Parametric Instability in Gravitational Wave Detectors

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Abstract

In 2000 Vladimir Braginsky and co-workers predicted that high power laser light in very long baseline gravitational wave detectors could give rise to parametric instability consisting of uncontrolled laser powered amplification of acoustic vibrations of the mirrors. In 2005 Chunmoung Zhao and co-workers published a detailed simulation which predicted that the planned Advanced LIGO gravitational wave detectors would experience such instabilities and that these would occur at a small fraction of design power. This would prevent full sensitivity from being achieved.

This thesis describes experimental studies of parametric instability. The first experiments took place at Gingin, where the author made the first observations of free running parametric instability using an 80 m high power cavity. The second set of experiments took place at LIGO Livingston after parametric instability was first observed there, these experiments focused on refining thermal detuning, a technique that had been proposed and tested at Gingin. Thermal detuning enabled the laser power to be raised sufficiently for the first detection of gravitational waves on Sept 14, 2015. The third set of experiments demonstrated the electrostatic damping technique in preparation for increased laser power: the initial demonstration was performed at LIGO Livingston, further work was carried out at LIGO Hanford.

The control of parametric instability was crucial to the first detection of gravitational wave signals from the coalescence of a binary black hole system. It will be of increasing importance in the new era of gravitational wave astronomy as detector sensitivity is increased. There is a realistic prospect of being able to detect gravitational waves from the very edge of the observable universe but this will require higher optical power and improved parametric instability control.

The first three chapters of this thesis provide introductions to gravitational waves, optomechanics and parametric instability respectively. They aim
to engage an audience that includes postgraduate students. Chapter 1 includes; a brief history of the field, fundamentals of gravitational wave detection, laser interferometer physics, an introduction to the research center at Gingin and a brief discussion about future gravitational wave detectors.

Gravitational wave detectors are high power optomechanical devices in which mechanical motion and resonances interact with optical resonances. Chapter 2 reviews some of the optical and mechanical resonances and discusses optomechanics in gravitational wave detectors. This introduction to optomechanics provides a foundation for the remainder of the thesis which focuses on the three-mode optomechanical interactions that lead to parametric instability. These interactions couple one elastic eigenmode of the mirror to two optical eigenmodes.

Chapter 3 reviews the theory of parametric instability and the history of experimental studies, followed by a review of proposed mitigation strategies.

Chapters 4 and 5 focus on experimental confirmation of the theory of parametric instability. Chapter 4 describes an experiment in the low power regime at Gingin, which confirmed the optical drive mechanism. It provided a method for measuring parametric gains well below the instability threshold.

Chapter 5 describes the characterisation of parametric instability at LIGO Livingston immediately prior to Observation Run 1, and the method used for avoiding instability. Instability was observed in LIGO in November 2014 when the laser power first exceeded the instability threshold. I was invited on the LIGO visitor program to assist in optimising the thermal detuning technique. Thermal detuning allowed the optical power to be increased by 150%, which soon afterwards resulted in the momentous discovery of the first two gravitational wave signals. Chapter 5 also discusses the requirements of parametric instability control at full design optical power, which is eight times higher than the power used in the first observation run.

Chapter 6 addresses electrostatic control of parametric instability and transverse optical mode modulation suppression of parametric instability.
LIGO Livingston prior to Observation Run 1. The second is a suppression mechanism observed at the Gingin facility. It leads to a proposed control scheme.

Three mode parametric interactions have useful applications because they are very sensitive to many different degrees of freedom such as; temperatures, temperature gradients and mirror figure errors. They also provide a means of sensing a large number of acoustic modes. In Chapter 7 schemes are proposed that can translate eigenmode degrees of freedom into control parameters that can allow greatly improved control of both high power cavities and full dual recycled Michelson interferometers. As a proof of principle, the testing of two specific monitoring schemes are reported.

As an experimental field parametric instability and three mode interaction monitoring is in its infancy. There are numerous possible applications and improvements. In Chapter 8 a selection of research directions that would greatly improve our understanding of three mode interactions in gravitational wave detectors are discussed. Several improvements made when commissioning parametric instability control at LIGO Hanford prior to Observation Run 2 are also presented along with the challenges encountered.

Finally Chapter 9 concludes the thesis with a summary of the main results and a discussion of discrepancies and research opportunities that could greatly benefit the operations of present and future gravitational wave detectors.
Thesis Declaration

I, Carl Blair, certify that:

This thesis has been substantially accomplished during enrolment in the degree. This thesis does not contain material which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution. No part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of The University of Western Australia and where applicable, any partner institution responsible for the joint-award of this degree. This thesis does not contain any material previously published or written by another person, except where due reference has been made in the text. The work(s) are not in any way a violation or infringement of any copyright, trademark, patent, or other rights whatsoever of any person.

This thesis contains published work and/or work prepared for publication, some of which has been co-authored. The bibliographic details of published works appearing in a modified form within this thesis are given below with an estimation of my contribution.


My contribution to the research was:
(i) 50% of the experimental work,
(ii) 70 % of the manuscript preparation.

Carl Blair "The next detectors for gravitational wave astronomy" Section 5 "Three mode parametric instabilities and their control for advanced gravitational wave detectors" Science China, Physics, Mechanics & Astronomy, 58 120405 (2015), (Ref. Chap. 5).

My contribution to the research was:
(i) 95% of the experimental work,
(ii) 90 % of the manuscript preparation.

My contribution to the research was:
(i) 95% of the experimental work,
(ii) 70 % of the manuscript preparation.


My contribution to the research was:
(i) 40% of the experimental work,
(ii) 20 % of the manuscript preparation.


My contribution to the research was:
(i) 30% of the experimental work,
(ii) 10 % of the manuscript preparation.


My contribution to the research was:
(i) 50% of the experimental work,
(ii) 30 % of the manuscript preparation.


My contribution to the research was:
(i) 50% of the experimental work,
(ii) 50 % of the manuscript preparation.

Student’s Signature : Date:

Supervisor’s Signature: Date:
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I would like to thank my supervisors Professor Chunnong Zhao, Professor David Coward and Professor Jie Pan. Particularly Zhao for taking the time to teach me, I envy your knowledge of the way the world works. I would like to thank members of the Australian Consortium for Gravitational Research particularly those at UWA. Present members gave so much to the investigations at Gingin, particularly my contemporaries Fang Qi, Liu Jian and Jiayi Qin. Others built the foundation on which my experiments were carried out: particularly Ju Li and John Winterflood for designing the suspension systems, Jean Charles Dumas, Andrew Woolley and Fan Yaohui for building the suspension and control systems and David Hosken for getting the lasers working. Without this work the investigations reported here would not have been possible. Also thanks to the Australian Research Council for supporting gravitational wave research and the Australian Government Research Training Program Scholarship that supported this research.

I would like to thank Yuri Levin for his encouragement to continue working on a challenging project and for several very educational meetings with patient and enlightening explanations.

My trips to LIGO were supported by the LSC fellows program and LIGO Visitor Program. These are amazing programs that give students the opportunity to work on operational gravitational wave detectors. I can not praise the people who advocate for these programs enough. These visits brought me into contact with many amazing scientists. The LIGO Scientific Collaboration is an open, welcoming collaboration. I was always surprised how willingly help was provided from experts, for whom I have the greatest admiration. In thanking the LIGO Scientific Collaboration I must thank the National Science Foundation of the USA for funding LIGO. LIGO was constructed by the California Institute of Technology and Massachusetts Institute of Technology with funding from the National Science Foundation, and operates under Cooperative Agreement No. PHY-0757058. Advanced LIGO was built under Grant No. PHY-0823459. Congratulations to Caltech and MIT for managing such an amazing facility.

To my girlfriend Camille Buononato, your support through some very challenging times was selfless. Finally I’d like to thank my family for their support. My mother Eve Blair, with her experience of writing two theses, she always had good advice. My father David Blair has, throughout my life, provided motivation and drive to understand how things work. Some of my first memories are science experiments in the back yard.

Thanks to you all. It’s been an amazing journey.
Preface - The journey of the last four years It has been a journey of struggles against the myriad of instabilities that made achieving free running parametric instability at Gingin such an enormous challenge. A journey of excitement when the fabled parametric instability became a reality first at Gingin, then shortly after at Advanced LIGO. A Journey of Adventure - invited to LIGO Livingston as a LIGO visitor, new people, new places and a much bigger laboratory. A journey of awe - dumbfounded by the magician commissioners who operate the Advanced LIGO detectors from the control room, with access to 100,000 channels of information but completely blind to reality, a bit of code could be moving a few gram mirror or the 6 tonne payload on the pre-isolation system. Finally, it was a journey of learning - sometimes painful, sometimes exhilarating.

