\text{def}}{=} \{x \in M \mid d(x, q) \leq \delta, q \in Q\}$, then $K$ is a bounded and closed subset of a complete Riemannian manifold. Hence $K$ is compact and non-empty because $x_0 \in K$.

We see that $\Phi_Q(x) \leq \delta^2$, for $x \in K$ and $\Phi_Q(x) > \delta^2$, for $x \not\in K$. Therefore $\Phi_Q$ attains its minimum in $K \subset M$.

The Riemannian mean is defined in terms of the Riemannian structure $(M, g)$ and thus is an intrinsic geometrical quantity invariant under isometries: if $f : M \to M$ is
an isometry then
\[ \Phi_{f(Q)}(f(x)) = \frac{1}{\#Q} \sum_{q \in Q} d(f(x), f(q))^2 = \frac{1}{\#Q} \sum_{q \in Q} d(x, q)^2 = \Phi_Q(x), \]
and \( f(Q) = f(Q) \).

**Remark 4.1.2** Definition 4.1.1 can be generalized to define a p-mean for metric spaces as follows. Let \( E \) be a metric space with metric \( d \). For a finite set \( Q \subset E \) and \( p \geq 1 \), define \( \Phi^p_E : E \to \mathbb{R} \) by
\[ \Phi^p_Q(x) \overset{\text{def}}{=} \frac{1}{\#Q} \sum_{q \in Q} d(x, q)^p. \]
The p-variance \( \sigma^p_Q \) is a global minimum value of \( \Phi^p_Q \) and the p-mean \( Q^p \in E \) is the set of points at which \( \Phi^p_Q = \sigma^p_Q \).

Let \( E \) be an Euclidean space \( \mathbb{R}^n \) with the Euclidean norm
\[ ||x|| = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}, \]
and the metric \( d(x, y) = ||x - y|| \). Setting \( p = 2 \) gives the standard definition of the mean value being the center of a mass (2.7) of a set \( Q \).

Denote the p-norm of a finite sequence \( \xi = \{ \xi^i \} \), where \( \xi^i \in \mathbb{R} \), by
\[ ||\xi||_p \overset{\text{def}}{=} \left( \sum_{i} |\xi^i|^p \right)^{1/p}. \]
The p-norm is monotonically decreasing with increasing \( p \) (cf. [79])
\[ ||\xi||_{\alpha} \leq ||\xi||_{\beta}, \text{ where } 1 \leq \beta < \alpha. \]

Hence if \( 1 \leq \beta < \alpha \) then
\[ (\#Q \cdot \Phi^\alpha_Q(x))^{1/\alpha} \leq (\#Q \cdot \Phi^\beta_Q(x))^{1/\beta}, \text{ for any } x \in E. \]
In particular we have
\[ (\#Q \cdot \sigma^\alpha_Q)^{1/\alpha} \leq (\#Q \cdot \sigma^\beta_Q)^{1/\beta}, \]
because for any \( \bar{Q}_\beta \in \bar{Q}_\beta \) there is
\[ (\#Q \cdot \sigma^\alpha_Q)^{1/\alpha} \leq (\#Q \cdot \Phi^\alpha_Q(\bar{Q}_\beta))^{1/\alpha} \leq (\#Q \cdot \Phi^\beta_Q(\bar{Q}_\beta))^{1/\beta} = (\#Q \cdot \sigma^\beta_Q)^{1/\beta}. \]
For example, taking \( \alpha = 2 \) and \( \beta = 1 \) in the above inequality yields
\[ \Phi^2_Q(x)/\#Q \leq \left( \Phi^1_Q(x) \right)^2, \text{ for any } x \in E. \]
Let $E$ be the unit $n$-sphere $S^n$ with the spherical distance $d(p,q)$ defined as the length of the shortest great-arc joining $p,q \in S^n$. Since $1 + z^p \leq (1 + z)^p$, for $z \geq 0$ and $p \geq 1$, we get $d(x,q)^p + d(-x,q)^p \leq \pi^p$. Applying this inequality to the definition of $\Phi_Q^p$ we get

$$\Phi_Q^p(x) + \Phi_Q^p(-x) \leq \pi^p$$

and hence $\Phi_Q^p(x) \leq \pi^p/2$ or $\Phi_Q^p(-x) \leq \pi^p/2$, which brings the following property of the spherical $p$-variance: $\sigma_Q^p \leq \pi^p/2$.

**Remark 4.1.3** A number of authors define a mean value of a continuous density $\mu$ as a critical point of the energy functional $J: M \to \mathbb{R}$

$$J(\omega) \overset{\text{def}}{=} \int_M d(w,x)^2 \mu(dx). \quad (4.2)$$

When $\mu$ has discrete support the functional (4.2) coincides with (4.1). Corcuera & Kendall [28] extend this definition to allow for non-minimality of geodesics. The authors define the Riemannian barycenter of a probability measure $\mu$ on $M$ to be a point $w \in M$ for which there exists a probability measure $\tilde{\mu}: \mathbb{T}_wM \to \mathbb{R}$ such that $\tilde{\mu}(\exp^{-1}_w A) = \mu(A)$, for all Borel $A \subseteq M$, and satisfying

$$\int_{\mathbb{T}_wM} V \tilde{\mu}(dV) = 0$$

(cf. Lemma 4.3.6 and Remark 4.3.9). The choice of $\tilde{\mu}$ depends on the choice of a family of geodesics from $w$ to each of the points in the support of $\mu$.

Although the Riemannian mean is a set of points where $\Phi_Q$ attains its global minimum, it is easier to investigate local minima of $\Phi_Q$. The following example shows that even for a simple space as the Lie group $\mathbb{SO}(n)$ the problem of deriving local minima of $\Phi_Q$ may be difficult.

**Example 4.1.4** Let $\mathcal{G} = \mathbb{SO}(n)$ be the special orthogonal group. The space $\mathbb{SO}(n)$ consists of all real $n \times n$ matrices $A$ such that $A \cdot A^t = 1$ and $\det A = 1$, where $A^t$ is the transpose of $A$. The tangent space at the identity $g = T_1 \mathbb{SO}(n)$ is the Lie algebra $\mathfrak{so}(n)$ of $n \times n$ skew-symmetric matrices (cf. [69, 83]). Any minimizing geodesic $\gamma: [0,1] \to \mathbb{SO}(n)$ from the identity $1$ to $A = \exp(W)$, where $W \in \mathfrak{so}(n)$, can be written uniquely as

$$\gamma(t) = \exp(tW).$$

To calculate the distance between $1$ and $\exp(W)$ it suffices to find the length of the velocity vector of the minimizing geodesic at the starting point $1$ (cf. Lemma 3.5.3). There is

$$\dot{\gamma}(0) = \frac{d}{dt} \bigg|_{t=0} \gamma(t) = W$$
hence
\[ d(1, \exp(W))^2 = \|W\|^2 = \langle W, W \rangle, \]
(4.3)

using the standard inner product
\[ \langle A, B \rangle = \text{trace} (A \cdot B^T), \]
(4.4)
cf. [69]. Let \( J \) be any point of \( SO(n) \). Since the metric (4.4) is bi-invariant, both the left and the right group multiplications in \( SO(n) \) are isometries (cf. [69]).

For a finite set \( Q = \{A_1, A_2, \ldots, A_m\} \subset SO(n) \), let \( \{V_1, V_2, \ldots, V_m\} \subset so(n) \) be a set of skew-symmetric real matrices defined as \( V_i \triangleq \log(A_i) \), for all \( i = 1, 2, \ldots, m \), with their elements bound to be in the interval \( (-\pi, \pi) \). For any \( A \in SO(n) \) we have
\[ d(A_i, A)^2 = d(1, A_i^{-1} \cdot A)^2 = \langle W_i, W_i \rangle, \]
(4.5)

where \( W_i = \log(A_i^{-1} \cdot A) = \delta_1(-V_i, V) \), \( V = \log(A) \) and \( \delta_1 : g \times g \rightarrow g \) is the Hausdorff series (cf. [15, 94]). The function \( \Phi_Q \) defined by (4.1) can be expressed then as
\[ \Phi_Q(A) = \frac{1}{m} \sum_{i=1}^{m} \langle W_i, W_i \rangle. \]

For \( A = \exp(V) \) to be a local minimum of \( \Phi_Q \) it has to be \( d\Phi_Q(V)(U) = 0 \), for all \( U \in so(n) \), where
\[ d\Phi_Q(V)(U) = \frac{2}{m} \sum_{i=1}^{m} \langle \delta_1(-V_i, V), D_2 \delta_1(-V_i, V)(U) \rangle. \]

The above equation is difficult to solve directly and a more practical iterative scheme will be presented in Section 4.4.

The definition of the Riemannian mean depends on the distance function and the metric. Thus it is no surprise that the curvature of the space plays a fundamental role in properties of the Riemannian mean. It will be helpful to consider first the class of complete manifolds with non-positive sectional curvature because it will be possible to draw conclusions from Euclidean space. However, in the general case of a complete manifold, the exponential mapping \( TM \rightarrow M \) fails to be 1-1. This poses difficulties in studying minima of \( \Phi_Q \). Properties of \( \Phi_Q \) in such situations are investigated in Section 4.6.

4.1.1 Complete manifolds with non-positive sectional curvature

This section investigates the special case of complete manifolds with non-positive sectional curvature. The following results are based on Kobayashi & Nomizu [54], see Remark 4.1.7. Roughly speaking, positive curvature causes nearby geodesics to converge, while negative curvature causes nearby geodesics to diverge (cf. [58]). As
will become apparent, for the class of manifolds with non-positive sectional curvature the investigations of the Riemannian mean simplifies considerably. One reason is that Riemannian manifolds with non-positive sectional curvature have no conjugate points, cf. Section 3.5. In this case if the manifold $M$ is simply connected\(^2\) then the exponential mapping $\exp_x$ is a diffeomorphism of $T_x M$ onto $M$, cf. [54]. On the other hand, the exponential mapping $\exp_x$ increases the distance. As a consequence, the function $\Phi_Q$ has exactly one critical point which is a point of global minimum.

**Example 4.1.5** Let us consider Euclidean space $M = \mathbb{R}^n$. The function $\Phi_Q$ in then given by

$$\Phi_Q(x) = \frac{1}{\#Q} \sum_{q \in Q} (x^i - q^i)^2.$$  

It is easy to see that $\Phi_Q$ takes its only minimum at $x = (a^1, a^2, \ldots, a^n)$, where

$$a^i = \frac{1}{\#Q} \sum_{q \in Q} q^i.$$  

We will show that for a complete simply connected Riemannian manifold $M$ with non-positive curvature the function $\Phi_Q$ has exactly one point of minimum.

Let $p \in M$ be a point of minimum of $\Phi_Q$. Since $M$ simply connected, the exponential map $\exp_p: T_p M \to M$ is a diffeomorphism (see [54]), and so there is a unique $V_q \in T_p M$ such that

$$\exp_p(V_q) = q, \quad \text{for all } q \in Q.$$  

Define $\tilde{\Phi}_Q: T_p M \to \mathbb{R}$ as

$$\tilde{\Phi}_Q(X) \equiv \frac{1}{\#Q} \sum_{q \in Q} d_E(X, V_q)^2,$$  

where $d_E$ is the Euclidean distance in $T_p M$. Let $0$ be the origin of $T_p M$. We will show that

$$\Phi_Q(p) = \tilde{\Phi}_Q(0) < \tilde{\Phi}_Q(X) \leq \Phi_Q(\exp_p X), \quad \text{for } X \neq 0. \quad (4.6)$$  

Since at the origin of $T_p M$ the Riemannian distance is equal to the radial distance (Section 3.6) then $d_E(0, V_q) = d(p, \exp_p V_q) = d(p, q)$ and so $\Phi_Q(p) = \tilde{\Phi}_Q(0)$. Since $T_p M$ is an Euclidean space, $\tilde{\Phi}_Q$ has the only point of minimum at $0$ (cf. Example 4.1.5), i.e.,

$$\tilde{\Phi}_Q(0) < \tilde{\Phi}_Q(X), \quad \text{for } X \neq 0.$$  

Finally, since $\exp_p$ is a distance increasing mapping, cf. [54], we have

$$d_E(X, V_q) \leq d(\exp_p X, \exp_p V_q) = d(\exp_p X, q).$$

---

\(^2\)A manifold $M$ is called *simply connected* if it is connected and any continuous function $f: S^1 \to M$ is continuously contractible to a point. For example $S^n$ is simply connected for $n \geq 2$ but $S^1$ is not simply connected, cf. [92].
and

\[ \Phi_Q(X) \leq \Phi_Q(\exp_p X), \text{ for all } X \in T_pM, \]

what proves (4.6). Thus the point \( p \in M \) is the only minimum of \( \Phi_Q \).

**Corollary 4.1.6** Let \( M \) be a complete simply connected Riemannian manifold of non-positive curvature. Then \( \Phi_Q \) has exactly one point of minimum.

**Remark 4.1.7** A more general result can be found in [54]. Let \( \mu: \mathcal{C}(A) \to \mathbb{R} \) be a positive measure on a compact topological space \( A \). Let \( f: A \to M \) be a continuous mapping from \( A \) into a complete, simply connected Riemannian manifold \( M \) with non-positive curvature. Then \( J: M \to \mathbb{R} \) given by

\[ J(x) \overset{\text{def}}{=} \int_A d(x, f(a))^2 d\mu(a) \]

takes its minimum at precisely one point.

We have seen that for the complete manifolds with non-positive curvature the Riemannian mean is a single point of local minimum of \( \Phi_Q \). To study more general spaces we will investigate critical points of \( \Phi_Q \) in Section 4.3.

### 4.2 Properties of the Riemannian Mean

The aim of this section is to present fundamental properties of the Riemannian mean following directly from its definition (Definition 4.1.1).

Let manifold \( X = M \times N \) be a cartesian product of two manifolds. Then \( X \) has a natural Riemannian metric \( g = g_M \otimes g_N \), called the *product metric* (cf. [58]) defined by

\[ g(U_M + V_M, U_N + V_N) \overset{\text{def}}{=} g_M(U_M, V_M) + g_N(U_N, V_N), \]

under the natural identification \( T_{(p_1,p_2)}M \times N = T_{p_1}M \oplus T_{p_2}N \). Thus a geodesic \( \gamma: I \to X \) is a product of geodesics \( \gamma_M: I \to M \) and \( \gamma_N: I \to N \) as \( \gamma(t) = (\gamma_M(t), \gamma_N(t)) \).

Therefore the product metric induces the distance function \( d_X: X \times X \to \mathbb{R} \) given by

\[ d_X^2 = d_M^2 + d_N^2, \text{ where } d_M: M \times M \to \mathbb{R} \text{ and } d_N: N \times N \to \mathbb{R}. \]

We have the following fact concerning Riemannian means on cartesian product of manifolds.
**Proposition 4.2.1** The Riemannian mean $Q_{M \times N}$ of $Q \subset M \times N$ is the cartesian product of the means of $Q_M$ and $Q_N$

$$Q_{M \times N} = Q_M \times Q_N.$$  

**Proof:** The proof easily follows from the fact that the sum of squares of distances calculated on the manifold $X = M \times N$ is equal to the sum of sums of squares of distances calculated on subcomponents $M$ and $N$. Namely, for any $p = (p_M, p_N) \in M \times N$

$$\Phi_X(p) = \frac{1}{\#Q} \sum_{q \in Q} d_X(p, q)^2 = \frac{1}{\#Q} \sum_{q \in Q} (d_M(p_M, q_M)^2 + d_N(p_N, q_N)^2)$$

$$= \Phi_M(p_M) + \Phi_N(p_N).$$

Q.E.D.

In [74] we find the following Proposition 4.2.2 which is applicable to the Riemannian mean.

**Proposition 4.2.2 (Noakes [74])** If $Q_1, Q_2 \subset M$ then $Q_1 \cap Q_2$ is either empty or $Q$ where $Q = Q_1 \cup Q_2$.

The following example illustrates an application of Proposition 4.2.2.

**Example 4.2.3** Let $Q = \{q_1, -q_1, q_2, -q_2\}$ be a set of two pairs of antipodal points in a sphere $S^2$. As the Riemannian mean of any two antipodal points is a great circle (cf. Example 2.2.2) then it follows that the Riemannian mean $Q = \{q, -q\}$ is either a pair of antipodal points, if $q_1 \neq q_2$ and $q_1 \neq -q_2$, or a great circle.

### 4.3 Critical Points and the Riemannian Mean

This section investigates properties of critical points of $\Phi_Q$. We start by deriving an approximate formulae for the Riemannian geodesics, the distance and $\Phi_Q$ in local coordinates. Theorem 4.3.1 gives an estimate for a critical point of $\Phi_Q$, for a large class of spaces. Based on this we investigate the existence and properties of critical points of $\Phi_Q$. These points are candidates for the Riemannian mean. In Section 4.3.2 we will study the Hessian of $\Phi_Q$ and its relationship with sectional curvature. Example 4.3.12 derives upper and lower bounds of sectional curvature for symmetric spaces. The formulae for the Riemannian mean for the $n$-sphere $S^n$ and the hyperbolic space $H^n_R$ are derived in Section 4.3.3, followed by more examples in Section 4.3.4.
4.3.1 Approximating the Riemannian mean

This section derives an approximate formula for a local minimum of $\Phi_Q$.

Let $M$ be complete and $(U, (x^i))$ be any normal coordinate chart centered at $p \in M$ containing all data points $Q \subset U \subset M$. Denote $Q_E$ to be the Euclidean mean calculated in local coordinates

$$Q_E^k \overset{\text{def}}{=} \frac{1}{\#Q} \sum_{q \in Q} q^k.$$  \hspace{1cm} (4.7)

Note that $Q_E$ depends on the choice of coordinates and thus it is not a geometric quantity.

The rest of this section will be devoted to the proof of Theorem 4.3.1. We shall use the Taylor series [96] of a geodesic and the metric tensor in normal coordinates.

**Theorem 4.3.1** Let $(M, g)$ be complete and $(U, (x^i))$ a normal coordinate chart centered at $p \in M$. Let $B_\delta(p) = \exp_p(B_\delta(0)) \subset U$ be a geodesic ball around $p \in M$. Then there exists $\delta > 0$ such that if $Q \subset B_\delta(p)$ then $\Phi_Q$ has a local minimum at $q \in U$, where $q$ in these coordinates is given by

$$q^k = Q_E^k + \frac{1}{3\#Q} \sum_{q \in Q} R_{iklj} Q_E^l q^l q^j + \frac{1}{12\#Q} \sum_{q \in Q} \partial_r R_{iklj} Q_E^l q^l q^i (q^r + Q_E^r) + \frac{1}{24\#Q} \sum_{q \in Q} \partial_k R_{irij} Q_E^l q^l q^i q^j + O(\delta^5),$$  \hspace{1cm} (4.8)

and $R_{iklj} = \langle R(E_i, E_k) E_l, E_j \rangle$ are the components of the Riemannian curvature tensor at $p$ in terms of basis $\{E_i\}$.

In order to prove Theorem 4.3.1 we need some results concerning calculations of geodesics in normal coordinates. The following Lemma 4.3.2 gives an expression for the initial velocity vector of a geodesic joining two nearby points. Since geodesics are the curves of constant speed (cf. Lemma 3.5.3), the length of a geodesic, and therefore the distance, can be found from the length of the velocity vector. This is used in Lemma 4.3.3 to find the distance between two points near the origin of normal coordinates.

**Lemma 4.3.2** Let $(M, g)$ be complete and $(U, (x^i))$ a normal coordinate chart centered at $s \in M$. Let $\gamma : [0, 1] \to U \subset M$ be a geodesic from $p$ to $q$, i.e., $\gamma(0) = p$ and $\gamma(1) = q$. If the points $p$ and $q$ are sufficiently close to $s$ then the velocity vector $\dot{\gamma}(0)$ expressed in
normal coordinates has the following approximation

\[ \gamma^k(0) = d^k + \frac{1}{2!} \Gamma^k_{ij}(p) d^i d^j + \frac{1}{3!} \partial_t \Gamma^k_{ij}(p) d^i d^j d^t \]

\[ + \frac{1}{4!} \partial_t \partial_m \Gamma^k_{ij}(p) d^i d^j d^m + O\left(\{p,q\}^5\right), \tag{4.9} \]

where \( d^k = q^k - p^k \) and \( \Gamma^k_{ij} \) are the Christoffel symbols.

**Proof:** Let \( V \in T_p M \) be the initial velocity vector defined by (4.9) without the trailing term \( O\left(\{p,q\}^5\right) \). At first we will verify that the geodesic \( \gamma_V : [0,1] \rightarrow M \), whose initial velocity vector at the point \( p \) is \( V \) satisfies \( \gamma_V(1) = q + O\left(\{p,q\}^5\right) \).

Write \( \gamma(t) \) as the Taylor series

\[ \gamma(t) = \gamma(0) + t\dot{\gamma}(0) + \frac{t^2}{2!} \ddot{\gamma}(0) + \frac{t^3}{3!} \gamma^{(iv)}(0) + \frac{t^4}{4!} \gamma^{(v)}(0) + \frac{t^5}{5!} \gamma^{(v)}(\xi), \tag{4.10} \]

where \( \xi \in (0,t) \). The geodesic equation in local coordinates has the following form:

\[ \dot{\gamma}^k(t) = -\Gamma^k_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t), \tag{4.11} \]

for all \( t \in [0,1] \). By differentiating (4.11) by \( t \) and substituting right hand side of (4.11) for \( \ddot{\gamma} \) we obtain

\[ \gamma^{(iv)} = -\partial_t \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j + 2 \Gamma^k_{ij} \Gamma^m_{lm} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^l \dot{\gamma}^m, \tag{4.12} \]

and, repeating this process

\[ \gamma^{(v)} = -\partial_t \partial_m \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^l \dot{\gamma}^m \dot{\gamma}^p + 2 \Gamma^k_{ij} \partial_p \Gamma^l_{jm} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^m \dot{\gamma}^p + 4 \Gamma^i_{jl} \partial_p \Gamma^k_{im} \dot{\gamma}^j \dot{\gamma}^l \dot{\gamma}^m \dot{\gamma}^p \]

\[ + \Gamma^i_{jl} \partial_p \Gamma^k_{mp} \dot{\gamma}^j \dot{\gamma}^l \dot{\gamma}^m \dot{\gamma}^p - 2 \Gamma^i_{jl} \Gamma^k_{im} \Gamma^m_{pq} \dot{\gamma}^j \dot{\gamma}^l \dot{\gamma}^p \dot{\gamma}^q - 4 \Gamma^i_{jl} \Gamma^k_{im} \Gamma^l_{pq} \dot{\gamma}^j \dot{\gamma}^l \dot{\gamma}^p \dot{\gamma}^q. \tag{4.13} \]

It is clear that \( \gamma^{(v)} = O\left(\||\gamma||^5\right) \) and in general \( \gamma^{(n)} = O\left(\||\gamma||^n\right) \), cf. Appendix A.1. Replacing \( \ddot{\gamma} \), \( \gamma^{(iv)} \) and \( \gamma^{(v)} \) in the Taylor series (4.10) with the identities (4.11), (4.12), and (4.13) yields

\[ \gamma^k(1) = \gamma^k + V^k - \frac{1}{2} \Gamma^k_{ij} V^i V^j - \frac{1}{6} \partial_t \Gamma^k_{ij} V^i V^j V^l + \frac{1}{24} \partial_t \partial_m \Gamma^k_{ij} V^i V^j V^l V^m + O\left(\{p,q\}^5\right), \tag{4.14} \]

where the terms on the right hand side are calculated at the point \( t = 0 \). Since in normal coordinates \( \Gamma(p) \) is of order of \( O(p) \) and because \( V \) is of order \( O(\{p,q\}) \), formula (4.14) simplifies to

\[ \gamma^k(1) = \gamma^k + V^k - \frac{1}{2} \Gamma^k_{ij} V^i V^j - \frac{1}{6} \partial_t \Gamma^k_{ij} V^i V^j V^l - \frac{1}{24} \partial_t \partial_m \Gamma^k_{ij} V^i V^j V^l V^m + O\left(\{p,q\}^5\right). \]
Direct calculations verify that \( \gamma_V(1) = q + O(\{p,q\}^5) \).

Consider \( \gamma_U(1) \) to be the **exponential mapping** \( \exp_p : \mathcal{T}_p M \to M \) defined as

\[
\exp_p(U) \overset{\text{def}}{=} \gamma_U(1),
\]

where \( U \in \mathcal{T}_p M \) is the initial velocity vector of the unique geodesic \( \gamma_U : [0,1] \to M \) starting at \( p \), see Figure 4.1. If \( q \) is sufficiently close to \( p \) then \( q \) is not conjugate to \( p \) along the geodesic \( \gamma_V(t) = \exp_p(t\widetilde{V}), t \in [0,1], \) (cf. [58]). Hence there is a neighbourhood \( \mathcal{W} \subset \mathcal{T}_p M \) of the vector \( \tilde{V} \in \mathcal{T}_p M \) such that the exponential map \( \exp_p : \mathcal{W} \to M \) is a diffeomorphism (cf. Lemma 3.6.2). Let \( \mathcal{V} = \exp_p(\mathcal{W}) \). From the way we chose \( \mathcal{W} \) it is clear that \( q \in \mathcal{V} \). If \( p \) and \( q \) are sufficiently close to the origin of the normal neighbourhood we may assume that \( \gamma_V(1) = q + O(\{p,q\}^5) \in \mathcal{V} \) and so \( \mathcal{V} \subset \mathcal{W} \). To finish the proof it remains to show that \( \|\widetilde{V} - V\| \) is of order \( O(\{p,q\}^5) \).

From (4.14) it follows that \( d\exp_p(V) = 1 + O(\{p,q\}^2) \). By the inverse function theorem

\[
d\exp_p^{-1}(\gamma_V(1)) = (d\exp_p(V))^{-1} = 1 + O(\{p,q\}^2),
\]

where \( 1 \) denotes the identity matrix. Hence there is a constant \( k > 0 \) such that

\[
\|d\exp_p^{-1}(x)\| \leq 1 + k(\|p\|^2 + \|q\|^2),
\]

for all \( x \in \mathcal{V} \). Finally, the mean value theorem applied to the component functions of \( d\exp_p^{-1} \) gives the bounds of the following form

\[
\|\widetilde{V} - V\| = \|\exp_p^{-1}(q) - \exp_p^{-1}(q + O(\{p,q\}^5))\| \\
\quad \leq (1 + k(\|p\|^2 + \|q\|^2)) \cdot O(\{p,q\}^5) = O(\{p,q\}^5).
\]

This completes the proof. \( \text{Q.E.D.} \)
The bonus of the above proof is the formula (4.14), which shows the first few terms of the Taylor series of the exponential map in any normal coordinates. In particular, when the initial point \( p \in M \) is the center of a normal coordinates, the Christoffel symbols vanish (but not their derivatives) and the Taylor series simplify to

\[
\exp_p(V)^k = V^k - \frac{1}{6} \partial^k_i V^i V^j V^l - \frac{1}{24} \partial^k_i \partial^k_m V^i V^j V^l V^m + O(\|\gamma\|^5),
\]

(4.15)

where all the terms but the first one \((V^k)\) vanish at the center of normal coordinates.

Next Lemma 4.3.3 allows us to estimate the distance between two nearby points. We use the property of minimizing geodesics parametrized by arc-length, for which their length is equal to their speed, Remark 3.5.4.

**Lemma 4.3.3** Let \((M, g)\) be complete and \((U, (x^i))\) be any normal coordinate chart centered at \( s \in M \). If the points \( p \) and \( q \) are sufficiently close to \( s \) then the square of distance \( d(p, q)^2 \) can be written as

\[
d(p, q)^2 = \delta_{ij} (p^i - q^i)(p^j - q^j) - \frac{1}{3} R^{kl}_{ijkj} p^i q^j + \frac{1}{12} \partial_t R^{kl}_{ijkj} p^i q^j (p^r + q^r) + O\left(\{p, q\}^6\right),
\]

where the components of the Riemannian curvature tensor \( R^{kl}_{ijkj} \) and its derivatives are calculated at \( s \).

**Proof:** From Remark 3.5.4 it follows that to find out the distance between \( p \) and \( q \) it suffices to calculate the length of the initial velocity vector \( \gamma(0) \) of the geodesic \( \gamma: [0, 1] \to M \) from \( p \) to \( q \). By Lemma 4.3.2

\[
\|\gamma\|^2 = g_{ij} \dot{\gamma}^i \dot{\gamma}^j = g_{ij} \dot{d}^i \dot{d}^j + g_{ij} \dot{\Gamma}^i_{js} \dot{d}^s \dot{d}^s + \frac{1}{3} g_{ij} \partial_t \dot{\Gamma}^i_{js} \dot{d}^s \dot{d}^s \dot{d}^s + \frac{1}{12} g_{ij} \partial_t \partial_t \dot{\Gamma}^i_{js} \dot{d}^s \dot{d}^s \dot{d}^s \dot{d}^s + O\left(\{p, q\}^6\right),
\]

(4.16)

where the metric \( g \) and the Christoffel symbols \( \Gamma \) are calculated at \( p \). We derive the Christoffel symbols \( \Gamma \) from the metric \( g \) according to the formula

\[
\dot{\Gamma}^k_{ij} = \frac{1}{2} g^{kl} \left( \partial_k g_{jl} + \partial_j g_{kl} - \partial_l g_{ij} \right).
\]

(4.17)

Recall that the Riemannian metric \( g \) in normal coordinates (cf. [96]) can be written as

\[
g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{iklj} x^i x^j - \frac{1}{6} \partial_m R_{iklj} x^i x^j x^m + O\left(\|x\|^4\right).
\]

(4.18)

The second term of (4.16) simplifies to

\[
g_{ij} \dot{d}^i \dot{\Gamma}^j_{js} \dot{d}^s \dot{d}^s = \frac{1}{2} g_{ij} g^{jl} \left( \partial_r g_{sl} + \partial_s g_{rl} - \partial_l g_{rs} \right) \dot{d}^r \dot{d}^s \dot{d}^s
\]

\[
= \frac{1}{2} \left( \partial_r g_{sl} + \partial_s g_{rl} - \partial_l g_{rs} \right) \dot{d}^r \dot{d}^s \dot{d}^s = \frac{1}{2} \partial_r g_{sl} \dot{d}^r \dot{d}^s \dot{d}^s.
\]
By (4.18) it follows that \( \partial_t g_{jk} \) is of order \( O(p) \) so the third term of (4.16) simplifies to

\[
\frac{1}{3} g_{ij} d^i \partial_t \Gamma^j_{rs} d^r d^s d^t = \frac{1}{6} g_{ij} \partial_l \left( g^{jk} \left( \partial_r g_{sk} + \partial_s g_{rk} - \partial_k g_{rs} \right) \right) d^i d^r d^s d^t
\]

\[
= \frac{1}{6} g_{ij} \partial_l g^{jk} \left( \partial_r g_{sk} + \partial_s g_{rk} - \partial_k g_{rs} \right) d^i d^r d^s d^t
\]

\[
+ \frac{1}{6} g_{ij} g^{jk} \left( \partial_l t g_{sk} + \partial_l s g_{rk} - \partial_l k g_{rs} \right) d^i d^r d^s d^t
\]

\[
= \frac{1}{6} \left( \partial_l t g_{si} + \partial_l g_{ti} - \partial_l g_{st} \right) d^i d^r d^s d^t + O\left( \{p, q\}^6 \right)
\]

\[
= \frac{1}{6} \partial_l \partial_t g_{si} d^i d^r d^s d^t + O\left( \{p, q\}^6 \right)
\]

and similarly for the last term of (4.16)

\[
\frac{1}{12} g_{ij} d^i \partial_l \partial_m \Gamma^j_{rs} d^r d^s d^t d^m = \frac{1}{24} g_{ij} \partial_l \partial_m \left( g^{jk} \left( \partial_r g_{sk} + \partial_s g_{rk} - \partial_k g_{rs} \right) \right) d^i d^r d^s d^t d^m
\]

\[
= \frac{1}{24} \partial_l \partial_m \partial_t g_{si} d^i d^r d^s d^t d^m + O\left( \{p, q\}^6 \right).
\]

The expression (4.16) of the square of the length of the velocity vector \( \dot{\gamma} \) becomes

\[
||\dot{\gamma}||^2 = g_{ij} \dot{d}^i \dot{d}^j + \frac{1}{2} \partial_r g_{ij} \dot{d}^i \dot{d}^j + \frac{1}{6} \partial_s \partial_r g_{ij} \dot{d}^i \dot{d}^j \dot{d}^s
\]

\[
+ \frac{1}{24} \partial_l \partial_m \partial_t g_{ij} \dot{d}^i \dot{d}^j \dot{d}^s \dot{d}^t + O\left( \{p, q\}^6 \right).
\]

From (4.18) and symmetries of the Riemannian curvature tensor (Theorem 3.4.2) we derive

\[
\partial_r g_{ij} \dot{d}^i \dot{d}^j = -\frac{1}{6} \partial_r R_{ijkl} \dot{p}^k \dot{p}^l \dot{q}^i \dot{q}^j + O\left( \{p, q\}^6 \right),
\]

\[
\partial_s \partial_r g_{ij} \dot{d}^i \dot{d}^j \dot{d}^s = 0 + O\left( \{p, q\}^6 \right) \quad \text{and}
\]

\[
\partial_q \partial_s \partial_r g_{ij} \dot{d}^i \dot{d}^j \dot{d}^s \dot{d}^t = 0 + O\left( \{p, q\}^6 \right).
\]

Plugging the above expressions into (4.19) will complete the proof. \( Q.E.D. \)

The result of Lemma 4.3.3 allows us to estimate the function \( \Phi_Q \). In the following proof of an estimate of a critical point of \( \Phi_Q \) we will need the following result.

