Maximal linear groups induced on the Frattini quotient of a p-group

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Dedicated to the memory of our distinguished colleague L.G. (Laci) Kovács

Abstract. Let $p > 3$ be a prime. For each maximal subgroup $H \leqslant \mathrm{GL}(d,p)$ with $|H| \geqslant p^{3d+1}$, we construct a d-generator finite p-group G with the property that $\mathrm{Aut}(G)$ induces H on the Frattini quotient $G/\Phi(G)$ and $|G| \leqslant p^{\frac{d^4}{2}}$. A significant feature of this construction is that $|G|$ is very small compared to $|H|$, shedding new light upon a celebrated result of Bryant and Kovács. The groups G that we exhibit have exponent p, and of all such groups G with the desired action of H on $G/\Phi(G)$, the construction yields groups with *smallest nilpotency class*, and in most cases, the *smallest order*.

1. Introduction

The number of groups of prime power order is dauntingly large: Higman and Sims [15, 25] showed that there are as many as $p^{2m^3(1+\mathrm{O}(m^{-1/3}))/27}$ groups of order p^m. This suggests that properties of p-groups should be investigated statistically. Given a property of p-groups, one may ask: What is the *range* of possibilities? What is the *frequency* distribution? What are the *mean* and *variance*?

Some questions concerning 'ranges' were considered in the 1970's. For example, one may ask which groups can arise as the group induced by the automorphism group $\mathrm{Aut}(G)$ acting on $G/\mathrm{Z}(G)$, for a p-group G. Heineken and Liebeck [11] showed that the range is as large as possible, namely for any finite group H and any prime $p > 2$, there exists a p-group G of nilpotency class 2, and exponent p^2 such that $\mathrm{Aut}(G)$ induces H on $G/\mathrm{Z}(G)$. Later this result was generalised to $p = 2$, see [17, 28]. The group G constructed in [11] is a d-generator p-group where $d = |H|\binom{k+2}{2}$ and H is k-generated. Soules and Woldar [26] reduce the number of generators of G to $d = |H|$ when H is a sporadic simple group. These examples have $|G| > p^{|H|}$ so it is unclear whether one sees such wild behaviour in practical examples, or whether $|G|$ is always huge compared to $|H|$. Is wildness of theoretical interest only?

2010 *Mathematics Subject Classification.* 20D45, 20D15, 20B25.

A result addressing the *frequency* is due to Helleloid and Martin. They show in [**13**, Theorem 3] that the group $A(G)$ induced on $G/\Phi(G)$ by the automorphism group of some d-generator p-group G, is 'almost always' the trivial subgroup of $\mathrm{GL}(d,p)$. In light of this result, a natural question about *ranges* is: Which subgroups $H \leqslant \mathrm{GL}(d,p)$ are conjugate[1] to $A(G)$, for some d-generator p-group G? Thus groups for which $A(G)$ is non-trivial are rare. However, Bryant and Kovács [**4**] prove a striking result: given *any* $H \leqslant \mathrm{GL}(d,p)$ where $d > 1$, there exists a d-generator p-group G such that $\mathrm{Aut}(G)$ induces on $G/\Phi(G)$ the linear group H. An alternative proof of this celebrated result is given in [**18**, Chapter VIII, §13]. Whilst the methods of the proof of [**4**, Theorem 1] are natural, utilising the Lie ring associated to a p-group, the conclusion is not constructive: it is an existence result bounding neither $|G|$, nor the nilpotency class of G, nor the exponent of G.

Inspired by the above results, given $H \leqslant \mathrm{GL}(d,p)$, we ask: Is it possible to find relatively small groups G (compared to $|H|$) satisfying $A(G) = H$? For certain classes of H, of order at least p^{3d+1}, we construct a d-generator finite p-group G with the property that $A(G) = H$ and $|G| \leqslant p^{\frac{d^4}{2}}$. Thus, our construction shows that 'small' p-groups G with $A(G) = H$ do in fact occur. Our methods for constructing G from H involve representation theory; our constructions are geometric, and we believe, also very natural. We hope that they contribute to a deeper understanding of automorphism groups of p-groups and their construction, as even the very efficient algorithms [**7**] to compute $\mathrm{Aut}(G)$ struggle when G is large, for example, when G is one of the groups we construct in Table 6.1. For more information on automorphism groups of p-groups, we refer the reader to the survey of Helleloid [**12**].

To state our main result we require the following definition. The *lower exponent-p central series*[2] for a group X is defined inductively by $X_0 = X$, $X_k = [X, X_{k-1}]X_{k-1}^p$ for $k \geqslant 1$. The smallest integer n for which $X_n = \{1\}$ (when it exists) is called the *lower p-length* of X, and we write $n_p(X) = n$. If X is a group of exponent p, the lower p-length of X is equal to the *nilpotency class* of X (or *class* for short). With our numbering convention ($X_0 = X$), we have $[X_i, X_j] \leqslant X_{i+j+1}$ for all $i, j \geqslant 0$. We alert the reader that for some authors X_i denotes the $(i+1)$st term of the lower central series for X.

THEOREM 1. *Let $p > 3$ be a prime, and let $d > 1$ be an integer. Suppose that H is a maximal subgroup of $\mathrm{GL}(d,p)$ with $\mathrm{SL}(d,p) \not\leqslant H$ and that $|H| \geqslant p^{3d+1}$. Then there exists a d-generator p-group G of exponent p, class at most 4, order at most $p^{\frac{d^4}{2}}$ and such that $\mathrm{Aut}(G)$ induces H on the Frattini quotient $G/\Phi(G)$. The nilpotency class, order and structure of G is given in Table 6.1.*

1.1. Strategy and outline of the paper. We address the problem: given $H \leqslant \mathrm{GL}(d,p)$ find G such that $A(G) = H$. To ensure that G is interesting, we choose H to be a

[1] After a basis has been chosen for $G/\Phi(G)$, we may regard $A(G)$ as a subgroup of $\mathrm{GL}(d,p)$, we thus speak of conjugacy to mean 'up to change of basis'. We will write $A(G) = H$ to mean a basis may be chosen to effect this equality.

[2] Properties of this series are given in Huppert and Blackburn [**19**, 16, Chapter VIII]. However, their definition differs from ours as it starts with $X_1 = X$.

maximal subgroup of $\mathrm{GL}(d,p)$, and insist that $|G|$ is *minimal* subject to having exponent p (cf. Remark 6.2). To avoid trivialities, we assume that $p > 2$ (as 2-groups of exponent 2 are elementary abelian). In Section 4 we summarise the maximal subgroups of $\mathrm{GL}(d,p)$ that we consider and explain the notation in Columns 1-4 of Table 6.1.

Our strategy for constructing G is to examine the freest d-generator group B of exponent p. The quotient $\Gamma(d,p,n) = B/B_n$ (the quotient of B by the nth term of its lower central series) is the universal d-generator p-group of exponent p and class n. Our results depend critically on a practical description of $\Gamma(d,p,n)$. In §2 we describe $\Gamma(d,p,n)$ using a new data structure which we call *Lie n-tuples*. The problem of constructing our desired group G is reduced in §2 to determining the H-submodule structure of a certain Lie power L^nV of the natural H-module V, see Theorem 2.2. In §3 we consider the irreducible submodules of Lie powers, keeping the prerequisites to a minimum. Aschbacher's classes $\mathcal{C}_i$ of maximal subgroups H of $\mathrm{GL}(d,p)$ are listed in §4 before we determine class-by-class the H-submodule structure of L^nV in §5. The proof of Theorem 1 appears in §6, and we conclude in §7 with some open questions and directions for future research.

Notation. Throughout the paper, V will denote a vector space of dimension d over a (possibly infinite) field $\mathbb{F}$. The precedence of the operators[3] A^n, S^n, T^n is greater than $\otimes$ which is greater than $\oplus$. For example, $A^nU \otimes V \oplus W$ means $((A^nU)\otimes V)\oplus W$.

2. Universal groups of exponent p

We fix integers d and n and a prime p. In this section, we discuss the universal group in the category of finite d-generator p-groups of class n and exponent p. First, we approach this group from an abstract point of view, and later realise this group concretely. We set the following notation:

- $F(d)$, the free group of rank d,
- $B(d,p) = F(d)/F(d)^p$, the relatively free group of rank d and exponent p,
- $\Gamma(d,p,n) = B(d,p)/B(d,p)_n$, the relatively free group of rank d, exponent p and class n.

Note that the group $\Gamma(d,p,n)$ is finite, having bounded rank, exponent and class. Moreover, $\Gamma(d,p,n)$ is universal, in the sense that each finite p-group of rank d, exponent p and class n is an image of $\Gamma(d,p,n)$. An explicit formula for the order for $\Gamma(d,p,n)$ was given by Witt; to describe this formula we require some additional knowledge of Lie rings.

Higman describes in [**14**] how to associate a graded Lie ring $L_{(N_i)}$ to a normal series $G = N_1 \trianglerighteqslant N_2 \trianglerighteqslant \cdots$ for a group G provided $[N_i, N_j] \leqslant N_{i+j}$ and $\bigcap_{i=1}^{\infty} N_i = \{1\}$ hold. The N_i/N_{i+1} are abelian as $[N_i,N_i] \leqslant N_{2i} \leqslant N_{i+1}$. We view the N_i/N_{i+1} as *additive* groups, and then form the abelian group $L_{(N_i)} = \bigoplus_{i=1}^{\infty} N_i/N_{i+1}$. The following multiplication rule $(g_iN_i)(g_jN_j) = [g_i,g_j]N_{i+j}$ turns $L_{(N_i)}$ into a graded Lie ring. The Hall-Witt identity for G (see [**27**]) gives rise to the Jacobi identity for $L_{(N_i)}$. The sections N_i/N_{i+1} are called *homogeneous components* of the Lie ring.

[3] The nth alternating, symmetric and tensor powers of V are denoted A^nV, S^nV and T^nV, respectively.

Returning now to $F(d)$ and $B(d,p)$, both the lower central series[4] of $F(d)$ (taking $N_i = \gamma_i(F(d))$) and the lower exponent-p central series of $B(d,p)$ (taking $N_i = B(d,p)_{i-1}$) satisfy the conditions $[N_i, N_j] \leqslant N_{i+j}$ and $\bigcap_{i=1}^{\infty} N_i = \{1\}$. This gives two related Lie rings which we denote simply by $\mathcal{L}$ and L:

$$(1) \qquad \mathcal{L} := L_{(\gamma_i(F))} = \bigoplus_{i=1}^{\infty} \mathcal{L}^i \quad \text{and} \quad L := L_{(B_{i-1})} = \bigoplus_{i=1}^{\infty} L^i,$$

where $\mathcal{L}^k$ and L^k are the kth homogeneous components of $\mathcal{L}$ and L, respectively.

It turns out that $\mathcal{L}^k$ is a free abelian group, and L^k is a vector space over the prime field $\mathbb{F}_p$. Witt [**31**, Satz 3] gave formulas for the rank $f(d,k)$ of $\mathcal{L}^k$, and dimension $f_p(d,k)$ of L^k. Indeed,

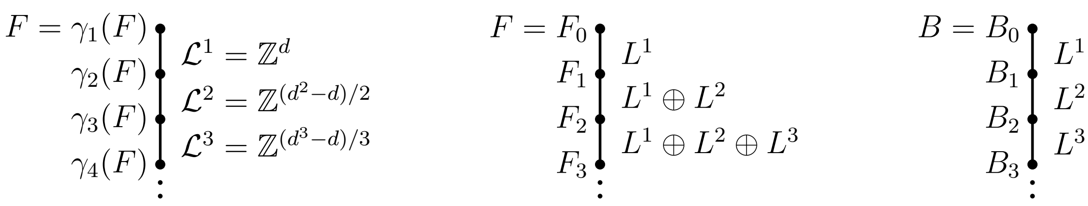
$$(2) \qquad \mathcal{L}^k \cong \mathbb{Z}^{f(d,k)} \quad \text{where} \quad f(d,k) = \frac{1}{k}\sum_{i|k} \mu(i) d^{\frac{k}{i}},$$

and μ is the number theoretic Möbius function. Also, by [**32**, p.209 (6p)], we have

$$(3) \qquad L^k \cong (\mathbb{F}_p)^{f_p(d,k)} \quad \text{where} \quad f_p(d,k) = \frac{1}{k}\sum_{i|k} \mu(i_0)\varphi(p^h) d^{\frac{k}{i}} \quad (i = i_0 p^h, p \nmid i_0),$$

and φ is Euler's totient function. Note that $f(d,k) = f_p(d,k)$ if $p > k$, and $F_{k-1}/F_k = \bigoplus_{i=1}^{k} L^i$ by [**13**, Theorem 16]. This is illustrated in Figure 1.

