REALISTIC ROCK MASS MODELLING AND ITS ENGINEERING APPLICATIONS

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ABSTRACT

Rock masses in nature contain various discontinuities of different forms. These discontinuities play a dominant role in the determination of the mechanical behaviours of rock masses. Due to the presence of the discontinuities, there exist a number of bottlenecks in order to realistically model the deformation and stability of a rock mass when a numerical approach is adopted. For example, a rock mass containing a discrete fracture network should be numerically represented by a three dimensional model, as a two dimensional model is not possible to give out accurate deformation or stability of a discrete three-dimensional block system. There are also various uncertainties related to the discontinuities. The fracture networks in rock masses are stochastic and a realistic numerical simulation should be able to consider the randomness of the discontinuity size, shape and distribution. Besides, numerical simulation of the deformation and stability of such a discrete rock mass system is extremely time consuming, especially when an open-close iteration algorithm is used to all the contacts of the three-dimensional blocks. An efficient numerical tool with a realistic computational cost when simulating a discrete rock mass system is highly desired.

This thesis intends to overcome the obstacles in realistically modelling the stability and deformation of discontinuous rock masses. A three-dimensional robust geological modelling tool is developed to generate blocky rock masses. It can deal with planar or non-planar, finite or infinite, convex or concave discontinuities. To ensure robustness, careful tolerance management, adoption of robust algorithms and other techniques have been implemented to make the developed program reliable. Measures such as usage of compact data structure, avoidance of unnecessary calculations are also taken to improve efficiency. In addition, several methods are used to verify the developed algorithm.

The traditional key block theory, which is considered as the best alternative in stability analysis of a rock mass with a huge number of rock blocks, is then extended to carry out progressive failure analysis of rock masses by considering the interactions of adjacent batches of key blocks. A computer program is developed and it is capable of searching different batches of key blocks progressively. A force transfer algorithm is proposed to consider the effects of the key blocks in later batches batch to batch to the first batch key blocks. After reinforcement, a two-step safety check on the stability of the
reinforced rock blocks is carried out. After checking the stability of surface key blocks, a second step safety check is used to find out whether larger instable key blocks are formed. Results from the examples show that the rock support design based on extended key block analysis is more realistic and rational.

Moreover, in order to cover the uncertainties, a stochastic key block analysis method is developed. With Monte Carlo simulations, the geological modelling tool dealing with non-persistent discontinuities is employed to generate a number of realizations of the blocky rock mass from the corresponding number of realizations of the discontinuity network for stability analysis. Two examples are used to show the applicability of the stochastic analysis to rock engineering problems. The discontinuity size effect on stability has been studied as well and the persistence of discontinuities has been shown to be critically important for the accurate prediction of key block statistics. Thus representing the discontinuity size more accurately in the rock mass model is important for the accurate prediction of the key block statistics.

Finally, a Discrete Boundary Element Method (DBEM) is developed by implementing contact algorithm into the dual reciprocity BEM with stepwise updating. In this method, the boundary of each block in the discrete blocky system is discretized with boundary elements while the domain of each block is divided into internal cells only for the domain integral of initial stresses. The open-close iterations are employed to ensure the computational accuracy of block interactions and an implicit time integration scheme is adopted for numerical stability. Some examples are used to verify and show the ability of the developed DBEM dealing with discontinuous contact problems.
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LIST OF ABBREVIATIONS

2D: Two Dimensional
3D: Three Dimensional
3DEC: Three Dimensional Distinct Element Code
BEM: Boundary Element Method
DBEM: Discrete Boundary Element Method
DDA: Discontinuous Deformation Analysis
DEM: Discrete Element Method
DFN: Discrete Fracture Network
DIPS: Data Interpretation Package using Stereographic projection
DQM: Differential Quadrature Method
DRBEM: Dual Reciprocity Boundary Element Method
FDM: Finite Difference Method
FEM: Finite Element Method
ID: Identification
KBT: Key Block Theory
K-S: Kolmogorov-Smirnov
MCS: Monte Carlo Simulation
NADIS: Numerical Analysis of Discontinuity Size
NMM: Numerical Manifold Method
PDF: Probability Density Function
RBF: Radial Basis Function
TC: Three Dimensional Cutting
UDEC: Universal Distinct Element Code
CHAPTER 1. INTRODUCTION

1.1 BACKGROUND

Rock mass, as a typical natural geological material, are complex with different extents of discontinuities, such as bedding planes, faults, shear zones, joints, etc. These discontinuities play a dominant role in the determination of the mechanical behaviours of the rock mass. Due to the special nature of the rock mass, a realistic description is difficult, which causes various inaccuracies in numerical modelling of rock mass deformation and stability.

![Figure 1-1 Fractured rock masses](The photo on the left was taken by Guoyang Fu at Dairy Farm Quarry in Singapore; the photo on the right was taken by Da Huang at Jinping I hydropower station, China)

Problems of fractured rocks (Figure 1-1) are rarely two dimensional, simply because of the three-dimensional nature of the fracture system geometry and the anisotropy and inhomogeneity of the rock matrix and fracture properties. Two-dimensional simplifications have been extensively used and they do have significant theoretical roles to play in research and development. However, it is the three dimensional models and their solutions that are the ultimate objectives for numerical solutions of rock engineering problems. The achievement of this goal depends significantly on the representation of three dimensional fracture-block systems.

Uncertainties are very common in rock engineering (Jing, 2003; Nelson, 2010; Ma, 2011). The most critical uncertainty comes from the difficulty in accurately describing...
the spatial distribution of the discontinuities inside the rock mass. Since the discontinuities cannot be observed or measured completely at a site, they are usually inferred from one- or two-dimensional data collected from field survey assuming orientation, dimensions and locations following a certain distribution respectively. Thus, the current prevailing deterministic numerical analysis is infeasible to cover the uncertainties. In addition, the complexity of the mechanical properties of the rocks and discontinuities in a rock mass also caused uncertainty. Usually rock and rock discontinuity properties are obtained from laboratory tests on samples of small sizes, which may not be able to reflect the real discontinuities well. On the other hand, in-situ experiments are difficult and expensive.

Key block theory has been widely used in rock engineering due to its effectiveness in stability analysis. Although progressive failure analysis can be conducted by searching the key blocks batch by batch (Wibowo, 1997; Thompson, 2002), the key blocks in later batches will affect the stability of key blocks in earlier batches. This phenomenon was not considered in previous studies.

For accurate analysis of fractured rock masses, the discrete element methods are still required. Under external loadings, some blocks may be broken into a number of sub-blocks. In order to simulate this fracturing process, an accurate description of the stress field in the block is needed. The traditional way is to employ the finite difference or finite element mesh into discrete element methods (Itasca, 2004; 2007; Munjiza, 2004; Shyu, 1993), in which domain mesh is used for each block. Computational cost is an important issue in discontinuous numerical modelling even when deformation of blocks is not considered. Therefore, every measure should be taken to improve efficiency. Due to its advantage of reducing dimensions by one, the Boundary Element Method (BEM) could be a good alternative to determine the stress distribution of each block in the discrete element methods.

The goal of this thesis is to make some attempts towards realistic modelling of fractured rock masses. A three dimensional modelling tool for fractured rock masses is developed to take into consideration of the finiteness of discontinuities first. Then the traditional key block method is extended to consider the interaction of blocks in adjacent batches for stability analysis. Probabilistic models have also been used to cover the uncertainties
in rock masses. At last, a discrete boundary element method is developed by implementing the contact mechanics with open-close iterations into the dual reciprocity boundary element method for efficiency improvement.

1.2 OBJECTIVES AND SCOPE OF THE THESIS

The main objective of this thesis is to develop a realistic modelling tool for fractured rock masses. The specific targets are outlined as follows:

- To develop a three-dimensional robust geological modelling tool by taking into account non-persistent discontinuities to generate blocky systems:
  - Formation of excavation surfaces including slope and tunnel geometries to facilitate engineering analysis;
  - Development of the three dimensional modelling tool to form discrete blocky systems using the block geometrical identification techniques;

- To extend the Key Block Theory by considering interactions of adjacent batches of key blocks for practical rock stability analysis and support design:
  - To develop a force transfer algorithm to consider the interactions of adjacent batches of key blocks;
  - To improve the safety check by checking whether surface key blocks are anchored by rock bolts to inner blocks, forming larger instable blocks;

- To implement stochastic methods in Key Block Theory in order to cover the uncertainties in rock engineering
  - Uncertainties in discontinuity geometry
  - Uncertainties in properties of rock and discontinuities

- To develop a Discrete Boundary Element Method (DBEM) for possible efficiency improvement of discrete element methods:
  - To further develop the Dual Reciprocity Boundary Element Method (DRBEM) with stepwise updating for large displacement analysis;
  - To develop a Discrete Boundary Element Method by implementing contact algorithm with open-close iterations into the DRBEM.

1.3 STRUCTURE OF THE THESIS

This thesis is divided into eight chapters.
In Chapter 1, a general introduction of the thesis is given, including the challenges encountered in rock engineering and their requirements on realistic rock mass modelling. In addition, the objectives and scope of this research are discussed.

Chapter 2 focuses on the literature review on different aspects of rock mass modelling. The geological models of rock masses are reviewed first including procedure from field mapping methods to data processing to the generation of three-dimensional discontinuity networks, and the block identification algorithms. Furthermore, the key block method is reviewed in terms of progressive failure analysis, followed by the uncertainties in rock engineering and methods to cover uncertainties in key block analysis. Discrete element methods used in rock engineering are also briefly reviewed, focusing on how continuum-based methods are combined with discontinuum based methods to enhance the deformability and improve the stress distribution field of rock blocks. At last, the developments of the Dual Reciprocity Boundary Element Method for elasto-dynamics are reviewed. Following the literature review, the issues and problems existing in the current rock mass modelling are listed and the possible solutions are discussed.

Chapter 3 presents an improved and generalized 3D block generation algorithm for engineering analysis. The basic theory of the algorithm and some modelling issues towards robustness and efficiency are discussed in detail. This algorithm can simulate both planar and non-planar discontinuities. The geometry of rock slope, represented by triangulated surfaces or general polygons, can be formed by either sequential cutting or contour map while tunnels with different shapes are modularized. In the resultant block system, there can be tens of thousands of blocks and the blocks can be convex, concave, or blocks with cavities or holes. Several methods are also adopted to verify the generated data in the program. An extensive number of examples are presented at the end to show the capacity of the developed algorithm.

In Chapter 4, the traditional key block methods are extended by considering the interactions of adjacent batches of key blocks. A computer program of key block method using vector analysis is developed to analyse the stability of blocky rock masses and it is capable of searching different batches of key blocks progressively. A force transfer algorithm is proposed and then implemented to consider the effects of the key
blocks in later batches batch to batch to the first batch key blocks. After reinforcement, a two-step safety check on the stability of the reinforced rock blocks is carried out. The first step checks the stability of individual surface key blocks while the second step performs a further safety check to find out whether surface key blocks are anchored into inner blocks, forming larger unstable key blocks. The extended key block analysis is applied to a three-dimensional tunnel example as well as an underground powerhouse project.

**Chapter 5** focuses on stochastic key block analysis based on realistic representation of blocky rock masses to cover uncertainties. With Monte Carlo simulations, the geological model generation tool dealing with non-persistent discontinuities is employed to generate a number of realizations of blocky rock mass from the corresponding number of realizations of the discontinuity network for stability analysis. Therefore, more accurate and reliable results of key blocks statistics from progressive failure analysis can be obtained. This method is demonstrated by the application to a hypothetical horseshoe-shaped tunnel. Some modelling issues are also discussed. In order to investigate how the discontinuity size will affect the stability of blocky rock masses, three scenarios of discontinuity network with different mean sizes are utilized for key block predictions. In the end, the stochastic key block analysis is applied to analyse the stability of the entrance of a tailrace tunnel of the Jinping I hydropower station.

**Chapter 6** introduces a further development of the Dual Reciprocity Boundary Element Method (DRBEM) with stepwise updating in order to be combined with the contact algorithm in the DDA. In the analysis, at the end of each time step, the domain geometry is updated and then considered as a new problem at the beginning of the next time step. The stresses, velocities and accelerations at all nodes are also calculated and used as initial conditions for the next time step. The initial stress involved in the analysis will lead to a domain integral in the governing equations. Internal cells are used for the integration of the initial stress term. Several examples are used to verify the stepwise-updated model.

**Chapter 7** describes the development of a Discrete Boundary Element Method (DBEM). In this method, the contact algorithm with open-close iterations in the DDA is implemented into the DRBEM. The DBEM is suitable for simulating discrete block
systems under different loading conditions. In this method, the boundary of each block in the discrete blocky system is discretized with boundary elements, while internal cells are used in each block only for the domain integral of the initial stress term. In this way, the deformation of blocks becomes flexible and the stress distribution inside the blocks can be more accurately depicted. The open-close iterations are used to ensure the computational accuracy of block interactions and the implicit time integration scheme is adopted for numerical stability. Three examples are used to validate the proposed method and this method is applied to some problems to show its capability of simulating the behaviour of blocky systems.

In the last chapter, **Chapter 8**, the main achievements of the whole PhD work are summarized and the future research works are discussed.
CHAPTER 2. LITERATURE REVIEW

2.1 INTRODUCTION

The discontinuities inside distinguish rocks from other engineering materials and they have a significant effect on the behaviours of the rock masses. The geometry of a rock mass is unique in existence. However, the discontinuities in the rock mass cannot be observed or measured directly, leading to uncertainties in both geometry and mechanical properties. This has caused difficulties in numerical modelling.

In order to model the behaviours of the rock masses realistically, the numerical model should be able to capture the main characteristics of the rock mass and meanwhile control the computational cost within a reasonable range. The main characteristics of a rock mass include that its 3D nature and uncertainties in geometrical and mechanical properties. In terms of efficiency, key block analysis is a good alternative for stability analysis especially in preliminary design due to its simplicity and effectiveness. For accurate analysis, discrete element methods should be employed. Since the computational cost of discrete element methods is quite high especially when deformation of blocks are considered, measures are needed to be taken to improve efficiency. This may require parallel computing techniques or some strategies or techniques to reduce computational cost.

The current study focuses on the 3D geometrical modelling of fractured rock masses, stability analysis using key block analysis, probabilistic analysis to cover uncertainties and development of a discrete boundary element method. Therefore, this chapter attempts to obtain a general view of the present status of rock mass modelling in particular on these four aspects.

2.2 GEOLOGICAL MODELLING OF FRACTURED ROCK MASSES

2.2.1 Discontinuity network modelling

Discontinuities in reality are three-dimensional and represented by orientations, shapes, dimensions, locations and apertures in space. The discontinuities inside the rock mass
play a significant role in the mechanical and hydrological behaviours of the rock mass, so it is important to represent the geometry and the mechanical properties of the discontinuities accurately. It is a common practice to treat only the large-scale discontinuities usually in large numbers as deterministic, the small-scale ones like flaws, micro-cracks and small discontinuities as properties of the rock matrix. The discontinuities of intermediate scales (called joints) are usually described using stochastic methods in an assemblage rather than individually.

Generally, discontinuity modelling involves four phases: field survey of the rock mass; mathematical treatment of the collected data; generation of discontinuity network models; refinement of the simulated realizations by comparison with measured data.

Field survey of the rock mass

Geological survey or field mapping is a fundamental part of the study of the stability conditions of a rock mass. So far different field mapping methods have been suggested and applied in rock engineering. These methods can be classified into two types: traditional mapping methods and advanced mapping methods. The traditional mapping methods include scanline mapping, window mapping and borehole logging. They require direct access to the rock mass that in some cases may be difficult and dangerous. For borehole logging, the discontinuity information is collected from borehole cores or walls. Advanced mapping methods include photogrammetry and laser scanning. When the rock surfaces are not accessible (environmental conditions are severe), the advanced mapping methods (non-contact methods) based on images are appealing. By measuring a very dense point cloud on the rock surface, the orientation and the location of the discontinuities can be derived. In order to achieve more reliable and accurate results, it is suggested that different methods could be combined together to collect data and do the statistical analysis. Landmark and Villaescusa (1992) combined scanline and window mapping in the joint mapping while circular scan lines and windows are combined by Rohrbaugh et al. (2002).

Mathematical treatment of the collected data

The mathematical treatment or the processing of measured data is the most important part for the generation of discontinuity network, and many research works have been
done since 1970s. Only a general procedure is reviewed next. More details can be found in APPENDIX A.

Whether the processing methods are reliable or not determines the generation of the discontinuity network. The complete survey involves the measurements of the discontinuity orientation, spacing (for scanline mapping) or areal frequency (for window mapping), and trace length. Due to the various errors or biases (Baecher, 1983; Einstein et al., 1983; Kulatilake, 1988) involved in the mapping process, careful defined sampling or correction procedures are essential in order to minimize these effects. These biases mainly include orientation bias, censoring bias, truncation bias and bias caused by the finiteness of scanlines or windows. There are several assumptions involved in the processing of collected data from field survey for discontinuity parameters. All the discontinuities are considered to be planar and the discontinuity centres are randomly and independently distributed in space. The discontinuity size distribution is also assumed to be independent of spatial location (Baecher et al., 1977). In addition, one key assumption inherent to most works is that discontinuities occur in sets of primarily parallel discontinuities and that each set has its average characteristics (Grossenbacher et al., 1997). According to the selected sampling method, the estimation method of different parameters (orientation, trace length, etc.) varies.

Shape is one of the most difficult parameters to establish. The real shape of discontinuities is unknown since the rock mass is usually inaccessible in three dimensions. Therefore, when dealing with discontinuity shape, researchers assumed different shapes for different research and application purposes. Polygon representation seems the most general and realistic, but circular shape of discontinuities is commonly assumed mainly due to simplicity. Zhang and Einstein (2010) drew the following conclusion based on their analyses and investigation results: Joints not affected by adjacent geological structures such as bedding boundaries tend to be elliptical (or approximately circular but rarely), while joints affected by or intersecting geological structures such as bedding boundaries tend to be most likely rectangles or similarly shaped polygons. If a very large number of discontinuities are involved, the significance of the discontinuity shape decreases with an increase in the discontinuity population size (Jing and Stephansson , 2007).
The orientation data should be corrected for the orientation bias caused by the relative orientation between the discontinuities and the sampling scanlines or sampling windows. The orientation bias can be corrected by introducing a geometrical correction factor based on the observed angle between the sampling line and the normal to a particular discontinuity (Terzaghi, 1965; Baecher, 1983; Wathugala et al., 1990; Priest, 1993). Wathugala et al. (1990) suggested a general procedure to correct sampling bias on orientation using a vector approach for window mapping.

The relatively short sampling lines that are required where a rock face is of limited extent or where borehole core is logged in short runs. Sen and Kasi (1984) addressed this sampling bias based on the negatively exponential distributed spacing and there is a small error in Sen and Kasi’s original formulas. Kulatilake (1988) and Priest (1993) corrected the small errors. Grossenbacher et al. (1997) present a method for determining discontinuity frequencies form data collected along circular scanlines, which they suggest can be expanded and adapted to the more general case of irregularly curved scanlines. Peacock et al. (2003) advanced the method of Priest and Hudson (1981) to develop a method for a curved scanline to be used to predict the numbers of discontinuities that would be observed in any direction.

The size parameter is commonly determined by taking measurements of trace lengths along exposed rock faces using either scanline or window sampling techniques. Normally, this includes determination of true trace length distribution from measured trace length data and determination of size distribution from true trace length distribution by assuming the shape of the discontinuities. The measured trace length data need to be processed first and the related biases (orientation bias, size bias, truncation bias and censoring bias) should be corrected before the relationship between trace length and size is used to infer the distribution of discontinuity size. Different methods have been proposed to get the mean and standard deviation of corrected trace length from measured trace length data for scanline mapping (Laslett, 1982; Priest, 1993), rectangular (Pahl, 1981; Kulatilake and Wu, 1984) and circular window mapping (Anderson and Dverstorp, 1987; Mauldon, 1998; Zhang and Einstein, 1998; Song, 2006; Zhang and Ding, 2010). Regarding the relationship between size and true trace length, Warburton (1980a; b) derived the stereological relationship between size and trace length over the entire exposure for both straight scanline mapping and window
mapping assuming circular and parallelogram shaped discontinuities respectively. Following the method of Warburton (1980a, b) and assuming elliptical discontinuities, Zhang et al. (2002) derived a general stereological relationship between discontinuity size (expressed by the major axis length $a$ of the ellipse) and trace length over the entire exposure. Detailed procedures from measured trace length data to corrected trace length data and to size distribution have also been proposed (Priest, 1993; 2004; Zhang and Einstein, 2000; 2010; Zhang et al., 2002).

Discontinuity intensity has different definitions based on the dimensions of the measurement region and the discontinuity (Dershowitz and Herda, 1992). It can be defined as number of discontinuities per unit length, area or volume (also called density), length of traces per unit area of trace plane, area of discontinuities per unit volume, or volume of discontinuities per unit volume of rock. Kulatilake and Wu (1984) proposed an equation to estimate the number of trace mid points per unit area, starting from number of traces per unit area counted on a rectangular sampling domain. Based on the concepts (the principle of associated points) given in the paper by Parker and Cowen (1976), an unbiased trace density estimator for any shape window whether convex or not with an area of $A$ is given by Mauldon (1998). A 3D discontinuity intensity as a function of the mean diameter and the true linear intensity of the joint set was given by Oda (1982). Another relation among the 3D discontinuity intensity, areal areal intensity and mean diameter of the joint set was obtained by Kulatilake et al. (1993).

All the above parameters plus the geometry of the modelling volume and knowledge on the underlying geology are the required input for the three-dimensional joint system models.

*Generation of discontinuity network models*

Several joint geometry models have been proposed to represent the three-dimensional joint network. In different models the discontinuity characteristics have different relationships with each other. By capturing the relationships of discontinuity characteristics, a 3-D geometrical model can be generated which represents the rock mass geometry as an entity. Different geometry models may be suitable for different types of rock formations.
Orthogonal model

In this model (Dershowitz and Einstein, 1988), joints, the shape of which is rectangle, are assumed to be contained in two or three mutually perpendicular sets of parallel joints. Only very minor variations in orientation are permissible while the locations of joints in each set can be described by a Poisson or Markov Process.

The Poisson disc model

The Baecher’s model (Baecher, 1983, Dershowitz and Einstein, 1988) assumes circular or elliptical shape of joints with a Poisson process of locations. For circular shape, the radius follows exponential or lognormal distribution while for elliptical shape, the minimum and maximum chord length can be of any distribution form. The disc model idealizes joints belonging to a given set as parallel discs centered at random points in space and with radii distributed according to a lognormal density function. The number of joint centers in a given volume is a random variable with a Poisson distribution.

Veneziano Model

This model (Dershowitz and Einstein, 1988) assuming polygonal shape of joints is based on Poisson plane and Poisson line processes. This model requires three steps to generate the joints: 1) Poisson plane process of uniformly distributed locations and orientations following a specified distribution; 2) Division of each plane into random polygons by Poisson lines processes; 3) Random selection of a certain proportion of the polygons as joints while the remaining as intact rock.

Dershowitz Model

Dershowitz’s model (Dershowitz, 1984) is similar to Veneziano’s model, but it adopts a different procedure after the Poisson plane process. It uses the intersection lines of the planes to divide each plane into polygons instead of the lines from Poisson line process.

Mosaic Block Tessellation Model

In the model (Dershowitz and Einstein, 1988) based on Mosaic Block Tessellation, the modeling domain is divided into small regions by regular or stochastic grids of interlocking polyhedral. The formation of tessellation relies on the seed points and block growth all of which can be deterministic or stochastic.
Hierarchical model

The dependence between sets is established using statistical methods as well as considering the geologic sequence of fracturing (Lee et al., 1990, Ivanova et al., 1995). The primary sets are considered first while the other set are hierarchically related. In each set, the joints are generated similar to the Veneziano Model with one major difference that the polygon marking process is non-homogeneous. The hierarchical model probably is suitable for some applications such as layered beds.

The MIT Geologic Stochastic Model

This model (Meyer and Einstein, 2002) is a further development of Veneziano Model. After the Poisson plane and line processes, the polygons in each plane are retained or discarded according to their shape and size. Then the modeling volume is divided into zones, and the polygons in each zone are retained or discarded with a certain probability, which can vary from zone to zone. At last, the remaining polygons are translated and rotated from their original location and orientation.

Most of these models have been directly used or enhanced first and then used in commercial software such as FracMan (Dershowitz et al., 1998) and NAPSAC (Herbert, 1994) mainly for fluid flow analysis. After the distributions of network parameters are derived and a model is chosen, Monte Carlo simulation can be performed to generate a large number of realizations, which are statistically equivalent but geometrically different.

Refinement of the simulated realizations

According to the above different geological models, a large number of realizations of the discontinuity network can be generated by Monte Carlo simulation. To be able to apply the discontinuity network to practical rock engineering projects, it is necessary to validate the generated stochastic discontinuity network. The simulated trace map is generated for each realization and compared with the measured trace map.

2.2.2 Generation of discrete blocky system

At the early stage of the development of block generation algorithms, the successive block partition technique is adopted with persistent discontinuity assumption. Warburton (1984) developed a block identification algorithm using infinitely large
discontinuity faces. He also developed a computer program called BLOCKS, without taking the influence of finite-sized discontinuities into consideration. The BGL (Block Generation Language) by Heliot (1988) was also limited to generate convex blocks. This method proposed to include knowledge of the tectonic history of the rock mass in order to make maximum use of the usually limited data on the discontinuity sets. The above approach, conceptually simple and easy to program, has been adopted by the commercial software 3DEC (Itasca, 2007) as the pre-processor for generating rock blocks. Since not all discontinuities in rock masses are persistent, persistent discontinuity assumption may lead to over-fragmentation of the blocky rock mass models.

Currently, there are mainly two approaches to form a realistic representation of blocky rock mass. The first approach is called the block partition plus integration method. This approach mainly consists of two steps. Firstly, based on a single block cutting algorithm, this method uses sequentially introduced discontinuities which are all assumed to be infinite. Then the other step was introduced to consider the finiteness of the discontinuities (Yu et al., 2009; Fu et al., 2010; Zhang and Lei, 2012). The extents of discontinuities are distinguished from those of non-discontinuity faces and then the non-discontinuity faces are deleted in the block integration process. In this way, the realistic blocky rock mass can be produced. However, if the number of discontinuities is large and most of them in the model are finite, it would be quite time-consuming to produce the final realistic rock mass even though some special techniques can be adopted to improve efficiency.

The other approach uses block geometrical identification techniques, and all the finite or infinite discontinuities are introduced at once. Ikegawa and Hudson (1992) presented an algebraic approach for isolated block identification and a graph theory concept for directed blocks. Based on graph theory, Lin et al. (1987) proposed a clear topological definition of the system without combinatorial topology development, which was for the first time presented by Jing (2000). Lu (2002) further developed this approach by simplifying the blocky structure identification system. A block cutting code called TC (Shi, 2006) was developed and used in 3D DDA program. Elmouttie et al. (2010) also developed a program called structural modeller and made some efforts towards robustness (Elmouttie et al., 2013). There are mainly three steps involved: calculating
the intersections of the discontinuities and boundary planes, recording the new vertices into the edge list and searching the closed loops in each plane using the maximum right-handed angle rule, and detecting blocks according to the generalized right-handed angle criterion. This method can consider both finite and infinite discontinuities and the produced blocks can be convex or concave.

2.3 KEY BLOCK THEORY FOR ROCK ENGINEERING

The Key Block Theory, initiated by both Warburton (1981) and Goodman and Shi (1985), is a special method for stability analysis of rock structures. It only considers the geometric characteristics of the rock blocks formed by the discontinuities and free surfaces and the force equilibrium of the driving and resistant forces. Mainly due to its relative simplicity and cost-effective search of key blocks in the rock structures, the Key Block theory enjoys wide applications in rock engineering. But in Key Block approach, all the blocks are considered to be rigid.

There are in general two approaches to identify the key blocks in a rock mass. One approach is based on the block theory proposed by Goodman and Shi (1985). This approach needs only minimum information, i.e., the number of joints sets, the average orientation of each joint set and the excavation geometry. Finiteness and removability analyses are first carried out for surface blocks and stability assessment of each removable block is then performed based on the maximum size of the potential key block calculated from the limitations of an excavation span. The searched key blocks by this approach are assumed to occur everywhere along the excavation. Windsor (1997) later introduced trace length and spacing values to limit the maximum block size. The other approach is based on vector analysis developed by Warburton (1981). It deals with arbitrary polyhedral blocks with any number of free faces. Since blocks are formed from the detailed information of discontinuities, each block is finite and only removability and stability analyses are carried out for each surface block. These two approaches are both important and best suited to design and construction stages respectively.

Over the past three decades, great improvements have been made regarding probabilistic analysis (Chan and Goodman, 1983; Young and Hoerger, 1989; Shi and Goodman, 1989; 1990; Tyler et al., 1991; Hatzor, 1993; Kuszmaul, 1994; 1999; Mauldon, 1995; Chen et al., 1997; Song et al., 2001), rotational modes (Mauldon and
Goodman, 1996; Fulvio Tonon, 1998), in-situ stress (Mauldon et al., 1997; Rocscience, 2003), progressive failure analysis (Wibowo, 1997; Thompson, 2002; Yarahmadi-Bafghi and Verdel, 2003; Noroozi et al. 2011) and support design (Windsor, 1997; Windsor and Thompson, 1992; Thompson and Winsor, 2007).

With the traditional key block method, only the stability conditions of blocks on the exposed surfaces are assessed and the corresponding rock bolting system is designed according to the stability analysis results of the surface blocks by applying a certain safety factor (Goodman and Shi, 1985; Windsor, 1997). However, the stability of the blocks on the periphery of the excavation may be greatly affected by the inner blocks next to them. Wibowo (1997) took the secondary blocks into consideration based on the Goodman and Shi’s approach. The secondary key-block analysis was similar to the traditional one except that the first batch key blocks have been removed and new free surfaces have been created. Thompson (2002) conducted the progressive failure analysis for underground excavations by using the BLOCKS program (Warburton, 1984). Yarahmadi-Bafghi and Verdel (2003) further proposed a 2D key group method to take into account the effect of the adjacent blocks to each removable but stable block day lighting into the excavation. Noroozi et al. (2011) extended the key group method to three dimensions. Although the progressive failure analysis has been conducted and the key group method can identify more than one batch of key blocks, the interaction between the key blocks in different batches was not considered.

For support design based on key block analysis, an expansion of the block theory has also been done by including predictions of key block sizes and the selection and assessment of suitable reinforcement schemes (Windsor and Thompson, 1992; Thompson and Windsor, 2007). The stability assessments of the reinforced rock mass by Windsor (1997) include geometric and force equilibrium assessments, and safety check by an incremental force-displacement computer program for each reinforced key block (Thompson, 2002).

2.4 UNCERTAINTIES IN ROCK MASS MODELLING

Uncertainty is unavoidable in rock engineering due to the special nature of rock masses. For stability analysis of fractured rock masses, the uncertainty of the results comes from
the discontinuity network, evaluation of the mechanical parameters and the numerical model for a certain method (Bagheri et al., 2009).

2.4.1 Types of uncertainties in rock engineering analysis

Uncertainty in discontinuity geometry
Discontinuities in reality are three-dimensional and represented by orientations, shapes, dimensions, locations and apertures in space. The discontinuities inside the rock mass play a significant role in the mechanical and hydrological behaviors of the rock mass, so it is important to accurately represent the geometry. However, an accurate and complete field measurement of all the discontinuities is impossible with the current mapping techniques. The main reason is that only limited information about the discontinuities from field survey is available. In addition, those discontinuities completely inside the rock mass, which cannot be observed and measured directly, are unknown (Baecher et al., 1977).