It turned out though that even with all this excitement my journey had just begun. I signed up for the LIGO fellows program so that I could continue to work on parametric instability at Advanced LIGO. LIGO fellows have two responsibilities, their responsibility towards their personal project and their responsibility to check LIGO data quality. It was in this capacity that I was at LIGO Livingston on the night of the 14 September 2015. The night we heard and made sense of the first gravitational wave that humans have ever detected. As LIGO Fellows we pored over the data, with the 1000 other scientist, in complete disbelief. We were all looking for that tell tale channel that would indicate that it was all a false alarm. It was unbelievable that god could have been so kind as to give us such a loud gravitational wave at the beginning of the first Advanced LIGO Observation Run. But it turns out that was exactly what happened. Slowly environmental coupling, injection or even a malicious hack were ruled out and the elation grew to a crescendo at the announcement on the 12th February 2016.

I feel incredibly privileged to have taken part in the study of parametric instability at such an important period in the field and to have taken part in the detector noise characterization work at Advanced LIGO at such an exciting time for gravitational wave astronomy.
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### Glossary

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<th>Term</th>
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</thead>
<tbody>
<tr>
<td><strong>Aliasing</strong></td>
<td>is the down-conversion in frequency that happens when a signal with a frequency greater than the Nyquist frequency is sampled. aliasing makes signals from two different frequencies indistinguishable, i.e. aliases of each other</td>
</tr>
<tr>
<td><strong>Anti-aliasing</strong></td>
<td>A filter to remove frequency content from a signal before ADC that does not obey Shannon’s sampling criterion</td>
</tr>
<tr>
<td><strong>Anti-imaging</strong></td>
<td>A filter to remove frequency content after DAC that does not obey Shannon’s sampling criterion</td>
</tr>
<tr>
<td><strong>Blue detuned cavity</strong></td>
<td>Where the pump field applied to an optical cavity is a higher frequency than the cavity resonance</td>
</tr>
<tr>
<td><strong>Bode plot</strong></td>
<td>Transfer function representation with 2 plots, log-linear magnitude and linear-linear phase</td>
</tr>
<tr>
<td><strong>Cavity detuning</strong></td>
<td>Frequency of the pump light not equal to cavity resonance frequency</td>
</tr>
<tr>
<td><strong>Chirp mass</strong></td>
<td>Function of a binaries component masses that approximately defines gravitational wave amplitude and frequency evolution</td>
</tr>
<tr>
<td><strong>Clipping losses</strong></td>
<td>The approximation for the diffraction loss from an optical cavity with finite sized mirrors. The proportion of the ideal beam (with infinite mirrors) that falls outside the clear aperture of the finite mirrors</td>
</tr>
<tr>
<td><strong>Concentric cavity</strong></td>
<td>A cavity configuration that is at the limit of stability (g factor=1, infinite beam size) where the mirrors radii of curvature are equal to half the cavity length</td>
</tr>
<tr>
<td><strong>Confocal cavity</strong></td>
<td>The most stable cavity configuration where the mirrors radii of curvature are equal to the cavity length</td>
</tr>
<tr>
<td><strong>Contrast defect</strong></td>
<td>Residual light from destructively interfering beams. such residual light reduces the contrast ratio</td>
</tr>
<tr>
<td><strong>Contrast ratio</strong></td>
<td>Ratio of the maximum to minimum light intensity</td>
</tr>
<tr>
<td><strong>Degeneracy</strong></td>
<td>When resonances are indistinguishable</td>
</tr>
<tr>
<td><strong>Diffraction losses</strong></td>
<td>The loss from an optical cavity that is due to the finite size of the mirrors</td>
</tr>
<tr>
<td><strong>Dither</strong></td>
<td>Injected noise that performs a useful function</td>
</tr>
<tr>
<td><strong>Effective Q factor</strong></td>
<td>The quality factor of an object in an experiment with external influences such as clamping, gas damping and radiation pressure</td>
</tr>
<tr>
<td><strong>Figure error (mirror)</strong></td>
<td>Deviation from the nominal (surface profile)</td>
</tr>
<tr>
<td>---------------------------</td>
<td>---------------------------------------------</td>
</tr>
<tr>
<td><strong>Impedance matched</strong></td>
<td>An optical cavity where the losses in the cavity are equal to the input coupler transmission. Such a cavity has zero fundamental mode reflection</td>
</tr>
<tr>
<td><strong>Input coupler</strong></td>
<td>The mirror of an optical cavity where light is injected</td>
</tr>
<tr>
<td><strong>Instability threshold</strong></td>
<td>The optical power where the parametric gain $R_m = 1$</td>
</tr>
<tr>
<td><strong>Intrinsic Q</strong></td>
<td>The quality factor of an object due only to its mechanical properties</td>
</tr>
<tr>
<td><strong>Isothermal bath</strong></td>
<td>An approximate model, ambient temperature is assumed not to change due to thermal fluctuations in the system being measured</td>
</tr>
<tr>
<td><strong>Jitter noise</strong></td>
<td>Pointing noise at a cavity input coupler</td>
</tr>
<tr>
<td><strong>Junk light</strong></td>
<td>The light that is reflected from an optical cavity that is impedance matched. Consists of spatial modes and frequency content not transmitted by the cavity</td>
</tr>
<tr>
<td><strong>Lock/locking</strong></td>
<td>Process of bringing control loops into their linear range where they may be engaged to control the system to the desired set-point. Generally in this thesis it means bringing a cavity or interferometer into its operational state, usually on resonance</td>
</tr>
<tr>
<td><strong>Nyquist frequency</strong></td>
<td>The light incident on an optical cavity</td>
</tr>
<tr>
<td><strong>Pump field</strong></td>
<td>The light incident on an optical cavity</td>
</tr>
<tr>
<td><strong>Over-coupled</strong></td>
<td>An optical cavity with losses larger than the input coupler transmission. The reflected light from such a cavity has a 180 degree phase shift</td>
</tr>
<tr>
<td><strong>Output coupler</strong></td>
<td>A partially transmitting mirror of an optical cavity where light is not injected</td>
</tr>
<tr>
<td><strong>Red detuned cavity</strong></td>
<td>Where the pump field applied to an optical cavity is a lower frequency than the cavity resonance</td>
</tr>
<tr>
<td>**Resolved sideband inter-</td>
<td>A process where a mechanical mode scatters light from one frequency to another in different resonant modes of an optical cavity.</td>