FIGURE 1. The lower central series of $F := F(d)$ and the lower exponent-p central series for F and $B := B(d,p)$. The Lie algebras $\mathcal{L}$ and L have the sections in the first and third chains.



Below we summarise the above discussion.

LEMMA 2.1. *Suppose that $p > n$ and $1 \leqslant k \leqslant n$. Then we have*

$$|\gamma_k(\Gamma(d,p,n))/\gamma_{k+1}(\Gamma(d,p,n))| = p^{f(d,k)} \quad \text{where } f(d,k) = \frac{1}{k}\sum_{i|k} \mu(i) d^{\frac{k}{i}}.$$

Next we turn to the automorphism group of $\Gamma(d,p,n)$, and of certain quotients.

THEOREM 2.2. *Let $B = B(d,p)$. If $B_n \leqslant M < B_{n-1}$ and $G = B/M$, then $A(G) = K$ where $K = \mathrm{N}_{\mathrm{GL}(d,p)}(M/B_n)$, i.e., the group $A(G)$ of automorphisms induced by* Aut(G) *on $G/\Phi(G)$ is K. Furthermore, the nilpotency class of G is n.*

[4]The *lower central series* of a group X is defined by $\gamma_1(X) := X$ and $\gamma_{i+1}(X) := [\gamma_i(X), X]$ for $i \geqslant 1$.

PROOF. First, $B_n \leqslant M < B_{n-1}$ implies $n_p(G) = n$ as $G_{n-1} = B_{n-1}/M$ is non-trivial. Second, the proof relies on the fact that $A(B) \cong \mathrm{GL}(d,p)$ induces a well-defined action on the elementary abelian p-groups B_{n-1}/B_n, see [18, Chapter VIII, Lemma 13.3 and Theorem 13.4] and [13, §2.2]. For the remainder of the proof, see [13, Theorem 13]. ☐

In order to apply Theorem 2.2, it is useful to have a more explicit description of $\Gamma(d,p,n)$. Construction 2.3 below achieves this and it relates the action of automorphisms to linear actions in an explicit way. Let $V = \mathbb{F}^d$ be a d-dimensional module over a field of characteristic p. View V as a $\mathrm{GL}(V)$-module, and consider the tensor algebra $T(V) = \bigoplus_{n\geqslant 0} T^n V$ where each $T^n V = V^{\otimes n}$ is a $\mathrm{GL}(V)$-module. For $u, v \in T(V)$ define

$$[u,v] := u \otimes v - v \otimes u, \tag{4}$$

and let $L(V)$ be the closure of V under this bracket operation. Then $L(V) = \bigoplus_{n\geqslant 1} L^n V$ is a free Lie $\mathbb{F}$-algebra by Witt's Theorem, where $L^n V := T^n V \cap L(V)$ is called the n-th *Lie power* of V, see [4, 20]. Note that $L^1 V = V = T^1 V$ and $[L^i V, L^j V] \subseteq L^{i+j} V$ for $i, j \geqslant 1$.

CONSTRUCTION 2.3 (Lie n-tuples). Let V be a d-dimensional vector space over a field $\mathbb{F}$ of characteristic p, and assume that $p > n$. We set

$$\Gamma_n(V) := \prod_{i=1}^{n} L^i V.$$

We write typical elements of $\Gamma_n(V)$ as $g_n = (v_1, \dots, v_n)$, $g'_n = (v'_1, \dots, v'_n)$ and $g''_n = (v''_1, \dots, v''_n)$ where $v_i, v'_i, v''_i \in L^i V$. A binary operation $g_n g'_n = g''_n$ on $\Gamma_n(V)$ is a rule for writing the v''_k in terms of the v'_j and v_i.

The operation for $\Gamma_1(V) = V$ is addition. For $n = 2, 3, 4$ it is defined as follows:

$$g_2 g'_2 = (v_1 + v'_1, v_2 + v'_2 + [v_1, v'_1]), \tag{5}$$

$$g_3 g'_3 = (v_1 + v'_1, v_2 + v'_2 + [v_1, v'_1], v_3 + v'_3 + 3[v_1, v'_2] + 3[v_2, v'_1] + [v_1, v'_1, v'_1 - v_1]), \tag{6}$$

$$\begin{aligned} g_4 g'_4 = (&v_1 + v'_1, v_2 + v'_2 + [v_1, v'_1], v_3 + v'_3 + 3[v_1, v'_2] + 3[v_2, v'_1] + [v_1, v'_1, v'_1 - v_1], \\ &v_4 + v'_4 + [v_1, v'_3] + 3[v_2, v'_2] + [v_3, v'_1] \\ &\quad + [v_2, v'_1, v'_1 - v_1] + [v_1, v'_2, v'_1 - v_1] + [v_1, v'_1, v'_2 - v_2] - [v_1, v'_1, v_1, v'_1]). \end{aligned} \tag{7}$$

where for notational convenience, left-normed Lie brackets such as $[[[v,v'],v''],v''']$ are abbreviated by $[v,v',v'',v''']$. ♢

REMARK 2.4. When $n < p$, the Lazard correspondence applied to the finite nilpotent Lie ring $L(V)/\bigoplus_{i>n} L^i V$ of class n gives a group of the same order and class which turns out to be isomorphic to our p-group $\Gamma_n(V)$ when $n \leqslant 4$. This observation allows us to deduce a multiplication rule for $\Gamma_n(V)$ for $n > 4$ from the Baker-Campbell-Hausdorff formula (see [5] for a nice overview). The rules (5)–(7) above allow us to do practical computations with the automorphism group of $\Gamma_n(V)$, as will become apparent below. For example, we identify the Lie elements $x = x_1 + \frac{1}{2}x_2 + \frac{1}{12}x_3 + \frac{1}{24}x_4$ and $y = y_1 + \frac{1}{2}y_2 + \frac{1}{12}y_3 + \frac{1}{24}y_4$ with the group elements (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) where $x_i, y_i \in L^i(V)$ and then we substitute x, y into the left-normed BCH formula for $z(x,y)$ where $e^x e^y = e^{z(x,y)}$ (*cf.* [5, p. 432]):

$$z(x,y) = x + y + \frac{1}{2}[x,y] - \frac{1}{12}[x,y,x] + \frac{1}{12}[x,y,y] - \frac{1}{24}[x,y,x,y] + \cdots .$$

Expressing the answer in the form $z = z_1 + \frac{1}{2}z_2 + \frac{1}{12}z_3 + \frac{1}{24}z_4$ by expanding modulo $\bigoplus_{i>4} L^i V$ gives the rule (7).

Theorem 2.5. *Let $V = \mathbb{F}^d$ be a d-dimensional space over $\mathbb{F}$. Then*

(i) *$\Gamma_2(V)$ is a group of order $|\mathbb{F}|^{d(d+1)/2}$, and class 2 when* $\mathrm{char}(\mathbb{F}) \neq 2$.
(ii) *$\Gamma_3(V)$ is a group of order $|\mathbb{F}|^{d(d+1)(2d+1)/6}$, and class 3 when* $\mathrm{char}(\mathbb{F}) \neq 2, 3$.
(iii) *$\Gamma_4(V)$ is a group of order $|\mathbb{F}|^{d(d+1)(3d^2+d+2)/12}$, and class 4 when* $\mathrm{char}(\mathbb{F}) \neq 2, 3$.
(iv) *If $|\mathbb{F}| = p$, $p > n$ and $n \leqslant 4$, then*

$$\Gamma(d,p,n) \cong \Gamma_n(\mathbb{F}_p^d).$$

In particular, $\Gamma_n(\mathbb{F}_p^d)$ has exponent p and class n.
(v) *For $n \leqslant 4$ there is a monomorphism $\alpha : \mathrm{GL}(V) \to \mathrm{Aut}(\Gamma_n(V))$ defined by $\alpha : g \mapsto \alpha_g$, where α_g is as follows:*

$$(v_1, \dots, v_n)\alpha_g = (v_1 g, \dots, v_n g).$$

(vi) *Suppose $p > n$, $n \leqslant 4$, and $V = \mathbb{F}_p^d$, then*

$$\mathrm{Aut}(\Gamma_n(V)) = K \rtimes \mathrm{GL}(V),$$

where K is the kernel of the action of $\mathrm{Aut}(\Gamma_n(V))$ on the quotient $\Gamma_n(V)/\Phi(\Gamma_n(V))$.

Proof. (i)–(iii) The associative law $(g_n g_n')g_n'' = g_n(g_n' g_n'')$ follows from the Lazard correspondence when $\mathrm{char}(\mathbb{F}) > n$. It is noteworthy that associativity holds even when $\mathrm{char}(F) \leqslant n$. It holds for $n = 1$ because $(v_1 + v_1') + v_1'' = v_1 + (v_1' + v_1'')$, and it holds for $n = 2$ because $[\ ,\ ]$ is biadditive. Verifying associativity for $n = 3, 4$ involves complicated (though technically simple) calculations. For this reason we delegated the task to a Magma [3] computer program whose source can be found at [10]. The identity element is easily seen to be the all zeroes vector, written $1 = (0, \dots, 0)$, and the inverse of g_n is $g_n^{-1} = (-v_1, \dots, -v_n)$. This follows because $[v, 0] = [0, v] = [v, -v] = 0$. Hence $\Gamma_n(V)$ is a group for $n \leqslant 4$ and all vector spaces $V = \mathbb{F}^d$.

Properties of these groups depend on the characteristic of the field $\mathbb{F}$. For example, it is easy to see by induction on k that $g_n^k = (kv_1, \dots, kv_n)$ for $k \in \mathbb{Z}$. Hence $\Gamma_n(V)$ has exponent p if $\mathrm{char}(\mathbb{F}) = p > 0$, and is torsion-free otherwise. The following commutator calculations are too long for most humans (when $n = 3, 4$) and were done by the Magma [3] computer programs in [10]:

$$[g_2, g_2'] = (0, 2[v_1, v_1']), \tag{8}$$

$$[g_3, g_3', g_3''] = (0, 0, 12[v_1, v_1', v_1'']), \tag{9}$$

$$[g_4, g_4', g_4'', g_4'''] = (0, 0, 0, 24[v_1, v_1', v_1'', v_1''']), \tag{10}$$

where for notational convenience, left-normed group commutators such as $[[[g, g'], g''], g''']$ are abbreviated by $[g, g', g'', g''']$.

The order of $\Gamma_n(V)$ is $\prod_{i=1}^{n} |L^iV|$ and $|L^iV| = |\mathbb{F}|^{f(d,i)}$ by [32] where $f(d,i)$ is given by (2). Moreover, it follows from (8), (9), (10) that $\Gamma_n(V)$ has class n if $\mathrm{char}(\mathbb{F}) \notin \{2, \ldots, n\}$. This proves parts (i)–(iii).

(iv) Suppose now that $\mathbb{F} = \mathbb{F}_p$, and consider part (iv) for $n \leqslant 4$. As $p > n$, Lemma 2.1 shows that $n_p(\Gamma(d,p,n)) = n$ and $|\Gamma(d,p,n)| = \prod_{i=1}^{n} |L^iV|$. Thus it follows that $\Gamma(d,p,n) \cong \Gamma_n(\mathbb{F}_p^d)$, as desired.

(v) Each $g \in \mathrm{GL}(V)$ induces an action on L^nV. A significant advantage of the definitions (5), (6), (7) is that the map α_g is easily verified to be an endomorphism of $\Gamma_n(V)$. In fact, α_g is an automorphism with inverse $\alpha_{g^{-1}}$. Thus the map $\alpha\colon \mathrm{GL}(V) \to \mathrm{Aut}(\Gamma_n(V))$ with $\alpha(g) = \alpha_g$ is a monomorphism.