Uncertainty in mechanical parameters
The mechanical parameters of the rocks and discontinuities have great effect on the results of the stability analysis of fractured rock masses. These parameters may vary from point to point. Usually rock and rock joint properties are obtained from laboratory tests on samples of small size, which may not be able to reflect the real discontinuities well since scale effects are involved. On the other hand, in-situ experiments are difficult and expensive. Determination of the exact values of these mechanical parameters is impossible. Therefore, there are always uncertainties in the mechanical properties when conducting stability analysis. The measurement errors are also a source of uncertainty causing measured values different from actual ones (Vanmarcke, 1977).

Uncertainty in numerical models
Numerical models are often simplified from the actual rock masses and they can never fully represent the actual ones. However, these numerical models can provide some insight into the behaviour of the rock masses. Each numerical method has its own assumptions and limitations and the uncertainty in those numerical methods results from the imperfect assumptions and degree of simplifications in the model. This type of uncertainty is not considered in the current study.
2.4.2 Probabilistic key block analysis

Most of the previous key block analyses are based on deterministic stability analysis for an individual block exposed on a slope or tunnel surface and researchers mainly focused on deriving the maximum size of a key block based on the joint sets orientation and mechanical parameters (Goodman and Shi, 1985; Menendez-Diaz et al., 2009). A few further studies (Tyler et al., 1991; Esterhuizen and Streuders, 1998) also used deterministic spacing and trace length data to limit the maximum size of key blocks. Since only a single block is considered, the uncertainties of discontinuities are limited to the orientation and mechanical parameters only (Park and West, 2001; Ahn and Lee, 2004; Chen, 2010; Johari et al., 2013). Some work has also been done on the relative probabilities of joint intersections (Hatzor, 1993; Hatzor and Feintuch, 2005), which shows that some joint intersections are more likely than the others when a number of joint sets exist in the rock mass.

In order to perform stability analysis of a blocky rock mass system, Chan and Goodman (1983) proposed a simulated-trace-map approach based on discontinuity network modelling using Monte Carlo simulation. For each stochastically modelled discontinuity network, trace map simulation, closed loops searching, and key block analysis are conducted successively. This approach assumes that during key block analysis discontinuities extend sufficiently into the rock mass and only one batch of key blocks can be found out using this method (Young and Hoerger, 1989; Chern and Wang, 1993; Starzec and Andersson, 2002b; Grenon and Hadjigeorgiou, 2003).

In rock engineering, it is quite common that after surface unstable blocks are identified and removed from the rock mass, blocks close to the surface will also be exposed and some of them may also become key blocks. In this way, different batches of key blocks can be found. This process is defined as a progressive failure of blocky rock mass in the present key block analysis. The traditional key block methods cannot simulate the progressive failure of blocky rock mass. In order to overcome this limitation, the discrete-blocky-system approach is adopted by Merrien-Soukatchoff et al. (2012) to conduct progressive stability analysis statistically with the reconstruction of blocky rock mass. RESOBLOK, based on the work of Heliot (1988) and developed by LAEGO, is employed to generate discrete rock mass models. However, due to the limitation of the
successive-block-partition technique, this program cannot really handle all finite discontinuities. In each realization, it assumes that all the discontinuities are of infinite extent or the joint sets could stop at the boundary of another set. Block number limitation has also been reported in RESOBLOK (Gasc-Barbier et al., 2008; Merrien-Soukatchoff et al., 2012), which limits the application of the program in large scale problems. In addition, for statistical stability analysis, the random variables are limited to orientation and spacing only, and it is difficult to treat discontinuity size as a random variable for all discontinuity sets. A structural modeller was developed by Elmouttie et al. (2010) and it has been used for the estimation of in-situ block size distribution (Elmouttie and Poropat, 2012) and study of the effect of random discontinuities (Elmouttie et al., 2012).

2.5 NUMERICAL METHODS IN ROCK ENGINEERING

2.5.1 Discrete element methods

Discrete element methods have been widely applied in rock engineering due to its advantage of simulating discrete block systems under different loading conditions. In the discrete element methods, the problem domain is treated as an assemblage of blocks dissected by discontinuities. The contacts between blocks are identified and continuously updated during the entire analysis process and represented by proper constitutive models (Jing, 2003). According to the solution techniques, discrete element methods are divided into explicit and implicit categories.

The most representative explicit discrete element method is the Distinct Element Method (DEM). It was first proposed by Cundall (1971) and used to deal with rigid blocks. It uses a force-displacement law specifying interactions between blocks and the Newton’s second law of motion providing displacements induced within the rock mass. The DEM adopted an explicit integration scheme, which does not involve the assembly of global matrices, leading to less computational cost. However, it has been argued that accuracy may be sacrificed for efficiency in some particular cases. In addition, small enough time steps are required to ensure numerical stability.

The implicit discrete element method is mostly represented by the Discontinuous Deformation Analysis (DDA), developed by Shi (1988). In the DDA, a first order
polynomial displacement function is assumed for block deformation, leading to a constant stress and strain in each block. In the DDA, an iterative process called open-close iterations is enforced at each time step to ensure the contact accuracy. Due to the use of implicit time integration scheme, the DDA is unconditionally stable and is able to accommodate considerably larger time step (Doolin and Sitar, 2004).

Both the rigid blocks in the DEM and blocks with a constant stress/strain in the DDA have their demerits, particularly for large deformable blocks. In order to enhance the blocks’ deformability and refine the stress distribution field in the blocks, further developments of the above two discrete element methods have been made by coupling with continuum based methods. The deformation and stress of individual blocks are taken into account by discretizing each block into a number of meshes. The finite difference grids have been introduced into the DEM, leading to the well-known commercial software UDEC and 3DEC (Itasca, 2004; 2007). Finite element meshes have also been combined with the DEM by Munjiza (2004) and the DDA by Shyu (1993) and Bao and Zhao (2013) respectively. The Numerical Manifold Method (NMM) (Shi, 1991) is another type of combined continuum-discontinuum method, which combines the FEM and the DDA in a unified framework.

Apart from the FEM and the FDM, the Boundary Element Method (BEM) has also been coupled with the discrete element methods. So far, the BEM has mainly been used together with the discrete element methods to model the far-field rock as an equivalent continuum due to its ability to model infinite and semi-infinite domains. By dividing the domain of interest into two fields, Lorig et al. (1986) and Lin and Al-Zahrani (2001) used the DEM and the DDA respectively to model the discrete near field, and the BEM to model the continuum far field. Prochazka (2004; 2014) developed the free hexagon method, which had similar concepts to the Particle Flow Code (Itasca, 2004). In this method the domain of interest is composed of hexagonal particles. Each hexagonal particle is discretized into six boundary elements to consider the deformation and the lumped mass matrix is used for each hexagon assuming it is sufficiently small.

2.5.2 Dual reciprocity boundary element method for dynamic analysis

The Boundary Element Method (BEM) is an alternative numerical method to the Finite Element Method (FEM). Mainly due to its advantage of boundary-only discretization
and its ability to deal with infinite or semi-infinite domains, BEM has attracted the attention of many researchers. Various techniques have been incorporated into the boundary integral equation to treat different problems, leading to a variety of boundary element formulations.

In terms of dynamic analysis, all the available BEMs can generally be divided into two categories: the frequency domain approach and the time domain approach. In the frequency domain approach (Cruse and Rizzo, 1968; Dominguez, 1978), the dynamic behaviour of the problem is studied by changing from time domain to frequency domain, obtaining the response for each frequency and then returning to time domain. The use of integral transformation techniques requires complicated mathematics, leading to numerical inaccuracies. The time domain approach solves the dynamic problems in the time domain directly, and can be sub-divided into the dynamic-type fundamental solution approach (Mansur, 1983; Antes, 1985) and the static-type fundamental solution approach (Nardini and Brebbia, 1983). Since the time-dependant fundamental solution is used in the dynamic-type fundamental solution approach, the complexity of the formulation is quite high. The static-type fundamental solution approach employs particular solutions to deal with inertial forces. The domain integral resulting from inertial forces is transformed into a boundary integral with the aid of the reciprocity theorem applied for the second time, the first time being when the dynamic problem is formulated in its integral form. As the reciprocity theorem is applied twice, this method is called Dual Reciprocity Boundary Element Method (DRBEM). The DRBEM eliminates the necessity of domain discretization for inertial forces, thus retaining the advantage of boundary-only discretization for dynamic analysis.

The DRBEM, which has been considered as a major development in BEMs for dynamic analysis was first proposed by Nardini and Brebbia (1983). Various developments have been made since then. They have mainly focused on radial basis functions, time integration schemes, and effect of internal points on accuracy.

The DRBEM is distinguished from other BEM techniques in that it employs the radial basis function (RBF) (Golberg et al., 1998). At the early stage of developments, the conical radial basis function $C \pm r$ ($C$ is a constant and $r$ is an interpolation distance) used in Brebbia and Nardini (1983) was almost exclusively employed as the
approximation function in the literature. Later, it was recognized that $c \pm r$ was just a particular RBF and other RBFs could be investigated to improve the accuracy and efficiency of the DRBEM. Other alternative RBFs include polynomial RBF, augmented thin plate splines (Bridges and Wrobel, 1996), Gaussian radial basis function (Rashed, 2002), compact supported RBFs (Rashed, 2002), multiquadrics (Samaan and Rashed, 2007), and Fourier radial basis functions (Hamzeh Javaran et al. 2011).

There are debates over the use of different time integration schemes in DRBEM for elastodynamics. Loeffler and Mansur (1987) concluded in their study that the Houbolt method is preferred in the DRBEM mainly due to the inherent artificial damping which effectively depresses the effect of higher modes in the response. Tanaka and Chen (2001) argued that in the Houbolt method high artificial damping often impairs the accuracy of the solution if a large time step is used and lack of self-starting not only increases programming labour, but also causes some complexity in computing. They concluded in their investigation that the Newmark-β method should be in general preferred in elasto-dynamics compared with the Wilson-θ and Houbolt methods, and the differential quadrature method appears to be a promising technique in practical computations. The study by Chen and Tanaka (2002) found that the damped Newmark algorithm was the most efficient and accurate for impact analysis with the DRBEM after comparing the results from the Newmark- β, Wilson- θ, differential quadrature and precise integration methods. Bozkaya (2008) coupled DRBEM with Differential Quadrature Method (DQM) vibration problems and good results were obtained.

There has been a general agreement that inclusion of a reasonable number of internal points improves the accuracy of the method, even though DRBEM can produce results of satisfactory accuracy without internal points (Chirino et al., 1994; Agnantiaris et al., 1996; Agnantiaris et al., 1998). The internal points along with boundary elements should also be distributed as uniformly as possible into the domain and over the boundary respectively in order to get more accurate results. The study by Agnantiaris et al. (1998) shows that an increase in the number of interior points increases the accuracy of the results of free vibration or harmonic forced vibration analyses, while decreasing the accuracy of the results of transient forced vibration analysis, unless this increase of the interior points is small. Since inclusion of internal points increases the number of degrees of freedoms, the decrease in the accuracy of the results in transient forced
vibration analysis was attributed to the more inaccurately computed higher modes taken into account in the transient response.

So far the DRBEM has been applied by researchers to free and forced vibrations (Bridges and Wrobel, 1994; Agnantiaris et al., 1998; 2001; Samaan and Rashed, 2007), wave propagation (Kanarachos and Provatidis, 1987), structural dynamics (Dominguez, 1993), dynamic non-linear analysis (Kontoni and Beskos, 1993), semi-infinite (Tosecký et al., 2008) and infinite (Rashed, 2002) domain problems.

2.6 ISSUES AND CONCLUSIONS

The greatest challenge in rock engineering is how to determine the unique discontinuity network (both geometry and physical properties) in a rock mass. Success in numerical modelling for rock mechanics and rock engineering depends almost entirely on the quality of the characterization of the discontinuity network (Jing, 2003). Therefore, it is essential to represent the discontinuities more realistically. With consideration of 3D discontinuities locating in space, a probabilistic approach with a large number of realizations is more appropriate to deal with the uncertainties. However, this will lead to quite low efficiency due to the limitation of computer resources. Thus, efficiency is an issue for probability-based numerical analysis using discrete element methods, since the discrete element methods themselves are computational time intensive even only for deterministic analysis.

It can be seen that a probability based geological rock structure model based on data collected from field survey is more appropriate to deal with rock masses and cover the uncertainties. This requires a robust block generation program to account for the randomness and non-persistence of discontinuities inferred from field mapping data. In terms of efficiency, measures should be taken to control the computational time. For stability analysis, key block theory should be more preferable due to its simplicity and high efficiency especially for preliminary analysis. Therefore, even though a large number of realizations are involved, the analyses can still be conducted within a reasonable time range by key block analysis. However, for accurate analysis, discrete element methods are still required. For the further development of discrete element methods, as suggested by Jing (2003), meshless BEM may be a useful alternative to extend DEM’s capacity.
In this thesis, a three-dimensional robust geological modelling tool will be developed first to handle the randomly generated discontinuities. Then the key block method will be further developed to consider progressive failure of blocky rock masses as well as stochastic analysis to cover uncertainties. In addition, the contact algorithm with open-close iterations in the DDA method will be implemented into the DRBEM for possible efficiency improvement, due to the BEM's advantage of reducing dimensions by one.
CHAPTER 3. A REALISTIC THREE-DIMENSIONAL GEOLOGICAL MODELLING METHOD FOR COMPLEX ROCK MASSES

3.1 INTRODUCTION

The rock mass, essentially a discontinuous medium, consists of intact rock and discontinuities. Close to the excavations, the discontinuities often divided the rock mass into an assemblage of rock blocks. Given that all the discontinuities in the rock mass have been known, a robust block identification algorithm is demanded to divide the rock into a discrete blocky system for further stability analysis. When the discontinuities are relatively longer than the dimensions of the domain of interest, the algorithm is quite simple and all the generated blocks are convex. On the other hand, when the discontinuities are relatively shorter than the dimensions of the domain of interest, the algorithm could become quite complex due to the possible arbitrary shape of blocks. A review on different block generation algorithms has been conducted in Chapter 2.

Based on the directed and complete theorems (Ikegawa and Hudson, 1992) and the theoretical topological basis (Jing, 2000), this chapter presents an improved and generalized 3D block generation algorithm using the block geometrical identification technique for engineering analysis. A number of reliable techniques, including point in polygon/polyhedron test algorithms, tree cutting, careful tolerance management, etc., are adopted to ensure the robustness of the developed algorithm. Some measures, such as adoption of compact data structure, avoidance of one-to-all iterative processes and other unnecessary calculations are also taken to improve the efficiency. In addition, different methods are used to verify the generated data in the program. In order to facilitate the formation of rock excavations, rock slope geometry represented by triangulated surfaces or general polygons is formed by either sequential cutting or contour map while tunnels with different kinds of shapes are modularized. This algorithm can simulate both planar and non-planar discontinuities. With the introduction of all the discontinuities and rock mass profile of arbitrary morphology, the realistic rock mass is obtained after removal of blocks within the excavation domain. In the
resultant block system, there can be tens of thousands of blocks and the blocks can be convex, concave, or blocks with cavities or holes.

### 3.2 MATHEMATICAL REPRESENTATION OF BLOCKY ROCK MASSES

In practice, a three-dimensional block system is an assembly of many rock blocks separated by discontinuities and boundary planes. Each block, called a polyhedron in mathematics, has its own vertices, edges and faces. A geometrical block generation algorithm has been developed to generate three-dimensional block systems from discontinuities and boundary planes.

![Figure 3-1 Sketch map of directed edges, loops, faces and blocks](image)

**3.2.1 Two theorems**

There are two important theorems (Ikegawa and Hudson, 1992) for block identification algorithm: directed and complete theorems. The directed theorem means all the edges, loops, faces and blocks are of directions. As shown in Figure 3-1, the edges $E_{ba}$ from $a$
to \( b \) and \( E_{ba} \) from \( b \) to \( a \) are of opposite directions. A directed face consists of only one exterior directed loop and may have more interior loops. The exterior loop and interior are of opposite directions. One directed face with one exterior loop (anticlockwise) and one interior loop (clockwise) is shown in Figure 3-1. A positive block is defined as all the normal vectors of the faces pointing to the inside of the block. With the introduction of this theorem, the convex and concave blocks can be dealt with under the same framework. The complete theorem denotes that the sum of all external or internal face vectors of a directed polyhedron is zero and the sum of external or internal edge vectors forming a directed face is zero. These two theorems are very useful in both the block generation and the verification processes.

### 3.2.2 Procedure of the algorithm

![Diagram of block generation algorithm](image)

The procedure of the algorithm, which is similar to those (Lin et al., 1987; Jing, 2000), is described briefly as follows: Before introducing discontinuities to conduct the algorithm, the domain of interest and slope or tunnel excavations are required to be defined for analysis. All the polygons including all the discontinuities and boundary faces are pre-processed. The coplanar polygons which are adjacent to or overlap each other are merged. Then the intersection points of these polygons are calculated and stored accordingly, followed by a process of deleting those points outside the domain of interest.
interest. After that, the leftmost traversal technique is employed to search the directed loops and then faces on each polygon. At last, the generalized right-handed angle criterion is used to identify the blocks. Before the result is utilized, a refinement process is adopted to merge some coplanar faces of each identified block. The flow chart is shown in Figure 3-2.

Representation of excavation surfaces and curved discontinuities
Excavations are commonly encountered in civil and mining engineering. In order to create numerical models for stability analysis, the realistic representation of excavation geometries is an integral part of the analysis. A convenient engineering modelling tool is vita, especially when quite a number of big and complex excavations are involved.

Figure 3-3 Slope profiles represented by triangulated surfaces and general polygons

Figure 3-4 Different tunnel and discontinuity shapes
Two methods are utilized to generate slope profiles. After measurement of the coordinates of some critical points on the slope surface, Surfer (Golden Software, 1990) etc. can be used to generate contour map and then the triangulated surfaces. Some commercial software like Sirovision (CSIRO) and 3DM Analyst (Adam technology) using digital photogrammetry technique are also available. With these kinds of software, the surface geometry can be obtained directly. Another method is to use sequential excavation or Boolean operations to obtain required slope geometries. Two examples of slope profiles have been generated as shown in Figure 3-3. In terms of tunnel profiles, different shapes of tunnels have been modularized to facilitate the formation (Figure 3-4). Each tunnel model is defined by the orientation, location and the dimensions in 2D plane of the starting face, and tunnel length.

**Pre-processing of all input polygons**

Since the domain for the generation of discontinuities are normally larger than the domain for blocky rock mass. Among all those input discontinuities, some of them may be located completely outside the domain of interest. Therefore, it is necessary to take measures to exclude them for further operations.

In addition, it is possible that some of the input polygons are on the same plane, having one or more edges connected or even part overlapping. It is essential to pro-process those polygons first before going to the next step. Otherwise, the program will collapse.

Before calculating the intersection of input polygons, the co-planarity needs to be checked first and all the adjacent polygons, which are in the same plane, are integrated into a new large polygon. The coordinates of the vertices in the new polygon are adjusted accordingly in order to make sure the vertices are all in that plane. Then the plane equations of these polygons need to be recalculated for further utilization. Here, the newly formed polygons can be concave, so the traditional normal vector calculation method which computes the cross product of two coincident polygon edges is not robust (for concave polygons and two collinear edges). A numerically robust way of computing the plane equation of an arbitrary 3D polygon called Newell’s method is adopted.
**Intersection of polygons**

The intersection points of these polygons are calculated and the stored accordingly, followed by a process of deleting those points outside the domain of interest.

For any three faces \( f_i, f_j \) and \( f_k \), first, whether any two of the three faces are parallel or not needs to be judged. By introducing a tolerance, some faces too close to parallel to each other are considered as parallel. If any two faces among the selected three faces are parallel to each other, the next group of three faces are chosen from the \( n \) faces. If these three faces are non-parallel faces, the coordinates of the intersection are calculated by the following equation:

\[
\mathbf{v} = \left( n_i \cdot p_i, n_j \cdot p_j, n_k \cdot p_k \right) \cdot \left[ n_i^T; n_j^T; n_k^T \right]^{-1}
\]  

(3.1)

where \( \mathbf{v} = (v_x, v_y, v_z) \) is the position vector of the intersection point; \( n_i, n_j, n_k \) are unit normal vector of faces \( f_i, f_j \) and \( f_k \) respectively, \( p_i, p_j \) and \( p_k \) are the position vectors of the three points on faces \( f_i, f_j \) and \( f_k \) respectively.

For each calculated intersection point, the point-in-polygon test is conducted to check whether this point is inside those three polygons. If so, the coordinates of the new intersection and the corresponding three polygons are recorded.

In the current algorithm, the vertices of original input polygons are not recorded as tree cutting is done later to regularize the edges and loops. The details of tree cutting are discussed in section 3.2.

All the calculated vertices need to be refined to exclude those outside the domain of interest by a point-in-polyhedron test. Next, the vertices are stored along each intersection line of two input polygon planes and a simple ordering algorithm is applied to obtain ordered vertices. Then the edges formed by those calculated vertices are recorded for further use.

**Loop and face search**

All the identified edges are sorted into each polygon. Then the edges passing through the same vertex are stored and reordered according to the angles between them. Each edge will be used twice in opposite directions, and it will be marked after each use.
After that, the leftmost traversal technique shown in Figure 3-5 is employed to search the directed loops.

The detailed procedure for loop searching is as follows:

For each input polygon, an edge pool is formed:

a) Randomly select an edge to start and mark this edge as used;

b) Based on the leftmost traversal technique, select the next edge from the group of edges passing through the ending vertex of previous edge, and mark this edge as being used;

c) Repeat step b) until a closed loop is formed;

d) Check weather all edges have been used exactly twice. If not, repeat step a)-c).

![Figure 3-5 Principle for closed loop searching (leftmost traversal technique)](image)

After all the loops have been identified, the area of each loop is given by:

\[
A = \frac{1}{2} \sum_{i=1}^{N_v} \begin{vmatrix}
1 & x_0 & y_0 \\
1 & x_i & y_i \\
1 & x_j & y_j \\
\end{vmatrix}
\]

\[i = 1 \sim N_v, j = i + 1; j = 1 \text{ if } i = N_v\]  \(3.2\)

where \(N_v\) is the number of vertices of the block, \((x_0, y_0)\) is the coordinates of a reference point (first vertex of the loop used in the program).
The positive area $A$ indicates that the vertices of the loop are arranged anticlockwise forming an interior block while the negative area $A$ means that the vertices are ordered clockwise forming an exterior block or boundary of an interior void. Among all the search loops, loop containment check is conducted to detect all the faces containing more than one loop on each polygon. This step is the same as that used for two-dimensional DDA block identification.

**Block identification**

This step is the most time consuming part of the algorithm. After all the faces are obtained, the faces passing through the same edge are stored and reordered according to the angles between them. Same as edges, each face will be used twice in opposite directions, and it will also be marked after each use. At last, the generalized right-handed angle criterion is used to identify the blocks (Figure 3-6).

![Figure 3-6 Principle for closed block searching (generalized right-handed angle rule)](image)

The detailed procedure about block searching is as follows:

All the faces will form a face pool in order to identify blocks:

1. Randomly select a face and then an attaching edge in this face to start and mark this face as used;
b) Based on the generalized right-handed angle rule, select the next face from the group of faces passing through the selected edge, and mark this face as being used;

c) Repeat step b) until a closed block is formed, in which all the edges will be used twice in opposite directions, and also use Euler characteristics to check the correctness of the operation;

d) Check weather all faces have been used exactly twice. If not, repeat step a)-c).

After all the blocks have been identified, the volume of each block is given by:

\[
V = \sum_{i=1}^{n} V_i = \frac{1}{6} \sum_{i=1}^{n} \sum_{j=1}^{m_i-2} \begin{vmatrix} x_p & y_p & z_p & 1 \\ x_j & y_j & z_j & 1 \\ x_{j+1} & y_{j+1} & z_{j+1} & 1 \\ x_{j+2} & y_{j+2} & z_{j+2} & 1 \end{vmatrix}
\]

where \( n \) is the number of faces in the block, \( m_i \) is the number of vertices in the block, \((x_p, y_p, z_p)\) is the coordinates of a reference point (the first vertex of the block is used in the program). A polygonal face \( i \) of \( m_i \) vertices can be divided in to \( m_i - 2 \) tetrahedrons with the reference point.

Same as areas of loops, the volume of each block could also be negative or positive. Among the blocks with negative volume, the one with biggest absolute value is infinite boundary block, and it will be excluded from the identified blocky system. All the other block with negative volume are checked versus other blocks with positive volume to find out which block they belong to (block containment judgement).

Post-processing

Before the result is utilized for analysis, a refinement process is adopted to merge some coplanar faces of each identified block. The data could be output in the format of other programs (LS-DYNA, 3D-DDA, 3D NMM, etc.)

3.2.3 Features of the algorithm

There are three types of faces in the generated model: fixed faces, discontinuity faces and free faces. The free faces are the actual excavated faces or existed natural outcrops. The fixed faces are usually the faces that separate the domain of interest from the surrounding rock mass. The discontinuity faces are faces of the blocks cut by the input
discontinuities. Those different types of faces of blocks are distinguished with different index numbers and they are essential for stability analysis.

Figure 3-7 A non-planar discontinuity

Figure 3-8 Different shapes of selected blocks from generated blocky systems a) Convex blocks; b) Concave blocks.

There are several features of this block generation algorithm: The input discontinuities can be finite or infinite, planar or non-planar, convex or concave in shape; the rock
slope profile can be of arbitrary morphology represented by triangulated surfaces or
general polygons; the discontinuities, especially some large faults or bedding planes can
be non-planar (Figure 3-7); the blocks in the generated blocky system could be convex
or concave or blocks with cavities or holes. Some generated convex and concave blocks
are displayed in Figure 3-8.

3.3 VERIFICATION, EFFICIENCY AND ROBUSTNESS
IMPROVEMENT

3.3.1 Correctness checking of the generated blocky system

One method for verification is the Euler characteristic. The Euler characteristic $\chi$ was
classically defined for the surfaces of polyhedrons

$$\chi = V - E + F$$

(3.4)

where $V$, $E$, and $F$ are the numbers of vertices edges and faces respectively in the given
polyhedron. The surfaces of any convex polyhedron, as used in Euler polyhedron
formula, have the Euler characteristic of $\chi = 2$ while those of a concave polyhedron
have various Euler characteristics. For a torus with N holes in it, $\chi$ can be written in the
following form:

$$\chi = V - E + F = 2 - 2 \times N$$

(3.5)

The three-dimensional graphic display is first used to check the information of blocks
and faces. From the 3D graphical interface, the produced blocks and their faces can be
seen directly. The utility functions are also made full use of for checking. Taking the
grouping function as example, for a large-sized model, part of the generated blocks are
selected and displayed in the interface. In this way, the results of the code can be
checked intuitively.

Block volume summation check (Shi 2006) can also be used as a method of result
checking: the summation of the block volumes should be equal to the volume of the
target block.

Some regular blocky system tests (Figure 3-9) which generate blocks with known
shapes and numbers are also performed. If the results are identical with what are
expected, the accuracy of the program can also be verified to some degree. The graphic display can also be used to check the results. This includes selection of some blocks or faces as a group and display of the indices of blocks, faces and vertices etc.

Figure 3-9 Regular blocky system test examples a) a block system with 160 blocks, 1576 faces and 2344 vertices; b) a block system with 8000 blocks, 48000 faces and 64000 vertices; c) a blocky system with 64000 blocks, 384000 faces and 512000 vertices.

3.3.2 Efficiency

The polygons used to represent curved discontinuities or excavations are also used to calculate the intersection points. Since those points will still exist after the calculation of all the calculation points, it is a waste of time to do the calculation of every three polygons from curved discontinuities or excavations. In addition, some of the above
polygons may be almost parallel to one another, causing loss of intersection points or even failure of the algorithm later after calculation. In order to improve efficiency and ensure reliability, unnecessary calculations of the intersection of three of these types of polygons are avoided. When three polygons all come from the same curved discontinuity or excavation surface, the intersection point and the three polygons are recorded directly without conducting calculation.

In the block identification algorithm, there are plenty of one-to-all iterative processes for vertices, loops, faces and blocks, which if not dealt with will consume a lot of computation time and make the program very tedious and time consuming. Taking the intersection point calculation of three finite polygons for example, the centroid of each polygon and radius of its circumscribed circle are calculated first. Then for three polygons, before conducting the parallel test and then the intersection calculation, the distance between the centroids of each pair of polygons is compared with the summation of their radii of the circumscribed circle. In this way, unnecessary computations can be avoided. Some similar procedures have been adopted to avoid these one-to-all processes.

3.3.3 Tolerance management

In computer programming, tolerance is a commonly used approach due to the accumulation of rounding and cancellation errors. Normally, a small value epsilon is selected to control the required tolerance and the range of epsilon depends on the scale of the performed computations. For the co-planarity test and testing of newly generated points against the existing points, these two are especially important but difficult to control especially when the density of the discontinuities is very high. If not dealt with properly, they will cause instability of later calculations. The allowable angle limit between two nearly co-planar polygons and the minimum distance between two adjacent points should be set with great care.

3.3.4 Tree cutting

The process of removing dangling and isolated discontinuities from the network is called tree cutting or network regularization. Actually, tree cuttings are not compulsory and whether they should be done or not depends on the objectives of numerical analyses.
Tree cutting can be done for several reasons. 1) Just for simplification. If the dangling and isolated edges or faces are not deleted, this will make the description of the blocks more complicated. Thus it will cause the numerical analysis to be more time-consuming. 2) Due to the current conditions of numerical analysis. So far, most of the numerical methods for the analysis of rock mass mainly treat each block/element having a constant stress and strain especially for 3D problems. The discontinuities inside the block/element do not affect the simulation results much. Also, the model after tree cuttings is used mainly for stress and strain analysis and not for discontinuity propagation.

However, for discontinuity propagation analysis, the part of discontinuity inside a block has great influence on the failure pattern and process. So in this case, the tree cutting should not be done. In addition, although some discontinuities do not contribute to the formation of blocks, they form part of the fluid-conducting pathways. Thus these discontinuities should not be deleted for fluid flow analysis.

Both edge tree cutting and loop tree cutting should be done by an iterative process before the loop and block detection respectively in a 3D space rather than in each discontinuity plane.

3.3.5 **Point in polygon and polyhedron tests**

Since the general shapes of polygons and identified blocks are adopted, the point in polygon and polyhedron tests should also be generalized and not be limited to convex ones. Point in polygon test is a simple case comparing to point in polyhedron test, so here we mainly discuss point in polyhedron test. Two methods working for both convex and concave polyhedron can be employed to perform this test. One is the decomposition of a concave polyhedron into convex ones, then using the method for judging whether a point is inside a convex polyhedron. Detection of a point whether inside the region of a convex polyhedron is achieved by using a set of inequalities. If coordinates x, y and z of the point satisfy the inequalities, the point is inside the convex region. Another method is the random ray generation method. A ray is shot from the point along some direction and the number of times it crosses the polygon boundary is counted. In general, the point will be inside the polygon for an odd number of crossings and outside for an even
number of crossings. Care must be taken to properly handle cases where the ray passes through a vertex or coincides with an edge or inside one face plane of the polyhedron.

3.4 **EXAMPLES**

An extensive number of blocky systems or single complex shaped blocks have been generated by the developed program to validate the block generation algorithm as well as show the capacity of the program.

3.4.1 **Probabilistic rock mass model**

![Probability based rock mass systems](image)

Figure 3-10 Probability based rock mass systems

With the robust 3D block generation program available, the discrete discontinuity network can be the input to generate the blocky rock mass. In Figure 3-10, three realizations of one tunnel model from Monte Carlo simulation are generated. All the parameters of the discontinuity network follow the same distributions, but the produced blocky systems are not the same.

3.4.2 **Blocky rock mass model generation process**

Based on the discontinuity data shown in Figure 3-11 and Table 3-1, a blocky rock mass model shown in Figure 3-12 a) is generated from the discontinuity network. There are three joint sets. For each set, the shape of each discontinuity is assumed to be rectangular, the locations follow a Poisson distribution and the sizes follow normal distribution. There are 3166 blocks, 208 of which are concave blocks. The volumes of the maximum and minimum blocks are 10243 m$^3$ and 0.000001 m$^3$ respectively. After the bench geometry and the tunnel geometry (Figure 3-12 c), d)) are introduced, the final open pit model and tunnel model as shown in Figure 3-12 e), f) can be generated.
respectively. The detailed information of the open pit and tunnel geometry is shown in Figure 3-12 b).