**Theorem 4.3.4** ([8]) *Let \( \xi: B^n \to \mathbb{R}^n \) be a continuous vector field on the n-dimensional unit ball. Assume that on the boundary \( S^{n-1} \) of \( B^n \), this vector field points outward

\[
||x|| = 1 \quad \text{then} \quad \langle x, \xi(x) \rangle \geq 0.
\]

Then \( \xi \) has a zero

\[
x_0 \in B^n \quad \text{such that} \quad \xi(x_0) = 0.
\]*
**Proof:** (of Theorem 4.3.1) Given a finite set \( Q \subset \mathcal{U} \) define a function \( \tilde{\Phi}_Q : M \to \mathbb{R} \) to be

\[
\tilde{\Phi}_Q(x) \overset{\text{def}}{=} \frac{1}{\#Q} \sum_{q \in Q} \left( \delta_{ij}(x^i - q^i)(x^j - q^j) - \frac{1}{3} R_{ijkl} x^k x^l q^i q^j - \frac{1}{12} \partial_t R_{ijkl} x^k q^i q^j (q^r + x^r) \right),
\]

where the components of the Riemannian curvature tensor and its derivatives are calculated at the origin of the normal neighbourhood \( \mathcal{U} \). Let \( |R| \) and \( |\partial R| \) be the upper bounds of absolute values of the components of the Riemannian curvature tensor and its derivatives calculated at the origin of the normal coordinates

\[
|R| \overset{\text{def}}{=} \max_{ijkl} |R_{ijkl}(0)| \quad \text{and} \quad |\partial R| \overset{\text{def}}{=} \max_{ijklr} |\partial_t R_{ijkl}(0)|.
\]

Let us assume that \( \delta > 0 \) is small enough so that \( B_{3\delta} \subset \mathcal{U} \) and the following inequality holds

\[
\frac{1}{6} \geq \frac{n^{7/2}}{3} |R| \cdot \delta^2 + \frac{5n^{9/2}}{24} |\partial R| \cdot \delta^3.
\]

We will show that if \( Q \subset B_{\delta} \) then \( \tilde{\Phi}_Q \) has a unique in \( B_{3\delta} \) point of local minimum \( \tilde{x}_0 \) such that \( \|\tilde{x}_0 - x_0\| \) is of order \( \mathcal{O}(\delta^3) \), where \( x_0 \) in normal coordinates is given by

\[
x_0^k \overset{\text{def}}{=} \tilde{Q}_E^k + \frac{1}{3\#Q} \sum_{q \in Q} R_{ijkl} Q_{klq}^i q^j + \frac{1}{12\#Q} \sum_{q \in Q} \partial_t R_{ijkl} Q_{klq}^i q^j (q^r + \tilde{Q}_E^r) + \frac{1}{24\#Q} \sum_{q \in Q} \partial_k R_{irlj} Q_{klq}^i q^j q^j.
\]

For any \( x \in \mathcal{U} \) and any orthonormal basis \( \{E_i\} \) of the tangent space \( T_x M \)

\[
d\tilde{\Phi}_Q(x)(E_k) = \frac{2}{\#Q} \sum_{q \in Q} \left( x^k - q^k - \frac{1}{3} R_{ijkl} x^l q^i q^j - \frac{1}{12} \partial_t R_{ijkl} x^l q^i q^j (q^r + x^r) - \frac{1}{24} \partial_k R_{irlj} x^l q^i q^j q^j \right)
\]

and

\[
d^2 \tilde{\Phi}_Q(x)(E_i, E_k) = \frac{2}{\#Q} \sum_{q \in Q} \left( \delta_{ik} - \frac{1}{3} R_{ijkq} q^q q^j - \frac{1}{12} \partial_t R_{ijkq} q^j q^j q^r - \frac{1}{6} \partial_q R_{ijkt} q^q q^q q^r \right) - \frac{1}{12} \partial_t R_{ijkl} x^l q^i q^j (q^r + x^r) - \frac{1}{12} \partial_t R_{ijkl} q^i q^j q^j q^r.
\]

We will show that the Hessian \( d^2 \tilde{\Phi}_Q \) is positive definite on \( B_{3\delta} \). We have the following inequalities

\[
\left| d^2 \tilde{\Phi}_Q(x)(E_i, E_i) \right| \geq 2 \left( 1 - \frac{n^2}{3} |R| \cdot \delta^2 - \frac{n^3}{12} |\partial R| \cdot \delta^3 - \frac{n^3}{4} |\partial R| \cdot \delta^2 \|x\| \right) \quad \text{and}
\]

\[
\left| d^2 \tilde{\Phi}_Q(x)(E_i, E_j) \right| \geq 2 \left( \frac{n^2}{3} |R| \cdot \delta^2 + \frac{n^3}{12} |\partial R| \cdot \delta^3 + \frac{n^3}{4} |\partial R| \cdot \delta^2 \|x\| \right)
\]
for $i \neq j$. Hence for $\|x\| < 3\delta$ there is

$$\left| d^2 \Phi_Q(x)(E_i, E_i) \right| - \sum_{i \neq j} \left| d^2 \Phi_Q(x)(E_i, E_j) \right| \geq 2 \left( 1 - \frac{n^3}{3} |R| \cdot \delta^2 - \frac{n^4}{12} |\partial R| \cdot \delta^3 - \frac{n^4}{4} |\partial R| \cdot \delta^2 \|x\| \right)$$

$$\geq 2 \left( 1 - \frac{n^3}{3} |R| \cdot \delta^2 - \left( \frac{1}{12} + \frac{3}{4} \right) \cdot n^4 |\partial R| \cdot \delta^3 \right)$$

$$> 2 \left( 1 - \frac{4}{6} \right) = \frac{2}{3},$$

where we used (4.21). As a consequence the Hessian $d^2 \Phi_Q$ is diagonally dominant on $B_{3\delta}$ and therefore it is non-singular and positive definite. Hence, in order to prove that $\tilde{\Phi}_Q$ has a unique in $B_{3\delta}$ point of local minimum, it is enough to prove that $\tilde{\Phi}_Q$ has a critical point $x_0$ in $B_{3\delta}$.

Let us introduce a continuous vector field $\xi: B^n \to \mathbb{R}^n$ on the $n$-dimensional unit ball defined as $\xi^k(x) \overset{\text{def}}{=} d \tilde{\Phi}_Q(x_0 + c \cdot \delta^5 \cdot x)(E_k)$, where the constant $c = 4/9 |R| n^6 + 1$. We will show that for sufficiently small $\delta > 0$ there is $\langle x, \xi(x) \rangle > 0$, for $\|x\| = 1$. This will prove that the vector field $\xi$ points outward on the boundary $S^{n-1}$ of $B^n$ and by Theorem 4.3.4 $\xi$ has at least one zero on $B^n$.

Write $d \tilde{\Phi}_Q(x)(E_k)$ in the following form:

$$d \tilde{\Phi}_Q(x)(E_k) = 2 \left( x^k - Q_k^k \right) - A^k(x) - B^k(x, x),$$

where $A$ is a linear and $B$ is a bi-linear operator. Namely

$$A^k(x) = \frac{1}{6 \#Q} \sum_{q \in Q} \left( 4 R_{ikl}q^i q^j + \partial_r R_{ikl} q^i q^j q^r \right) x^l$$

and

$$B^k(x, y) = \frac{1}{12 \#Q} \sum_{q \in Q} \left( 2 \partial_r R_{ikl} q^i q^j + \partial_k R_{ikl} q^i q^j \right) x^l y^r.$$

Note that $q^i$ is of order $O(\delta)$, and $A$ and $B$ are of order $O(\delta^2)$. Confining ourselves with expressions up to the order of $O(\delta^5)$ we have

$$\xi^k(x) = d \tilde{\Phi}_Q(Q_E + A(Q_E)/2 + B(Q_E)^2/2 + c \cdot \delta^5 \cdot x)(E_k)$$

$$= A^k(Q_E) + B^k(Q_E, Q_E) + 2c \cdot \delta^5 \cdot x^k$$

$$+ A^k(Q_E) + A(Q_E)/2 + B(Q_E)^2/2 + c \cdot \delta^5 \cdot x$$

$$- B^k(Q_E + A(Q_E)/2 + B(Q_E)^2/2 + c \cdot \delta^5 \cdot x)^2$$

$$= 2c \cdot \delta^5 \cdot x^k - \frac{1}{2} A^k(A(Q_E)) + O(\delta^6).$$
Hence

$$\langle x, \xi(x) \rangle \geq 2c \cdot \delta^5 - \frac{1}{2} \left( \frac{2}{3} n^3 |R| \delta^3 \right)^2 \delta + O(\delta^5) = 4 \left( c - \frac{4n^6}{9} |R|^2 \right) \delta^5 + O(\delta^6).$$

Choosing $\delta > 0$ so small that the term $O(\delta^6)$ is negligible with the others, from the above inequality it follows that $\langle x, \xi(x) \rangle > 0$, for all $\|x\| = 1$. By Theorem 4.3.4 it is clear that $\xi$ has zero in $B^n$ and therefore $\Phi_Q$ has a critical point at $\bar{x}_0 \in B_{c \cdot \delta^5}(x_0)$.

Note that (4.21) gives the upper bounds for norm of $x_0$. Indeed

$$\|x_0\| \leq \|\Phi_Q\| + \left( \sum_{i=1}^{n} \left( \frac{n^3}{3} |R| \cdot \delta^3 + \frac{5n^4}{24} |\partial R| \cdot \delta^4 \right)^2 \right)^{1/2} \leq \delta + \delta \cdot \left( \frac{n^3}{3} |R| \cdot \delta^3 + \frac{5n^4}{24} |\partial R| \cdot \delta^4 \right) \sqrt{n} \leq \delta \cdot \left( 1 + \frac{1}{6} \right).$$

Hence $\bar{x}_0 \in B_{3\delta}$.

It remains to show that there is a point $x$ of local minimum of $\Phi_Q$ such that $\|x - \bar{x}_0\| = O(\delta^5)$. By Lemma 4.3.3 $\Phi_Q = \tilde{\Phi}_Q + O(\delta^5)$. Define a mapping $h_\Phi: \mathbb{R} \times M \to \mathbb{R}^n$ to be

$$h_\Phi(0,x) \overset{\text{def}}{=} (1 - a) \cdot d\tilde{\Phi}_Q(x) + a \cdot d\Phi_Q(x).$$

Note that $h_\Phi$ is linear in the first argument and smooth in the second argument, and $h_\Phi(0,\bar{x}_0) = 0$. For $\alpha \in [0,1]$ and $x \in [\bar{x}_0 - \delta^2, \bar{x}_0 + \delta^2]$ partial derivative $D_2h_\Phi(\alpha,x)$ is nonsingular since

$$\|D_2h_\Phi(\alpha,x)\| \geq \|d^2\tilde{\Phi}_Q(x)\| - a \|d^2\Phi_Q(x) - d^2\Phi_Q(x)\| = \|1 + xO(\delta^2)\| - aO(\delta^4)$$

is positive for $\delta$ sufficiently small. By the implicit function theorem there is a unique $\mathcal{C}^1$ map $u: I \to M$ such that $h_\Phi(u(\alpha), \alpha) = 0$, where $I$ is an open interval containing $[0,1]$. Thus $\bar{x} = u(1)$ is a singular point of $\Phi_Q$ and $\|\bar{x} - \bar{x}_0\| = \|u(1) - u(0)\| < \|d\tilde{\Phi}_Q(x) - d\Phi_Q(x)\| = O(\delta^5)$. Since $\Phi_Q$ is smooth $\Phi_Q$ has local minimum at $\bar{x}$. This completes the proof.

Q.E.D.

The formula (4.8) can be expressed in a more compact form. In statistical applications samples are often characterized by variances and covariances. The covariance measures uniformity of a distribution of a sample. We would expect therefore that the difference between the Riemannian mean and Euclidean (embedded) mean will increase with the increase of covariance calculated on the points of $Q$ and with the increase of the curvature of the manifold. This will become apparent from Corollary 4.3.5.
Let us define Euclidean covariance \( \text{Cov}_Q \) as
\[
\text{Cov}_Q^{ij} \overset{\text{def}}{=} \frac{1}{\# Q} \sum_{q \in Q} (q^i - \bar{Q}_E^i) (q^j - \bar{Q}_E^j) = \frac{1}{\# Q} \sum_{q \in Q} (q^i q^j - \bar{Q}_E^i \bar{Q}_E^j). \tag{4.22}
\]

Then
\[
\frac{1}{\# Q} \sum_{q \in Q} (q^i q^j) = \text{Cov}_Q^{ij} + \bar{Q}_E^i \bar{Q}_E^j.
\]

Let us expand the definition of Euclidean covariance \( \text{Cov}_Q \) to
\[
\text{Cov}_Q^{ijk} \overset{\text{def}}{=} \frac{1}{\# Q} \sum_{q \in Q} (q^i q^j q^k) = \text{Cov}_Q^{ij} \bar{Q}_E^i + \bar{Q}_E^i \text{Cov}_Q^{jk} - \bar{Q}_E^i \text{Cov}_Q^{ij}.
\]

We can rewrite the formula (4.8) for a local minimum of \( \Phi_Q \) in terms of covariance.

**Corollary 4.3.5** Local minimum of \( \Phi_Q \) can be expressed as
\[
\bar{q}^k = \bar{Q}_E^k + \frac{1}{3} \sum_{ijl} R_{ijkl} \bar{Q}_E^l \text{Cov}_Q^{ij} + \frac{1}{12} \sum_{ijklr} \partial_r R_{ijkl} \bar{Q}_E^l (\text{Cov}_Q^{ij} + \bar{Q}_E^i \text{Cov}_Q^{jr} + 2 \bar{Q}_E^r \text{Cov}_Q^{ij}) + \frac{1}{24} \sum_{ijklr} \partial_k R_{irlj} \bar{Q}_E^l \text{Cov}_Q^{ij} + O(\delta^5). \tag{4.23}
\]

**Proof:** This is just a matter of recalculating the terms of (4.8):
\[
\frac{1}{3 \# Q} \sum_{q \in Q} R_{ijkl} \bar{Q}_E^l q^i q^j = \frac{1}{3} R_{ijkl} \bar{Q}_E^l (\text{Cov}_Q^{ij} + \bar{Q}_E^i \text{Cov}_Q^{jr} + 2 \bar{Q}_E^r \text{Cov}_Q^{ij}) = \frac{1}{3} R_{ijkl} \bar{Q}_E^l \text{Cov}_Q^{ij},
\]

where we used the symmetries of the Riemannian curvature tensor, cf. Theorem 3.4.2.

In a similar way
\[
\frac{1}{12 \# Q} \sum_{q \in Q} \partial_r R_{ijkl} \bar{Q}_E^l q^i q^j q^r = \frac{1}{12} \partial_r R_{ijkl} \bar{Q}_E^l \bar{Q}_E^r \text{Cov}_Q^{ij},
\]

and
\[
\frac{1}{24 \# Q} \sum_{q \in Q} \partial_k R_{irlj} \bar{Q}_E^l \bar{Q}_E^r q^i q^j = \frac{1}{24} \partial_k R_{irlj} \bar{Q}_E^l \bar{Q}_E^r \text{Cov}_Q^{ij},
\]

and finally
\[
\frac{1}{12 \# Q} \sum_{q \in Q} \partial_r R_{ijkl} \bar{Q}_E^l q^i q^j q^r
= \frac{1}{12} \partial_r R_{ijkl} \bar{Q}_E^l (\text{Cov}_Q^{ijr} + \bar{Q}_E^i \bar{Q}_E^r \bar{Q}_E^r + \bar{Q}_E^i \text{Cov}_Q^{jr} + \bar{Q}_E^r \text{Cov}_Q^{ij} + \bar{Q}_E^i \text{Cov}_Q^{jr} + \bar{Q}_E^r \text{Cov}_Q^{ij})
= \frac{1}{12} \partial_r R_{ijkl} \bar{Q}_E^l (\text{Cov}_Q^{ijr} + \bar{Q}_E^i \text{Cov}_Q^{jr} + \bar{Q}_E^r \text{Cov}_Q^{ij}).
\]
4.3 critical points and the riemannian mean

Figure 4.2: The variation $\Gamma(s, t)$

Adding these up completes the proof. \[ Q.E.D. \]

Corollary 4.3.5 suggests that if there is such a normal neighbourhood $\mathcal{U}$ centered at $p \in M$ that $Q \subset \mathcal{U}$ and the Euclidean mean is at the origin $\overline{Q}_E = 0$ then $\Phi_Q$ has a critical point near $p$. The following result, whose proof is based on [49], shows that the critical points of $\Phi_Q$ are precisely the points where $\overline{Q}_E = 0$.

**Lemma 4.3.6** Let $(M, g)$ be complete. Suppose there is a normal neighbourhood $\mathcal{U}$ of $p \in M$ so that the finite set $Q \subset \mathcal{U}$. Let $\overline{Q}_E$ be Euclidean mean calculated in normal coordinates centered at $p$. Then $p$ is a critical point of $\Phi_Q$ if and only if $\overline{Q}_E = 0$.

**Proof:** We will show first, that for any point $x$ sufficiently close to $p$

$$d\Phi_Q(x) = -\frac{2}{\#Q} \sum_{q \in Q} \exp^{-1}_x q,$$  \hspace{1cm} (4.24)

where $\exp^{-1}_x q$ is the tangent vector of the geodesic from $x$ to $q$.

In this proof we shall use the notation introduced in Section 3.5. Let $\gamma: (-\varepsilon, \varepsilon) \to M$ be a geodesic through the point $x \in \mathcal{U}$, i.e., $\gamma(0) = x$. Let $q \in \mathcal{U}$ be any point of $Q$. Consider a family of geodesics $\Gamma: [0, 1] \times (-\varepsilon, \varepsilon) \to M$ from $q$ to $\gamma(t)$ defined as

$$\Gamma(s, t) \overset{\text{def}}{=} \exp_q \left( s \cdot \exp^{-1}_q (\gamma(t)) \right),$$

where $\exp^{-1}_q (\gamma(t))$ denotes the tangent vector of the geodesic from $q$ to $\gamma(t)$, Figure 4.2. The family $\Gamma$ defines two collections of curves: $\Gamma_s(t) \overset{\text{def}}{=} \Gamma(s, t)$ defined on $(-\varepsilon, \varepsilon)$ by setting $s =$ constant and $\Gamma_t(s) \overset{\text{def}}{=} \Gamma(s, t)$ defined on $[0, 1]$ by setting $t =$ constant. Since $\mathcal{U}$ is the normal neighbourhood of $p$ then $p$ and $q$ are non conjugate points along a
unique minimizing geodesic from \( q \) to \( p \). Then for a sufficiently small \( \varepsilon > 0 \) there is a unique minimizing geodesic \( \Gamma_t(s) \) from \( q \) to \( \gamma(t) \), for \( x \) close enough to \( p \), cf. [45]. To shorten the notation, let us denote

\[
T(s, t) \overset{\text{def}}{=} \frac{d}{dt} \Gamma_t(s) \quad \text{and} \quad S(s, t) \overset{\text{def}}{=} \frac{d}{ds} \Gamma_t(s),
\]

cf. Section 3.5. For any \( q \in Q \), using the above notation, we may write

\[
d(q, \gamma(t))^2 = \langle S(s, t) , S(s, t) \rangle = \int_0^1 \langle S(s, t) , S(s, t) \rangle ds,
\]

since \( \langle S(s, t) , S(s, t) \rangle \) is a square of the norm of the velocity vector of a geodesic and hence it does not depend on \( s \). By the compatibility of the connection, the Symmetry Lemma (Lemma 3.5.5) and from the fact that \( D_s S(s, t) = 0 \) (since \( T_t(s) \) is a geodesic) it follows that

\[
\frac{d}{dt} d(q, \gamma(t))^2 = 2 \int_0^1 \langle D_t S(s, t) , S(s, t) \rangle ds = 2 \int_0^1 \langle D_s T(s, t) , S(s, t) \rangle ds = 2 \int_0^1 \langle T(s, t) , S(s, t) \rangle ds.
\]

Finally, from \( T(0, t) = 0 \) clearly

\[
\frac{d}{dt} d(q, \gamma(t))^2 = 2 \langle T(1, t) , S(1, t) \rangle = -2 \langle \dot{\gamma}(t) , \exp_{\gamma(t)}^{-1} q \rangle,
\]

(4.25)

where \( S(1, t) = -\exp_{\gamma(t)}^{-1} q \) is the tangent vector of the geodesic from \( q \) to \( \gamma(t) \). From the definition of \( \Phi_Q \) we obtain the following

\[
\frac{d}{dt} \Phi_Q(\gamma(t)) = \frac{1}{\#Q} \sum_{q \in Q} \frac{d}{dt} d(q, \gamma(t))^2 = -\frac{2}{\#Q} \sum_{q \in Q} \langle \dot{\gamma}(t) , \exp_{\gamma(t)}^{-1} q \rangle.
\]

Because \( \dot{\gamma}(0) \) is an arbitrary vector in \( T_x M \), putting \( t = 0 \) proves (4.24). Therefore

\[
d\Phi_Q(p)^i = -\frac{2}{\#Q} \sum_{q \in Q} q^i,
\]

in terms of the normal coordinates centered at \( p \). Thus \( p \) is a critical point of \( \Phi_Q \) if and only if \( \sum_{q \in Q} q^i = 0 \), for all \( i \). This completes the proof. \( \text{Q.E.D.} \)

The hypothesis of Lemma 4.3.6 allow points of the neighbourhood \( \mathcal{U} \) to be conjugate to the points of \( Q \) in \( \mathcal{U} \). Therefore the critical point is not necessarily a point of local minimum. This is clearly demonstrated by the following example.

**Example 4.3.7** Consider a unit sphere \( S^2 \subset \mathbb{R}^3 \) with a normal neighbourhood centered at the north pole \( p = (0, 0, 1) \). Let \( Q = \{q_1, q_2\} \), where \( q_1 = (\sqrt{1 - z^2}, 0, z) \) and \( q_2 = (-\sqrt{1 - z^2}, 0, z) \) are two opposite points at the same longitude, Figure 4.3 on the next page. Point \( p \) is a critical point of \( \Phi_Q \) and \( Q_E = 0 \) calculated in the normal coordinates centered at \( p \):
4.3 critical points and the riemannian mean

Figure 4.3: For a set of points on a sphere which are symmetrically distributed at the same longitude the north pole is a critical point of $\Phi_Q$

- if $0 < z < 1$ (northern hemisphere) then $\Phi_Q$ has minimum at $p$;
- if $-1 < z < 0$ (southern hemisphere) then $\Phi_Q$ has maximum at $p$, and
- if $z = 0$ (points lie on the equator) then $\Phi_Q$ has minimum on the great circle passing through $p$.

As an immediate consequence of the Lemma 4.3.6 one has the following.

**Corollary 4.3.8** If there is such a normal chart $(U, (x^i))$ centered at $p \in \overline{Q}$ such that $Q \subset U$, then the Euclidean mean $\overline{Q}_E$ calculated in these coordinates coincides with the origin. In particular, if the Riemannian mean $\overline{Q}$ consists of a single point then $\overline{Q} = \{\overline{Q}_E\}$, where $\overline{Q}_E$ is calculated in the normal coordinates centered at $\overline{Q}$.

**Remark 4.3.9** In view of Lemma 4.3.6 the connection between the Riemannian mean and the Riemannian barycenter introduced by Corcuera & Kendall [28] becomes clear (cf. Remark 4.1.3). Set the probability measure $\mu: M \to \mathbb{R}$ to be zero everywhere except for a finite set $Q \subset M$

$$
\mu(x) = \begin{cases} 
\frac{1}{\#Q} & \text{if } x \in Q; \\
0 & \text{otherwise.}
\end{cases}
$$
In this case the energy functional (4.2) becomes

$$J(w) = \int_M d(w, x)^2 \mu(dx) = \frac{1}{\#Q} \sum_{q \in Q} d(w, q)^2 = \Phi_Q(w).$$

Suppose there is a normal neighbourhood $U$ of $w \in M$ such that $Q \subset U$. By (4.24) in the proof of Lemma 4.3.6 $\Phi_Q$ has a critical point at $w$ if and only if

$$\sum_{q \in Q} \exp^{-1} q = 0.$$

Note that the barycenter condition holds since from the definition of the probability measure $\tilde{\mu} : T_w M \to \mathbb{R}$ we have $\tilde{\mu}(\exp^{-1} x) = \mu(x)$ and setting $V = \exp^{-1} x$ yields

$$\int_{T_w M} V \tilde{\mu}(dV) = \frac{1}{\#Q} \sum_{q \in Q} \exp^{-1} q = 0.$$

Thus we have proven that the Riemannian mean is a barycenter. The converse is not true. In fact the Riemannian barycenter is defined even if the exponential map fails to be 1–1, for example in the cut locus. We shall further investigate the points of local minimum of $\Phi_Q$ in Section 4.6. We will prove that the Riemannian mean has no points in common with the cut locus of $Q$, see Theorem 4.6.2.

By requesting the upper limit to the diameter of $Q$, depending on the sectional curvature on $U \subset M$, in the hypothesis of Lemma 4.3.6 it can be ensured that $\Phi_Q$ has exactly one point of minimum, for any Riemannian complete manifold. This agrees with our earlier results, e.g., Theorem 4.3.1, where we assume that the diameter of $Q$ is small. In order to study the critical points of $\Phi_Q$ we shall investigate its second derivative—the Hessian.

### 4.3.2 Second derivatives—the Hessian of $\Phi_Q$

To further investigate the properties of $\Phi_Q$ we again study its second derivative $d^2 \Phi_Q$. The estimates of the upper bounds can be found in [49]. The formula (4.31), needed for the proof of Theorem 4.4.1, was derived by the author.

We shall use the same notation as in Section 3.5 and the proof of Lemma 4.3.6.

By (4.25) it follows

$$\frac{d^2}{dt^2} d(q, \gamma(t))^2 = 2 \frac{d}{dt} \langle T(1, t), S(1, t) \rangle = 2 (\langle D_t T(1, t), S(1, t) \rangle + \langle T(1, t), D_t S(1, t) \rangle).$$

The $D_t T(1, t) \equiv 0$ clearly follows from the way $T(s, t)$ was defined, since $T(1, t) = \gamma(t)$ is a geodesic. By the Symmetry Lemma (Lemma 3.5.5) the above expression simplifies
Since $\Gamma(s,t)$ is a variation of geodesics, the mapping $s \mapsto T(s,t)$ is a family of Jacobi fields, (cf. [69]). Let $J(s) \overset{\text{def}}{=} T(s,t)$ denote the Jacobi field along the geodesic $\Gamma_t(s)$, where $J(0) = 0$ and $J(1) = T(1,t) = \dot{\gamma}(t)$. The above equality can now be expressed in terms of $J$ as follows

$$\frac{d^2}{dt^2}d(q, \gamma(t))^2 = 2 \langle T(1,t), D_s T(1,t) \rangle.$$ (4.26)

Let $J^\top$ denote the tangential component of the Jacobi field $J$ and $J^\perp$ denote the normal component of the Jacobi field $J$. We shall consider the two components of $J$ separately. By the Jacobi equation (3.6) and symmetries of the Riemannian curvature tensor (Theorem 3.4.2)

$$0 = \langle D_s^2 J, S(s,t) \rangle + Rm(J, S(s,t), S(s,t), S(s,t)) = \langle D_s^2 J, S(s,t) \rangle.$$

Hence the tangential component $J^\top$ is a linear function of $s$:

$$J^\top(s) = J^\top(0) + s D_s J^\top(0).$$

In our case $J(0) = 0$ therefore $D_s J^\top(1) = D_s J^\top(0) = J^\top(1)$ and

$$\langle J^\top(1), D_s J^\top(1) \rangle = \langle J^\top(1), J^\top(1) \rangle.$$ (4.27)

From (4.27) we conclude that if $V \in T_p M$ is parallel to a minimizing geodesic from $q$ to $p$ then $d^2 \Phi_Q(V,V) = 2 \langle V, V \rangle$.

The normal component $J^\perp$ is more difficult to analyse. We will use Comparison Theorems (cf. Section 3.7) to find bounds for the normal Jacobi fields. The inequalities (3.8) and (3.9) give an upper and lower bounds, respectively. These estimates allow us to find bounds for the Hessian $d^2 \Phi_Q$. We shall start with the lower estimate.

Let the upper bound of the sectional curvature along $\gamma$ be positive, $\Delta > 0$, then $S_{\Delta}(s) = \Delta^{-1/2} \sin \Delta^{1/2} s$ and $S_{\Delta}(s) > 0$, for $s \in (0, \frac{1}{2} \pi \Delta^{-1/2})$. Let $\gamma: [0,1] \to M$ be a geodesic of length $\ell$ then $\tilde{\gamma}: [0,\ell]$ defined as $\tilde{\gamma}(s) = \gamma(s/\ell)$ is a unit speed $\|\tilde{\gamma}\| = \ell^{-1} \|\dot{\gamma}\| = 1$ so that

$$\tilde{J}(s) = J(s/\ell) \quad \text{and} \quad \tilde{T}(s) = T(s/\ell).$$

For any $0 < s_0 < \frac{1}{2} \pi \Delta^{-1/2}$ the hypothesis of Theorem 3.7.1 are satisfied and from (3.9) we get

$$\langle \overline{T'}(s) \cdot u, \overline{T}(s) \cdot u \rangle \geq \ell \cdot \frac{S'_{\Delta}(s)}{S_{\Delta}(s)} \cdot \langle \overline{T}(s) \cdot u, \overline{T}(s) \cdot u \rangle.$$

Note also that $S'_{\Delta}(s) > 0$, for $s \in (0, s_0)$. Applying the above inequality to the normal Jacobi field, yields

$$\langle D_s J^\perp(1), J^\perp(1) \rangle \geq \ell \cdot \frac{S'_{\Delta}(\ell)}{S_{\Delta}(\ell)} \cdot \langle J^\perp(1), J^\perp(1) \rangle.$$
Let $B_\rho(p)$ be an open Riemannian ball centered at $p$ and let $\gamma$ be a minimizing geodesic in $B_\rho(p)$. Hence the length of $\gamma$ is less than $2\rho$ we get an analogue of (4.27) for the normal component of the Jacobi field

$$\left\langle J^\perp(1), D_s J^\perp(1) \right\rangle \geq 2\rho \cdot \frac{S'_\Delta(2\rho)}{S_\Delta(2\rho)} \cdot \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle.$$  \hfill (4.28)

Finally, (4.26) and (4.27) combined with (4.28) imply that the Hessian $d^2\Phi_Q(p)$ is positive definite

$$d^2\Phi_Q(p)(V, V) > 0, \quad \text{for any non-zero } V \in T_pM,$$ \hfill (4.29)

in the open ball $B_\rho(p)$, where $\rho < \frac{1}{4}\pi \Delta^{-1/2}$.

For the upper estimate we will proceed as before but use (3.8) instead. Suppose that the lower bound of the sectional curvature is negative $C > -\delta$, where $\delta > 0$. For any $0 < s_0 < \frac{1}{2}\pi \Delta^{-1/2}$ the hypothesis of Theorem 3.7.1 are satisfied and from (3.8) we get

$$\left\langle \mathbf{T}(s) \cdot u, \mathbf{T}(s) \cdot u \right\rangle \leq \ell \cdot \frac{S'_\delta(s)}{S_\delta(s)} \cdot \left\langle \mathbf{T}(s) \cdot u, \mathbf{T}(s) \cdot u \right\rangle,$$

where $S_\delta(s) = \delta^{-1/2}\sinh \delta^{1/2}s$. Note that $S_\delta(s) > 0$, for any $s > 0$. As in the previous case

$$\left\langle D_s J^\perp(1), J^\perp(1) \right\rangle \leq \ell \cdot \frac{S'_\delta(\ell)}{S_\delta(\ell)} \cdot \left\langle J^\perp(1), J^\perp(1) \right\rangle$$

and in the ball $B_\rho(p)$ we get the upper bound for the normal component of the Jacobi field

$$\left\langle J^\perp(1), D_s J^\perp(1) \right\rangle \leq 2\rho \cdot \frac{S'_\delta(2\rho)}{S_\delta(2\rho)} \cdot \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle.$$ \hfill (4.30)

Note that the function $s \cdot S'_\delta(s)/S_\delta(s)$ is increasing, for positive $s$. Direct calculations show that if $0 < \rho < \frac{1}{4}\pi \delta^{-1/2}$ then $\left\langle J^\perp(1), D_s J^\perp(1) \right\rangle \leq 2 \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle$. Finally, by (4.26) and from (4.27) combined with (4.30) implies that the Hessian $d^2\Phi_Q(p)$ satisfies

$$d^2\Phi_Q(p)(V, V) < 4 \left\langle V, V \right\rangle, \quad \text{for any non-zero } V \in T_pM,$$ \hfill (4.31)

in the open ball $B_\rho(p)$, where $\rho < \frac{1}{4}\pi \Delta^{-1/2}$ and $\rho < \frac{1}{4}\pi \delta^{-1/2}$.

As a consequence of (4.29) one has the following.

**Corollary 4.3.10 (Karcher [49])** If $Q \subset B_\varrho \subset M$, where $\varrho > 0$ is sufficiently small so that the open Riemannian ball $B_\varrho$ is convex and if $\varrho < \pi/4 \cdot C^{-1/2}$ in case of positive sectional curvature, $C > 0$, then $\Phi_Q$ is a convex function on $B_\varrho$ and therefore $\Phi_Q$ has unique point of local minimum in $B_\varrho$. 
Kendall [51] studies convexity of $\Phi_Q$ and improves the above result as follows. Let $B_R$ be a closed Riemannian ball of radius $R$ and centered at $p$ in a complete Riemannian manifold and let $C^2$ be a supremum of sectional curvatures in $B_R$ (zero if the supremum is negative). Then $B_R$ is a regular geodesic ball if: (i) $CR < \pi/2$ and (ii) the cut locus of $p$ does not meet $B_R$. Kendall shows that if $Q$ is contained in a regular geodesic ball $B_R$ then $\Phi_Q$ has unique point of local minimum in $B_R$.

As an illustration of the use of Corollary 4.3.10 and methods of calculating the sectional curvature, we will briefly discuss some examples of symmetric spaces.

**Remark 4.3.11** It should be noted that symmetric spaces are a special class of homogeneous spaces—$\mathfrak{G}/H$, where $\mathfrak{G}$ is a connected Lie group (isometry group) and $H$ a closed subgroup of $\mathfrak{G}$. If the metric on $\mathfrak{G}/H$ is induced by the bi-invariant metric on $\mathfrak{G}$, then

$$Rm(X, Y, Y, X) = \frac{1}{4} \|[X, Y]\|_p^2 + \|[X, Y]\|_h^2,$$

(4.32)

where $g = p \oplus h$ is a canonical decomposition. In particular, the sectional curvature is non-negative (cf. [27]).

**Example 4.3.12** If $M$ is a compact (non-compact) symmetric Riemannian manifold then the sectional curvature, with respect to the Riemannian metric, is positive (non-positive), cf. [61]. In particular:

**The sphere** $S^R_n$. The diameter of the $n$-sphere $S^R_n$ is $\pi R$ (cf. [58]). From the symmetry of the sphere, it is also clear that for any two points $p, q \in S^R_n$ with $d(p, q) < \pi R$ there is exactly one minimizing geodesic in $S^R_n$. By the Whitehead's Theorem 3.8.3 it follows that any open ball $B_{\pi R/2} \subset S^R_n$ is convex (Definition 3.8.2). Since the sectional curvature $C$ of $S^R_n$ is constant and equal to $1/R^2$, we get $\pi/4 \cdot C^{-1/2} = \pi R/4$. Hence, by Corollary 4.3.10 if

$$Q \subset B_{\pi R/4} \subset S^R_n$$

then $\Phi_Q$ has a unique point of local minimum in $B_{\pi R/4}$. However, Buss & Fillmore [22] and Kendall [51] improve Karcher’s result for spheres and prove the existence and uniqueness of the minimum of $\Phi_Q$ in the open hemisphere $B_{\pi R/2}$.

**Complex projective space** $\mathbb{C}P^n$, cf. [27]. Geometrically, $\mathbb{C}P^n$ can be thought of as the collection of one-dimensional complex subspaces of $\mathbb{C}^{n+1}$. Complex projective space is the homogenous space $\mathbb{C}P^n = SU(n+1)/U(n)$, where $U(n)$ is the unitary group and $SU(n+1)$ is the subgroup of $U(n+1)$. The Lie algebra $u(n)$ can be identified with the space of $n \times n$ skew-hermitian matrices, while the Lie algebra $su(n+1)$ can be identified with the space of $(n + 1) \times (n + 1)$ skew-hermitian matrices having trace equal to zero.
The bi-invariant metric on $SU(n+1)$, defined by $(A,B) = -1/2 \text{trace } A \cdot B$, gives rise to the canonical decomposition $su(n+1) = p + u(n)$, where $p$ can be identified with the space of matrices of the form

$$
\begin{bmatrix}
0 & \cdots & 0 & \bar{x}_1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 & \bar{x}_n \\
\bar{x}_1 & \cdots & \bar{x}_n & 0
\end{bmatrix}.
$$

The multiplication by $i$ is a linear transformation $J: p \to p$. Clearly $J^2 = -1$, so $(X,Y) = (J(X), J(Y))$. Since $[p,p] \subset u(n)$ then $\mathbb{C}P^n$ is a symmetric space.

For any orthonormal $X,Y \in p$, $(\|X\| = 1 = \|Y\|$ and $(X,Y) = 0)$, as a consequence of Remark 4.3.11 one has the following

$$
Rm(X,Y,Y,X) = \frac{1}{4} \|[X,Y]\|^2_p = \frac{1}{4} + \frac{3}{4} (J(X), Y)^2.
$$

We conclude that

$$
Rm(X,Y,Y,X) = \begin{cases} 
\frac{1}{4} & \text{if } (J(X), Y) = 0, \text{ and;} \\
1 & \text{if } (J(X), Y) = 1.
\end{cases}
$$

By Theorem 3.8.3 any open ball $B_{\pi/2} \subset \mathbb{C}P^n$ is convex, see also [53]. Since the sectional curvature of $\mathbb{C}P^n$ is no greater than 1, by Corollary 4.3.10, if $Q \subset B_{\pi/4} \subset \mathbb{C}P^n$

then $\Phi_Q$ has a unique point of local minimum in $B_{\pi/4}$.