(vi) The action of $\mathrm{Aut}(\Gamma_n(V))$ on the Frattini quotient $\Gamma_n(V)/\Phi(\Gamma_n(V)) \cong V$ induces a homomorphism $\mathrm{Aut}(\Gamma_n(V)) \to \mathrm{GL}(V)$, which is surjective by part (v). We have now shown that $\mathrm{GL}(V)$ is a subgroup (and a quotient group) of $\mathrm{Aut}(\Gamma_n(V))$. Hence $\mathrm{Aut}(\Gamma_n(V))$ splits as $\mathrm{Aut}(\Gamma_n(V)) = K \rtimes \mathrm{GL}(d,p)$ for $n \leqslant 4$, with K as in the statement above. In fact, K is a normal p-subgroup of $\mathrm{Aut}(\Gamma_n)$ by a theorem of Hall. □

Remark 2.6. The constants appearing in the commutator relations given in (8), (9) and (10) are denominators appearing in the Baker-Campbell-Hausdorff formula. The connection is related to the Lazard correspondence as explained in Remark 2.4.

Remark 2.7. One may guess that rules (5)–(7) for multiplying Lie n-tuples do no more than encode a pc-presentation[5] for $\Gamma_n(V)$. This turns out *not* to be the case. For example, consider a special group $\Gamma_2(V) = G$ of order $p^{\binom{m}{1}+\binom{m}{2}}$ and exponent $p > 2$ where $V = (\mathbb{F}_p)^m$. Let G have generators g_i, $1 \leqslant i \leqslant m$, and $h_{k,j}$, $1 \leqslant j < k \leqslant m$, and define a pc-presentation for G by $g_i^p = h_{k,j}^p = 1$, and $g_j^{g_k} = g_j h_{k,j}$ for $1 \leqslant j < k \leqslant m$. This pc-presentation gives rise to the symbolic multiplication rule

$$\left(\prod_{i=1}^{m} g_i^{x_i} \prod_{j<k} h_{k,j}^{y_{k,j}}\right)\left(\prod_{i=1}^{m} g_i^{x_i'} \prod_{j<k} h_{k,j}^{y_{k,j}'}\right) = \prod_{i=1}^{m} g_i^{x_i+x_i'} \prod_{j<k} h_{k,j}^{y_{k,j}+y_{k,j}'+x_j x_k'}. \tag{11}$$

Indeed when $m = 1$, *every* pc-presentation for G (with different composition series or transversals) has the same rule. It is much easier to prove that $\mathrm{GL}(V)$ is a subgroup of $\mathrm{Aut}(G)$ using the more geometric 'Lie' rule (5), than using (11). We return to this point in Remark 5.7.

3. Some representation theory

Bryant and Kovács proved [4, Theorem 1] by considering regular submodules of a certain sum of Lie powers [4, Theorem 2]. In this section, we consider the relevant Lie representation theory for our results. A good introduction to this topic is [20]. As noted

[5]The abbreviation 'pc' stands for 'power-conjugate', 'power-commutator' or 'polycyclic', see [16].

in §2, the action of $\mathrm{GL}(V)$ on V induces an action on the tensor algebra $T(V)$, and on $L(V)$ (which is a subset of T^nV containing V, closed under the Lie bracket $[\ ,\ ]$).

Our aim in this section is to describe the $\mathrm{GL}(V)$-modules L^iV for $1 \leqslant i \leqslant 4$ and to show that they are irreducible. We note that the representation theory of $\mathrm{GL}(V)$ on T^nV is known when $\mathrm{char}(\mathbb{F}) = 0$ (see [8]) and the irreducible $\mathrm{GL}(V)$-modules are described by the representation theory of the symmetric group S_n of degree n. We require the analogous results when $\mathbb{F}$ is a finite field and $\mathrm{char}(\mathbb{F}) > n$, which we have been unable to locate in the literature.

The action of $g \in \mathrm{GL}(V)$ on the nth tensor power $T^nV = V^{\otimes n}$ is

$$(v_1 \otimes \cdots \otimes v_n)g = (v_1g) \otimes \cdots \otimes (v_ng) \qquad \text{where } v_1, \ldots, v_n \in V,$$

and the following action of the symmetric group of degree n commutes with that of $\mathrm{GL}(V)$:

$$(v_1 \otimes \cdots \otimes v_n)\sigma = (v_{1\sigma^{-1}}) \otimes \cdots \otimes (v_{n\sigma^{-1}}) \qquad \text{where } v_1, \ldots, v_n \in V, \text{ and } \sigma \in \mathsf{S}_n.$$

Suppose now that $\mathrm{char}(\mathbb{F}) \notin \{2, \ldots, n\}$ so that S_n acts completely reducibly on T^nV. There exist primitive central orthogonal idempotents[6] $e_1, \ldots, e_r \in \mathbb{F}\mathsf{S}_n$ which satisfy

$$T^nV = \bigoplus_{i=1}^{r} (T^nV)e_i. \tag{12}$$

Since the actions of $\mathrm{GL}(V)$ and S_n commute, this is a $\mathrm{GL}(V)$-invariant decomposition of T^nV. The primitive idempotents

$$e_1 = \frac{1}{n!}\left(\sum_{\sigma \in \mathsf{S}_n} \sigma\right) \quad \text{and} \quad e_2 = \frac{1}{n!}\left(\sum_{\sigma \in \mathsf{S}_n} \mathrm{sgn}(\sigma)\sigma\right)$$

give rise to the *symmetric* and *alternating* powers S^nV and A^nV, respectively. For vectors $v_1, \ldots, v_n \in V$ we define

$$v_1 \odot \cdots \odot v_n = n!(v_1 \otimes \cdots \otimes v_n)e_1 \quad \text{and} \quad v_1 \wedge \cdots \wedge v_n = n!(v_1 \otimes \cdots \otimes v_n)e_2.$$

The symmetric and alternating powers are spanned by vectors of the form $v_1 \odot \cdots \odot v_n$ and $v_1 \wedge \cdots \wedge v_n$ respectively, and their dimensions are

$$\dim(S^nV) = \binom{d+n-1}{n} \quad \text{and} \quad \dim(A^nV) = \binom{d}{n}. \tag{13}$$

For the case $n = 2$, this gives

$$T^2V = V \otimes V = A^2V \oplus S^2V \qquad \text{if } \mathrm{char}(\mathbb{F}) \neq 2. \tag{14}$$

We now relate L^nV for $n \leqslant 3$, to more familiar modules. We have $L^1V = V$ and $L^2V = A^2V$ because $v_1 \wedge v_2 = [v_1, v_2]$ (see (4)). We warn the reader that $[v_1, v_2, v_3] \neq v_1 \wedge v_2 \wedge v_3$; the left hand side term has four summands while the right hand side term has six summands.

LEMMA 3.1. *Suppose that* $\mathrm{char}(\mathbb{F}) \neq 2, 3$. *The following hold.*

[6] This means $\sum_{i=1}^{r} e_i = 1$, $e_i^2 = e_i \in \mathsf{Z}(\mathbb{F}\mathsf{S}_n)$ for $1 \leqslant i \leqslant r$, and $e_ie_j = 0$ for $1 \leqslant i < j \leqslant r$.

(i) *If* $d \geqslant 3$, *then* $L^3V = X_1 \oplus X_2$ *is a sum of irreducible* H*-modules, where* H *is the group* $\mathrm{GL}(1,\mathbb{F}) \wr \mathsf{S}_d$ *of all monomial matrices, and* $\dim(X_1) = 2\binom{d}{2}$ *and* $\dim(X_2) = 2\binom{d}{3}$.

(ii) *If* $d > 1$, *then* L^3V *is an irreducible* $\mathrm{GL}(V)$*-module.*

(iii) *There are isomorphisms* $A^2V \otimes V \cong L^3V \oplus A^3V$, *and* $S^2V \otimes V \cong S^3V \oplus L^3V$ *of* $\mathrm{GL}(V)$*-modules. Hence* $T^3V \cong S^3V \oplus L^3V \oplus L^3V \oplus A^3V$.

Proof. (i) Suppose that H preserves a decomposition $V = V_1 \oplus \cdots \oplus V_d$ where the 1-dimensional subspaces $V_i = \langle v_i \rangle$ are permuted transitively. Let $K := G_1 \times \cdots \times G_r$ be the base group of $H = \mathrm{GL}(V_1) \wr \mathsf{S}_d$ where $G_i = \mathrm{GL}(V_i)$. For i, j, k there are three possibilities for the dimension of $V_i + V_j + V_k$, depending on the cardinality of the set $\{i, j, k\}$. For $A, B \subseteq V$ let $[A, B] := \langle [a, b] \mid a \in A, b \in B \rangle$. Then $A^2V \otimes V$ has two obvious H-submodules:

$$W_1 = \sum_{i<j} [V_i, V_j] \otimes (V_i + V_j), \qquad \text{and} \qquad W_2 = \sum_{k \notin \{i,j\}} [V_i, V_j] \otimes V_k.$$

It is clear that $A^2V \otimes V = W_1 + W_2$. Since

$$\dim(W_1) + \dim(W_2) \leqslant 2\binom{d}{2} + d\binom{d-1}{2} = d\binom{d}{2} = \dim(A^2V \otimes V),$$

the inequality above is an equality and $A^2V \otimes V = W_1 \oplus W_2$ is an H-module decomposition. Now

$$v_1 \wedge v_2 \wedge v_3 = [v_1, v_2] \otimes v_3 + [v_2, v_3] \otimes v_1 + [v_3, v_1] \otimes v_2, \tag{15}$$

so W_2 contains A^3V and $\dim(W_2/A^3V) = d\binom{d-1}{2} - \binom{d}{3} = 2\binom{d}{3} > 0$ since $d \geqslant 3$. We claim that W_1 and W_2/A^3V are irreducible H-modules.

We may write each 2-dimensional subspace $[V_i, V_j] \otimes (V_i + V_j)$ of W_1 as the sum of two 1-dimensional K-invariant subspaces, which are isomorphic to $[V_i, V_j] \otimes V_i$ and $[V_i, V_j] \otimes V_j$ respectively. Hence W_1 can be written as the sum of $2\binom{d}{2}$ 1-dimensional subspaces that are pairwise non-isomorphic as K-modules. As these are permuted transitively by H, we find that W_1 is an irreducible H-module.

For W_2, let Δ be the set of 3-subsets of $\{1, \ldots, d\}$. For each $\delta = \{i, j, k\}$ in Δ define

$$U_\delta := [V_i, V_j] \otimes V_k + [V_j, V_k] \otimes V_i + [V_k, V_i] \otimes V_j.$$

Then $W_2 = \bigoplus_{\delta \in \Delta} U_\delta$. The diagonal matrix $t = (\alpha_1, \ldots, \alpha_d) \in K$ acts on the 3-dimensional space U_δ as the scalar matrix $\alpha_i \alpha_j \alpha_k I$. Hence, if $\delta \neq \delta'$, then U_δ and $U_{\delta'}$ are non-isomorphic K-modules. Let $M \cong \mathsf{S}_3$ be the setwise stabiliser of δ. As $M \leqslant \mathsf{S}_d \leqslant H$, we may view U_δ as an M-module. Since $p > 3$, U_δ is a sum of 1- and 2-dimensional irreducible M-submodules. By (15) the 1-dimensional submodule is

$$A^3V \cap U_\delta = \langle v_i \wedge v_j \wedge v_k \rangle = \langle\, [v_i, v_j] \otimes v_k + [v_j, v_k] \otimes v_i + [v_k, v_i] \otimes v_j \,\rangle.$$

Now A^3V is the direct sum of $\binom{d}{3}$ pairwise non-isomorphic 1-dimensional K-submodules, one for each 3-set $\{i, j, k\} \in \Delta$. These K-submodules are permuted transitively by S_d, and so A^3V is an irreducible H-module. Now suppose that N is an H-submodule where $A^3V < N \leqslant W_2$. Choose $x \in N \setminus A^3V$ and write $x = \sum_{\delta \in \Delta} u_\delta$ where $u_\delta \in U_\delta$. Then there exists $\delta \in \Delta$ for which $u_\delta \notin A^3V$. In order to prove that $N = W_2$ it suffices to show that $U_\delta \leqslant N$, as S_d is transitive on Δ.