![Figure 3-11 Stereographic projection of orientation data](image)

Table 3-1 Discontinuity parameters

<table>
<thead>
<tr>
<th>Orientation (°)</th>
<th>Location</th>
<th>Shape</th>
<th>Size (m)</th>
<th>Friction angle (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dip</td>
<td>Dip direction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40±(15)</td>
<td>0±(15)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60±(15)</td>
<td>30±(15)</td>
<td>Poisson distribution</td>
<td>rectangular</td>
<td>Normal distribution</td>
</tr>
<tr>
<td>75±(15)</td>
<td>320±(15)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.4.3 Sequential excavation

In most of the cases, key blocks tend to fall into the opening upon the excavation done. As a result, underground failure normally occurs before any reinforcement is installed to the rock mass. Solution to that suggests the blasting of the tunnel face to be carried out in sequence specifically for sections with very large unstable key blocks and high resultant driving force. This method has been practiced in the past and effectively
prevents the falling blocks to allow for partial installation of the reinforcements. The excavation orders vary with regard to the location of the worst key blocks (i.e. roof or side wall).

![Figure 3-12 Rock mass model generation](image)

Figure 3-12 Rock mass model generation a) One blocky rock mass model; b) Geometry information; c) Four-bench open pit geometry; d) Tunnel geometry; e) Four-bench open pit model; f) Tunnel model

The developed programs are capable of simulating the excavation process. The key block analysis will be discussed in Chapter 4. Based on the previous rock mass model, one horseshoe shaped tunnel was excavated sequentially in four steps, as shown in Figure 3-13. The key blocks were identified in each step. In this way, the best excavation sequence can be approximately selected after comparing different
excavation processes. Then before each step of the excavation, the key block especially the large ones can be pre-anchored in advance.

Figure 3-13 Sequential excavation of a horseshoe shaped tunnel with key block analysis

3.4.4 Other examples

Single block generation

The Great Wall, the longest human-made structure in the world, is a symbol of China's ancient civilization and a testament to construction techniques that have existed for centuries. Figure 3-14 shows a simplified shape of the great wall profile with the current developed program. If the detained structure of the solid wall is known explicitly, the more detailed model can be generated.

Figure 3-14 Part of a great wall

A numerical model of the underground mining is obtained through the input geometries and locations of those shafts. Figure 3-15 a) shows how the real underground mining
structure looks like, and Figure 3-15 b) shows the generated 3-D numerical model obtained by the current algorithm.

![Figure 3-15 Inclined Shafts in Underground Mine Development](image)

This program can also be used in underground engineering for military purposes. Some military weapons are stored inside specific underground caverns, and sometimes people can hide inside the underground caverns during the wars. Normally, stable passages are required to connect the underground cavern and the ground, and heavy transportations may be needed. In addition, oil tanks can also be stored underground. Definitely, safety is the most important issue here. The underground cavern as well as the structures for underground tank storage must be able to withstand earthquake, bomb attack etc. As shown in Figure 3-16, the numerical models for underground cavern with passages and underground oil storage structure are generated respectively in Figure 3-16 a) and b).

![Figure 3-16 Underground excavations a) Underground cavern with passages; b) Underground oil storage tank](image)
Figure 3-17 shows a slope and multi-bench open pit profile respectively. Figure 3-18 displays two models with a number of intersecting horseshoe or rectangular shaped tunnels.

![Figure 3-17 Complex slope profile and multi-bench open pit surface](image1)

![Figure 3-18 Intersecting tunnel examples](image2)

It should be mentioned that each of the above geometries (profile of the problem) is only a single block. This shows the generality of the current algorithm dealing with arbitrary shaped blocks (convex, concave or blocks with one or more holes). When detailed discontinuity network is known for those problems, more detailed models can be generated to conduct the stability analysis or other related analyses.

**Two or multi-block system**

Figure 3-19 presents a model with 9 different shapes and orientations of tunnels. These tunnels are able to intersect with one another. In addition, a curved fault passes through the intersection of two circular intersecting tunnels dividing this model into totally two blocks. In Figure 3-20, a curved horse shoe-shaped tunnel model is simulated. After three set of joints are introduced, the discrete blocky rock mass is obtained as shown in Figure 3-20 b).
Figure 3-19 A couple of intersecting tunnels with one curved fault and horse-shoe shaped tunnel

Figure 3-20 Curved tunnel a) Curved tunnel profile; b) Generated blocky rock mass model

Before a dam is built, whether it is stable under the water pressure needs to be carefully analyzed. With the help of the current program, the dam model (Figure 3-21) can be generated by considering the structures of the dam and attaching foundation. Thus further stability analysis can be conducted to ensure safety.

Figure 3-21 A dam model
Figure 3-22 Discontinuous rock mass a) one complex slope b) half of an open pit model; c) Old masonry bridge model

Figure 3-22 and Figure 3-23 display some discontinuous rock masses. A quite complex slope model is generated in Figure 3-22 a). Thirteen major discontinuities are considered in the generation of the blocky rock mass region. There are totally 107 convex or concave blocks in the final rock mass. For the open pit model in Figure 3-22 b), eight boundary faces and 37 discontinuities are introduced and 838 blocks are generated. Old masonry bridges are often required to sustain heavier loads than in the past, so it is necessary to assess the safety conditions of them. An old masonry bridge model is generated in Figure 3-22 c). It is made up of a semi-circular arc brick vault and spandrel walls and there are 136 bricks in total in this arc bridge. A blocky rock mass model (Figure 3-23 a)) is obtained when three randomly generated joint sets are generated in a domain with cuboid shape. There are totally 8261 block of which 206 blocks are concave. A rock slope model is also generated by four randomly generated joints sets (Figure 3-23 b)), and 7522 blocks are identified in the blocky system with 152 concave blocks among them.
Figure 3-23 Rock mass models a) A blocky system with 8261 blocks and 206 concave blocks; b) A rock slope model with 7522 blocks and 152 concave blocks

Figure 3-24 Applications of block generation program a) ANSYS/LS-DYNA model; b) NMM-3D model

So far, the three-dimensional block program has been used as a pre-processor for ANASYS/LS-DYNA (Hallquist 1999), 3DEC (Itasca, 2007) and NMM-3D (He and Ma, 2010). In Figure 3-24, two slope models have been generated by the block generation program and imported into ANSYS/LS-DYNA and NMM-3D respectively.

3.5 CONCLUSIONS

In this chapter, an improved algorithm for the generation of three-dimensional geological models has been presented for engineering analysis. The discontinuities are treated more realistically with finite extent of different shapes. The block generation process involves mainly three steps, namely, forming zone of interest, formation of rock mass profile, generation of rock mass system by introducing all the discontinuities. The
rock slope profile can be represented by triangulated surfaces or general polygons; the rock tunnel can be straight or curved with different kinds of cross-sectional shapes. The results from this algorithm are verified by several methods to ensure robustness. The generated blocks can be convex or concave, which will certainly compose complex block systems. Both simply connected blocks and multi-connected blocks can be treated. Theoretically, it can deal with any blocky rock system in rock engineering. With this robust algorithm, discontinuity network by Monte Carlo simulation with a large number of realizations can be used to generate probability based rock mass models. The geometrical rock mass modelling is useful, not only in investigating the stability of rock slope and tunnels and their support designs, but also in giving input information for more advanced numerical simulations (DDA, DEM, NMM, etc.). Since detailed information of the discontinuities and rock blocks has been obtained from the geometrical modelling, seepage analysis of underground water and its effect on the rock mass behaviour can also be studied.
CHAPTER 4. AN IMPROVED KEY BLOCK METHOD FOR PROGRESSIVE FAILURE ANALYSIS OF BLOCKY ROCK MASSES

4.1 INTRODUCTION

In Chapter 3, a three-dimensional geological modelling tool has been developed to generate blocky systems. Then a suitable method is required to conduct stability analysis for engineering design. The discrete element methods, which are able to accurately simulate the interactions of blocks in the blocky system, should be used. However, the computational cost of these methods is highly expensive.

Key block method (Goodman and Shi, 1985; Warburton, 1981) is a good alternative for stability analysis of blocky rock masses. It does not perform time-consuming stress/strain analysis but only effectively identifies removable and instable blocks (termed as key blocks), considering the geometric information and equilibrium of each block on the periphery of the excavation. Compared with numerical methods (both continuum- and discontinuum-based stress analysis methods), the key block method has the advantage of being simple and highly efficient. Therefore, it has wide applications in rock engineering. The key block method is most suitable for hard and blocky rock masses. It can be used to high porous, weathered and fissured rocks and has the potential to be applied to some soils (Goodman, 1995).

However, the force interactions of the adjacent batches of key blocks have been ignored in all existing key block analysis methods. This may result in a dangerous rock support design since the forces transferred from the inner batches key blocks to the first batch ones may lead to larger sliding forces of the surface key blocks. Furthermore, a larger key block may be formed after the rock bolts fix the surface blocks to the inner blocks.

In this chapter, the traditional key block methods (Goodman and Shi, 1985; Warburton, 1981) are extended by considering the interactions of adjacent batches of key blocks. Based on the reconstructed blocky rock masses, a computer program of key block method using vector analysis is developed to analyse the stability of blocky rock masses and it is capable of searching different batches of key blocks progressively. A force
transfer algorithm is developed and then implemented to consider the effects of the key blocks in later batches batch to batch to the first batch key blocks. After a suitable reinforcement scheme is selected, a two-step safety check on the stability of the reinforced rock blocks is carried out. The first step is similar to the force equilibrium assessment by Windsor (1997) which computes and compares the sliding force and the supporting force of the reinforced surface key blocks. The second step performs a further safety check to find out whether larger instable key blocks are formed. This second step safety check has been ignored in the previous studies, which may result in an unsafe rock support design. The extended key block analysis is applied to a three-dimensional tunnel example as well as an underground powerhouse project.

4.2 KEY BLOCK METHOD USING VECTOR ANALYSIS

In order to consider the effects of the blocks adjacent to the surface rock blocks, a blocky rock mass system needs to be firstly constructed. A three-dimensional block generation program has been developed to generate blocky rock masses from networks of discontinuities with finite extents. The discontinuity network must be as real as possible based on geological survey, which is not included in the discussion scope of this study. With the reconstructed rock mass, a progressive failure analysis is conducted which is necessary for the following force transfer process.

In rock mass stability analysis, all forces acting on an individual block can be divided into two groups, i.e., driving and resisting forces (Rocscience, 2003). The driving forces are from the self-weight of each block, water pressure, and dynamic forces from blasting or an earthquake. The resisting forces are provided by shear strength of discontinuities, induced stresses and forces from rock support.

4.2.1 Definition of key block

Key blocks have been defined differently by researchers. Goodman and Shi (1985) defined key blocks as the instable blocks intersected by both excavation surfaces and discontinuities. Mauldon (1995) and Yarahmadi-Bafghi and Verdel (2003) treated both stable and instable removable blocks as key blocks. The definition of key stone by Warburton (1987) described that the block was key for the movements of other blocks. In the present study, the definition of key blocks given by Goodman and Shi (1985) is
adopted. Under this definition, although removal of a key block does not definitely lead to progressive failure of the rock mass, support of all the key blocks around the excavation surfaces does ensure stability.

4.2.2 Assumption

There are several assumptions involved in the key block analysis. All the blocks are assumed to be approximately rigid. Although the blocks can fail with both translational modes and rotational modes, only translational modes including free falling or uplifting, single plane sliding and double plane sliding are considered in the current study. According to Warburton (1993), the assumption of translation only is often reasonable, at least for the important early part of the movement. Due to the only consideration of translational modes, the resultant force acts through the centre of the corresponding rock blocks. Each block is treated as if its neighbouring blocks are temporally fixed. The discontinuities can be planar or non-planar with finite extents. If not planar, they are represented by a set of general polygons or triangulated surfaces. The fixed neighbouring blocks assumption is considered to be removed after the force transfer process.

4.2.3 Stability analysis of a rock mass

Single block stability analysis

The single block stability analysis is the basis of the extended key block analysis and is applied at the three stages of the analyses: progressive failure analysis to search different batches of key blocks; force transfer process to transfer sliding forces of key blocks in later batches to key blocks in earlier batches; safety check of the combined larger blocks connected by rock bolts.

The computational procedure suggested by Warburton (1981) is implemented to analyse the stability of a single 3D block. For reason of completeness, the procedure is repeated here. For each selected potential key block with at least one free face (day-lighting block), the removability and stability analyses are conducted as follows (please refer to Warburton (1981) for details):

The removability analysis of each day-lighting block is achieved by checking all the possible sliding modes. This is in essence a trial and error method and the geometrical
configuration of all the discontinuity faces which are in contact with the block are checked whether they permit the block to move or not. Firstly, the resultant driving force vector \( \mathbf{R} \) with unit vector \( \mathbf{r} \), and the dot product of the unit vector \( \mathbf{r} \) and the unit normal vector \( \mathbf{n}_i \) of each discontinuity face are calculated as follows:

\[
T_i = \mathbf{n}_i \cdot \mathbf{r} \tag{4.1}
\]

Then each of the following possible states is checked in order: stable, free falling, single plane sliding (Figure 4-1 c)) and double plane sliding (Figure 4-1 d)). In Figure 4-1, \( S_d \) is the sliding direction of that key block.

If there exists one discontinuity-face \( i \) and it satisfies \( T_i = -1 \), this means the direction of the resultant force is normal to face \( i \) and the movement of this block is totally prevented by discontinuity-face \( i \). Thus this block is stable (Figure 4-1 a)).

If for any discontinuity-face \( T_i \geq 0 \), this denotes that this block can fall freely along the direction of its resultant driving force (Figure 4-1 b)).

If there exists one discontinuity-face \( i \) satisfying \( T_i < 0 \) and \( T_i \neq -1 \), the assumption that this block may slide on this plane is made. Then the vector \((\mathbf{n}_i \times \mathbf{r}) \times \mathbf{n}_i\) is calculated and checked if any other discontinuity prevents the movement of this block along this direction. If not, the conclusion is made that this block can slide along this direction on this plane (Figure 4-1 c)). The sliding direction \( S_d \) can be determined as follows:

\[
S_d = (\mathbf{n}_i \times \mathbf{r}) \times \mathbf{n}_i \text{ or } S_d = \mathbf{r} - (\mathbf{r} \cdot \mathbf{n}_i)\mathbf{n}_i \tag{4.2}
\]

If this block does not satisfy the conditions of free falling and sliding on any discontinuity plane, the next step will be conducted by checking each pair of discontinuity faces. If there exist two discontinuity-faces \( i \) and \( j \) satisfying \(-1 < T_m < 0 \) \( (m = i, j) \), the failure mode is assumed to be sliding along the intersection of these two faces. The other discontinuity faces will be checked to see if they will block the movement of this block along this direction. If not, the assumption is verified and this block can move along the intersection of faces \( i \) and \( j \) (Figure 4-1 d)). The sliding direction is determined by:
\[ S_d = \text{sign}[(n_i \times n_j) \cdot r](n_i \times n_j) \]

(4.3)

Otherwise this block is stable.

After the determination of the failure mode, the stability analysis is performed to check if the block is stable or not under the loading system. The resisting forces by considering friction and cohesion of the discontinuities are calculated, and then the factor of safety, which is defined as the resisting force divided by the driving force, is determined. The net sliding force \( F \) of each block under each failure mode is calculated using the formulae by Goodman and Shi (1985):

For free falling:

\[ F = R \]

(4.4)

For sliding on one plane:

\[ F = |n_i \times R| - |(n_i \times R)|\tan \theta_i - c_i A_i \]

(4.5)

where \( \theta_i \) is friction angle, \( c_i \) is the cohesion and \( A_i \) is the area of face \( i \).
For sliding on the intersection of two planes:

$$ F = \frac{1}{|n_i \times n_j|^2} [ |R \cdot (n_i \times n_j)||n_i \times n_j| - |(R \times n_i) \cdot (n_i \times n_j)| \tan \phi_i - c_i A_i - |(R \times n_i) \cdot (n_i \times n_j)| \tan \phi_j - c_j A_j $$

(4.6)

where $\phi_i$, $c_i$ and $A_i$ are friction angle, cohesion and the area of face $i$ respectively while $\phi_j$, $c_j$ and $A_j$ are friction angle, cohesion and the area of face $j$.

The whole procedure is shown in the flow chart in Figure 4-2. It should be mentioned that for the stability analysis of each single key block during force transfer process, the sliding forces transferred from key blocks in adjacent later batch are included in the resultant driving force.

![Flow chart of single block stability analysis](image)

**Figure 4-2 Flow chart of single block stability analysis**

**Progressive failure analysis of blocky rock mass**

A two-dimensional tunnel is shown in Figure 4-3. The key blocks numbered 1 belong to the first batch. Their removal will allow the movement of key blocks in the second batch into the new excavation space. Successively, key blocks in batches 3-7 will slide into the excavating tunnel.
With the generation of rock mass model using the block generation program from a discontinuity network, the above progressive failure of the discrete blocky rock mass can be simulated by the key block methods (Goodman and Shi, 1985; Warburton, 1981).

![Figure 4-3 Progressive failure of a tunnel](image)

After the blocky rock mass has been reconstructed, the blocks around the excavation surfaces with one or more free faces (day-lighting blocks) are selected as they are potential key blocks. The centroid and volume of each potential key block are calculated using Simplex integration (Shi, 1997), and the resultant driving force is determined by vector analysis. Then the single block stability analysis is employed to check whether they are key blocks or not.

With the searching and then removal of a key block around the excavation surfaces, each of the contact surfaces between this key block and its adjacent blocks becomes free surface, which means that the constraints of the adjacent block have been reduced. The adjacent block which is originally stable could become instable. In this way, the progressive failure of the rock mass can be simulated with all the batches of key blocks searched and then removed until there are no key blocks remaining in the rock mass as shown in Figure 4-4.

The above analysis has no difference compared to the traditional key block methods (Goodman and Shi, 1985; Warburton, 1981) except that a progressive failure analysis of the blocky rock mass is performed. This analysis is necessary for the following stability assessment of the key blocks by considering their interactions.
4.3 AN IMPROVED KEY BLOCK METHOD BY CONSIDERING INTERACTIONS OF KEY BLOCKS

In the traditional key block analysis, each block is treated as if its adjacent blocks are fixed (Warburton, 1987). Actually, the resultant sliding forces of the neighbouring key blocks in the later batch are imposed on key blocks in the earlier batch. Unfortunately, the current key block methods did not consider such force transfer and it may lead to an unsafe estimation of the sliding forces of those surface instable key blocks. In order to consider the effects between adjacent batches of key blocks, a force transfer algorithm is developed. Thus the assumption that when each block is analysed, its neighbouring blocks are treated as fixed is relaxed. For support design, only the key blocks exposed to the excavation are critical for the stability of the rock mass. If these blocks are supported and held in position, the overall stability of the rock mass can be assured. By transferring the sliding forces of key blocks in later batches batch by batch to the key blocks in earlier batches, the required supporting forces of the surface key blocks can then be more accurately calculated.

Figure 4-5 shows an example, which has totally 5 batches of key blocks. There is only one key block existing in the first batch, which is critical for the stability of the rock mass. By the means of a progressive process, the out-of-balance forces of all the inner key blocks will be transferred to that day-lighting key block.
In the force transfer process, the effects of key blocks in a later batch to key blocks in its adjacent earlier batch can be classified into destabilising and stabilising effects. In Figure 4-6, blocks \( b_1 \) and \( b_2 \) both are key blocks, and belong to the later batch \( i \) and an earlier batch \( j \) respectively, but block \( b_2 \) have different effects on block \( b_1 \) in Figure 4-6 a) and b). The out-of-balance force of block \( b_2 \) contributes to the movement of block \( b_1 \) in Figure 4-6 a) while that of block \( b_2 \) prevents the movement of block \( b_1 \) in Figure 4-6 b).

Here, one two-dimensional example is used to demonstrate the principle of the force transfer algorithm. In Figure 4-7, block 3 belongs to batch \( i + 1 \) while block 1 and 2 belong to batch \( i \). Blocks 3 has one contact face with block 1 and 2 respectively, and the out-of-balance force is \( F \). After analysis, these two faces are also force transfer faces (effective contact faces), thus the force \( F \) will be transferred to blocks 1 and 2 via these two contact faces separately. Since the stiffness of each rock block is quite high, the deformation will be very small. During the force transfer process, blocks 1 and 2 are
assumed to be held in position. In addition, due to the translational modes assumption, it is reasonable to further assume that the magnitude of each transferred force $F_i$ is proportional to the projected area $A'_i$ of contact faces to the plane perpendicular to the sliding force of block 3.

$$F_i = F \times \frac{A'_i}{\sum_i A'_i} \quad i = 1, 2 \quad (4.7)$$

For a blocky rock mass, several batches of key blocks may be searched. For the key blocks in a later batch, their out-of-balance forces will be transferred to key blocks in the earlier batch via the effective contact faces. The flow chart in Figure 4-8 shows the detailed procedure of the force transfer process:

1) For each key block $B_m$ in a later batch $i + 1$ and one of its neighbouring key block $B_n$ in an earlier batch $i$, the number of contact faces between these two blocks are identified;

2) The effective contact faces are determined from the contact faces according to the relationship of the sliding direction $n$ ($n_1 \ n_2 \ n_3$) of block $B_m$ and the contact faces with normal vector $n_{\text{nk}}$ ($u_k \ v_k \ w_k$) in
block $B_n$. If $\mathbf{n} \cdot \mathbf{n}_{vk} > 0$, the face $k$ is one of the force transfer face;

3) If the force will be transferred to more than one block, the effective contact faces will be projected to be perpendicular to the sliding direction, and the forces transferred to each block will be proportional to the area of the projected faces;

4) After the sliding forces of key blocks in batch $i + 1$ are transferred to key blocks in batch $i$, the stability of the key blocks in the earlier batch $i$ is re-assessed by key block analysis using the vector analysis method;

The processes 1)-4) are repeated, until the out-of-balance forces of all the key blocks in later batches are transferred to the key blocks in the first batch. Then the resultant forces of key blocks in the first batch are the final ones used for support design.

![Flow chart for force transfer algorithm](image-url)

**Figure 4-8 Flow chart for force transfer algorithm**
4.4 REINFORCEMENT EFFECTS AND SAFETY CHECK

After stability analysis of the blocky rock mass, a suitable rock bolting system needs to be selected. The support design can be done on the basis of the results from the above key block analysis (Goodman et al., 1982; Windsor, 1997). All the key blocks identified by the key block analysis should be reinforced or supported by a suitable support scheme. The traditional rock mass classification systems (RMR and Q system) (Bieniawski, 1973; Barton et al., 1974) can also be employed for support design. The rock bolting system can be systematic, spot bolting system or a combination of these two systems. The systematic rock bolting system is normally used to control the overall deformation of the entire tunnel and it can form an arc effect in the tunnel roof leading to self-supporting (Li, 2006). Due to the uncertainties in rock mass geometry, the systematic rock bolting system can help reduce the possibility of instability of local key blocks. The spot bolting system is mainly for large key blocks at local areas. Fibre reinforced shotcrete is usually used to support those small key blocks which are not reinforced by the above two bolting systems. If these small blocks are not critical for the movements of other blocks or not leading to a wider instability of the rock mass, they can also be removed without disturbing adjacent blocks (Wu et al., 2011).

After reinforcement by rock bolts, the stability of the reinforced rock mass needs to be re-checked. According to the principle of key block methods, only the key blocks on the periphery of the opening have to be supported. If these are held in position by the support system, the blocks behind will remain in place (Goodman, 1995). However, after rock bolts have been applied to the rock mass, some of the key blocks may be anchored to their neighbouring blocks and form large instable blocks. In order to check if these kind of large key blocks are formed, a two-step further safety check is suggested in this study.

First, each key block is re-assessed by considering the support force from the rock bolts. For each key block in the first batch, the number of rock bolts which intersect this block is counted. Then whether the provided support forces to each key block is sufficient or not is assessed. The strengthening effect of a bolt is divided into an additional cohesion term related to the parallel component of the force and confining stress term connected to the normal component of the force in the bolt. They both depend on bolt inclination,
strength of the rock-bolt-grout system as well as friction angle of joints (Pellet and Egger, 1996). The current analysis adopted the simplified approach used by Grenon and Hadjigeorgiou (2003), and the above factors as well as the bolt deformation process are not taken into account. More accurate analyses can be conducted to calculate the bolt contribution using empirical equations established by Bjurstrom (1974) and Spang and Egger (1990) or analytical solutions developed by Pellet and Egger (1996). The factor of safety of each reinforced key block is defined as follows:

(a) Free falling:

\[
\text{Factor of Safety} = \frac{\sum_{k=1}^{n} RB_{g}^{k}}{G}
\]

(b) Single plane sliding:

\[
\text{Factor of Safety} = \frac{G_{N_{i}} tan \phi_{i} + A_{i} c_{i} + \sum_{k=1}^{n} (RB_{nl}^{k} tan \phi_{i} + RB_{sl}^{k})}{G_{s_{i}}}
\]

(c) Double plane sliding:

\[
\text{Factor of Safety} = \frac{G_{N_{i}} tan \phi_{i} + A_{i} c_{i} + G_{N_{j}} tan \phi_{j} + A_{j} c_{j}}{G_{s_{ij}}}
\]

\[+ \frac{\sum_{k=1}^{n} (RB_{nl}^{k} tan \phi_{i} + RB_{nj}^{k} tan \phi_{j} + RB_{sl}^{k})}{G_{s_{ij}}}
\]

where \(n\) is the number of rock bolts, \(G\) is block gravity, \(G_{N_{i}}\) and \(G_{s_{i}}\) and are the normal and tangential components of gravity to sliding plane \(i\) respectively, \(G_{s_{ij}}\) is the component of block gravity in the direction of the intersection line between plane \(i\) and \(j\), \(RB_{g}\) is the component of bolt capacity in the gravity direction, \(RB_{nl}\) and \(RB_{sl}\) are the normal and tangential components of bolt capacity to plane \(i\), \(RB_{s_{ij}}\) is the component of bolt capacity in the direction of the intersection line between plane \(i\) and \(j\), \(\phi_{i}\), \(c_{i}\) and \(A_{i}\) are the friction angle, cohesion and area of plane \(i\) respectively.

For a large block with the area of free face larger than a certain threshold value (for example 0.5 m\(^2\) used in this study), if the systematic bolting system is not enough to support it, additional rock bolts will be required. For small blocks with the area of free
face smaller than the threshold value, fibre reinforced shotcrete with/without wire mesh can be used to provide the support.

Secondly, the rock bolts are assumed to be strong enough to connect the blocks intersected by them. All the key blocks connected by rock bolts are added together to form larger blocks and the removability and stability of the added blocks are re-checked. It is noted that the length of rock bolts inside the blocks should satisfy the minimum anchorage length. Otherwise, the rock bolts could not hold the blocks together. As shown in Figure 4-9, one two-dimensional rock tunnel model is used to show the principle of safety check. Before reinforcement, the rock mass is instable and the rock blocks will fall down one after another and a progressive failure will occur. Then support design is needed for the stabilization of the rock tunnel. After the rock bolting system has been selected and installed, the rock bolts will connect the rock blocks into several larger blocks. In Figure 4-9 a), the rock bolts are installed perpendicular to the tunnel excavation surfaces and distributed evenly along the surfaces. Each rock bolt is almost parallel to one another. After reinforcement, the rock bolts connected rock blocks around the tunnel surfaces into five large blocks as shown in Figure 4-9 b). Blocks ①, ③ and ⑤ are tapered blocks and cannot move as long as their adjacent blocks are held in position. However, the formation of blocks ② and ④ are a special case in which the key blocks in the first batch is anchored to key blocks in

Figure 4-9 Safety check of the reinforced rock mass: a) a horseshoe shaped tunnel reinforced by rock bolts; b) connected rock blocks by rock bolts
the second batch or even higher batches thus forming larger removable blocks. If the stabilizing force along the bottom discontinuity of any of the two connected new block is not large enough to counter the sliding force, this tunnel is not stable under the current rock bolting system. Thus the rock support scheme should be replaced or re-adjusted and further analyses should be performed until the stability of all the blocks in both the first step and second step is achieved.

4.5 VERIFICATIONS AND APPLICATIONS

A computer program using C language has been developed to implement the force transfer algorithm and two-step safety check into the vectorial key block program and two examples are analysed to demonstrate the proposed algorithm.

4.5.1 Verification of removability analysis

A wedge was formed by two joints with orientations 60/045 and 60/135 respectively on a slope surface (Figure 4-10). The developed key block program was employed to find out the possible failure mode of this wedge. It was found that this block was removable with double plane sliding mode and the sliding direction is (0.63, 0, -0.77).

![Figure 4-10 A wedge formed on a slope surface](image)

In order to verify the results, the UNWEDGE (Rocscience, 2003) was used to analyse the same problem. Error! Reference source not found. shows the stereographic projection of the input planes. This block would slide along the intersection of the two joints. This verifies the developed key block method for stability of blocky rock masses.
Figure 4-11 Analysis results by UNSWEDGE (Rocscience, 2003): a) stereonet view of input planes; b) display of the wedge formed on the slope

4.5.2 A 3D tunnel example

Figure 4-12 Rock mass model generation: a) one blocky rock mass model; b) discontinuity orientation information; c) tunnel geometry; d) rock tunnel model
A tunnel example is used to demonstrate the concept of extended key block analysis for support design. A cuboid of rock mass with dimensions 80m×80m×60m (Figure 4-12 a)) is simulated containing a horseshoe shaped tunnel of interest. There are mainly three joint sets, and the detailed information is shown in Table 4-1. For each set, the shape of each discontinuity is assumed to be rectangular while the locations and sizes follow a Poisson distribution and normal distribution respectively. The orientation information is shown using the equal angle lower hemisphere projection in Figure 4-12 b). Based on the above information, the discontinuity network is generated using the Monte Carlo simulation. Then the blocky rock mass model as shown in Figure 4-12 a) is generated using the three-dimensional block generation program. The horseshoe shaped tunnel is 10m wide, 80m long and max. 10m/min. 5m high. With the introduction of the tunnel geometry (Figure 4-12 c)) and the removal of the blocks in the excavation, the final tunnel model is generated as shown in Figure 4-12 d). In this model, there are totally 4458 rock blocks including both convex and concave blocks. This is a shallow tunnel, so the in-situ stress is ignored in the analysis. The cohesion is also neglected. The density of the rock is set to be 2700 kg/m³.

<table>
<thead>
<tr>
<th>Orientation (°)</th>
<th>Location</th>
<th>Shape</th>
<th>Size (m)</th>
<th>Friction angle (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dip Dip direction</td>
<td>Location</td>
<td>Shape</td>
<td>Size (m)</td>
<td>Friction angle (°)</td>
</tr>
<tr>
<td>30±(20) 030±(20)</td>
<td>Poisson distribution</td>
<td>Rectangle</td>
<td>Normal distribution: each side length: (µ=100, σ=30)</td>
<td>25(±5)</td>
</tr>
<tr>
<td>45±(20) 150±(20)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60±(20) 270±(20)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then the proposed key block analysis is carried out to identify the instable blocks. There are five batches of key blocks in total and the first batch included ten key blocks. Figure 4-13 a) illustrates all the batches of key blocks in this model while Figure 4-13 b) displays only the first batch of key blocks. Based on the extended key block analysis, the net sliding forces of the key blocks in later batches are transferred batch by batch to the key blocks in the first batch. Both the stabilising and de-stabilising effects of key
blocks in later batches are reflected in the results. The comparison is made between the key blocks in the first batch before and after the force transfer (Table 4-2). From Table 4-2, it can be seen that blocks 3043 and 4446 become stable when the effects of key blocks in later batches are taken into consideration. The failure mode of block (ID 4449) changes from double plane sliding to single-plane sliding. The net-sliding forces of blocks 4420 and 4449 increase.