The orthogonal group $SO(n)$. Let $\mathfrak{g}$ be a Lie group with a bi-invariant Riemannian metric. The sectional curvature

$$
Rm(X,Y,Y,X) = \frac{1}{4} \|[X,Y]\|^2
$$

is always non-negative (cf. [27, 69, 68]). In a case of $SO(3)$, the bi-invariant metric has constant sectional curvature $C = 1/4$. Therefore, by the previous discussion it is clear that if $Q \subset B_{\pi/2} \subset SO(3)$

then $\Phi_Q$ has a unique point of local minimum in $B_{\pi/2}$.

The only simply connected symmetric spaces having positive curvature are the spheres, complex and quaternionic projective spaces, and the Cayley plane. Except for the spheres, they have canonical metrics with sectional curvature varying between $\frac{1}{4}$ and 1, cf. [27].
4.3.3 The Riemannian mean on spheres and hyperbolic spaces

We will discuss local minima of $\Phi_Q$ for the n-sphere $S^n_R$ and the hyperbolic space $H^n_R$. Note that the sectional curvature $C$ of both of these manifolds is constant, cf. [58]. In such a case the expression for the Riemannian curvature tensor has a particularly simple form

$$R_{ijkl} = C (g_{ik}g_{jk} - g_{ik}g_{jl}).$$  \hspace{1cm} (4.33)

Formula (4.23) for the local minimum of $\Phi_Q$ simplifies to

$$q_k = \bar{Q}_E + \frac{1}{3} \sum_{ijl} R_{iklj} \bar{Q}_E^{ij} \text{Cov}^{ij}_Q + \mathcal{O}(\delta^5).$$

We can express the above result in a coordinate-free way

$$\bar{q} = \bar{Q}_E + \frac{C}{3} (\bar{Q}_E \text{trace Cov}_Q - \text{Cov}_Q \cdot \bar{Q}_E) + \mathcal{O}(\delta^5),$$  \hspace{1cm} (4.34)

where the ‘·’ is the standard multiplication of a matrix by a vector, whose components are calculated in normal coordinates. We determine a local minimum of $\Phi_Q$ for the two classes of model Riemannian manifolds.

- Firstly let us consider the sphere $S^n_R$ of radius $R$. Its sectional curvature is $1/R^2$, thus according to (4.34) $\Phi_Q$ has a local minimum at

$$\bar{q} = \bar{Q}_E + \frac{1}{3 R^2} (\bar{Q}_E \text{trace Cov}_Q - \text{Cov}_Q \cdot \bar{Q}_E) + \mathcal{O}(\delta^5).$$  \hspace{1cm} (4.35)

- Next consider the hyperbolic space $H^n_R$. Its sectional curvature is $-1/R^2$, thus according to (4.34) $\Phi_Q$ has a local minimum at

$$\bar{q} = \bar{Q}_E - \frac{1}{3 R^2} (\bar{Q}_E \text{trace Cov}_Q - \text{Cov}_Q \cdot \bar{Q}_E) + \mathcal{O}(\delta^5).$$  \hspace{1cm} (4.36)

By Corollary 4.3.6 this is the only minimum of $\Phi_Q$.

Formulae (4.35) and (4.36) give an expression for local minima of $\Phi_Q$. In view with Corollary 4.1.6 it is clear that (4.36) is an approximation of the Riemannian mean for the hyperbolic space. By Corollary 4.3.10 if the set $Q \subset S^n_R$ lies in the Riemannian
ball \( B_\rho \), where the radius \( \rho < \pi R/4 \), then the Riemannian mean contains just a single point. Thus is this case (4.35) gives approximate expression for the Riemannian mean. While Fisher [41] takes \( \bar{Q}_E \) as a mean value for spherical data, formula (4.35) shows that the mean related to the spherical geometry, the Riemannian mean, differs from \( \bar{Q}_E \) by terms proportional to the covariance \( \text{Cov}_Q \) of a sample \( Q \) and sectional curvature. This is illustrated by examples in the next section.

### 4.3.4 Examples

We present two examples of application of the Riemannian mean to illustrate how the results of the previous sections can be applied in practice. We chose experimental data published in Fisher [41] and Anderson & Sclove [6], see Appendix B. A brief comparison between mean values calculated in different ways is summarised in Table 4.1 on the facing page and Table 4.2 on page 76. In both examples below we used the approximations (4.35) and (4.36) to the Riemannian means disregarding the error terms.

#### Example 4.3.13 \((S^2)\)

As an illustration of the use of the results of Section 4.3.3 we derived the Riemannian mean for a spherical data \( Q \) of 26 points of the magnetic remanence\(^3\) measurements, see Table B.1 on page 158. The covariance (4.22) calculated in the normal coordinates centered in the south pole \( p = (0,0,-1) \) is

\[
\text{Cov}_Q = \begin{bmatrix}
0.0178 & 0.0049 \\
0.0049 & 0.0117
\end{bmatrix}.
\]

The calculated difference between the Riemannian mean \( \bar{Q} \) and the embedded mean \( \bar{Q}_E \) is \((-0.0013,-0.0012)\). A comparison of three different means is presented in Table 4.1 on the next page. The Fisher’s mean (projected mean), the Riemannian mean and embedded mean are expressed in normal, geological and Euclidean coordinates. The Fisher’s mean is derived according to (2.2). The Riemannian mean was obtained by minimizing \( \Phi_Q \) and its value agrees with (4.35) disregarding the error term. Figure 4.4 on the facing page shows the data points, the Riemannian mean is where the shaded lines intersect and the Fisher’s mean is marked by ‘●’.

#### Example 4.3.14 \((\mathbb{H}^2)\)

We want to compare the embedded and the Riemannian means for a Poincaré half-plane. To derive transformation to normal coordinates in \( \mathbb{H}^2 \) it will be helpful to consider the three following models of hyperbolic spaces, cf. [82].

\(^3\)When certain materials are exposed to a powerful magnetic field, traces of that influence may remain in the object, even after the field has been removed. These traces are termed remanence.
Figure 4.4: Lambert-Schmidt (equal-area) projection of the magnetic remanence measurements

<table>
<thead>
<tr>
<th></th>
<th>Normal $x^1$</th>
<th>Normal $x^2$</th>
<th>Geological dec.</th>
<th>Geological inc</th>
<th>Euclidean $x$</th>
<th>Euclidean $y$</th>
<th>Euclidean $z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fisher's mean</td>
<td>-0.4643</td>
<td>-0.3349</td>
<td>144.20</td>
<td>57.20</td>
<td>-0.4394</td>
<td>-0.3169</td>
<td>-0.8406</td>
</tr>
<tr>
<td>Riemannian mean</td>
<td>-0.4697</td>
<td>-0.3393</td>
<td>144.16</td>
<td>56.80</td>
<td>-0.4439</td>
<td>-0.3206</td>
<td>-0.8368</td>
</tr>
<tr>
<td>Embedded mean</td>
<td>-0.4685</td>
<td>-0.3381</td>
<td>144.18</td>
<td>56.90</td>
<td>-0.4429</td>
<td>-0.3196</td>
<td>-0.8377</td>
</tr>
</tbody>
</table>

Table 4.1: Comparison of the Fisher's, embedded and the Riemannian mean in the sphere $S^2$
The Poincaré half-plane model: \((x, y)\), where \(-\infty < x < \infty\) and \(0 < y < \infty\), the metric is given by

\[
h = \frac{(dx)^2 + (dy)^2}{y^2}.\]

The polar-coordinates model: \((r, \theta)\), where \(0 \leq r < \infty\) and \(-\pi < \theta \leq \pi\), the metric is given by

\[
h = (dr)^2 + (\sinh r)^2(d\theta)^2.\]

The Poincaré disc model: \((u, v)\), where \(u^2 + v^2 < 1\), the metric is given by

\[
h = 4 \frac{(du)^2 + (dv)^2}{(1-u^2-v^2)^2}.\]

The transformation from the polar-coordinates to the Poincaré disc is

\[
(r, \theta) \mapsto (u, v), \quad \text{where} \quad u = \tanh \frac{r}{2} \cdot \cos \theta \quad \text{and} \quad v = \tanh \frac{r}{2} \cdot \sin \theta.
\]

Similarly, from the Poincaré disc to the half-plane we shall use

\[
w = u + iw \mapsto z = x + iy, \quad \text{where} \quad z = \frac{i + w}{i - w}.
\]

Composing the two mappings above we obtain the transformation from the polar-coordinates to the Poincaré half-plane as follows

\[
(r, \theta) \mapsto (x, y), \quad \text{where} \quad x = \frac{\cos \theta \cdot \sinh r}{\cosh r - \sin \theta \cdot \sinh r} \quad \text{and} \quad y = \frac{1}{\cosh r - \sin \theta \cdot \sinh r}.
\]

Note that the center of the polar-coordinates is mapped to \((0, 1)\) of the half-plane.
Data of 10 samples from the normal distribution (cf. Table B.2 on page 159) was transformed to the standard scores by

\[ (\mu, \sigma) \mapsto \left( \frac{\mu - \hat{\mu}}{\hat{\sigma}\sqrt{n}}, \frac{\sigma}{\sqrt{n}} \right), \]

where \( \hat{\mu} \) and \( \hat{\sigma} \) is the mean and standard deviation of the whole population, respectively. Both of them are known. Note that this transformation maps the parameters of the normal distribution \((\mu, \sigma)\) to the neighbourhood of the center of the polar-coordinates \((0, 1)\), Figure 4.6. Hence this transformation is well suited for calculation of the Riemannian mean.

A comparison of three different means is presented on Table 4.2 on the next page. The Euclidean mean and the embedded mean were calculated in the half-plane and normal coordinates, respectively. The Riemannian mean was derived from (4.36) setting \( R = 1 \), where the covariance calculated in the normal coordinates centered at \( p = (0, 1) \) is

\[
\text{Cov}_Q = \begin{bmatrix}
0.0032 & 0.0035 \\
0.0035 & 0.0441
\end{bmatrix}.
\]

By (4.36) the difference between the Riemannian mean \( \bar{Q} \) and the embedded mean \( \bar{Q}_E \) is \((0.0014, -0.0001)\).

Figure 4.6 illustrates the data points, the Riemannian mean is where the shaded lines intersect and the Euclidean mean is marked by ‘•’.

Remark 4.3.15 Let \( M \) be the space of normal distributions on \( \mathbb{R} \) with the density function

\[
f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left( -\frac{(x - \mu)^2}{2\sigma^2} \right),
\]
Table 4.2: Comparison of the Euclidean, embedded and the Riemannian mean in the Poincaré half-plane

<table>
<thead>
<tr>
<th>Normal</th>
<th>Radial</th>
<th>half-plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^1$</td>
<td>$q^2$</td>
<td>$r$</td>
</tr>
<tr>
<td>Euclidean mean:</td>
<td>0.0015</td>
<td>-0.0125</td>
</tr>
<tr>
<td>Embedded mean:</td>
<td>0.0939</td>
<td>-0.0280</td>
</tr>
<tr>
<td>Riemannian mean:</td>
<td>0.0925</td>
<td>-0.0278</td>
</tr>
</tbody>
</table>

where $\mu$ and $\sigma$ is the mean and standard deviation, respectively. If we introduce the Riemannian metric $g$ defined as the inverse of the Fisher information matrix then $(M, g)$ is the Riemannian structure, where the metric is

$$g = \left(\frac{d\mu}{\sigma^2} + 2\frac{d\sigma}{\sigma^2}\right)^2.$$

Hence the space of normal distributions on $\mathbb{R}$ is the Poincaré half-plane with $\mu$ and $\sigma$ as the coordinates, cf. [17, 70]. The factor 2 can be easily eliminated by the transformation $\sigma \mapsto \sigma^{\sqrt{2}}$.

4.4 An Iterative Method of Calculating the Riemannian Mean

In this section an iterative method of deriving local minima of $\Phi_Q$ is investigated. The convergence of the iteration is proved and in Section 4.4.1 computational examples are given. Let us start with the following observation. The formula (4.23) tells us that if we choose normal coordinates such that the Euclidean mean is zero then the Riemannian mean is located at the origin. In such a coordinate system the two means coincide. This property of the Riemannian mean is a foundation for the two iterative algorithms for computing the spherical weighted average introduced in Buss & Fillmore [22]. (Section 2.3.4 gives a brief description of the spherical weighted averages.) The first Buss & Fillmore’s algorithm, investigated in this section in a more general setting, can be described as follows:

**Step 1:** choose the Euclidean mean to be the first approximation, say $x_0$;

**Step 2:** for any $k \geq 0$, choose normal coordinates centered at $x_k$;

**Step 3:** calculate the Euclidean mean in this coordinate system;

**Step 4:** take the Euclidean mean as the next iteration $x_{k+1}$ and go to Step 2.
Experimental results obtained by Buss & Fillmore show linear convergence rate of the above algorithm when applied to small samples on $S^2$ and $S^3$. The second algorithm of Buss & Fillmore, based on the Newton descent, is of quadratic convergence. The following Theorem 4.4.1 specifies the necessary conditions for $\{x_k\}$ to converge to a point of local minimum of $\Phi_Q$ on a complete Riemannian manifold. We shall conclude this section with examples of symmetric spaces where the above algorithm successfully calculates points of local minima.

**Theorem 4.4.1** Let $M$ be a complete Riemannian manifold. If there exists a Riemannian ball $B_\rho$ of radius $\rho > 0$ such that $Q \subset B_\rho \subset M$ and the following conditions are satisfied:

i) $\rho < \pi/4 \cdot |C|^{-1/2}$, for all non zero sectional curvatures $C$ in $B_\rho$ (if all $C$ in $B_\rho$ are zero this condition is satisfied automatically), and

ii) there is a point $x_0 \in B_\rho$ such that $\Phi_Q(x_0) < \Phi_Q(y)$, for any $y \in \partial B_\rho$,

then the series $\{x_k\}$ converges to $\bar{x} \in B_\rho$, and $\bar{x}$ is a point of local minimum of $\Phi_Q$.

**Proof:** From the hypothesis (i), by (4.29) and (4.31), for any $p \in B_\rho$ and any non-zero $V \in T_p M$ the Hessian $d^2\Phi_Q$ satisfies $0 < d^2\Phi_Q(p)(V, V) < 4 \langle V, V \rangle$. Set

$$M = \inf_{y \in \partial B_\rho} \Phi_Q(y),$$

where $\partial B_\rho$ is the boundary of the Riemannian ball $B_\rho$. $M$ is well defined since $\Phi_Q$ is continuous and $\partial B_\rho$ is compact. From the hypothesis (ii) it is also clear that $\Phi_Q$ has at least one point of local minimum in the interior of $B_\rho$. Denote $x_{k+1} = \phi(x_k)$, where $\phi$ is a step of iteration. We will show, that if there is $x_0 \in B_\rho$ such that $\Phi_Q(x_0) < M$, then for any $k \geq 0$ point $x_k \in B_\rho$ and $\Phi_Q(x_k) \geq \Phi_Q(x_{k+1})$, where equality holds if and only if $x_k = x_{k+1} = x$ is the point of local minimum of $\Phi_Q$. From the compactness of the Riemannian ball $\overline{B}_\rho$ the series $\{\Phi_Q(x_k)\}$ converges to $\Phi_Q(\bar{x})$, for some $\bar{x} \in B_\rho$. Finally, we will show that $\{x_k\}$ converges to $\bar{x}$ and $\bar{x}$ is a point of local minimum of $\Phi_Q$.

For any $k > 0$ define a function

$$f_k : [0, 1] \to \mathbb{R}, \text{ where } f_k(t) \overset{\text{def}}{=} \Phi_Q(\gamma_k(t)),$$

where $\gamma_k : [0, 1] \to M$ is a Riemannian geodesic from $x_k$ to $x_{k+1}$ defined as $\gamma_k(t) \overset{\text{def}}{=} \exp_{x_k}(t \cdot \exp_{x_k}^{-1}(x_{k+1}))$. From the Taylor series, for any $s \in [0, 1]$

$$f_k(s) = f_k(0) + sf'_k(0) + \frac{s^2}{2}f''_k(\xi), \text{ where } \xi \in (0, s). \quad (4.37)$$
Choose normal coordinates centered at $x_k \in B_e$. The Riemannian geodesic $\gamma_k$ is a radial geodesic $t \mapsto t \cdot x_{k+1} + (1-t) \cdot x_k$. In these coordinates $x^i_k = 0$ and

$$x^i_{k+1} = \phi(x_k)^i = \frac{1}{\#Q} \sum_{q \in Q} q^i.$$  \hfill (4.38)

If $x_k$ is a point of local minimum then by Lemma 4.3.6 $x_{k+1} = 0 = x_k$ and so the series $\{x_k\}$ converges to $x \in B_e$. Suppose that $x_k$ is not a critical point of $\Phi_Q$, then $x_{k+1} \neq 0$. By (4.24) we get

$$f'_k(0) = \left. \frac{d}{dt} \right|_{t=0} \Phi_Q(t \cdot x_{k+1} + (1-t) \cdot x_k) = d\Phi_Q(x_k)(x_{k+1})$$

$$= - \frac{2}{\#Q} \delta_{ij} \sum_{q \in Q} q^i \frac{1}{\#Q} \sum_{q \in Q} q^j = -2 \|x_{k+1}\|^2$$

and, for any $\xi \in (0, s)$

$$f''_k(\xi) = \frac{d^2}{dt^2} \bigg|_{t=\xi} \Phi_Q(t \cdot x_{k+1} + (1-t) \cdot x_k) = \xi^2 d^2\Phi_Q(\gamma_k(\xi))(x_{k+1})^2.$$  

For any geodesic segment $\gamma([0, s]) \subset \overline{B}_e$ by (4.31) we get

$$d^2\Phi_Q(\gamma_k(\xi))(x_{k+1})^2 < 4 \|x_{k+1}\|^2,$$

for any $\xi \in (0, s)$ and from the Taylor series (4.37)

$$f_k(s) < f_k(0) - 2s \|x_{k+1}\|^2 + 2s^2 \xi^2 \|x_{k+1}\|^2$$

$$= f_k(0) - 2s(1 - s\xi^2) \|x_{k+1}\|^2$$

$$< f_k(0).$$  \hfill (4.39)

Note, that (4.39) implies that the geodesic $\gamma_k: [0, 1] \to M$ lies entirely in $B_e$. Suppose the opposite, that $\gamma_k$ intersects the boundary of the Riemannian ball $\partial B_e$. Let $\gamma_k(s_0) \in \partial B_e$ be the first point of intersection. From the way we defined $\mathcal{M}$ and from the hypothesis $f_k(0) \leq \mathcal{M} \leq f_k(s_0)$, which contradicts (4.39). This shows that $x_{k+1} = \gamma_k(1) \in B_e$ and $\Phi_Q(x_{k+1}) = f_k(1) < f_k(0) = \Phi_Q(x_k)$. Therefore, the series $\{\Phi_Q(x_k)\}$ is non-increasing and since it is bound by zero, it converges to $\Phi_Q(\bar{x})$, for some $\bar{x} \in B_e$. It remains to show that $\bar{x}$ is a point of local minimum of $\Phi_Q$ and the series $\{x_k\}$ converges to $\bar{x} \in B_e$. Suppose, that $\bar{x}$ is not a point of local minimum of $\Phi_Q$ in $B_e$, then $\phi(\bar{x}) \neq \bar{x}$ and since $\Phi_Q(\bar{x}) < \mathcal{M}$ then $\phi(\bar{x}) \in B_e$ and $\Phi_Q(\phi(\bar{x})) < \Phi_Q(\bar{x})$. This contradicts that $\{\Phi_Q(x_k)\}$ converges to $\Phi_Q(\bar{x})$. Therefore $\{x_k\}$ converges to a point of local minimum of $\Phi_Q$ what was to show.

Q.E.D.

Note that the above algorithm does not accumulate any roundoff errors. Each step is independent as the new coordinates of each data point have to be calculated again. The proof of Theorem 4.4.1 does not give any indication about how fast $\{x_k\}$ converges. We will compare the convergence on symmetrical spaces of low dimensions, $\text{SO}(3)$ and $\text{SO}(4)$, in the following examples.
4.4 an iterative method of calculating the riemannian mean

4.4.1 Examples: Lie groups

In this section the iterative method of finding local minima of $\Phi_Q$, introduced in Section 4.4, is demonstrated and shown in practice. In the case when $(M, g)$ is a Lie group with left-invariant metric, the series $\{x_k\}$ can be defined as:

$$x_{k+1} = x_k \cdot \exp \left( \frac{1}{\#Q} \sum_{q \in Q} \log \left(x_k^{-1} \cdot q \right) \right),$$  \hspace{1cm} \text{(4.40)}

where, for classical groups, exp and log are calculated on the matrices representing group elements. It is worth noting that (4.40) converges to $x$ if and only if

$$\sum_{q \in Q} \log \left(x^{-1} \cdot q \right) = 0.$$

The following examples are numerical applications of (4.40) to classical groups.

**Example 4.4.2** Choose $\mathfrak{g}$ to be $SO(3)$, the group of orthogonal matrices of the determinant equal to one. We randomly generated five matrices:

$$s_1 = \begin{bmatrix} 0.797686 & 0.593766 & 0.104152 \\ -0.602831 & 0.786087 & 0.136606 \\ -0.000760396 & -0.17178 & 0.985135 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0.88913 & 0.399243 & 0.223724 \\ -0.420265 & 0.905805 & 0.0537903 \\ -0.181175 & -0.14185 & 0.973167 \end{bmatrix},$$

$$s_3 = \begin{bmatrix} 0.405352 & 0.48169 & 0.776959 \\ -0.707638 & 0.703401 & -0.0669007 \\ -0.578739 & -0.522687 & 0.625987 \end{bmatrix}, \quad s_4 = \begin{bmatrix} 0.565098 & 0.509979 & 0.648526 \\ -0.648948 & 0.760147 & -0.0322887 \\ -0.509441 & -0.402613 & 0.760508 \end{bmatrix},$$

$$s_5 = \begin{bmatrix} 0.829972 & 0.146253 & 0.53829 \\ -0.293241 & 0.935307 & 0.198017 \\ -0.474506 & -0.322197 & 0.819166 \end{bmatrix}.$$

We used Mathematica, setting precision to $10^{-19}$, to perform calculations according to (4.40). Starting with $x_0$ set to the identity, the following results show fast convergence:

**step 1**

$$x_1 = \begin{bmatrix} 0.747775 & 0.452718 & 0.485673 \\ -0.548518 & 0.833394 & 0.0676904 \\ -0.374113 & -0.317018 & 0.871516 \end{bmatrix}, \quad x_1 - x_0 = \begin{bmatrix} -0.252225 & 0.452718 & 0.485673 \\ -0.548518 & -0.166606 & 0.0676904 \\ -0.374113 & -0.317018 & -0.128484 \end{bmatrix};$$

**step 2**

$$x_2 = \begin{bmatrix} 0.745715 & 0.455201 & 0.48652 \\ -0.551746 & 0.831239 & 0.0679606 \\ -0.373479 & -0.319115 & 0.871022 \end{bmatrix}, \quad x_2 - x_1 = \begin{bmatrix} -0.00206058 & 0.00248279 & 0.000846381 \\ -0.00322778 & -0.00215547 & 0.000270175 \\ 0.000633978 & -0.00209687 & -0.000493495 \end{bmatrix};$$

**step 3**

$$x_3 = \begin{bmatrix} 0.745703 & 0.45522 & 0.486519 \\ -0.551777 & 0.831222 & 0.0679683 \\ -0.373465 & -0.319131 & 0.871022 \end{bmatrix}, \quad x_3 - x_2 = \begin{bmatrix} -0.0000112359 & 0.0000195003 & -1.02364 \times 10^{-6} \\ -0.0000245994 & -0.0000169643 & 7.6515 \times 10^{-6} \\ 0.0000139078 & -0.0000163454 & -2.527 \times 10^{-8} \end{bmatrix};$$
Table 4.3: Convergence of the iterative algorithm applied to points in $SO(4)$

step 4

$$x_4 = \begin{bmatrix} 0.745703 & 0.45522 & 0.486519 \\ -0.551771 & 0.831222 & 0.0679684 \\ -0.373465 & -0.319131 & 0.871022 \end{bmatrix}$$

$$x_4 - x_3 = \begin{bmatrix} -7.33144 \times 10^{-8} & 1.51409 \times 10^{-7} & -2.92968 \times 10^{-8} \\ -2.02881 \times 10^{-7} & -1.4346 \times 10^{-7} & 1.07456 \times 10^{-7} \\ 1.53356 \times 10^{-7} & -1.57688 \times 10^{-7} & 7.97892 \times 10^{-9} \end{bmatrix}$$

step 5

$$x_5 = \begin{bmatrix} 0.745703 & 0.45522 & 0.486519 \\ -0.551771 & 0.831222 & 0.0679684 \\ -0.373465 & -0.319131 & 0.871022 \end{bmatrix}$$

$$x_5 - x_4 = \begin{bmatrix} -5.20599 \times 10^{-10} & 1.15283 \times 10^{-9} & -2.80727 \times 10^{-10} \\ -1.73709 \times 10^{-9} & -1.26117 \times 10^{-9} & 1.32172 \times 10^{-9} \\ 1.52696 \times 10^{-9} & -1.64046 \times 10^{-9} & 0 \end{bmatrix}$$

step 6

$$x_6 = \begin{bmatrix} 0.745703 & 0.45522 & 0.486519 \\ -0.551771 & 0.831222 & 0.0679684 \\ -0.373465 & -0.319131 & 0.871022 \end{bmatrix}$$

$$x_6 - x_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The iterative method took only 6 steps to derive local minimum. The comparison of the differences between any two successive terms $x_{k+1} - x_k$ indicates that the series converged faster than linearly.

The above example is very encouraging as far as convergence is concerned. We are able to derive the Riemannian mean for $SO(3)$ within just few steps. But with the increase of the dimension of the manifold, the number of steps required for the series to converge is likely to increase as well, as the following example demonstrates.

Example 4.4.3 $SO(4)$. We generated three elements of $SO(4)$ by deriving $\exp$ of the following three skew-symmetric matrices

$$s_1 = \begin{bmatrix} 0 & 0.6 & -2.5 & 1.6 \\ -0.6 & 0 & 2.1 & 1.1 \\ 2.5 & -2.1 & 0 & -3.1 \\ -1.6 & -1.1 & 3.1 & 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 & -1.4 & 1.8 & 0.2 \\ 1.4 & 0 & -2.1 & -3. \\ -1.8 & 2.1 & 0 & 1. \\ -0.2 & 3. & -1. & 0 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 0 & 2.8 & 1.1 & 0.8 \\ -2.8 & 0 & 0.9 & -0.7 \\ -1.1 & -0.9 & 0 & -0.5 \\ -0.8 & 0.7 & 0.5 & 0 \end{bmatrix}$$
4.4 an iterative method of calculating the riemannian mean

\[
\ln \left( \|x_{n+1} - x_n\|^2 \right)
\]

regression line: \( y = 0.549 - 1.706 \cdot x; \)
\( R^2 = 1 \)

Figure 4.7: Rate of convergence of the iterative algorithm applied to points in SO(4)

On this occasion, the iteration took 26 steps. We concluded that the series \( \{x_k\} \) converge to a point of local minimum of \( \Phi_Q \)

\[
x = \begin{bmatrix}
0.307882 & -0.385181 & 0.188274 & -0.849351 \\
0.929145 & 0.207376 & 0.137799 & 0.273307 \\
-0.127873 & -0.416653 & 0.837992 & 0.328356 \\
0.159835 & -0.796888 & -0.493288 & 0.309982
\end{bmatrix}
\]

The Figure 4.7 and Table 4.3 on the facing page illustrate the rate of convergence of the iterative algorithm applied to the points of this example. Brief analysis performed for other sets of data suggests that the rate of convergence also depends on the distance of the data points from the origin (bounds of the absolute values of entries in the matrices) and possibly the covariance matrix (4.22). A method involving (4.34) applied to the points of this example converges in 18 steps instead of 26. More research is needed to investigate efficiencies of the above iterative algorithms and to compare them with the algorithms developed by Buss & Fillmore [22].

Recall that in Example 4.1.4 we derived a formula for \( \Phi_Q \) in \( \text{SO}(n) \) involving the Hausdorff series. We are now able to compare the results of the two methods of calculating local minima.

Example 4.4.4 We randomly generated three elements of \( \text{SO}(3) \):

\[
s_1 = \exp \begin{bmatrix}
0 & 1.60066 & -0.17133 \\
-1.60066 & 0 & -0.45014 \\
0.17133 & 0.45014 & 0
\end{bmatrix},
\]

\[
s_2 = \exp \begin{bmatrix}
0 & 2.14505 & 1.83805 \\
-2.14505 & 0 & 1.19673 \\
-1.83805 & -1.19673 & 0
\end{bmatrix},
\]

\[
s_3 = \exp \begin{bmatrix}
0 & 0.6088 & -0.06088 \\
-0.6088 & 0 & 1.28155 \\
0.06088 & 1.28158 & 0
\end{bmatrix}.
\]
The method based on Example 4.1.4 and approximate expression for the Hausdorff series gives

\[
\mathcal{S}_a = \begin{bmatrix}
-0.678119 & 0.732873 & 0.0552459 \\
-0.658327 & -0.572283 & -0.488976 \\
-0.326741 & -0.367954 & 0.870546
\end{bmatrix}
\]

as a critical point of $\Phi_Q$. The iterative method gives

\[
\mathcal{S}_i = \begin{bmatrix}
-0.632066 & 0.774755 & -0.015757 \\
-0.682778 & -0.566415 & -0.461506 \\
-0.366479 & -0.280944 & 0.886997
\end{bmatrix}
\]

Calculations of values of function $\Phi_S$ at this points brings:

$\Phi_S(\mathcal{S}_a) = 8.14574$ and $\Phi_S(\mathcal{S}_i) = 8.08724$.

The iterative method is more accurate since it does not accumulate roundoff errors. It is however more computationally expensive because it involves the exp and log operations. One would want to employ a hybrid method combining these two. In this approach $\mathcal{S}_a$ instead of the Euclidean mean would be taken as a starting point of the iterative method.

4.5 Convexity

Recall that a subset $\Omega$ of a Riemannian manifold $M$ is convex if for any $p, q \in \Omega$, there is a unique (in $M$) minimizing geodesic from $p$ to $q$ lying entirely in $\Omega$, cf. Definition 3.8.2. This section proves that for a complete Riemannian 2-manifold any convex set containing $Q$ that lies inside some normal neighbourhood of a critical point of $\Phi_Q$ also contains that critical point. The proof is geometrical and is based on facts of plane geometry:

- the center of mass of $Q$ lies inside the convex hull of $Q$, and
- a line passing through any interior point of a connected set $\Omega$ divides $\Omega$ into two or more separate connected subsets.

Recently Corcuera & Kendall [28] showed that for any geodesically convex set $\Omega \subseteq M$, where $M$ is a Riemannian 2-manifold $M$, such that: $\Omega$ has smooth boundary of codimension 1, and $\mu$ is a probability measure supported in $\Omega$, then every barycenter of $\mu$ (cf. Remark 4.1.3) is contained in $\Omega$. They also showed a counter example of this fact for a 3-manifold, cf. Remark 4.5.2. Our Lemma 4.5.1 and its proof arose independently of Corcuera & Kendall.
In Euclidean space the center of mass (2.7) of a set of points $Q$ lies inside the convex hull of $Q$. From Lemma 4.3.6 we conclude the Riemannian version of the above statement for a complete 2-manifold. The idea of the proof is to show that in the normal coordinates centered at the Riemannian mean, all radial geodesics from the center to the points of $Q$ lie entirely in any, geodesically convex set containing $Q$.

**Lemma 4.5.1** Let $(M, g)$ be a complete 2-manifold and $\Omega$ be a convex set containing $Q$. Suppose $\Phi_Q$ has a non-degenerate critical point $x \in M$. If there is a normal neighborhood $U$ of $x$ containing $\Omega$ such that any Riemannian geodesic in $U$ is minimizing then $x \in \Omega$.

**Proof:** Choose any normal coordinates on $U$ centered at $x$. In these coordinates $\Omega$ is a connected set in the plane. Any line segment in $U$ passing through the origin of the normal coordinates is a Riemannian geodesic and, by hypothesis, is minimizing.

Since $x$ is a non-degenerate critical point of $\Phi_Q$ then by Lemma 4.3.6 the Euclidean mean $\overline{Q}_E$ lies at the origin of the normal coordinates. Hence in these coordinates $x = \overline{Q}_E$. We will show that $\overline{Q}_E \in \Omega$. Consider the three separate cases.

If $Q = \{q\}$ contains just a single point then $\overline{Q}_E = q \in Q \subset \Omega$ and there is nothing to prove.
Suppose now that $Q = \{p, q\}$, where $p \neq q$. Let $\gamma_{pq}$ be a minimizing geodesic from $p$ to $q$ and let $\overline{pq}$ be a line segment from $p$ to $q$. From the plane geometry, the Euclidean mean $\overline{Q}_E$ is a middle point of $\overline{pq}$ and thus $\overline{Q}_E \in \overline{pq}$. Hence the line segment $\overline{pq}$ passes through the origin of the normal coordinates and therefore it is a Riemannian geodesic. From the convexity of $\Omega$ and the uniqueness of the minimizing geodesic it must be that $\overline{pq} = \gamma_{pq} \subset \Omega$, since $p, q \in \Omega$. This proves that $\overline{Q}_E \in \Omega$ for the case $Q = \{p, q\}$.

Finally, suppose that $Q$ has three or more points and $\overline{Q}_E \notin \Omega$. If all the points of $Q$ are co-linear then clearly $\overline{Q}_E$ is a point of a line segment $\overline{pq}$, for some $p, q \in Q$. By the same argument as for the two point case $\overline{Q}_E \in \Omega$, a contradiction. We may assume then that there are at least three non-co-linear points of $Q$. Let $H$ be the Euclidean convex hull of $Q$. From plane geometry, the Euclidean mean $\overline{Q}_E = x$ belongs to the interior of $H$. Choose any vertex $s$ of $H$ and draw the line $\ell_{sx}$ through $s$ and the origin $x$, Figure 4.8 on the page before. If the line $\ell_{sx}$ passes through any other vertex $q$ of $H$ then $\overline{Q}_E \in \overline{sq} \subset \Omega$, since $\overline{sq}$ is a Riemannian geodesic, $s, q \in \Omega$ and $\Omega$ is convex. This contradicts that $\overline{Q}_E \notin \Omega$. We may assume then that the line $\ell_{sx}$ crosses exactly one side $\overline{pq}$ of the convex hull $H$ and thus $\ell_{sx}$ separates $p$ and $q$ in $\Omega$. Let $\gamma_{pq}$ be a minimizing geodesic from $p$ to $q$. Since $p, q \in \Omega$ and $\Omega$ is convex then $\gamma_{pq} \subset \Omega$. Therefore, $\ell_{sx}$ and $\gamma_{pq}$ must intersect in at least one point. Let $s' \in \ell_{sx} \cap \gamma_{pq} \subset \Omega$ be a point of intersection. Then the line segment $\overline{s's}$ either contains $x$ or it does not. If $x \in \overline{s's}$ then $\overline{s's}$ is a Riemannian geodesic. But because $s', s \in \Omega$ we have $\overline{Q}_E \in \overline{s's} \subset \Omega$, which contradicts the assumption $\overline{Q}_E \notin \Omega$. Otherwise, since the line $\ell_{px}$ separates $s$ and $q$ in $\Omega$, $\ell_{px}$ also separates the two points $s', q \in \Omega$, and thus the line $\ell_{px}$ and the geodesic segment $\gamma_{s'q}$ must intersect. Let $p' \in \Omega$ be a point of intersection $p' \in \ell_{px} \cap \gamma_{s'q}$. But the line segment $\overline{pp'}$ is passing through the origin $\overline{Q}_E$ and thus it is a minimizing geodesic from $p$ to $p'$. Again, by uniqueness of minimizing geodesics there is $\overline{Q}_E \subset \overline{pp'} = \gamma_{pp'} \subset \Omega$, which leads to a contradiction. \(\text{Q.E.D.}\)

**Remark 4.5.2** Corcuera & Kendall [28] consider a projection of a measure $\mu$ on the tangent space at the center of mass. In particular, the problem of whether barycenters of probability measures supported by $\Omega$ also lie in $\Omega$ is investigated. Corcuera & Kendall obtain positive answer for 2-manifolds and present a counterexample for a 3-manifold.
4.6 Conjugate Points and Cut Loci

Recall that the conjugate points are precisely the images of singularities of the exponential map, Lemma 3.6.2. Noakes’ research [74] (unpublished) has shown that the spherical mean $\bar{Q}$ is not conjugate to any point of $Q$, i.e., $\bar{Q} \cap -Q$ is empty.