We claim that $u_\delta \in N$. Assuming the claim is true, then the M-submodule $U_\delta \cap N$ satisfies $U_\delta \cap A^3V < U_\delta \cap N \leqslant U_\delta$ and by the above remarks, the only M-submodule of U_δ properly containing the 1-dimensional submodule $U_\delta \cap A^3V$ is U_δ itself. Hence $U_\delta \leqslant N$ and $N = W_2$.

We now prove the claim. Because S_d is transitive on Δ, we may assume that $\delta = \{1, 2, 3\}$. Let

$$a := (-1, 1, 1, \ldots, 1), \quad b := (-1, -1, 1, \ldots, 1), \quad c := (-1, -1, -1, 1, \ldots, 1)$$

be elements of K. For $\delta' \in \Delta$, observe that if $1 \in \delta'$, then $u_{\delta'}a = -u_{\delta'}$ and if $1 \notin \delta'$, then $u_{\delta'}a = u_{\delta'}$. Thus $y := \frac{1}{2}(x - xa) = \sum_{\delta' \in \Delta, 1 \in \delta'} u_{\delta'}$, and $y \in N$. Now observe that if $1 \in \delta'$ and $2 \notin \delta'$ then $u_{\delta'}b = -u_{\delta'}$, and if $\{1, 2\} \subset \delta'$ then $u_{\delta'}b = u_{\delta'}$. Setting $z := \frac{1}{2}(y + yb)$, we have $z = \sum_{\delta' \in \Delta, \{1,2\} \subset \delta'} u_{\delta'}$ and $z \in N$. Now for all δ' such that $\{1, 2\} \subset \delta'$ we have $u_{\delta'}c = u_{\delta'}$ unless $\delta' = \{1, 2, 3\}$. Hence we obtain $u_{\{1,2,3\}} = \frac{1}{2}(z - zc)$. Thus $u_{\{1,2,3\}} \in N \backslash A^3V$, as desired. In summary, we have shown that the only H-submodule of W_2 properly containing A^3V is W_2 itself. Hence W_2/A^3V is indeed irreducible as an H-module. Thus $L^3V = (A^2V \otimes V)/A^3V = X_1 \oplus X_2$, where $X_1 \cong W_1$ and $X_2 \cong W_2/A^3V$ are irreducible.

(ii) When $d = 2$, part (i) shows that $L^3V = X_1$ is an irreducible H-module, and hence an irreducible $\mathrm{GL}(V)$-module. When $d \geqslant 3$, there is a non-monomial matrix in $\mathrm{GL}(V)$ which maps a non-zero element of X_1 into X_2. This proves that L^3V is an irreducible $\mathrm{GL}(V)$-module.

(iii) The map $\phi\colon A^2V \otimes V \to L^3V$ given by $\phi([u, v] \otimes w) = [[u, v], w]$ is a (well-defined) $\mathrm{GL}(V)$-module homomorphism. Furthermore, it follows from (15) and the Jacobi identity in L^3V that $A^3V \leqslant \ker(\phi)$. It is clear that ϕ is surjective. We observe that

$$\dim\left(\frac{A^2V \otimes V}{A^3V}\right) = d\binom{d}{2} - \binom{d}{3} = \frac{d^3 - d}{3} = \dim(L^3V)$$

using (2), and hence $\ker(\phi) = A^3V$.

The group algebra $A := \mathbb{F}\mathsf{S}_3$ can be written as $A = Ae_1 \oplus Ae_2 \oplus Ae_3$ where e_1, e_2, e_3 are primitive central orthogonal idempotents where

$$e_1 = \frac{1}{6}\sum_{\sigma \in \mathsf{S}_3} \sigma, \quad e_2 = \frac{1}{6}\sum_{\sigma \in \mathsf{S}_3} \mathrm{sign}(\sigma)\sigma, \quad \text{and} \quad e_3 = 1 - e_1 - e_2.$$

Then $T := T^3V$ equals TA, and hence $T = T_1 \oplus T_2 \oplus T_3$, where $T_i = Te_i$. However, $T_1 = S^3V$ and $T_2 = A^3V$, and

$$T = T^2V \otimes V = (S^2V \oplus A^2V) \otimes V = (S^2V \otimes V) \oplus (A^2V \otimes V).$$

By the previous paragraph, $A^2V \otimes V$ has two composition factors: A^3V and L^3V. It follows from the equation $T_1 \oplus T_2 \oplus T_3 = (S^2V \otimes V) \oplus (A^2V \otimes V)$ that $T_3 \cap (A^2V \otimes V) = L^3V$. A similar argument shows that $(V \otimes A^2V) \cap T_3 = L^3V$. However, $A^2V \otimes V \cong V \otimes A^2V$ and

$(A^2V \otimes V) \cap (V \otimes A^2V) = A^3V$. Thus $T_3 = L^3V \oplus L^3V$ and so $S^2V \otimes V = S^3V \oplus L^3V$ holds, as desired. □

Finally, we must understand the structure of L^4V when $\dim(V) = 2$.

Lemma 3.2. *Suppose that $d = 2$ and $\mathrm{char}(\mathbb{F}) \neq 2, 3$. Then $L^4V \cong A^2V \otimes S^2V$ is an irreducible $\mathrm{GL}(V)$-module.*

Proof. Fix a basis $\{e_1, e_2\}$ for V. It is well-known that the left-normed vectors $[v_1, v_2, v_3, v_4] := [[[v_1, v_2], v_3], v_4]$ span L^4V. Indeed, $\{s_1, s_2, s_3\}$ is a basis for L^4V where

$$s_1 = [e_1, e_2, e_1, e_1], \quad s_2 = [e_1, e_2, e_1, e_2] \quad \text{and} \quad s_3 = [e_1, e_2, e_2, e_2].$$

Note that $s_2 = [e_1, e_2, e_2, e_1]$. Define the map $\phi\colon A^2V \otimes S^2V \to L^4V$ by:

$$\phi([e_1, e_2] \otimes (e_1 \odot e_1)) = s_1, \quad \phi([e_1, e_2] \otimes (e_1 \odot e_2)) = s_2, \quad \phi([e_1, e_2] \otimes (e_2 \odot e_2)) = s_3.$$

Since $s_2 = [e_1, e_2, e_2, e_1]$, ϕ is well-defined. It follows from the linearity of $[,]$ and the universality property of the exterior square, symmetric square and the tensor product, that ϕ is a linear map. Since ϕ is surjective, and the dimensions of the respective spaces are equal, we see that ϕ is an isomorphism. Moreover, a direct calculation shows that ϕ is a $\mathrm{GL}(V)$-module isomorphism. Since S^2V is irreducible as a $\mathrm{GL}(V)$-module and $\dim(A^2V) = 1$, it follows that L^4V is irreducible as a $\mathrm{GL}(V)$-module. □

4. Aschbacher's Theorem

An idea pervading Felix Klein's *Erlanger Programm* is that there is a correspondence between geometry and group theory. A group gives rise to a geometry, and 'interesting' subgroups give rise (via stabilisers) to 'interesting' geometric substructures. Our group will be $\mathrm{GL}(d, q)$, where $q = p^a$, and its 'interesting' subgroups will be its *maximal* subgroups H. A celebrated result of Aschbacher relates maximal subgroups of the classical groups to geometry. For $\mathrm{GL}(V) \cong \mathrm{GL}(d, q)$, the geometric subgroups fall into eight classes of subgroups which we now define:

- $\mathcal{C}_1$ stabilisers of proper non-zero subspaces of V;
- $\mathcal{C}_2$ stabilisers of an equidimensional direct sum decomposition $V = V_1 \oplus \cdots \oplus V_r$;
- $\mathcal{C}_3$ stabilisers of an extension field structure $\mathbb{F}_{q^r}$ where r is prime;
- $\mathcal{C}_4$ stabilisers of an unequal dimensional tensor decomposition $V = V_1 \otimes V_2$;
- $\mathcal{C}_5$ subgroups conjugate (modulo scalars) to a linear group over $\mathbb{F}_{q^{1/r}}$ where r is prime;
- $\mathcal{C}_6$ normalisers of an r-subgroup of symplectic type where $r \neq p$ is prime;
- $\mathcal{C}_7$ stabilisers of an equidimensional tensor product decomposition $V = V_1 \otimes \cdots \otimes V_r$;
- $\mathcal{C}_8$ stabilisers of non-degenerate forms on V.

The following statement of Aschbacher's Theorem follows [**21**, Theorem 1.2.1]. An alternative form of the theorem is given in [**29**, §3.10.3].

Theorem 4.1 (Aschbacher, [**1**]). *Let q be a power of p and suppose $H \leqslant \mathrm{GL}(d, q)$ and $\mathrm{SL}(d, q) \nleqslant H$. Then*

(i) *H is contained in a member of (at least one) of the classes $\mathcal{C}_1$ – $\mathcal{C}_8$, or*

(ii) *$H/\mathrm{Z}(H)$ is almost simple and H acts absolutely irreducibly on the natural module for* $\mathrm{GL}(d,q)$.

The subgroups H in Theorem 4.1 satisfying $H \notin \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_8$ are said to be of type $\mathcal{C}_9$. The size of a maximal subgroup H varies by class: the classes $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_5 \cup \mathcal{C}_8$ all contain a 'large' subgroup, that is, a subgroup H with $|H| \geqslant q^{3d+1}$. On the other hand, for $H \in \mathcal{C}_6 \cup \mathcal{C}_7$, we have $|H| < q^{3d+1}$. To understand the order of groups in the class $\mathcal{C}_9$, we use the following theorem of Liebeck.

Theorem 4.2 (Liebeck [24]). *Let T be a simple classical group with natural projective module V of dimension d over $\mathbb{F}_q$, and let X be a group such that $T \trianglelefteqslant X \leqslant \mathrm{Aut}(T)$. If H is any maximal subgroup of X, then one of the following holds:*

(i) *H is a known group (and $H \cap T$ has a well-described (projective) action on V);*
(ii) $|H| < q^{3d}$.

For $T \cong \mathrm{PSL}(d,q)$, the remarks in [24] show that the projective actions of the groups in part (i) of the theorem above are those of groups from classes $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_5 \cup \mathcal{C}_8$. Since every maximal subgroup of $\mathrm{GL}(d,q)$ not containing $\mathrm{SL}(d,q)$ must contain $\mathrm{Z}(\mathrm{GL}(d,q))$, we obtain:

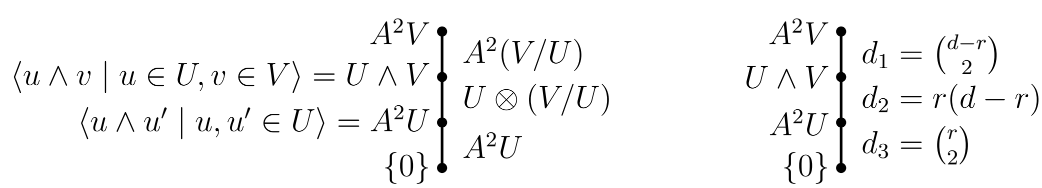
Corollary 4.3. *Let H be a maximal subgroup of* $\mathrm{GL}(d,q)$ *not containing* $\mathrm{SL}(d,q)$. *Then $H \in \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_5 \cup \mathcal{C}_8$, or $|H| < q^{3d+1}$.*

5. Representation theory of maximal subgroups on Lie powers

We now assume that $\mathrm{char}(\mathbb{F}) = p$ is an odd prime and that $\mathbb{F}$ is finite. Recall that V is a d-dimensional vector space over $\mathbb{F}$. The aim of this section is to determine the reducibility of L^2V, L^3V and L^4V (where necessary) as H-modules, for a maximal subgroup H of $\mathrm{GL}(V)$. In the cases where the modules are reducible, we also aim to determine the smallest quotient modules.

5.1. The reducible $\mathcal{C}_1$ case.

Figure 2. The $\mathrm{GL}(V)_U$ composition factors of A^2V and their respective dimensions.