</tr>
<tr>
<td>action**</td>
<td></td>
</tr>
<tr>
<td><strong>Set-point</strong></td>
<td>Offset in feedback control loop to maintain a non-zero control point</td>
</tr>
<tr>
<td><strong>Three mode interaction</strong></td>
<td>An interaction between two resonant optical modes and one resonant mechanical mode</td>
</tr>
<tr>
<td><strong>Under-coupled</strong></td>
<td>An optical cavity with total loss smaller than the input coupler transmission. The reflected light from such a cavity has a 0 degree phase shift</td>
</tr>
<tr>
<td>**Unresolved sideband in-</td>
<td>A process where a mechanical mode scatters light from one frequency to another, both within one linewidth on an optical cavity.</td>
</tr>
<tr>
<td>terection**</td>
<td></td>
</tr>
<tr>
<td><strong>Violin modes</strong></td>
<td>The resonant modes of suspension fibers</td>
</tr>
<tr>
<td><strong>Whitening filter</strong></td>
<td>A filter that makes the signal spectrum flat (white)</td>
</tr>
<tr>
<td>Acronyms</td>
<td>Description</td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
</tr>
<tr>
<td>3MI</td>
<td>Three Mode Interaction</td>
</tr>
<tr>
<td>ADC</td>
<td>Analogue to Digital Conversion</td>
</tr>
<tr>
<td>ASD</td>
<td>Amplitude Spectral Density function</td>
</tr>
<tr>
<td>aLIGO</td>
<td>Advanced Laser Interferometer Gravitational-wave Observatory (USA)</td>
</tr>
<tr>
<td>BS</td>
<td>Beam Splitter</td>
</tr>
<tr>
<td>CDS</td>
<td>Control and Data System (aLIGO)</td>
</tr>
<tr>
<td>DAC</td>
<td>Digital to Analogue Conversion</td>
</tr>
<tr>
<td>dB</td>
<td>Decibel (unit of relative amplitude and power)</td>
</tr>
<tr>
<td>DC</td>
<td>Direct Current ie non-oscillating component</td>
</tr>
<tr>
<td>ESD</td>
<td>Electro-Static Drive</td>
</tr>
<tr>
<td>ETM</td>
<td>End Test Mass</td>
</tr>
<tr>
<td>ETMX/Y</td>
<td>End Test Mass of the X/Y arm of the interferometer</td>
</tr>
<tr>
<td>FEA</td>
<td>Finite Element Analysis</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite Element Modeling</td>
</tr>
<tr>
<td>FWHM</td>
<td>Full Width Half Maximum</td>
</tr>
<tr>
<td>GPS</td>
<td>Global Positioning System</td>
</tr>
<tr>
<td>HG_{pq}</td>
<td>Hermite Gaussian mode of order p horizontal and q vertical</td>
</tr>
<tr>
<td>HOTEM</td>
<td>Higher Order Transverse Electromagnetic Mode</td>
</tr>
<tr>
<td>HWHM</td>
<td>Half Width Half Maximum</td>
</tr>
<tr>
<td>ITM</td>
<td>Input Test Mass</td>
</tr>
<tr>
<td>ITMX/Y</td>
<td>Input Test Mass of the X/Y arm of the interferometer</td>
</tr>
<tr>
<td>LG_{pq}</td>
<td>Laguerre Gaussian mode of order p/2 radial and q azimuthal</td>
</tr>
<tr>
<td>LIGO</td>
<td>Laser Interferometer Gravitational-wave Observatory (USA)</td>
</tr>
<tr>
<td>OMC</td>
<td>Output Mode Cleaner</td>
</tr>
<tr>
<td>O1</td>
<td>Observation run 1 (of Advanced LIGO Sept2015-Jan2016)</td>
</tr>
<tr>
<td>PD</td>
<td>Photo Detector</td>
</tr>
<tr>
<td>PI</td>
<td>Parametric Instability</td>
</tr>
<tr>
<td>PG</td>
<td>Parametric Gain represented by symbol $R_m$</td>
</tr>
<tr>
<td>PLL</td>
<td>Phase Locked Loop</td>
</tr>
<tr>
<td>PMC</td>
<td>Pre-Mode Cleaner</td>
</tr>
<tr>
<td>PRC</td>
<td>Power Recycling Cavity</td>
</tr>
<tr>
<td>PSD</td>
<td>Power Spectral Density function</td>
</tr>
<tr>
<td>QPD</td>
<td>Quadrant Photo Detector</td>
</tr>
<tr>
<td>RH</td>
<td>Ring Heater (test-mass thermal actuator)</td>
</tr>
<tr>
<td>RM</td>
<td>Reaction Mass (to the test mass for ESD actuation)</td>
</tr>
<tr>
<td>RoC</td>
<td>Radius of Curvature</td>
</tr>
<tr>
<td>SRC</td>
<td>Signal Recycling Cavity</td>
</tr>
<tr>
<td>TEM_{pq}</td>
<td>Transverse Electromagnetic Mode of Hermite Gaussian type of order p horizontal and q vertical</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>α</td>
<td>Thermal expansion coefficient (K⁻¹)</td>
</tr>
<tr>
<td>α_{coat}</td>
<td>Optical absorption coefficient of surface coating (W/W_{cavity})</td>
</tr>
<tr>
<td>α_{RH}</td>
<td>Optical absorption by ring heater (W/W_{cavity})</td>
</tr>
<tr>
<td>α_{ext}</td>
<td>Optical absorption of empirical term (W/W_{cavity})</td>
</tr>
<tr>
<td>α_Q</td>
<td>ESD quadrant force coefficient (N/V²)</td>
</tr>
<tr>
<td>β</td>
<td>Chree, Lamb [66] spatial parameter (meters)</td>
</tr>
<tr>
<td>b_m</td>
<td>Effective mass scaled ESD overlap factor for mechanical mode m</td>
</tr>
<tr>
<td>c</td>
<td>Speed of light 2.99792×10⁸ m/s</td>
</tr>
<tr>
<td>C</td>
<td>Damping coefficient</td>
</tr>
<tr>
<td>C_{ij}</td>
<td>Temperature frequency conversion matrix</td>
</tr>
<tr>
<td>C_{mnkl}</td>
<td>Elasticity tensor</td>
</tr>
<tr>
<td>C_p</td>
<td>Heat capacity (J/(kg.K))</td>
</tr>
<tr>
<td>□</td>
<td>D’Alembertian operator</td>
</tr>
<tr>
<td>d</td>
<td>Distance to gravitational wave signal source</td>
</tr>
<tr>
<td>d_{orbit}</td>
<td>Distance between orbiting objects</td>
</tr>
<tr>
<td>E</td>
<td>Young’s Modulus (material elastic constant)</td>
</tr>
<tr>
<td>E_i</td>
<td>Amplitude of electromagnetic wave 'i'</td>
</tr>
<tr>
<td>E_{pq}</td>
<td>Amplitude of the electric field of an optical mode of Hermite Gaussian type of order p horizontal and q vertical</td>
</tr>
<tr>
<td>E^{S}_{pq}</td>
<td>Amplitude of the electric field of a scattered optical mode of Hermite Gaussian type of order p horizontal and q vertical</td>
</tr>
<tr>
<td>ε</td>
<td>Relative permittivity</td>
</tr>
<tr>
<td>ε_e</td>
<td>Surface emissivity</td>
</tr>
<tr>
<td>ε_{kl}</td>
<td>Shear tensor</td>
</tr>
<tr>
<td>η</td>
<td>Quantum efficiency (photodiodes)</td>
</tr>
<tr>
<td>η</td>
<td>Minkowski tensor (flat spacetime)</td>
</tr>
<tr>
<td>f</td>
<td>Frequency of oscillation (Hz)</td>
</tr>
<tr>
<td>F</td>
<td>Finesse of optical cavity</td>
</tr>
<tr>
<td>f_s</td>
<td>Sample frequency (samples/s)</td>
</tr>
<tr>
<td>F</td>
<td>Force function (N)</td>
</tr>
<tr>
<td>G</td>
<td>Gravitational constant 6.67408 × 10¹¹ m³kg⁻¹s⁻²</td>
</tr>
<tr>
<td>G</td>
<td>Einstein’s curvature tensor</td>
</tr>
<tr>
<td>g</td>
<td>Gravitational acceleration at the earth’s surface</td>
</tr>
<tr>
<td>g_E</td>
<td>End mirror geometry factor</td>
</tr>
<tr>
<td>g_{factor}</td>
<td>g-factor product g_I·g_E indicates geometric stability of optical cavity</td>
</tr>
<tr>
<td>g_I</td>
<td>Input mirror geometry factor</td>
</tr>
<tr>
<td>G_{s}</td>
<td>Shear (elastic) modulus</td>
</tr>
<tr>
<td>G_{o}</td>
<td>Miao’s[197] optomechanical coupling constant</td>
</tr>
<tr>
<td>γ_{pq}</td>
<td>Half line-width of TEM_{pq} optical mode</td>
</tr>
<tr>
<td>γ_m</td>
<td>Half line-width of mechanical resonant mode</td>
</tr>
<tr>
<td>γ_0</td>
<td>Line-width of fundamental optical mode</td>
</tr>
</tbody>
</table>
\( \hbar \) The Plank constant \( 6.62607 \times 10^{-34} \text{ m}^2\text{kg/s} \)