**Theorem 4.6.1 (Conjugate Points Theorem, Noakes)** Let $\bar{q} \in Q \subset S^n$. Then $-\bar{q}$ is not a point of local minimum of $\Phi_Q$.

The significance of Theorem 4.6.1 is not only computational. It is of interest to understand what geometrical conditions imposed on the finite set $Q$ ensure that the Riemannian mean does not have points in common with cut loci of the points of $Q$, cf. [28]. This section investigates cut points (cf. Section 3.8) in relation to the Riemannian mean. The result of this section is Theorem 4.6.2, the proof of is a generalization of the proof of Theorem 4.6.1 and it is based on non-uniqueness of minimizing geodesics between cut points (Theorem 3.8.1).

**Theorem 4.6.2** Let $M$ be a complete n-manifold, $Q$ be a finite set of points in $M$ and $p$ be a local minimum of $\Phi_Q$. If for all $q \in Q$, $p$ is not conjugate to $q$ along any minimizing geodesic from $p$ to $q$ then $Q$ has no common points with the cut locus of $p$.

**Proof:** (of Theorem 4.6.2) Suppose that there is a point $\bar{q} \in Q$ such that $\bar{q}$ belongs to the cut locus of $p$. We will show that $p$ is not a point of local minimum of $\Phi_Q$. 

\[ \begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.9.png}
\caption{Fundamental property of the cut locus: geodesics $\gamma_2 \neq \gamma_1$ are both minimizing}
\end{figure} \]
Let $\tilde{Q} = Q \setminus \{q\}$. Since $p$ is not conjugate to any point of $Q$ then (4.24) holds for $\tilde{Q}$ and we have

$$d\Phi_{\tilde{Q}}(p) = -\frac{2}{\#\tilde{Q}} \sum_{q \in \tilde{Q}} \exp^{-1}_p q,$$

where $\exp^{-1}_p q$ is the tangent vector of the geodesic from $p$ to $q$. By Theorem 3.8.1 there are two minimizing geodesics $\gamma_1, \gamma_2 : [a, b] \to M$ from $p$ to $\tilde{q}$ and $\dot{\gamma}_1(a) \neq \dot{\gamma}_2(a)$. Let $\gamma_1$ and $\gamma_2$ be given unit speed parametrization and set $V = \dot{\gamma}_1(a) - \dot{\gamma}_2(a)$, $V \in T_p M$ and $V \neq 0$, Figure 4.9 on the preceding page. Let $\exp_p tV$ be a geodesic defined on $[-\epsilon, \epsilon]$ whose initial velocity is $V$. Since $p$ is not conjugate to $\tilde{q}$ along $\gamma_1$ the formula (4.25) holds and thus for a small $\epsilon > 0$ we have

$$d(p, \exp_p (\epsilon V))^2 - d(p, \tilde{q})^2 \leq -2 \langle \epsilon V, \dot{\gamma}_1(a) \rangle + O(\epsilon^2) = -2 \epsilon (1 - \langle \dot{\gamma}_2(a), \dot{\gamma}_1(a) \rangle) + O(\epsilon^2),$$

where the inequality comes from the fact that a variation of $\gamma_1$ may not be minimizing even to the first order. Similarly moving away from $p$ by $-\epsilon V$ we see that

$$d(p, \exp_p (-\epsilon V))^2 - d(p, \tilde{q})^2 \leq -2 \langle -\epsilon V, \dot{\gamma}_2(a) \rangle + O(\epsilon^2) = -2 \epsilon (1 - \langle \dot{\gamma}_1(a), \dot{\gamma}_2(a) \rangle) + O(\epsilon^2).$$

And finally we obtain

$$\#Q \left( \Phi_Q(\exp_p (\epsilon V)) - \Phi_Q(p) \right) \leq -2 \epsilon (1 - \langle \dot{\gamma}_1(a), \dot{\gamma}_2(a) \rangle) + \epsilon \cdot \#Q \cdot d\Phi_{\tilde{Q}}(p) + O(\epsilon^2)$$

and

$$\#Q \left( \Phi_Q(\exp_p (-\epsilon V)) - \Phi_Q(p) \right) \leq -2 \epsilon (1 - \langle \dot{\gamma}_1(a), \dot{\gamma}_2(a) \rangle) - \epsilon \cdot \#Q \cdot d\Phi_{\tilde{Q}}(p) + O(\epsilon^2).$$

Choose $\epsilon$ so small that the $O(\epsilon^2)$ term is negligible in comparison with the others. Then $\Phi_Q(\exp_p (\epsilon V)) - \Phi_Q(p) < 0$ or $\Phi_Q(\exp_p (-\epsilon V)) - \Phi_Q(p) < 0$ and therefore $p$ is not a point of local minimum of $\Phi_Q(p)$. This completes the proof. \textit{Q.E.D.}

For spheres with the spherical distance Theorem 4.6.1 implies that $\Phi_Q$ is differentiable at the points of local minimum. Similarly, Theorem 4.6.2 states that for any complete manifold, either a point of local minimum of $\Phi_Q$ is conjugate to a point $q \in Q$ along some minimizing geodesic, or $\Phi_Q$ is differentiable. For example, on the cylinder $S^1 \times \mathbb{R}$, there are no conjugate points along any geodesic. By Theorem 4.6.2 $\Phi_Q : S^1 \times \mathbb{R} \to \mathbb{R}$ is differentiable at the points of local minimum.
4.7 The Riemannian Mean in Statistics

There is a growing interest in Riemannian geometry applied to statistics since parametric statistical models have a natural Riemannian structure given by Fisher information. Oiler & Corcuera [78] study notions of mean value and moments extended to random object taking values on a manifold. They obtain an intrinsic version of the Cramér-Rao lower bound depending on the curvature of the statistical model. As a consequence, Oiler & Corcuera obtain geometrical versions of the Rao-Blackwell and Lehmann-Scheffé theorems.

In this section we will discuss the relevance of the Riemannian mean to parametric statistical models. After a short discussion of one-parameter distributions in Section 4.7.1, we will derive the Riemannian mean in the space of the Poisson distributions, Section 4.7.2.

In statistical applications of the Riemannian geometry Amari [4] defines the $\alpha$-connection as:

$$
\Gamma_{ijm}^{\alpha} \overset{\text{def}}{=} E_p (\partial_j \partial_i \ell) + \frac{1-\alpha}{2} E_p (\partial_j \ell \partial_i \ell \partial_m \ell),
$$

where $\Gamma_{ijk} = \Gamma_{ij}^{m}g_{mk}$ are the Christoffel symbols and $\ell$ the log-likelihood (cf. Definition 3.9.1). The case $\alpha = 0$ corresponds to the Information Connection (which actually is the only Riemannian connection). Different values of $\alpha$ produces different link functions [50], the functions which relate parameters of the model to the mean. In the 1-dimension case Hougaard [46] investigates the transformation $\psi = f(\theta)$ for the following values of $\alpha$:

**Mixture Connection** corresponds to $\alpha = -1$. It is shown that $\psi$ parameterization can be estimated with zero bias asymptotically in a curved exponential family $E_p (\hat{\psi}) = \psi + O(n^{-1})$, where $n$ is the size of the sample.

**Skewness Connection** corresponds to $\alpha = -1/3$.

**Information Connection** corresponds to $\alpha = 0$. Let $Y = f(X)$ then the variance $V(Y) = \sigma^2(X) (f'(\mu_x))^2$. For $\psi = f(\theta)$ to have constant variance it has to be $f'(\theta) \propto \sqrt{\det J}$ and $\det J \propto \sigma^2(\theta)^{-1}$, where $J$ is the Fisher information matrix.

**Normal Connection** corresponds to $\alpha = 1/3$.

**Exponential Connection** corresponds to $\alpha = 1$.

The summary of link functions for the most common distributions are reproduced after [39] on Table 4.4 on the next page.
**Table 4.4:** Link functions corresponding to the Amari's α-connections for various distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>mean (μ)</th>
<th>canonical</th>
<th>constant variance</th>
<th>normal</th>
<th>skewness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>0</td>
<td>= μ</td>
<td>μ</td>
<td>μ</td>
<td>μ</td>
</tr>
<tr>
<td>Poisson</td>
<td>e^θ</td>
<td>ln(μ)</td>
<td>2√μ</td>
<td>3√μ</td>
<td>3/2μ²/³</td>
</tr>
<tr>
<td>Binomial</td>
<td>e^θ/(1 + e^θ)</td>
<td>ln(μ/(1 - μ))</td>
<td>2 sin⁻¹√μ</td>
<td>I_μ(1/3,1/3)</td>
<td>I_μ(2/3,2/3)</td>
</tr>
<tr>
<td>Gamma</td>
<td>-1/θ</td>
<td>= -1/μ</td>
<td>2 ln(μ)</td>
<td>-3μ⁻¹/³</td>
<td>-3μ⁻¹/³</td>
</tr>
<tr>
<td>Inverse</td>
<td>1/√⁻²θ</td>
<td>= 1/2μ²</td>
<td>-2/√μ</td>
<td>-1/√μ</td>
<td>ln(μ)</td>
</tr>
<tr>
<td>Gaussian</td>
<td>²⁻¹θ</td>
<td>= 1/2μ²</td>
<td>-2/√μ</td>
<td>-1/√μ</td>
<td>ln(μ)</td>
</tr>
</tbody>
</table>

\[ I_μ(α,b) = \int_0^x x^{-α}(1 - x)^{b-1}dx, \text{ for } μ ≤ π, α, b > 0 \text{ and } 0 ≤ x ≤ 1. \]

Section 4.7.2 investigates the connection between the Riemannian mean and the constant variance link function for the Poisson distribution. But first let us see why the **Information Connection** leads to the constant variance link function.

### 4.7.1 One-parameter estimator

We will show the relevance of the Riemannian mean to the constant variance link function. From a geometrical point of view, one-parameter estimator corresponds to one-dimensional geometry. Recall that if \( Z = f(Y) \) is a random variable, where \( f \) and \( Y \) are known, then by the Taylor series:

\[
f(Y) = f(μ) + (Y - μ)f'(μ) + \frac{(Y - μ)^2}{2}f''(μ) + O(|Y - μ|^3),
\]

where \( μ = E_p(Y) \). Calculating expected values of both sides we get

\[
E_p(f(Y) - f(μ))^2 \approx (f'(μ))^2 E_p(Y - μ)^2.
\]

(4.41)

For an efficient estimator, the Riemannian metric \( g \) is equal to the inverse of variance, hence (4.41) yields

\[
(V(Y))^{-1} \approx (V(f(Y)))^{-1} (f'(μ))^2 = g(f(μ)) (f'(μ))^2 = \|f'(μ)\|^2.
\]

Choose \( f: I \rightarrow ℝ \) to be a geodesic in a 1-manifold with the metric provided by the Fisher information matrix (3.10). Since all Riemannian geodesics are constant speed curves (cf. Lemma 3.5.3), then \( \|f'(μ)\|^2 \) is constant.

**Remark 4.7.1** Let \( λ \in ℝ \) be an efficient estimator, \( g \) be the metric defined by Fisher information and \( f: I \rightarrow ℝ \) is a geodesic on 1-manifold \((ℝ, g)\). Then the map \( f^{-1} \) is a variance stabilizing function.
We will make use of the above ideas in the next section, where the Poisson one-parameter distribution provides a good example.

### 4.7.2 Poisson distribution

We will illustrate the relation of the Riemannian mean to the expected value of a single variable with the example of the Poisson one-parameter distribution.

Let \( p(y) \) be the Poisson probability distribution in the form

\[
p(y) = \frac{\lambda^y e^{-\lambda}}{y!} ,
\]

where \( \lambda \) is the average value of \( y \). The characteristic (moment generating) function for the Poisson distribution is \( \exp(\lambda(e^t - 1)) \), cf. [64].

Suppose \( Y_1, Y_2, \ldots, Y_n \) is a random sample from the Poisson distribution with mean \( \lambda \), then

- the maximum likelihood estimator \( \hat{\lambda} \) for \( \lambda \) is \( \bar{Y} \);
- expected value of \( \hat{\lambda} \) is \( \lambda \) and variance of \( \hat{\lambda} \) is \( \lambda / n \).

For one estimated parameter the inverse of the Fisher information (3.10) is equal to

\[
I(\theta) = \left( nE_p \left( -\frac{\partial^2}{\partial \theta^2} \ln f(y) \right) \right)^{-1} ,
\]

where \( f(y) \) is a probability density function, cf. [64]. In the case of the Poisson distribution:

\[
\frac{\partial}{\partial \lambda} \frac{\lambda^y e^{-\lambda}}{y!} = \frac{1}{y!} \left( y \lambda^{y-1} e^{-\lambda} - \lambda^y e^{-\lambda} \right)
\]

so that

\[
\frac{\partial}{\partial \lambda} \ln p(y) = y \lambda^{-1} - 1 \quad \text{and} \quad \frac{\partial^2}{\partial \lambda^2} \ln p(y) = -y \lambda^{-2}
\]

thus

\[
I(\lambda) = \left( nE_p (y \lambda^{-2}) \right)^{-1} = \frac{\lambda^2}{nE_p(y)} = \frac{\lambda}{n}
\]

is equal to the variance. Such an estimator is said to be efficient.

We will derive the Riemannian mean in the manifold whose metric is induced by the Fisher information matrix (4.43). Let \( M \) be a Riemannian 1-manifold with a metric \( g \). In this case \( g: M \to \mathbb{R}_+ \) is a function. To find out the Riemannian distance we will
investigate Riemannian geodesics in $M = \mathbb{R}$. The only component of the Christoffel symbol $\Gamma(x)$, according to the formula (3.2) is equal to:

$$\Gamma(x) = \frac{1}{2} g(x) \left( \frac{dg(x)}{dx} + \frac{dg(x)}{dx} - \frac{dg(x)}{dx} \right) = \frac{1}{2} \frac{dg(x)}{dx}.$$

The equation (A.1) for a geodesic becomes:

$$\dot{x} = -\Gamma(x) \dot{x} = -\frac{1}{2g(x)} \frac{dg(x)}{dx} \cdot \dot{x}^2.$$

Thus by (4.44) the Riemannian metric for the Poisson distribution is $g(x) = n/x$. Hence the Riemannian geodesics satisfy the following differential equation

$$\ddot{x} = \frac{\dot{x}^2}{2x}.$$

It is easy to check that $x(t) = c^2(t+a)^2$ is a solution of (4.45) and the geodesic $\gamma: I \to \mathbb{R}$ from $p$ to $q$ can be expressed as

$$\gamma(t) = ((\sqrt{q} - \sqrt{p}) t + \sqrt{p})^2.$$

To derive the Riemannian distance it is enough to find the norm of the velocity vector $\dot{\gamma}(0)$

$$d(p, q)^2 = g(p)(\dot{\gamma}(0))^2 = \frac{n}{p} (2(\sqrt{q} - \sqrt{p})\sqrt{p})^2 = 4n(\sqrt{q} - \sqrt{p})^2.$$

The Riemannian mean minimizes the following function

$$\Phi_Q(x) = \frac{4n}{\#Q} \sum_{q \in Q} (\sqrt{x} - \sqrt{q})^2.$$

By standard methods of calculus we find that $\Phi_S$ has minimum at

$$\overline{Q} = \frac{1}{\#Q^2} \left( \sum_{q \in Q} \sqrt{q} \right)^2.$$

This shows that the Riemannian mean for $\lambda$ is Euclidean mean for $\sqrt{\lambda}$. Thus we have two probability measures $P \cong \mathbb{R}_+$ and $P' \cong \mathbb{R}_+$: $P$ estimates $\lambda$ through the Riemannian metric and $P'$ estimates $\sqrt{\lambda}$ through the Euclidean metric.

We shall conclude this section by investigating the estimator $\theta = \sqrt{\lambda}$. According to Definition 3.9.1, the log-likelihood function $\ell$ for $\theta$ is

$$\ell(p_1, p_2, \ldots, p_n) = 2 \ln \theta \sum_i y_i - n\theta^2 - \sum_i \ln y_i!$$

$$\frac{\partial}{\partial \theta} \ell(p_1, p_2, \ldots, p_n) = 2 \theta \sum_i y_i - 2n\theta.$$
Thus the maximum likelihood estimate of \( \sqrt{\lambda} \) for the Poisson distribution is

\[
\hat{\theta} = \sqrt{\frac{1}{n} \sum y_i} = \sqrt{\lambda}.
\]

Recall the following definition of a biased estimator.

**Definition 4.7.2 (Unbiased estimator, cf. [64])** Let \( \hat{\theta} \) be a point estimator of a parameter \( \theta \). Then \( \hat{\theta} \) is an unbiased estimator if \( E(\hat{\theta}) = \theta \). Otherwise \( \hat{\theta} \) is said to be biased.

We shall show that the estimate \( \hat{\theta} \) is biased by calculating the expected value

\[
E_p\left( \sqrt{\sum_i y_i} \right).
\]

Let \( Z = f(Y) \) be a random variable, where \( f \) and \( Y \) are known. By the Taylor series:

\[
f(Y) = f(\mu) + (Y - \mu)f'(\mu) + \frac{(Y - \mu)^2}{2}f''(\mu) + O\left(|Y - \mu|^3\right),
\]

where \( \mu = E_p(Y) \). The expected value of random variable \( Z \) can be expressed as

\[
E_p(Z) = f(\mu) + \frac{1}{2}V(Y)f''(\mu) + O\left(|Y - \mu|^3\right).
\]

Let us consider the square root transformation \( Z = f(Y) = \sqrt{Y} \) (cf. [52, page 253]). We have

\[
E_p(Z) = \mu^{1/2} + \frac{\mu^{-1/2}}{2} (Y - \mu) - \frac{\mu^{-3/2}}{4} (Y - \mu)^2 + O\left(|Y - \mu|^3\right).
\]

In the case of the Poisson distribution of the random variable \( Y \), the expected value \( E_p(Y) = \lambda \) and the variance \( V(Y) = \lambda \). Recall that for the Poisson distribution we have \( E_p(\bar{Y}) = \lambda \) and \( V(\bar{Y}) = \lambda/n \) (cf. [64]). From (4.46) it is easy to see that

\[
E_p\left( \sqrt{\bar{Y}} \right) = \sqrt{\lambda} - \frac{\lambda^{-3/2}}{4} \frac{\lambda}{n} + O(\lambda^3) = \sqrt{\lambda} \left(1 - \frac{1}{4n\lambda}\right) + O(\lambda^3).
\]

We conclude that the bias \( B \) of the estimator \( \sqrt{\lambda} \) for the Poisson distribution has the following approximation

\[
B = E_p\left( \sqrt{\lambda} \right) - \sqrt{\lambda} + O(\lambda^3) = -\frac{1}{4n\sqrt{\lambda}} + O(\lambda^3).
\]

Finally we shall find an estimate for the standard deviation \( \sigma_Z \). From (4.46) we get

\[
Z - E_p(Z) = (Y - \mu)f'(\mu) - \frac{(Y - \mu)^2}{2}f''(\mu) + O\left(|Y - \mu|^3\right).
\]
and hence
\[ \sigma_Z^2 = \sigma_Y^2 \left( \frac{1}{2} \mu^{-1/2} \right)^2 + O(\|Y - \mu\|^3) = \frac{1}{4} + O(\|Y - \mu\|^3). \]  

(4.47)

The above result (4.47) is an estimate of the variance of the Poisson distribution modified by the root transformation.

We have seen already in Example 4.3.14 followed by Remark 4.3.15 that the geometric approach to the space of normal distributions leads to the hyperbolic space—Poincaré half-plane. In this section we looked at one-parameter distributions and their relation with the one-dimensional geometry. We showed that the root transformation applied to the Poisson distribution coincides with the variance stabilizing link function, cf. Table 4.4 on page 88. This is no surprise in view with Remark 4.7.1 which explains the connection between Riemannian geodesics on one-dimensional manifolds and the variance stabilizing link functions for one parameter distributions.

### 4.8 Conclusion

The Riemannian mean provides an intrinsic quantity inferred from experimental data that represents the central tendency of a sample. This central tendency is defined in terms of the geometry of the space and is not related to any particular distribution. Given a family of samples, the Riemannian mean provides a geometrically intrinsic (i.e., preserved by isometries) discriminant.

The purpose of this chapter was to demonstrate how the simple concept of a mean value can be applied to samples of points in geometrical spaces. Starting from spherical means and the center of mass we proved a number of properties of the Riemannian mean. We showed how the Riemannian mean depends on the curvature tensor and on the covariance of the sample points. This phenomenon is illustrated with examples of manifolds of constant curvature: the sphere and the hyperbolic space. We introduced and proved the convergence of an iterative method of deriving the local minima of \( \Phi_Q \). This method can be used to derive points of the Riemannian mean for classical Lie groups and other abstract spaces. We also proved another important property of the Riemannian mean in the cut locus theorem. The cut locus theorem together with the Noakes' conjugate point theorem have both theoretical and practical significance. For example, they specify geometrical conditions that ensure the Riemannian mean has no common points with cut loci of the points of \( Q \). Finally, we proved the convexity property of the Riemannian mean for a complete 2-manifold. In brief, this property says that under certain conditions, any convex set containing \( Q \) also contains the Riemannian mean. The investigation of the Riemannian mean is
linked to other research of the Riemannian barycenter and the center of mass, especially in the area of statistics, cf. [70]. For example, the parametric statistical models have a natural Riemannian manifold structure. In a number of examples we illustrated applications of the Riemannian mean to one-parameter and two-parameters statistical models. Recent research, e.g., [17, 66, 78], shows growing interest in developing intrinsic methods of statistical estimations. These are geometrical methods related to Riemannian manifolds whose metric is induced from the Fisher information matrix.

The next part of the thesis investigates curves in Riemannian manifolds that are critical to some functionals. In that respect these Riemannian variational curves generalize Riemannian geodesics. After studying properties of the Riemannian variational curves, Chapter 6 presents their application to the problem of interpolation tangential directions in the plane \( \mathbb{R}^2 \).
definition of the riemannian mean and its properties
Part III

Curves and Projective Spaces
Chapter 5

Variational Curves in
Riemannian Spaces and Their
Properties

The notion of polynomial curves in Riemannian geometry is not well defined. Since a choice of coordinates may change the form of a polynomial to a more complicated function, it is clear that classification depending on the order of variables is not geometrically invariant. Consider the analogy between straight lines in Euclidean spaces and geodesics in Riemannian spaces. The form of a geodesic can be quite complex. One way of looking at geodesics is through their minimizing properties. These properties are independent of the choice of coordinates. We say that these properties are geometrically intrinsic, for example, the length minimizing property is preserved under isometries. Similarly, one can classify other types of curves by their minimizing (or variational) properties. Such a variational approach enables a geometrically intrinsic classification and provides scope for new theoretical developments.

This chapter extends the research on variational curves in Riemannian spaces. The subject of cubic splines in Riemannian geometry has been introduced in [77]. This was subsequently generalized to polynomial curves [23] and geometric splines [24, 90]. The new results presented here concern properties of these curves in particular spaces: the unit sphere $S^n$ and the space of rotations $SO(3)$ endowed with left-invariant metric.

This chapter is set out to achieve the following objectives:

* to introduce the subject of variational curves in Riemannian geometry,
to provide a variety of examples of variational curves in non-Euclidean spaces, and

to present new results which will be needed to study the interpolation in the projective space $\mathbb{R}P^2$, Chapter 6.

The chapter is organized as follows. Section 5.1 provides an overview of current research on variational curves in a geometrical setup. In particular, Section 5.1.1 describes the first results that led to the generalization of Riemannian geodesics. This new family of curves, having certain minimizing properties, was first studied by Noakes et al. [77]. Further research in this area has been continued by Brunnett & Crouch [19], Camarinha [23] and Silva et al. [90], to name just a few. Results concerning such generalizations are presented in Section 5.1.2. There seems to be a variety of names given to the class of curves satisfying minimizing properties in Riemannian spaces: splines or polynomial curves. Section 5.2 introduces a formal definition of a class of $\mathcal{D}^k$-curves, which is in a tradition with de Boor [33], a founder of studies of minimizing curves in Euclidean spaces. The $\mathcal{D}^k$-curves satisfy a system of Euler-Lagrange equations, obtained by Camarinha, Silva & Crouch [24]. We remark on important differences between splines in $\mathbb{R}^n$ and $\mathcal{D}^k$-curves in Riemannian spaces. Section 5.3 presents results concerning the simplest non-trivial class of $\mathcal{D}^1$-curves. In particular, Section 5.3.1 presents the result by Camarinha [23] addressing the problem of existence and uniqueness of $\mathcal{D}^1$-curves. We derive a system of ordinary differential equations of fourth-order satisfied by $\mathcal{D}^1$-curves in Section 5.3.2. This system of equations reduces to a simple form at the origin of normal coordinates. The remaining part of this chapter describes properties of $\mathcal{D}^1$-curves in the unit sphere, Section 5.3.3, and presents new results concerning $\mathcal{D}^1$ and $\mathcal{D}^2$-curves on Lie groups, Section 5.3.4.

Throughout this chapter we will assume that $M$ is a smooth manifold endowed with a fixed Riemannian metric $g$. All covariant derivatives are understood to be with respect to the Riemannian connection of $g$.

### 5.1 Geometric Splines

This section presents results concerning variational curves in non-Euclidean spaces investigated in [77, 23, 19, 90, 30, 24]. Section 5.3.2 and Section 5.3.4 provide the author's contribution to the variational curves on Lie groups. Section 5.3.3 describes properties of variational curves in the unit $n$-sphere, Lemma 5.3.4.
Let \((M, g)\) be a Riemannian structure. Let us consider the class of regular curves with non-vanishing derivatives. A regular curve is a smooth curve \(\gamma: I \to M\) such that \(\dot{\gamma}(t) \neq 0\), for \(t \in I\). For a regular curve \(\gamma: [a, b] \to M\) and \(k \geq 0\) define a functional

\[
\Phi^{(k)}(\gamma) \overset{\text{def}}{=} \int_{a}^{b} \left\langle D_{t}^{k} \dot{\gamma}, D_{t}^{k} \dot{\gamma} \right\rangle dt,
\]

where \(D_{t}^{0} \dot{\gamma}\) is assumed to be just \(\dot{\gamma}\). We will investigate a class of regular curves \(\gamma: [a, b] \to M\) which are critical for the functional \(\Phi^{(k)}\). For \(k = 0\) we have the classical problem of Riemannian geodesics.

In the literature, a number of variations to this problem were studied, cf. [19, 77, 23, 24, 30, 90]. The most significant are described in Section 5.1.1 and Section 5.1.2.

### 5.1.1 Cubic splines

Noakes et al. [77] define a cubic spline on a Riemannian manifold \(M\) as a series of smooth curves \(\gamma_i: [0, 1] \to M\) each minimizing the functional \(\Phi^{(1)}\). This definition is justified by considering Euclidean space \(M = \mathbb{R}\). Then the covariant derivative \(D_t\) along a curve \(\gamma\) becomes an ordinary derivative by the parameter \(t\) (often called time) and the functional \(\Phi^{(1)}\) becomes

\[
\Phi^{(1)}(\gamma) = \int_{0}^{1} (\dot{\gamma}(t))^2 dt,
\]
which is minimized by a cubic spline, cf. [1] and Example 5.2.3. Later on, in Section 5.3.2, we will show another property of curves critical for $\Phi^{(1)}$, demonstrating an analogy between cubics splines and polynomial curves of order three.

Noakes et al. derive the necessary conditions for a critical curve to be a cubic spline in the form of Euler-Lagrange ordinary differential equations

$$D_3^2 \dot{\gamma} + R(D_t \dot{\gamma}, \gamma) \dot{\gamma} = 0,$$

where $R$ is the Riemannian curvature endomorphism (Definition 3.4.1). In the special case of the Lie group $\mathbb{M} = \text{SO}(3)$, the necessary condition (5.1) can be expressed as $\dot{\gamma}(t) = \gamma(t) v(t)$, where $v$ satisfies:

$$\ddot{v}(t) = \dot{v}(t) \times v(t) + c,$$ (5.2)

$c$ is an arbitrary vector in $\mathbb{R}^3$ and $\times$ denotes the vector product in $\mathbb{R}^3$.

Camarinha [23] investigates generalized cubics over a more general class of piecewise smooth curves in Riemannian manifolds. The generalized cubics, referred to as elastic curves, are the curves minimizing functional $\mathcal{F}$, where

$$\mathcal{F}(\gamma) = \frac{1}{2} \int_a^b ((D_t \dot{\gamma}, D_t \gamma) + \tau^2 \langle \dot{\gamma}, \ddot{\gamma} \rangle) dt.$$ (5.3)

The parameter $\tau$ is called the elastic parameter. The necessary conditions for a piecewise smooth curve to be an elastic curve is the following set of Euler-Lagrange ordinary differential equations

$$D_3^2 \dot{\gamma} + R(D_t \dot{\gamma}, \gamma) \dot{\gamma} - \tau^2 D_t \dot{\gamma} = 0.$$ (5.3)

When the elastic parameter vanishes, $\tau = 0$, the elastic curve becomes a geometric cubic, when $\tau \to \infty$ the elastic curve becomes a geodesic. This is why the elastic curves are also called the hybrid curves, cf. [19]. Camarinha notes, that a geodesic

$$\tilde{\gamma} = \left( \gamma, \frac{1}{\tau} \right), \quad \tau \neq 0,$$

in $\mathcal{T}\mathbb{M}$ endowed with the Sasaki metric, cf. [84, 85], is the elastic curve in $\mathbb{M}$. This follows from the following property of the Sasaki metric

$$\left\langle \dot{\gamma}, \ddot{\gamma} \right\rangle = \langle \dot{\gamma}, \ddot{\gamma} \rangle + \langle D_t V, D_t V \rangle,$$ (5.4)

where $\tilde{\gamma} = (\gamma, V)$. Silva et al. [90] remark on the stronger result:

$$\left( \gamma, \frac{1}{\tau} D_t^k \gamma \right), \quad k \geq 0,$$

is a geodesic in $\mathcal{T}\mathbb{M}$ if and only if $\gamma$ is a geodesic in $\mathbb{M}$.
They also note that the elastic problem with zero elastic parameter $\tau$ does not reduce to the constrained geodesic problem, because a metric in $T\mathcal{M}$ satisfying

$$\left\langle \frac{d}{dt} (\gamma, \dot{\gamma}), \frac{d}{dt} (\gamma, \dot{\gamma}) \right\rangle = \langle D_t \dot{\gamma}, D_t \dot{\gamma} \rangle$$

is not a Riemannian metric, cf. Remark 5.3.1.

Crouch & Silva [30] consider the special case of Riemannian cubics in symmetric spaces. Let $\mathcal{G}$ be a connected Lie group, $K$ compact subgroup of $\mathcal{G}$, then $\mathcal{G}/K$ is a symmetric homogeneous space. Let $\pi: \mathcal{G} \to \mathcal{G}/K$ denote the canonical projection. The Lie algebra $\mathcal{L}$ of $\mathcal{G}$ admits canonical decomposition $\mathcal{L} = \mathcal{S} \oplus \mathcal{M}$, where $\mathcal{M}$ can be identified with the tangent space to $\mathcal{G}/K$ at the identity $1 = \pi(1_{\mathcal{G}})$. If $1 = \gamma(0)$ is the initial point of a curve $\gamma: [0, b] \to \mathcal{G}/K$ and $W \in T_1(\mathcal{G}/K)$ is an arbitrary tangent vector, then there exists a unique parallel vector field $W_t$ along $\gamma$ having the value $W$ at $1$, cf. [69]. Therefore one can consider a unique parallel orthonormal frame $\{E_i\}$ along $\gamma$ that coincides with an orthonormal frame $\{E_i\}$ at $1$. Hence if $W_t = w^i(t)E_i(\gamma(t))$ is an arbitrary vector field along $\gamma$, then its pullback from $\gamma(t)$ to $1$ is the vector field $W(t) = w^i(t)E_i \in T_1\mathcal{M}$.

Now, let $\gamma: [0, b] \to \mathcal{G}/K$ be a critical curve for $\Phi^{(1)}$, $\gamma(0) = 1$ and $V_t = v^i(t)E_i(\gamma(t))$ be the velocity vector field along $\gamma$. Crouch & Silva prove that

$$D_t^3 V_t + R(D_t V_t, V_t)V_t = 0 \quad \text{if and only if} \quad V^{(t)}(t) + \left[ V(t), \left[ V^{(0)}(t), V(t) \right] \right] = 0,$$

where

$$V(t) = v^i(t)E_i, \quad V^{(0)}(t) = \frac{d}{dt} v^i(t)E_i, \quad \text{and} \quad V^{(t)}(t) = \frac{d^3}{dt^3} v^i(t)E_i.$$

For example, one can obtain Riemannian cubic spline equation in the unit sphere $S^2$, by applying the above result to the velocity vector $V(t)$ of a curve $\gamma(t)$ in $T_{x_0}S^2$ starting at the point $x_0$.

### 5.1.2 Higher order geometric splines

Camarinha et al. [24] investigate higher order geometric splines over a class of smooth curves in $\mathcal{M}$. The splines are the critical curves of $\Phi^{(k)}$ over the class of $\mathcal{G}^{2k-1}$-curves $\gamma: [a, b] \to \mathcal{M}$, such that $\gamma|_{[t_{i-1}, t_i]}$ is smooth, with $\gamma(t_i) = p_i$, for a set of distinct points $p_i \in \mathcal{M}$ and fixed times $a = t_0 < t_1 < \cdots < t_l = b$, and satisfying the following boundary conditions

$$D_t^j \dot{\gamma}(a) = V_{a j} \quad \text{and} \quad D_t^j \dot{\gamma}(b) = V_{b j}, \quad \text{for } j = 0, \ldots, k - 1,$$
where $V_{aj}$ and $V_{bj}$ are fixed vectors. Camarinha et al. prove that if $\gamma$ is a critical curve for the functional $\Phi^{(k)}$, then $\gamma$ is $\mathcal{C}^{2k}$ and satisfies the Euler-Lagrange equations (5.7).

The geometric splines for a connected compact abelian Lie group $\mathcal{G}$ are given by

$$\gamma(t) = \exp[f_1(t)X_1] \cdot \exp[f_2(t)X_2] \cdots \cdot \exp[f_m(t)X_m],$$

where $\{X_1, X_2, \ldots, X_m\}$ is a frame of left (or right) invariant vector fields and $f_i(t)$ are polynomial splines of degree $2k + 1$.

### 5.2 Definition and Examples

In this section we propose a definition of $\mathcal{C}^k$-curves, a class of variational curves in Riemannian manifolds. The definition, which in the view of Theorem 5.2.2, equivalent to the definition of geometric splines in [24], emphasizes the variational genesis of the curves. The important differences of $\mathcal{C}^k$-curves and spline functions are illustrated by Remark 5.2.4 and Remark 5.3.1. The interpolation properties of $\mathcal{C}^k$-curves in Riemannian spaces were studied in [30, 97]. The problem of interpolation in the projective spaces is studied in Chapter 6.