Lemma 5.1. *Suppose that $H = \mathrm{GL}(V)_U \in \mathcal{C}_1$ is the stabiliser of an r-dimensional subspace U of V where $0 < r < d := \dim(V)$.*

(i) *If $d > 2$, then L^2V is a reducible H-module and the dimension of the smallest quotient module is r if $d - r = 1$, and $\binom{d-r}{2}$ otherwise.*

(ii) *If $p > 3$ and $d = 2$, then L^2V is an irreducible H-module, and L^3V is a uniserial, reducible 2-dimensional H-module.*

Proof. (i) We first show that we have a composition series for the H-module A^2V as in Figure 2. Define $\pi_1 : A^2V \to A^2(V/U)$ by $\pi_1(v \wedge w) = (v + U) \wedge (w + U)$. This map is a surjective H-module homomorphism, with kernel

$$U \wedge V := \langle u \wedge v \mid u \in U, v \in V\rangle. \tag{16}$$

Observe that A^2U is an H-invariant subspace of $U \wedge V$. We claim that $\{0\} \subseteq A^2U \subseteq U \wedge V \subseteq A^2V$ is the desired composition series. Note that $\mathrm{GL}(V/U)$ and $\mathrm{GL}(U)$ act irreducibly on $A^2(V/U)$ and A^2U respectively. We construct an H-module isomorphism $U \otimes (V/U) \cong (U \wedge V)/A^2U$ as follows. We define $\phi\colon U \otimes (V/U) \to (U \wedge V)/A^2U$ to be the linear extension of the following map:

$$u \otimes (v + U) \mapsto u \wedge v + A^2U.$$

It is straightforward to check that ϕ is well-defined, surjective and an H-module homomorphism. Comparing dimensions reveals that ϕ is an H-module isomorphism and therefore shows that $(U\wedge V)/A^2U$ is also irreducible. Hence A^2V has a composition series as depicted in Figure 2, where the factors are irreducible or zero.

To prove that A^2V is a uniserial H-module, we must show that $\{0\}, A^2U, U \wedge V, A^2V$ are the only H-submodules of A^2V (some may coincide). Using the fact that invertible matrices of the form $\left(\begin{smallmatrix} I & 0 \\ * & * \end{smallmatrix}\right)$ lie in H, fix U elementwise and are transitive on $V \setminus U$, it follows that there is no H-invariant complement to A^2U in $U \wedge V$. A similar argument shows that there is no H-invariant complement to $U \wedge V$ in A^2V. Thus A^2V is uniserial as claimed and the dimensions of the composition factors are as shown in Figure 2. Note that, since $d > 2$, there are at least two composition factors, so A^2V is reducible. If $d_1 = 0$, then $r = d - 1$ is the dimension of the smallest quotient module.

(ii) Suppose now that $d = 2$, $r = 1$ and $p > 3$. Then A^2V is an irreducible 1-dimensional H-module and $A^3V = \{0\}$. Since V is a reducible H-module, $L^3V \cong A^2V \otimes V$ is a uniserial 2-dimensional H-module with unique non-trivial submodule $A^2V \otimes U$. ☐

5.2. The imprimitive $\mathcal{C}_2$ case.

Lemma 5.2. *Suppose that $H = \mathrm{GL}(V_1) \wr \mathsf{S}_r \in \mathcal{C}_2$ fixes an equidimensional decomposition*

$$V = V_1 \oplus \cdots \oplus V_r \quad \text{where } 1 < r \leqslant d \text{ and } \mathrm{char}(\mathbb{F}) = p > 2.$$

(i) *If $1 < r < d$, then $L^2V = U_1 \oplus U_2$ where U_1 and U_2 are irreducible H-modules satisfying $0 < \frac{d}{2}(\frac{d}{r} - 1) = \dim(U_1) < \dim(U_2)$.*

(ii) *If $p > 3$ and $2 < r = d$, then H acts irreducibly on L^2V, and L^3V is a sum of two irreducible H-modules of dimensions $2\binom{d}{2}$ and $2\binom{d}{3}$.*

(iii) *If $p > 3$ and $2 = r = d$, then H acts irreducibly on L^2V and L^3V, and $L^4V \cong A^2V \otimes S^2V \cong X_1 \oplus X_2$ where $\dim(X_1) = 2$ and $\dim(X_2) = 1$.*

Proof. (i) Consider the base group $K := G_1 \times \cdots \times G_r$ of H where $G_i = \mathrm{GL}(V_i)$. For each i we identify A^2V_i with the obvious subspace of A^2V. Furthermore, for $i \neq j$ set

$V_i \wedge V_j := \langle u \wedge w \mid u \in V_i, w \in V_j\rangle$ mimicking the notation in (16). Then $V_i \wedge V_j = V_j \wedge V_i$, and we note that $V_i \wedge V_j$ is isomorphic as a K-module to $V_i \otimes V_j$ if $i \neq j$. Hence we have the following K-module decomposition:

$$A^2V = A^2\left(\bigoplus_{i=1}^{r} V_i\right) = U_1 \oplus U_2 \qquad \text{where } U_1 \cong \bigoplus_{i=1}^{r} A^2V_i \text{ and } U_2 \cong \bigoplus_{i<j} V_i \otimes V_j.$$

Observe that A^2V_i and $V_i \otimes V_j$ are irreducible K-modules. Thus $A^2V_1, \dots, A^2V_r$ are pairwise non-isomorphic K-submodules of U_1, and the $V_i \otimes V_j$ with $i < j$ are pairwise non-isomorphic K-submodules of U_2 (witnessed by the differing kernels of the action of K). However, S_r permutes these non-isomorphic K-modules transitively. It follows from Clifford's Theorem [**6**, pp. 343–344] that both U_1 and U_2 are irreducible H-modules. We have $\dim(U_1) = r\binom{d/r}{2}$, $\dim(U_2) = \binom{r}{2}\frac{d^2}{r^2}$, and $0 < \dim(U_1) < \dim(U_2)$. Hence when $r < d$ we have that A^2V is a reducible H-module.

(ii) Suppose now that $2 < r = d$. By part (i), $U_1 = \{0\}$ and $A^2V = U_2$ is an irreducible H-module. By Lemma 3.1(ii), $L^3V = X_1 \oplus X_2$ is a sum of irreducible H-submodules of dimensions $2\binom{d}{2}$ and $2\binom{d}{3}$, respectively.

(iii) Finally, consider the case that $2 = r = d$. Then $L^2V = A^2V$ is 1-dimensional and $A^3V = \{0\}$. Hence $L^3V \cong A^2V \otimes V$ is the tensor product of an irreducible H-module with a 1-dimensional module, and is therefore irreducible.

Restricting the $\mathrm{GL}(V)$-isomorphism $L^4V \cong A^2V \otimes S^2V$ in Lemma 3.2, gives an H-isomorphism. Now H is generated by matrices of the form $g = \left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right)$, where x and y are non-zero, and the action of these matrices on L^4V is understood using the map ϕ defined in the proof of Lemma 3.2. It follows that the following is an H-module decomposition:

$$A^2V \otimes S^2V \cong \langle v_1 \wedge v_2 \otimes v_1 \odot v_1, v_1 \wedge v_2 \otimes v_2 \odot v_2\rangle \oplus \langle v_1 \wedge v_2 \otimes v_1 \odot v_2\rangle.$$

These 2- and 1-dimensional H-submodules are irreducible, as desired. □

5.3. The extension field $\mathcal{C}_3$ case. We assume that $\mathbb{F} = \mathbb{F}_q$ is finite, $\mathrm{char}(\mathbb{F}) = p$ and let $\mathbb{E} = \mathbb{F}_{q^r}$ with r a prime. In this case, $\Gamma\mathrm{L}(1, \mathbb{E}/\mathbb{F})$ is a maximal subgroup of $\mathrm{GL}(r, p)$ by [**21**, Theorem 1.2.1]. The rth cyclotomic polynomial $\Phi_r(t)$ factors over $\mathbb{F}_q$ as a product of equal-degree irreducibles by [**23**, Theorem 2.47(ii), p. 61]. This common degree divides $r - 1$.

Lemma 5.3. *Let $\mathbb{E} = \mathbb{F}_{q^r}$ and $\mathbb{F} = \mathbb{F}_q$ where r is a prime and q a power of the prime p. Let V be an irreducible $\Gamma\mathrm{L}(1, \mathbb{E}/\mathbb{F})$-module over $\mathbb{F}$. Then $\dim(V)$ equals r, or divides $r - 1$. In particular, the maximum dimension of an irreducible $\Gamma\mathrm{L}(1, \mathbb{E}/\mathbb{F})$-module over $\mathbb{F}$ is r.*

Proof. Observe that $\Gamma\mathrm{L}(1, \mathbb{E}/\mathbb{F})$ is isomorphic to the metacyclic group

$$H = \langle \phi, \mu \mid \phi^r = \mu^{q^r-1} = 1, \mu^\phi = \mu^q\rangle.$$

Consider $V^{\mathbb{E}} = V \otimes_{\mathbb{F}} \mathbb{E}$ as an $\mathbb{E}M$-module where $M = \langle\mu\rangle$. As $|M| = q^r - 1$ is coprime to p, it follows that $V^{\mathbb{E}}$ is a completely reducible M-module by Maschke's Theorem. Let W be an irreducible $\mathbb{E}M$-submodule of $V^{\mathbb{E}}$. Thus $\dim_{\mathbb{E}}(W) = 1$, as $\mathbb{E}$ is a splitting field for M. Hence μ acts as a non-zero scalar, $\lambda(\mu) \in \mathbb{E}$ say on W.

CASE: $\lambda(\mu) = \lambda(\mu^q)$. Since $\lambda(\mu) = \lambda(\mu)^q$, we have $\lambda(\mu) \in \mathbb{F}$. Then $\langle\mu\rangle$ acts on V as the matrix $\lambda(\mu)I$. It follows that V is an irreducible $\mathbb{F}H$-module if and only if V is an irreducible $\langle\phi\rangle$-module. Thus, by the remarks preceding this lemma, $\dim(V)$ divides $r-1$.

CASE: $\lambda(\mu) \neq \lambda(\mu^q)$. Let $U = \bigoplus_{i=0}^{r-1} W^{\phi^i}$. Note that W is not isomorphic to W^ϕ as an $\mathbb{E}M$-module by assumption. Hence U is the sum of pairwise non-isomorphic $\mathbb{E}M$-submodules, which are permuted transitively by H. Thus U is an irreducible $\mathbb{E}H$-module. Note also that U is a summand of $V^{\mathbb{E}}$. By [2, 26.6(1)] we have that V is a summand of the restriction $U_{\mathbb{F}}$, of U to $\mathbb{F}$. By [18, VII Theorem 1.16(e)], $U_{\mathbb{F}}$ is a direct sum of isomorphic modules, each of dimension $\dim_{\mathbb{E}}(U) = r$. Hence $\dim_{\mathbb{F}}(V) = r$. □

The computational algebra systems [3] and [9] were used to investigate the submodule structure of Lie powers for $\mathcal{C}_3$ groups H. The first n for which L^nV was H-reducible turned out to be completely reducible. From the data we collected, we could guess, but not prove, the dimension of the smallest quotient H-module of L^nV. Thus we suspect that the three inequalities that appear in Table 6.1 (in the $\mathcal{C}_3$ rows) are in fact equalities.

LEMMA 5.4. *Suppose that* $H = \mathrm{GL}(d/r, \mathbb{F}_{q^r}) \rtimes \mathrm{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q) \in \mathcal{C}_3$ *is a subgroup of* $\mathrm{GL}(V)$ *where* $V = (\mathbb{F}_q)^d$*, and* r *is a prime, and suppose* $\mathrm{char}(\mathbb{F}_q) = p > 2$.

- (i) *If* $1 < r < d$ *then* H *acts reducibly on* L^2V*, preserving a quotient of dimension* $\binom{d/r}{2}r$.
- (ii) *If* $3 < r = d$ *then* H *acts reducibly on* L^2V*, with a minimal quotient of dimension* d.
- (iii) *If* $3 = r = d$ *and* $p > 3$*, then* H *acts irreducibly on* L^2V*, and reducibly on* L^3V.
- (iv) *If* $2 = r = d$ *and* $p > 3$*, then* H *acts irreducibly on* L^2V *and* L^3V*, and reducibly on* L^4V.