\( \mathbf{h} \) Metric perturbation tensor

\( h \) small perturbation to Minkowski metric tensor

\( h_+ \) + polarised gravitational wave strain

\( h_\times \) \( \times \) polarised gravitational wave strain

\( J_n \) \( n^{th} \) Bessel function of the first kind

\( k \) Spring constant \( (\text{N/m}) \)

\( k_B \) The Boltzmann constant \( 1.38065 \times 10^{-23} \text{ m} \text{kg/s}^2 \text{K}^{-1} \)

\( \kappa \) Thermal conductivity \( (\text{W/(m.K)}) \)

\( L \) Cavity length

\( \hat{L} \) Restoring force operator

\( L_G \) Gravitational wave luminosity

\( \lambda \) Wavelength of light

\( \lambda_0 \) Wavelength of fundamental optical mode in cavity

\( \omega_{fsr}/2\pi \) Cavity free spectral range in Hz

\( \Lambda \) General overlap parameter (Braginsky’s notation)

\( M \) Mass \( (\text{kg}) \)

\( M_\odot \) Mass of the sun

\( M_{\text{chirp}} \) Chirp mass of coalescing binary system

\( \mu_m \) Effective mass of mode \( m \)

\( N \) Number of (photons, samples, etc)

\( \nu \) Poisson’s ratio

\( \omega \) Radial oscillation frequency \( (\text{rad/sec}) \)

\( \omega_{fsr}/2\pi \) Cavity free spectral range in Hz

\( \omega_g/2\pi \) Gravitational wave frequency \( (\text{Hz}) \)

\( \omega_i/2\pi \) Set of \( i \) mechanical mode resonance frequencies \( (\text{Hz}) \)

\( \omega_{or} \) Orbital Frequency \( (\text{Hz}) \)

\( \omega_{rot}/2\pi \) Frequency or rotation \( (\text{Hz}) \)

\( \omega/2\pi \) Frequency of mechanical mode \( m \) resonance \( (\text{Hz}) \)

\( \omega_0/2\pi \) Frequency of observed resonances \( (\text{Hz}) \)

\( P \) Optical Power \( (\text{W}) \)

\( P_{opt} \) Optimal Optical Power for Quantum noise limited sensitivity \( (\text{W}) \)

\( \pi \) Ratio of circle circumference to diameter \( 3.14159 \)

\( \phi_{eff} \) Effective loss angle

\( \phi_m(\omega) \) Loss factor for mode \( m \) at frequency \( \omega \)

\( \phi \) Loss angle

\( \phi_{rtg} \) Round trip Gouy phase

\( Q_m \) Q factor of mechanical mode of frequency \( m \)

\( Q_{pq} \) Q factor of optical mode with transverse order \( p, q \)

\( R_m \) Parametric gain of mode \( m \) (open loop gain)

\( R_c \) Reflection coefficient

\( R_{\text{mirror}} \) Radius of curvature of ’mirror’

\( r \) radial distance in polar coordinates

\( \mathbf{r} \) Three dimensional vector \( \in \mathbb{R}_3 \)
$r_s$ Schwarzschild radius
$
\rho$
Density (kg/m$^3$)
$
\sigma_{mn}$
Elastic stress tensor
$
S^\nu_{\text{noise}}$
Displacement spectral density of 'noise'
$t$
time
$T$
Temperature (K)
$\mathbf{T}$
Stress energy tensor
$T_c$
Transmission coefficient
$U(\mathbf{r})$
Energy density of elastic deformation
$v$
Velocity
$V_{\text{bias}}$
Bias voltage on ESD
$V_{\text{drv}}$
Measured drive amplitude in QPD volts (Ch. 4)
$V_Q$
ESD control voltage on Q
$V_{\text{rd}}$
Measured readout amplitude in QPD volts (Ch. 4)
$w$
Beam size ($1/e^2$)
$w_0$
Beam size at beam waist ($1/e^2$)
$W_i$
Total energy for set of $i$ modes
$x$
Spatial unit vector
$x_m$
Amplitude of acoustic mode $m$
$\dot{x}_m$
Velocity of acoustic mode $m$
$\ddot{x}_m$
Acceleration of acoustic mode $m$
y
Spatial unit vector
$z$
Spatial unit vector - Distance along optic axis
$z_R$
Rayleigh distance
$Z_s$
Impedance of spacetime
Chapter 1

Gravitational Waves, from Laplace to LIGO and Beyond

The fields of gravitational wave detection and optomechanics have been on a collision course since the first proposals for interferometric gravitational wave detectors in the 1960s and 1970s [126, 257]. This fate is due to the inherent quantum nature of light which dictates quantum uncertainty. Interferometric gravitational wave detectors use light to probe the geometry of spacetime. This light is split by a beam-splitter sent down two orthogonal paths where it interacts with several mirrors. The superposition of the two returning beams detected with a photodetector provide the measure of the difference in spacetime distortion experienced on the two orthogonal paths. Measurements with photons are intrinsically controlled by their discreteness and their momentum. Their discreteness along with the statistical nature of quantum physics means that there are random fluctuations in the number of photons arriving in any measurement time. In common with many statistical phenomena, the fluctuations depends on the square root of the number of photons $\sqrt{N}$, while the fractional fluctuations is given by $\sqrt{N}/N = 1/\sqrt{N}$. This is called photon counting noise. Generally to reduce a counting noise we increase the number of counts, this is what creates the demand for more photons or more optical power. But we neglect the photon momentum. When we measure with light in gravitational wave detectors we bounce photons off a mirror, as we increase the number of photons we increase the total momentum transfer. This too is statistical producing a $\sqrt{N}$ fluctuating disturbance on to the mirror we call radiation pressure noise. The photon counting noise is a quantum statistical white noise - flat in frequency. The fluctuating disturbance on the mirror is also quantum statistical white noise however the dynamics of the mirror introduce frequency dependence. This results in there being a minimum in the sum of these two fundamental noises, for a particular detector design, at a particular frequency and
the frequency of this minimum changes with the number of photons or optical power. Gravitational wave detectors have been designed for optimum sensitivity at \( \sim 100 \text{ Hz} \) requiring an optical power approaching 1 MW, that corresponds to \( N \approx 5 \times 10^{24} \) photons per second. These photons exert a force of \( \sim 7 \text{ mN} \) when reflected off a mirror. That is about the weight of a light feather or hair. It doesn’t sound like much. But in this thesis I will see how tiny forces exerted by photons can radically change the sensitivity of gravitational wave detectors, push mirrors out of alignment and the main subject of this thesis - cause vibrational modes in mirrors to be excited in an unstable exponential ring-up.

The vibrational modes of gravitational wave detector mirrors generally ring with ultra-sonic frequencies. The first instability observed at LIGO had a frequency \( \approx 15.5 \text{ kHz} \). Listening to such an instability at the interferometer output by turning the light signal into an audio signal sounds like a cross between a mosquito slowly flying into your ears and being run down by a train. It is not good for detecting gravitational waves.

In this thesis we will study gravitational waves and detector designs in Chapter 1 with a special focus of the exciting first detection of gravitational waves in 2015. Optomechanics theory is the subject of Chapter 2 and parametric instability theory and a review of experimental results is the subject of Chapter 3. In Chapter 4 we study how the parametric gain can be estimated in a low power regime where the light power would have to be increased by a factor of \( \sim 1000 \) to produce instability. Chapter 5 is a characterisation of the Advanced LIGO arm cavities in respect to their susceptibility to parametric instability, it is described how thermal tuning was optimised to avoid instability allowing the first detection of gravitational waves. In Chapter 6 the two additional techniques that have been demonstrated for controlling parametric instability are presented. Then in Chapter 7 the optomechanics of these instabilities in turned into a tool. A general overview on what could be done with such a tool is presented and two specific implementations are used to obtain valuable information about the thermal state of the Advanced LIGO mirrors.

The control of parametric instability is ongoing research, I spent two months at LIGO Hanford toward the end of PhD candidacy helping to commission parametric instability control. The majority of the Chapter 8 explores the detail and difficulties in commissioning such a system. However there are also several sections that elucidate where further study could greatly help build understanding that would help combat parametric instability. The thesis concludes with a summary of the main achievements.
The ultimate aim of these efforts is the detection of gravitational waves. In that aim the study of parametric instability has so far been successful, parametric instability has not been a limiting factor on gravitational wave detector performance. However it will become increasingly difficult to maintain this claim. I spent eight months at LIGO Livingston and two months at LIGO Hanford helping to establish monitoring and control schemes. I was lucky enough to be at LIGO Livingston the day the first gravitational wave was detected. I saw first hand the way the instrument was operating on the day that humankind opened our ears to the songs of the universe.

Humans have studied the skies with the electromagnetic (EM) spectrum for millennia. The last century has seen the observed EM spectrum expanded from one octave of the visible light spectrum to a massive 80 octave spectrum spanning radio-waves to gamma rays. The detection of gravitational waves is the first sample of a whole new spectrum. Like we’ve been deaf our whole lives and suddenly heard a young girl scream. In the first detection we heard two whole octaves, but we expect that 80 octaves are awaiting discovery.

In the near future we hope to explore gravitational wave frequencies in the entire ‘audio’ spectrum - bass to soprano - with ground based gravitational wave detectors. Then hopefully, infra-sound gravitational wave frequencies will be captured by space interferometers. Ultra-infrasound gravitational wave frequencies may be inferred using some our galaxy’s clocks - a radio millisecond pulsar timing array, as our detector. Finally at the lowest possible frequencies with period comparable to the age of the universe we hope to detect gravitational waves as frozen perturbations in the cosmic microwave background. Then who knows what the spectrum of gravitational wave in ultrasound frequencies might bring.