We shall define the class of minimizing curves over the family of piecewise smooth curves as follows, cf. [23]. A smooth curve $\gamma: [a, b] \to M$ is an admissible curve, if there exists a finite subdivision $a = t_0 < t_1 < \cdots < t_l = b$, such that $\gamma|_{[t_i-t_{i-1}]}$ is a regular curve, for $i = 1, 2, \ldots, l$. We will assume the following formal definition.

**Definition 5.2.1** For a given $k \geq 1$ we define a class $\mathcal{C}^k$ of admissible curves $\gamma: [a, b] \to M$ that are critical points for the functional

$$\Phi^{(k)}(\gamma) = \int_a^b \left\langle D_t^k \dot{\gamma}, D_t^k \dot{\gamma} \right\rangle dt,$$

given $2k + 2$ boundary conditions

$$\gamma(a) = p, \quad \gamma(b) = q, \quad D_t^i \dot{\gamma}(a) = V_{ai}, \quad D_t^i \dot{\gamma}(b) = V_{bi},$$

for $i = 0, 1, \ldots, k - 1$, where $V_{ai}$ and $V_{bi}$ are fixed vectors.

It is clear that the class $\mathcal{C}^k$ depends on the Riemannian structure $(M, g)$ and the boundary conditions. The class $\mathcal{C}^k$ is often referred to as a class of polynomial curves or splines, notably in [77, 24]. However, there are important differences between the polynomial curves in the Euclidean space and a curved space. For example, a curve of class $\mathcal{C}^k$ does not necessarily belongs to $\mathcal{C}^{k+1}$, this is to say $\mathcal{C}^k \not\subseteq \mathcal{C}^{k+1}$. Similarly, the
Riemannian metric is not suitable to express the approximation properties of curves in Riemannian manifolds, similar to those obtained by de Boor [33], for polynomial splines in $\mathbb{R}^n$, cf. Remark 5.2.4.

It should be noted that Definition 5.2.1 cannot be applied to the even-degree geometric splines. This is a consequence of the Euler-Lagrange equations one gets from (5.5). It turns out, for example, that quantity under the integral

$$\int_a^b \langle D_t^k \dot{\gamma}, D_t^{k+1} \dot{\gamma} \rangle \, dt, \quad k \geq 0$$

is not positive definite, similarly as in $\mathbb{R}^n$.

We have the following theorem generalizing the result (5.1) of Noakes et al. [77].

**Theorem 5.2.2 (Camarinha, Silva & Crouch [24])** Let a $\mathcal{C}^{2k-1}$-curve $\gamma: [a, b] \to M$ be a critical curve for the functional $\mathcal{E}^{(k)}$ such that $\gamma|_{[t_i-1, t_i]}$ is smooth, $\gamma(t_i) = p_i$, for a set of distinct points $p_i \in M$ and satisfies the boundary conditions (5.6), then $\gamma$ is $\mathcal{C}^{2k}$ and on every interval $[t_{i-1}, t_i]$ it satisfies the following Euler-Lagrange ordinary differential equations

$$D_t^{2k+1} \dot{\gamma} - \sum_{j=1}^{k} (-1)^j R(D_t^{2k-j} \dot{\gamma}, D_t^{j-1} \dot{\gamma}) \dot{\gamma} = 0, \quad (5.7)$$

where $R$ is the Riemannian curvature endomorphism.

Theorem 5.2.2 provides a system of ordinary differential equations satisfied by $\mathcal{D}^k$-curves on a Riemannian manifold. Note that the opposite is also true: a $\mathcal{C}^{2k-1}$-curve satisfying (5.7) is a critical point for the functional $\mathcal{E}^{(k)}$. It is important to see why the Riemannian variational curves are important. Since the Riemannian connection, curvature tensor and curvature endomorphism are isometrically invariant, cf. [58, 53], the equality (5.7) is also isometrically invariant. This implies that $\mathcal{D}^k$-curves are preserved by isometries, i.e., an isometry takes $\mathcal{D}^k$-curves to $\mathcal{D}^k$-curves.

Throughout this chapter we focus on the simplest non-trivial case of $\mathcal{D}^1$-curves—the (Riemannian) cubics. Although the Euler-Lagrange equations (5.1) in this case are quite simple, one cannot expect to find a solution in a closed form, cf. [80].

**Example 5.2.3** Setting $k = 1$ in the Euler-Lagrange equations (5.7) we get (5.1).

In an Euclidean space the Riemannian curvature vanishes everywhere and (5.1) becomes

$$\frac{d^4}{dt^4} \gamma(t) = 0$$
satisfied by a cubic curve
\[ \gamma(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0, \]
for any constant vectors \( a_i \in \mathbb{R}^n \).

**Remark 5.2.4** In \( \mathbb{R}^n \) splines have the property of best approximation, cf. de Boor [33]. Let \( f \in C^k([a, b]) \), take \( n + 1 \) points \( t_i \in \mathbb{R} \), \( i = 0, 1, \ldots, n \), and let
\[ \gamma(t_i) = f(t_i) \quad \text{and} \quad \gamma^{(j)}(t_i) = f^{(j)}(t_i), \]
for \( i = 0, 1, \ldots, n \) and \( j = 1, 2, \ldots, k - 1 \). Introduce the inner product
\[ \langle f, g \rangle_k \overset{\text{def}}{=} \int_a^b \left( f^{(k)}(t), g^{(k)}(t) \right) dt, \]
defined for any two \( f, g: [a, b] \to M \), which have square-integrable \( k \)-th derivatives on \([a, b]\) and a pseudo-norm
\[ \|f\|_k \overset{\text{def}}{=} \langle f, f \rangle_k^{1/2}. \]

The best approximation property of spline functions of odd degree is characterized in [33] with respect to the above measure, i.e., \( \sigma \) best approximates \( f \) if \( \sigma \) minimizes \( \|f - \sigma\|_k \). The above result does not extend in terms of Riemannian metric, since the inner product \( \langle D^k f(t), D^k g(t) \rangle \) makes sense only for vectors in the same tangent space. Let alone, \( f - \sigma \) is not well defined.

Note that a \( \mathcal{D}^k \)-curve is not necessarily a \( \mathcal{D}^{k+1} \)-curve, as it would be in the case of ordinary polynomial curves in \( \mathbb{R}^n \), as the following remark explains.

**Remark 5.2.5** For polynomials we have \( \Pi_n \subset \Pi_{n+1} \) in the sense that
\[ \gamma(t) \in \Pi_n \iff \frac{d^{n+1}}{dt^{n+1}} \gamma(t) \equiv 0. \]

With the classes \( \mathcal{D}^n \) the relation is more complex. For simplicity, we will confine ourselves with manifolds of constant sectional curvature \( C \). It is easy to check that in this case
\[ D_t (R(X, Y)Z) = R(D_t X, Y)Z + R(X, D_t Y)Z + R(X, Y)D_t Z, \]
for any vector fields \( X, Y, Z \) along \( \gamma \). We will show that there exists a \( \mathcal{D}^1 \)-curve that is not a \( \mathcal{D}^2 \)-curve. From the equation (5.1) satisfied by a \( \mathcal{D}^1 \)-curve we can calculate \( D_t^2 \gamma \) as follows
\[ D_t^2 \gamma = -D_t (R(D_t \dot{\gamma}, \dot{\gamma})\dot{\gamma}) = -R(D_t^2 \dot{\gamma}, \dot{\gamma}) \dot{\gamma} - R(D_t \dot{\gamma}, \dot{\gamma}) D_t \dot{\gamma}, \]
where we used the symmetries of the curvature tensor, Theorem 3.4.2. Differentiating the above expression once more yields
\[ D_t^5 \gamma = -R(D_t^4 \gamma, \gamma) \gamma - R(D_t^3 \gamma, D_t \gamma) \gamma - 2R(D_t^2 \gamma, \gamma) D_t \gamma - R(D_t \gamma, \gamma) D_t^2 \gamma. \]

Applying this computation to a $\mathcal{D}^2$-curve equation (5.24) we see that if $\gamma$ is a $\mathcal{D}^1$-curve, then
\[ D_t^5 \gamma - R(D_t^4 \gamma, D_t \gamma) \gamma + R(D_t^3 \gamma, \gamma) \gamma = -2R(D_t^2 \gamma, D_t \gamma) \gamma - 2R(D_t^2 \gamma, \gamma) D_t \gamma - R(D_t \gamma, \gamma) D_t^2 \gamma. \]

Calculating the inner product of the above identity and $\gamma$, by the symmetries of the curvature tensor (Theorem 3.4.2) we see that the right hand side of the above equality becomes $-3 Rm(D_t^2 \gamma, \gamma, D_t \gamma, \gamma)$. Because $\gamma$ is a $\mathcal{D}^1$-curve, we may freely set the initial conditions $\gamma, D_t \gamma$ and $D_t^2 \gamma$ at $t = 0$ so that
\[ Rm(D_t^2 \gamma(0), \gamma(0), D_t \gamma(0), \gamma(0)) \neq 0. \]

Hence, such a $\mathcal{D}^1$-curve does not satisfy (5.24) and therefore is not a $\mathcal{D}^2$-curve.

### 5.3 Simple Variational Curves

This section provides an overview of the $\mathcal{D}^1$-curves. Let us start with the following Remark 5.3.1 to justify the investigations of these curves.

**Remark 5.3.1** The same variational problem expressed in different space may yield different results. By changing metric, for example, geodesics that arise as curves minimizing energy, can have a different form. Since geodesics were investigated by many mathematicians for centuries they are well understood ([29, 92, 69, 27]). It is interesting to ask if $\mathcal{D}^1$-curves in one space are geodesic in another space. Namely, let $\gamma: I \to M$ be a $\mathcal{D}^1$-curve in a Riemannian manifold $(M, g)$. The question is, whether there exists a manifold $(\bar{M}, \bar{g})$ such that $\langle D_t \gamma(t), D_t \gamma(t) \rangle_{\bar{g}}$ is equal to $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\bar{g}}$.

We shall consider the following case more closely. Let be given a family of curves $\Gamma_{(a,b)}$ in $T\mathbb{R}^2$ intersecting at the origin $(0,0)$. Namely let
\[ \Gamma_{(a,b)}(t) = (\gamma(t), \dot{\gamma}(t)), \]
where $\gamma(t) = at^3 + bt^2$ is a cubic polynomial. Then $\dot{\gamma}(t) = 3at^2 + 2bt$ is a parabola in the plane and $\Gamma_{(a,b)}(0) = (0,0)$, for all $a, b \in \mathbb{R}$. Since in normal coordinates centered at 0 geodesics passing through the origin are straight lines, there must be a diffeomorphism $\varphi_{(a,b)}: \mathbb{R}^4 \to \mathbb{R}^4$ that maps $\Gamma_{(a,b)}$ onto straight lines. Let $\varphi_{(a,b)} \circ \Gamma_{(a,b)}(t) = \sigma$.
loss, we may assume that \( \sigma(t) = tK \), where \( K \in \mathbb{R}^4 \) is a constant vector. Differentiating the above condition by \( t \) and setting \( t = 0 \) yields

\[
K = \varphi_{(a,b)} \circ \tilde{\Gamma}_{(a,b)}(0) = \varphi_{(a,b)} \circ (0, 2b).
\]

This is impossible, since \( \varphi_{(a,b)} \) is a diffeomorphism that does not depend on \( a \).

### 5.3.1 Existence and uniqueness of \( D^1 \)-curves

Camarinha [23] investigates the existence and uniqueness conditions of \( D^1 \)-curves. The main results are presented below.

**Theorem 5.3.2 (Camarinha [23])** For every point \((p, v, y, z)\) in an open set \( U \in \mathcal{T}^3M \), there exists a neighbourhood \( \mathcal{U} \) of \((p, v, y, z)\), \( \mathcal{U} \subseteq U \), and a positive real number \( \varepsilon \) such that, for each \((p_0, v_0, y_0, z_0) \in \mathcal{U} \), there exists a unique cubic \( \gamma: (-\varepsilon, \varepsilon) \to \mathcal{U} \) satisfying the initial conditions

\[
\gamma(0) = p_0, \quad \gamma' = v_0, \quad D_t^2 \gamma = y_0 \quad \text{and} \quad D_t^3 \gamma = z_0.
\]

Furthermore, \( \gamma \) depends smoothly on the point \((p_0, v_0, y_0, z_0) \in \mathcal{U} \). Any two cubics, satisfying the initial conditions, coincide in some open interval around \( t = 0 \).

The above result is derived by considering the map \( \text{Pol}: \mathcal{T}M \times \mathcal{T}M \to \mathcal{T}M \times \mathcal{T}M \) which is a generalization of the exponent map \( \exp: \mathcal{T}M \to M \). Below, we derive the condition (5.5) for \( D^1 \)-curves as a system of differential equations in local coordinates. This system may be also used to derive the above result concerning the existence and uniqueness of \( D^1 \)-curves.

### 5.3.2 Differential equations of \( D^1 \)-curves

In this section we will derive the equations of \( D^1 \)-curves in local coordinates involving the Christoffel symbols and their derivatives. This is a fourth-order system of nonlinear ordinary differential equations. At the origin of normal coordinates this system simplifies significantly.

Let \( \gamma: I \to M \) be a \( D^1 \)-curve and let \( t_0 \in I \). Choose coordinates on an open set \( \gamma(t_0) \in U \subset M \) and set \( \{E_i\} \) to be coordinate basis vectors. These vectors have the property that their commutators vanish identically \([E_i, E_j] \equiv 0\). For any vector field \( V \) along \( \gamma \)

\[
(D_t V(t_0))^k = \frac{d}{dt} V^k(t_0) + V^i(t_0) \hat{\Gamma}^k_{ij} \gamma(t_0),
\]

(5.8)
cf. [58]. From (5.8) we will calculate covariant derivatives according to the following scheme

\[ (D^1 \gamma)^k = \frac{d}{dt} (D \gamma)^k + (D \gamma)^l \Gamma^k_{lj} \]

\[ (D^2 \gamma)^k = \frac{d}{dt} (D^1 \gamma)^k + (D^1 \gamma)^l \Gamma^k_{lj} \]

\[ (D^3 \gamma)^k = \frac{d}{dt} (D^2 \gamma)^k + (D^2 \gamma)^l \Gamma^k_{lj} \]

where we omitted arguments of the functions, for clarity. We also need to calculate coefficients of the Riemannian curvature tensor

\[ R_{ijkl} = (R(E_i, E_j)E_k, E_l) \]

\[ = \langle \nabla_{E_i} \nabla_{E_j} E_k, E_l \rangle - \langle \nabla_{E_j} \nabla_{E_i} E_k, E_l \rangle - \langle \nabla_{[E_i, E_j]} E_k, E_l \rangle \]

\[ = \langle \partial_i \Gamma^m_{jk} E_m + \Gamma^m_{jk} \nabla_{E_i} E_m, E_l \rangle - \langle \partial_j \Gamma^m_{ik} E_m + \Gamma^m_{ik} \nabla_{E_j} E_m, E_l \rangle \]

\[ = \partial_i \Gamma^m_{jk} g_{ml} + \Gamma^m_{jk} \Gamma^s_{lm} g_{sl} - \partial_j \Gamma^m_{ik} g_{ml} - \Gamma^m_{ik} \Gamma^s_{jm} g_{sl} \]

Now, we raise the last index of the Riemannian tensor to get \( R_{ijkl} \) in the standard way, where \( g^{ql} \) are the components of the inverse matrix \((g_{ql})^{-1}\)

\[ R^{ijkl} = R_{ijkl} g^{ql} \]

\[ = \partial_i \Gamma^m_{jk} g_{mq} g^{ql} + \Gamma^m_{jk} \Gamma^s_{lm} g^{ql} - \partial_j \Gamma^m_{ik} g_{mq} g^{ql} - \Gamma^m_{ik} \Gamma^s_{jm} g_{ql} \]

\[ = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm} \]

Applying these calculations and the formula for the covariant derivative (5.8) to (5.1) we derive a system of ordinary differential equations for the component functions \( \gamma(t) = (\gamma^1(t), \gamma^2(t), \ldots, \gamma^n(t)) \) in local coordinates

\[ 0 = \partial_k \partial_i \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l + 2 \Gamma^m_{ip} \partial_i \gamma^p \gamma^j \gamma^k \gamma^l + \Gamma^i_{jk} \partial_i \Gamma^m_{lp} \gamma^p \gamma^j \gamma^k \gamma^l \]

\[ + \Gamma^i_{jk} g_{pq} \gamma^j \gamma^k \gamma^l \gamma^q + \Gamma^i_{jk} \Gamma^j_{lp} \Gamma^m_{iq} \gamma^j \gamma^k \gamma^l \gamma^q - \Gamma^i_{jk} \Gamma^j_{lp} \gamma^p \gamma^j \gamma^k \gamma^l \gamma^q \]

\[ + 2 \partial_k \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l + 2 \partial_k \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l + 3 \Gamma^j_{ik} \Gamma^m_{lp} \gamma^j \gamma^l \gamma^p \gamma^q \]

\[ - \Gamma^j_{ik} \Gamma^m_{lp} \gamma^j \gamma^k \gamma^l \gamma^q + 3 \Gamma^j_{ik} \gamma^j \gamma^k \gamma^l \gamma^q + 2 \partial_k \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l \gamma^q \]

\[ + 2 \Gamma^j_{ik} \Gamma^m_{lp} \gamma^j \gamma^k \gamma^l + \Gamma^j_{ik} \Gamma^m_{lp} \gamma^j \gamma^k \gamma^l + \Gamma^j_{ik} \Gamma^m_{lp} \gamma^j \gamma^k \gamma^l \]

\[ + 3 \Gamma^j_{ik} \gamma^j \gamma^k \gamma^l \gamma^q + \Gamma^j_{ik} \gamma^j \gamma^k \gamma^l \gamma^q + \gamma^m \gamma^k \gamma^l \gamma^q \]

Since we assumed that the connection is Riemannian, it is torsion free and the Christoffel symbols \( \Gamma \) are symmetric, i.e., \( \Gamma^k_{ij} = \Gamma^k_{ji} \) (cf. [58]), hence the above reduces to

\[ 0 = \partial_k \partial_i \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l + 2 \Gamma^m_{ip} \partial_i \gamma^p \gamma^j \gamma^k \gamma^l + \Gamma^i_{jk} \partial_i \Gamma^m_{lp} \gamma^p \gamma^j \gamma^k \gamma^l \]
\begin{align*}
+ \Gamma^i_{jk} \Gamma^l_{il} \gamma^j \gamma^k \gamma^l 
+ 4 \partial_k \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l 
+ 3 \Gamma^m_{ij} \gamma^j \gamma^l
\end{align*}
\begin{align*}
+ 2 \partial_i \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l 
+ 4 \Gamma^i_{jk} \Gamma^m_{il} \gamma^j \gamma^k \gamma^l 
+ 2 \Gamma^i_{jk} \Gamma^m_{il} \gamma^j \gamma^k \gamma^l
\end{align*}
\begin{align*}
+ 4 \Gamma^m_{ij} \gamma^i \gamma^j \gamma^k \gamma^l + \gamma^{(ij)} m. \tag{5.9}
\end{align*}

Hence in terms of local coordinate system the equation (5.1) satisfied by a \( \mathcal{D} \)-curve \( \gamma : I \rightarrow M \) takes the form (5.9).

We will conclude this section by showing that, locally, a \( \mathcal{D} \)-curve looks like a polynomial cubic curve. Let us introduce normal coordinates in \( U \). Since the Christoffel symbols vanish at the origin of normal coordinates (cf. [58]) equality (5.9) becomes

\begin{align*}
0 &= \partial_k \partial_i \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l + 2 \partial_k \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l 
+ 2 \partial_k \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l 
+ 2 \partial_k \Gamma^m_{ij} \gamma^j \gamma^k \gamma^l 
+ \gamma^{(ij)} m 
\end{align*}

after renaming dummy indices. To show that the above equality can be simplified further will need the following fact.

**Lemma 5.3.3** At the origin of normal coordinates

\begin{align*}
\partial_j \Gamma^m_{ki}(0) + \partial_i \Gamma^m_{jk}(0) + \partial_k \Gamma^m_{ij}(0) = 0 \quad \text{and} \quad \partial_i \partial_m \Gamma^k_{ij}(0) \gamma^i \gamma^j \gamma^k \gamma^l = 0. \tag{5.10}
\end{align*}

**Proof:** Simple calculations show that by the formula (4.17) for Christoffel symbols and by the Taylor series for a Riemannian metric (4.18) the following property of partial derivatives of the Christoffel symbols at the origin of normal coordinates holds

\begin{align*}
\partial_i \Gamma^k_{ij}(0) = \frac{1}{3} (R_{ijk} + R_{ijjk}). \tag{5.11}
\end{align*}

Applying algebraic Bianchi's identity (Theorem 3.4.2) to the Riemannian curvature tensor yields

\begin{align*}
\partial_j \Gamma^m_{ki} + \partial_i \Gamma^m_{jk} + \partial_k \Gamma^m_{ij} &= \frac{1}{3} (R_{jkim} + R_{jikm}) + \frac{1}{3} (R_{ijkm} + R_{ikjm}) + \frac{1}{3} (R_{kijm} + R_{kjim}) \\
&= \frac{1}{3} (R_{jkim} + R_{jikm} + R_{ijkm} + R_{kijm}) + \frac{1}{3} (R_{jkim} + R_{ikjm} + R_{kjim}) = 0.
\end{align*}

To prove the second equality, note that by the Taylor series for a Riemannian metric (4.18) and by the symmetries of the curvature tensor (Theorem 3.4.2) there is

\begin{align*}
\partial_i \partial_m \Gamma^k_{ij}(0) \gamma^i \gamma^j \gamma^k \gamma^l 
= \frac{1}{12} (3 \partial_k R_{ijim} + 3 \partial_k R_{ijim} + 2 \partial_l R_{kijm} + 2 \partial_l R_{kijm} + \partial_j R_{lmik} + \partial_i R_{lmjk}) \gamma^i \gamma^j \gamma^k \gamma^l 
= 0.
\end{align*}

**Q.E.D.**
Applying (5.10) to the above calculations we see that at the origin of normal coordinate system $\gamma^{(0)}(0) = 0$, cf. Example 5.2.3.

In conclusion, in local coordinates, a Riemannian cubic (a $\mathcal{D}^1$-curve) satisfies the system of ordinary differential equations (5.9). In the center of normal coordinates (5.9) becomes the system defining the polynomial cubic splines. Hence locally, the $\mathcal{D}^1$-curves should have the same properties as cubics. The above result also provides another way of looking at some approximation methods of calculating the Riemannian cubics in Lie groups, see [48, 80] for more examples.

### 5.3.3 The $\mathcal{D}^1$-curves in $S^n$

The unit $n$-sphere $S^n$ is probably the simplest non-trivial manifold so we will start our investigations in that space. Elastic curves (class of curves more general than $\mathcal{D}^1$-curves) in $S^2$ were already investigated by Brunnett & Crouch [19]. The equations satisfied by the spherical elastic curves derived by Brunnett & Crouch are more suitable for computations. Our approach to study $\mathcal{D}^1$-curves (Riemannian cubics) in $S^n$ is different as we are mostly concerned with establishing a set of necessary and sufficient conditions satisfied by these curves. The necessary and sufficient conditions derived in this section are the basis of our investigations in Chapter 6.

Let $\gamma: I \to S^n$ be a $\mathcal{D}^1$-curve in $n$-sphere. Consider $S^n$ embedded in $\mathbb{R}^{n+1}$. We have the following lemma.
Lemma 5.3.4 A $\mathcal{D}^1$-curve $\gamma: I \to S^n$ satisfies the following condition

$$\frac{d}{dt} \left( \gamma^{(ii)}(t) - 2 \langle \gamma(t), \gamma(t) \rangle \gamma(t) \right) = \lambda(t) \gamma(t), \quad (5.12)$$

where $\lambda: I \to \mathbb{R}$ is a scalar function and the inner product $\langle \cdot, \cdot \rangle$ is the 'dot' product in the Euclidean space $\mathbb{R}^{n+1}$.

**Proof:** To shorten the notation, let us denote the covariant derivative $D_t \gamma$ of $\gamma$ along $\gamma$ by $\alpha$. Then, the system of ordinary differential equations (5.1) becomes

$$D^2_t \alpha + R(\alpha, \gamma) \gamma = 0.$$ 

The covariant derivative $\alpha$ can be easily derived by the Gauss formula (3.3) applied to $S^n$ embedded in $\mathbb{R}^{n+1}$

$$\alpha = \nabla_\gamma \gamma = \gamma - \langle \gamma, \gamma \rangle \gamma,$$

where the inner product is the *dot* product in the Euclidean space $\mathbb{R}^{n+1}$, that plays a role of the *ambient manifold*. The second covariant derivative of $\alpha$ along $\gamma$ is equal to

$$D^2_t \alpha = \nabla_\gamma (\dot{\alpha} - \langle \dot{\alpha}, \gamma \rangle \gamma)$$

$$= \ddot{\alpha} - \langle \ddot{\alpha}, \gamma \rangle \gamma - \langle \dot{\alpha}, \dot{\gamma} \rangle \gamma - \langle \dot{\alpha}, \gamma \rangle \dot{\gamma}$$

$$- \langle \dot{\alpha}, \gamma \rangle \gamma + \langle \dot{\alpha}, \gamma \rangle \langle \gamma, \gamma \rangle \gamma + \langle \dot{\alpha}, \dot{\gamma} \rangle \gamma \gamma + \langle \dot{\alpha}, \gamma \rangle \langle \dot{\gamma}, \gamma \rangle$$

$$= \ddot{\alpha} - \langle \ddot{\alpha}, \gamma \rangle \gamma - \langle \dot{\alpha}, \gamma \rangle \dot{\gamma}, \quad (5.13)$$

where we used $\langle \gamma, \gamma \rangle = 1$ and $\langle \dot{\gamma}, \gamma \rangle = 0$. Since the unit sphere $S^n$ is a Riemannian manifold of constant sectional curvature $C = 1$, by Lemma 3.4.3 we get

$$R(\alpha, \gamma) \gamma = \langle \gamma, \gamma \rangle \alpha - \langle \alpha, \gamma \rangle \dot{\gamma}. \quad (5.14)$$

Adding (5.13) to (5.14) we obtain the following formula for a $\mathcal{D}^1$-curve in the unit $n$-sphere

$$\ddot{\alpha} - \langle \ddot{\alpha}, \gamma \rangle \gamma + \langle \dot{\gamma}, \gamma \rangle \alpha = 0, \quad (5.15)$$

where we used the property $\langle \alpha, \gamma \rangle = 0$. Using the explicit formula for $\alpha$ in the last term of (5.15) we get $\ddot{\alpha} + \langle \dot{\gamma}, \gamma \rangle \gamma - \langle \dot{\alpha} + \langle \dot{\gamma}, \gamma \rangle \gamma, \gamma \rangle = 0$. We may now express the above equality in the following form

$$\ddot{\alpha} + \langle \dot{\gamma}, \gamma \rangle \gamma = c_1 \gamma, \quad (5.16)$$

where $c_1: I \to \mathbb{R}$ is given by $\langle \ddot{\alpha} + \langle \dot{\gamma}, \gamma \rangle \gamma, \gamma \rangle$. On the other hand

$$\ddot{\alpha} = \frac{d^2}{dt^2} \left( \dot{\gamma} - \langle \dot{\gamma}, \gamma \rangle \gamma \right)$$

$$= \frac{d}{dt} \left( \gamma^{(iv)} - \langle \gamma^{(iv)}, \gamma \rangle \gamma - \langle \dot{\gamma}, \dot{\gamma} \rangle \gamma - \langle \gamma, \gamma \rangle \dot{\gamma} \right)$$

$$= \gamma^{(iv)} - \langle \gamma^{(iv)}, \gamma \rangle \gamma - 2 \langle \gamma^{(iv)}, \dot{\gamma} \rangle \gamma - 2 \langle \gamma^{(iv)}, \gamma \rangle \dot{\gamma}$$

$$\gamma - \langle \dot{\gamma}, \dot{\gamma} \rangle \gamma - 2 \langle \dot{\gamma}, \gamma \rangle \dot{\gamma} - \langle \dot{\gamma}, \gamma \rangle \dot{\gamma}$$

$$= \gamma^{(iv)} - 2 \langle \gamma^{(iv)}, \gamma \rangle \dot{\gamma} - 2 \langle \dot{\gamma}, \gamma \rangle \dot{\gamma} - \langle \dot{\gamma}, \gamma \rangle \dot{\gamma} - c_2 \gamma,$$
where \( c_2 : I \rightarrow \mathbb{R} \) is given by \( \langle \gamma^{(iv)} , \gamma \rangle + 2 \langle \gamma^{(iii)} , \dot{\gamma} \rangle + \langle \ddot{\gamma} , \dot{\gamma} \rangle \). Applying above calculations to (5.16) we finally get

\[
c_1 \gamma + c_2 \ddot{\gamma} = \gamma^{(iv)} - 2 \langle \gamma^{(iii)} , \gamma \rangle \dot{\gamma} - 2 \langle \dot{\gamma} , \gamma \rangle \dot{\ddot{\gamma}} + \langle \gamma , \dot{\gamma} \rangle \ddot{\gamma}
\]

\[
= \gamma^{(iv)} - 2 \langle \gamma^{(iii)} , \gamma \rangle \dot{\gamma} - 2 \langle \dot{\gamma} , \gamma \rangle \dot{\ddot{\gamma}} - 2 \langle \dot{\gamma} , \gamma \rangle \ddot{\gamma}
\]

\[
= \frac{d}{dt} \left( \gamma^{(ii)} - 2 \langle \dot{\gamma} , \gamma \rangle \dot{\gamma} \right), \tag{5.17}
\]

where we used \( \langle \dot{\gamma} , \gamma \rangle + \langle \dot{\gamma} , \dot{\gamma} \rangle = 0 \). Taking \( \lambda(t) = c_1(t) + c_2(t) \) proves (5.12). \textbf{Q.E.D.}

Note that Lemma 5.3.4 gives \( n \) independent equations, enough to find a numerical solution. Case \( n = 2 \), a \( \mathcal{D}^1 \)-curve in \( S^2 \), is illustrated in Figure 5.2 on page 109. Figure 5.3 shows the same curve expressed in spherical coordinates.

It is not difficult to derive explicit form of the function \( \lambda \) of (5.12). Because \( \langle \gamma , \dot{\gamma} \rangle = 0 \), taking the inner product of (5.12) and \( \dot{\gamma} \) yields

\[
0 = \left\langle \frac{d}{dt} \left( \gamma^{(iv)} - 2 \langle \dot{\gamma} , \gamma \rangle \dot{\gamma} \right), \dot{\gamma} \right\rangle
\]

\[
= \frac{d}{dt} \left( \left\langle \gamma^{(iii)} - 2 \langle \dot{\gamma} , \gamma \rangle \dot{\gamma} \right\rangle \right) - \left\langle \gamma^{(iv)} - 2 \langle \dot{\gamma} , \gamma \rangle \dot{\gamma} \right\rangle, \dot{\gamma}
\]

\[
= \frac{d}{dt} \left( \left\langle \gamma^{(ii)} , \dot{\gamma} \right\rangle - 2 \langle \dot{\gamma} , \gamma \rangle \left\langle \dot{\gamma} , \gamma \right\rangle \right) - \left\langle \gamma^{(iv)} , \ddot{\gamma} \right\rangle + 2 \langle \dot{\gamma} , \gamma \rangle \left\langle \dot{\gamma} , \ddot{\gamma} \right\rangle
\]

\[
= \left\langle \gamma^{(iv)} , \ddot{\gamma} \right\rangle + \left\langle \gamma^{(iii)} , \dot{\gamma} \right\rangle - 2 \left\langle \gamma^{(iii)} , \gamma \right\rangle \left\langle \dot{\gamma} , \dot{\gamma} \right\rangle - 2 \langle \dot{\gamma} , \dot{\gamma} \rangle \left\langle \ddot{\gamma} , \gamma \right\rangle - 4 \langle \dot{\gamma} , \gamma \rangle \left\langle \ddot{\gamma} , \dot{\gamma} \right\rangle
\]

\[
- \left\langle \gamma^{(ii)} , \ddot{\gamma} \right\rangle + 2 \langle \dot{\gamma} , \gamma \rangle \left\langle \ddot{\gamma} , \dot{\gamma} \right\rangle
\]

\[
= \left\langle \gamma^{(iv)} , \ddot{\gamma} \right\rangle - 2 \left\langle \gamma^{(iii)} , \gamma \right\rangle \left\langle \dot{\gamma} , \dot{\gamma} \right\rangle,
\]
where we used \( \langle \dot{\gamma}, \gamma \rangle + \langle \gamma, \dot{\gamma} \rangle = 0 \). Applying the above result to the inner product of (5.12) and \( \gamma \) brings

\[
\left\langle \frac{d}{dt} \left( \gamma^{(ii)} - 2 \langle \dot{\gamma}, \gamma \rangle \dot{\gamma} \right), \gamma \right\rangle = \frac{d}{dt} \left( \left\langle \gamma^{(ii)} - 2 \langle \dot{\gamma}, \gamma \rangle \dot{\gamma}, \gamma \right\rangle \right) - \left\langle \gamma^{(ii)} - 2 \langle \dot{\gamma}, \gamma \rangle \dot{\gamma}, \dot{\gamma} \right\rangle
\]

\[
= \frac{d}{dt} \left( \left\langle \gamma^{(ii)}, \gamma \right\rangle - 2 \langle \dot{\gamma}, \gamma \rangle \langle \dot{\gamma}, \gamma \rangle \right) - \left\langle \gamma^{(ii)}, \dot{\gamma} \right\rangle + 2 \langle \dot{\gamma}, \gamma \rangle \langle \dot{\gamma}, \dot{\gamma} \rangle
\]

\[
= \langle \gamma^{(ii)}, \gamma \rangle + \langle \gamma^{(ii)}, \dot{\gamma} \rangle - \langle \gamma^{(ii)}, \dot{\gamma} \rangle + 2 \langle \dot{\gamma}, \gamma \rangle \langle \dot{\gamma}, \dot{\gamma} \rangle
\]

\[
= \langle \gamma^{(ii)}, \gamma \rangle + 2 \langle \dot{\gamma}, \gamma \rangle \langle \dot{\gamma}, \dot{\gamma} \rangle.
\]

Hence \( \lambda = \langle \gamma^{(ii)}, \gamma \rangle + 2 \langle \dot{\gamma}, \gamma \rangle \langle \dot{\gamma}, \dot{\gamma} \rangle \) is the explicit expression for \( \lambda \) in (5.12). We can conclude:

**Corollary 5.3.5** A smooth curve \( \gamma: I \to S^n \) is a \( D^1 \)-curve in the unit sphere if and only if \( \gamma \) satisfies (5.12), where

\[
\lambda(t) = \left\langle \gamma^{(ii)}(t), \gamma(t) \right\rangle + 2 \langle \dot{\gamma}(t), \gamma(t) \rangle \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle
\]

(5.18)

for all \( t \in I \).

**Proof:** This clearly follows from the equation (5.17) for a \( D^1 \)-curve in the unit sphere and from the above formula for \( \lambda \). \( \Box \)

**5.3.4 \( D^1 \) and \( D^2 \)-curves on Lie groups**

We will investigate the \( D^k \)-curves, for \( k = 1, 2 \), on Lie groups. Let us begin with some facts needed to calculate the Euler-Lagrange equations (5.7) for Lie groups, cf. [58, 68]. Let \( \mathcal{G} \) be a Lie group with Lie algebra \( \mathfrak{g} \). A Riemannian metric \( g \) on \( \mathcal{G} \) is said to be *left invariant* if it is invariant under all left translations:

\[
L_p^* g = g,
\]

(5.19)

for all \( p \in \mathfrak{g} \). Similarly, \( g \) is *right invariant* if it is invariant under all right translations, and is *bi-invariant* if it is both left and right invariant (see Section 3.2).