PROOF. (i) As above write $\mathbb{E} = \mathbb{F}_{q^r}$ and $\mathbb{F} = \mathbb{F}_q$. We think of H as acting semilinearly on $V' = \mathbb{E}^{d/r}$, and view V as $(V')_{\mathbb{F}}$, i.e., V' with scalars restricted to $\mathbb{F}$. Thus $\dim_{\mathbb{E}}(V') = d/r$ and $\dim_{\mathbb{F}}(V) = d$. Similarly, let $T' = A^2V'$, and let $T = (T')_{\mathbb{F}}$. Since $d/r > 1$, we have $\dim_{\mathbb{F}}(T) = r\dim_{\mathbb{E}}(T') = r\binom{d/r}{2} > 0$. We construct a surjective $\mathbb{F}H$-module homomorphism $\eta\colon A^2V \to T$. Certainly $\ker(\eta)$ is a *proper* submodule of A^2V because $\dim(T) > 0$, and $\ker(\eta)$ is non-zero because

$$\dim(\ker(\eta)) = \dim(A^2V) - \dim(T) = \binom{d}{2} - r\binom{d/r}{2} = \frac{d(d-d/r)}{2} > 0.$$

Fix a basis $\alpha_1, \ldots, \alpha_r$ for $\mathbb{E}$ over $\mathbb{F}$ and a basis $v_1, \ldots, v_{d/r}$ for V'. Then V has a basis

$$\{\alpha_i v_j \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant d/r\}$$

and T has a basis

$$\{\alpha_i v_j \wedge v_k \mid 1 \leqslant i \leqslant r, 1 \leqslant j < k \leqslant d/r\}.$$

Furthermore, A^2V has a basis consisting of vectors of the form $\alpha_i v_k \wedge \alpha_j v_\ell$. As $\alpha_i\alpha_j \in \mathbb{E}$, we may write $\alpha_i\alpha_j = \sum_{s=1}^r \lambda_s\alpha_s$ where $\lambda_s \in \mathbb{F}$. Define $\eta\colon A^2V \to T$ by

$$\eta(\alpha_i v_k \wedge \alpha_j v_\ell) = (\alpha_i\alpha_j)v_k \wedge v_\ell = \left(\sum_{s=1}^r \lambda_s\alpha_s\right) v_k \wedge v_\ell = \sum_{s=1}^r \lambda_s(\alpha_s v_k \wedge v_\ell).$$

Certainly η is a $\mathrm{GL}(V')$-homomorphism, and $\eta(\beta v_k \wedge \gamma v_\ell) = \beta\gamma v_k \wedge v_\ell$ for all $\beta, \gamma \in \mathbb{E}$. As $\theta \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})$ maps $\alpha_i v_k$ to $(\alpha_i^\theta)v_k$, we see that $\eta((\alpha_i v_k \wedge \alpha_j v_\ell)^\theta)$ equals

$$\eta(\alpha_i^\theta v_k \wedge \alpha_j^\theta v_\ell) = (\alpha_i^\theta \alpha_j^\theta)v_k \wedge v_\ell = \sum_{s=1}^{r} \lambda_s(\alpha_s^\theta v_k \wedge v_\ell) = \eta(\alpha_i v_k \wedge \alpha_j v_\ell)^\theta.$$

Hence η is an H-homomorphism as desired. Since η is a surjective $\mathbb{F}H$-homomorphism, and $0 < \dim(\ker(\eta)) < \dim(A^2V)$, H acts reducibly on A^2V. As $\mathrm{GL}(V')$ acts irreducibly on A^2V', it follows that H acts irreducibly on T.

(ii) Suppose that $d = r$ is prime and $r > 3$. Then $H \cong \mathrm{C}_{q^d-1} \rtimes \mathrm{C}_d$. We adopt the notation in the proof of Lemma 5.3 and write $H = \langle \phi, \mu \mid \phi^d = \mu^{q^d-1} = 1, \mu^\phi = \mu^q \rangle$. Let $e_0, e_1, \dots, e_{d-1}$ be a basis for V over $\mathbb{F} = \mathbb{F}_q$. Let A and C be the $d \times d$ matrices over $\mathbb{F}$ corresponding to the action of ϕ and μ on V. Now let $\mathbb{E} = \mathbb{F}_{q^d}$ and set $V^\mathbb{E} = V \otimes_\mathbb{F} \mathbb{E}$. Since C is irreducible over $\mathbb{F}$, its characteristic polynomial has distinct roots $\zeta, \zeta^q, \dots, \zeta^{q^{d-1}}$ in $\mathbb{E}$. Thus C is conjugate in $\mathrm{GL}(d, \mathbb{E})$ to the diagonal matrix $C^\mathbb{E} := \mathrm{diag}(\zeta, \zeta^q, \dots, \zeta^{q^{d-1}})$. Let $A^\mathbb{E}$ be the matrix with $e_i A^\mathbb{E} = e_{i+1}$ where the subscripts are read modulo d. Then $A^\mathbb{E}$ satisfies $(C^\mathbb{E})^{A^\mathbb{E}} = (C^\mathbb{E})^q$, and it follows that there exists a matrix in $\mathrm{GL}(d, \mathbb{E})$ that conjugates A to $A^\mathbb{E}$ and C to $C^\mathbb{E}$. The matrices A, C in $\mathrm{GL}(V)$ induce matrices a, c in $\mathrm{GL}(A^2V)$ and $A^\mathbb{E}$, $C^\mathbb{E}$ in $\mathrm{GL}(V^\mathbb{E})$ induce matrices $a^\mathbb{E}$, $c^\mathbb{E}$ in $\mathrm{GL}(A^2V^\mathbb{E})$. The induced matrices $a, c \in \mathrm{GL}(A^2V)$ and $a^\mathbb{E}, c^\mathbb{E} \in \mathrm{GL}(A^2(V^\mathbb{E}))$ are (simultaneously) conjugate in $\mathrm{GL}(A^2(V^\mathbb{E}))$.

The action of $a^\mathbb{E}$ and $c^\mathbb{E}$ relative to the basis $e_i \wedge e_j$, $0 \leqslant i < j < d$, for $A^2V^\mathbb{E}$ is given by $e_i \wedge e_j a^\mathbb{E} = e_{i+1} \wedge e_{j+1}$ and $e_i \wedge e_j c^\mathbb{E} = \zeta^{q^i+q^j} e_i \wedge e_j$. We show that a typical eigenvalue $\xi_{i,j} = \zeta^{q^i+q^j}$ of $c^\mathbb{E}$ does not lie in $\mathbb{F}$. Indeed, suppose that $\xi_{i,j} \in \mathbb{F}$, then $\xi_{i,j}^{q^{j-i}} = \xi_{i,j}$ and $\zeta^{q^j+q^{2j-i}} = \zeta^{q^i+q^j}$. Since $\zeta^{q^{2(j-i)}} = 1 = \zeta^{q^d-1}$, and $q^d - 1$ is coprime to $q^{2(j-i)}$, it follows that ζ has order 1, a contradiction. As $\xi_{i,j}$ is an eigenvalue of c, it follows that c does not fix an $\mathbb{F}$-subspace of dimension less than d. The d-dimensional $\mathbb{E}$-subspace $U = \langle e_i \wedge e_{i+1} \mid 0 \leqslant i < d \rangle$, is invariant under $a^\mathbb{E}$ and $c^\mathbb{E}$. The restrictions of $a^\mathbb{E}$ and $c^\mathbb{E}$ to U have matrices $a_U^\mathbb{E} = A$ and $c_U^\mathbb{E} = \mathrm{diag}(\xi_{0,1}, \xi_{0,1}^q, \dots, \xi_{0,1}^{q^{d-1}})$, respectively. The subgroup $\langle a_U^\mathbb{E}, c_U^\mathbb{E} \rangle$ is irreducible by Clifford's Theorem [6, pp. 343–344]. A simple calculation shows that the character values of the monomial group $\langle a_U^\mathbb{E}, c_U^\mathbb{E} \rangle$ lie in $\mathbb{F}$, so by a theorem of Brauer [18, VII Theorem 1.16(e)], the subgroup $\langle a_U^\mathbb{E}, c_U^\mathbb{E} \rangle$ of $\mathrm{GL}(d, \mathbb{E})$ is conjugate to an irreducible subgroup of $\mathrm{GL}(d, \mathbb{F})$. In summary, we have proved that every non-zero H-submodule of A^2V has $\mathbb{F}$-dimension at least d, and one has dimension precisely d. As H can be shown to act completely reducibly on A^2V, it follows that the smallest dimensional proper quotient module of A^2V has dimension d.

(iii) Suppose that $d = r = 3$. The argument in part (ii) shows that H preserves an irreducible 3-dimensional subspace of $A^2V = L^2V$. Thus H acts irreducibly on L^2V. By (2), $\dim(L^3V) = (3^3 - 3)/3 = 8$ so by Lemma 5.3, H acts reducibly on L^3V preserving a submodule of codimension at most 3.

(iv) Suppose that $d = r = 2$. Then H acts irreducibly on the 1-dimensional space L^2V, and on the 2-dimensional space $L^3V \cong A^2V \otimes V$. Finally, H acts reducibly on L^4V by

Lemma 5.3 as $\dim(L^4V) = (2^4 - 2^2)/4 = 3$, and H preserves a submodule of codimension at most 2. ☐

5.4. The tensor reducible $\mathcal{C}_4$ case.

LEMMA 5.5. *Suppose that $H = \mathrm{GL}(V_1)\circ\mathrm{GL}(V_2) \in \mathcal{C}_4$ where $2 \leqslant \dim(V_1) < \dim(V_2)$ and $\mathrm{char}(\mathbb{F}) \neq 2$. Then $L^2(V_1 \otimes V_2) = U_1 \oplus U_2$ where $U_1 \cong A^2V_1 \otimes S^2V_2$ and $U_2 \cong S^2V_1 \otimes A^2V_2$ are irreducible H-modules satisfying $0 < \dim(U_1) < \dim(U_2) < \dim(L^2(V_1 \otimes V_2))$.*

PROOF. Let $H = \mathrm{GL}(V_1) \circ \mathrm{GL}(V_2)$ preserve the decomposition $V = V_1 \otimes V_2$ where $2 \leqslant \dim(V_1) < \dim(V_2)$. By (14), we have the following H-module isomorphisms

$$\begin{aligned} T^2V &= (V_1 \otimes V_2) \otimes (V_1 \otimes V_2)\\ &\cong (V_1 \otimes V_1) \otimes (V_2 \otimes V_2)\\ &\cong \big(S^2V_1 \oplus A^2V_1\big) \otimes \big(S^2V_2 \oplus A^2V_2\big)\\ &\cong \big(S^2V_1 \otimes S^2V_2 \;\oplus\; A^2V_1 \otimes A^2V_2\big) \;\oplus\; \big(S^2V_1 \otimes A^2V_2 \;\oplus\; A^2V_1 \otimes S^2V_2\big)\\ &\cong S^2V \oplus A^2V. \end{aligned}$$

Equating symmetric and anti-symmetric parts gives the following H-module isomorphisms:

$$\begin{aligned} S^2V &\cong S^2V_1 \otimes S^2V_2 \;\oplus\; A^2V_1 \otimes A^2V_2, \quad \text{and}\\ A^2V &\cong S^2V_1 \otimes A^2V_2 \;\oplus\; A^2V_1 \otimes S^2V_2. \end{aligned}$$

In particular, we see that $A^2V \cong L^2V$ is reducible as an H-module. Since S^2V_i and A^2V_i are irreducible $\mathrm{GL}(V_i)$-submodules (for $i = 1, 2$), it follows that $S^2V_1 \otimes A^2V_2$ and $A^2V_1 \otimes S^2V_2$ are irreducible modules for $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ and hence for $H = \mathrm{GL}(V_1) \circ \mathrm{GL}(V_2)$. Since $2 \leqslant d_1 < d_2$ where $d_1 = \dim(V_1)$ and $d_2 = \dim(V_2)$, it is easy to see that $0 < \binom{d_1}{2}\binom{d_2+1}{2} < \binom{d_1+1}{2}\binom{d_2}{2} < \binom{d_1d_2}{2}$, and hence $0 < \dim(U_1) < \dim(U_2) < \dim(L^2(V_1 \otimes V_2))$. ☐

5.5. The tensor induced case $\mathcal{C}_7$. The classes $\mathcal{C}_i$ considered so far all contain 'large' maximal subgroups of $\mathrm{GL}(d, p)$, i.e., ones with $|H| \geqslant p^{3d+1}$. By contrast, none of the $\mathcal{C}_7$ subgroups H are large in this sense; indeed Corollary 4.3 shows that $|H| < p^{3d+1}$. Intuitively, the smaller $|H|$ is compared to $|\mathrm{GL}(d, p)|$ the less likely it is that modules with dimensions much larger than d remain irreducible, when restricted to H. Thus one might expect that our desired p-group G (with $A(G) = H$) has small nilpotency class, and that it is not too hard to construct. The first expectation is true, but not the second, as the small dimensional modules such as L^2V and L^3V turn out to be hard to handle.