![Figure 4-13 Key block search: a) key blocks in all batches; b) key blocks in the first batch](image)

<table>
<thead>
<tr>
<th>Block ID</th>
<th>Sliding direction</th>
<th>Net sliding force (KN)</th>
<th>Failure mode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>before force transfer</td>
<td>after force transfer</td>
<td>before</td>
</tr>
<tr>
<td>2745</td>
<td>0.58</td>
<td>-0.14</td>
<td>0.58</td>
</tr>
<tr>
<td>3043</td>
<td>0.30</td>
<td>0.32</td>
<td>-0.90</td>
</tr>
<tr>
<td>3081</td>
<td>0.22</td>
<td>-0.53</td>
<td>0.22</td>
</tr>
<tr>
<td>3460</td>
<td>-0.39</td>
<td>-0.65</td>
<td>-0.39</td>
</tr>
<tr>
<td>3463</td>
<td>0.22</td>
<td>-0.53</td>
<td>0.22</td>
</tr>
<tr>
<td>4009</td>
<td>0.27</td>
<td>0.55</td>
<td>0.27</td>
</tr>
<tr>
<td>4420</td>
<td>0.03</td>
<td>-0.01</td>
<td>0.03</td>
</tr>
<tr>
<td>4446</td>
<td>0.19</td>
<td>-0.38</td>
<td>0.19</td>
</tr>
<tr>
<td>4448</td>
<td>-0.17</td>
<td>-0.75</td>
<td>-0.17</td>
</tr>
<tr>
<td>4449</td>
<td>-0.27</td>
<td>-0.54</td>
<td>-0.27</td>
</tr>
</tbody>
</table>

Then the number of rock bolts intersected with each key block in the first batch is calculated. Figure 4-15 shows the unrolled map of the tunnel surfaces including the systematic rock bolting system and the free faces of the key blocks in the first batch. From this figure, it is clearly seen that only blocks 4420 and 4009 are reinforced by the
systematic bolting system with two and one rock bolt(s) respectively while the other six key blocks are all relatively small in volume or area of free faces and located between the evenly distributed rock bolts.

![Figure 4-14 Blocky rock mass with rock bolting system](image)

After careful check, key blocks 4420 and 4009 in the first batch are stable. No additional rock bolts are needed. In this example, 7 cm fibre reinforced shotcrete is sufficient to support the other small key blocks.

There are totally 288 rock bolts in the systematic bolting system. All the blocks connected by those rock bolts are defined as the range in this rock mass model affected by the bolting system (Figure 4-16). After the blocks intersected by each rock bolt are calculated, 16 connected large rock blocks within the range affected by the bolting system are formed around the tunnel surfaces. Each of these blocks is composed by a number of blocks in the original rock mass model connected together by the rock bolts. Some of the connected blocks are selected and shown in Figure 4-17. The removability and stability analyses of these integrated blocks are performed, and the results show that all these blocks are stable.
Figure 4-16 Blocks reinforced by rock bolts

Figure 4-17 Selected blocks connected by rock bolts
4.5.3 Powerhouse example

An underground powerhouse project is carried out in a fractured blocky rock mass. It consists of a series of underground structures. Three main parts of the powerhouse are analysed in this study: powerhouse cavern, transformer cavern and access tunnel. The domain of rock mass with dimensions 130m×114m×110m shown in Figure 4-18 a) is considered.

Three joint sets are identified based on the analysis of the collected data from field survey, and the detailed information is shown in Table 4-3. The shape of each discontinuity is assumed to be elliptical. For each set, the locations and sizes follow a Poisson distribution and exponential distribution respectively. Next the discontinuity network is generated using the Monte Carlo simulation and validated using goodness-of-fit tests by comparison with the measured trace map. Then the blocky rock mass
model shown in Figure 4-18 a) is generated. The profile of three excavations is shown in Figure 4-18 b). The main dimensions are listed in Table 4-4 with cross sectional parameters defined in Figure 4-18 c). After the geometry of the excavations is introduced and the blocks in the excavations are removed, the final powerhouse model is generated as shown in Figure 4-18 d). In this model, there are totally 7511 rock blocks with different shapes. The in-situ stress and cohesion are not considered in this analysis. The density of the rock is set to be 2500 kg/m$^3$.

Table 4-3 Information about the discontinuity network

<table>
<thead>
<tr>
<th>Orientation(°)</th>
<th>Location</th>
<th>Shape</th>
<th>Size (m)</th>
<th>Friction angle (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dip</td>
<td>Dip direction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40±(15)</td>
<td>070±(15)</td>
<td>Poisson</td>
<td>Ellipse</td>
<td>Exponential distribution: major and minor axis length: ($\lambda = 0.004$)</td>
</tr>
<tr>
<td>65±(15)</td>
<td>150±(15)</td>
<td></td>
<td></td>
<td>28±(2)</td>
</tr>
<tr>
<td>50±(15)</td>
<td>230±(15)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4-4 Dimensions of excavations

<table>
<thead>
<tr>
<th></th>
<th>Width (m)</th>
<th>Side wall height (m)</th>
<th>Arch height (m)</th>
<th>Length (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Powerhouse Cavern</td>
<td>20</td>
<td>24</td>
<td>5</td>
<td>47</td>
</tr>
<tr>
<td>Transformer Cavern</td>
<td>10</td>
<td>10</td>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>Access tunnel</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>43</td>
</tr>
</tbody>
</table>

All the instable blocks are identified by the key block analysis. There are six batches of key blocks in total and the first batch included 18 key blocks. Figure 4-19 a) illustrates all the batches of key blocks in this model while Figure 4-19 b) displays only the first batch of key blocks. The extended key block analysis, in which the net sliding forces of the key blocks in later batches are transferred batch by batch to the key blocks in the first batch, is conducted. Error! Not a valid bookmark self-reference. shows the key block data in the first batch before and after the force transfer. Among those key blocks, the information of 7 blocks does not change since there are no key blocks from later batches next to them. There are 3 blocks (block ID 2348, 7455 and 7462) which become stable when the effects of key blocks in later batches are considered.
### Table 4-5 Key block information before and after force transfer

<table>
<thead>
<tr>
<th>Block ID</th>
<th>Sliding direction before force transfer</th>
<th>Sliding direction after force transfer</th>
<th>Net sliding force (KN) before</th>
<th>Net sliding force (KN) after</th>
<th>Failure mode before</th>
<th>Failure mode after</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>before</td>
<td>after</td>
<td>before</td>
<td>after</td>
<td>before</td>
<td>after</td>
</tr>
<tr>
<td>2348</td>
<td>-0.33</td>
<td>-0.56</td>
<td>-0.76</td>
<td></td>
<td>124.69</td>
<td>double stable</td>
</tr>
<tr>
<td>2349</td>
<td>0.57</td>
<td>-0.15</td>
<td>-0.81</td>
<td>0.57</td>
<td>-0.152</td>
<td>0.81</td>
</tr>
<tr>
<td>2988</td>
<td>0.23</td>
<td>-0.56</td>
<td>-0.80</td>
<td>0.23</td>
<td>-0.56</td>
<td>-0.80</td>
</tr>
<tr>
<td>3556</td>
<td>0.23</td>
<td>-0.56</td>
<td>-0.80</td>
<td>0.23</td>
<td>-0.56</td>
<td>-0.80</td>
</tr>
<tr>
<td>5070</td>
<td>0.19</td>
<td>-0.38</td>
<td>-0.91</td>
<td>0.22</td>
<td>-0.41</td>
<td>-0.89</td>
</tr>
<tr>
<td>5707</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>5914</td>
<td>0.19</td>
<td>-0.38</td>
<td>-0.91</td>
<td>0.21</td>
<td>-0.41</td>
<td>-0.89</td>
</tr>
<tr>
<td>5923</td>
<td>0.13</td>
<td>-0.51</td>
<td>-0.85</td>
<td>0.13</td>
<td>-0.51</td>
<td>-0.85</td>
</tr>
<tr>
<td>5924</td>
<td>0.36</td>
<td>-0.68</td>
<td>-0.64</td>
<td>0.36</td>
<td>-0.68</td>
<td>-0.64</td>
</tr>
<tr>
<td>6503</td>
<td>0.30</td>
<td>-0.49</td>
<td>-0.82</td>
<td>0.30</td>
<td>-0.49</td>
<td>-0.82</td>
</tr>
<tr>
<td>6569</td>
<td>0.70</td>
<td>0.44</td>
<td>-0.56</td>
<td>0.92</td>
<td>-0.04</td>
<td>-0.40</td>
</tr>
<tr>
<td>6615</td>
<td>0.70</td>
<td>0.44</td>
<td>-0.56</td>
<td>0.77</td>
<td>0.48</td>
<td>-0.42</td>
</tr>
<tr>
<td>7168</td>
<td>-0.59</td>
<td>-0.59</td>
<td>-0.56</td>
<td>0.70</td>
<td>0.44</td>
<td>-0.56</td>
</tr>
<tr>
<td>7399</td>
<td>0.70</td>
<td>0.15</td>
<td>-0.70</td>
<td>0.84</td>
<td>0.18</td>
<td>-0.52</td>
</tr>
<tr>
<td>7455</td>
<td>-0.29</td>
<td>-0.71</td>
<td>-0.64</td>
<td></td>
<td>54.04</td>
<td>double stable</td>
</tr>
<tr>
<td>7462</td>
<td>-0.19</td>
<td>-0.69</td>
<td>-0.70</td>
<td></td>
<td>32.36</td>
<td>double stable</td>
</tr>
<tr>
<td>7498</td>
<td>-0.73</td>
<td>-0.16</td>
<td>-0.67</td>
<td>-0.89</td>
<td>0.19</td>
<td>-0.42</td>
</tr>
<tr>
<td>7501</td>
<td>-0.56</td>
<td>-0.14</td>
<td>-0.82</td>
<td>-0.67</td>
<td>-0.17</td>
<td>-0.72</td>
</tr>
</tbody>
</table>

Figure 4-19 Key block search: a) key blocks in all batches; b) key blocks in the first batch
A systematic bolting system with a length of 3m and a spacing of 2m is chosen for this powerhouse. Figure 4-20 a) shows both the rock bolting system and the blocky rock mass.

![Image](image_url)

Figure 4-20 Reinforcement of the blocky rock mass: a) blocky rock mass with rock bolting system; b) blocks reinforced by rock bolts

For each surface key block, the number of rock bolts that reinforce that block is counted. After careful check, all the key blocks with the sliding force larger than 20 KN in the first batch are reinforced by one or a number of rock bolts and thus stable. For the support of all the other key blocks in the first batch, the 15cm thick shotcrete is utilized by considering the free surface area, sliding direction and force of each key block.

There are 1315 rock bolts in total in the systematic bolting system. The range in this rock mass model affected by the bolting system is shown in Figure 4-20 b). After the blocks intersected by each rock bolt are calculated, 35 connected large rock blocks within the range affected by the bolting system are formed around those excavation surfaces. Some of the connected blocks are selected and shown in Figure 4-21. After the performance of the removability and stability analyses for these connected blocks, they are all stable.
4.6 CONCLUSIONS

The extension of the traditional key block method has been presented in this chapter. The rock mass model is generated by considering the spatial distribution of the discontinuities inside a rock mass. It is convenient to identify different batches of key blocks by a progressive failure process. A force transfer algorithm has been proposed in order to take into consideration the interactions between key blocks in different batches. Based on the extended key block method, the corresponding support design is more rational. After a rock bolting system is selected, a two-step safety check procedure is employed for the assessment of the stability of the bolted rock mass. The tunnel and powerhouse examples showed that the later batches key blocks did affect significantly
the sliding force of some surface key blocks. If this effect is ignored, the required supporting force for some key blocks will be significantly underestimated. The proposed method also has the advantage of finding out whether larger key block will be formed after the rock bolting system is applied.

However, it should be realized the present study is based on a deterministic rock mass model. Due to the difficulties to accurately measure the spatial distribution of the discontinuities and the properties of the intact rock and discontinuities, probabilistic methods should be used to reflect these uncertainties in the rock mass and a large number of realizations of discontinuity network and then the blocky rock mass statistically equivalent to the actual rock mass need to be generated. Future work is required to take into consideration the statistical nature of the discontinuities and application of the proposed key block analysis using probabilistic methods for support design.
CHAPTER 5. A STOCHASTIC KEY BLOCK ANALYSIS METHOD FOR ROCK MASSES CONTAINING GEOLOGICAL UNCERTAINTIES

5.1 INTRODUCTION

Due to the discontinuous nature of rock masses, rock engineering practices are significantly affected by the uncertainties of geological and geometrical parameters of the inherent discontinuities (Goodman, 1995). Uncertainties in rock engineering also include mechanical properties of discontinuities as well as the loading conditions (water pressure, seismic forces). All these parameters as well as their uncertainties affect the stability of the blocky rock masses (Starzec et al., 2002a; Park and West, 2001; Johari et al., 2013). A stochastic analysis is more appropriate to deal with the effect of these uncertainties.

In this chapter, we focus on the development of a stochastic key block analysis method based on realistic stochastic representation of blocky rock masses. With Monte Carlo simulations, the developed robust block generation program dealing with non-persistent discontinuities is employed to generate a number of realizations of a blocky rock mass from the corresponding number of realizations of the discontinuity network for stability analysis. Therefore, more accurate and reliable results of key blocks statistics from progressive failure analysis can be obtained. This method is demonstrated by the application to a hypothetical horseshoe-shaped tunnel. Some modelling issues are also discussed. In order to investigate how the discontinuity size will affect the stability of blocky rock mass, three scenarios of discontinuity network with different mean sizes are utilized for key block predictions. The entrance of a tailrace tunnel at the Jinping I hydropower station is analysed by the stochastic key block analysis as well.

5.2 METHODOLOGY OF STOCHASTIC KEY BLOCK ANALYSIS

The stochastic key block analysis is based on three successive modules: discontinuity network generation using Monte Carlo simulations, the developed block generation program (Chapter 3) and vectorial key block analysis (Chapter 4). With Monte Carlo
simulation, a number of Discrete Fracture Network (DFN) realizations are utilized to cover the uncertainties in discontinuity geometry, so they are statistically equivalent to the actual geometry within the rock mass. The robust block generation program, which can handle complex excavations and non-persistent discontinuities, lays the foundation for the generation of blocky systems from randomly generated finite discontinuities. After generation of all blocky system realizations, key block analysis by considering the uncertainties in mechanical parameters of discontinuities and loading conditions is conducted for each realization and results are output for statistical analysis.

For engineering design, care should be taken for the determination of domain size since it affects the stability analysis results. For a tunnel model with a small domain, after analysis, all the blocks above the tunnel except those with boundary faces (fixed faces) are possibly key blocks. If the domain is larger, more key blocks will probably be searched. Thus in the larger domain case, the volume of key blocks is underestimated. For stability analysis, the boundaries of the domain should be far enough from the excavation. This normally does not apply to the distance between the lower boundary and the opening floor except that the water pressure and/or other forces have to be considered and they are so high that some removable blocks on the floor can also become key blocks. In addition, experience and engineering judgement should also be exercised to determine the size of the domain based on the geological data.

5.2.1 Stability analysis of a single realization

A typical stability analysis procedure is shown in Figure 5-1 for both a rock slope and tunnel. It is actually one realization of the stochastic analysis. The rock mass model with complex excavations can also be handled. Four horseshoe shaped tunnels with the geometries depicted in Figure 5-2 a) are excavated from a blocky rock mass. One realization of the rock mass model is shown in Figure 5-2 b). After key block analysis, all batches (41 batches in total and 247 key blocks) and first batch (19 key blocks) of key blocks are identified and presented in Figure 5-2 c) and d). Among those key blocks in the first batch, one concave key block with complex shape is generated at the intersection of two intersecting tunnels (Figure 5-2 e)). After checking the detailed information of this key block, it is found that the failure mechanism is double-plane
sliding with sliding direction (0.28 -0.86 -0.43). The existence of this key block with complex morphology demonstrates the capability of stability analysis by this approach.

5.2.2 Modelling with Monte Carlo simulation

After field mapping, subsequent data processing and laboratory tests, the distributions for all the parameters with uncertainties are estimated, and a suitable geological model with the consideration of interdependence of discontinuity characteristics are determined. Then Monte Carlo simulation can be utilized to generate random numbers for each parameter in the model. The Monte Carlo Simulation essentially is a random number generator that is useful for prediction, estimation and risk analysis.

Random number generation
A number of methods including Inverse transformation, Composition, Convolution, Acceptance-rejection, Sampling and Data–Driven Techniques etc. have been used to generate random numbers (Saucier, 2000). All of these methods are based on a supply
of uniformly distributed random numbers in the half-closed unit interval \([0, 1)\) and they are only concerned with transforming the uniform random numbers on the unit interval into random numbers following the prescribed form. Among those methods, the inverse transformation technique is the most commonly used one.

Figure 5-2 Stability analysis of a complex underground excavation: a) Intersecting tunnels; b) Final tunnel model; c) Key blocks in all batches; d) Key blocks in the first batch; e-f) Key block 646 formed at the intersection of two horseshoe shaped tunnels
1) Fisher distribution (Saucier, 2000)

\[ f_f(x) = \frac{K \sin x e^{K \cos x}}{e^K - e^{-K}}, \quad (x \in [0, \infty)) \]  

(5.1)

\( x \) is the angular deviation from the mean orientation, and \( K \) is Fisher’s constant.

To simplify fisher distribution, the following approximate expression for the cumulative distribution is used:

\[ F_f \approx 1 - e^{K(\cos x - 1)} \]  

(5.2)

Random numbers \( x_f \) following Fisher distribution can be obtained as follows:

\[ x_f = \arccos \left( \frac{\ln(1 - r)}{K} + 1 \right), \quad r \in \text{rand}[0, 1) \]  

(5.3)

The dip \( \beta \) and dip direction \( \alpha \) of the generated discontinuity can be evaluated from \( \theta \) by the following equations proposed by Leung and Quek (1995):

\[ \sin \beta = \frac{\sin(\beta_a - \theta) \sin \theta + 8 \cos \beta_a \sin^2 \frac{\theta}{2} \sin^2 \frac{\lambda}{2}}{\sin \theta} \]  

(5.4)

\[ \cos(\alpha_a - \alpha) = \frac{\cos \theta - \sin \beta_a \sin \beta}{\cos \beta_a \cos \beta} \]  

(5.5)

where \( \lambda \) is an arbitrary angle that can be randomly generated following a uniform distribution over the interval \((0, 2\pi)\), \( \beta_a \) and \( \alpha_a \) are the known mean orientation of a discontinuity.

2) Uniform distribution (Saucier, 2000)

The probability density function of uniform distribution is expressed as:

\[ f_u(x) = \frac{1}{(x_{\text{max}} - x_{\text{min}})} \quad (x \in [x_{\text{max}}, x_{\text{min}}]) \]  

(5.6)

For random numbers from a uniform distribution over the interval \([a, b]\), the cumulative distribution function is given by:

\[ F_u(x) = \int_a^x f_u(x)dx = \int_a^x \frac{1}{b - a}dx = \frac{x - a}{b - a} \]  

(5.7)
Let \( r = F_u(x) \), \( r \) is a random number between 0 and 1, then the uniformly distributed random number \( x_u \) on \([a, b]\) is given by:

\[
x_e = r(b - a) + a \quad r \in \text{rand}[0, 1) \tag{5.8}
\]

3) Negative exponential distribution (Saucier, 2000)

For random numbers from a negative exponential distribution:

\[
f_e(x) = \lambda e^{-\lambda x}, \quad (x > 0) \tag{5.9}
\]

The cumulative distribution function is given by:

\[
F_e(x) = \int_{-\infty}^{x} f_e(x)dx = 1 - e^{-\lambda x} \tag{5.10}
\]

Assume \( r = F_e(x) \), \( r \) is a random number between 0 and 1, then the negatively distributed random number \( x_e \) is given by the inverse transformation:

\[
x_e = -\frac{1}{\lambda} \ln(1 - r), \quad r \in \text{rand}[0, 1) \tag{5.11}
\]

4) Normal distribution

\[
f_n(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad (\sigma > 0; \ x \in (-\infty, \infty)) \tag{5.12}
\]

Since the probability density function cannot be integrated in closed form, the central limit theorem is applied to approximate and the random numbers following a normal distribution is given by (Jing and Stephansson, 2007):

\[
x_n = \mu + \sigma \sqrt{\frac{12}{M}} \left( \sum_{t=1}^{M} \rho_t - \frac{M}{2} \right), \rho \in \text{rand}[0, 1] \tag{5.13}
\]

Usually, \( M = 10 - 12 \) can provide reasonably good results.

5) Lognormal distribution

\[
f_l(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-\frac{-(\ln(x) - \mu)^2}{2\sigma^2}}, \quad (x \in (0, \infty)) \tag{5.14}
\]

The random numbers for a lognormal distribution can be obtained by the following simple mappings (Jing and Stephansson, 2007)
\[ x' = \ln x, \mu' = \ln \mu - \frac{\sigma^2}{2}, \sigma' = \sqrt{\ln \left( \left( \frac{\sigma}{\mu} \right)^2 + 1 \right)} \] (5.15)

**Stochastic rock mass model generation**

By repeatedly picking values from a probability distribution for the random variables of discontinuity geometrical and mechanical properties and loading conditions, a series of realizations can be obtained.

A series of distributions for orientation, spacing and trace length (Table A.1 in APPENDIX A) have been reported in the literature (Baecher, 1983; Priest, 1993; Jing and Stephansson, 2007). If spacing follows exponential distributions, joint centres tend to be randomly located in space (Priest and Hudson, 1976; Baecher et al., 1977).

![Flow chart of probabilistic key block analysis](image)

**Figure 5-3 Flow chart of probabilistic key block analysis**

After a large number of realizations for the discontinuity network are generated, the same number of blocky system realizations can be obtained by employing the block generation program. Each realization of the blocky system is only a partial representation of the real blocky rock mass. The collection of a sufficient number of realizations is thus a better representation of the in-situ rock mass. The random numbers
for mechanical properties of discontinuities and loading conditions are only used in key block analysis. Figure 5-3 presents the flow chart of the probabilistic key block analysis in our study.

When Monte Carlo simulation is used, one question will naturally arise: how many realizations are sufficient for a reliable statistical analysis of the stability of a blocky rock mass? Monte Carlo simulation normally requires a large number of realizations in order to achieve desired accuracy. However, adequate results can be achieved with a sufficient number of realizations, if the purpose is to estimate the mean and variance of the response variable rather than acquire detailed information on the response distribution (Starzec and Tsang, 2002; Hammah et al., 2009).

5.3 APPLICATION TO ONE HORSESHOE SHAPED TUNNEL

5.3.1 Stochastic key block analysis of a horseshoe shaped tunnel

In this study, a hypothetical tunnel model is used to demonstrate the modelling approach. Currently only discontinuity geometry parameters are considered as random variables while mechanical properties of rock matrix and discontinuities are treated as deterministic values.

![Figure 5-4 Model geometry: a) Domain geometry; b) Tunnel geometry]

A cuboid of rock mass with dimensions 70m × 70m × 50m shown in Figure 5-4 a) is simulated containing a horseshoe shaped tunnel of interest (See geometrical data in Figure 5-4 b)). The Baecher model (Baecher, 1983) is adopted for the discontinuity network. There are mainly three joint sets with dip and dip directions (30 ± (10) 150 ± (10)), (45 ± (10) 024 ± (10)), (75 ± (10) 250 ± (10)) respectively. The number of
discontinuities is 13 in each set. The location of these discontinuities in each set along x,
y and z axes follows uniform distribution while the size of the side length of the
discontinuities assumed to be rectangular follows negative exponential distribution with
mean value $\lambda=0.0125$. It should be emphasized that when size distribution is inferred
from trace length data according to the stereological relationship, all the discontinuities
in one set are assumed to be geometrically similar (Zhang et al., 2002). No matter which
shape (ellipse or parallelogram) the rectangle is further simplified from, this assumption
always applies when the discontinuity parameters are transformed from observed field
data. However, this assumption is ignored in this example in order to generate a more
general discontinuity network. The two adjacent two sides of the rectangles are treated
independently and each of the two side lengths is assumed to follow a negative
exponential distribution. The rotational angle of each rectangle in its discontinuity plane
is set to be 0. Furthermore, when discontinuities are small, they do not form blocks and
will be deleted due to the tree cutting process in the block generation program. In the
key block analysis, all blocks are treated as rigid bodies. So the existence of the
discontinuities partially intersecting with or inside a block does not affect the simulation
results much. Also, the model after tree cuttings is mainly used for stability analysis
rather than for discontinuity propagation. However, for discontinuity propagation
analysis, the discontinuity partly intersecting with or inside a block has great influence
on the failure pattern as well as the process. So in this case, the tree cutting should not
be done. In addition, although some discontinuities do not contribute to the formation of
blocks, they form part of the fluid-conducting pathways. Thus these discontinuities
should not be deleted for fluid flow analysis. In order to generate a reasonable number
of blocks, a truncated negative exponential distribution is used with specified upper
limit (120 m) and lower limit (40 m) to avoid small size value. This is a shallow tunnel,
so the in-situ stress can be ignored in the analysis. The density of the rock is set to be
2500 kg/m$^3$ for a granitic rock mass. The friction angle and cohesion for all
discontinuities are 30° and 0 KPa respectively.

Three realizations of the blocky system model are shown in Figure 5-5. Although
appearing different geometrically, these three blocky systems are statistically
equivalent. After key block analysis is conducted for each realization, the unstable
blocks with free falling and sliding modes are output for further analysis. It should be
emphasized that the configuration of key blocks identified in each realization are only a possible case of the analysed actual tunnel model in space. All the key block information from all realizations together can approximately represent the average key block data.

![Figure 5-5 Three realizations of the tunnel model](image)

![Figure 5-6 Change in the number and volume of key blocks with the increase of the number of realizations](image)
In order to determine the number of realizations, which is sufficient for the probabilistic key block analysis, 110 realizations are generated first. Among those parameters of key blocks, the total number of key blocks alone is not a complete indicator of the rock excavation performance. However, it provides information about the mean size of those key blocks combined with other information (Chan and Goodman, 1983). The total key block volume is considered in (Starzec and Andersson, 2002a; Starzec and Tsang, 2002) as the representative parameter for stability assessment. Figure 5-6 presents how the mean number and volume of key blocks in our example change respectively with the increase of the number of realizations. It can be seen that for this tunnel example after about 60 simulations, both the mean number and volume of key blocks tend to be stable. Therefore, it is concluded that 60 realizations of blocky rock mass model are sufficient and utilized to conduct the stability analysis. After all realizations are analysed, it is found that the predicted mean number of generated blocks is 2925 with standard deviation 411 and the predicted mean volume of a single block is $83 \, m^3$ with standard deviation $12 \, m^3$. Table 5-1 lists the predicted information of key blocks. It is worth mentioning that the number of realizations to achieve convergent statistics of key blocks will be different for different joint set distributions and persistence conditions.

From this example, it can be clearly seen that if all the parameters in the discontinuity network are considered as random numbers following different distributions, the variations in the volume and number of key blocks are quite significant (Table 5-1).

### Table 5-1 Predicted key block information

<table>
<thead>
<tr>
<th></th>
<th>All batches</th>
<th>First batch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of key blocks</td>
<td>Mean 112</td>
<td>Mean 41</td>
</tr>
<tr>
<td>Volume of key blocks</td>
<td>STDEV 126</td>
<td>STDEV 3</td>
</tr>
<tr>
<td>(m$^3$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean volume of a key</td>
<td>Mean 1879</td>
<td>Mean 42</td>
</tr>
<tr>
<td>block (m$^3$)</td>
<td>STDEV 3448</td>
<td>STDEV 3</td>
</tr>
<tr>
<td>Number of key blocks</td>
<td>Mean 13</td>
<td>Mean 13</td>
</tr>
<tr>
<td>(m$^3$)</td>
<td>STDEV 13</td>
<td>STDEV 6</td>
</tr>
<tr>
<td>Volume of key blocks</td>
<td>Mean 13</td>
<td>Mean 6</td>
</tr>
<tr>
<td>(m$^3$)</td>
<td>STDEV 13</td>
<td>STDEV 13</td>
</tr>
<tr>
<td>Mean volume of a key</td>
<td>Mean 6</td>
<td>Mean 3</td>
</tr>
<tr>
<td>block (m$^3$)</td>
<td>STDEV 4</td>
<td>STDEV 4</td>
</tr>
</tbody>
</table>

A study of the morphology of the key blocks is conducted. Normally, the name of a polyhedron is determined by the number of faces (e.g. tetrahedron has four faces). In
this study, in order to make the definition for key blocks consistent with those in the literature (Menendez-Diaz et al., 2009), the tetrahedral key blocks include those with two (corresponding to at least two free faces) and three discontinuity faces (corresponding to at least one free face) while for other polyhedral key blocks all the free surfaces are treated as one free surface to determine the order of the polyhedron. With this definition, each pentahedral key block has four discontinuity faces and at least one free faces. In this example, tetrahedron is the most common type of key blocks for all simulations, and this is in agreement with the experience (Grenon and Hadjigeorgiou, 2003; Hatzor and Goodman, 1993; Hatzor, 1993) that tetrahedral blocks are the most common type observed in the field. It should be mentioned that the mathematical explanation of this phenomenon was also provided by Mauldon (1992) and Hatzor and Feintuch (2005). In most of the simulated cases, the tetrahedral and pentahedral key blocks accounts for around 80% of the key blocks.

The failure mechanisms are also characterized. Free falling, single-plane sliding and double-plane sliding are the three possible failure modes of the key blocks when translational failures only are considered. The failure type statistics can provide guidance for support design. In this case, both plane and wedge failures accounting for 48% and 43% respectively among all the key blocks occur more possibly than free falling failure.

The prediction of the size of the key blocks is important as it indicates the kind and amount of supports that are required to be applied (Gasc-Barbier et al., 2008). The key block size information from all realizations is combined together and the relationship between the percentage of key block volume and various cut-off sizes of key blocks for all batches and first batch is depicted in Figure 5-7. Since the frequency of each key block with volume larger than 6 $m^3$ is quite low, the horizontal scale is shrunk to 6 $m^3$ for the benefit of the clarity of the plot. The largest volume among the key blocks from all 60 realizations is 1143 $m^3$, but the probability of occurrence of such a block is only 0.01% which is extremely low. It can be seen that the majority of the key blocks in all batches and first batch are relatively small, which means that small key blocks will be most frequently encountered in the field. As shown in the cumulative percentage curve (Figure 5-7), 68% of the key blocks in all batches have a volume of smaller than 6 $m^3$ while the percentage is 87% for the key blocks in the first batch.
Figure 5-7 The simulated size distribution of key blocks: a) First batch; b) All batches

Figure 5-8 Selected cases from the 60 simulations: a) worst case (421 key blocks with a total volume of 15858 m$^3$); b) best case (Only 5 key blocks with a total volume of 3 m$^3$)

It should be noted that the progressive failure of blocky rock mass actually overestimates the number and thus the volume of key blocks, because some key blocks
may become self-supporting by rotation or stable with the stabilizing effect of in-situ stress especially for underground excavations. However, these effects are not taken into account. This means the analysis results will be conservative. The worst cases with larger total key block volume can be further analysed using static/dynamic failure process simulation or stress and deformation analysis. Figure 5-8 depicts the worst and best cases selected from the 60 realizations.

Based on the stochastically generated rock mass model, the modelling strategy suggested by making full use of different methods, such as Key Block Method, Discontinuous Deformation Analysis (DDA) method and Numerical Manifold Method (NMM), can be utilized for a realistic rock mass stability design. Figure 5-9 schematically shows the concept towards a realistic and cost-effective rock mass modelling strategy.