**Bi-invariant metric**

In this section we will assume that the Riemannian metric \( g \) is bi-invariant. By Theorem 3.2.3, for any two left invariant vector fields \( X, Y \) on \( \mathcal{G} \)

\[
\nabla_X Y = \frac{1}{2} [X, Y].
\]

(5.20)
5.3 simple variational curves

By Theorem 3.4.4, for any left invariant vector fields $X, Y, Z$ on $\mathfrak{g}$ the Riemannian curvature endomorphism $R$ of $g$ is

$$R(X, Y)Z = \frac{1}{4} [Z, [X, Y]].$$

(5.21)

Choose basis $\{E_i\}$ in $\mathfrak{g}$. The following lemma will be helpful to calculate covariant derivatives on Lie groups equipped with a bi-invariant metric.

**Lemma 5.3.6** Let $g$ be a bi-invariant metric on a Lie group $\mathfrak{g}$ and $\gamma: I \rightarrow \mathfrak{g}$ be a regular curve. Let $V = V^i E_i$ be a left invariant vector field along $\gamma$ and $D_t$ be a covariant derivative along $\gamma$, then

$$D_t V = \dot{V} - \frac{1}{2} V^i \dot{\gamma}^j [E_i, E_j].$$

(5.22)

**Proof:** From the definition of the covariant derivative and by (5.20) we get

$$D_t V = D_t (V^i E_i) = \dot{V}^i E_i + V^i D_t E_i = \dot{V} + V^i \dot{\gamma}^j \nabla_{E_j} E_i$$

$$= \dot{V} - \frac{1}{2} V^i \dot{\gamma}^j [E_i, E_j].$$

Q.E.D.

Applying (5.22) to the velocity vector $\dot{\gamma}$ yields

$$D_t \dot{\gamma} = \ddot{\gamma} - \frac{1}{2} \dot{\gamma}^i \dot{\gamma}^j [E_i, E_j] = \ddot{\gamma},$$

(5.23)

since the Lie bracket is antisymmetric the second term sums to zero. As calculations of higher order covariant derivatives become more and more involved, it is useful to introduce the following notation:

- $(p)$ denotes the $p$-th $t$ derivative of $\gamma$: $\gamma^{(p)} i E_i$;

- $[(i_1), [(i_2), \ldots, (i_m)]]$ denotes the sequence of Lie brackets:

$$\gamma^{(p_1)} i_1 \gamma^{(p_2)} i_2 \ldots \gamma^{(p_m)} i_m [E_{i_1}, [E_{i_2}, \ldots [E_{i_m}]]].$$

Note that from the antisymmetry of the Lie bracket $[(p), (p)] = 0$, for any $p \geq 0$. By Lemma 5.3.6 we get the following rules for covariant differentiation along $\gamma$:

$$D_t (p) = (p + 1) + \frac{1}{2} [(1), (p)]$$
and

\[ D_t \left[ (i_1), (i_2), \ldots, (i_m) \right] \]

\[ = \left[ (i_1 + 1), (i_2), \ldots, (i_m) \right] + \left[ (i_1), (i_2 + 1), \ldots, (i_m) \right] \]

\[ + \cdots + \left[ (i_1), (i_2), \ldots, (i_m + 1) \right] + \frac{1}{2} \left[ (i_1), (i_2), \ldots, (i_m) \right]. \]

Using this notation we may write (5.23) as

\[ D_t \gamma = D_t(1) = (2) + \frac{1}{2} [(1), (1)] = (2) = \dot{\gamma}. \]

Continuing in the same manner

\[ D_t^2 \gamma = D_t(2) = (3) + \frac{1}{2} [(1), (2)], \]

\[ D_t^3 \gamma = D_t \left( (3) + \frac{1}{2} [(1), (2)] \right) \]

\[ = (4) + \frac{1}{4} [(1), (3)] + \frac{1}{2} \left( [(2), (2)] + [(1), (3)] + \frac{1}{2} [(1), (2)] \right) \]

\[ = (4) + [(1), (3)] + \frac{1}{4} [(1), (2)] \]

\[ D_t^4 \gamma = (5) + \frac{1}{2} [(1), (4)] + [(2), (3)] + [(1), (4)] + \frac{1}{2} [(1), (3)] \]

\[ + \frac{1}{4} \left( [(2), (1), (2)] + [(1), (2), (2)] + [(1), (1), (3)] + \frac{1}{2} [(1), (1), (2)] \right) \]

\[ = (5) + \frac{3}{2} [(1), (4)] + [(2), (3)] + \frac{3}{4} [(1), (1), (3)] \]

\[ + \frac{1}{4} [(2), (1), (2)] + \frac{1}{8} [(1), (1), (1), (2)], \]

\[ D_t^5 \gamma = (6) + \frac{1}{2} [(1), (5)] \]

\[ + \frac{3}{2} \left( [(2), (4)] + [(1), (5)] + \frac{1}{2} [(1), (1), (4)] \right) \]

\[ + [(3), (3)] + [(2), (4)] + \frac{1}{2} [(1), (2), (3)] \]

\[ + \frac{3}{4} \left( [(2), (1), (3)] + [(1), (2), (3)] + [(1), (1), (4)] + \frac{1}{2} [(1), (1), (3)] \right) \]

\[ + \frac{1}{4} \left( [(3), (1), (2)] + [(2), (2), (2)] + [(2), (1), (3)] + \frac{1}{2} [(1), (2), (1), (2)] \right) \]

\[ + \frac{1}{8} \left( [(2), (1), (1), (2)] + [(1), (2), (1), (2)] + [(1), (1), (1), (2)] \right) \]

\[ + [(1), (1), (1), (3)] + \frac{1}{2} [(1), (1), (1), (2)] \]
5.3 simple variational curves

\[ + \frac{1}{2} [(1), (1), (1), (3)]] + \frac{1}{4} [(1), (2), (1), (2)]] + \frac{1}{8} [(2), (1), (1), (2)]]\n\[ + \frac{1}{16} [(1), (1), (1), (2)]\].

We will use the above computations to derive formula for the \( \mathcal{G}^2 \)-curve on the Lie group endowed with the bi-invariant metric, see the example below.

**Example 5.3.7** The Euler-Lagrange equations (5.7) for \( k = 2 \) becomes

\[
D_t^2 \gamma - R(D_t^2 \gamma, D_t \gamma) \gamma + R(D_t^3 \gamma, \gamma) \gamma = 0. \tag{5.24}
\]

As the Riemannian endomorphism is a tensor in each argument, we may use the notation above. Thus by (5.21)

\[
R(D_t^2 \gamma, D_t \gamma) \gamma = \frac{1}{4} \left( (1), \left[ (3) + \frac{1}{2} [(1), (2)], (2)] \right) \right.
\]

\[
= -\frac{1}{4} [(1), (2)], (3)] - \frac{1}{8} [(1), (2), (1), (2)]\]

and

\[
R(D_t^3 \gamma, \gamma) \gamma = \frac{1}{4} \left( (1), \left[ (4) + [(1), (3)] + \frac{1}{4} [(1), (1), (2)], (1)] \right) \right.
\]

\[
= -\frac{1}{4} [(1), (1), (4)] - \frac{1}{4} [(1), (1), (3)] - \frac{1}{16} [(1), (1), (1), (2)].
\]

Plugging in these expressions into (5.24) we get the following condition for a \( \mathcal{G}^2 \)-curve

\[
0 = (6) + 2 [(1), (3)] + \frac{5}{2} [(2), (4)]
\]

\[
+ \frac{5}{4} [(1), (1), (4)] + \frac{3}{2} [(1), (2), (3)] + [(2), (1), (3)] + \frac{1}{4} [(3), (1), (2)]\]

\[
+ \frac{1}{4} [(1), (1), (1), (3)] + \frac{3}{8} [(1), (2), (1), (2)] + \frac{7}{8} [(2), (1), (1), (2)].
\]

In case of the group of rotations \( \text{SO}(3) \) the above equality can be expressed in a standard way:

\[
0 = v^{(v)} + 2v \times v^{(v)} + \frac{5}{2} \dot{v} \times v^{(v)}
\]

\[
+ \frac{5}{4} v \times (v \times v^{(v)}) + \frac{3}{2} v \times (\dot{v} \times \ddot{v}) + \frac{3}{2} \dot{v} \times (v \times \ddot{v}) + \frac{1}{4} v \times \dot{v} \times (v \times \dot{v})
\]

\[
+ \frac{1}{4} v \times (v \times (v \times \dot{v})) + \frac{3}{8} v \times (\dot{v} \times (v \times \dot{v})) + \frac{1}{8} \dot{v} \times (v \times (v \times \dot{v}))
\]

\[
= \frac{d}{dt} \left( v^{(v)} + 2v \times v^{(v)} + \frac{5}{2} \dot{v} \times v^{(v)} + \frac{1}{2} \dot{v} \times \ddot{v} + \frac{7}{4} v \times (v \times \dot{v}) + \frac{1}{4} v \times (v \times (v \times \dot{v})) \right),
\]

where \( v = \dot{\gamma} \).

Thus, from Example 5.3.7 we obtain the following property of \( \mathcal{G}^2 \)-curves in \( \text{SO}(3) \).
Corollary 5.3.8 If $\gamma: I \to SO(3)$ is a $\mathcal{H}^2$-curve in $SO(3)$ equipped with the bi-invariant metric, $v = \dot{\gamma}$, then the expression

$$v^{(iv)} + 2v \times v^{(iii)} + \frac{1}{2} v \times \ddot{v} + \frac{5}{4} v \times (v \times \dot{v}) + \frac{1}{4} v \times (v \times (v \times \dot{v}))$$

is constant along $\gamma$.

Existence of $\mathcal{H}^1$-curves in $SO(3)$

In this section we investigate the existence and uniqueness of $\mathcal{H}^1$-curves in $SO(3)$. A $\mathcal{H}^1$-curve equation can be expressed in terms of an orthonormal frame $\{E_i\}$ along the curve that coincides with an orthonormal frame $\{E_i\}$ at the identity 1, cf. Section 5.1.1. Suppose that the mapping $v: I \to \mathbb{R}^3$ satisfies

$$\dot{v} = \ddot{v} \times v + c,$$

(5.25)

where $c$ is a constant, cf. [77]. We will find a minimal interval, where a solution of (5.25) exists. First, some facts about analytic functions and ordinary differential equations need to be considered.

A function is called analytic in a domain $I$, if it can be expanded in a power series at any point of $I$, which is convergent in some neighbourhood of that point, cf. [12]. We have the following fact underlying our investigations of a solution of (5.25), which establishes the existence of solutions in the neighbourhood of a given initial value.

Theorem 5.3.9 (Analytic Equations, cf. [12]) Given ordinary differential equations

$$\frac{dx}{dt} = X(x, t),$$

if $X(x, t)$ is an analytic real function of the real variables $(x_1, x_2, \ldots, x_n)$ and $t$, then every solution of the equations is analytic.

It is clear that (5.25) is analytic since it is multilinear. Therefore, by Theorem 5.3.9, we may confine ourselves to analytic solutions. Let $v$ be analytic on an interval $I \subset \mathbb{R}$ then $v$ can be expressed as a Taylor series, converging for all $t \in I$

$$v(t) = \sum_{i=0}^{\infty} v_i \frac{t^i}{i!},$$

(5.26)

where we denoted the $i$-th $t$ derivative of $v$ by $v_i = v^{(i)}(0)$. For (5.26) to satisfy (5.25) it must be $v_2 = v_1 \times v_0 + c$ and

$$v_{i+2} = \sum_{j=0}^{i} \binom{i}{j} v_{j+1} \times v_{i-j},$$

(5.27)
for $i > 0$. We will show that under certain conditions the series (5.26) uniformly converges on $t \in [-\delta, \delta]$, for any $v_0$, $v_1$ and $c$.

**Lemma 5.3.10** Given $v_0$, $v_1$ and $c$ let there be two constants $0 < M/3 \leq l$ that satisfy

\[ ||v_i|| < M^i i! \tag{5.28} \]

for $i = 0, 1, 2$. Then (5.28) holds for all $i \geq 0$.

**Proof:** Induction. Let (5.28) holds for any $i \leq k + 1$, where $k \geq 1$. We will show that (5.28) also holds for $k + 2$. Note first, that for any $i > 0$, the number of terms in (5.27) is always odd. Therefore, we can write (5.27) in the following way

\[
v_{i+2} = \sum_{j=0}^{i} \binom{i}{j} v_{j+1} \times v_{i-j}
\]

\[
= v_{i+1} \times v_0 + \sum_{j=0}^{\lfloor (i-1)/2 \rfloor} \binom{i}{j} v_{j+1} \times v_{i-j} + \sum_{j=1}^{\lfloor (i+1)/2 \rfloor} \binom{i}{j-1} v_j \times v_{i-j+1}
\]

where $\lfloor \cdot \rfloor$ is thought as rounding down to a whole number. From (5.28) and the assumption that $M/3 \leq l$ we get

\[
||v_{k+2}|| \leq ||v_{k+1}|| \cdot ||v_0|| + \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} \left| \begin{pmatrix} k \\ j-1 \end{pmatrix} - \begin{pmatrix} k \\ j \end{pmatrix} \right| \cdot ||v_j|| \cdot ||v_{k-j+1}||
\]

\[
< M^2 k! l^{k+1} (k+1)! + \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} \left( \begin{pmatrix} k \\ j \end{pmatrix} - \begin{pmatrix} k \\ j-1 \end{pmatrix} \right) M! j! M^{k-j+1} (k-j+1)!
\]

\[
= M^2 k! l^{k+1} \left( k + 1 + \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} (k-j+1-j) \right) = M^2 k! l^{k+1} \sum_{j=0}^{\lfloor (k+1)/2 \rfloor} (k+1-2j)
\]

\[
\leq M^2 k! l^{k+1} \frac{(k+1+0)(k+3)/2}{2} = M^2 (k+2)! l^{k+1} \frac{k+3}{4(k+2)} \leq M^2 (k+2)! l^{k+1} \frac{M}{3} \leq M l^{k+2} (k+2)!.
\]

This proves that the (5.28) holds for any $i \geq 0$. \textbf{Q.E.D.}
Take $0 < \delta < 1/l$, where $l$ is satisfies the hypothesis of Lemma 5.3.10. Then for any $t \in [-\delta, \delta]$ the series (5.26) converges, since by Lemma 5.3.10
\[
\|v(t)\| \leq \sum_{i=0}^{\infty} \|v_i\| \frac{\delta^i}{i!} < M \sum_{i=0}^{\infty} (\delta)^i = \frac{M}{1-\delta} < \infty.
\]
Hence we have the following.

**Corollary 5.3.11** For any set of initial conditions $v(0)$, $\dot{v}(0)$ and for any $c$, let $M$ and $l$ satisfy (5.28). Then there exists a unique analytic solution $v: [-1/l, 1/l] \to \mathbb{R}^3$ of (5.25).

**Left invariant metric**

Let $g$ be left-invariant Riemannian metric on a Lie group $\mathfrak{g}$, cf. (5.19), and let $g$ be its corresponding Lie algebra. Let us introduce the bilinear operator $B: g \times g \to g$ defined by the following property
\[
\langle B(Z, X), Y \rangle \equiv \langle [X, Y], Z \rangle, \tag{5.29}
\]
for all $Y \in g$, cf. [7]. It is known ([58, 91]) that for a bi-invariant metric $g$
\[
\langle [X, Y], Z \rangle = \langle [Z, X], Y \rangle,
\]
thus $B(Z, X) = [Z, X]$, in this case.

Let $\{E_i\}$ be an orthonormal basis of $\mathfrak{g}$ at the identity $1$. Denote $\tilde{E}_i = L^*_p E_i$, then $\{\tilde{E}_i\}$ is an orthonormal basis at $p$.

By Lemma 3.2.2 and because $\langle X, Y \rangle$ is constant for any $X, Y$, we have
\[
2 \langle \nabla_{\tilde{E}_i} \tilde{E}_j, \tilde{E}_k \rangle = \langle \tilde{E}_i, [\tilde{E}_k, \tilde{E}_j] \rangle + \langle \tilde{E}_j, [\tilde{E}_k, \tilde{E}_i] \rangle + \langle \tilde{E}_k, [\tilde{E}_i, \tilde{E}_j] \rangle
= \langle E_i, [E_k, E_j] \rangle + \langle E_j, [E_k, E_i] \rangle + \langle E_k, [E_i, E_j] \rangle
= -B(E_i, E_j), E_k) - \langle B(E_j, E_i), E_k \rangle + \langle [E_i, E_j], E_k \rangle,
\]
where we used the fact that the metric and vectors are all left-invariant, and the definition of the $B$ operator (5.29). Thus from the linearity of all involved operators we get a formula for the covariant derivative
\[
\nabla_X Y = \frac{1}{2} ([X, Y] - B(X, Y) - B(Y, X)), \tag{5.30}
\]
where $X$ and $Y$ on the left are left-invariant vector fields and on the right are their values at the identity. We shall consider the $\mathfrak{m} = \mathfrak{so}(3)$ case more closely.
Group of rotations \( SO(3) \)

Since calculations in the case of left invariant metric become complex, we will examine one particular group: the group of rotations in \( \mathbb{R}^3 \), i.e., \( M = SO(3) \).

Any left invariant metric in \( SO(3) \) can be represented as a \( 3 \times 3 \)-matrix \( A \). Note that by choosing the basis \( \{E_i\} \) to be the eigenvectors of \( A \) satisfying \( AE_i = I_i E_i \), the metric \( A \) in this basis is a diagonal matrix

\[
A = \begin{bmatrix}
I_1 & 0 & 0 \\
0 & I_2 & 0 \\
0 & 0 & I_3
\end{bmatrix}.
\]

If \( A \) is bi-invariant then clearly \( I_1 = I_2 = I_3 \).

**Lemma 5.3.12** Let \( A \) be left-invariant metric of \( SO(3) \), then the bilinear operator \( B \) defined by (5.29) can be expressed as

\[
B(X, Y) = A^{-1} ((AX) \times Y).
\]  

(5.31)

**Proof:** We will verify that the property (5.29) is satisfied. For any vectors \( X, Y, Z \in \mathbb{R}^3 \) and a symmetric nonsingular \( 3 \times 3 \)-matrix \( A \) we have

\[
\langle [X, Y], Z \rangle - \langle B(Z, X), Y \rangle
= Z^T A (X \times Y) - Y^T A (A^{-1} ((AZ) \times X))
= (AZ)^T (X \times Y) + Y^T (X \times (AZ))
= 0
\]

from the property of the vector product \( (u \times v)^T w = u^T (v \times w) \), cf. Lemma 6.5.3. 

\( Q.E.D. \)

To shorten the notation, let us introduce operator \( \mathcal{R}(X, Y) \) \( \overset{\text{def}}{=} B(X, Y) + B(Y, X) \). Since \( B \) is bilinear, it is clear that the operator \( \mathcal{R} \) is bilinear and symmetric. For \( SO(3) \), by Lemma 5.3.12, \( \mathcal{R}: \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) has the following form

\[
\mathcal{R}(X, Y) = A^{-1} ((AX) \times Y + (AY) \times X).
\]

Note that for the bi-invariant metric (\( A \) is the identity matrix up to a constant factor) \( \mathcal{R}(X, Y) \equiv 0 \). Now, the general formula for the covariant derivative (5.30) can be written as

\[
\nabla_X Y = \frac{1}{2} (X \times Y - \mathcal{R}(X, Y)).
\]

(5.32)
Denote $\gamma(t) = V(t) = V^i(t)E_i$, then for any vector field $X(t) = X^i(t)E_i$ the covariant derivative $D_t$ along $\gamma$ can be expressed as

$$D_tX = \dot{X} + \frac{1}{2}(V \times X - \mathcal{R}(V, X))$$  \hspace{1cm} (5.33)

because from the properties of $D_t$ there is

$$D_tX = \dot{X}^i E_i + X^i V^j \nabla_{E_j} E_i$$

$$= \dot{X}^i E_i + X^i V^j \frac{1}{2}(E_j \times E_i - \mathcal{R}(E_j, E_i))$$

$$= \dot{X} + \frac{1}{2}(V \times X - \mathcal{R}(V, X)).$$

Applying (5.33) to the vector field $V$, we derive

$$D_tV = \dot{V} - \frac{1}{2} \mathcal{R}(V, V);$$

$$D_t^2 V = \frac{d}{dt} (D_tV) + \frac{1}{2} (V \times (D_tV) - \mathcal{R}(V, D_tV))$$

$$= \dot{V} - \frac{3}{2} \mathcal{R}(V, \dot{V}) + \frac{1}{2} V \times \dot{V} - \frac{1}{4} V \times \mathcal{R}(V, V) + \frac{1}{4} \mathcal{R}(V, \mathcal{R}(V, V));$$

$$D_t^3 V = \frac{d}{dt} (D_t^2 V) + \frac{1}{2} (V \times (D_t^2 V) - \mathcal{R}(V, D_t^2 V))$$

$$= V^{(iii)} - \frac{3}{2} \mathcal{R}(V, \dot{V}) + \frac{3}{2} \mathcal{R}(V, \ddot{V}) + \frac{1}{2} V \times \ddot{V} - \frac{1}{4} V \times \mathcal{R}(V, V)$$

$$- \frac{1}{2} V \times \mathcal{R}(V, \dot{V}) + \frac{1}{4} \mathcal{R}(V, \mathcal{R}(V, V)) + \frac{1}{2} \mathcal{R}(V, \mathcal{R}(V, \dot{V}))$$

$$+ \frac{1}{2} V \times \left( \dot{V} - \frac{3}{2} \mathcal{R}(V, \dot{V}) + \frac{1}{2} V \times \dot{V} - \frac{1}{4} V \times \mathcal{R}(V, V) + \frac{1}{4} \mathcal{R}(V, \mathcal{R}(V, V)) \right) - \frac{1}{2} \mathcal{R}(V, \ddot{V})$$

$$+ \frac{3}{4} \mathcal{R}(V, \mathcal{R}(V, V)) - \frac{1}{4} \mathcal{R}(V, V \times \dot{V}) + \frac{1}{8} \mathcal{R}(V, V \times \mathcal{R}(V, V)) - \frac{1}{8} \mathcal{R}(V, \mathcal{R}(V, V))$$

$$= V^{(iii)} + V \times \ddot{V} - \frac{1}{4} V \times \mathcal{R}(V, V) - \frac{5}{4} V \times \mathcal{R}(V, \dot{V})$$

$$+ \frac{1}{4} V \times (V \times \dot{V}) - \frac{1}{8} V \times (V \times \mathcal{R}(V, V)) + \frac{1}{8} V \times \mathcal{R}(V, \mathcal{R}(V, V))$$

$$- 2 \mathcal{R}(V, \ddot{V}) - \frac{3}{2} \mathcal{R}(V, \dot{V}) + \frac{1}{4} \mathcal{R}(V, \mathcal{R}(V, V)) + \frac{5}{4} \mathcal{R}(V, \mathcal{R}(V, \dot{V}))$$

$$- \frac{1}{4} \mathcal{R}(V, V \times \dot{V}) + \frac{1}{8} \mathcal{R}(V, V \times \mathcal{R}(V, V)) - \frac{1}{8} \mathcal{R}(V, \mathcal{R}(V, \mathcal{R}(V, V))).$$

In order to derive (5.1) we need to find a formula of the Riemannian curvature endomorphisms $R(D_tV, V)V$ in $\mathbb{SO}(3)$ endowed with left-invariant metric. Because the curvature endomorphisms is a tensor, at first we will calculate its components $R_{ijk}^l$ in local coordinates. From the definition of the curvature endomorphisms (Definition 3.4.1) and from the formula of covariant derivative (5.32) we get

$$R_{ijk}^l E_l = R(E_i, E_j)E_k$$

$$= \nabla_{E_i} \nabla_{E_j} E_k - \nabla_{E_j} \nabla_{E_i} E_k - \nabla_{E_i \times E_j} E_k.$$
\[
\begin{align*}
&= \frac{1}{2} \nabla_{E_j} (E_j \times E_k - \mathcal{R}(E_j, E_k)) - \frac{1}{2} \nabla_{E_j} (E_i \times E_k - \mathcal{R}(E_i, E_k)) \\
&\quad - \frac{1}{2} (E_i \times E_j) \times E_k + \frac{1}{2} \mathcal{R}(E_i \times E_j, E_k) \\
&\quad - \frac{1}{4} (E_j \times (E_i \times E_k - \mathcal{R}(E_i, E_k)) - \mathcal{R}(E_j, E_i \times E_k - \mathcal{R}(E_i, E_k))) \\
&\quad + \frac{1}{2} E_k \times (E_i \times E_j) + \frac{1}{2} \mathcal{R}(E_k, E_i \times E_j) \\
&\quad \mathcal{R}(E_i, \mathcal{R}(E_j, E_k)) - \mathcal{R}(E_i, \mathcal{R}(E_j, E_k)) \\
&\quad \mathcal{R}(E_i, \mathcal{R}(E_j, E_k)) - \mathcal{R}(E_i, \mathcal{R}(E_j, E_k)) \\
&\quad \mathcal{R}(E_i, \mathcal{R}(E_j, E_k)) - \mathcal{R}(E_i, \mathcal{R}(E_j, E_k)).
\end{align*}
\]

From that we find
\[
R(DtV, V)V = \frac{1}{4} (V \times (DtV \times V) + V \times \mathcal{R}(DtV, V) - DtV \times \mathcal{R}(V, V) \\
+ \mathcal{R}(V, DtV \times V) - \mathcal{R}(DtV, V \times V) + 2 \mathcal{R}(V, DtV \times V) \\
+ \mathcal{R}(DtV, \mathcal{R}(V, V)) - \mathcal{R}(V, \mathcal{R}(DtV, V))) \\
= \frac{1}{4} (V \times \left( \left( \dot{V} - \frac{1}{2} \mathcal{R}(V, V) \right) \times V \right) + V \times \mathcal{R}(\dot{V} - \frac{1}{2} \mathcal{R}(V, V), V) \\
- \left( \dot{V} - \frac{1}{2} \mathcal{R}(V, V) \right) \times \mathcal{R}(V, V) + \mathcal{R}(V, \left( \dot{V} - \frac{1}{2} \mathcal{R}(V, V) \right) \times V) \\
+ 2 \mathcal{R}(V, \left( \dot{V} - \frac{1}{2} \mathcal{R}(V, V) \right) \times V) + \mathcal{R}(\dot{V} - \frac{1}{2} \mathcal{R}(V, V), \mathcal{R}(V, V)) \\
- \mathcal{R}(V, \mathcal{R}(\dot{V} - \frac{1}{2} \mathcal{R}(V, V), V))) \\
= \frac{1}{4} V \times (\dot{V} \times V) - \frac{1}{8} V \times (\mathcal{R}(V, V) \times V) + \frac{1}{4} V \times \mathcal{R}(V, \dot{V}) - \frac{1}{8} V \times \mathcal{R}(V, \mathcal{R}(V, V)) \\
- \frac{1}{4} \dot{V} \times \mathcal{R}(V, V) - \frac{3}{8} \mathcal{R}(V, \mathcal{R}(V, V) \times V) \\
+ \frac{3}{4} \mathcal{R}(V, \dot{V} \times V) + \frac{1}{4} \mathcal{R}(\dot{V}, \mathcal{R}(V, V)) - \frac{1}{8} \mathcal{R}(\mathcal{R}(V, V), \mathcal{R}(V, V)) \\
- \frac{1}{4} \mathcal{R}(V, \mathcal{R}(\dot{V}, V)) + \frac{1}{8} \mathcal{R}(V, \mathcal{R}(V, V), V)).
\]

Thus the equation (5.1) for a \( \mathcal{G}^1 \)-curve in \( SO(3) \) endowed with left-invariant metric assumes the following form
\[
0 = V^{(m)} + V \times \dot{V} - \frac{1}{2} \dot{V} \times \mathcal{R}(V, V) - V \times \mathcal{R}(V, \dot{V})
\]
\[-2\mathcal{H}(V, \dot{V}) - \frac{3}{2} \mathcal{H}(\dot{V}, \dot{V}) + \frac{1}{2} \mathcal{H}(\dot{V}, \mathcal{H}(V, V)) + \mathcal{H}(V, \mathcal{H}(V, \dot{V}))
- \mathcal{H}(V, V \times \dot{V}) + \frac{1}{2} \mathcal{H}(V, V \times \mathcal{H}(V, V)) - \frac{1}{8} \mathcal{H}(\mathcal{H}(V, V), \mathcal{H}(V, V)).\]

The above equality can be written in a more compact form
\[
0 = \frac{d}{dt} \left( \dot{V} + V \times \dot{V} - \frac{1}{2} V \times \mathcal{H}(V, V) - 2 \mathcal{H}(V, \dot{V}) + \frac{1}{2} \mathcal{H}(V, \mathcal{H}(V, V)) \right) 
- \mathcal{H}(V, V \times \dot{V}) + \frac{1}{2} \mathcal{H}(V, V \times \mathcal{H}(V, V)) + \frac{1}{2} \mathcal{H}(\dot{V}, \dot{V}) - \frac{1}{8} \mathcal{H}(\mathcal{H}(V, V), \mathcal{H}(V, V)).
\]

Note, that if the metric is bi-invariant, then \(\mathcal{H} \equiv 0\) and the above condition becomes (5.2). Suppose now, that the metric \(A\) is close to the identity: \(A = 1 + \delta\), where \(\|\delta\|\) is small. Then
\[
\mathcal{H}(X, Y) = (1 + \delta)^{-1} (((1 + \delta) X) \times Y + ((1 + \delta) Y) \times X)
- (\delta X) \times Y + (\delta Y) \times X + O(\|\delta\|^2).
\]
Thus the condition (5.1) satisfied by \(\mathcal{D}^1\)-curves in \(SO(3)\) endowed with left-invariant metric becomes
\[
0 = \frac{d}{dt} \left( \dot{V} + V \times \dot{V} - V \times ((\delta V) \times V) - 2 (\delta V) \times \dot{V} - 2 \left( \delta \dot{V} \right) \times V \right) 
- (\delta V) \times \left( V \times \dot{V} \right) - \left( \delta \left( V \times \dot{V} \right) \right) \times V + \left( \delta \dot{V} \right) \times \dot{V} + O(\|\delta\|^2).
\]
In particular, when a \(\mathcal{D}^1\)-curve is close to geodesic, then \(\dot{V}\) is small. Disregarding terms involving \(\dot{V}\) in the above equality yields
\[
0 = V^{(iii)} + V \times \dot{V} - 2 (\delta V) \times \dot{V} - 2 \left( \delta \dot{V} \right) \times V + O(\|\delta\|^2).
\]
Moreover, disregarding terms of the order of \(O(\|\delta\|^2)\), we see that a \(\mathcal{D}^1\)-curve in \(SO(3)\), which is close to geodesic, satisfies the following approximate equality
\[
0 = \frac{d}{dt} \left( \dot{V} + V \times \dot{V} - 2 \frac{d}{dt} ((\delta V) \times V) \right).
\]

### 5.4 Conclusion

An attractive way to extend the classical theory of interpolating curves in Euclidean space is through studying curves as solutions to variational problems. Variational properties of curves naturally extend to non-Euclidean spaces. This chapter presents an overview of the current research on variational curves in Riemannian geometry. The first published results concerning cubic splines on curved spaces appeared in [77]. This paper inspired a number of extensions to the theory of variational curves, i.e.,
elastic curves \([23, 19, 90]\), and more general, higher order geometric splines \([24]\). This family of curves has a number of applications, for example in robotics and computer aid design. One considers higher order variational problems when higher smoothness is required.

In this chapter the author contributed the following to existing research:

- a system of Euler-Lagrange equations in local coordinates satisfied by \(\mathcal{D}^1\)-curves (Section 5.3.2) was derived; this system simplifies significantly at the center of normal coordinates—it becomes a set of ordinary differential equations satisfied by a polynomial function of order of 3,
- a necessary condition for a \(\mathcal{D}^1\)-curve in the unit sphere \(S^n\) in a form of non-linear ordinary differential equation of third order (Section 5.3.3) was derived,
- results concerning \(\mathcal{D}^1\) and \(\mathcal{D}^2\)-curves in the group \(\mathbb{SO}(3)\) of rotations in \(\mathbb{R}^3\) and other Lie groups (Section 5.3.4) were as follows:
  - the minimal interval, depending on the initial conditions, where a unique analytic \(\mathcal{D}^1\)-curve in \(\mathbb{SO}(3)\) exists was determined, and
  - the \(\mathcal{D}^1\)-curves in \(\mathbb{SO}(3)\) endowed with left-invariant metric we investigated.

The next chapter addresses the problem of interpolation in the projective space \(\mathbb{RP}^2\) by means of envelopes, where the \(\mathcal{D}^1\)-curves in the unit sphere \(S^2\) play an important role.
Chapter 6

Interpolation with the Riemannian Variational Curves

In the previous chapter we presented a theory of variational curves in Riemannian spaces originated by Noakes et al. [77] and further developed by Camarinha et al. [24], Crouch & Silva Leite [30] and others, cf. Section 5.1.1. The aim of this chapter is to demonstrate how the variational curves in Riemannian spaces can be applied to the interpolation of tangential lines, Section 6.5. This extends current theory and research in the field of approximation in non-Euclidean spaces.

6.1 Introduction

Splines have applications in Mathematics (approximation theory, statistics and numerical analysis), engineering (design and computer-aided methods in manufacturing) and computer science (computer vision). The variational curves in Riemannian spaces have a similar scope for applications. For example, Noakes et al. [77] investigate interpolation in the space of rotations in $\mathbb{R}^3$, the problem found in robotics. The configuration of the rigid body in $\mathbb{R}^3$ is given by a point in $\mathbb{R}^3 \times SO(3)$, where the first component describes position of a reference point of the body (for instance the center of mass) in $\mathbb{R}^3$ and the second describes orientation of a fixed frame. The problem is to find a curve interpolating the set of points in $\mathbb{R}^3 \times SO(3)$. One may consider interpolation of the components in $\mathbb{R}^3$ and $SO(3)$ independently. Noakes et al. derive the equations satisfied by Riemannian cubics in a complete Riemannian manifold and for the special case of the Lie group $SO(3)$. A further example is provided by Crouch & Silva Leite [30] who investigate the dynamic interpolation for nonlinear control sys-
tems in Riemannian manifolds. Such situations arise in motion planning problems and tracking problems for nonlinear systems.

This chapter presents an application of the Riemannian variational curves to approximation of a shape from ray-based information. Given a set of tangential lines (rays) to a contour of a convex body we approximate the shape of the body by reformulating the problem to the projective plane. In the projective plane, which is a smooth and compact Riemannian manifold, we interpolate the points corresponding to the tangential lines, with the Riemannian variational curves. The envelope of the family of plane lines corresponding to the interpolating curve in the projective plane\(^1\) is the curve interpolating the tangential lines, and thus it is the approximation of the shape of the convex body.