1.1 Gravitational Waves

1.1.1 History of Gravitational Waves
In 1805 Laplace, in his famous Traité de Mécanique Céleste observed that, “if gravitation is produced by the impulse of fluid directed towards the center of an attracting body” with finite speed, the force of this rotating system does not follow the “right line” connecting the bodies, “tending to decrease the rectangular coordinate to the” attracting body. (quotes are taken from translation by Nathaniel Bowditch [179]). However he concludes the section with the supposition “that the gravitating fluid has
velocity which is at least 100 million times greater than that of light” and so for all intents an purposes is infinite.

This was an observation well before its time with a conclusion that discourages investigation into ‘slow gravity’. It took more than a hundred years for a theory of gravity that explained the delay in the transmission of the force of gravity to be introduced. Published in 1915, the General Theory of Relativity [105] describes such a ‘slow gravity’. The theory not only gave us equations that beautifully described anomalous astronomical observations, ending the need for the imagined planet Vulcan\(^1\). It also predicted gravitational lensing - where light is bent by a strong gravitational field. Gravitational lensing was confirmed in the expedition in 1922 to Wallal Station in Western Australia where light was observed to bend around the sun during a solar eclipse [65]. Finally and most importantly General Relativity gave a conceptual understanding of gravity - curved space entwined with distorted time dictating the way things move. Or as Wheeler put it more eloquently “Matter tells spacetime how to curve and spacetime tells matter how to move” [259]. Einstein had given us a whole new view of the universe.

General Relativity describes gravity as a distortion in spacetime that propagates at the speed of light. As the speed of Laplace’s “gravitating fluid” is finite - orbiting gravitationally bound objects must loose angular momentum just as Laplace described. The loss of angular momentum or energy results in the in-spiral of orbiting objects.

In 1916, the year after General Relativity was unleashed on the world Einstein published “On Gravitational Waves” [105] which described a means of dissipating energy. This was the first description of gravitational waves. As Weyl described [258], Einstein’s description produced gravitational waves of three flavours 1) longitudinal-longitudinal; 2) longitudinal- transverse; 3) transverse-transverse. The story that follows Einstein’s first publications on Gravitational waves is full of drama and controversy.

General relativity describes a 4-D geometry, and gravitational waves an oscillation in this geometry. Could it be that gravitational waves are just an artifact of the co-ordinate system? In 1922 Sir Arthur Stanley Eddington, with the aim of clarifying the speed of gravitational waves, wrote “The Propagation of Gravitational Waves” [103]. He found that not all of Einstein’s solutions existed in different co-ordinate systems. He concludes “that transverse-transverse waves are propagated with the speed of

\(^1\)Vulcan was a hypothesised (and searched for) planet that was required to explain the procession of the perihelion of Mercury under Newtonian gravity
light in all systems of co-ordinates. Waves of the first and second types have no fixed velocity - a result which rouses suspicion as to their objective existence.” Which prompted his comment that “gravitational waves travel and the speed of thought”. Einstein scored one out of three. Gravitational waves appeared to exist and travel at the speed of light, but there was only one flavour.

The story continues in 1936. In an attempt to produce an exact solution rather than the approximations used in 1916, Einstein and his assistant Rosen were convinced that gravitational waves did not exist. This really shows a healthy skepticism! The argument revolved around a singularity that appeared consistently in the calculations [165]. But upon submission to *Physical Review* in his new home of the USA Einstein received his first anonymous editorial review that claimed he was in error (in a detailed 10 page review). He was not impressed, that someone would question his work\(^2\). In a not so healthy show of disrespect for his peers he ignored the review, withdrew the paper and never published in *Physical Review* again [164].

Howard Percy Robertson who had reviewed Einstein and Rosen’s paper had a point. The singularities that were produced were a result of the co-ordinate system, much like the singularity of planetary coordinates at the north and south pole. Einstein later published the paper, with corrections based on suggestions to use cylindrical coordinates relayed to him from Robertson via Einstein’s new assistant Infeild. The argument in the paper dramatically changed, describing what is now called the Einstein-Rosen metric.

Ideas of gravitational wave detectors really came out of the next chapter in the history of gravitational waves. Rosen claimed that gravitational waves do not carry any energy and hence have no objective existence. At the Chapel Hill conference in 1957 [97] Pirani neatly described how gravitational wave make particles move. In arguments between Bondi Feynman and Pirani Rosen’s argument was finally resolved by Feynman’s balls on stick argument. If gravitational waves move particles relative to each other and a stick attached to one of these particle rubs on the other particle then the heat generated through friction proves that gravitational waves do have objective existence. Soon afterwards Bondi [57] published a Nature paper where the balls and stick became the famous rings on stick or sticky bead thought experiment depicted in Figure 1.1.

\(^2\)This was not the norm in Germany at the time, and may have been Einstein’s only editorial review
Figure 1.1: Bondi’s thought experiment to demonstrate gravitational wave carry energy. A passing gravitational wave distorts a stick with rings at the marked positions. (a) the gravitational wave strain is zero, no distortion occurs. (b) the peak negative strain compresses the stick and rings, however the elasticity of the stick provides some reaction force to the induced strain, whereas there is no reaction force for the rings, as a result the rings move relative to the marks on the stick. (c) The peak positive strain elongates the stick, the same argument applies. (d) Once the gravitational wave passes the stick returns to its normal condition.

In this thought experiment the notion that the rings move relative to the stick means that with some friction heat will be generated and energy will have been transferred from the gravitational wave to heat. The motion of the ring relative to the stick is due to the fact the gravitational wave pushing on the rings faces no impedance, while when pushing on the stick it faces the impedance of the elastic restoring force of the stick.

This is the first proposal for a gravitational wave detector. It is a proposal that could only be realised in ones head as its inventors understood. The nature of matter and spacetime makes such a measurement impossible in real life. In the diagram the ring moves twice as far as the mark, this indicates that the stick and spacetime have the same elasticity, or impedance. In reality the impedance of spacetime can be calculated \[ Z_s = \frac{c^3}{G} = 4 \times 10^{35} \text{ kg/s}. \] Spacetime is $10^{22}$ times stiffer than diamond. So the stick and the ring will be distorted in an almost identical manner. In addition the amplitude of the gravitational wave strain $h$ was known to be very
At the time of these discussions, the densest objects known to exist in the cosmos were white dwarf stars with an incredible density of one hundred thousands tonnes/m$^3$. Even two such exotic objects colliding would not produce gravitational waves that, at the time, could have been detected from any appreciable distance. For an observer in line with the orbital axis of two identical white dwarves in a circular orbit, the gravitational wave induced strain $h_+$ measured far from the orbiting stars is:

$$h_+ = -\frac{4GM^2c^2}{d^3} \left(\frac{v^2}{c^2}\right) \cos(\omega_gt).$$  \hspace{1cm} (1.1)

Here $G$ is the gravitational constant $6.67408 \times 10^{-11}$m$^3$kg$^{-1}$s$^{-2}$ and $c$ is the speed of light. The stars of mass $M$ are orbiting at a radial frequency $\omega_{\text{orbit}}$ which is half the radial frequency of the gravitational wave $\omega_g$, they are separated by a distance $d$ from the observer and $t$ is time. This strain describes the relative change in position of two point masses that are otherwise in freefall. If we imagine the the closest known white dwarf at Sirius B is a binary with two 0.5 solar mass white dwarfs of diameter $\sim 10,000$km. The Roche limit indicates tidal forces pull the stars apart when they are separated by 1.3 (for solids) to 2.44 (for fluids) times their diameter so we set $s = 20,000$km. The orbital velocity is $v = \sqrt{2MG/s}$. The gravitational wave strain seen here on earth $d = 8.6$ light years away would be $\sim 10^{-18}$ m/m at coalescence with a frequency of $\sim 1$ Hz. Such measurement at the time would have appeared far outside the realm of possibility.

Feynman and Bondi’s arguments did away with doubts about the existence of gravitational waves. Wheeler who was at the Chappel Hill conference with Feynman and Bondi went on to influence the maverick experimental physicist Joseph Weber who was the first to seriously contemplate the detection of gravitational wave. Discussions on this topic cannot proceed without first giving a brief formal description of gravitational waves.

1.1.2 Description of Gravitational Waves

General relativity describes space as an elastic medium. This elasticity of space is evident from the Einstein field equations[81]:

$$G = \frac{8\pi G}{c^4} T.$$  \hspace{1cm} (1.2)

Changes in the stress, energy tensor, $T$ result in changes in Einstein’s curvature tensor $G$, here $G$ is the gravitational constant and $c$ the speed of light. $G$ is a function of the
metric tensor $g$ and its derivatives. This equation allows the curvature of spacetime that is defined by the metric tensor $g$ to be calculated for a given distribution of matter and energy.