![Diagram of realistic rock mass modelling](image)

**Figure 5-9 Toward realistic rock mass modelling**

To consider the randomness of the discontinuities with numerous realizations of the 3-D geological model, key block analysis is able to not only capture unstable blocks, but also greatly reduce the computational cost to a tractable extent. Those critical cases, which result in larger collapsible zone, can be identified. 3D DDA (Shi, 2001) treating each block with a constant stress and strain can then be applied to simulate the mechanical behaviour more accurately. For those most critical cases from DDA
analysis, 3D NMM (He and Ma, 2010) or other coupled Discrete Element Methods (DEM)
should be applied. It will give more accurate results of rock mass deformation and stress
distribution. With this strategy, the rapid stability analysis by key block theory and advanced
discrete element methods can be used together in a complementary way to achieve more
accuracy while the computational cost is controlled to an acceptable level.

5.3.2 Investigation on the size effect of discontinuities

Figure 5-10 One realization in each scenario: (Left) scenario 1; (Middle) scenario 2; (Right)
Scenario 3

Three scenarios with different discontinuity size data are investigated. The previous
tunnel example is considered as scenario 2 referred to as medium. Another two
scenarios are generated by reducing the mean size of discontinuities to \( \lambda = 0.0167 \) (small)
and increased the size to be persistent (infinite) respectively while the orientation,
location and density are fixed. Each of the side length of the rectangular discontinuities
is still considered as independent and follows a truncated negative exponential
distribution with the same upper and lower limits as those in scenario 2. Figure 5-10
shows three realizations of rock mass model from each scenario.

From Table 5-2, clearly, if the persistent discontinuity model is adopted, the block
model is very fragmented. The number of blocks in scenario 3 is almost 5 times
compared with scenario 1 and doubled compared with scenario 2. At the mean time, the
mean block volume is greatly underestimated compared with the non-persistent
discontinuity models. Significant differences in the number and volume of key blocks
have been found as the mean discontinuity size varies (Table 5-3). Some realizations in
scenario 1 are key block free. The mean volume of key blocks in scenario 3 is almost doubled compared with that in scenario 2, and 12 times larger than that in scenario 2.

Table 5-2 Comparison of results on the predicted blocks in those three scenarios

<table>
<thead>
<tr>
<th>Statistical parameters of the predicted blocks</th>
<th>Scenario 1 (Small)</th>
<th>Scenario 2 (Medium)</th>
<th>Scenario 3 (Infinite)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of blocks</td>
<td>Mean: 1295</td>
<td>2925</td>
<td>6642</td>
</tr>
<tr>
<td></td>
<td>STDEV: 257</td>
<td>411</td>
<td>701</td>
</tr>
<tr>
<td>Mean volume of a block (m$^3$)</td>
<td>Mean: 191</td>
<td>83</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>STDEV: 37</td>
<td>12</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 5-3 Comparison of results on the predicted key blocks in those three scenarios

<table>
<thead>
<tr>
<th>Statistical information of the predicted key blocks</th>
<th>Scenario 1 (Small)</th>
<th>Scenario 2 (Medium)</th>
<th>Scenario 3 (Infinite)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of key blocks</td>
<td>Mean: 29</td>
<td>112</td>
<td>280</td>
</tr>
<tr>
<td></td>
<td>STDEV: 24</td>
<td>126</td>
<td>342</td>
</tr>
<tr>
<td>All batches</td>
<td>Mean: 313</td>
<td>1879</td>
<td>3798</td>
</tr>
<tr>
<td>Volume of key blocks (m$^3$)</td>
<td>STDEV: 382</td>
<td>3448</td>
<td>6519</td>
</tr>
<tr>
<td>Mean volume of a key block (m$^3$)</td>
<td>Mean: 10</td>
<td>13</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>STDEV: 13</td>
<td>13</td>
<td>7</td>
</tr>
<tr>
<td>No. of key blocks</td>
<td>Mean: 7</td>
<td>13</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>STDEV: 5</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>First batch</td>
<td>Volume of key blocks (m$^3$)</td>
<td>STDEV: 35</td>
<td>42</td>
</tr>
<tr>
<td>Mean volume of a key block (m$^3$)</td>
<td>Mean: 5</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>STDEV: 5</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
After combining the key block size information from all Monte Carlo simulations for each scenario, it was found that the majority of blocks have a very small volume in all those three scenarios. A close inspection on the key block volume data reveals that the largest key block in scenario 3 is only $792 \text{ m}^3$ while the values are $1022 \text{ m}^3$ for scenario 2 and $1143 \text{ m}^3$ for scenario 1 respectively. This means that the persistent scenario underestimates the maximum volume of single key blocks quite significantly. This highlights the necessity for the non-persistent representation of discontinuities in the stability analysis of blocky rock mass. The discontinuity persistence should be estimated carefully from collected field data and also modelled as close and accurate as possible to that inferred from field data.

5.4 STABILITY ANALYSIS OF THE ENTRANCE OF THE TAILRACE TUNNEL AT JINPING I HYDROPOWER STATION

Jinping I hydropower station is located upstream of the great Jinping river bend. The power generation complex (Figure 5-11), located on the right bank of the Yalong river, consists of an underground powerhouse, two surge chambers, a main transformer chamber, two U-shaped tail-race tunnels, six bus tunnels connecting the underground powerhouse and the transformer chamber and other buildings (Huang and Huang, 2010).

![Figure 5-11 Layout of underground powerhouse in Jinping I hydropower station (Wu et al., 2010)](image)

The two tailrace tunnels are used to link the hydroelectric facility to the Yalong river, and they are 350m and 490.49m long respectively. The stability of the entrance of the
tunnels is very important for the normal performance of the hydropower station. In this example, the entrance of one of the tailrace tunnel is analysed. The dimensions of the domain of interest are shown in Figure 5-12. The tunnel orientation is NW and the tunnel dimension is $18 \times 18$ m$^2$. The free surfaces of the domain of interest include both the slope surface and tunnel excavation surfaces. The rock in this region is marble and the dip and dip direction of the slope surface are around 75 and 295 respectively. The discontinuities near the entrance of the tailrace tunnel consist of a major fault, one set of bedding planes and two joint sets (Huang and Huang, 2010; Huang et al., 2007). The fault with dip 80 and dip direction 165 is located at a distance of 15m from the entrance of the trailrace tunnel.

Chi square/K-S tests were used to determine the suitable distributions of the measured data and then the size distribution of discontinuities was inferred from trace length data using the stereological relationship (Villaescusa and Brown, 1992). The dip angle and dip direction of the bedding planes is observed to vary from 25 to 40 and 300 to 310 respectively on site. In the simulation, the dip angle and dip direction follow uniform distribution within their range. Spacing is found to follow exponential distribution with mean value 2.5m. All the bedding planes are assumed to be persistent. The orientations of the joint sets 1 and 2 both follow Fisher distribution with $k = 20$ and $k = 75$ respectively. The average values of the dip and dip direction for joint set 1 are 75 and 140 while those for joint set 2 are 70 and 035. The sizes of the two joint sets follow exponential distribution with mean values 15m and 20m respectively. Fixed density is used (totally 80 joints in the domain) for the joints in set 1. Compared with
joint set 1, the joints in set 2 are not well developed (totally 30 joints in the domain). The spacing of the two joint sets follows exponential distribution, so it is reasonable to assume the locations the joints follow random distribution in the domain of interest.

The density of the marble rock is 2760 kg/m$^3$. The friction angle of the joints follows normal distribution with average 25° and variance 5°. The cohesion is assumed to be zero for all discontinuities. Since the domain of interest is close to the slope surface, the in-situ stress is quite low and is ignored in the analysis. 120 realizations are generated and three realizations with corresponding identified key blocks are displayed in Figure 5-13. Figure 5-14 shows the average volume of unstable blocks with the increase of the number of realizations. In this example, it can be seen that after around 90 realizations, the average volume reaches stable values.

After analysis, the predicted mean volume of generated blocks in the domain is 2358 with standard deviation 475, and the predicted mean volume of a single block is 116 m$^3$ with standard deviation 28 m$^3$. The predicted information of key blocks from both the tunnel and slope surfaces is shown in Table 5-4.
Among the key blocks from all realizations, tetrahedral, pentahedral and other type of blocks account for 68%, 21% and 11% respectively. In this example, tetrahedral blocks are still the most common type of key blocks. In terms of failure modes, a close inspection on the key blocks in all realizations reveals that 65% of the key blocks fail with single plane sliding mode. This is mainly caused by the set of the bedding planes. This failure mode has also been widely observed on site as shown in Figure 5-15.

**Table 5-4 Predicted key block information at the entrance of the tailrace tunnel**

<table>
<thead>
<tr>
<th></th>
<th>All batches</th>
<th>First batch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of key blocks</td>
<td>Mean 418</td>
<td>Mean 47</td>
</tr>
<tr>
<td>Volume of key blocks (m³)</td>
<td>Mean 276</td>
<td>Mean 47</td>
</tr>
<tr>
<td>Mean volume of a key block (m³)</td>
<td>Mean 15698</td>
<td>Mean 580</td>
</tr>
<tr>
<td>Number of key blocks</td>
<td>Mean 10095</td>
<td>Mean 760</td>
</tr>
<tr>
<td>Volume of key blocks (m³)</td>
<td>Mean 41</td>
<td>Mean 14</td>
</tr>
<tr>
<td>Mean volume of a key block (m³)</td>
<td>Mean 16</td>
<td>Mean 14</td>
</tr>
</tbody>
</table>

The key block volume information from all realizations is combined together, and the number of key blocks with volume between different size intervals is counted. The relationship between the percentage of key block volume and various cut-off sizes is plotted in Figure 5-16 for first batch and all batches respectively. As quite low
frequency of blocks with volume larger than 8 m³ has been observed, only blocks with volume smaller than 8 m³ are considered. Among all the key blocks, the largest block volume is 1976 m³. After careful checking, it is found that the blocks with volume larger than 500 m³ are all located on the slope surface, which has also been observed on site. From Figure 5-16, it is seen that most of the key blocks are quite small. The percentage of key blocks with volume smaller than 8 m³ is 61% for all batches and 82% for first batch.

Figure 5-15 Observed failure on the slope surface around the entrance of the tailrace tunnel
5.5 CONCLUSIONS

A discrete blocky system approach for stochastic key block analysis has been presented. Based on Monte Carlo simulations of discontinuity network and a reliable blocky rock mass generator, a close-to-nature representation of the blocky rock mass can be simulated and the block and key block predictions can thus be obtained more accurately. The proposed approach is capable of dealing with non-persistent discontinuities, complex excavations and progressive analysis of blocky rock mass. The stability analysis of a horseshoe shaped tunnel is conducted. The key block characteristics (failure mechanism, the number, volume, shape etc. of key blocks) are statistically analysed to provide guidance for support design. The critical (or worst) cases with larger total key block volume are found out and selected for further more accurate and rigorous numerical analysis with DDA or NMM to achieve more accuracy within a reasonable computational cost. In this way, the stochastic key block analysis can be considered as a pre-processor for DDA or NMM.

Three scenarios with different mean discontinuity size data are employed to investigate the size effect of discontinuities. The persistence of discontinuities has been shown to be critically important for the accurate prediction of key block statistics. Utilization of persistent discontinuities leads to over-estimation of fragmentation of the rock mass, if most of the discontinuities are not much greater than the dimensions of the domain.

Figure 5-16 The simulated key block size distribution at the entrance of the tailrace tunnel: a) First batch; b) All batches
Subsequently, more key blocks are estimated and the maximum volume of single key blocks is underestimated for the persistence scenario. Thus representing the discontinuity size more accurately in the blocky rock mass model is important for the accurate prediction of the key block statistics.

A case study application to the entrance of a tailrace tunnel at Jinping Hydropower station has also been worked out. Results show that key blocks statistics provides relatively more complete information for support design.

Future work will involve checking the sensitivity of other discontinuity network parameters (orientation and spacing etc.) and mechanical parameters to the key block prediction by the current modelling approach. Engineering applications for support design and verification by in-situ observations will also be conducted in the near future.
CHAPTER 6. AN IMPROVED DUAL RECIPROCIDY
BOUNDARY ELEMENT METHOD FOR LARGE
DISPLACEMENT ELASTO-DYNAMIC ANALYSIS

6.1 INTRODUCTION

In the traditional Dual Reciprocity BEM methodology (Dominguez, 1993), small
deformation and displacement are assumed. A constant stiffness matrix obtained from
the initial geometry is used in the whole analysis process. The displacements in the
solutions from a local coordinate system attached to each node are calculated.

In the discrete element methods, due to the interactions among the blocks, a block may
have large displacement or even large deformation in each time step. This means that
the geometry of the blocky system (coordinates of nodes) changes with time. Therefore,
the traditional BEM methodology is not applicable. The BEM with stepwise updating is
required. At the end of each time, if the geometry is updated, the stresses should also be
transferred to the next step as initial stress, since it affects the displacement field of the
blocks.

In this chapter, the DRBEM is further developed to consider geometry updating in each
time step in order to be combined with the DDA. During the analysis, at the end of each
time step, the domain geometry is updated and then considered as a new problem at the
beginning of the next time step. The stresses, velocities and accelerations at all the
nodes are also calculated and used as initial conditions for the next time step. The initial
stress involved in the analysis will lead to a domain integral in the governing equations
and internal cells are used for the integration of this domain term. This development
will enable the DRBEM to be able to be combined with DDA as well as conduct
analysis of large displacement and large deformation. Several examples are also used to
verify the developed algorithm.

6.2 DUAL RECIPROCIDY BOUNDARY ELEMENT METHOD
FOR ELASTO-DYNAMICS CONSIDERING INITIAL STRESS
The detail formulation of the DRBEM presented in Dominguez (1993) and Kontoni and Beskos (1993) is followed in the section.

In many problems, initial stresses may be present due to in-situ stresses, thermal strains, or elastic-plastic behaviour. The presence of these stresses causes modification to the displacement field. Expressions for internal values of stresses or strains are also of fundamental importance for the stepwise dynamic analysis with geometry updating, even though initial stresses do not exist in the initial conditions.

The stress in an elastic body is defined as:

$$\sigma_{ij} = \sigma^t_{ij} - \sigma^0_{ij}$$  \hspace{1cm} (6.1)

where $\sigma^t_{ij}$ and $\sigma^0_{ij}$ are the total stress and initial stress respectively.

The differential equations of motion of an elastic homogeneous medium $\Omega$ enclosed by a boundary $\Gamma$ are expressed in Cartesian tensor notation as:

$$\sigma_{ij,j} + b_i + c\dot{u}_i - \rho \ddot{u}_i = 0$$  \hspace{1cm} (6.2)

where $\sigma_{ij}$ is the Cartesian stress tensor, $b_i$ is volume body force, $u_i$ is the displacement on the boundary, $\rho$ is the mass density, $c$ is damping ratio.

Since body force does not change with time, its effect can be obtained from static analysis, and then applied to the dynamic solution by superposition method to get the overall response. So it can be omitted in the following analysis. In addition, if the damping effect is also ignored, the equation of motion can be simplified as follows:

$$\sigma_{ij,j} - \rho \ddot{u}_i = 0$$  \hspace{1cm} (6.3)

with boundary conditions:

$$u_i = \bar{u}_i(t) \text{ on } \Gamma_1$$

$$p_i = \bar{p}_i(t) \text{ on } \Gamma_2$$  \hspace{1cm} (6.4)

where $p_i$ is the traction on the boundary, $\bar{u}_i$ and $\bar{p}_i$ are the prescribed values of displacements and tractions on the boundary.

Application of the method of weighted residuals to equation (6.3) leads to the weak form of the equation of motion:
If the product differentiation statement is considered,

\[(\sigma_{ij} u^*_i)_j = \sigma_{ij,j} u^*_i + \sigma_{ij} u^*_{i,j} = \sigma_{ij,j} u^*_i + \sigma_{ij} \varepsilon^*_{ij} \]  
\tag{6.6}

Equation (6.6) is rearranged and the term \(\sigma_{ij,j} u^*_i\) is substituted into equation (6.5),

\[\int_{\Omega} \left((\sigma_{ij} u^*_i)_j - \sigma_{ij} \varepsilon^*_{ij}\right)d\Omega - \int_{\Omega} \rho \ddot{u}_i u^*_i d\Omega = 0 \]  
\tag{6.7}

Then the divergence theorem is used to transform the volume integral to a boundary integral and the following equation is obtained:

\[\int_{\Omega} \sigma_{ij} u^*_i n_j d\Omega - \int_{\Omega} \sigma_{ij} \varepsilon^*_{ij} d\Omega - \int_{\Omega} \rho \ddot{u}_i u^*_i d\Omega = 0 \]  
\tag{6.8}

On a boundary, with unit-outward normal \(\mathbf{n}\), the surface tractions must be in equilibrium with the internal stresses, leading to the equilibrium condition:

\[p_i = \sigma_{ij} n_j \]  
\tag{6.9}

Substituting equation (6.1) and (6.9) into (6.8),

\[\int p_i u^*_i d\Omega - \int \sigma^*_{ij} \varepsilon^*_{ij} d\Omega + \int \sigma^0_{ij} \varepsilon^*_{ij} d\Omega - \int \rho \ddot{u}_i u^*_i d\Omega = 0 \]  
\tag{6.10}

The reciprocity theorem is applied:

\[\int \sigma^*_{ij} \varepsilon^*_{ij} d\Omega = \int \sigma^0_{ij} \varepsilon^*_{ij} d\Omega \]  
\tag{6.11}

Then the integration by parts is carried out on the second term again

\[\int p_i u^*_i d\Omega + \int (\sigma^*_{i,j}) u_i d\Omega - \int p^* u_i d\Omega + \int \sigma^0_{ij} \varepsilon^*_{ij} d\Omega - \int \rho \ddot{u}_i u^*_i d\Omega = 0 \]  
\tag{6.12}

Substitute the subscript \(j\) and \(k\) in place of the subscript \(i\) and \(j\) respectively throughout, equation (6.12) is rewritten as:
\[
\int p_j u_j^* d + \int_\Omega (\sigma^*_{j,k,k}) u_j d\Omega - \int p_j^* u_j d + \int_\Omega \sigma^0_{j,k} \varepsilon^*_{j,k} d\Omega - \int_\Omega \rho \ddot{u}_j u_j^* d\Omega = 0
\] (6.13)

Assume the following equation:
\[
\sigma^*_{j,k,k} = -b^*_j = -\delta e_j = -\delta_{ij} e_i \delta
\] (6.14)

where \(\delta^P_i\) is the Dirac delta function and \(e_j\) represents a unit load acting in direction \(j\).

If each point load is taken as independent, the starred displacements and tractions can be written in the following form:
\[
\begin{align*}
\varepsilon^*_{j,k} &= \varepsilon^*_{ij}(\xi, x) e_i \\
p^*_j &= p^*_{ij}(\xi, x) e_i \\
u^*_j &= u^*_{ij}(\xi, x) e_i
\end{align*}
\] (6.15)

where \(\xi\) and \(x\) denote the source point and field point respectively. The source point \(\xi\) is where the unit forces are assumed to be acting. \(u^*_{ij}, p^*_{ij}\) and \(\varepsilon^*_{ijk}\) are the displacement, traction and strain fundamental solutions of the Navier’s equation (Telles, 1983). The displacement fundamental solution represents the displacement at field point \(x\) in the direction of \(i\) resulting from a Dirac delta function force at point \(\xi\) in the direction of \(j\). \(p^*_{ij}\) and \(\varepsilon^*_{ijk}\) have the similar meanings. These three terms are expressed as follows:
\[
\begin{align*}
u^*_{ij} &= \frac{1}{8\pi G (1 - \nu)} \left[ (3 - 4\nu) \ln \left( \frac{1}{r} \right) \delta_{ij} + r_j r_i \right] \\
p^*_{ij} &= -\frac{1}{4\pi (1 - \nu) r} r_n \left[ (1 - 2\nu) \delta_{ij} + 2r_i r_j \right] + \frac{1 - 2\nu}{4\pi (1 - \nu) r} \left( r_j n_i - r_i n_j \right) \\
\varepsilon^*_{ijk} &= \frac{u^*_{ijk} + u^*_{ik,j}}{2}
\end{align*}
\] (6.16) (6.17) (6.18)

where \(r_n = r_m n_m, r\) is the distance between the source point \(\xi\) and the field point \(x\). It can be observed that \(u^*_{ij}\) is symmetrical with respect to \((i, j)\) and \((\xi, x)\) while \(p^*_{ij}\) is unsymmetrical in both.

It is observed that the unit force vector \(e_i\) is common to all integrals. By cancelling this vector and after some rearrangement, the following equation is obtained:
where $X$ is the field point in the domain. As $\xi$ goes to the boundary, the singularity is dealt with properly. The boundary integral equation can be obtained as follows:

\begin{equation}
\begin{aligned}
c_{ij}(\xi)u_j(\xi) + \int \mathbf{p}_{ij}(\xi, x)u_j(x)d\Omega \\
= \int \mathbf{u}_{ij}(\xi, x)p_j(x)d\Omega + \int \mathbf{e}_{ijk}(\xi, X)\mathbf{\sigma}_{jk}(X)d\Omega \\
- \int \rho \ddot{u}_j(x, t)u_{ij}(\xi, x)d\Omega
\end{aligned}
\end{equation}

The field displacement $u_j(x)$ is approximated by the following collocation scheme:

\begin{equation}
u_j(x, t) = \alpha_i^p(t)f_i^p(x)
\end{equation}

where the sum on $p = 1, \cdots, m$, $\alpha_i^p(t)$ are the unknown time-dependent functions, and $f_i^p(x)$ are a series of coordinate functions.

Taking the derivative of equation (6.21) twice with respect to time, the approximated accelerations are obtained:

\begin{equation}
\ddot{u}_j(x, t) = \ddot{\alpha}^p_i(t)f_i^p(x)
\end{equation}

Substituting equation (6.22) into the last term in equation (6.20) and this term becomes:

\begin{equation}
\int_\Omega \rho \dddot{u}_j(\xi, y_k)u_{ij}(\xi, x)d\Omega = \rho \dddot{\alpha}_i^p(t)\int_\Omega f_i^p(x)u_{ij}(\xi, x)d\Omega
\end{equation}

The reciprocity theorem is used again, and the above term is further expressed as follows:
\[
\int_\Omega \rho \ddot{u}_j(\xi, y_k)u_{ij}^*(\xi, x)d\Omega = \rho \ddot{a}_i^T(t) \left( -c_{ij}(\xi)\psi_{ij}^P(\xi, y_k) - \int p_{ij}^*(\xi, x)\psi_{ij}^P(\xi, y_k)d \right) + \int u_{ij}^*(\xi, x)\eta_{ij}^P(\xi, y_k)d
\]

(6.24)

where \(\psi_{ij}^P\) and \(\eta_{ij}^P\) are the displacements and tractions satisfying the Navier’s equation with \(-f_j^P(x)\) as the body force. If the radial basis function \(f_j^P(x) = 1 - r\) (Brebbia and Nardini, 1983) is used, the expressions of \(\psi_{ij}^P\) and \(\eta_{ij}^P\) are as follows:

\[
\psi_{ij}^P = \frac{1 - 2\nu}{(5 - 4\nu)G}r_i r_j r^2 + \frac{1}{30(1 - \nu)G} \left[ \left( \frac{3 - 10\nu}{3} \right) \delta_{ij} - r_i r_j \right] r^3
\]

(6.25)

\[
\eta_{ij}^P = \left\{ \begin{array}{l}
2\nu \left( \frac{3}{5 - 4\nu} r^2 + \frac{1}{30(1 - \nu)} \right) r_i \delta_{ij} \\
+ \left( \frac{1 - 2\nu}{5 - 4\nu} r^2 + \frac{1}{30(1 - \nu)} \right) r_i \delta_{ij} \\
+ \left( \frac{1 - 2\nu}{5 - 4\nu} r^2 + \frac{10\nu - 9}{90(1 - \nu)} r^2 \right) (r_i \delta_{ij} \\
+ r_j \delta_{ij} + 2 \frac{1}{30(1 - \nu)} r^2 r_i r_j \right) \right. \\
\end{array} \right.
\]

(6.26)

Therefore, the equilibrium equation of the DRBEM formulation for dynamic analysis can be written as follows:

\[
c_{ij}(\xi)u_j(\xi) + \int p_{ij}^*(\xi, x)u_j(x)d = \int u_{ij}^*(\xi, x)p_j(x)d \\
+ \rho \left( c_{ij}(\xi)\psi_{ij}^P(\xi, y_k) + \int p_{ij}^*(\xi, x)\psi_{ij}^P(x, y_k)d \right) \\
- \int u_{ij}^*(\xi, x)\eta_{ij}^P(x, y_k)d \right) \ddot{a}_i^T(t) + \int \varepsilon_{ijk}(\xi, X)\sigma_{jk}^0(X)d\Omega
\]

(6.27)

where \(c_{ij}\) is a constant determined from the shape of the boundary, \(u_j\) and \(p_j\) are the displacement and tractions at the boundary respectively, \(p_{ij}^*\) and \(u_{ij}^*\) are the traction and
displacement fundamental solutions respectively, $\rho$ is the material density, $\alpha_i^p$ is a set of unknown coefficients approximating the displacement $u_j$.

If $\xi$ is moved to the internal nodes, $c_{ij}$ become $\delta_{ij}$, and the displacements of internal nodes can be calculated by the following equation:

$$u_i(\xi) = \int u_{ij}^*(\xi, x)p_j(x)d + \int p_{ij}^*(\xi, x)u_j(x)d$$

$$+ \rho \left( \psi_{il}^p(\xi, y_k) + \int p_{ij}^*(\xi, x)\psi_{jl}^p(x, y_k)d \right)$$

$$- \int u_{ij}^*(\xi, x)\eta_{jl}^p(x, y_k)d \right) \tilde{\alpha}_i^p(t) + \int_{\Omega} \sigma_{jk}^0(X)\varepsilon_{ijk}(\xi, X)d\Omega$$

(6.28)

### 6.3 STRESSES AT INTERNAL AND BOUNDARY NODES

From the Hooke’s law and the strain-displacement relationship, we have

$$\sigma_{ij} = \lambda \delta_{ij}u_{k,k} + G(u_{i,j} + u_{j,i})$$

(6.29)

The displacement derivatives in equation (6.29) are obtained from equation (6.28), and then the stresses of internal points can be expressed in the following equation:

$$\sigma_{ij} = \int_{\Gamma} u_{ijk}^p \rho d\Gamma - \int_{\Gamma} p_{ijk}^p u_k d\Gamma$$

$$- \rho \left( \tilde{\sigma}_{ijl}^p + \int_{\Gamma} p_{ijkl}^p \psi_{kl}^p d\Gamma - \int_{\Gamma} u_{ijkl}^p \eta_{kl}^p d\Gamma \right) \tilde{\alpha}_i^p$$

$$+ \int_{\Omega} E_{ijkl} \sigma_{kl}^0 d\Omega + J_{ijkl}^\sigma \sigma_{kl}^0$$

(6.30)

where the definition of the terms (Telles, 1983) are as follows:

$$\tilde{\sigma}_{ijl}^p = \lambda \delta_{ij} \psi_{kl,m} + G(\psi_{ij,l}^p + \psi_{jl,i}^p)$$

(6.31)

$$u_{ijk}^p = \frac{1}{4\pi(1 - \nu)r} \left( 1 - 2\nu \right) \left( r_i \delta_{jk} + r_j \delta_{kl} - r_k \delta_{ij} \right) + 2r_i r_j r_k$$

(6.32)
The above equation is not valid on the boundary or for the points very close to it since strong singularity is involved in the integrals.

The stress components at a boundary point can be determined purely based on the boundary data. The so-called “Traction Recovery Method” by Ribeiro et al. (2008) is used to calculate the stresses at the boundary nodes.

The global stress components are obtained from the local stress components using the following transformation:

\[
\sigma_{mn} = L_{rm}L_{sn}\sigma_{rs}
\]  
(6.36)

If the contribution of displacements, tractions and initial stresses, the global stress components are written as:

\[
\sigma_{mn} = A_{mnja}u_j^a + B_{mnj}P_j + C_{mnkl}\sigma_{kl}^0
\]  
(6.37)

The terms \(A_{mnja}, B_{mnj}\) and \(C_{mnkl}\) are given by:

\[
A_{mnja} = \frac{2G}{1-v}L_{1m}L_{1n}L_{2j}\frac{d\xi}{dx_1}\frac{dN_a}{d\xi}
\]

\[
B_{mnj} = (L_{1m}L_{2n} + L_{2m}L_{1n})L_{1j} + \left(\delta_{mn} - \frac{1-2v}{1-v}L_{1m}L_{1n}\right)L_{2j}
\]  
(6.38)

\[
C_{mnkl} = -L_{1m}L_{1n}\left(\delta_{kl} - \frac{1}{1-v}L_{2k}L_{2l}\right)
\]

with the subscripts \(l, m, n, j, k\) ranging from 1 to 2,
In plane strain, the above coefficients in equation (6.38) must be supplemented by the following additional terms:

\[
\frac{d\xi}{dx_1} = \frac{1}{\sqrt{\left(\frac{dx_1}{d\xi}\right)^2 + \left(\frac{dx_2}{d\xi}\right)^2}} = \frac{1}{f(\xi)}
\]

\[L_{22} = -L_{11} = n_2\]

\[L_{21} = L_{12} = n_1\]

It is important to note that if the stresses are calculated at a node shared by several elements and the stress field is continuous at that point, the results obtained from each element should be averaged. In case that the stress field is discontinuous, the stresses of the same point at different elements should be calculated separately.

\[A_{33ji} = \frac{2G\nu}{1-\nu}L_{1j} \frac{d\xi}{dx_1} \frac{dN_a}{d\xi}\]

\[B_{33j} = \frac{\nu}{1-\nu}L_{2j}\]

\[C_{33kl} = \frac{\nu}{1-\nu}L_{2k}L_{2l} - \delta_{3k}\delta_{3l}\]

(6.39)

6.4 NUMERICAL IMPLEMENTATION

6.4.1 Solution procedure

Figure 6-1 Representation of elements: a) Quadratic element; b) Bi-quadrilateral cell

In order to solve the derived integral equations numerically, the boundary of the domain needs to be discretized into a number of boundary elements. In addition, the domain is
also discretized into internal cells for the integration of the initial stress domain integral. Quadratic boundary elements and bi-quadratic internal cells (Figure 6-1) are used.

The shape functions of quadratic boundary elements for 2D problems are given by:

\[
N_\alpha = \frac{\xi (\xi + \xi_\alpha)}{2} \quad \alpha = 1 \& 2
\]

\[
N_3 = 1 - \xi^2
\]

(6.40)

For 2D internal cells, the shape function is:

\[
N_\alpha = \frac{1}{4} (1 + \xi_\alpha \xi)(1 + \eta_\alpha \eta)(-1 + \xi_\alpha \xi + \eta_\alpha \eta) \quad \alpha = 1 \text{ to } 4
\]

\[
N_\alpha = \frac{1}{4} (1 + \xi_\alpha \xi + \eta_\alpha \eta)(1 - (\xi_\alpha \eta)^2 - (\eta_\alpha \xi)^2) \quad \alpha = 5 \text{ to } 8
\]

(6.41)

where \(\xi_\alpha\) and \(\eta_\alpha\) are the \(\xi\) and \(\eta\) coordinates of the element node \(\alpha\).

Thus the coordinates, displacements, tractions and initial stresses are described by the following relations:

\[
x_i = \sum_{\alpha=1}^{m} N_\alpha x_i^\alpha
\]

\[
u_i = \sum_{\alpha=1}^{m} N_\alpha u_i^\alpha
\]

\[
p_i = \sum_{\alpha=1}^{8} N_\alpha p_i^\alpha
\]

\[
\sigma_{ij}^0 = \sum_{\alpha=1}^{8} N_\alpha \sigma_{ij}^{0\alpha}
\]

(6.42)

where \(x_i^\alpha\), \(u_i^\alpha\), \(p_i^\alpha\) and \(\sigma_{ij}^{0\alpha}\) are the values of coordinates, displacements, tractions and initial stress at the node \(\alpha\) of a boundary element or an internal cell, \(m\) is the number of nodes (3 for boundary elements and 8 for internal cells).