This chapter is organized as follows. Section 6.2 outlines the problem of interpolation of tangential directions in the plane. A background on the projective plane and the envelopes is provided in Section 6.3. In particular, Section 6.3.1 gives a brief introduction to the projective plane and Section 6.3.2 describes envelopes. Envelopes of a family of lines in the plane are explained by Example 6.3.2. Further properties of the envelopes of lines in the plane are outlined in Section 6.4. We note on a duality of the plane and the projective plane induced by envelopes in Corollary 6.4.1. This duality clarifies that interpolation with the Riemannian geodesics in the projective plane produces a linear interpolation in the plane and therefore justifies the approach of interpolation with higher order Riemannian variational curves described in Chapter 5. This section is concluded by a classic method of deriving an envelope of a family of lines in the plane defined by a smooth curve in the projective plane. The main results of this chapter are presented in Section 6.5, where we derive the system of ordinary differential equations satisfied by the envelope of lines defined by the Riemannian cubic in the projective plane, Lemma 6.5.2. Lemma 6.5.4 states the necessary set of conditions for a plane curve to be an envelope of a \(\mathcal{C}^1\)-curve in the projective plane. Therefore the results of Lemma 6.5.4 may be applied to derive a class of interpolating curves in the plane induced by the Riemannian cubics. Two numerical examples demonstrate two methods of interpolation of tangent directions by the Riemannian cubics in the projective plane that show different rates of convergence as a function of density of samples. Example 6.5.1 illustrates interpolation with segments of the Riemannian cubics in the projective plane. Each segment is uniquely determined by four tangent lines in the plane. The error estimation indicates convergence of order \(O(n^{-2.18})\), where \(n\) is a number of segments. Example 6.5.6 interpolates a curve given

\(^1\)We will often refer to the envelope of a family of lines defined by a curve in the projective space as the envelope of the curve, for brevity.
6.2 The Problem

This section states the problem of interpolation of a shape from ray-based information. Consider a convex contour $C$ in the plane, Figure 6.1. Given a set of lines $\{\ell_i\}$ tangent to $C$ we want to approximate the contour $C$. Since every line $\ell_i$ can be associated with a point in the projective plane, the above problem is equivalent to the interpolation of points in $\mathbb{RP}^2$. The key contribution of this chapter is to show how the Riemannian variational curves, studied in Chapter 5, can be applied to interpolate the points in the projective plane. We will also study the envelopes of the variational curves in $\mathbb{RP}^2$ that approximate the contour $C$.

Another approach to a problem of recovering shape from a set of rays with application to computer vision was taken by Kutulakos [55]. Planar slices of a three dimensional object are represented by the light field boundary. The reconstruction of the shape turns out to be simpler by considering the problem in the oriented projective sphere ($T^2$), where the set of rays—samples of the light field boundary—is used to derive a convex set in $T^2$, the conical polyhedron ([21]). Such a polyhedron whose vertices correspond to the set of rays and whose edges are arcs of great circles, is an approximate representation of a visual hull of the planar slice. One of the differences between Ku-
tulakos' approach and the one presented in this chapter is that we smooth the polyg­
onal curve approximating the visual hull with Riemannian cubics in $\mathbb{R}P^2$. Further
developments in reconstructing 3-dimensional shapes from multiple photographs is
presented in Kutulakos & Seitz [56]. To compute photorealistic shapes, Kutulakos &
Seitz develop the space curving algorithm that derives a photo hull. The space curv­
ing algorithm is a further refinement of the visual hull deriving technique, taking
into account the radiance constrains from non-background pixels. The reconstructed
scene is a subset of the visual hull and can contain concavities. Our approach does
not use this kind of information.

Splines provide a commonly used technique of interpolation in Euclidean spaces,
cf. [32]. Depending on required smoothness, one applies polynomial splines of a de­
sired degree, cf. [1, 33]. For instance, the cubic polynomial splines are of class $C^2$,
making the interpolating curve differentiable. In the problem stated above, however,
we are given the tangent lines without points of tangency. Therefore, it is reasonable
to transfer the above problem to another space, namely to the projective plane ($\mathbb{R}P^2$).
Each line in the plane corresponds to a single point in the projective plane. The prob­
lem then reduces to interpolation of points in the projective plane $\mathbb{R}P^2$, which is a
smooth compact manifold locally isomorphic to the unit sphere $S^2$. To address the
above problem, we shall apply the theory of variational curves in the Riemannian
spaces, described in Chapter 5. The interpolating curve, for example the Riemannian
cubic, is then mapped back to the plane, producing the curve interpolating tangential
directions.

6.3 Preliminaries

As background to the following discussion we begin by introducing the concept of the
real projective plane and then a definition of the envelope. The variational curves in
the Riemannian spaces were introduced in Chapter 5.

6.3.1 The real projective plane

Let us start with an intuitive description of the projective plane. For our purposes
we shall think of the projective plane $\mathbb{R}P^2$ as of a completion the space of unoriented
lines in $\mathbb{R}^2$. An unoriented line (a direction) $l$ in the plane is a set of points $(x, y) \in \mathbb{R}^2$
satisfying $ax + by + c = 0$, where $a, b, c \in \mathbb{R}$ and $a, b$ are not simultaneously equal to
zero. For any $\lambda \neq 0$, the triples $(\lambda a, \lambda b, \lambda c)$ and $(a, b, c)$ define the same line. Therefore
a line can be uniquely identified by the homogeneous coordinates $(a, b, c)$, cf. [13, 72].
The projective plane is the quotient of $\mathbb{R}^3 \setminus \{0\}$ by the equivalence relation identifying $p$ and $q$ if they are scalar multiples of one another, i.e., $p \sim q$ if and only if there exists $\lambda \in \mathbb{R}$, such that $p = \lambda q$. Since every $p \in \mathbb{R}^3 \setminus \{0\}$ gives rise to $p/\|p\| \in S^2$, the quotient set of $\mathbb{R}^3 \setminus \{0\}/\sim$ coincides with the quotient of $S^2$ by the equivalence relation identifying antipodal points, i.e., $p \sim q$ if and only if $p = \pm q$. To be concise, $\mathbb{R}P^2 = S^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the isometric action $p \mapsto -p$ (antipodal point of $p$). Since the unit sphere $S^2$ is a smooth compact manifold, so is the projective space, and the canonical map $\pi: S^2 \to \mathbb{R}P^2$ is continuous and surjective, cf. [11]. Any atlas of $S^2$ in which each chart $(\mathcal{U}, \varphi)$ has as domain $\mathcal{U} \subset S^2$ an open hemisphere (or part of it) induces an atlas for $\mathbb{R}P^2$, where $\left(\pi(\mathcal{U}), \varphi \circ (\pi|_\mathcal{U})^{-1}\right)$ is a chart in $\mathbb{R}P^2$, cf. [53]. Hence, it is clear that the projective plane is locally isomorphic to the unit sphere, whose metric is the standard metric on the sphere $S^2 \subset \mathbb{R}^3$ induced by the Euclidean metric, cf. [21, 58]. The projective plane is an example of non-orientable surface\(^2\). Although $\mathbb{R}P^2$ cannot be embedded in $\mathbb{R}^3$ (cf. [11, 16]), it can be embedded in $\mathbb{R}^4$. An example of an embedding is the function $\tilde{f}: \mathbb{R}P^2 \to \mathbb{R}^4$ defined by

$$\tilde{f}(p) \overset{\text{def}}{=} (f \circ \pi^{-1})(p), \quad \text{where} \quad f: S^2 \to \mathbb{R}^4 \quad \text{is} \quad f(x, y, z) = (yz, xz, xy, x^2 + 2y^2 + 3z^2),$$

see [92, volume 1] for the proof of this fact.

To sum up briefly, we may regard the projective plane as the unit sphere with antipodal points identified. Or, instead of the whole sphere, we may consider the projective plane to be a hemisphere, say the "southern" hemisphere, with identification of diametrically opposite points on the peripheral equator, cf. [29, 63]. There is a bijection $\mathbb{R}P^2 \setminus \{\pm (0,0,1)\}$ to the space of directions:

$$(a, b, c) \mapsto \ell = \{(x, y) \in \mathbb{R}^2 \mid ax + by + c = 0\}$$

with the inverse

$$\ell = \{(x, y) \in \mathbb{R}^2 \mid ax + by + c = 0, (a, b) \neq (0, 0)\}$$

$$\mapsto \pm \left[\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}\right].$$

These properties of $\mathbb{R}P^2$ play a crucial role in Section 6.5, where the methods of interpolation with the Riemannian cubics in the projective plane are studied.

### 6.3.2 Envelopes

This section briefly describes envelopes, another ingredient needed to apply the theory of variational curves in Riemannian spaces to the problem of interpolation of tan-

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\(^2\)In general, the projective space $\mathbb{R}P^n$ is orientable if and only if $n$ is odd, cf. [53].
gent directions in the plane. In the projective geometry one often speaks of a curve in the plane as the envelope of its tangent lines, cf. [59]. This is a consequence of the duality of points and lines in the plane. We shall explore this duality in a geometrical context more in Section 6.4. Recall the definition of the envelope:

**Definition 6.3.1 (The envelope, cf. [14, 16])** For a family of curves, a curve that is tangent to all curves of the family and which is tangent at each of its points to some curve of the family is called the envelope of the given family of curves. The envelope can consist of more than one curve.

Let \( f: \mathbb{R}^3 \to \mathbb{R} \) and a family of curves be given by

\[
(f(x, y, \alpha) = 0)
\]

then it is easy to see that every point of the envelope of the family (6.1) satisfies

\[
\begin{align*}
 f(x, y, \alpha) & = 0 \quad \text{and} \\
 f_\alpha(x, y, \alpha) & = 0.
\end{align*}
\]

(6.2)

If we eliminate parameter \( \alpha \) from (6.2) we get a necessary condition for an envelope, cf. [14]. In general, if a family of curves is given by

\[
(f(x, y, \alpha_1, \alpha_2, \ldots, \alpha_m) = 0 \quad \text{and} \\
g_1(\alpha_1, \alpha_2, \ldots, \alpha_m) = 0, \\
\vdots \\
g_{m-1}(\alpha_1, \alpha_2, \ldots, \alpha_m) = 0,
\]

(6.3)

then every point of the envelope satisfies

\[
\det \begin{bmatrix}
 f_{\alpha_1} & f_{\alpha_2} & \cdots & f_{\alpha_m} \\
 (g_1)_{\alpha_1} & (g_1)_{\alpha_2} & \cdots & (g_1)_{\alpha_m} \\
 \vdots & \vdots & \ddots & \vdots \\
 (g_{m-1})_{\alpha_1} & (g_{m-1})_{\alpha_2} & \cdots & (g_{m-1})_{\alpha_m}
\end{bmatrix} = 0.
\]

(6.4)

The curve obtained by eliminating the parameters from (6.4) is called the discriminant curve. The discriminant curve contains envelope but also it contains the locus of all singular points on the curves of the family of curves, cf. [14]. The concept of envelope is illustrated by the following example.

**Example 6.3.2** Suppose that the ends of a line segment of constant length \( l \) slide along two perpendicular lines. The equation of the family of lines is given by

\[
\alpha y + \beta x - \alpha \beta = 0, \quad \text{where} \quad \alpha^2 + \beta^2 = l^2.
\]
Thus in this case the system (6.3) consists of two functions

\[
\begin{align*}
    f(x, y, \alpha, \beta) &= \alpha y + \beta x - \alpha \beta, \\
    g(\alpha, \beta) &= \alpha^2 + \beta^2 - l^2.
\end{align*}
\]

By (6.4) a discriminant curve is given by

\[
\det \begin{bmatrix}
    y - \beta & x - \alpha \\
    2\alpha & 2\beta
\end{bmatrix} = 2 (\beta y - \beta^2 - \alpha x + \alpha^2) = 0.
\]

Eliminating parameters \(\alpha\) and \(\beta\) from the above equality we obtain an equation

\[x^{2/3} + y^{2/3} = l^{2/3}\]

defining an astroid, Figure 6.2.

Sometimes it is easier to work with the following property defining the envelopes. The envelope is the limit of intersections of nearby curves, cf. [18]. This property is used in Section 6.4 to show that the envelope of the Riemannian geodesic in \(\mathbb{RP}^2\) is a single point in \(\mathbb{R}^2\).

### 6.4 Envelopes as a Mapping

In the previous sections we gave an overview of the projective plane and envelopes. This section investigates the relationship between curves in the projective plane and the curves in the plane induced by envelopes. Since a curve in the projective plane
interpolation with the riemannian variational curves

generates a family of lines in the plane it makes sense to think of envelopes as a mapping between curves of the two spaces. As it will become apparent in this section (see in particular Corollary 6.4.1), envelopes induce a duality between geodesics and points in these spaces.

Let us start with an elementary but important observation. Consider the unit sphere \( S^2 \) embedded in \( \mathbb{R}^3 \). It is clear that for a pair of orthonormal vectors \( A, B \in \mathbb{R}^3 \), the curve \( \gamma \) defined by \( \gamma(t) = A \sin t + B \cos t \) satisfies \( ||\gamma(t)|| = 1 \) and \( ||\dot{\gamma}(t)|| = 1 \).

Since, by its definition, \( \gamma \) lies in the plane \( AB0 \), it is an arc of a great circle, therefore \( \gamma: [0, \pi/2] \to S^2 \) is the unit speed geodesic in the sphere.

If the north pole \((0,0,1)\) does not belong to the plane \( AB0 \), then any point of \( \gamma \) uniquely identifies a line in the plane, cf. Section 6.3.1. For \( \gamma(t_0) = (a, b, c) \), where \( t_0 \in [0, \pi/2] \), let \( \ell_t = \{(x,y) \in \mathbb{R}^2 | ax + by + c = 0\} \). We will show that the whole family of lines \( \ell_t, t \in [0, \pi/2] \), intersects at precisely one point.

Take any \( t_1, t_2 \in [0, \pi/2], t_1 \neq t_2 \), and set \( (a, b, c) = \gamma(t_1) \) and \( (a', b', c') = \gamma(t_2) \). The point of intersection \( \ell_{t_1} \cap \ell_{t_2} \) is determined by system of two linear equations, whose only solution can written as

\[
x = \frac{W_x}{W}, \quad y = \frac{W_y}{W},
\]

where \( W_x = -cb' + c'b, W_y = -ac' + a'c \) and \( W = ab' - a'b \neq 0 \). Applying the expressions for \( (a, b, c) \) and \( (a', b', c') \) it is easy to verify that the above solution becomes

\[
x = \frac{A^3B^2 - A^2B^3}{A^2B^1 - A^1B^2} \quad \text{and} \quad y = \frac{A^1B^3 - A^3B^1}{A^2B^1 - A^1B^2} \quad (6.5)
\]

depending only on \( A \) and \( B \). Since (6.5) holds for any two lines of the family \( \ell_t \), the point \((x, y) \in \mathbb{R}^2\) is the point of intersection of the whole family \( \ell_t \). Hence we have the following.

**Corollary 6.4.1** Let \( \gamma: I \to \mathbb{R}P^2 \) be a Riemannian geodesic in the projective plane. Then the envelope of \( \gamma \) is either an empty set or a single point.

**Proof:** As noted in Section 6.3.1 the projective plane is locally isomorphic to the unit sphere in \( \mathbb{R}^3 \), therefore any small enough segment of \( \gamma \) can be identified with an arc of a great circle in \( S^2 \). By (6.5), the family of lines in \( \mathbb{R}^2 \) defined by this arc is either a set of parallel lines or a set of lines intersecting in a single point. Therefore the envelope of this family of lines is either empty or a point of intersection, cf. [18]. Since the Riemannian geodesic in \( \mathbb{R}P^2 \) is a differentiable and continuous curve, the above property clearly extends onto entire \( \gamma \).

Q.E.D.
For further calculations of envelopes we shall use the following classical construction.
Let $\gamma: I \to \mathbb{R}P^2$ be a regular curve in the real projective plane (differentiable and such that $\dot{\gamma}(t) \neq 0$, for all $t \in I$) and let $\tilde{\gamma}: I \to S^2$ be a preimage of $\gamma$ in the unit sphere, i.e., $\tilde{\gamma}(t) = \pi^{-1}(\gamma(t))$, for all $t \in I$, where $\pi: S^2 \to \mathbb{R}P^2$ is the canonical projection, introduced in Section 6.3.1. Let $\tilde{\omega}(t) = \tilde{\gamma}(t) \times \tilde{\gamma}(t)$, where 'x' denotes the vector product in $\mathbb{R}^3$. We will show that $\omega: I \to \mathbb{R}^2$ defined by $\omega(t) \overset{\text{def}}{=} \left(\frac{\tilde{\omega}^1(t)}{\tilde{\omega}^3(t)}, \frac{\tilde{\omega}^2(t)}{\tilde{\omega}^3(t)}\right)$ is the envelope of $\gamma$, provided $\tilde{\omega}^3(t) \neq 0$.

For every $t_0 \in I$ the point $\gamma(t_0) = (a(t_0), b(t_0), c(t_0)) \in \mathbb{R}P^2$ corresponds to a line $l(t_0)$ in $\mathbb{R}^2$. By Definition 6.3.1 it is enough to show that $\omega$ is tangent to $l(t_0)$ at $\omega(t_0)$, for all $t_0 \in I$. To show that the incidence condition at $\omega(t_0)$ holds, we note that

$$\omega^1(t_0) \cdot a(t_0) + \omega^2(t_0) \cdot b(t_0) + c(t_0) = \frac{1}{\tilde{\omega}^3(t_0)} \langle \tilde{\omega}(t_0), \gamma(t_0) \rangle = 0$$

by the definition of $\tilde{\omega}$, since $\tilde{\omega}$ is orthogonal to $\tilde{\gamma}(t_0)$ and therefore it is also orthogonal to $\gamma(t_0)$. Similarly, to show that the tangency condition at $\omega(t_0)$ holds, we write

$$\tilde{\omega}^1(t_0) \cdot a(t_0) + \tilde{\omega}^2(t_0) \cdot b(t_0) = \frac{1}{\tilde{\omega}^3(t_0)} \langle \tilde{\omega}(t_0), \gamma(t_0) \rangle - \frac{\tilde{\omega}^3(t_0)}{(\tilde{\omega}^3(t_0))^2} \langle \tilde{\omega}(t_0), \gamma(t_0) \rangle = 0,$$

where we used the incidence condition and the definition of $\tilde{\omega}$. Since the above equalities hold for every $t_0 \in I$, this proves that $\omega$ is the envelope of $\gamma$. We shall use the above result in the next section, where we derive the conditions satisfied by the envelope of the Riemannian cubic in $\mathbb{R}P^2$. For a generalization of the above construction to higher dimensions see, for example [71].

Let $\Psi: \mathcal{C}(\mathbb{R}P^2) \to \mathcal{C}(\mathbb{R}^2)$ denote the mapping from the space of continuous curves in $\mathbb{R}P^2$ to the space of continuous curves in $\mathbb{R}^2$ defined as follows. If $\gamma: I \to \mathbb{R}P^2$ is a continuous curve then $\omega = \Psi(\gamma)$ is the envelope of the family of lines in $\mathbb{R}^2$ corresponding
Figure 6.4: Duality of $\mathbb{R}^2$ and $\mathbb{R}P^2$: geodesics are mapped into points

to $\gamma$. We have seen that $\Psi$ maps Riemannian geodesics to points (constant curves). It is clear therefore that the envelope of a piecewise geodesic in the projective plane is a piecewise linear curve in the plane. We conclude this section with the following remark.

**Remark 6.4.2** By Corollary 6.4.1, $\Psi$ identifies points of the plane with geodesics in the projective plane. Instead of considering Riemannian geodesics in $\mathbb{R}P^2$ let us consider a reversed situation. Namely, let $\ell$ be a line in the plane. Let $P$ and $P'$, $P \neq P'$, be any two distinct points of $\ell$, and $\gamma$ and $\gamma'$ be two geodesics satisfying $\Psi(\gamma) = P$ and $\Psi(\gamma') = P'$. By the discussion above, each point of $\gamma$ and $\gamma'$ corresponds to a line in the plane passing through $P$ and $P'$, respectively. Therefore, the point of intersection $\gamma \cap \gamma'$ must correspond to the line $\ell$. Since this is true for any two points of $\ell$, it is clear that all geodesics in $\mathbb{R}P^2$, corresponding to the points of $\ell$, intersect at the point $\Psi^{-1}(\ell)$.

Consider a mapping $\Phi: \mathscr{C}(\mathbb{R}^2) \to \mathscr{C}(\mathbb{R}P^2)$ from the space of continuous curves in $\mathbb{R}^2$ to the space of continuous curves in $\mathbb{R}P^2$ assigning to a plane curve $\omega: I \to \mathbb{R}^2$ an envelope of geodesics $\gamma_t$ in $\mathbb{R}P^2$, where $\gamma_t = \Psi^{-1}(\omega(t))$. For any line $\ell$ in $\mathbb{R}^2$ there is $\Phi(\ell) = \Psi^{-1}(\ell)$. Similarly, for any point $P$ in $\mathbb{R}^2$ there is $\Phi(P) = \Psi^{-1}(P)$. The envelopes map geodesics in the projective plane $\mathbb{R}P^2$ to points in $\mathbb{R}^2$, and vice versa, envelopes map lines in $\mathbb{R}^2$ to points in $\mathbb{R}P^2$, cf. Figure 6.4. Consequently, envelopes induce a duality between the plane and the projective plane, where geodesics correspond to points and points correspond to geodesics.

The results of this section indicate a relationship, induced by envelopes, between curves in the real projective plane and curves in the plane. We have seen that the linear interpolation in one space corresponds to the linear (or piecewise geodesic) interpolation in another space. It is interesting to ask about the relationship between
higher order variational curves in the projective plane and the plane. The next section investigates such a relation in the case of the Riemannian cubics.

### 6.5 Interpolation in the Space of Directions

This section addresses the problem outlined in Section 6.2. The interpolation of tangent directions in the plane ($\mathbb{R}^2$) is now formulated in terms of the interpolation of points in the projective plane ($\mathbb{RP}^2$). The envelope of an interpolating curve in $\mathbb{RP}^2$ is the interpolating curve of the original problem in $\mathbb{R}^2$. Suppose that for the interpolating curve in $\mathbb{RP}^2$ we chose a piecewise geodesic, i.e., a curve consisting of geodesic segments joining the control points, cf. Figure 6.4 on the facing page. Let $\gamma_i$ and $\gamma_{i+1}$ be any two adjacent segments of the interpolating curve. By Corollary 6.4.1 geodesies in the projective plane correspond to points in the plane, therefore the two segments $\gamma_i$ and $\gamma_{i+1}$ in $\mathbb{RP}^2$ determine a line segment in $\mathbb{R}^2$. It is clear then, that the envelope of the piecewise geodesic is a plane polygon whose vertices correspond to the Riemannian geodesics in $\mathbb{RP}^2$. Hence, the interpolation by geodesics in $\mathbb{RP}^2$ leads to the linear interpolation in the plane. If we require an interpolating curve to be smooth, we need to consider higher order variational curves in $\mathbb{RP}^2$, for example the class of curves described in Section 5.2.

This section investigates an interpolation with the $\mathcal{D}^1$-curves (the Riemannian cubics) in the projective plane, as a solution to the problem described in Section 6.2. We begin with Example 6.5.1 showing how the interpolation works in practice. Since the Riemannian cubic is a solution of the system of ordinary differential equations of the fourth order, cf. Section 5.1, we require four boundary values to determine the cubic uniquely. The boundary conditions are derived from the four tangent lines. It turns out that the envelope of the Riemannian cubic in the projective plane satisfies a simple system of ordinary differential equations (6.6) given in Lemma 6.5.2. These equations resemble the formulae (5.2) for the Riemannian cubics in the space of rotations in $\mathbb{R}^3$. Lemma 6.5.4 proves the sufficient conditions for a plane curve to be the envelope of the Riemannian cubic. An application of this result to the interpolation is outlined in Example 6.5.6.

The following example illustrates the main idea and presents numerical results of the interpolation with the Riemannian cubics in the projective space.

**Example 6.5.1** We consider a plane curve limacons, cf. [18], $f: I \rightarrow \mathbb{R}^2$, defined by

$$f(t) = (\lambda \cos t + \cos 2t + 1, \lambda \sin t + \sin 2t),$$
Figure 6.5: Riemannian cubic interpolating four points in $\mathbb{RP}^2$

Figure 6.6: The resulting curve (dashed) interpolates a curve (solid gray) given four tangent lines
6.5 interpolation in the space of directions

\[
\begin{array}{ccc}
\text{segments} & \text{error} & \ln \text{error} \\
2^2 & 3.061 \times 10^{-2} & -3.486 \\
2^3 & 3.108 \times 10^{-3} & -5.774 \\
2^4 & 3.602 \times 10^{-4} & -7.923 \\
2^5 & 4.439 \times 10^{-5} & -10.023 \\
\end{array}
\]

Table 6.1: Convergence of the interpolation of tangent directions

\[
\ln \text{error}
\]

Figure 6.7: Error of interpolation of tangent directions as a function of density of samples

where we set \( \lambda = 3 \). Given four tangent directions in the plane \( \ell_i \), \( i=0,\ldots,3 \), we will find an interpolating function \( g: I \rightarrow \mathbb{R}^2 \) by transferring the problem to the projective space as follows. Each of \( \ell_i \) represents a control point in \( \mathbb{P}^2 \). Since the Riemannian cubic in the projective plane is a solution of the system of two ordinary differential equations of the third order, four points is a sufficient number of boundary conditions determining uniquely the Riemannian cubic \( \gamma: [t_0, t_3] \rightarrow \mathbb{P}^2 \), satisfying \( \gamma(t_i) = \ell_i \), for \( i=0,\ldots,3 \), cf. Figure 6.5 on the preceding page. The image of the cubic \( g = \Psi(\gamma) \) is the interpolating curve \( g: [t_0, t_3] \rightarrow \mathbb{R}^2 \), such that \( g \) at \( t_i \) is tangent to \( \ell_i \), \( i=0,\ldots,3 \), cf. Figure 6.6 on the facing page. By incrementing the number of tangent lines, the interpolating curve \( g \) becomes a piecewise envelope of the Riemannian cubics, where each segment \( g|_{[t_i, t_{i+1}]} \) is determined by four tangent lines: \( \ell_{i-1}, \ell_i, \ell_{i+1} \) and \( \ell_{i+2} \), except of course for the first and the last segments. The results of the interpolation are presented in Table 6.1. The linear regression analysis results, Figure 6.7, \( Y = 0.814 - 2.176 \cdot X ; \ R^2 = 1 \), indicate the interpolation rate \( O(n^{-2.18}) \), where as a measure of accuracy of the interpolation we have used the following error formula

\[
\text{error} = \left( \int_a^b \| f(t) - g(t) \|^2 \, dt \right)^{1/2}.
\]

We have seen in Example 6.5.1 that given a set \( \{\ell_i\} \) of four or more ordered un-
oriented lines in $\mathbb{R}^2$, one may interpolate a contour tangent to these lines with the envelopes of variational curves in the projective plane. We shall now study the plane interpolating curves derived in this way. It is a bit surprising to find that these curves satisfy a system of ordinary differential equations very similar to (5.2) satisfied by the Riemannian cubics in $\mathbb{S}O(3)$.

Lemma 6.5.2 Let $\gamma: I \rightarrow \mathbb{S}^2$ be a $C^1$-curve in the unit sphere. Set $\tilde{\omega}: I \rightarrow \mathbb{R}^3$ to be defined by the formula $\tilde{\omega}(t) = \tilde{\gamma}(t) \times \tilde{\gamma}(t)$, then $\tilde{\omega}$ satisfies the following system of differential equations

$$\frac{d}{dt} \left( \tilde{\omega} - 2\tilde{\omega} \times \tilde{\omega} \right) = 0,$$

for all $t \in I$.

In the following proof we will use the following properties of vector calculus in $\mathbb{R}^3$, cf. [91].

Lemma 6.5.3 (Properties of vector calculus in $\mathbb{R}^3$) Let $u, v, w \in \mathbb{R}^3$ be three arbitrary vectors. Then

$$\langle u \times v, w \rangle = \langle u, v \times w \rangle,$$

$$u \times (v \times w) = \langle u, w \rangle v - \langle u, v \rangle w$$

and

$$\|u \times v\|^2 = \langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2.$$

Proof: To shorten the notation, let us denote $z(t) = \tilde{\omega}(t) - 2\tilde{\omega}(t) \times \tilde{\omega}(t)$. We will prove that $z(t) = \tilde{\omega}^{(iii)}(t) - 2\tilde{\omega}(t) \times \tilde{\omega}(t) = 0$ by showing that the inner product of $z$ and three non-zero orthogonal vectors is zero, namely: $\langle z, \tilde{\gamma} \rangle = 0$, $\langle z, \tilde{\gamma} \rangle = 0$ and $\langle z, \tilde{\gamma} \times \tilde{\gamma} \rangle = 0$.

We will start by showing that $\tilde{\omega}^{(iii)} \times \tilde{\gamma} + 2\tilde{\omega} \times \tilde{\gamma}$ is parallel (in $\mathbb{R}^3$) to $\tilde{\gamma}$. Differentiating $\tilde{\omega}$ with respect to $t$ we have the following relations

$$\dot{\tilde{\omega}} = \tilde{\gamma} \times \tilde{\gamma}, \quad \ddot{\tilde{\omega}} = \tilde{\gamma}^{(iii)} \times \tilde{\gamma} + \tilde{\gamma} \times \tilde{\gamma}, \quad \text{and} \quad \tilde{\omega}^{(iii)} = \tilde{\gamma}^{(iii)} \times \tilde{\gamma} + 2\tilde{\gamma}^{(iii)} \times \tilde{\gamma}.$$

Applying the above calculations to $\tilde{\omega}^{(iii)} \times \tilde{\gamma}$ and $\tilde{\omega} \times \tilde{\gamma}$, and by the properties of the vector calculus in $\mathbb{R}^3$, cf. Lemma 6.5.3, we see that

$$\tilde{\omega}^{(iii)} \times \tilde{\gamma} = -\tilde{\gamma}^{(iii)} + 2 \langle \tilde{\gamma}^{(iii)}, \tilde{\gamma} \rangle \tilde{\gamma} + \langle \tilde{\gamma}^{(iii)}, \tilde{\gamma} \rangle \tilde{\gamma}$$

and

$$\tilde{\omega} \times \tilde{\gamma} = -\langle \tilde{\gamma}, \tilde{\gamma} \rangle \tilde{\gamma} + \langle \tilde{\gamma}, \tilde{\gamma} \rangle \tilde{\gamma} + \langle \tilde{\gamma}^{(iii)}, \tilde{\gamma} \rangle \tilde{\gamma}.$$
where we used $\langle \gamma, \gamma \rangle = 1$ and $\langle \hat{\gamma}, \hat{\gamma} \rangle = 0$. By the hypothesis, $\gamma$ is a $\mathcal{D}^1$-curve and by Lemma 5.3.4 the (5.12) holds, therefore

$$\omega^{(ii)} \times \gamma + 2\omega \times \hat{\gamma}$$

$$= -\gamma^{(iv)} + 2 \left( \langle \gamma^{(iv)}, \gamma \rangle \hat{\gamma} + 2 \langle \gamma, \gamma \rangle \hat{\gamma} \right) + 2 \left( \langle \gamma^{(iv)}, \hat{\gamma} \rangle \hat{\gamma} + 2 \langle \gamma^{(iv)}, \hat{\gamma} \rangle \hat{\gamma} \right) \gamma = c\gamma,$$

(6.7)

where $c = -\lambda + \langle \gamma^{(iv)}, \gamma \rangle + 2 \langle \gamma^{(iv)}, \hat{\gamma} \rangle = 2 \left( \langle \gamma^{(iv)}, \hat{\gamma} \rangle + \langle \gamma^{(iv)}, \hat{\gamma} \rangle \right) = 0$.

We will show first that $\langle \dot{e}, \hat{\gamma} \times \gamma \rangle = \langle \omega^{(ii)} \times \omega \rangle = 0$. By the properties of the vector product

$$0 = \langle \omega^{(ii)} \times \gamma + 2\omega \times \hat{\gamma}, \dot{\gamma} \rangle$$

Next, we will show that $\langle \dot{e}, \hat{\gamma} \times \gamma \rangle = 0$. By (6.7) and because $\gamma \times (\dot{\gamma} \times \gamma) = \hat{\gamma}$

$$\langle \omega^{(ii)} \times \gamma + 2\omega \times \hat{\gamma}, \dot{\gamma} \rangle = \langle \omega^{(ii)} \times \gamma, \dot{\gamma} \rangle + \langle 2\omega \times \hat{\gamma}, \dot{\gamma} \rangle = \langle \omega^{(ii)} \times \gamma, \dot{\gamma} \rangle - \langle \omega^{(ii)} \times \gamma, \omega \rangle$$

It remains to show that $\langle \dot{e}, \hat{\gamma} \times \gamma \rangle = 0$.

$$\langle \omega^{(ii)} \times \gamma + 2\omega \times \hat{\gamma}, \dot{\gamma} \rangle = 2 \langle \gamma^{(ii)} \times \gamma, \dot{\gamma} \rangle - 2 \langle \gamma^{(ii)} \times \gamma + \dot{\gamma} \times \dot{\gamma}, \omega \times \gamma \rangle$$

what proves (6.6).

Q.E.D.

Lemma 6.5.2 specifies the necessary conditions satisfied by the curve $\omega : I \rightarrow \mathbb{R}^3$ defined by $\omega = \dot{\gamma} \times \gamma$, where $\gamma : I \rightarrow S^2$ is a $\mathcal{D}^1$-curve in the unit sphere. Recall from Section 6.3.1 that the projective plane can be identified as the quotient $S^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the isometric action $p \mapsto -p$. Let $\gamma = \pi(\dot{\gamma})$ be a curve in the projective plane. The equivalent class of $\gamma$ consists of the two elements $\{\dot{\gamma}, -\dot{\gamma}\}$. Because $\mathbb{Z}_2$ is an isometry, both $\dot{\gamma}$ and $-\dot{\gamma}$ share the same geometrical (intrinsic) properties. In particular, if $\dot{\gamma}$ is a $\mathcal{D}^k$-curve in the unit sphere so is the $-\dot{\gamma}$, see the introduction to Chapter 5. Furthermore, the formula $\omega = \dot{\gamma} \times \gamma$ holds for the $-\dot{\gamma}$ as well. Hence, if $\dot{\gamma}$ is a $\mathcal{D}^1$ curve in the unit sphere, then $\gamma = \pi(\dot{\gamma})$ is the $\mathcal{D}^1$-curve in the projective plane and Lemma 6.5.2 holds for $\gamma$.

We shall now prove the sufficient set of conditions for a plane curve to be an envelope of a $\mathcal{D}^1$-curve in the projective plane. In this way one may derive plane interpolating curves directly from the system of ordinary differential equations (6.8), subject to additional conditions (6.9).
**Lemma 6.5.4** Let \( \tilde{\omega}: I \to \mathbb{R}^3 \) be a smooth regular curve in \( \mathbb{R}^3 \) satisfying:

- the following system of ordinary differential equations
  \[
  \ddot{\omega}(t) = 2\dot{\omega}(t) \times \omega(t) + c, \quad \text{where } c \in \mathbb{R}^3 \text{ is a constant}
  \]  \hspace{1cm} (6.8)

  and
  \[
  \langle c, \dot{\omega}(t_0) \times \omega(t_0) \rangle = -\|\dot{\omega}(t_0) \times \omega(t_0)\|^2, \quad \text{for some } t_0 \in I,
  \]  \hspace{1cm} (6.9)

- \( \|\dot{\omega}(t) \times \omega(t)\| \neq 0, \text{ for all } t \in I, \) and

- \( \omega^3(t) \neq 0, \text{ for all } t \in I. \)

Then \( \omega(t) = (\omega^1(t)/\omega^3(t), \omega^2(t)/\omega^3(t)) \) is the envelope of some \( S^1 \)-curve \( \gamma: I \to \mathbb{R}P^2 \) in the projective plane.

**Proof:** Let \( \tilde{\gamma}: I \to S^2 \) be a curve in the unit sphere defined by

\[
\tilde{\gamma}(t) = \frac{\dot{\omega}(t) \times \omega(t)}{\|\dot{\omega}(t) \times \omega(t)\|}.
\]

The function \( \tilde{\gamma} \) is well defined since by the hypothesis \( \|\dot{\omega}(t) \times \omega(t)\| \neq 0. \) Then \( \gamma = \pi(\tilde{\gamma}) \) is a well defined curve in the projective plane. We will show that if \( \tilde{\omega} \) satisfies hypothesis of Lemma 6.5.4 then \( \gamma: I \to \mathbb{R}P^2 \) is the \( S^1 \)-curve.