THEOREM 5.6. *Let $H = \mathrm{GL}(V_1) \wr \mathsf{S}_r \leqslant \mathrm{GL}(V)$ preserve the tensor decomposition $V = V_1 \otimes \cdots \otimes V_r$, so $H \in \mathcal{C}_7$. Suppose that $p := \mathrm{char}(\mathbb{F}) > 2$, $r \geqslant 2$, and $t := \dim(V_1) = \cdots = \dim(V_r) \geqslant 2$.*

- (i) *If $p > 2$ and $r > 2$, then L^2V is reducible and the smallest quotient module of L^2V has dimension $\binom{t}{2}^r$ if r is odd, and $r\binom{t}{2}^{r-1}\binom{t+1}{2}$ if r is even.*
- (ii) *If $p > 3$ and $r = 2$, then L^2V is an irreducible H-module, and L^3V is a reducible H-module. The smallest dimension of a quotient module of L^3V is 4 if $t = 2$, and $(t + 1)t^2(t - 1)^2(t - 2)/9$ if $t > 2$.*

PROOF. As $H \in \mathcal{C}_7$, we have $H = \mathrm{GL}(V_1) \wr \mathsf{S}_r \leqslant \mathrm{GL}(V_1^{\otimes r})$ where $t \geqslant 2$ and $r \geqslant 2$.

(i) Suppose now that $V = V_1 \otimes \cdots \otimes V_r$ where $r \geqslant 2$ and $p > 2$. Rearranging tensor factors, and using (14) shows that

$$T^2V = T^2V_1 \otimes \cdots \otimes T^2V_r = (A^2V_1 \oplus S^2V_1) \otimes \cdots \otimes (A^2V_r \oplus S^2V_r).$$

Expanding gives 2^r summands. We show that collecting these summands into S_r-orbits gives $T^2V = \bigoplus_{k=0}^r U_k$ where the U_k are pairwise non-isomorphic irreducible H-submodules satisfying

$$A^2V = \bigoplus_{k \text{ odd}} U_k, \quad S^2V = \bigoplus_{k \text{ even}} U_k, \quad \text{and} \quad \dim(U_k) = \binom{r}{k}\binom{t}{2}^k\binom{t+1}{2}^{r-k}.$$

We identify the 2^r summands with the elements of the vector space $C = (\mathbb{F}_2)^r$. The orbits of S_r on the vectors of C are $C_0, \dots, C_r$ where C_k comprises the $\binom{r}{k}$ vectors with precisely k ones. Define

$$U_k = \bigoplus_{(\varepsilon_1,\dots,\varepsilon_r)\in C_k} X^{\varepsilon_1}(V_1) \otimes \cdots \otimes X^{\varepsilon_r}(V_r) \quad \text{where} \quad X^{\varepsilon_i}(V_j) = \begin{cases} A^2V_j & \text{if } \varepsilon_i = 1, \\ S^2V_j & \text{if } \varepsilon_i = 0. \end{cases}$$

The summands of U_k are pairwise non-isomorphic irreducible modules for the base group $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_r)$ of H, so by Clifford's Theorem [**6**, pp. 343–344], U_k is an irreducible H-submodule. The formula for $\dim(U_k)$ is now clear as $\dim(A^2V_i) = \binom{t}{2}$ and $\dim(S^2V_j) = \binom{t+1}{2}$ by (13).

The number of irreducible H-submodules U_k of A^2V is the number of odd k satisfying $0 \leqslant k \leqslant r$, namely $\lceil r/2 \rceil$. Hence A^2V is reducible precisely when $r > 2$. Suppose that k_0 is odd and $\dim(U_{k_0}) \leqslant \dim(U_k)$ for all odd k satisfying $0 \leqslant k \leqslant r$. Observe first that $r - k < k$ implies that $\dim(U_{r-k}) > \dim(U_k)$ so we may assume $r/2 \leqslant k_0 \leqslant r$. Second, note that if k, ℓ are odd and $r/2 \leqslant \ell < k$, then it follows that $\dim(U_\ell) > \dim(U_k)$ because $\binom{r}{\ell} > \binom{r}{k}$. Hence $k_0 = r$ when r is odd, and $k_0 = r-1$ when r is even. This proves part (i).

(ii) Suppose now that $p > 3$, $r = 2$, and $V = V_1 \otimes V_2$. By part (i), L^2V is irreducible. We use Lemma 3.1 to investigate the K-module structure of $A^2V \otimes V$ where $K = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ is normal in H of index 2. It follows from part (i) that we have the following K-module decomposition: $A^2V = (A^2V_1 \boxtimes S^2V_2) \oplus (S^2V_1 \boxtimes A^2V_2)$ where $\boxtimes$ denotes 'outer tensor product' for K. Consider the following K-module decomposition:

$$\begin{aligned} A^2V \otimes V &\cong \big((A^2V_1 \boxtimes S^2V_2) \oplus (S^2V_1 \boxtimes A^2V_2)\big) \otimes (V_1 \boxtimes V_2) \\ &\cong (A^2V_1 \otimes V_1) \boxtimes (S^2V_2 \otimes V_2) \oplus (S^2V_1 \otimes V_1) \boxtimes (A^2V_2 \otimes V_2). \end{aligned}$$

Lemma 3.1(ii) gives $A^2V_i \otimes V_i \cong L^3V_i \oplus A^3V_i$ and $S^2V_i \otimes V_i \cong S^3V_i \oplus L^3V_i$, so

$$\begin{aligned} A^2V \otimes V &\cong (L^3V_1 \oplus A^3V_1) \boxtimes (S^3V_2 \oplus L^3V_2) \oplus (S^3V_1 \oplus L^3V_1) \boxtimes (L^3V_2 \oplus A^3V_2) \\ &\cong (B_1 \oplus C_1) \boxtimes (A_2 \oplus B_2) \oplus (A_1 \oplus B_1) \boxtimes (B_2 \oplus C_2) \end{aligned}$$

where $A_i = S^3V_i$, $B_i = L^3V_i$, and $C_i = A^3V_i$. Expanding shows that $A^2V \otimes V$ is a sum of 8 irreducible K-modules as follows:

$$A^2V \otimes V \cong P \oplus Q \oplus R \oplus S \tag{17}$$

where

$$P = A_1 \boxtimes B_2 \oplus B_1 \boxtimes A_2, \qquad Q = A_1 \boxtimes C_2 \oplus C_1 \boxtimes A_2,$$
$$R = B_1 \boxtimes C_2 \oplus C_1 \boxtimes B_2, \qquad S = B_1 \boxtimes B_2 \oplus B_1 \boxtimes B_2.$$

By Clifford's Theorem [6, pp. 343–344], P, Q and R are pairwise non-isomorphic irreducible H-modules, whilst S is the sum of two irreducible H-modules, S_1 and S_2 say, each isomorphic to $B_1 \otimes B_2$. Using Lemma 3.1(iii), we reconcile the H-decompositions

$$A^2V \otimes V = L^3V \oplus A^3V \quad \text{and} \quad A^2V \otimes V = P \oplus Q \oplus R \oplus S_1 \oplus S_2.$$

TABLE 5.1. Dimensions of irreducible H-submodules of $A^2V \otimes V$.

U	P	Q	R	S_1	S_2	$a = \dim(A_i)$	$b = \dim(B_i)$	$c = \dim(C_i)$	d
$\dim(U)$	$2ab$	$2ac$	$2bc$	b^2	b^2	$\frac{(t+2)(t+1)t}{6}$	$\frac{(t+1)t(t-1)}{3}$	$\frac{t(t-1)(t-2)}{6}$	t^2

The dimensions of the modules P, Q, R, S_1 and S_2 are displayed in Table 5.1. Since L^3V is a completely reducible H-module, there exist $p, q, r, s_1, s_2 \in \{0, 1\}$ such that

$$\dim L^3V = \frac{t^6 - t^2}{3} = p \dim P + q \dim Q + r \dim R + s_1 \dim S_1 + s_2 \dim S_2.$$

The above gives rise to 32 polynomial equations in t. If $t \neq 4$, then the only solutions are $(p, q, r, s_1, s_2) = (1, 0, 1, 1, 0)$ or $(p, q, r, s_1, s_2) = (1, 0, 1, 0, 1)$. If $t = 4$, then there are two additional possibilities since $\dim R = \dim Q$, namely that $(p, q, r, s_1, s_2) = (1, 1, 0, 0, 1)$ or $(p, q, r, s_1, s_2) = (1, 1, 0, 1, 0)$. Renumbering if necessary, assume that $S_1 \leqslant L^3V$ and thus $S_2 \leqslant A^3V$. Hence, if $t \neq 4$ we obtain $L^3V \cong P \oplus R \oplus S_1$. When $t = 4$ the additional possibility that $L^3V \cong P \oplus Q \oplus S_1$ arises. As L^3V is completely reducible, the smallest non-zero quotient H-module is isomorphic to the smallest irreducible H-submodule of L^3V. If $t = 2$ then $c = 0$ and $L^3V \cong P \oplus S_1$ and the minimal dimension of an H-submodule of L^3V is 4. If $t > 2$ then $c > 0$ and the dimensions of the minimal H-submodules of L^3V are $2ab$, $2bc$ and b^2. Since $a > c$ and $b > 2c$, the smallest dimension of a minimal submodule of L^3V in this case is $2bc = (t + 1)t^2(t - 1)^2(t - 2)/9$. ☐

5.6. The $\mathcal{C}_8$ case, classical groups in natural action. As our primary interest is in the field $\mathbb{F}_p$, we do not consider the unitary groups here. The following remark elucidates the symplectic case in Lemma 5.8(i).

REMARK 5.7. The extraspecial group G of order p^{1+2m} with exponent $p > 2$ has a pc-presentation

$$G = \left\langle g_1, \dots, g_{2m+1} \mid g_1^p = \cdots = g_{2m+1}^p = 1,\ g_{2k}^{g_{2k-1}} = g_{2k}g_{2m+1},\ 1 \leqslant k \leqslant m \right\rangle \tag{18}$$

where $g_j^{g_i} = g_j$ for $1 \leqslant i < j \leqslant 2m+1$ and $(i,j) \notin \{(2k-1, 2k) \mid 1 \leqslant k \leqslant m\}$. Using collection, we can symbolically multiply

$$\left(g_1^{x_1} g_2^{y_1} \cdots g_{2m-1}^{x_m} g_{2m}^{y_m} g_{2m+1}^{z}\right)\left(g_1^{x_1'} g_2^{y_1'} \cdots g_{2m-1}^{x_m'} g_{2m}^{y_m'} g_{2m+1}^{z'}\right) = g_1^{x_1+x_1'} g_2^{y_1+y_1'} \cdots g_{2m-1}^{x_m+x_m'} g_{2m}^{y_m+y_m'} g_{2m+1}^{z+z'+\sum_{k=1}^m x_k y_k'}.$$

However, writing $v_1 = (x_1, y_1, \ldots, x_m, y_m)$ and $v_1' = (x_1', y_1', \ldots, x_m', y_m')$, we have a more symmetric multiplication rule on pairs in $\mathbb{F}_p^{2m} \times \mathbb{F}_p$:

$$(v_1, v_2)(v_1', v_2') = (v_1 + v_1', v_2 + v_2' + \beta(v_1, v_1'))$$

where $\beta(v_1, v_1') = \sum_{k=1}^m (x_k y_k' - x_k' y_k) \pmod p$. This rule is a 'quotient' of the Lie 2-tuple rule in Example 2.7, and it helps to show that the conformal symplectic group $\mathrm{CSp}(\beta)$ is a subgroup of $\mathrm{Aut}(G)$. If $g \in \mathrm{CSp}(\beta)$ satisfies $\beta(v_1 g, v_1' g) = \beta(v_1, v_1')\delta_g$ where $\delta_g \in \mathbb{F}$ is non-zero, then the map $(v_1, v_2)\alpha_g = (v_1 g, v_2 \delta_g)$ lies in $\mathrm{Aut}(G)$, and $g \mapsto \alpha_g$ is a monomorphism $\mathrm{CSp}(\beta) \to \mathrm{Aut}(G)$. This proves that $\mathrm{Aut}(G)$ *splits* over $\mathrm{Inn}(G)$, cf. [**30**, Theorem 1(a)].