After the spatial discretization, equation (6.27) can be written in matrix form if only inertial forces are considered in the formulation:

\[
H_{BB} U_B - G_{BB} P_B - \phi (H_{BB} \psi_{BB} - G_{BB} \eta_{BB}) \bar{\alpha}_B - E_{BT}^d \sigma^0 = 0
\]

(6.43)
where $H_{BB}$ and $G_{BB}$ are the influence matrices corresponding to the boundary terms in $p_{ij}$ and $u_{ij}$, plus the addition in the case of the $H_{BB}$ matrices of the diagonal terms in $c_{ij}$, $\Psi_{BB}$ and $\eta_{BB}$ are both $N \times N$ matrices. $N$ is the number of boundary nodes. $\bar{\alpha}_B$ is a vector with its dimension $N$. It is evident that $H_{BB}$ is assembled from individual element matrices $\int_{\Gamma} p_{ij} u_j d\Gamma$ and the diagonal sub-matrices $c_{ij}$. The global matrix $G_{BB}$ is obtained from individual matrices $\int_{\Gamma} u_{ij} p_j d\Gamma$ while the matrices $\Psi_{BB}$ and $\eta_{BB}$ simply contain the values of functions $\Psi_{BB}$ and $\eta_{BB}$ at nodal points. $E^d_{BT}$ is the coefficient matrix of initial stress for displacements. Among those terms, $B$ refers to boundary nodes and $T$ refers to all nodes. For example, $H_{BB}$ contains the integrals over the boundary elements when collocation points are the boundary nodes.

For internal nodes, the displacement equation (6.28) is written in matrix form:

$$U_I = G_{IB} P_B - H_{IB} U_B + \varrho(\psi_{IB} + H_{IB} \Psi_{BB} - G_{IB} \eta_{BB}) \bar{\alpha}_B + E^d_{IT} \sigma^0$$

(6.44)

where $I$ in the subscript refers to internal nodes. $H_{IB}$ contains the integrals over the boundary elements when the collocation points are the internal nodes. The other terms with subscripts $I$ and $B$ have the same meanings.

Equations (6.27)-(6.28) can be assembled into the same equation as follows:

$$H U - G P - \varrho(\Psi - G \eta) \bar{\alpha} - E^d \sigma^0 = 0$$

(6.45)

where $H = \begin{bmatrix} H_{BB} & 0 \\ H_{IB} & I \end{bmatrix}$, $I$ is unit matrix, $G = \begin{bmatrix} G_{BB} \\ G_{IB} \end{bmatrix}$, $\Psi = \begin{bmatrix} \psi_{BB} & \psi_{BI} \end{bmatrix}$, $\eta = \begin{bmatrix} \eta_{BB} & \eta_{BI} \end{bmatrix}$, $E^d = \begin{bmatrix} E^d_{IT} \\ E^d_{BT} \end{bmatrix}$.

Transformation between $U$ and $\bar{\alpha}$

$$\dot{U} = F_f \ddot{\alpha}$$

(6.46)

$$\ddot{\alpha} = F_f^{-1} \dot{U}$$

(6.47)

This approach can be used to deal with either known or unknown body forces, which will result in domain integrals in the formulation. For known body forces such as gravity, the vector $\alpha$ may be obtained explicitly using Gauss elimination while for unknown body forces such as inertial forces, the matrix $F$ must be inverted.
Assuming \( E = F_f^{-1}, F_f = [F_{fBB} F_{fBI}], M = -q(H \psi - G \eta)E, g \) is gravitational acceleration, \( \dot{U} = [\dot{U}_B] \).

\[
HU - GP + M \ddot{U} + Mg - E^d\sigma^0 = 0
\] (6.48)

After the governing equations of motion are obtained, a suitable time marching scheme to represent the relationship among displacement, velocity, and acceleration is required to solve the equation. The velocity and acceleration are usually expressed in terms of the displacement. The system equations are not satisfied at every time instant, but only at the individual time step of the time interval. The Houbolt method was preferred (Nardini and Brebbia, 1983; Loeffler and Mansur, 1987) mainly due to the introduced numerical damping to filter the high mode response. Later, Tanaka and Chen (2001) concluded that the Newmark method should be in general preferred in the context of the DRBEM formulations of elastodynamics problems.

The generalized Newmark-\( \beta \) method is as follows:

\[
U_{n+1} = U_n + h\ddot{U}_n + \frac{h^2}{2}[(1 - 2\beta)\dddot{U}_n + 2\beta\dddot{U}_{n+1}]
\] (6.49)

\[
\dddot{U}_{n+1} = \dddot{U}_n + h[(1 - \gamma)\dddot{U}_n + \gamma\dddot{U}_{n+1}]
\] (6.50)

\( h \) (or \( \Delta t \)) is the time step size for an incremental dynamic formulation, \( U_n \) and \( U_{n+1} \) denote the approximation to the values \( U(t) \) and \( U(t + h) \) for a time step \( h \); \( \beta \) is acceleration weighting and \( \gamma \) is velocity weighting. \( \gamma \) with a value larger than 0.5 will result in numerical damping proportional to \( (\gamma - 0.5) \).

Newmark-\( \beta \) method with two parameters \( \beta = 0.5 \) and \( \gamma = 1.0 \) used in Dr. Shi’s DDA (Doolin and Sitar, 2004) is adopted in the DRBEM and equation (6.49)-(6.50) will be simplified as follows:

\[
\dddot{U}_{n+1} = \dddot{U}_n + h\dddot{U}_{n+1} = \frac{2}{h}U_{n+1} - \dot{U}_n
\] (6.51)

\[
\dddot{U}_{n+1} = \frac{2}{h^2}U_{n+1} - \frac{2}{h}\dot{U}_n = \dddot{U}_n
\] (6.52)
It should be mentioned that numerical damping is essential to not only ensure the stability of the DRBEM analysis but also allow the oscillations caused by contact forces to dissipate rapidly, resulting in a stable state, which ultimately allows open-close iteration to converge rapidly. The amount of numerical damping is also proportional to the time step size.

With the consideration of initial stress, the governing equations can be written in the following form:

\[
\left( H + \frac{2}{h^2} M \right) U_{n+1} = GP + Mg + \frac{2}{h} M \dot{U}_n + E^d \sigma^0 \tag{6.53}
\]

To avoid numerical problems caused by poor matrix conditioning, it is desirable to make the magnitudes of tractions and displacements broadly equal. This is achieved by dividing the tractions by a representative stress measure (e.g. shear modulus) and the displacements by a representative length measure (typically the mean element length).

After rearranging the equilibrium equations according to the boundary conditions, the final form can be obtained:

\[
AX = F \tag{6.54}
\]

where \( A \) is the new coefficient matrix after the boundary conditions are considered, \( X \) is a combination of unknown nodal displacements and surface tractions and \( F \) is the force vector.

The equation (6.30) for stresses at internal nodes can be written in matrix form as follows:

\[
\sigma^l = \tilde{G} P - \tilde{H} U + \tilde{M} \dot{U} + E^\sigma \sigma^0 \tag{6.55}
\]

where \( \tilde{H} \) and \( \tilde{G} \) contains the integrals over the boundary elements when the collocation points are the internal nodes, \( P \) and \( U \) are the surface tractions and displacements at the boundary nodes respectively, \( \tilde{M} \) is the equivalent mass matrix, \( E^\sigma \) is the coefficient matrix for stresses and \( \sigma^0 \) is the initial stresses in the domain.

While the equation (6.37) for stresses at boundary nodes can be written in matrix form as follows:
\[ \sigma^I = A U + B P + C \sigma^0 \] (6.56)

where \( A, B \) and \( C \) are the coefficient matrices for displacements, tractions and initial stresses respectively.

It should be noted that in large deformation analysis, the internal cells may be distorted during the deformation process, leading to poor integration of the domain integrals. Therefore, for large deformation analysis with occurrence of cell distortions, the internal cells should also be regenerated to improve the quality of the domain integrations. In addition, the initial stresses at the internal nodes from the previous time step should be interpolated to the new internal nodes in the newly generated internal cells accordingly using the mapping techniques. In the current analysis, large deformation causing deterioration of the quality of the cells is not considered, so the regeneration of internal cells is not considered.

### 6.4.2 Evaluation of singular integrals

Consider a general function \( f(x^p, x) \) with \( x \) representing the Cartesian coordinates and \( x^p \) denoting the value of \( x \) at source point \( p \) in domain \( \Omega \) with dimension \( Dim \) (\( Dim = 1 \) for one dimensional problems, \( Dim = 2 \) for two dimensional problems and \( Dim = 3 \) for three dimensional problems). The domain integrals in boundary element method can be written in the following form:

\[ I(x^p) = \int_{\Omega} f(x^p, x) d\Omega = \int_{\Omega} \frac{\bar{f}(x^p, x)}{r^\beta(x^p, x)} d\Omega \] (6.57)

where \( \bar{f}(x^p, x) \) is bounded everywhere and \( r(x^p, x) \) is the distance between \( x^p \) and \( x \). The level of singularity of domain integral \( I(x^p) \) is defined by Gao (2005):

\[
\begin{align*}
& \beta = 0 \quad \text{regular} \\
& 0 < \beta < Dim \quad \text{weakly singular} \\
& \beta = Dim \quad \text{strongly singular} \\
& \beta = Dim + 1 \quad \text{hypersingular} \\
& \beta = Dim + 2 \quad \text{supersingular}
\end{align*}
\] (6.58)

**Weakly singular integrals**

\[ \int_{\Gamma} u_{ij}^p p_j d\Gamma \]
This boundary integral is weakly singular and can be handled by the Logarithmic Gauss quadrature (Stroud and Secrest, 1966), which takes the following form:

$$ I = \int_0^1 \log_e(1/x) f(x) dx \approx \sum_{k=1}^{n} w_k f(x_k) $$

(6.59)

where $x_k$ and $w_k$ are the nodes and weights respectively. The nodes and weights for 10-point Logarithmic Gauss quadrature are shown in Table 6-1.

<table>
<thead>
<tr>
<th>Nodes $x_k$</th>
<th>weights $w_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0090426309</td>
<td>0.1209551319</td>
</tr>
<tr>
<td>0.0539712662</td>
<td>0.1863635425</td>
</tr>
<tr>
<td>0.1353118246</td>
<td>0.1956608732</td>
</tr>
<tr>
<td>0.2470524162</td>
<td>0.1735771421</td>
</tr>
<tr>
<td>0.3802125396</td>
<td>0.1356956729</td>
</tr>
<tr>
<td>0.5237923179</td>
<td>0.0936467585</td>
</tr>
<tr>
<td>0.6657752055</td>
<td>0.0557877273</td>
</tr>
<tr>
<td>0.7941904160</td>
<td>0.0271598109</td>
</tr>
<tr>
<td>0.8981610912</td>
<td>0.0095151826</td>
</tr>
<tr>
<td>0.9688479887</td>
<td>0.0016381576</td>
</tr>
</tbody>
</table>

This integral is also weakly singular in the two dimensional domain. An element subdivision technique employed by Lachat and Watson (1976) is an effective means to deal with the weak singularity. In this technique, elements containing the source node are further divided into triangular sub-elements. As shown in Figure 6-2, if source node P is located at a mid-side node, three triangular sub-elements (①-③) are subdivided.
while if source node P is located at a corner node, only two triangular elements (①-②) are further divided. Then the sub-elements are mapped into square intrinsic element space (Figure 6-3). As a result of this degeneracy, the Jacobian of the transformation is of order $r$, where $r$ is the distance from the source node. Consequently, the weak singularity is eliminated and this integral can be evaluated by normal Gauss quadrature.

![Figure 6-2 Sub-elements for singular point P](image)

**Figure 6-2 Sub-elements for singular point P**

![Figure 6-3 Mapping of sub-elements](image)

**Figure 6-3 Mapping of sub-elements**

*Strongly singular integrals*

\[
\int_{\Gamma} p_{ij}^*(\xi, x) u_j d\Gamma
\]

This strongly singular boundary integral is handled by the use of the rigid-body-motion constraint. The $c_{ij}(\xi)u_j(\xi)$ term is also solved together with the above term.
For a finite domain, assuming a rigid body displacement in the direction of one of the Cartesian coordinates, the traction and body force vector must be zero, thus:

\[ H^{lq} = 0 \]  \hspace{1cm} (6.60)

where \( I^{lq} \) is a vector that for all nodes has unit displacement along \( q \) direction (\( q=1, 2 \)) and zero displacement in any other direction. The diagonal sub-matrix can be written:

\[ H_{ij}^{kk} = (\delta_{km} - 1) \sum_{m=1}^{N} H_{ij}^{km} \]  \hspace{1cm} (6.61)

where \( N \) is the number of nodes, the subscripts \( i \) and \( j \) range from 1 to 2 and the superscripts \( k \) and \( m \) denote the nodes.

For infinite domain, the diagonal sub-matrix can be calculated from the equation:

\[ H_{ij}^{kk} = \delta_{ij} + (\delta_{km} - 1) \sum_{m=1}^{N} H_{ij}^{km} \]  \hspace{1cm} (6.62)

For semi-infinite domain, the diagonal sub-matrix can be calculated from the equation:

\[ H_{ij}^{kk} = \frac{1}{2} \delta_{ij} + (\delta_{km} - 1) \sum_{m=1}^{N} H_{ij}^{km} \]  \hspace{1cm} (6.63)

\[ \int_{\Omega} \varepsilon_{ijkl}^{\sigma} \sigma_{kl}^{0} \, d\Omega \]

This strong singularity in this domain integral is dealt with by the singularity isolation method by Gao and Trevor (2000). The essence of this technique is the isolation of the singularity and its transformation into a local boundary integral. The singularity arises in the domain integral is isolated by rewriting the integral in the form:

\[ \int_{\Omega} \varepsilon_{ikjn}^{\sigma}(p,q) \sigma_{ik}^{0}(q) \, d\Omega(q) = \int_{\Omega} \varepsilon_{ikjn}^{\sigma}(p,q) [\sigma_{ik}^{0}(q) - \sigma_{ik}^{0}(p)] \, d\Omega(q) \]

\[ + \sigma_{ik}^{0}(p) \int_{\Omega} \varepsilon_{ikjn}^{\sigma}(p,q) \, d\Omega(q) \]  \hspace{1cm} (6.64)
The first integral on the right hand side is weakly singular and can be integrated numerically using the same technique in section 4.1. The second term is strongly singular and can be further transformed as:

$$\sigma_{ik}^0(p) \int_\Omega \varepsilon_{kjn}^\sigma(p, q) \, d\Omega(q) = \sigma_{ik}^0(p) \int_{\Gamma_c} \varepsilon_{ikjn}^\sigma(p, q) \tau_m n_m \log r \, d\Gamma$$

(6.65)

### 6.4.3 Traction discontinuity

Although displacements are uniquely defined at corners, the tractions could be multi-valued, because each element has different outward normal vectors. Since the boundary integral equations can lead to only equation per degrees of freedom at each node, the number of equations at corner nodes is generally insufficient. Several methods could be adopted to deal with the traction discontinuity.

One method of solving this problem is to use the double node approach. In this approach (Figure 6-4), an additional node 2'' with the same coordinates as 2' is added into the corner. The node 2' belongs to the element 1-2' while the node 2'' belongs to the element 2''-3. Thus different tractions \((n_1' n_2')\) and \((n_1'' n_2'')\) can be considered for the corner. An additional equation is required for this approach to take into account the displacement continuity.

Using discontinuous elements is another alternative. The traction discontinuity is achieved by displacing inside the element the nodes that meet or that would meet at corners. This treatment can be used for any type of corners and generally give accurate results. However, it has been noted by Wilde (1998) that using discontinuous elements
has significant disadvantages in terms of solution stability, computational effort and accuracy.

A much simpler way out of this difficulty is to compute the G matrix on element level, rather than on node level (Brebbia and Domingiz, 1992). Namely, the G matrix should never be assembled, but its element sub-matrices should be multiplied by the element vectors of \( P \). In that way the external tractions are expressed, and taken into account, element wise, which permits the discontinuities of tractions at element boundaries without any constraints. However, it should be mentioned that the approach does not work for the case in which displacements are prescribed on both sides of the corner. This approach avoids the use of extra nodes and discontinuous elements and it is adopted in the developed program.

### 6.5 STEPWISE UPDATING FOR LARGE DISPLACEMENT ANALYSIS

In both finite and boundary element methodology, the geometry of domain is not required to be updated in each time step for linear systems. They both just calculate the displacements of each nodes corresponding to the initial geometry of the problem domain. Constant coefficient matrices of the initial geometry are used throughout the analysis process only the force vector is updated at the end of each time step used as initial conditions for next step. Since the un-deformed geometry is used during the whole analysis process, this methodology is not suitable for large-displacement and large deformation analysis. Furthermore, the motion in discrete element methods is geometrically nonlinear due to the continuously updated contacts during the entire analysis process. Therefore, those coefficient matrices are function of the positions, and the geometry updating of the blocky system is required at the end of each time step for the combination of BEM with DDA.

The traditional DRBEM methodology is compared with the stepwise updating algorithm for the DRBEM as shown in Table 6-2. The Newmark-\( \beta \) method with parameters \( \gamma = 1 \) for velocity weighting and \( \beta = 1/2 \) for acceleration weighting is adopted here rather than the Houbolt method used in Dominguez (1993).
Table 6-2 Comparison of time integration algorithms in traditional DRBEM and the current DRBEM with stepwise updating

<table>
<thead>
<tr>
<th>Traditional BEM methodology</th>
<th>Stepwise updating</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Dominguez, 1993)</td>
<td></td>
</tr>
<tr>
<td>a) Data initialization: ( \mathbf{U}_0, \dot{\mathbf{U}}_0, \mathbf{F}_0 )</td>
<td>a) Data initialization: ( \mathbf{U}_0, \dot{\mathbf{U}}_0, \mathbf{F}_0 )</td>
</tr>
<tr>
<td>b) Formation of coefficient matrices: ( \mathbf{H}, \mathbf{G} ) and ( \mathbf{M} )</td>
<td>b) for each ( t_i, i = 1, 2, \ldots, n ) time step, do</td>
</tr>
<tr>
<td>c) Formation of ( \mathbf{\tilde{H}} = \mathbf{H} + \frac{2}{h_t} \mathbf{M} ) ( \mathbf{F} = \mathbf{G} \mathbf{P} )</td>
<td>c) Coefficient matrix updating: ( \mathbf{H}, \mathbf{G} ) and ( \mathbf{M} )</td>
</tr>
<tr>
<td>d) Rearrangement of equations ( \mathbf{A} \mathbf{X} = \mathbf{F} )</td>
<td>d) Formation of ( \mathbf{\tilde{H}} = \mathbf{H} + \frac{2}{h_t} \mathbf{M} ) ( \mathbf{F} = \mathbf{G} \mathbf{P} )</td>
</tr>
<tr>
<td>e) Inversion of ( \mathbf{A} )</td>
<td>e) Rearrangement of equations ( \mathbf{A} \mathbf{X} = \mathbf{F} )</td>
</tr>
<tr>
<td>f) for each ( t_i, i = 1, 2, \ldots, n ) time step, do</td>
<td>f) Inversion of ( \mathbf{A} )</td>
</tr>
<tr>
<td>g) Force vector updating ( \mathbf{\tilde{F}} = \mathbf{F} + \frac{2}{h_t} \mathbf{U}_n + \frac{2}{h_t} \mathbf{\dot{U}}_n + \mathbf{g} )</td>
<td>g) Force vector updating ( \mathbf{\tilde{F}} = \mathbf{F} + \mathbf{M} \mathbf{g} + \mathbf{E}^t \sigma_0 + \frac{2}{h} \mathbf{M} \mathbf{\dot{U}}_n )</td>
</tr>
<tr>
<td>h) Equation solving: ( \mathbf{X}_{n+1} = \mathbf{A}^{-1} \mathbf{\tilde{F}} )</td>
<td>h) Equation solving: ( \mathbf{X}_{n+1} = \mathbf{A}^{-1} \mathbf{\tilde{F}} )</td>
</tr>
<tr>
<td>i) Velocity updating ( \mathbf{U}<em>{n+1} = \frac{2}{\Delta t} (\mathbf{U}</em>{n+1} - \mathbf{U}_n) - \mathbf{\dot{U}}_n )</td>
<td>i) Geometry updating</td>
</tr>
<tr>
<td>j) end for</td>
<td>j) Velocity updating ( \mathbf{U}<em>{n+1} = \frac{2}{h_t} \mathbf{U}</em>{n+1} - \mathbf{U}_n )</td>
</tr>
<tr>
<td>k) Stress updating</td>
<td>k) Stress updating</td>
</tr>
<tr>
<td>l) end for</td>
<td>l) end for</td>
</tr>
</tbody>
</table>

6.6 NUMERICAL EXAMPLES

A computer program using C language based on the stepwise updating based DRBEM have been developed. Three examples are used here to verify the correctness of the DRBEM with updated geometry at each time step for dynamic analysis.

6.6.1 Free fall example

A free falling object is an object that is falling under the sole influence of gravity. Any object that is being acted upon only by the force of gravity is said to be in a state of free fall. There are two important motion characteristics that are true of free-falling objects: Free-falling objects do not encounter air resistance; all free-falling objects accelerate downwards at a rate of 9.8 m/s.
Free falling is a typical example with large displacements and no deformation occurring during the analysis process. This example here is used to check the formation of all the coefficient matrices in the DRBEM.

The displacement increment at each time step is calculated by the following formula:

\[ d = \frac{1}{2} \times g(t_{n+1}^2 - t_n^2) \]  (6.66)

The velocity at each time step is determined by:

\[ v = \frac{1}{2} gt \]  (6.67)

Free falling objects with different shapes and sizes, different time steps and different number of elements have been used to test the developed program. It has been found that the results from each case that has been tried are in good agreement with the analytical solution as shown in Figure 6-5 and Figure 6-6.

Figure 6-5 Displacement increment of a free falling block
6.6.2 Cantilever beam subjected to bending

A rectangular cantilever beam of dimensions \((2m \times 4m)\) in Figure 6-7 is loaded by a flexural impulsive loading with triangular shape as shown in Figure 6-8. The shear modulus is 40000 \(N/m^2\), the Poisson’s ratio is 0.25 and the density is 1 \(kg/m^3\). This beam is discretized into 12 quadratic boundary elements, each of which is 1m in length. In order to consider the domain integral of the initial stress term, the beam domain is also discretized into 8 quadratic internal cells, which are compatible with the quadratic boundary elements. In this example, the time step is taken as 0.0002s.
The time history of the horizontal displacements of the middle point A at the end of the cantilever beam is demonstrated in Figure 6-9. The results are plotted together with those obtained from the traditional BEM methodology (Dominguez 1993) without geometry updating. It can be seen that there is very good agreement between those two methods.

6.6.3 A rectangular block subjected to tension

A rectangular strip with the same dimensions as the cantilever beam in the previous example is studied. It is clamped at the bottoms and the top side is uniformly subjected to a Heaviside step function representing a suddenly applied unit load at time $t = 0$ (Figure 6-10). The same material properties, time step size and discretization of boundary and domain as the previous example are used.
The analytical solution of this elasticity problem can be derived in the following form (Wang et al., 1996):

\[
\begin{align*}
    u(x, t) &= \varepsilon_s x - \\
    &8\varepsilon_s x \sum_{n=0}^{\infty} \frac{1}{((2n+1)\pi)^2} \cos \left( \frac{E A (2n + 1) \pi t}{M 2L} \right) \sin \left( \frac{2n + 1}{2} \pi \right) \sin \left( \frac{2n + 1}{2L} x \right)
\end{align*}
\]  (6.68)

where \( \varepsilon_s \) is the static strain, \( L \) is the length, \( E \) is Young’s modulus, \( A \) is the cross sectional area and \( M \) is mass per unit length.

The vertical displacement history of point A and the normal traction history of the middle point B at the fixed end are plotted in Figure 6-11 and Figure 6-12 respectively with different time steps using the same Newmark-\( \beta \) scheme with \( \beta = 0.5 \) and \( \gamma = 1 \) against analytical solutions. From those two figures, it can be seen that when the time step size is 0.0005s, a certain amount of numerical damping still exists and smooth the curve a little bit. As the numerical damping is proportional to the time step size, further reducing the time step size to 0.0002s leads to better results. In addition, better results can also be achieved by decreasing the \( \gamma \) value, which also controls the amount of numerical damping.
6.7 CONCLUSIONS

In order to be combined with DDA, the stepwise updating of the DRBEM is necessary. This chapter presents the DRBEM with stepwise updating. The detailed formulation has been derived with the consideration of initial stress, which leads to weak singularity in the displacement equation and strong singularity in the stress equation. Internal cells are adopted for the initial stress domain integral. The geometry of domain is updated at the end of each time and used as the new geometry at the beginning of the next step. Furthermore, the stresses and velocities are required to be calculated at the end of each time step and used as initial conditions in the next step. Three examples have been used to test the developed DRBEM program with stepwise updating and satisfactory results have been achieved.
The stepwise updating lays the groundwork for the development of the discrete boundary element method in the next chapter. In addition, with the consideration of geometry updating, large displacement and large deformation can also be simulated. In the current algorithm, internal cells are used to deal with the domain integral of initial stresses. Future work will include adopting a robust and reliable algorithm to transform this domain integral to a boundary integral, making the DRBEM really meshless to be coupled with the DDA.
CHAPTER 7. A DISCRETE BOUNDARY ELEMENT METHOD (DBEM) FOR BLOCKY ROCK STABILITY ANALYSIS

7.1 INTRODUCTION

A Discrete Boundary Element Method (DBEM) is developed by implementing an open close iteration contact algorithm into an improved dual reciprocity boundary element method. This developed DBEM is capable of simulating both the deformation and movement of blocks in blocky systems. Based on geometry updating, it adopts an incremental dynamic formulation taking into consideration initial stresses and dealing with external concentrated and contact forces conveniently. The boundary of each block in the discrete blocky system is discretized with boundary elements while the domain of each block is divided into internal cells only for the integration of the domain integral of the initial stress term. The contact forces among blocks are treated as concentrated forces and the open-close iterations are applied to ensure the computational accuracy of block interactions. In the current method, an implicit time integration scheme is adopted for numerical stability. Three examples are used to validate the algorithm and also show the effectiveness of the algorithm in simulating block movement, sliding, deformation and interaction of blocks. At last, masonry arc, block toppling and tunnel stability examples are conducted to demonstrate that the DBEM is applicable for simulation of blocky systems.

7.2 FORMULATION OF THE DISCRETE BOUNDARY ELEMENT METHOD

In order to deal with concentrated force, the following term is added to the right hand side of the governing equilibrium equation (6.27) and equation (6.28) for the determination of the displacements at internal points for each block:

$$\hat{u}_{ik}(\xi, P)^q F_k(P)^q$$

where the sum on $q = 1, \cdots, M, M$ is the number of concentrated forces, $\hat{u}_{ik}(\xi, P)$ is the produced displacement in $k$ direction at the location of the concentrated force when a
unit force is acting at the source point $\xi$ in the $i$ direction, $F_k(P)$ is the $kth$ concentrated force. The matrix form of this term is $\hat{G}F_c$. When the concentrated forces are acting at the nodes of elements, singularity will occur. A simplified procedure is adopted by setting the elements of $\hat{G}$ corresponding to the source point $\xi$ coinciding with the field point $P$. This means that the local shear and squeezing effect of the concentrated force at point $P$ are ignored.

For each individual block, the coefficient matrices are formed and then assembled into the global matrices. The governing equation of the discrete boundary element method for discrete blocky system in matrix form is written as follows:

$$HU - GP + M\dot{U} + Mg - E^d\sigma^0 - \hat{G}F_c = 0$$ (7.1)

A corresponding term $\hat{u}^{\sigma}_{ijk} q F_k q$ should be added into the right hand side of the equation (6.30) to calculate the stresses at boundary and internal nodes and $\hat{u}^{\sigma}_{ijk}$ is defined as follows:

$$\hat{u}^{\sigma}_{ijk} = \lambda\delta_{ij}\hat{u}_{mk,m} + G(\hat{u}_{ik,j} + \hat{u}_{jk,i})$$ (7.2)

### 7.3 CONTACT ALGORITHM

In the DBEM, each block in the system is treated as a BEM domain and it contributes to the simultaneous equations according to the equation (7.1). These blocks are also related with neighbouring blocks through contacting and separating, and their movement must obey the contact criterion, i.e., ‘no penetration, no tension’ (Shi, 1988). The effect of the contact can be represented by applying two stiff contact springs in the normal and shear directions or frictional forces along the sliding edge. The normal and shear contact springs are added if the blocks are in contact and not sliding relative to each other, and deleted if the blocks separate or the normal contact force is tensile. If the blocks are in contact and sliding relative to each other, a normal spring is added together with the frictional forces.

The solution of the analysis requires the exact number of contacts and their relevant information. However, the total number of contacts is unknown prior to the solution of the problem. This phenomenon is particularly serious in case of multi-body contact problems. For such a problem, a solution is to use a trial-and-error iteration procedure,
which is named open-close iteration in 2D DDA/NMM (Shi, 1988). Open-close iteration is applied to identify the contact and arrange the correct locations of contact springs for each time step. After determining the contact points and the associated contact forces, formulation of normal contact and shear contact sub-matrices or friction force sub-matrices are then calculated and added to the global simultaneous equations.

7.3.1 Contact detection

Figure 7-1 Vertex-to-edge contact and vertex-to-vertex contact between two blocks

Two vertex-to-vertex contacts

One vertex-to-vertex contact and one vertex-to-edge

○ Vertex-to-vertex contact

○ Vertex-to-edge

Figure 7-2 Edge-to-edge contacts between two blocks
Since blocks interact at their boundaries, for two-dimensional blocks, only three types of contact exist: vertex-to-vertex, vertex-to-edge (Figure 7-1), and edge-to-edge (Figure 7-2). Among these contacts, the vertex-to-edge contact and vertex-to-vertex contact are the two basic types and the edge-to-edge can be converted as the combination of the two basic types. The edge-to-edge contact may exist in four forms (Figure 7-2), and each form could be treated as different combinations of vertex-to-vertex and vertex-to-edge contacts.

The vertex-to-vertex contact and vertex-to-edge contact are then transformed to point-line crossing inequalities (Shi, 1988). A vertex-to-edge contact has the edge as the only entrance line. The vertex-to-edge contact occurs when the vertex passes the edge. The vertex-to-vertex contact can be further divided into two types: the one between a convex angle and a concave angle (Figure 7-3 a), and the one between two convex angles (Figure 7-3 b-d)). Both types have two entrance lines. For the first type (the contact between a convex angle and a concave angle), the two edges of the concave angle are the entrance lines. For the contact between two convex angles, the criteria in Table 7-1 are used to determine the entrance lines. The vertex-to-vertex contact occurs when one of the entrance lines is passed by the corresponding vertex.

<table>
<thead>
<tr>
<th>Ranges of these two angles</th>
<th>Entrance lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha \leq 180^\circ )</td>
<td>( \beta \leq 180^\circ )</td>
</tr>
<tr>
<td>( \alpha \leq 180^\circ )</td>
<td>( \beta &gt; 180^\circ )</td>
</tr>
<tr>
<td>( \alpha &gt; 180^\circ )</td>
<td>( \beta \leq 180^\circ )</td>
</tr>
<tr>
<td>( \alpha &gt; 180^\circ )</td>
<td>( \beta &gt; 180^\circ )</td>
</tr>
</tbody>
</table>

Contacts are determined by calculating the penetration distance \( d_n \) and the stiff springs are applied to ensure no-penetration. For the contact between two convex angles, a normal spring is used between the vertex and its corresponding entrance line, which has a smaller entrance distance. For the contact between a convex angle and a concave angle, a stiff spring is applied if only one of the two entrance lines is passed, and two
stiff springs are applied if both entrance lines are passed. For the vertex-to-edge contact, only one stiff spring is used.