Note first that if \( \tilde{\omega} \) satisfies (6.8) and (6.9) holds for some \( t_0 \in I, \) then (6.9) holds for all \( t \in I. \) This can be easily seen by calculating the difference of both sides of (6.9) and differentiating the result with respect to \( t \) as follows

\[
\frac{d}{dt} \left( \langle \dot{\omega} \times \omega + c, \dot{\omega} \times \omega \rangle \right) = \langle \dot{\omega} \times \omega, \dot{\omega} \times \omega \rangle + \langle \dot{\omega} \times \omega + c, \dot{\omega} \times \omega \rangle = \langle 2\dot{\omega} \times \omega + c, \dot{\omega} \times \omega \rangle = 0,
\]

where we used (6.8). Hence \( \langle \dot{\omega} \times \omega + c, \dot{\omega} \times \omega \rangle \) is independent of \( t \) and therefore (6.9) holds for all \( t \in I. \) As a consequence one has the following equality \( \langle \dot{\omega}, \dot{\omega} \times \omega \rangle = \|\dot{\omega} \times \omega\|^2, \text{ for all } t \in I. \)

In the next step we will show that \( \dot{\gamma} \times \tilde{\gamma} = \tilde{\omega}, \text{ for all } t \in I. \) Indeed, since

\[
\dot{\gamma} = \frac{\ddot{\omega} \times \dot{\omega}}{\|\ddot{\omega} \times \dot{\omega}\|} \times \left( \ddot{\omega} \times \omega, \dot{\omega} \times \omega \right) = \|\dot{\omega} \times \omega\|^{-1} \cdot \left( \ddot{\omega} \times \omega - \langle \ddot{\omega} \times \omega, \tilde{\gamma} \rangle \tilde{\gamma} \right)
\]
\[ \dot{\gamma} \times \dot{\gamma} = \left\| \ddot{\omega} \times \ddot{\omega} \right\|^2 \cdot \left( \ddot{\omega} \times \ddot{\omega} \right) \times \left( \ddot{\omega} \times \ddot{\omega} \right) = \left\| \ddot{\omega} \times \ddot{\omega} \right\|^2 \cdot \left( \dot{\omega} \times \ddot{\omega} \right) \cdot \ddot{\omega} = \ddot{\omega}. \]

Since, by the hypothesis, \( 0 \neq \|\ddot{\omega}\|^2 = \|\dot{\gamma} \times \dot{\gamma}\|^2 = \|\ddot{\gamma}\|^2 \), we see that \( \dot{\gamma} \) is a regular curve in \( S^2 \).

In the last step we will show that if \( \ddot{\omega} \) satisfies (6.8) and \( \dot{\gamma} \times \dot{\gamma} = \ddot{\omega} \), where \( \|\dot{\gamma}\| = 1 \), then \( \dot{\gamma} \) satisfies (5.12). Since \( \dot{\gamma} \times \ddot{\omega} = \dot{\gamma} \times \left( \dot{\gamma} \times \dot{\gamma} \right) = \dot{\gamma} \), therefore \( \ddot{\gamma} = \dot{\gamma} \times \ddot{\omega} + \gamma \times \dot{\omega} \) and \( \ddot{\gamma}^{(iii)} = \dot{\gamma} \times \ddot{\omega} + 2 \dot{\gamma} \times \ddot{\omega} + \ddot{\gamma} \times \left( 2 \ddot{\omega} \times \ddot{\omega} + c \right) \), where we used (6.8). Because \( \|\ddot{\gamma}\| = 1 \) then \( \left( \dot{\gamma} \times \ddot{\gamma} \right) = 0 \) and by the properties of the vector product (Lemma 6.5.3) we have \( \ddot{\gamma}^{(iii)} = \ddot{\gamma} \times c - 3 \left( \hat{\gamma}, \ddot{\gamma} \right) \ddot{\gamma} + \left( \hat{\gamma}, \ddot{\gamma} \right) \ddot{\gamma} \). Therefore the left hand side of (5.12) becomes

\[
\frac{d}{dt} \left( \dot{\gamma} \times c - 3 \left( \hat{\gamma}, \ddot{\gamma} \right) \ddot{\gamma} + \left( \hat{\gamma}, \ddot{\gamma} \right) \ddot{\gamma} \right)
= \dot{\gamma} \times c - 3 \left( \left( \dot{\gamma} \times \ddot{\gamma} \right) + \left( \hat{\gamma}, \ddot{\gamma} \right) \right) \ddot{\gamma} - 3 \left( \hat{\gamma}, \ddot{\gamma} \right) \ddot{\gamma}
- \left( \left( \dot{\gamma} \times \ddot{\gamma} \right) + \left( \hat{\gamma}, \ddot{\gamma} \right) \right) \ddot{\gamma} - \left( \hat{\gamma}, \ddot{\gamma} \right) \ddot{\gamma} + \left( \hat{\gamma}, \ddot{\gamma} \right) \ddot{\gamma}.
\] (6.10)

To verify (5.12) we must prove that (6.10) is equal to \( \left( \left( \ddot{\gamma}, \ddot{\gamma} \right) + 2 \left( \dot{\gamma}, \ddot{\gamma} \right) \left( \hat{\gamma}, \ddot{\gamma} \right) \right) \ddot{\gamma} \). As in the proof of Lemma 6.5.2 we shall calculate the inner product of (6.10) with the three nonzero pairwise orthogonal (in \( \mathbb{R}^3 \)) vectors \( \hat{\gamma}, \dot{\gamma} \times \ddot{\gamma} \) and \( \ddot{\gamma} \).

- Since \( \left( \dot{\gamma}, \ddot{\gamma} \right) = 0 \) the first and the second term of (6.10) are orthogonal to \( \dot{\gamma} \).
  Then clearly the inner product of (6.10) and \( \dot{\gamma} \) is zero.

- By the same reasoning, the inner product of (6.10) and \( \dot{\gamma} \times \ddot{\gamma} \) is equal to

\[
\left( \left( \dot{c}, \ddot{\gamma} \right) - \left( \dot{\gamma}, \ddot{\gamma} \times \ddot{\gamma} \right) \right) \left( \ddot{\gamma}, \dot{\gamma} \right).
\] (6.11)

Since \( \ddot{\omega} = \dot{\gamma} \times \ddot{\gamma} \) then \( \ddot{\omega} = \ddot{\gamma} \times \ddot{\gamma} \) and by the properties of the vector product

\[
\dot{\omega} \times \ddot{\omega} = \left( \dot{\gamma} \times \ddot{\gamma} \right) \times \left( \ddot{\gamma} \times \ddot{\gamma} \right) = \left( \ddot{\gamma}, \dot{\gamma} \times \ddot{\gamma} \right) \ddot{\gamma}.
\]

Applying these calculations to (6.9) we see that

\[
\left( \dot{c}, \ddot{\gamma} \right) \left( \ddot{\gamma}, \ddot{\gamma} \times \ddot{\gamma} \right) = - \left( \ddot{\gamma}, \ddot{\gamma} \times \ddot{\gamma} \right)^2 = - \left( \ddot{\gamma}, \ddot{\gamma} \times \ddot{\gamma} \right) \left( \ddot{\gamma}, \ddot{\gamma} \times \ddot{\gamma} \right).
\]

By the hypothesis \( \dot{\omega} \times \ddot{\omega} \neq 0 \) therefore \( \left( \ddot{\gamma}, \ddot{\gamma} \times \ddot{\gamma} \right) \neq 0 \) and the above equality implies that (6.11) is equal to zero.
Lastly, the inner product of (6.10) and \( \bar{\gamma} \) is equal to
\[
- \left( c, \dot{\bar{\gamma}} \times \bar{\gamma} \right) - \left( \bar{\gamma}, \dot{\bar{\gamma}} \right)^2 - 3 \left( \bar{\gamma}^{(iii)}, \dot{\bar{\gamma}} \right) - 3 \left( \bar{\gamma}, \ddot{\bar{\gamma}} \right).
\] (6.12)

It is a matter of simple calculations to verify that \( \left( c, \dot{\bar{\gamma}} \times \bar{\gamma} \right) = \left( \dot{\bar{\gamma}}, \bar{\gamma} \right)^2 + \left( \bar{\gamma}^{(iii)}, \dot{\bar{\gamma}} \right) \). Applying this result to (6.12) we see that the inner product of (6.10) and \( \bar{\gamma} \) is equal to \(-2 \left( \dot{\bar{\gamma}}, \bar{\gamma} \right)^2 - 4 \left( \bar{\gamma}^{(iii)}, \dot{\bar{\gamma}} \right) - 3 \left( \bar{\gamma}, \ddot{\bar{\gamma}} \right) = 2 \left( \dot{\bar{\gamma}}, \bar{\gamma} \right) \left( \ddot{\bar{\gamma}}, \bar{\gamma} \right) + \left( \bar{\gamma}^{(iii)}, \bar{\gamma} \right)\), and hence \( \bar{\gamma} \) satisfies the hypothesis of Corollary 5.3.5. Thus we have proven that \( \bar{\gamma} \) is a \( \mathcal{Q}_1 \)-curve in the unit sphere.

The proof of Lemma 6.5.4 is now straightforward. Let \(-\bar{\gamma}\) be an image of \( \bar{\gamma} \) under the isometric action \( \mathbb{Z}_2 \), see Section 6.3.1. Since \( \left( -\dot{\bar{\gamma}} \right) \times ( -\bar{\gamma} ) = \dot{\bar{\gamma}} \times \bar{\gamma} = \tilde{\gamma} \), all the above calculations for \( \bar{\gamma} \) remain valid for \(-\bar{\gamma}\) and we conclude that \(-\bar{\gamma}\) is also a \( \mathcal{Q}_1 \)-curve in the unit sphere. Therefore \( \gamma = \pi(\bar{\gamma}) = \pi(-\bar{\gamma}) \) is the \( \mathcal{Q}_1 \)-curve in the projective plane. From the definition of \( \bar{\gamma} \) it is clear that \( \langle \bar{\gamma}, \tilde{\omega} \rangle = 0 \) and \( \langle \dot{\bar{\gamma}}, \tilde{\omega} \rangle = 0 \), therefore \( \langle \bar{\gamma}, \dot{\tilde{\omega}} \rangle = 0 \) and, by the results of Section 6.4, \( \omega \) (well defined since by the hypothesis, \( \bar{\omega}^3(t) \neq 0 \), for all \( t \in I \)) is the envelope of \( \gamma \).

Q.E.D.

Lemma 6.5.2 specifies the sufficient conditions for a plane curve to be the envelope of lines defined by the Riemannian cubic in the projective plane. Combining Lemma 6.5.2 and Lemma 6.5.4 we have the following.

**Theorem 6.5.5** Let \( \tilde{\omega}: I \to \mathbb{R}^3 \) be a smooth regular curve in \( \mathbb{R}^3 \), such that:

i) \( \tilde{\omega}^3(t) \neq 0 \), for all \( t \in I \) and

ii) \[ \left\| \dddot{\tilde{\omega}}(t) \times \ddot{\omega}(t) \right\| \neq 0 \), for all \( t \in I \).

Then the plane curve given by \( \omega(t) = (\tilde{\omega}^1(t)/\tilde{\omega}^3(t), \tilde{\omega}^2(t)/\tilde{\omega}^3(t)) \) is the envelope of some \( \mathcal{Q}_1 \)-curve \( \gamma: I \to \mathbb{R}P^2 \) in the projective plane if and only if
\[
\dddot{\omega}(t) = 2\ddot{\omega}(t) \times \dddot{\omega}(t) + c,
\] (6.13)

where \( c \in \mathbb{R}^3 \) is a constant satisfying
\[
\left( c, \dddot{\omega}(t_0) \times \dddot{\omega}(t_0) \right) = -\left\| \dddot{\omega}(t_0) \times \dddot{\omega}(t_0) \right\|^2, \quad \text{for some } t_0 \in I.
\] (6.14)

**Proof:** Suppose that \( \tilde{\omega}: I \to \mathbb{R}^3 \) satisfies the hypothesis of Theorem 6.5.5. If (6.13) and (6.14) hold then by Lemma 6.5.4 \( \omega: I \to \mathbb{R}^2 \) is the envelope of some \( \mathcal{Q}_1 \)-curve in the projective plane.
segments | error    | ln error |
----------|----------|----------|
2^0       | 6.351 x 10^-1 | -0.454   |
2^1       | 1.588 x 10^-2 | -4.142   |
2^2       | 8.605 x 10^-4 | -7.058   |
2^3       | 5.216 x 10^-5 | -9.861   |
2^4       | 3.236 x 10^-6 | -12.641  |
2^5       | 2.022 x 10^-7 | -15.414  |

Table 6.2: Convergence of the interpolation

To prove the second part of the theorem, let \( \gamma: I \to \mathbb{R}^2 \) be a \( \mathcal{D}^1 \)-curve in the projective plane. Then any regular curve \( \widetilde{\gamma}: I \to S^2 \) such that \( \pi(\widetilde{\gamma}) = \gamma \) is a \( \mathcal{D}^1 \)-curve in the unit sphere. Take \( \tilde{\omega} = \tilde{\gamma} \times \tilde{\gamma} \) and if the hypothesis (i) holds, then \( \omega \) is the envelope of \( \widetilde{\gamma} \), and therefore also of \( \gamma \). By Lemma 6.5.2 the equality (6.13) holds. To complete the proof, we will show that (6.14) holds for all \( t \in I \). Differentiating the expression for \( \tilde{\omega} \) we have:

\[
\tilde{\omega} = \tilde{\gamma} \times \tilde{\gamma}, \quad \text{and} \quad \tilde{\omega} = \tilde{\gamma}^{(iii)} \times \tilde{\gamma} + \tilde{\gamma} \times \tilde{\gamma}.
\]

Hence, by the properties of vector calculus in \( \mathbb{R}^3 \),

\[
\tilde{\omega} \times \tilde{\omega} = \left( \tilde{\gamma} \times \tilde{\gamma} \right) \times \left( \tilde{\gamma} \times \tilde{\gamma} \right) = -\left( \tilde{\gamma} \times \tilde{\gamma}, \tilde{\gamma} \right) \tilde{\gamma}.
\]

By (6.13) it follows that

\[
\mathbf{c} + \tilde{\omega} \times \tilde{\omega} = \tilde{\omega} - \tilde{\omega} \times \tilde{\omega} = \tilde{\gamma}^{(iii)} \times \tilde{\gamma} + \tilde{\gamma} \times \tilde{\gamma} + \left( \tilde{\gamma} \times \tilde{\gamma}, \tilde{\gamma} \right) \tilde{\gamma}.
\]

Therefore, the product

\[
\left( \mathbf{c} + \tilde{\omega} \times \tilde{\omega}, \tilde{\omega} \times \tilde{\omega} \right) = \left( \tilde{\gamma} \times \tilde{\gamma}, \tilde{\gamma} \right) \cdot \left( \tilde{\gamma} \times \tilde{\gamma}, \tilde{\gamma} \right) + \left( \tilde{\gamma} \times \tilde{\gamma}, \tilde{\gamma} \right) + \left( \tilde{\gamma} \times \tilde{\gamma}, \tilde{\gamma} \right) = 0
\]

what verifies (6.14) and completes the proof. Q.E.D.

**Example 6.5.6** We consider a plane curve limaçons defined in Example 6.5.1. Given \( n + 1 \) function's values \( f(t_i) \) and its derivatives \( f(t_i) \), where \( a = t_0 < t_1 < \cdots < t_n = b \), we shall employ Lemma 6.5.4 to find a piecewise interpolation of \( f \) on the interval \([a, b]\), cf. Figure 6.8 on the next page.

On each interval between \( t_{i-1} \) and \( t_i \), \( i = 1, \ldots, n \), we determine the interpolating function \( g: [t_{i-1}, t_i] \to \mathbb{R}^2 \) in the following way. Let the components of \( \omega \) of Lemma 6.5.4 be \( \omega(t) = \omega^3(t) \cdot (g^1(t), g^2(t), 1) \). Then from (6.8) we obtain a system of five ordinary differential equations for the five unknown functions: \( g^1, g^i, v^1, v^2 \).
Figure 6.8: Interpolation of a plane curve (gray thick line) with one, two and four segments of the envelopes (thin black lines) of the $S^1$-curves
In error

regression line: \( y = -0.898 - 2.946 \cdot x; \)
\[ R^2 = 0.997 \]

Figure 6.9: Error of interpolation as a function of density of samples

and \( w^3 \)

\[
\begin{align*}
\omega^3(t) \dot{\omega}^1(t) + 2v^1(t) \dot{\omega}^3(t) - 2v^2(t) \left( \omega^3(t) \right)^2 + g^1(t) \dot{\omega}^3(t) &= c_1, \\
\omega^3(t) \dot{\omega}^2(t) + 2v^2(t) \dot{\omega}^3(t) - 2v^1(t) \left( \omega^3(t) \right)^2 + g^2(t) \dot{\omega}^3(t) &= c_2, \\
\ddot{\omega}^3(t) + 2g^1(t) v^2(t) \left( \omega^3(t) \right)^2 - 2g^2(t) v^1(t) \left( \omega^3(t) \right)^2 &= c_3, \\
\dot{g}^1(t) &= v^1(t), \\
\dot{g}^2(t) &= v^2(t).
\end{align*}
\]

We find \( g = (g^1, g^2) \) by solving the above system, subject to the boundary conditions at the endpoints of the interval \([t_{i-1}, t_i]\), i.e., \( g(t_j) = f(t_j) \) and \( v(t_j) = \dot{g}(t_j) = \dot{f}(t_j) \), for \( j = i - 1, i \). In addition, (6.9) translates to one more boundary condition for \( w^3 \). The results of the interpolation of the limacons on the interval \([-1.89, -0.4]\) are illustrated on Figure 6.9. The linear regression analysis results, \( Y = -0.898 - 2.946 \cdot X; R^2 = 0.997 \), indicate the interpolation rate \( O(n^{-0.95}) \), where as a measure of accuracy of the interpolation we have used the following error formula

\[
\text{error} = \left( \int_a^b \| f(t) - g(t) \|^2 \, dt \right)^{1/2},
\]

cf. Table 6.2 on page 143. It is worth noting that extending the length of a single segment interval \([t_{i-1}, t_i]\) may cause the function \( w^3 \) to pass through zero, what in turn makes the interpolating curve singular, see the hypothesis of Lemma 6.5.4.
6.6 Conclusion

Riemannian variational curves provide a geometrical setup for approximation extending beyond the Euclidean spaces. This chapter investigated the interpolation of undirected lines in the plane. The problem of interpolation may be reformulated to the projective plane—the Riemannian manifold locally isomorphic to the unit sphere $S^2$—as an interpolation of points. For example, the interpolation with piecewise geodesics in the projective plane ($\mathbb{R}P^2$) produces piecewise linear interpolation in the plane ($\mathbb{R}^2$). The interpolation with the Riemannian cubics in $\mathbb{R}P^2$, the case investigated in this chapter, produces a differentiable curve in $\mathbb{R}^2$ determined by four adjacent directions. The interpolating curve, which is an envelope of the Riemannian cubic in $\mathbb{R}P^2$, satisfies a system of ordinary differential equations (6.6). Two numerical examples, Example 6.5.1 and Example 6.5.6, demonstrate the effectiveness of the above interpolation method.

In summary, this chapter contributed the following to existing research:

- it derived a system of ordinary differential equations satisfied by the envelope of the Riemannian cubic in the unit sphere (Lemma 6.5.2), and
- it derived a sufficient conditions for a plane curve to be the envelope of the Riemannian cubic in $\mathbb{R}P^2$ (Lemma 6.5.4).

The next chapter summarizes the research undertaken by the author and provides a number of open questions for future investigations related to the approximation with the Riemannian variational curves.
Chapter 7

Summary

The aim of this chapter is to provide a concise summary of the main discoveries and new findings that emerged from the research conducted. In addition, this chapter demonstrates how the methodologies adopted are appropriate tools of investigation and have resulted in new and significant findings that extend existing research. The effectiveness of the methods of investigation are shown by referring to the results obtained in Section 4.3.4, Section 4.4, and Section 6.5 of the thesis. While the research is informed by the work of Noakes [74] and Noakes et al. [77] significant new ground is covered in relation to the Riemannian mean and the Riemannian variational curves, see Section 4.6, Section 5.3.3 and Section 5.3.4. With respect to the research undertaken, the author acknowledges the limitations and signals further areas for research, which were not within the scope of the thesis.

7.1 Introduction

In this thesis, geometrical methods of inference have been studied. These methods allow for generalizations to non-Euclidean spaces. Both the geometrically defined mean value—*the Riemannian mean* as well as a class of minimizing curves—*the $D^k$-curves*, are the quantities defined in terms of minimizations of certain functions and functionals on complete Riemannian manifolds. Such variational definitions ensure isometric invariance of these quantities, as shown in Section 4.1 and Section 5.2. In the first instance, the Riemannian mean $\overline{Q}$, studied in Part II, is a set of points that minimize $\Phi_Q: M \rightarrow \mathbb{R}$ defined as a sum of squares of distances from the data points $Q$. The means defined in this way directly depend on Riemannian geodesics, because the
Riemannian distance is defined in terms of minimizing curves—geodesics.\(^1\) Hence, studies of the Riemannian mean in this thesis and elsewhere (cf. [49]) largely coincide with studies of geodesics. Secondly, the Riemannian variational curves studied in Part III are generalizations of the Riemannian geodesics in the following sense. While Riemannian geodesics are the critical curves to the functional of energy, cf. [69], variational curves, studied in this thesis, are the critical curves to functionals defined in terms of higher order covariant derivatives. For example, the Riemannian cubics are the critical curves to the functional \(\int_I \| D_t \gamma(t) \|^2 dt \), where \(D_t\) is the covariant derivative along \(\gamma : I \to M, I \subset \mathbb{R}\) is some interval and \(M\) is a complete manifold. The main contribution of this thesis is outlined below, in Section 7.2. We conclude with directions for further research in Section 7.3.

### 7.2 Principal Findings of the Research

In Part II we investigate the Riemannian mean in complete Riemannian manifolds. The allowance for the mean to be a set, rather than a single point, solves the problem of ambiguity in symmetric spaces, for example for symmetrically distributed points in the unit circle \(S^1\). The global minimum admits a singleton, for example when in the case of any two non-antipodal \((q_1 \neq -q_2)\) points in \(S^1\). Our approach contrasts with the center of mass defined for continuous densities of subsets of a complete manifold \(M\), cf. [28], where the center is a critical point of the energy functional, see Remark 4.1.3. Another approach is taken by Corcuera & Kendall [28], who define the Riemannian barycenter of a probability measure \(\mu\) on a Riemannian manifold by lifting \(\mu\) to a measure on the tangent bundle \(TM\). This generalization allows for situations where geodesics are not unique and are no longer minimizing the length, i.e., at and beyond the cut points along geodesics. We prove the Cut Locus Theorem (Theorem 4.6.2) ensuring conditions upon no cut point along any Riemannian geodesic from the data point belongs to the Riemannian mean. Theorem 4.3.1 and Corollary 4.3.5 demonstrate the dependence of the local minimum of \(Q\) on the curvature tensor and the covariance of a sample \(Q\). It is particularly useful for regular spaces with a constant sectional curvature, for example the unit sphere \(S^n\) or the group of rotations \(SO(3)\). Based on the convexity property of the center of mass in Euclidean spaces, Lemma 4.5.1 proves that the Riemannian mean belongs to any geodesically convex set containing \(Q \subset M\), if \(M\) is a 2-dimensional complete manifold. An iterative fast converging method of derivation of the Riemannian mean, whose convergence is proven by Theorem 4.4.1, is illustrated by Example 4.4.2 for the case \(M = SO(3)\).

\(^1\)Every minimizing curve, given a unit speed parametrization, is a geodesic, cf. [58]
Part III is devoted to the method of inference through variational curves. Our motivation is the problem of approximation of a convex shape in the plane, given a set of tangential lines. We reformulate this problem in terms of interpolation of points in the projective plane ($\mathbb{RP}^2$) by Riemannian variational curves. Variational curves were investigated in Euclidean spaces in many contexts, cf. [26], but little attention has been given to the non-Euclidean case. The first published paper, known to the author, regarding the Riemannian cubics was Noakes, Heinzinger & Paden [77]. The subject of variational curves in Riemannian manifolds was further studied in [23, 19, 90, 24].

In Chapter 5 we investigate the class of $\mathcal{D}^k$-curves $\gamma : I \rightarrow M$ defined as the critical curves to the functional $\int_{\gamma} \left\| D_{\gamma(t)}^{k} \gamma(t) \right\|^2 dt$. We estimate the minimal interval on which a unique analytic $\mathcal{D}^1$-curve, often referred to as the Riemannian cubics, in $\mathbb{SO}(3)$ exists (local existence and uniqueness was proven in Camarinha [23]). Furthermore, we investigate the Riemannian cubics in $\mathbb{SO}(3)$ endowed with left-invariant metric in Section 5.3.4. With clearly defined applications in mind, in Section 5.3.3, we derive the necessary and sufficient conditions for a $\mathcal{D}^1$-curve in the unit sphere $S^n$. These conditions are the key to our approach, where we regard the projective plane as a quotient of the unit sphere $S^2$ by the equivalence relation $Z_2$, that identifies antipodal points. The metric on $\mathbb{RP}^2$ is the standard metric of the unit sphere $S^2 \subset \mathbb{R}^3$ induced by the Euclidean metric. Therefore, the results concerning geometry of Riemannian cubics on the unit sphere also hold in the projective plane. In Section 6.4 we show that the interpolation with piecewise geodesics in $\mathbb{RP}^2$ produces piecewise linear interpolation in the plane. Section 6.5 investigates the interpolation with the Riemannian cubics in $\mathbb{RP}^2$. The envelope of the cubic is a differentiable curve in $\mathbb{R}^2$ determined by four adjacent directions. The interpolating curve, which is an envelope of the Riemannian cubic in $\mathbb{RP}^2$, satisfies a system of ordinary differential equations (6.6). Two numerical examples, Example 6.5.1 and Example 6.5.6, demonstrate the effectiveness of the interpolation method with Riemannian cubics.

### 7.3 Limitations and Future Directions

The author's interest in studying the Riemannian mean was influenced by the Noakes' research ([74], unpublished) of spherical means. It soon became apparent that studying the global minima of functions defined on general non-Euclidean spaces is a complex task. We took the approach of studying critical points of $\Phi_Q$ because the methods of investigation of local problems are well established. It also became apparent that there was a strong connection between the critical points of the function $\Phi_Q : M \rightarrow \mathbb{R}$ and properties of the exponential map. Some authors, notably Corcuera
& Kendall [28], generalize the notion of the mean to the cases where the exponential map is not well defined. As shown by Theorem 4.6.2, under certain conditions, the cut points are not the points of local minimum of $\Phi_Q$. Therefore such points do not belong to the Riemannian mean. It is interesting to know, if the same is true for conjugate points, i.e., if the Riemannian mean can contain a conjugate point to any point of a sample. Although this fact has been already proven for spheres by Noakes, were the conjugate (and cut) points are precisely the antipodal points, in general non-symmetrical spaces, the notion of conjugate points seems to be too general. The answer to this particular question requires different methods and is beyond the scope of this thesis. Future research would need to study the conjugate points in the case of symmetric spaces, for example symmetric Lie groups and complex projective spaces.

The approach adopted in this thesis was to study the Taylor series of the function resulted in deriving the formula (4.23) of Corollary 4.3.5, showing the dependence of the critical points of $\Phi_Q$ on the curvature tensor and the covariance of the sample points $Q$. Future author's investigations will concern higher order functions generalizing statistical moments. The studies of moments of random objects on Hausdorff and connected manifolds can be found in Oiler & Corcuera [78]. Another extension of the Riemannian mean is an analog of linear regression, i.e., the problem of fitting a geodesic, or more general, a subspace, into sample points. For example, fitting a subgroup of a Lie group given a sample. Noakes [74] investigates fitting the great circle into a set of sample points in the unit sphere.

The author's study of the Riemannian variational curves was motivated by, and positively solved in Chapter 6, the problem of interpolation in the projective plane. Intensive studies of variational curves, other than geodesics, in non-Euclidean spaces can be found in [77, 24, 30, 19, 90, 23]. In particular, Camarinha [23] investigates the Riemannian cubics. Camarinha shows complexities involved when one wants to determine minimizing properties of the variational curves. Therefore, similar to [24], the author defines the class of $D^k$-curves as critical curves to certain functionals. As shown in Camarinha [23] and Silva et al. [90], the $D^k$-curves do not reduce to geodesics (although the elastic curves do) in $TM$. Hence, the $D^k$-curves represent a new class of curves, which are distinct from geodesics. Current research investigates whether some properties of the variational curves in the Euclidean spaces also hold in the non-Euclidean context. For example, the corner cutting algorithm that generates polynomial curves in $\mathbb{R}^n$ was studied in Noakes [73, 76] for a case of Riemannian manifolds. An interesting question to ask is whether the diminishing property of splines generalizes for the Riemannian cubics. The studies of this problem by the methods similar to those found in Lane & Riesenfeld [57] are in progress. A natural continuation of the studies of interpolation by the Riemannian cubics would involve
the envelopes of planes in $\mathbb{R}^3$ (or hyperplanes in higher dimensions) instead of lines in $\mathbb{R}^2$ and interpolating functions of several variables. It is clear that most of the questions related to higher order variational curves in the non-Euclidean context are still to be answered.

This research demonstrates that the adoption of geometrical methods of inference to investigate averages and approximation have brought new insight into well known quantities like mean values and interpolation. In addition, the research indicates new areas for investigation: fitting subspaces (subgroups) into samples and interpolation in higher dimensions in non-Euclidean spaces. Nevertheless, the geometrical generalizations are complex and require further studies. The results are significant as they show that it is possible to derive the geometrically intrinsic average value of sample points in a complete Riemannian manifold and apply variational methods in Riemannian spaces to the interpolation of undirected lines.
Part IV

Appendices
Appendix A

Riemannian Metric

A.1 Taylor Series Derivation

In this section we derive first few terms of the Taylor series of components of a geodesics expressed in local coordinates. These results are used to derive an approximations in Section 4.3.1 of this thesis.

Let $\gamma : I \to M$ be a geodesic. By (3.4) in Section 3.3 the component functions $\gamma(t) = (\gamma^1(t), \gamma^2(t), \ldots, \gamma^n(t))$ satisfy the the geodesic equation

$$\dot{\gamma}^k(t) = -\Gamma^k_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t), \quad (A.1)$$

for all $t \in I$, where $\Gamma$ are the Christoffel symbols. In order to calculate higher time-derivatives of geodesics we differentiate (A.1) in respect with parameter $t$, replacing terms $\dot{\gamma}$ with (A.1). Continuing in this way we are able to find relations between higher order derivatives of $\gamma$ and its velocity $\dot{\gamma}$.

A.1.1 Non-symmetric connection

Assume at first a more general case, when the connection is not symmetric, therefore the Christoffel symbols are not necessarily symmetric in lower indices.

By differentiating (A.1) in respect with parameter $t$ and replacing $\dot{\gamma}$ with (A.1) we get

$$\gamma^{(ii)}(t) = -\partial_i \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^l + \Gamma^k_{ij} \Gamma^l_{im} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^m + \Gamma^k_{ij} \Gamma^l_{mi} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^m, \quad (A.2)$$

where we omitted parameters. Differentiating (A.2) in respect with parameter $t$ and replacing $\ddot{\gamma}$ with (A.1) we get

$$\gamma^{(iii)}(t) = -\partial_i \partial_j \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^l + \Gamma^k_{ij} \Gamma^l_{im} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^m + \Gamma^k_{ij} \Gamma^l_{mi} \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^m \dot{\gamma}^p$$
Similarly one can calculate higher order terms but this complicates very rapidly. Note that $\tilde{\gamma}$ is a sum of terms of order $O(||\gamma||^2)$, $\gamma^{(iii)}$ is a sum of terms of order $O(||\gamma||^3)$. In general $\gamma^{(n)}$ is a sum of terms of order $O(||\gamma||^n)$ and higher.

A.1.2 Symmetric connection

In the case of the Riemannian connection, which is symmetric (Section 3.2), the Christoffel symbols are symmetric in lower indices. Calculations in Section A.1.1 simplify: an analogue of (A.2) is

$$\gamma^{(ii)k} = -\partial_k \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j \gamma^m + 2\Gamma^m_{ij} \Gamma^l_{im} \dot{\gamma}^i \dot{\gamma}^j \gamma^m \gamma^p,$$

and an analogue of (A.3) is

$$\gamma^{(iii)k} = -\partial_k \partial_m \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j \gamma^m \gamma^p + 2\Gamma^k_{ij} \partial_p \Gamma^l_{jm} \dot{\gamma}^i \dot{\gamma}^j \gamma^m \gamma^p + 4\Gamma^m_{ij} \partial_p \Gamma^l_{im} \dot{\gamma}^i \dot{\gamma}^j \gamma^m \gamma^p + 4\Gamma^k_{ij} \partial_m \Gamma^l_{im} \dot{\gamma}^i \dot{\gamma}^j \gamma^m \gamma^p - 4\Gamma^k_{ij} \partial_p \Gamma^l_{mp} \dot{\gamma}^i \dot{\gamma}^j \gamma^m \gamma^p.$$

As a consequence, discarding higher order terms of the Taylor series, one has the following:

$$\gamma^k(1) = \gamma^k + \dot{\gamma}^k - \frac{1}{2} \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j \gamma^m - \frac{1}{6} \partial_l \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j \gamma^l + \frac{1}{3} \Gamma^m_{ij} \Gamma^k_{jm} \dot{\gamma}^i \dot{\gamma}^j \gamma^m \gamma^p + \frac{1}{12} \Gamma^k_{ij} \partial_m \Gamma^l_{jm} \dot{\gamma}^i \dot{\gamma}^j \gamma^m \gamma^p + \frac{1}{24} \Gamma^m_{ij} \partial_p \Gamma^k_{jm} \dot{\gamma}^i \dot{\gamma}^j \gamma^m \gamma^p + \frac{1}{12} \Gamma^m_{ij} \partial_p \Gamma^k_{jm} \dot{\gamma}^i \dot{\gamma}^j \gamma^m \gamma^p,$$

where the terms on the right hand side are calculated at the point $t = 0$. 
Appendix B

Data Sets

B.1 Spherical Data

Data from [41] of magnetic remanence in specimens of Paleozoic red-beds from Argentina. The original data is presented in second and third columns of Table B.1 on the following page in terms of geological coordinates: inclination (inc.) and declination (dec.). The inclination is the angle between the vector and the horizontal plane, and its declination (azimuth) is the angle between North and the projection of the vector onto the horizontal plane. The relations between geological (dec., inc.), spherical ($\phi, \theta$), polar ($\rho, \psi$) and normal coordinates ($q^1, q^2$) is summarized below (see Example 4.3.13)

\[
\begin{align*}
\phi &= 360 - \text{dec. in degrees} \\
\theta &= 90 + \text{inc. in degrees} \\
x^1 &= \cos \phi \cdot \sin \theta \\
x^2 &= \sin \phi \cdot \sin \theta \\
x^3 &= \cos \theta \\
\rho &= 2 \sin(90 - \theta/2) \text{ in degrees} \\
\psi &= \phi \\
q^1 &= (\pi - \theta) \cdot \cos \phi \text{ in radians} \\
q^2 &= (\pi - \theta) \cdot \sin \phi \text{ in radians}
\end{align*}
\]
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<th>inc.</th>
<th>(\phi)</th>
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Table B.1: Spherical Data
Table B.2: Ten samples of size 10 from a normal population

### B.2 Normal Population Samples

Data from [6]: ten samples of size $n = 10$ from a normal population with a mean $\mu = 69$ and standard deviation of $\sigma = 3$. We transformed the original data using:

$$
\mu \mapsto \frac{\mu - \hat{\mu}}{\sigma/\sqrt{n}} \quad \text{and} \quad \sigma \mapsto \frac{\sigma}{\sqrt{n}}.
$$

This data is represented in Table B.2 in three different coordinate systems (see Example 4.3.14):

- **Poincaré half-plane**: $x = \mu$ and $y = \sigma$;
- **polar-coordinates**: $(r, \theta)$; and
- **normal coordinates**: $(q^1, q^2)$.  

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160 data sets
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