LEMMA 5.8. *Suppose that $H \in \mathcal{C}_8$ is the stabiliser of a non-degenerate form on $V = (\mathbb{F}_q)^d$, where q is an odd prime power and $d > 2$.*

(i) *If H preserves an alternating form, then H acts reducibly on L^2V, and the smallest dimension of a quotient module is 1.*
(ii) *If H preserves a quadratic form, then H acts irreducibly on L^2V, and reducibly on L^3V. Moreover, the smallest dimension of a quotient module of L^3V is d or 1.*

PROOF. (i) Suppose that $H = \mathrm{CSp}(\beta)$ is the conformal symplectic group preserving the alternating form $\beta\colon V \times V \to \mathbb{F}_q$ up to scalar multiples. Recall that $\mathrm{CSp}(\beta)/\mathrm{Sp}(\beta) \cong \mathbb{F}_q^\times \cong \mathrm{C}_{q-1}$. The linear map $\pi\colon L^2V \to \mathbb{F}_q$ satisfying $\pi([v,w]) = \beta(v,w)$ is well-defined precisely because β is alternating. Moreover, since β is an H-invariant form we have that π is an H-module homomorphism, and $\mathrm{CSp}(\beta)$ acts non-trivially on $\mathbb{F}_q$ with kernel $\mathrm{Sp}(\beta)$. Clearly π is onto, therefore $\dim(L^2V/\ker(\pi)) = 1$. As $\dim(L^2V) = \binom{d}{2} > 1$ for $d > 2$, we see that L^2V is reducible as claimed.

(ii) Suppose that H preserves the symmetric form $\beta\colon V \times V \to \mathbb{F}_q$ up to non-zero scalar multiples. Since p is odd, H acts irreducibly on A^2V, see [**24**, Table 1]. Define $\pi\colon T^3V \to V \otimes \mathbb{F}_q$ by $\pi(u \otimes v \otimes w) = u \otimes \beta(v,w)$. Since H preserves β up to scalars, we see that π is an H-module homomorphism. Moreover, since

$$u \wedge v \wedge w = u \otimes v \otimes w - u \otimes w \otimes v + v \otimes w \otimes u - v \otimes u \otimes w + w \otimes u \otimes v - w \otimes v \otimes u$$

we have

$$\pi(u \wedge v \wedge w) = u \otimes (\beta(v,w) - \beta(w,v)) + v \otimes (\beta(w,u) - \beta(u,w)) + w \otimes (\beta(u,v) - \beta(v,u)).$$

Thus $\pi(A^3V) = \{0\}$ since β is symmetric. Now choose vectors u, v and w of V so that $u \otimes v \otimes w$ is a fundamental tensor and such that $f(u,w) = 0$ and $\beta(v,w) \neq 0$ (such a choice is always possible since β is non-degenerate). Then $x := u \otimes v \otimes w - v \otimes u \otimes w \in A^2V \otimes V$ and $\pi(x) = u \otimes \beta(v,w) \neq 0$. Hence

$$A^3V \leqslant \ker(\pi) \cap (A^2V \otimes V) < A^2V \otimes V$$

and the quotient $(A^2V \otimes V)/(\ker(\pi) \cap (A^2V \otimes V))$ is isomorphic to a submodule of $V \otimes \mathbb{F}_q$. Since the latter is an irreducible H-module, we have that the smallest quotient module of L^3V has dimension d or 1. □

Remark 5.9. We do not consider the case when H is a maximal subgroup of $\mathrm{GL}(d,p)$ containing $\mathrm{SL}(d,p)$. In this case the irreducible $\mathrm{GL}(V)$-submodules of L^nV with $p > n$, are likely to restrict to irreducible $\mathrm{SL}(V)$-modules. In the case $d = 2$ excluded in Lemma 5.8, H contains $\mathrm{Sp}(2,p) = \mathrm{SL}(2,p)$.

6. Proof of the main theorem

In this section we complete the proof of Theorem 1. In fact, we prove a stronger theorem from which Theorem 1 follows, after an application of Corollary 4.3.

Theorem 6.1. *Let $p \geqslant 5$ be a prime, and let $d \geqslant 2$ be an integer. Suppose that H is a maximal subgroup of $\mathrm{GL}(d,p)$ with $\mathrm{SL}(d,p) \nleqslant H$ and that H lies in one of the Aschbacher classes $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_5 \cup \mathcal{C}_7 \cup \mathcal{C}_8$. Then there exists a d-generator p-group G of exponent p, class at most 4, order at most $p^{\frac{d^4}{2}}$ and $A(G) = H$. The nilpotency class, order and structure of G is given in Table 6.1.*

Proof. Let H be as in the statement of the theorem and let $V = \mathbb{F}_p^d$. Note that H cannot be in class $\mathcal{C}_5$ and cannot be in class $\mathcal{C}_8$ preserving a unitary form. We seek a d-generator p-group G of exponent p and minimal class such that $A(G) = H$. Now $\mathrm{GL}(V)$ (and hence H) acts on the sections of the lower exponent-p central series of the d-generator Burnside group $B = B(d,p)$. By Lemmas 5.1, 5.2, 5.4, 5.5, 5.8 and Theorem 5.6 there exists an $n \leqslant 4$ such that H acts irreducibly on $L^1V, \ldots, L^{n-1}V$ (with the exception that if H is of class $\mathcal{C}_1$ then H is reducible on L^1V), and there is a maximal H-submodule, say M/B_n, of $B_{n-1}/B_n \cong L^nV$ which is not $\mathrm{GL}(V)$-invariant. Set $G := B/M$. We claim that G is the desired p-group.

Since $B_n < M < B_{n-1}$ is H-invariant, G is a proper quotient of the finite group $\Gamma_n(V) = \Gamma(d,p,n)$. In particular, G has class n. Now $H \leqslant \mathrm{N}_{\mathrm{GL}(V)}(M/B_n) \leqslant \mathrm{GL}(V)$ and since H is maximal in $\mathrm{GL}(V)$, our choice of M gives $\mathrm{N}_{\mathrm{GL}(V)}(M/B_n) = H$. Hence Theorem 2.2 gives $A(G) = \mathrm{N}_{\mathrm{GL}(V)}(M/B_n) = H$.

It remains to bound $|G|$. By construction, G is a quotient of $\Gamma(d,p,n)$, and the order of the latter group is given in Theorem 2.5. From this it easily follows that $|G| \leqslant p^{\frac{d^4}{2}}$ as claimed. □

Remark 6.2. For a given $H \leqslant \mathrm{GL}(d,p)$, we let $\mathcal{G}(H)$ be the category of all finite d-generator p-groups P with $A(P) = H$. Then the group G appearing in Theorem 6.1 is the minimal element of $\mathcal{G}(H)$ with respect to exponent and nilpotency class. In fact, if $H \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_7$ or H is a $\mathcal{C}_8$ subgroup preserving a symplectic form, we have also found the groups in $\mathcal{G}(H)$ of minimal order.

Remark 6.3. Let H be the $\mathcal{C}_1$ maximal subgroup $\mathrm{GL}(V)_U$ which fixes a proper non-zero subspace U of V. Let $r = \dim(U)$ and let $P = (C_p)^r \times (C_{p^2})^{d-r}$. Then P is abelian

TABLE 6.1. The exponent-p groups G of class n in Theorem 1 for different Aschbacher classes $\mathcal{C}_i$ where $|G| = p^m$ and $m = \sum_{i=1}^{n-1} f(d,i) + \dim(G_{n-1})$.

$\mathcal{C}_i$	$V = G_0/G_1$	H	conditions	n	$p\geqslant$	$\dim(G_{n-1})$	G_{n-1}
$\mathcal{C}_1$	$0 < U < V$	$\mathrm{GL}(V)_U$	$1<r<d-1$	2	3	$\binom{d-r}{2}$	$A^2(V/U)$
		$r := \dim(U)$	$1<r=d-1$	2	3	r	$U \otimes (V/U)$
			$(d,r) = (2,1)$	3	5	1	$A^2V \otimes (V/U)$
$\mathcal{C}_2$	$\bigoplus_{i=1}^r V_i$	$\mathrm{GL}(V_1) \wr \mathsf{S}_r$	$1 < r < d$	2	3	$\binom{d/r}{2}r$	U_1
		$d = r\dim(V_1)$	$4 < r = d$	3	5	$d(d-1)$	W_1
			$3, 4 = r = d$	3	5	$2\binom{d}{3}$	W_2/A^3V
			$2 = r = d$	4	5	1	Lemma 5.2
$\mathcal{C}_3$	$(\mathbb{F}_{p^r})^{d/r}$	$\Gamma\mathrm{L}(d/r, \mathbb{F}_{p^r})$	$1 < r < d$	2	3	$\leqslant \binom{d/r}{2}r$	Lemma 5.4(i)
			$3 < r = d$	2	3	d	Lemma 5.4(ii)
			$3 = r = d$	3	5	$\leqslant 3$	Lemma 5.4(iii)
			$2 = r = d$	4	5	$\leqslant 2$	Lemma 5.4(iv)
$\mathcal{C}_4$	$V_1 \otimes V_2$	$\mathrm{GL}(V_1) \circ \mathrm{GL}(V_2)$	$1 < d_1 < d_2$	2	3	$\binom{d_1}{2}\binom{d_2+1}{2}$	$A^2V_1 \otimes S^2V_2$
		$d_i := \dim(V_i)$	$d = d_1d_2$				Lemma 5.5
$\mathcal{C}_7$	$\bigotimes_{i=1}^r V_1$	$\mathrm{GL}(V_1) \wr \mathsf{S}_r$	$2 < r$	2	3	5.6(i)	$U_{2\lfloor (r-1)/2 \rfloor}$
		$d = \dim(V_1)^r$	$2 = r$	3	5	5.6(ii)	R if $t > 2$
$\mathcal{C}_8$		$\mathrm{CSp}(\beta)$	$2 < d$	2	3	1	det
		$\mathrm{GO}(\beta)$	$2 < d$	3	5	$1, d$	Lemma 5.8

and of exponent p^2, and it is easy to check that $A(P) = H$. The group P has smaller order than the corresponding group listed in Table 6.1, but the exponent is p^2 rather than p.

7. Some open questions

Aschbacher's Theorem [21, Theorem 1.2.1] and the results of Sections 3, 4, 5 work over an arbitrary finite field $\mathbb{F}_q$. There is no definition of 'q-groups' where $q = p^f$ and $f > 1$.

However, taking a group $\Gamma_n(\mathbb{F}_q^d)$ defined in Construction 2.3 results in a group that has a Frattini quotient isomorphic to $\mathbb{F}_q^d$. Unfortunately, these groups are not relatively free since they are df-generator groups and the lower central series of $\Gamma_n(\mathbb{F}_q^d)$ is not the same as that of $\Gamma_n(\mathbb{F}_p^{df})$.

How must our results be modified when $p = 2$? How large must the nilpotency class of G be in the cases $\mathcal{C}_6$ and $\mathcal{C}_9$ which contain no 'large' subgroups? How do the multiplication rules (5)–(7) for the universal groups $\Gamma_n(\mathbb{F}^d)$ generalise for $n > 4$? To what extent can collection in groups of exponent p given by pc-presentations be replaced by *symbolic* computations in Lie n-tuple groups? (This type of question is explored in [**22**], for example.)

Suppose that H is a maximal subgroup of $\mathrm{GL}(V)$ and the irreducible $\mathrm{GL}(V)$-submodules of $L^1V, \dots, L^{n-1}V$ restrict to irreducible H-submodules, and n is maximal with this property. Our examples lead us to ask: Is L^nV, viewed as an H-module, always either completely reducible or uniserial?

Acknowledgements. The first author acknowledges the support of the Australian Research Council Future Fellowship FT120100036, the second and fourth authors acknowledge the support of the Australian Research Council Discovery Grant DP140100416 and the third author acknowledges the ARC grants DP120100446 and DE160100081. We would like to thank Kay Magaard, Csaba Schneider and Oihana Garaialde Ocanã for helpful comments, and Saul Freedman for carefully reading the manuscript.

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