For each vertex-entrance line contact pair, there are three possible contact modes: open, sliding and locked. At the beginning of each time step, the contact modes for all contact pairs are assumed to be locked except those contact pairs inherited from the previous time step. For a locked contact pair, a normal spring is applied to push the vertex away from the entrance line in the normal direction and a shear spring is applied to avoid the tangential displacement between the vertex and the entrance line.

Figure 7-3 Entrance lines of an angle-to-angle contact (Shi, 1988)

All contact forces are treated as concentrated forces in the DBEM. After the contact forces between a pair of blocks in contact are calculated, those contact forces are applied to both blocks at the same contact point in opposite directions.
7.3.2 Contact force calculation

Normal contact

The normal spring contact displacement (Figure 7-4) at the end of the current time step is:

$$d_n = \frac{\Delta}{l}$$  \hspace{1cm} (7.3)

If $P_1$ passes through edge $P_2P_3$, $d_n$ should be negative. $\Delta$ is the area of triangle $P_1P_2P_3$, and $l$ is the length of edge $P_2P_3$ at the end of the current time step:

$$l = \sqrt{(x_2 + u_2 - x_3 - u_3)^2 + (y_2 + v_2 - y_3 - v_3)^2}$$

$$\approx \sqrt{(x_2 - x_2)^2 + (y_2 - y_3)^2}$$ \hspace{1cm} (7.4)

$$\Delta = \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix} \approx S_{n0} + \begin{vmatrix} y_2 - y_3 & x_3 - x_2 \end{vmatrix} \begin{vmatrix} u_1 \\ v_1 \end{vmatrix} +$$

$$\begin{vmatrix} y_3 - y_1 & y_1 - y_3 \end{vmatrix} \begin{vmatrix} u_2 \\ v_2 \end{vmatrix} + \begin{vmatrix} y_1 - y_2 & y_2 - y_1 \end{vmatrix} \begin{vmatrix} u_3 \\ v_3 \end{vmatrix}$$ \hspace{1cm} (7.5)

$$S_{n0} = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$ \hspace{1cm} (7.6)

So the produced normal contact force is:
Shear contact

Assume \( P_0 \) with coordinates \((x_0, y_0)\) is the projection of vertex \( P_1 \) on the edge \( \overline{P_2P_3} \), the shear displacement is:

\[
d_s = \frac{1}{l} \overline{P_0P_1} \cdot \overline{P_2P_3} = \frac{1}{l} (x_1 + u_1 - x_0 - u_0, y_1 + v_1 - y_0)
\]

\[
= \frac{1}{l} (x_3 + u_3 - x_2 - u_2, y_3 + v_3 - y_2 - v_2)
\]

Ignoring the second-order infinite small terms and rearranging equation (7.8) give

\[
d_s = \frac{S_{s0}}{l} + \frac{1}{l} (x_3 - x_2, y_3 - y_2) \begin{pmatrix} u_1 - u_0 \\ v_1 - v_0 \end{pmatrix} = \frac{S_{s0}}{l} + \frac{1}{l} (x_3 - x_2, y_3 - y_2) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}
\]

\[
+ \frac{1}{l} (x_2 - x_3, y_2 - y_3) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}
\]

\[
S_{s0} = (x_3 - x_2, y_3 - y_2) \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix}
\]

So the produced normal contact force is:

\[
F_n = p_n d_n = \frac{p_n}{l} (S_{s0} + (x_3 - x_2, y_3 - y_2) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + (x_2 - x_3, y_2 - y_3) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix})
\]

(7.7)

Frictional force

For the sliding mode, besides the normal spring, a pair of frictional forces instead of a shear spring should be added. Based on the Coulomb’s friction law, the frictional force is calculated as:

\[
F_f = p_n d_n s \tan(\varphi) + c
\]

(7.12)

where \( p \) is normal spring stiffness, \( s \) is the sign of the displacement of \( P_1 \) relative to \( P_0 \) in the direction of \( \overline{P_2P_3} \), \( \varphi \) is the friction angle, \( c \) is cohesion.
7.3.3 Assembly of the H matrix and force vectors

The formation of coefficient matrices in the governing equations is similar to that in DDA (Shi, 1988). A numbering system of blocks and boundary elements is properly designed. The blocks are first numbered as 1, 2, …, N consecutively. Then the boundary elements and nodes in each block are also numbered consecutively with no gap between the blocks:

Block 1: elements: 1, 2, …, $N_e^1$; nodes: 1, 2, …, $N_n^1$;

Block 2: elements: $N_e^1 + 1, N_e^1 + 2, …, N_e^2$; nodes: $N_n^1 + 1, N_n^1 + 2, …, N_n^2$;

Block $i$: elements: $N_e^{i-1} + 1, N_e^{i-1} + 2, …, N_e^i$; nodes: $N_n^{i-1} + 1, N_n^{i-1} + 2, …, N_n^i$;

Block N: elements: $N_e^{N-1} + 1, N_e^{N-1} + 2, …, N_e^N$; nodes: $N_n^{N-1} + 1, N_n^{N-1} + 2, …, N_n^N$.

All the coefficient matrices are formed separately for each block and then assembled into the global matrices according to the degrees of freedom. The contact matrix of each block is also needed to be put into the global H matrix and F vector.

All the contact forces are treated as concentrated forces and handled by the following equation:

$$F_p = \tilde{G}_p^{pq} F_C$$

(7.13)

where $F_p$ is the resultant force vector corresponding to block $p$ ($p = i, j$), $F_C$ is the concentrated force and it could be the normal contact force ($F_n$), shear contact force ($F_s$) or frictional force ($F_f$), $\tilde{G}_p^{pq}$ is the coefficient matrix for the concentrated force $F$ applied at point $P_q$ ($q = 1, 2, 3$) in block $p$.

Since the contact forces may contain unknown displacements of contact points, some terms are required to be rearranged and moved to the H coefficient matrix accordingly. The details of how each term of contact matrices is derived and then put into H matrix and the resultant force vector are described next:

Normal contact matrix

The resultant force vector $F_{ni}$ contributed by normal contact forces $F_n$ applied to block $i$ is written as:
\[ F_{ni} = \hat{G}_i^{P} F_n * \begin{pmatrix} \frac{y_2 - y_3}{l} \\ \frac{x_3 - x_2}{l} \end{pmatrix} \] (7.14)

while the resultant force vector \( F_{nj} \) contributed by normal contact forces applied to block \( j \) is

\[ F_{nj} = \hat{G}_j^{P} F_n * \begin{pmatrix} \frac{y_3 - y_2}{l} \\ \frac{x_2 - x_3}{l} \end{pmatrix} = \hat{G}_j^{P^2} F_n * \begin{pmatrix} \frac{y_3 - y_1}{l} \\ \frac{x_1 - x_3}{l} \end{pmatrix} + \hat{G}_j^{P^3} F_n * \begin{pmatrix} \frac{y_1 - y_2}{l} \\ \frac{x_2 - x_1}{l} \end{pmatrix} \] (7.15)

After the corresponding rearrangement, the normal contact matrices can be added into \( H \) and \( F \) as follows respectively:

\[
P_n \hat{G}_i^{P_1} \begin{pmatrix} (y_2 - y_3)(y_2 - y_3) & (x_3 - x_2)(y_3 - y_3) \\ (y_2 - y_3)(x_3 - x_2) & (x_3 - x_2)(x_3 - x_2) \end{pmatrix} \rightarrow H_i^{P_1}
\]

\[
P_n \hat{G}_i^{P_2} \begin{pmatrix} (y_3 - y_1)(y_3 - y_1) & (x_3 - x_2)(y_3 - y_3) \\ (y_3 - y_1)(x_3 - x_2) & (x_3 - x_2)(x_3 - x_2) \end{pmatrix} \rightarrow H_i^{P_2}
\]

\[
P_n \hat{G}_i^{P_3} \begin{pmatrix} (y_1 - y_2)(y_1 - y_2) & (x_3 - x_2)(y_1 - y_2) \\ (y_1 - y_2)(x_3 - x_2) & (x_3 - x_2)(x_3 - x_2) \end{pmatrix} \rightarrow H_i^{P_3}
\]

\[
P_n \hat{G}_j^{P_1} \begin{pmatrix} (y_3 - y_1)(y_3 - y_1) & (x_3 - x_2)(y_3 - y_1) \\ (y_3 - y_1)(x_3 - x_2) & (x_3 - x_2)(x_1 - x_3) \end{pmatrix}
+ \hat{G}_j^{P_2} \begin{pmatrix} (y_2 - y_3)(y_2 - y_3) & (x_3 - x_2)(y_1 - y_2) \\ (y_2 - y_3)(x_3 - x_2) & (x_3 - x_2)(x_1 - x_3) \end{pmatrix} \rightarrow H_j^{P_1}
\]

\[
P_n \hat{G}_j^{P_2} \begin{pmatrix} (y_3 - y_1)(y_3 - y_1) & (x_1 - x_3)(y_3 - y_1) \\ (y_3 - y_1)(x_1 - x_3) & (x_1 - x_3)(x_1 - x_3) \end{pmatrix}
+ \hat{G}_j^{P_3} \begin{pmatrix} (y_3 - y_1)(y_3 - y_1) & (x_1 - x_3)(y_1 - y_2) \\ (y_3 - y_1)(x_1 - x_3) & (x_1 - x_3)(x_2 - x_1) \end{pmatrix} \rightarrow H_j^{P_2}
\]

\[
P_n \hat{G}_j^{P_1} \begin{pmatrix} (y_1 - y_2)(y_1 - y_2) & (x_2 - x_1)(y_3 - y_1) \\ (y_1 - y_2)(x_2 - x_1) & (x_2 - x_1)(x_1 - x_3) \end{pmatrix}
+ \hat{G}_j^{P_2} \begin{pmatrix} (y_1 - y_2)(y_1 - y_2) & (x_2 - x_1)(y_3 - y_1) \\ (y_1 - y_2)(x_2 - x_1) & (x_2 - x_1)(x_1 - x_3) \end{pmatrix} \rightarrow H_j^{P_3}
\]

\[
- \frac{p_n S_0 l_i^{P_1}}{l^2} \begin{pmatrix} y_2 - y_3 \\ x_3 - x_2 \end{pmatrix} \rightarrow F_i
\]
The resultant force vector $F_{si}$ contributed by shear contact forces $F_s$ applied to block $i$ is written as:

$$F_{si} = \hat{G}_i^{p_i}F_s * \left\{ \begin{array}{c} \frac{x_3 - x_2}{l} \\
\frac{y_3 - y_2}{l} \end{array} \right\}$$  \hspace{1cm} (7.17)$$

while the resultant force vector $F_{sj}$ contributed by normal contact forces applied to block $j$ is

$$F_{sj} = \hat{G}_j^{p_j}F_s * \left\{ \begin{array}{c} \frac{x_2 - x_3}{l} \\
\frac{y_2 - y_3}{l} \end{array} \right\} = \hat{G}_j^{p_j}F_s * \left\{ \begin{array}{c} a_3 \\
\frac{l}{a_4} \\
\frac{l}{a_5} \end{array} \right\} + \hat{G}_j^{p_3}F_s * \left\{ \begin{array}{c} \frac{a_3}{l} \\
\frac{l}{a_4} \\
\frac{l}{a_5} \end{array} \right\}$$  \hspace{1cm} (7.18)$$

After the corresponding rearrangement, the shear contact matrices can be added into $H$ and $F$ as follows respectively:

$$\frac{p_s}{l^2} \hat{G}_i^{p_i} \left( (x_3 - x_2)(x_3 - x_2) \frac{(y_3 - y_2)(x_3 - x_2)}{(y_3 - y_2)(y_3 - y_2)} \right) \rightarrow H_i^{p_i}$$

$$\frac{p_s}{l^2} \hat{G}_i^{p_j} \left( a_3(x_3 - x_2) \frac{a_4(x_3 - x_2)}{a_3(y_3 - y_2) a_4(y_3 - y_2)} \right) \rightarrow H_i^{p_2}$$

$$\frac{p_s}{l^2} \hat{G}_i^{p_j} \left( a_5(x_3 - x_2) \frac{a_6(x_3 - x_2)}{a_5(y_3 - y_2) a_6(y_3 - y_2)} \right) \rightarrow H_i^{p_3}$$

$$\frac{p_s}{l^2} \hat{G}_j^{p_2} \left( (x_3 - x_2)a_3 \frac{(y_3 - y_2)a_3}{(y_3 - y_2)a_4} \right) + \hat{G}_j^{p_3} \left( (x_3 - x_2)a_5 \frac{(y_3 - y_2)a_5}{(y_3 - y_2)a_6} \right) \rightarrow H_j^{p_1}$$

$$\frac{p_s}{l^2} \hat{G}_j^{p_2} \left( a_3a_3 \frac{a_4a_3}{a_3a_4} \frac{a_4a_3}{a_4a_4} \right) + \hat{G}_j^{p_3} \left( a_3a_5 \frac{a_4a_5}{a_3a_6} \frac{a_4a_5}{a_4a_6} \right) \rightarrow H_j^{p_2}$$
Frictional force matrix

The resultant force vector $F_{fi}$ contributed by frictional forces $F_f$ applied to block $i$ is written as:

$$F_{fi} = \hat{G}_i^p F_f \star \left( \begin{array}{c} \frac{x_3 - x_2}{l} \\ \frac{y_3 - y_2}{l} \end{array} \right) \quad (7.19)$$

while the resultant force vector $F_{fj}$ contributed by frictional forces applied to block $j$ is

$$F_{fj} = \hat{G}_j^p F_f \star \left( \begin{array}{c} \frac{x_2 - x_3}{l} \\ \frac{y_2 - y_3}{l} \end{array} \right)$$

$$= \hat{G}_j^p F_f \star \left( \begin{array}{c} \frac{(1 - t)(x_3 - x_2)}{l} \\ \frac{(1 - t)(y_3 - y_2)}{l} \end{array} \right) + \hat{G}_j^p F_f \star \left( \begin{array}{c} \frac{t(x_3 - x_2)}{l} \\ \frac{t(y_3 - y_2)}{l} \end{array} \right) \quad (7.20)$$

After the corresponding rearrangement, the frictional force vectors can be added into $F$ as follows:

$$-\frac{F_f}{l} \hat{G}_i^p \left( \begin{array}{c} x_3 - x_2 \\ y_3 - y_2 \end{array} \right) \rightarrow F_i$$

$$\frac{F_f}{l} \left( \hat{G}_j^p \left( \begin{array}{c} (1 - t)(x_3 - x_2) \\ (1 - t)(y_3 - y_2) \end{array} \right) + \hat{G}_j^p \left( \begin{array}{c} t(x_3 - x_2) \\ t(y_3 - y_2) \end{array} \right) \right) \rightarrow F_j$$

To illustrate how the global matrices are assembled, a two-block system is shown in Figure 7-5. Figure 7-6 shows how contact matrices are assembled into the global $H$ matrix, each small square represents a $2 \times 2$ matrix corresponding to orthogonal displacement components at a node.
Figure 7-5 A two-block system

Figure 7-6 Assembly of contact matrix into H matrix of a two-block system
7.4 FRAMEWORK OF THE DISCRETE BOUNDARY ELEMENT METHOD

Figure 7-7 Flow chart of the Discrete Boundary Element Method
Same as other discrete element methods, the discrete boundary element method also adopts an incremental dynamic formulation. In order to satisfy the assumption of infinitesimal displacements within a time step, a sufficient number of time steps are required when the total deformation and displacements are large. The results calculated from the previous time step are used as the new reference state to predict the behaviours of the current state.

The governing equation can then be obtained after substituting Newmark -$\beta$ scheme (Doolin and Sitar, 2004) into equation (7.1):

$$
\left( H + \frac{2}{h^2} M \right) U_{n+1} = G P + M g + \frac{2}{h} M U_n + E^d \sigma^0 + \hat{G} F_c
$$  \hspace{1cm} (7.21)

After rearranging the governing equations according to the boundary conditions, the final form can be obtained:

$$
AX = F
$$  \hspace{1cm} (7.22)

where $A$ is the new coefficient matrix after the boundary conditions are considered, $X$ is a combination of unknown nodal displacements and surface tractions and $F$ is the force vector.

After solving equation (7.22), equation (6.28) with term $\hat{u}_{ik}^{*} (\xi, P)^q F_k (P)^q$ is used to update the displacements of the internal points in each block. Equations (6.30) and (6.36) with term $\hat{u}_{ijk}^{*} F_k^q$, and equation (6.51) are used to calculate the stresses and velocities of both the internal points and boundary points.

Since the Newmark-$\beta$ scheme with $\beta = 0.5$ and $\gamma = 1$ is used, accelerations do not need to be updated in each time step. For each time step, there are initial velocity components and initial stresses together with the contact forces existed in each block to form the updated governing equations. For static analysis, the velocity at the beginning of each time step is assumed to be zero.

The flow chart of the proposed DBEM is shown in Figure 7-7.
7.5 VERIFICATION EXAMPLES

The stepwise updating based BEM has been implemented into Dr. Shi’s original DDA source code using C language.

A very important aspect of numerical analysis is validation using simple examples for which analytical or semi-analytical solutions exist. Such investigations are also essential for the validation of the implementation of the algorithm into a computer program. Before applying the developed discrete boundary element method as a predicting tool like other discrete element methods, some studies are conducted to validate the correctness of the developed code.

7.5.1 Verification of the term dealing with concentrated forces

Since contact forces are treated as concentrated forces, so it is important to check the correctness of this term dealing with concentrated forces.

The cantilever beam example subjected to bending in chapter 6 is used here. The traditional dual reciprocity boundary element methodology is adopted. There are 12 quadratic elements along the boundary and no internal cells. Five cases (Figure 7-8) of concentrated forces are considered with the distributed force is simplified as concentrated forces applied at different nodes. The locations of those nodes are indicated in Figure 7-8 at a certain distance from the left hand side. The displacements of point A located at the middle of the top free end are compared with the case with surface tractions at the free end. It is noted that $F = q(t) \times W$.

![Diagram of cases 1 to 5 with concentrated forces at different locations](image-url)
Figure 7-8 Five cases of concentrated forces applied at the free end of the cantilever beam

The results in Figure 7-9 show that the displacement curves in all cases are in good agreement, and this verifies the correctness of the term for concentrated forces in DRBEM.

Figure 7-9 Comparison of the displacements of Point A from all cases

7.5.2 A block sliding on an incline

Figure 7-10 Problem geometry of a block sliding on the inclined surface
The single-block sliding example, which was used to validate the DDA method by MacLaughlin (1997) and MacLaughlin and Doolin (2006), is used to verify the developed DBEM.

A block with dimensions of $0.1m \times 0.4m$ is sliding under the gravity along an incline with dimensions of $0.1m \times 2.6m$ and bottom fixed (Figure 7-10).

The analytical solution for the displacement $d$ of the block relative to its initial at-rest position as a function of time $t$ is derived based on the Newton’s second law as follows:

$$
    d = \begin{cases} 
        0 & \phi > \alpha \\
        \frac{1}{2}at^2 = \frac{1}{2}(gsin\alpha - gcosatan\phi)t^2 & \phi < \alpha 
    \end{cases}
$$

(7.23)

where $\phi$ is the friction angle between the block and the inclined surface, $\alpha$ is the angle of the incline, $g$ is the gravitational acceleration.

![Graph comparing displacements versus time for different values of friction angle](image)

**Figure 7-11** Comparison of displacements versus time for different values of friction angle using the DBEM

The sliding block is divided into 10 quadratic elements and 4 internal cells while the incline is divided into 28 quadratic elements and 13 internal cells.

Six cases with different values of friction angle ($\phi = 0, 5, 10, 15, 20, 25$) are simulated by the developed discrete boundary element method. The analytical solution
and simulated results of the displacements are plotted in Figure 7-11. It can be seen that the DBEM simulation results are in good agreement with the analytical solutions.

7.5.3 Layered rock deflection

Stratified rock is a common feature in mining and civil engineering. When an opening is excavated in a stratified rock mass, the thinner layer of the rock above the opening tends to detach from the main rock mass and form separated beams (Figure 7-12). The layers of rocks at the roof of the opening are often assumed to act like clamped beams.

![Figure 7-12 Layered rock above the openings (Louisville Mega Cavern)](image)

![Figure 7-13 Geometry of the two layer system](image)

This example analyses a two-layer system as shown in Figure 7-13. It can be considered as a composite beam with both ends clamped consisting of a thin beam with Young's modulus $E_1$, thickness $t_1$ and unit weight $\gamma_1$, and a thick beam with Young's modulus $E_2$, thickness $t_2$ and unit weight $\gamma_2$. The thin beam is overlying above the thick beam. Assuming only under gravity, if the deflection of the thin beam is larger than that of the thick beam, the thin beam will push down the bottom thick beam.
The maximum deflection of a single clamped, an elastic beam under gravity is given by (Goodman, 1989)

\[ u_{\text{max}} = \frac{\gamma L^2}{32Et^2} \]  

(7.24)

where \( \gamma \) is the unit weight, \( E \) is the Young’s modulus, \( L \) and \( t \) are the length and thickness of the beam respectively. As indicated in the above equation, the maximum deflection of a beam is inversely proportional to the square of the thickness of the beam.

An analytical solution for a two-layer system with both ends clamped was given by Goodman (1998) assuming only self weight is considered. The deflection of the bottom thick beam is calculated by assigning it an increased unit weight \( \gamma_e \) given by:

\[ \gamma_e = \frac{E_2t_2^2(\gamma_2t_2 + \gamma_1t_1)}{E_2t_2^3 + E_1t_1^3} \]  

(7.25)

In this example, the length of the composite beam is 36 m, and the thicknesses of the upper beam and the lower beam are \( t_1 = 2m \) and \( t_2 = 3m \) respectively. Both beams are assumed to have the same material properties: Young's modulus \( E_1 = E_2 = 4.8\text{GPa} \), Poisson's ratio \( \nu_1 = \nu_2 = 0.25 \), density \( \rho_1 = \rho_2 = 2580\text{ kg/m}^3 \) and unit weight \( \gamma_1 = \gamma_2 = 25300\text{ N/m}^3 \). The interface between two layers is assumed to be smooth with zero friction angle and cohesion.

![DBEM elements and internal cells](image)

In the DBEM model, the boundary of each of the two beams is divided into 34 quadratic boundary elements while each block domain is discretized into 30 internal cells (Figure 7-14). For each beam, the corner nodes of the internal cells are used to improve accuracy (14 nodes for each beam). In order to measure the deflection of the bottom layer, 17 measured points are selected along the longitudinal axis with a spacing of 2 m. After analysis, the deformed geometry of the composite beam is shown in Figure 7-15.
In order to make comparison with the results by the DBEM, the NMM (Shi, 1992) is also used to simulate the deflection of the composite beam. In the NMM model, 1859 manifold elements are used and 12 fixed points are applied to each end of the clamped double-layer beam. The deformed geometry by the NMM is shown in Figure 7-16.

**Figure 7-15 Deformed geometry of the composite beam using the DBEM**

**Figure 7-16 Deformed composite beam model using NMM**

**Figure 7-17 Comparison of the deflection results from DBEM and NMM**

**Figure 7-18 The contour of stress along x axis of the lower beam by the DBEM**
According to the analytical solution in Goodman (1989), the maximum deflection of the composite beam is 0.041m. After checking the displacements of the measured points from the two models, it is found that the maximum deflections of the bottom are both quite close to the analytical solution with 0.0414m in the DBEM model and 0.0408m in the NMM model. The vertical displacements of these measured points from the DBEM are also plotted in Figure 7-17 against those from the NMM. It can be seen that quite good agreement has been achieved. The $\sigma_x$ and $\sigma_y$ contour of the lower beam obtained from the DBEM is presented in Figure 7-18 and Figure 7-19 respectively.

### 7.5.4 Stability of the masonry arch

The Couplet-Heyman benchmark problem is one of the well-known classical solutions in the theory of masonry arches. In this problem, the minimum thickness $t$ of a semi-circular arch is related to the radius $a$ required to support the weight. The analytical solution (Heyman, 1982) is derived with the assumptions of no sliding, no crushing and no tension between voussoirs. If the ratio $t/a$ is smaller than a critical value 0.105869, the arch will fail as a four-hinge mechanism. If the ratio $t/a$ is larger than this critical value, the arch is stable.
In this example, a semicircular arch with $t=0.1\text{m}$ and $a=1\text{m}$ (Figure 7-20 a)) is simulated with the developed DBEM. The ratio is 0.1, which is just below the critical value. The density is $2500\ \text{kg/m}^3$, Young's modulus is $2\ \text{GPa}$, Poisson's ratio is 0.25, and the penalty spring stiffness is assumed to be $20E$. In order to ensure that sliding does not occur, high friction angle and cohesion values are used.
Each block is divided into 12 quadratic boundary elements and 8 internal cells as shown in Figure 7-20 b) and the corner nodes of the internal cells inside each block (3 nodes for each block) are considered in the governing equations to improve accuracy. The failure process of the arch is presented in Figure 7-21. It is clearly observed that the arch is not stable and collapse as a four-hinge mechanism as predicted by the analytical solution.
Furthermore, the thickness of the arch is increased to 0.112m to make the ratio just above the critical value. The arch model with mesh is shown in Figure 7-22. Same parameters are used in this case. After simulation using the DBEM, it is found that this arch is stable.

![Figure 7-22 The arc model with t=0.112m and a=1m](image)

### 7.5.5 Toppling failure simulation

In rock slope engineering, toppling is a failure mode, which differs significantly from sliding along an existing or induced slip plane. It is a mass-movement process, which is characterized by forward rotation and overturning of rock columns or plates separated by closely spaced and steeply dipping joints.

![Figure 7-23 Three basic types of toppling failure (Agliardi, 2012)](image)

Three basic types of toppling failure mechanisms (Figure 7-23) are defined by Goodman and Bray (1976) including flexural, block, and block flexural. The toppling failures has been studied by Kieffer (1998) and Goodman and Kieffer (2000).
A toppling example is designed and used to show the capacity of the developed DBEM. A blocky system with several parallel rock columns sitting on a slope of $35^\circ$ is simulated. A toe block is placed fixed to the slope and it is treated as part of the slope in the simulation. The model geometry with and without mesh is shown in Figure 7-24 a). The boundary of each block is divided into a number of boundary elements and the domain is divided into a number of internal cells compatible to the boundary elements as shown in Figure 7-24 b). To improve accuracy, the corner nodes of the internal cells inside each block are considered in the governing equation. The Young’s modulus of the blocks is $20 \text{ GPa}$ while the Poisson’s ratio is 0.25. The friction angle and cohesion are both assumed to be 0. The penalty spring stiffness is assumed to be $40E$. Only gravity is considered.

For a single block with thickness $b$ and $h$ on an incline with dipping angle $\psi$ (Figure 7-25), it topples when the following condition is satisfied (Hoek and Bray, 1981):

$$ W \cos \psi - W \sin \psi > W h $$
Using this criteria, all the blocks (rock columns) topple if they are put separately on a slope of $35^\circ$. Due to the existence of the toe block on the slope, all the blocks cannot slide down and the two short blocks 1 and 2 next to it tend to stable. However, the long blocks with larger moments such as 3, 7 and 10 still tend to topple and they interact with their neighbouring blocks and force them to rotate together. The simulated toppling process using the developed DBEM is presented in Figure 7-26. During the interacting process, the top part of the tall blocks deforms like a cantilever beam. A combination of rotational and flexural failure can be observed.
7.5.6 Stability analysis of a tunnel roof in layered rock

Yeung (1993) and Yeung and Blair (2000) analysed the behaviours of a mine roof in layered rock using DDA. In this study, a tunnel roof is designed and analysed by the DBEM.

The stability of a rectangular tunnel roof in a rock mass consisting of horizontal strata and vertical joints is investigated using the developed DBEM. The roof is $4m \times 12m$ consisting of five layers with the same layer thickness, and in each layer, there are a number of vertical joints. The roof model is supported by two L-shaped abutments as shown in Figure 7-27 a). Each block is divided into a number of boundary elements and internal cells (Figure 7-27 b)) and the corner nodes of the internal cells inside each block are considered in the governing equations to improve accuracy. Five points are
selected to output their vertical displacements with time. The material properties are listed in Table 7-2.

![Diagram of a tunnel roof model showing geometry and model with boundary elements and internal cells.]

**Figure 7-27 Tunnel roof model: a) geometry; b) model with boundary elements and internal cells**

**Table 7-2 Material properties**

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's modulus</td>
<td>1 GPa</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>0.25</td>
</tr>
<tr>
<td>Density</td>
<td>2450 kg/m³</td>
</tr>
<tr>
<td>Friction angle</td>
<td>40°</td>
</tr>
<tr>
<td>Cohesion</td>
<td>0 kPa</td>
</tr>
<tr>
<td>Contact normal spring stiffness</td>
<td>50E</td>
</tr>
</tbody>
</table>
From the simulated results, it can be seen that the central bottom part of the roof slides into the opening first. The shear resistance between blocks develops and induces the rotation and deformation of the blocks close to the abutments, forming an arch as shown in Figure 7-28. Thus the roof becomes stable. The vertical displacements of the selected five points in Figure 7-27 a) are plotted in Figure 7-29.

![Figure 7-28 Failure of the tunnel roof at time t=5.35s](image)

**Figure 7-28 Failure of the tunnel roof at time t=5.35s**

![Figure 7-29 Vertical displacements of the selected points](image)

**Figure 7-29 Vertical displacements of the selected points**

7.6 CONCLUSIONS

A discrete boundary element method is developed by implementing an open close iteration contact algorithm into the dual reciprocity boundary element method. The open-close iterations are applied for accurate contact analysis and the implicit time integration scheme is used for numerical stability. The results of validation examples show that it can obtain results in good agreement with the analytical or numerical solutions, and this shows the effectiveness of the DBEM in simulating deformation,
movements and interactions of blocks. The masonry arc, toppling and tunnel examples demonstrate the applicability of the DBEM for blocky systems.

By taking advantages of the principles of the continuum mechanics in the BEM and the discontinuum mechanics in the DDA, the DBEM allows for the simulation of complex problems including fracturing analysis, fluid flow analysis and dynamic analysis.

In the proposed method, internal cells are used for the integration of the initial stress term. This requires updating of the displacements, stress and velocity in the internal nodes, thus increasing the amount of computational time. In the near future, robust techniques will be sought to transform the domain integration of the initial stress term to the boundary. This will not only leading to easy mesh generation, but also improve the efficiency quite dramatically. Furthermore, the proposed DBEM is limited to individual blocks which are all finite domains. As most geotechnical problems are actually infinite or semi-infinite, this method needs to be extended to be able to solve problems with infinite or semi-infinite blocks involved.
CHAPTER 8. CONCLUSIONS AND FUTURE WORK

8.1 CONCLUSIONS

A realistic modelling of blocky rock masses requires capture of the main characteristics of the rock mass and a strategy to balance the computational efficiency and accuracy. Towards realistic rock mass modelling, five integrated, yet relatively independent works have been carried out and summarized as follows:

1) The algorithm for the generation of three-dimensional geological models is further developed for engineering analysis. Taking into consideration of the finiteness of the discontinuities, the current work mainly focus on the robustness, efficiency and convenience for engineering analysis. Different measures are taken in the algorithm and a number of methods are also used to verify the results for robustness. For rock engineering projects, the rock slope profile can be represented by triangulated surfaces or general polygons; while the rock tunnel can be straight or curved with different kinds of cross-sectional shapes. An extensive number of examples have been generated using the developed program to show its capacity. This robust geological model generation tool is essential since randomly generated discontinuities from Monte Carlo simulation are handled.