Consider what happens to this metric tensor $g$ in the case that a small perturbation $h$ ($h \ll 1$) is applied to an otherwise flat spacetime defined by the Minkowski tensor $\eta$

$$g = \eta + h.$$  

(1.3)

By choosing the right co-ordinate system one can arrive at a simple form for the linearised Einstein field equations[81]:

$$-\Box \bar{h} = \frac{16\pi G}{c^4} T.$$ 

(1.4)

Here $\bar{h}$ is the trace reversed metric perturbation and $\Box$ is the D’Alembertian operator which is the flat spacetime wave operator.

Consider the results of this small deformation far from any matter or energy such the $T = 0$. Then $\Box \bar{h} = 0$. An analysis of the Riemann tensor must be used to determine what solutions are real. Two independent solutions have the form of monochromatic plane wave metric perturbations

$$\bar{h} = A \cos k_\mu x^\mu.$$ 

(1.5)

Here $k_\mu x^\mu$ can be decomposed into a time varying component and $\omega y t$ and spatial component $\vec{h}\vec{x}$ where the $\vec{x}$ notation indicates a reduction to three spatial co-ordinates from four spacetime co-ordinates.

The spatial varying component has two independent plane wave forms $h_+$ and $h_\times$. The effect of these metric perturbations on the proper spacing between two test particles initially at rest, separated by $\zeta(0)$ is

$$\zeta(t) = \zeta(0) \left(1 + 0.5h_+ \sin^2 \theta \cos 2\phi + 0.5h_\times \sin^2 \theta \sin 2\phi\right).$$ 

(1.6)

Here $\theta$ is the angle of the vector separating the particles from the wave propagation direction vector. $\phi$ is the angle of the vector separating the particles around the wave propagation direction vector. So particles separated purely along the wave propagation vector ($\theta = 0, \pi$) will not change their separation. Particles separated in a direction transverse to the direction of wave propagation have maximum change in separation $|\zeta(t) - \zeta(0)|$ at four angles. Hence why gravitational waves are quadrapole fields.
Just as linear polarisations of light can be added with a phase delay to produce circularly, or more generally elliptically polarised light. $h_+$ and $h_\times$ gravitational waves can be added with a phase delay to produce elliptically polarised gravitational waves. The distortion of a ring of test particles is displayed in Figure 1.2 for a gravitational wave passing through the page.

![Figure 1.2: Gravitational wave of the $h_+$, $h_\times$ and a circular polarisation, acting on a ring of test particles through one cycle, with phase indicated below](image)

### 1.1.3 Astrophysical Sources of Gravitational Waves

A gravitational wave is a quadrapole field. Any objects with a quadrupole moment with a non-zero second derivative in time will produce gravitational waves. Orbiting point masses satisfy this condition. The quadrupole moment is calculated with Equation 1.7.

$$I_{i,j} = \sum_k m_k (3r_{ki}r_{kj} - |r|^2 \delta_{ij})$$

Here $k$ indexes the orbiting particles, with mass $m_k$ and position $r_k$ and $\delta$ is the Kronecker delta. The quadrupole moment has frequency terms $\omega_p$ twice the orbital frequency $\omega_{\text{orbit}}$. This is intuitive if we consider a pair of particles viewed side on, these particles produce a linearly polarised gravitational wave that can be viewed in
Figure 1.3. In this figure we imagine we are viewing a nearby ring of test particles and through them, far away, we see their source.

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Figure 1.3: Near and far views of an edge on binary, of gravitational waves and their source. In the close foreground a ring of test particles shows the gravitational wave strain. In the far distant background the binary stars can be seen in orbit edge on, periodically eclipsing each other. The deformation pattern is shown to have a phase $\phi_g$ that is advancing at double that of the orbital phase $\phi_{\text{orbit}}$. This wave is said to be in the $h_+$ polarisation.

Circularly polarised gravitational waves would result from orbiting astrophysical objects with an orbital plane that faces earth. Note in these figures the phase of the strain amplitude relative the the objects orientation is chosen for aesthetics. In general the relative phase between the gravitational wave and the electromagnetic signal is defined.

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Figure 1.4: Near and far views of an face on binary, of gravitational waves and their source. In the close foreground a ring of test particles shows the gravitational wave strain. In the far distant background the binary stars can be seen in orbit face on. The deformation pattern is shown to have a phase $\phi_g$ that is advancing at double that of the orbital phase $\phi_{\text{orbit}}$. The polarisation has two components separated in phase by 90 degrees $h_+ + jh_\times$.

In Figure 1.4 the circular polarised gravitational wave produced by orbiting astrophysical objects viewed face on is shown. In both cases the gravitational wave frequencies are twice the orbital frequency.
Orbiting astrophysical objects are by no means the only possible source of gravitational waves. Supernova - exploding stars that have been observed every hundred years or so since antiquity, may make significant gravitational waves if the explosion deviates enough from spherical symmetry to produce a significant quadrupole moment. Gravitational waves are produced mainly from the collapsing core of the supernova remnant as long as it is not spherically symmetric and from asymmetric neutrino emission. Recent simulations [124] indicate the short (∼ms) fairly wide frequency band (500 Hz to greater than 2500 Hz) gravitational wave signals from supernova could give valuable information about the supernova process.

Fast rotating neutron stars or white dwarves could also produce gravitational waves if \( R \)-mode instabilities [22] of accreting stars limit rotational speeds. The \( R \)-mode instability spatially distorts the star at a threshold rotational speed such that rotational energy is dissipated to gravitational wave radiation. Similarly bar \( mode \) instabilities have also been proposed [226]. It is also possible that fast spinning neutron stars could support surface structures [41, 196, 143] giving them a quadrupole moment.

Primordial gravitational waves would be created in the early universe if the matter energy density was not uniform. Such primordial gravitational waves would most likely have been generated in the inflationary period \( \sim 10^{-30} \) sec after the big bang. These primordial gravitational waves were claimed to have been detected in 2014 by the BICEP2 collaboration [16]. Very soon after their publication it was suggested that their results were due to the polarisation effect from space dust in magnetic fields. The claim was a based on a search for \( B \)-modes in the polarisation of the cosmic microwave background (CMB) which would be caused by gravitational waves. Data from the Plank mission in 2015 [17] found no conclusive evidence for these primordial gravitational waves in the CMB.

Other than the imprint these gravitational wave leave on the CMB, primordial gravitational waves may be directly detected. Like the cosmic microwave background, these gravitational waves would be red-shifted to very low frequencies. Most models predict a primordial gravitational wave spectrum rising inversely with frequency. The strength and characteristics of these gravitational waves depend critically on the process that produce them and the type of deviation from a uniform mass energy density assumed [78, 40, 175, 107]. This makes any detection of these gravitational waves very exciting as it will likely constrain models of the early universe considerably.

The reason for much skepticism about gravitational wave detection was due to the small amplitude of gravitational wave strain from orbiting stars (Equation 1.1).
However over the 1960’s evidence was found for more compact states of matter. Radio pulses from *pulsars* - rapidly rotation neutron stars, were discovered by Jocelyn Bell Burnell [37]. These stars, the densest known form of matter were analyzed by Bell Burnell during her doctorate, controversially, Hewish her supervisor won the Nobel prize [1] for inventing the aperture synthesis radio astronomy technique that enabled Bell Burnell’s discovery.

In 1962 X-ray emissions [127] from sources which are now known to consist of black holes accreting matter from binary companions [210] were identified. An artists impression of what this exotic duo might look like is shown in Figure 1.5. These objects were detected with the first X-ray telescopes that briefly visited space on board unguided Aerobee rockets.

![Figure 1.5: Artist’s impression of first black hole observations](image)

Black holes had been discussed since 1784 [200]. However this ultimately dense state of matter had only had a theoretical footing since Schwarzschild produced the first solutions to Einstein’s field equations [225].

These compact states of matter completely changed the maximum expected gravitational wave radiation luminosity. As Blair [55] recounts from a lecture given by
Weber in 1974 - normally for a rotating bar the gravitational wave luminosity is given by Equation 1.8

\[ L_G \approx \left( \frac{G}{c^5} \right) M^2 d^4 \omega_{\text{rot}}^6 \frac{2}{45}. \]  

(1.8)

Here \( L_G \) is the gravitational wave luminosity, \( M \) is the mass, \( d \) is the length of the bar and \( \omega_{\text{rot}} \) is the radial rotational frequency. The \( G/c^5 \) term warns us that gravitational waves from average mass average rotation speed sources have very low luminosity. However by replacing the mass term with the mass of a black hole \((r_s c^2 / 2G)\) and radial velocity with linear velocity Equation 1.9 ensues

\[ L_G \approx \left( \frac{c^5}{G} \right) \left( \frac{v}{c} \right)^6 \left( \frac{r_s}{r} \right)^2 \frac{8}{45}. \]  

(1.9)

Here \( r = d/2 \) is half the length of the bar and \( r_s \) is the Schwarzschild radius of the bar. The equation for the gravitational wave luminosity of a black hole bar has inverted the \( G/c^5 \) term. For size approaching the Schwartzchild radius, moving at close to the speed of light the luminosity approaches an incredible \( c^5/G \approx 10^{50} \) W. However I have made an error here by assuming that the quadrupole approximation of Equation 1.2 is still valid in this regime. Linearised general relativity is not the right tool for calculating peak luminosity of coalescing black holes, however as we will soon see this has turned out to be a surprisingly good approximation.

Numerical simulations of general relativity have been used to calculate the luminosity and energy dissipated from coalescing black holes [162]. The numbers are still astounding. For a pair of 30 solar mass black holes the peak luminosity is 200 M\( \odot c^2/sec = 10^{47} \) W, where M\( \odot \) is the mass of the sun. The calculated dissipated energy as they spiral from an infinite separation to coalescence is \( \sim 5\% \) of their rest mass, most of this energy is dissipated in the last 0.2 seconds.

As black holes are expected to range in size from 1.4 M\( \odot \) to 20 billion M\( \odot \) [238] gravitational wave frequencies from coalescing black holes are expected to range from \( \sim 1 \) kHz to \( \sim \mu \)Hz.

### 1.2 Gravitational Wave Detectors

Supernova and indications of black holes and neutron stars in the universe made the detection of gravitational waves plausible. However with little knowledge of the distribution of neutron stars and black holes and little knowledge of the processes that underpinned supernova it was a massive leap of faith to attempt to detect gravitational waves. Weber took that leap of faith, proposing gravitational wave detectors
and emitters in 1960 [253]. In this section Weber’s and subsequent gravitational wave detectors will be described.

1.2.1 Resonant Mass Detectors

The first gravitational wave detectors built by Weber in the late 1960’s were 1.5 ton suspended pieces of aluminium maintained in a vacuum. Strain signals were measured with quartz transducers. *Weber Bars* were thermal noise limited at frequencies of maximum sensitivity around the normal modes of the cylinder [254]. The principle was that of a bell. When a gravitational wave (with frequency close the a normal mode resonance) passes, it induces a strain in the bar. This strain is amplified by the resonance of the bar producing a change in the normal mode’s amplitude. Figure 1.6 is a reproduction of the image in Weber’s paper describing gravitational wave antenna [254].

![Gravitational Wave Antenna](image)

Figure 1.6: The first gravitational wave antenna, showing the suspended bar, quartz transducer and vacuum system. Copied from [254]

After two years of operation Weber reported that there were many unexplained coincidences between two antenna separated by 1000 km [255]. Two years later Weber claimed to have detected gravitational waves [256]. The science community took notice and many groups around the world built Weber Bars.

The excitement was such that the *Lunar Surface Gravimeter* (LSG) was sent aboard the Apollo 17 mission to be installed on the moon in 1972. The LSG’s primary objective was to measure lunar free-mode oscillations induced by gravitational waves [128]. Unfortunately the instrument partially failed due to a design flaw. LSG’s sensitive bandwidth was reduced and there was no evidence of lunar free-mode oscillations, only a string of lunar seismic events [161].
Groups around the world running bar detector experiments quickly began publishing their results. These publications [125, 244, 100, 109], indicating null results, cast doubt on Weber’s claims. Several of these groups had improved detector designs resulting in increased sensitivity [244, 181, 207].

As Weber noted his bar was thermal noise limited. To improve the noise performance proposals were made for cryogenic gravitational wave detectors. Cryogenic temperatures enabled better transducers such as capacitive or inductive superconducting quantum interference devices called squids [248] and superconducting parametric transducers [247]. Suspensions were improved and bar materials were chosen carefully to attain high Q factors at cryogenic temperatures to increase the resonant amplification of the bar. A network of 5 such detectors called Niobe, Nautilus, Allegro, Explorer and Auriga operated in the in the late 1980’s and 1990’s [240, 31, 193, 30]. Almost 5 orders of magnitude improvement in bar detector sensitivity had been achieved, however still gravitational waves eluded detection.

Figure 1.7: Three examples of resonant mass detectors. One of Weber’s bars now an Exhibit at LIGO Hanford, Niobe which is now on display at the Gravity Discovery Center Western Australia and Minigrail, still in operation in Denmark.

The Niobe detector shown in the center panel of Figure 1.7 used a parametric transducer to read out the bar resonant modes [52]. This system developed at the University of Western Australia was a very sophisticated optomechanical transducer based on microwave superconducting cavities. Like modern interferometric gravitational wave detectors it required extreme frequency stabilisation, extremely high Q factor electromagnetic and acoustic resonators (Introduced in Chapter 2 Sections 2.2 and 2.3). The bar itself had a Q factor of $2.3 \times 10^8$. The system made use of two mode optomechanical interactions (see Chapter 2 Section 2.4). It made use of detuning to
achieve optomechanical tuning and a problem of parametric instability [55] had to be solved. Tobar suggested using a refined microwave re-entrant cavity as the transducer [241] could allow measurements close to the quantum limit of 1 kHz resonators. This parametric optomechanical transducer is governed by the same interactions discussed in the next chapter.

Unfortunately in 1987 all the cryogenic detectors were off-line for upgrades. SN1987A, the bright supernova visible to the naked eye from a distance of 51 kpc was picked up by Neutrino detectors [150] and Amaldi and Weber reported coincident events [19, 263] between room temperature bar detectors and the Mont Blanc Neutrino detector 3 hours prior to the main neutrino event, however this has widely been discredited. It is not thought to be associated with gravitational waves from the supernova largely because no plausible physical explanation has been proposed [263].

After more than 10 years running state of the art cryogenic bar detectors no detection of gravitational waves had been detected. A new approach was required. Proposals and development for geometrically optimised spherical detectors were underway. One such spherical detector Minigrail operated for science runs in 2006 and 2008 [85]. There were also experiments with materials such a silicon and quartz [18]. However significant funding for resonant mass detectors evaporated with advances in interferometric techniques. In Figure 1.8 the remarkable improvements achieved with cryogenic resonant bar detectors is displayed. They were almost 5 orders of magnitude better than Weber’s original bar detectors.
The improvement achieved with the first generation of laser interferometer gravitational wave detectors is equally impressive. Since the construction of large scale interferometric gravitational wave detectors most resonant mass detectors have been decommissioned.

1.2.2 Laser Interferometers

Designs for Interferometric gravitational wave detectors had been proposed in the 1960s [126]. However many of the technologies required for such a project were in their infancy. In 1972 Rainer Weiss made a detailed proposal [257] for a suspended test mass Michelson interferometer with optical delay lines in place of the normal Michelson interferometer arms. He did a detailed noise analysis that led to the conclusion that a 1 km interferometer could be four orders of magnitude better than a typical Weber bar type detector.

A Michelson interferometer is a perfect instrument for detecting linearly polarized gravitational waves, as shown in Figure 1.9. An input light is split at a beam-splitter such that it is directed at two mirrors. These mirrors reflect the light back at the beam splitter such that the two beams recombine. The recombined light produces interference that moves with changes in differential arm length. Without special care this interference pattern appears as stripes of bright and dark. If a bright stripe moves
by the distance between one bright stripe and the next - a fringe, the differential arm length changed by half a wavelength.

Figure 1.9: A Michelson interferometer being distorted by a + polarised gravitational wave.

The rationale for proposing delay lines, as shown in Figure 1.10, was that a simple Michelson interferometer of any reasonable size was not sensitive enough. To give a sense of scale imagine a 1 km Michelson interferometer with resolution similar to what Michelson and Morley claimed [201] to achieve, \( \sim 1/50 \) of a fringe\(^3\). Such a detector can achieve a stain sensitivity \( \lambda_0/100L = 10^{-11} \) which is far worse than bar detectors.

\(^3\)Of course lasers and photodiodes now provide far better resolution than Michelson and Morley’s \( 1/50^{th} \) of a fringe.
Figure 1.10: Diagram of a delay line interferometer. The laser light is split by the beam splitter passing through holes in mirror 3 and 4, the light bounces back and forth in the delay line arms \((N=3\) times are shown) before passing through a second hole in mirrors 3 and 4 before being recombined at the beam splitter. The beam heading towards the photodiode destructively interferes (ideally no light) while the majority of the light directed toward the laser is wasted.