2) The traditional key block method has also been further extended for progressive failure analysis. Based on the reconstructed three dimensional blocky rock mass, the developed key block program is capable of identifying different batches of key blocks by a progressive failure process. A force transfer algorithm is proposed in order to take into consideration the interactions among key blocks in different batches. After a rock bolting system is selected, a two-step safety check procedure is employed for the assessment of the stability of the bolted rock mass. The tunnel and powerhouse examples both showed that the later batches key blocks did affect significantly the sliding force of some surface key blocks. If this effect is ignored, the required supporting force for some key blocks will be significantly underestimated. The proposed method also has the advantage of finding out whether larger key blocks are formed after the rock bolting system is applied.
3) In order to cover the uncertainties in geometric and mechanical parameters of discontinuities, stochastic key block analysis has been conducted as well. Based on Monte Carlo simulations of discontinuity network and the developed blocky rock mass generator, a close-to-nature representation of the blocky rock mass can be simulated and the block and key block predictions can thus be obtained more accurately. The proposed approach is capable of dealing with non-persistent discontinuities, complex excavations and progressive analysis of blocky rock mass. The stability analysis of a horseshoe shaped tunnel is conducted. The key block characteristics (failure mechanism, the number, volume, shape etc. of key blocks) are statistically analysed to provide guidance for support design. The critical (or worst) cases with larger total key block volume are found out and selected for further more accurate and rigorous numerical analysis with DDA or NMM to achieve more accuracy within a reasonable computational cost. Three scenarios with different mean discontinuity size data are employed to investigate the size effect of discontinuities. The persistence of discontinuities has been shown to be critically important for the accurate prediction of key block statistics. Utilization of persistent discontinuities leads to over-estimation of fragmentation of the rock mass, if most of the discontinuities are not much greater than the dimensions of the domain. Subsequently, more key blocks are estimated and the maximum volume of single key blocks is underestimated for the persistence scenario. Thus representing the discontinuity size more accurately in the blocky rock mass model is important for the accurate prediction of the key block statistics. In addition, a case study application to the entrance of a tailrace tunnel at Jinping I hydropower station has also been carried out which verified the applicability of the stochastic key block analysis method in rock engineering.

4) The stepwise updating of the DRBEM is carried out, which is necessary not only for analysis of large displacements but also for the combination with the DDA. The detailed formulation has been derived with the consideration of initial stress, which leads to weak singularity in the displacement equation and strong singularity in the stress equation. In the current algorithm, internal cells are used to deal with the domain integral of initial stresses for robustness. The geometry of domain is updated at the end of each time and used as the new geometry at the beginning of the next step. Furthermore, the stresses, velocities and accelerations are required to be calculated at
the end of each time step and used as initial conditions in the next step. Three examples have been used to test the developed dual reciprocity boundary element program with stepwise updating and satisfactory results have been obtained. The stepwise updating lays the groundwork for the development of the discrete boundary element method. In addition, with the consideration of geometry updating, large displacement and large deformation could be simulated.

5) The contact algorithm with open-close iterations in the DDA is implemented into the dual reciprocity boundary element method with stepwise updating. This method, called Discrete Boundary Element (DBEM), is capable of simulating the behaviour of blocky systems. In this method, each block is discretized with boundary elements and internal cells. The internal cells are only used for the domain integration of the initial stress term. For contact analysis, all contact forces are considered as concentrated forces and open-close iterations are applied to identify the correct contact location and force. Some examples are used to validate the correctness of the formulation and coding and also show the capacity of the proposed method.

It should be mentioned that the reliability of the analysis results from the developed key block method and DBEM is highly dependent on the quality of the characterization of the discontinuity network and the properties of both discontinuities and rock matrix. However, uncertainties in rock masses are quite difficult to quantify due to lack of measured data support, etc.

8.2 FUTURE RESEARCH

Special ideas for continuing research include the following:

Since the reliability of the discontinuity network significantly affects the results of stability analysis by key block method and numerical methods, more research is required on the discontinuity network including shape, size, density and location of joints. Reliable approaches need to be developed to accurately transform field mapping results to three-dimensional discontinuity network for engineering analysis of blocky rock masses.

The proposed discrete boundary element method is limited to individual blocks, which are all finite domains. As most geotechnical problems are actually infinite or semi-
infinite, this method needs to be extended to be able to solve problems with infinite or semi-infinite blocks involved.

In the discrete boundary element method, internal cells are required in each individual block for the domain integration of initial stress. A robust technique is required to transform the domain integral of initial stress and thus eliminate the internal cell leading to real meshless of the proposed DBEM and also improve the efficiency quite dramatically.

Introduction of failure criterion to enhance the DBEM with the ability of conducting fracture propagation is another possible improvement.

In addition, it is essential to extend the discrete boundary element method to three dimensions since all rock engineering problems are three dimensional due to the existence of discontinuities. This will require a robust and reliable three-dimensional contact algorithm dealing with arbitrary shaped blocks (convex or concave) and implementation of the popular computer technology parallel computing to improve efficiency.
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APPENDIX A. MATHEMATICAL TREATMENT OF MEASURED DATA

The complete survey involves the measurements of the discontinuity orientation, spacing and trace length. A series of probability distributions have been reported in the literature (Baecher, 1983; Priest, 1993; Jing and Stephansson, 2007) to represent the above three parameters (see Table A-1).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Distribution forms</th>
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<tbody>
<tr>
<td>Orientation</td>
<td>Fisher, Bingham, hemispherical uniform, bivariate Fisher/normal, uniform</td>
</tr>
<tr>
<td>Spacing</td>
<td>Log-normal, Negative exponential, Gamma, Power law</td>
</tr>
<tr>
<td>Trace Length</td>
<td>Log-normal, Negative exponential, Gamma, normal, Power law</td>
</tr>
</tbody>
</table>

Due to the various errors or biases (Baecher, 1983; Einstein et al., 1983; Kulatilake, 1988) involved in the mapping process, careful defined sampling or correction procedures are essential in order to minimize these effects. These biases mainly include orientation bias, censoring bias, truncation bias and bias caused by the finiteness of scanlines or windows. According to the selected sampling method, the estimation method of different parameters (orientation, trace length, etc.) varies.

There are several assumptions involved in the processing of collected data from field survey for discontinuity parameters. All the discontinuities are considered to be planar and centres are randomly and independently distributed in space. The size distribution is also assumed to be independent of spatial location. In addition, one key assumption, which is inherent to most works, is that discontinuities occur in sets of primarily parallel discontinuities and that each set has its average characteristics.

Shape

Shape is one of the most difficult parameters to establish. The real shape of discontinuities is unknown since the rock mass is usually inaccessible in three
dimensions. Thus when dealing with discontinuity shape, researchers assumed different shapes for different research and application purposes (Table A-2).

Polygon representation seems the most general and realistic, but circular shape of discontinuities is commonly assumed mainly due to simplicity. Zhang and Einstein (2010) drew the following conclusion based on their analyses and investigation results: Joints not affected by adjacent geological structures such as bedding boundaries tend to be elliptical (or approximately circular but rarely), while joints affected by or intersecting geological structures such as bedding boundaries tend to be most likely rectangles or similarly shaped polygons. If a very large number of discontinuities are involved, the significance of the discontinuity shape decreases with an increase in the discontinuity population size (Jing and Stephansson, 2007).

<table>
<thead>
<tr>
<th>Shape</th>
<th>Researchers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular</td>
<td>Baecher et al., 1977; Warburton, 1980a; Villaescusa and Brown, 1990; Kulatilake et al., 1993</td>
</tr>
<tr>
<td>Polygon</td>
<td>Dershowitz et al., 1998; Meyer and Einstein, 2002</td>
</tr>
<tr>
<td>Parallelogram</td>
<td>Warburton, 1980b</td>
</tr>
<tr>
<td>Elliptical</td>
<td>Zhang et al., 2002</td>
</tr>
<tr>
<td>Rectangular</td>
<td>Zhang et al., 2002</td>
</tr>
</tbody>
</table>

Orientation
The orientation bias is caused by the relative orientation between the discontinuities and the sampling scanlines or sampling windows.

The sampling line will tend to intersect preferentially those discontinuities whose normal vectors make a small angle to the sampling line. This bias can be corrected by introducing a geometrical correction factor based on the observed angle between the sampling line and the normal to a particular discontinuity (Terzaghi, 1965; Baecher, 1983; Wathugala et al., 1990; Priest, 1993). Terzaghi weighting is derived as follows:
\[ W = (\cos \delta)^{-1} = \left[ (\cos (\alpha_i - \alpha_s) \cos \beta_i \cos \beta_s + \sin \beta_i \sin \beta_s) \right]^{-1} \]

where \( \alpha_i, \beta_i \) are dip direction and dip angle of each discontinuity (set); \( \alpha_s, \beta_s \) are trend and plunge of the scanline; \( \delta \) is the acute angle between the sampling line and the set normal vector.

If \( \delta \) approaches 90°, \( W \) will becomes very large, to avoid this it is desirable to set a maximum allowable weighting. A maximum value of 10 corresponding to an angle of 84.3°, has been suggested by Priest (1993). DIPS (Data Interpretation Package using Stereographic projection) developed by Rocscience (2003) also suggested the maximum angle be in the range of 65° – 85° with a default value of 75°.

Wathugala (1990) suggested a general procedure to correct sampling bias on orientation using a vector approach for window mapping. This procedure is applicable for sampling planes of any orientation.

\[ W = \{whd_i[\cos^2 \beta_i + \sin^2 \beta_i \cos^2 (\alpha_r - \alpha_i)]^{0.5} + \frac{\pi}{4}d_i^2[w \sin \beta_i] \cos (\alpha_r - \alpha_i) + h \cos \beta_i \}^{-1} \]

where \( \alpha_r \) is the strike of the sampling window; \( d_i \) is the diameter of the \( i^{th} \) discontinuity; \( w \) and \( h \) are width and height of a rectangular window respectively.

The weighting, \( W \), can be applied to contour and rosette plots in DIPS, and is also used in the weighted mean vector calculations (Priest, 1993). To obtain the mean orientation of each set (Priest, 1993): calculate the weighting \( w_i \) for each discontinuity \( i \); calculate the total weighted sample size \( N_w = \sum_{i=1}^{N} w_i \) for a sample size \( N \); calculate the normalized weighting factor \( w_{ni} = \frac{w_i N}{N_w} \); calculate the corrected direction cosines for the normal of each discontinuity; the mean orientation of the \( N \) discontinuities is the orientation of the resultant vector.

**Spacing and linear frequency**

The relatively short sampling lines that are required where a rock face is of limited extent or where borehole core is logged in short runs.
Sen and Kasi (1984) addressed this sampling bias based on the negatively exponential distributed spacing and there is a small error in Sen and Kasi’s original formulas. Kulatilake (1988) and Priest (1993) corrected the small errors:

\[
u_{XL} = \frac{1 - e^{-\lambda L}(1 + \lambda L)}{\lambda(1 - e^{-\lambda L})}
\]

\[
\sigma_{XL} = \frac{2 - e^{-\lambda L}(2 + 2\lambda L + \lambda^2 L^2)}{\lambda^2(1 - e^{-\lambda L})} - (\nu_{XL})^2
\]

From these two formulae we can see that if L is large, the mean and standard deviation will be close to \(1/\lambda\) and \(1/\lambda^2\) respectively. The linear frequency is the reciprocal of the spacing.

Grossenbacher et al. (1997) present a method for determining discontinuity frequencies form data collected along circular scanlines, which they suggest can be expanded and adapted to the more general case of irregularly curved scanlines. Peacock et al. (2003) advanced the method of Priest and Hudson (1981) to develop a method for a curved scanline to be used to predict the numbers of discontinuities that would be observed in any direction.

**Trace length and size**

Despite the promise of geophysical methods to detect discontinuities, there is currently no direct technique for determining the size and shape of discontinuities in situ. The borehole, even of large diameter, provides minimal information on size and shape. To provide the size and shape information, more boreholes are needed which will cost a large amount of money. Therefore, the size parameter is commonly determined by taking measurements of trace lengths along exposed rock faces using either scanline or window sampling techniques. Normally, this includes determination of true trace length distribution from measured trace length data and determination of size distribution from true trace length distribution by assuming the shape of the discontinuities. The measured trace length data need to be processed first and the related biases should be corrected before the relationship between trace length and size is used to infer the distribution of discontinuity size.
There are several probability density functions (Priest and Hudson, 1981; Priest, 1993), which are defined for the inference of size distribution:

i(l): PDF of the measured semi-trace lengths trimmed below a level l and curtailed above a level c, and this function is only used for scanline mapping;

h(l): PDF of the true semi-trace lengths intersected by the scanline;

g(l): PDF of measured trace lengths on a finite exposure subjected to sampling biases;

f(l): PDF of true trace lengths on the entire rock surface;

g(D) or g(a): PDF of the size of discontinuities.

(1) Biases

Four biases should be considered (Baecher and Lanney, 1978; Priest and Hudson, 1981; Kulatilake and Wu, 1984; Mauldon, 1998; Zhang and Einstein, 1998; 2000; Priest, 2004):

**Orientation bias**: The probability of a discontinuity appearing in an outcrop or excavation surface depends on the relative orientation between the outcrop and the discontinuity.

**Size bias**: a) Large discontinuities are more likely to appear at an exposed surface than small ones; b) Long traces are more likely to appear in a mapping window or on a scanline than short ones.

**Truncation bias**: Trace lengths below some known cut-off length are not recorded due to the very short traces, which are difficult or sometimes impossible to measure.

**Censoring bias**: Long discontinuity traces may extend beyond the visible exposure so that one end or both ends of the traces cannot be seen. In the field survey, these long traces can be counted but they cannot be measured.

In order to determine the true trace length distribution, size bias b), truncation bias and censoring bias should be corrected. When determining the size distribution from the true trace length, the orientation bias and size bias a) need to be considered. Truncation bias
is relatively easy to deal with. Decreasing the truncation-level in surveys can reduce the
effects of truncation bias on trace length estimates. Warburton (1980a) proposed one
method to correct the truncation bias. The normalized probability density of observed
trace lengths is given by:

\[
g(l) = \begin{cases} 
0 & l < l_t \\
\frac{f(l)}{\int_l f(l)dl} & l \geq l_t 
\end{cases}
\]  

(184)

where \(l_t\) is the threshold value for trace length mapping (e.g. 0.1cm).

(2) Methods to get the mean and standard deviation of corrected trace length from
measured trace length data

In the stage from measured trace length to true trace length, the distribution of true trace
length will be determined. This actually needs to find the distribution form, mean and
standard deviation of corrected trace length. Here the methods for determining the mean
and standard deviation of true trace length by different mapping methods are
summarized below.

a) Scanline mapping

Laslett (1982) proposed the following likelihood for the observed (censored) joint trace
length distribution from line sampling:

\[
LF = \frac{\prod_{i=1}^{N_0} dF(X_i) \prod_{j=1}^{N_1} [1 - F(Y_j)] \prod_{k=1}^{N_2} \int_{Z_k} [1 - F(\nu)]d\nu}{\mu_L^{N_0+N_1+N_2}}
\]  

(185)

where \((X_1, \cdots X_{N_0})\) are the observed trace lengths with both ends exposed; \((Y_1, \cdots Y_{N_1})\)
are the observed trace lengths with one end exposed; \((Z_1, \cdots Z_{N_2})\) are the observed trace
lengths with no ends exposed, and \(\mu_L\) is the mean corrected trace length; \(N_0, N_1, N_2\) are
the number of traces with both ends censored, with one end censored and with both ends
observed respectively.

If the discontinuity traces follow negatively exponential distribution, the maximum
likelihood estimate for the mean trace length form can be obtained as follows:
This likelihood corrects for progressive censoring (edge effects) on the observed twodimensional data.

The relationships between measured semi-trace length, corrected semi-trace length, measured trace length and corrected trace length were given by Priest (1993). The relationship between observed trace length and corrected trace length:

\[ g(l) = \frac{lf(l)}{\mu_L} \]  \hspace{1cm} (A.8)

By assuming the traces follow a Poisson process, the observed semi-trace length comes from the following distribution:

\[ h(l) = \int_{l} \left( \frac{1}{m} \right) g(m) dm \]  \hspace{1cm} (A.9)

\[ i(l) = \frac{h(l)}{\int_{l}^{c} h(l) dl} \hspace{1cm} (t \leq l \leq c) \]  \hspace{1cm} (A.10)

b) Rectangular window mapping

Pahl (1981) suggested a technique to estimate mean trace length on an infinite surface and this method is restricted to a set of parallel discontinuities. This method is also based on Poisson line segment model (trace midpoints are randomly and uniformly distributed. This method corrects for censoring and size biases with the assumption that joints are convex which leads to line traces.

\[ \mu_L = \frac{wh(N + N_0 + N_2)}{(wcos\phi + hsin\phi)(N - N_0 + N_2)} \]  \hspace{1cm} (A.11)

where \( w, h \) are the height and width of the rectangular planar rock face window and \( \phi \) is the angle between the traces and the vertical; \( N \) is the total number of traces, and \( N = N_0 + N_1 + N_2. \)
Kulatilake and Wu (1984) extended Pahl’s technique to discontinuities whose orientation is described by a probabilistic distribution. This method provides corrections for censoring bias indirectly.

\[ \mu_L = \frac{wh(N + N_0 + N_2)}{(wB + hA)(N - N_0 + N_2)} \]  
\[ A = \int_{\alpha_l}^{\alpha_u} \int_{\theta_l}^{\theta_u} \frac{1}{(1 + \tan^2 \theta \cos^2 \delta)^{1/2}} f(\theta, \alpha) d\theta d\alpha \]  
\[ B = \int_{\alpha_l}^{\alpha_u} \int_{\theta_l}^{\theta_u} \frac{1}{(1 + \cot^2 \theta \sec^2 \delta)^{1/2}} f(\theta, \alpha) d\theta d\alpha \]

where \( f(\theta, \alpha) \) is the probability density function of discontinuity orientation with \( \theta_l \leq \theta \leq \theta_u \) and \( \alpha_l \leq \alpha \leq \alpha_u \), where subscripts \( l \) and \( u \) denote lower and upper limits respectively. \( \delta \) denotes the acute angle between the dip direction and the sampling plane.

c) Circular window mapping

Andersson and Dverstorp (1987) proposed three estimators for mean trace length with Laslett’s moments method:

\[ \mu_{L1} = \frac{2L_{total}}{N - N_0 + N_2} \]  
\[ \mu_{L2} = \frac{L_{total}}{N - N_0} \]  
\[ \mu_{L3} = \frac{\pi RL_{total}}{\pi NR - 2L_{total}} \]  

where \( L_{total} \) is the sum of the \( N \) measured trace lengths. If the number of the sample traces is very large, the three estimators above tend to the same value.

Mauldon (1998) advanced Pahl’s method for mean trace length and derived estimators of mean trace length for windows with arbitrary convex boundaries. Results for rectangular and circular windows are obtained as special cases of the general solutions for arbitrary convex windows. The equation for a rectangular window is the same as the one by Kulatilake and Wu (1984).
For circular window mapping, Zhang and Einstein (1998) and Mauldon (1998) assuming circular shaped discontinuities have derived the same formula independently (these two are identical to each other):

\[
\mu_L = \frac{\pi(N + N_0 - N_2)}{2(N - N_0 + N_2)} \times R
\]  
(A.14)

where \( R \) is the radius of the circular window.

This equation is based on the implicit assumption that each trace has the same chance of appearing on the sampling window. This assumption is applicable only for the deterministic orientation (i.e. parallel traces on the two-dimensional exposure) and it is not applicable in the usual case where the discontinuity orientation has a scatter and is probabilistically distributed. Zhang and Einstein’s approach used the general approach given by Kulatilake and Wu (1984).

Mauldon (1998) has also proposed the method for the use of multiple circular windows: \( m = N + N_0 - N_2, n = N - N_0 + N_2 \), \( \bar{m} \) and \( \bar{n} \) are the average of \( m \) and \( n \) for all the circles.

For multiple circles with same radius \( R \)

\[
\mu_L = \frac{\pi R}{2} \left( \frac{\bar{n}}{\bar{m}} \right)
\]  
(A.15)

For multiple circles with different radii or size

\[
\bar{n} = l \cdot (4R)
\]  
(A.16)

\[
\rho = \frac{1}{2\pi} \left( \frac{\sum m}{\sum R^2} \right)
\]  
(A.17)

\[
\mu_L = l / \rho
\]  
(A.18)

This estimator automatically corrects for the problems of censoring and length bias, and yield results with acceptable levels of accuracy (Mauldon et al., 1999b).

There are two special cases when equations (A.11)-(A.14) are applied (Zhang and Einstein, 1998): 1) If \( N_0 = N \), then \( \mu_L \to \). In this case, all the discontinuities
intersecting the sampling window have both ends censored. This implies that the area of the window for the discontinuity survey may be too small. 2) If \( N_2 = N \), then \( L \rightarrow 0 \).

In this case, all the discontinuities intersecting the sampling window have both ends observable. According to Pahl (1981), this result is due to violation of the assumption that the midpoints of traces are uniformly distributed in the two dimensional space. These two special cases can be addressed by increasing the sampling window size, and/or changing the sampling window position, or to use multiple windows of the same size but at different locations and then use the total numbers from these windows to estimate \( L \) (Zhang and Einstein, 1998).


From the above methods, the standard deviation cannot be determined. Thus we need to make some assumptions in order to get the PDF of corrected trace length.

Zhang and Ding (2010) derived the standard deviation expression of mean trace length estimator based on circular windows with reasonable accuracy:

\[
SD(\mu) \cong \frac{\pi R \sqrt{N[N(N_0 + N_2) - (N_0 - N_2)^2]}}{(N - N_0 + N_2)^2} \tag{A.19}
\]

The standard deviation of the mean trace length estimator decreases as the size of the sampling window increases.

(3) Relationship between size and true trace length

There are three assumptions when size information is inferred from trace length: a) The discontinuity centres obeyed a three dimensional Poisson process with a volumetric frequency \( \lambda_v \) (volume density has a Poisson distribution); b) The discontinuities in each set should be parallel or nearly parallel. c) Joint size was assigned an arbitrary distribution with no dependence on special location.

Based on the above assumptions, Warburton (1980a; b) derived the stereological relationship between size and trace length over the entire exposure for both straight scanline mapping and window mapping:
where \( f_A(l) \) (and \( f_L(l) \)) are \( f(l) \) for window mapping and scanline mapping respectively, \( D \) is the diameter of the discontinuities; \( l \) is the trace length of discontinuities; \( \mu_D \) is the mean of joint diameter; \( E(D^2) \) is the mean of \( D^2 \) or the second moment of the joint diameter distribution; \( k \) is the ratio of longer to short side length \( a \) of the parallelograms; \( \delta \) is the angle between the shorter side and the line parallel to the trace; and \( \epsilon \) is the angle between the longer side and the line parallel to the trace. For parallelogram discontinuities, the joints in a set are assumed to be represented by parallelograms of various sizes in which all similar sides are parallel; the joints are geometrically similar, which implies a constant ratio of longer to shorter sides for all parallelograms. It can be seen that the joints in the model differ from each other only by scale and by simple translation in space with no rotation.

Following the method of Warburton (1980a; b), Zhang et al. (2002) derived a general stereological relationship between discontinuity size (expressed by the major axis length \( a \) of the ellipse) and trace length over the entire exposure:

\[
f_A(l) = \frac{l}{M \mu_a} \int_{l/M}^{\infty} \frac{g(a)da}{\sqrt{(Ma)^2 - l^2}} \quad (l \leq aM) \quad \text{(Elliptical discontinuities)}
\]

\[
M = \frac{\sqrt{\tan^2 \beta + 1}}{k^2 \tan^2 \beta + 1}
\]
where $k$ is the aspect ratio of the length of the discontinuity major axis to that of the minor axis; $\beta$ is the angle between the discontinuity major axis and the trace line (note that $\beta$ is measured in the discontinuity plane).

(4) Procedures from measured trace length data to size distribution

a) True trace length distribution from the measured trace length data

First the mean and standard deviation of true trace length are estimated using the above formulas for different mapping methods. Then the Chi-square or Kolmogorov-Smirnov (K-S) goodness-of-fit test is employed to find the most suitable distribution form of $g(l)$ based on the common distributions forms for trace length reported in the literature. If no standard deviation of true trace length is obtained, $f(l)$ and $g(l)$ can be assumed to have the same standard deviation (Kulatilake et al., 1993) or coefficient of variation (COV) which is the ratio of mean to standard deviation (Zhang and Einstein, 2000).

b) Discontinuity size from true trace length

It is reported that the estimation of size distribution by the stereological relationship between trace length and size may not necessarily always be robust (Villaescusa and Brown, 1992). Baecher et al. (1977) and Billaux et al. (1989) have found that considerable changes in the nature of the underlying joint diameter distribution can give rise to very slight changes in the simulated trace length distribution and conversely. In order to solve this problem, Villaescusa and Brown (1992) used a stereological theorem by Crofton (1885) for scanline mapping. This theorem relates the expected values of observed chords and the expected values of theoretical areas of convex bodies, and it can be used to suggest the nature of the underlying joint size distribution. Under the assumption of joint convexity and circularity, the expected values of the observed joint trace lengths and the joint areas are related by the following formula:

$$\frac{E(S^2)}{E(S)} = \frac{\pi}{3} \frac{E(l^2)}{E(l)}$$  \hspace{1cm} (A.25)

where $S$ is the area of a discontinuity; $l$ is the measured trace length.
Zhang and Einstein (2000) and Zhang et al. (2002) extended the method by Villaescusa and Brown (1992) with the assumption of circular and elliptical discontinuities respectively for circular window mapping.

With the calculated true trace length distribution, the procedure for the above methods is as follows: If discontinuities are circular, for each assumed distribution form in Table A-1, the mean $D$ and standard deviation $\sigma_D$ of the diameter function $g(D)$ using equation (A.20) or (A.21) are calculated; if discontinuities are elliptical, an iterative process is needed to find the actual major axis orientation. For each assumed distribution form, by assuming a major axis orientation and aspect ratio, $a$ and $\sigma_a$ are computed using equation (A.24) and the curves relating $a$ and $\sigma_a$ to $k$ are drawn. The process repeated until the curves for different sampling windows intersect at one point, and the major axis for this case is the inferred actual major axis orientation. If discontinuities are rectangular, equation (A.22) or (A.23) should be used, the two adjacent sides should be perpendicular and the procedure is the same as elliptical discontinuities. Then the best distribution form of $g(a)$ or $g(D)$ by checking the equality of equation (equation (A.25)). The best distribution form is the form for which the left and right sides are the closest to each other.

A strategy has also been proposed by Priest (1993; 2004) for scanline mapping. Assume $g(D)$ conforms to one of the distribution forms in Table A-1, then numerical integration techniques will then be applied to process $g(D)$ through the biases from equations (A.8)-(A.10), (A.21) to predict the form of $i(l)$. The generated $i(l)$ can then be compared with the measured semi-trace length data. Optimisation techniques can be applied to determine the form and parameters of $g(D)$ that minimise the error between the observed and simulated $i(l)$. The results have also been validated against analytical or numerical simulations. An Excel spreadsheet NADIS (Numerical Analysis of Discontinuity Size) has been developed by Priest (2004) to conduct the above analysis.

**Intensity**

Discontinuity intensity has different definitions based on the dimensions of the measurement region and the discontinuity (Dershowitz and Herda, 1992). It can be defined as number of discontinuities per unit length, area or volume (also called
density), length of traces per unit area of trace plane, area of discontinuities per unit volume, or volume of discontinuities per unit volume of rock.

Kulatilake and Wu (1984) proposed the following equation to estimate the number of trace mid points per unit area, $\lambda$, starting from number of traces per unit area, $N_t$, counted on a rectangular sampling domain:

$$\lambda = \frac{N_0 + \sum_{i=1}^{N_1}[P_1(C)]_i + \sum_{j=1}^{N_2}[P_0(C)]_j}{N_0 + N_1 + N_2} \frac{N_t}{\lambda_0}$$  \hspace{1cm} (A.26)

where $i$ and $j$ denote the discontinuities with one end and both ends censored respectively; $P_1(C)$ and $P_0(C)$ denote the probability that the trace midpoint of the discontinuities with one end and two ends censored are within the window.

$$P_1(C) = \frac{\int_{-\infty}^{2\alpha} f(x)dx}{\int_{-\infty}^{\alpha} f(x)dx}, P_0(C) = \frac{\int_{-\infty}^{\alpha} f(x)dx}{\int_{-\infty}^{\alpha} f(x)dx} + \frac{\int_{\alpha}^{2\alpha} f(x)dx}{\int_{\alpha}^{\alpha} f(x)dx}$$

where $\alpha$ is the length of the intersected trace appearing within the window and $f(x)$ is the probability density function of trace length on the infinite two-dimensional plane. The traces are assumed to be parallel to each other.

Based on the concepts (the principle of associated points) given in the paper by Parker and Cowen (1976), an unbiased trace density estimator for circular windows of radius $r$ (Mauldon, 1998) is given by:

$$\rho = \frac{N_0 - N_2}{2\pi r^2} = \frac{m}{2\pi r^2}$$  \hspace{1cm} (A.27)

where $m = N_0 - N_2$ denotes the number of trace endpoints inside the circular window.

It is important to note that Parker and Cowen have assumed that the line segment orientation on a two-dimensional plane follows a uniform distribution. It is impossible to satisfy this assumption in dealing with rock discontinuity traces appearing from a distinct discontinuity set orientation.

Mauldon (1998) for any shape window whether convex or not with an area of $A$, the unbiased density estimator:
When several circular windows are used to obtain a single density estimate, the combined circle area can be treated as a single non-convex window. The unbiased density estimator for multiple windows, valid even if the circles overlap, is

$$\rho = \frac{N + N_0 - N_2}{2A} = \frac{m}{2A}$$  \hspace{1cm} (A.28)

Mauldon et al. (1999a) proposed an unbiased fracture trace intensity estimator based on the count of trace intersections with a circular scanline:

$$I = \frac{n}{4r}$$  \hspace{1cm} (A.30)

where \(n\) is the number of intersections between fracture traces and a circular scanline of radius \(r\). This estimator is valid regardless the orientation distribution of traces. The variance of the estimator decreases as the radius of the circular scanline increases. Also, as the radius increases, the estimated value of intensity approaches the average intensity calculated according to the window mapping method.

Mauldon et al. (2001) proposed to deploy several circular scanlines to obtain an improved intensity estimate and this practice is recommended. If circles are all of the same size, \(n\) counts are determined for each circle and then averaged. If circles of different radii are used, the individual point estimates of \(n\) are weighted by their respective perimeters to determine an average intensity. Thus, if circles of radius \(r_1, r_2, \ldots, r_k\), with perimeters \(p_1, p_2, \ldots, p_k\) yield counts \(n_1, n_2, \ldots, n_k\) and intensity estimates \(I_1, I_2, \ldots, I_k\), the length weighted intensity estimate is

$$I = \frac{I_1p_1 + I_2p_2 + \cdots + I_kp_k}{p_1 + p_2 + \cdots + p_k} = \frac{2\pi(I_1r_1 + I_2r_2 + \cdots + I_kr_k)}{2\pi(r_1 + r_2 + \cdots + r_k)} = \frac{1}{4} \frac{n_1 + n_2 + \cdots + n_k}{r_1 + r_2 + \cdots + r_k}$$  \hspace{1cm} (A.31)

A 3D discontinuity intensity as a function of the mean diameter and the true linear intensity of the joint set was given by Oda (1982):

$$\rho = \frac{N + N_0 - N_2}{2A} = \frac{m}{2A}$$  \hspace{1cm} (A.28)
where $(\lambda_v)_j$ is volumetric frequency of jth joint set; $(\lambda_l)_j$ is linear frequency of jth joint set along its mean normal vector; $D$ is diameter of circular joint; $\mathbf{n}_i = \mathbf{n} \cdot \mathbf{i}$; $\mathbf{n}$ is the unit normal vector of each joint; $\mathbf{i}$ is the unit vector in the direction of the scanline.

Another relation among the 3D discontinuity intensity, areal areal intensity and mean diameter of the joint set was obtained (Kulatilake et al., 1993):

$$E(\lambda_v) = \frac{E(\lambda_a)}{E(D)E|\sin\phi|} \quad (A.33)$$

where $\lambda_v$ is volumetric joint intensity; $\lambda_a$ is the areal joint intensity; $E(\lambda_a)$ is the mean number of joint centres per unit area (Kulatilake and Wu, 1984); $D$ is the diameter of the joint; $V$ is the angle between the joint plane and the sampling plane.