A delay line with \(N\) round trips in each arm can increase the effective length by \(N\), each round trip the light picks up the additional phase from the change in length created by a passing gravitational wave. The sensitivity can be increased in this manner until the storage time in the delay line approaches half the period of the gravitational wave. So to detect 100Hz gravitational wave the ideal effective length would be 1500 km.

Several prototype detectors were built on the ∼1-10 m scale. Notably Billing’s Munich argon interferometer [42] in 1981, followed by the Garching 30 m prototype in 1988 [227] and the 10 m prototype at Glasgow [209]. The Garching instrument and the Glasgow instrument did the first joint interferometer science run in 1989. These prototype detectors improved with technological advances. They demonstrated that it was technologically possible to realise suspended interferometer configurations. Suspension vibration isolation systems are critical to reducing ground vibration at the test mass.

Many improvements in laser technology and control systems were developed for the purpose of building these prototypes such Pound-Drever-Hall locking that will be discussed in the following chapter. Drever [99] extended the well known microwave resonator stabilization technique [215] to optical frequencies, he used this technique
to stabilize Fabry Pérot optical cavities in place of Weiss’s delay lines.

Shot noise limited sensitivity was achieved in these early prototypes [227]. Shot noise was introduced at the beginning of this chapter as photon counting noise, a fundamental Poisson noise associated with the measurement of photons. Shot noise is expressed as a displacement noise spectral density in Equation 1.10.

\[ S_{SN}^{x} = \frac{\hbar c \lambda}{\pi \eta P} \]  \hspace{1cm} (1.10)

Here \( S_{SN}^{x} \) is the displacement spectral density of shot noise, \( \eta \) is the photodiode quantum efficiency, \( P \) the optical power, \( \hbar \) Plank’s constant and \( c \) the speed of light. Shot noise is reduced by using shorter wavelength, high quantum efficiency diodes or increasing the optical power. One way to increase to optical power is to reuse light escaping the interferometer.

The idealised interferometer configuration for gravitational wave detection is a carefully designed interferometer where the wavefronts of beams are so perfect that there is complete destructive interference at the dark power (where the photodetector is in Figure 1.10) and complete constructive interference at the bright port (in the direction towards laser, the wasted light in Figure 1.10). When a gravitational wave passes the differential arm length changes and the signal can be measured as some light on the detector. Under this arrangement the light returning towards the laser is the wasted light we wish to reuse or recycle. To recycle the light a mirror is added in front of the laser, shown as PRM in Figure 1.11.
This turns the whole interferometer into a cavity. If tuned correctly light will resonate in the entire interferometer. This increases the optical power in the detector [195]. The power recycling configuration can easily lead to a 100-fold increase in optical power in the interferometer.

Early prototype detectors that used combinations of optical cavities, power recycling and delay lines quickly surpassed the sensitivity of Weber’s bar. This convinced the scientific community that investment in kilometer scale interferometric gravitational wave detectors was worthwhile [80]. The first generation of gravitational wave detectors were planned and constructed through the 1990’s. They were all Michelson interferometers. They include: two LIGO 4km power recycled (PR) Fabry Pérot (FP) arm cavity detectors in the USA, the VIRGO 3km PR, FP detector in Italy, GEO600 a 600 m PR folded arm interferometer in Germany and TAMA a 300 m PR FP in Japan. The principle of these detectors was all similar. (a) Isolate the mirrors from the ground with suspension systems. (b) Store light in the arms to get a long effective arm length and (c) add a power recycling mirror to accumulate light to reduce shot noise. Figure 1.11 shows a schematic of the general first generation detector configuration. These detectors operated in joint science runs from 2002 to 2011. In this configuration ∼10 W of input laser power is resonantly enhanced with the addition of the power recycling mirror (PRM) to produce ∼1 kW incident on the
beam splitter. An input mode cleaner is included in the diagram as it is essential to reduce the laser intensity and phase noise. The peak sensitivity achieved with the first generation of gravitational wave detectors approached $10^{-23}/\sqrt{\text{Hz}}$ for short duration gravitational waves bursts.

Plans for Advanced LIGO and Virgo were being made prior to constructing initial detectors. Upgrades to virtually every part of the detectors were required to increase sensitivity by an order of magnitude. Better suspensions were designed, higher power lasers were designed, bigger mirrors with smoother surfaces were required, soft electrostatic actuation replaced magnet coil actuators and signal recycling was introduced. The configuration of advanced LIGO is a dual recycled Michelson interferometer with DC readout. A schematic is shown in Figure 1.12. The dual recycling refers to the addition of a signal recycling mirror (SRM). This allows the detector to be optomechanically tuned as we will see in Section 2.5.2. The tuning used in Observation Run 1 was resonant sideband extraction [206] where the storage time of the signal sidebands is reduced. DC readout refers to the fact that the gravitational wave signal is readout from the DC level of the beam transmitted through the output mode cleaner (OMC). To get the amplitude and phase with the DC readout technique, a small amount of the carrier that is resonant in (and very well filtered by) the arm cavities is allowed to leave the dark port by detuning one arm cavity length with respect to the other. The carrier light beats with the signal sidebands transmitted through the OMC allowing the recovery of the gravitational wave amplitude and phase. This technique was demonstrated at the Caltech 40 m prototype in 2007 and was shown to improve sensitivity at GEO600 [149] and initial LIGO [121].
Figure 1.12: Advanced generation interferometric gravitational wave detector design, showing input mode cleaner (IMC) beam splitter (BS) input test masses (ITMX and ITMY) end test masses (ETMX and ETMY), Signal recycling mirror (SRM), output mode cleaner (OMC) and a photodetector for detecting changes in the relative length of the two arms.

The configuration shown in Figure 1.12 is referred to as the advanced detector configuration. There are several differences between different detectors. Advanced LIGO [142] has 40 kg fused silica test masses at room temperature. It is designed for 800 kW optical power contained in the arm cavities. The parameter that defines the laser spot size on the mirrors is called the cavity $g$ factor described in Chapter 2 Section 2.3.1. Advanced LIGO arm cavities have $g$ factors of 0.81. This $g$ factor tunes parametric instability as will be described in Chapter 3. Advanced VIRGO [75] has 42 kg fused silica test masses at room temperature. It is designed for 760 kW optical power contained in the arm cavities. These cavities have a $g$ factor 0.92 [90]. The Large Cryogenic Gravitational wave Telescope (LCGT) “nicknamed” KAGRA [28] is a 3 km under ground detector with 22.7 kg cryogenic sapphire test masses. It is designed for 900 kW optical power contained in the arm cavities. These cavities have a $g$ factor 0.34.

Several notable prototypes were constructed or revamped to test technologies for advanced and third generation gravitational wave detectors. These include the 40 m prototype at Caltech [251] where alignment and control techniques have been tested. The AIGO High Optical Power Test Facility described in Section 1.2.4 where issues
with the operation for high power, high g factor high finesse suspended optical cavities are investigated. Finally the 10 m prototype at the Albert Einstein Institute in Hanover [131] is constructed for testing next generation gravitational wave detector technologies.

In the next section I examine the noise sources that limit Advanced LIGO sensitivity and explain some of the design requirements that lead to optomechanical interactions such as parametric instability.

### 1.2.3 Noise Sources in Laser Interferometers

Commissioning advanced gravitational wave detectors is an endless hunt for sources of noise. A particular peak in the noise spectrum that does not fit the model is cause for investigation. When the source of the noise is identified it is either fixed and/or added to the model such that ideally the entire noise spectrum is understood. In that way efforts can be directed towards reducing the sources of noise that will produce the most benefit.

Martynov [190] was responsible for determining many of these noise contributions found during commissioning of Advanced LIGO Livingston. Two figures from Martynov’s thesis are reproduced here to demonstrate how detector sensitivity is limited by a variety of noise contributions. Figure 1.13 shows the noise spectrum of advanced LIGO during observation run 1 compared to modelled noise contributions (and their sum).
Figure 1.13: Sources of noise in advanced gravitational wave detectors. Reproduced from [190]

Comparing the measured noise spectrum (red) and the sum of modelled noises (green) demonstrates the detector sensitivity is well understood at frequency greater than 200 Hz. In this region the sensitivity is limited primarily by quantum noise. Under 200 Hz the sum of modelled noises deviates from the measured noise spectrum, indicating improved models are required. I will not investigate each trace in this figure in detail. Instead I refer to another figure from Martynov’s thesis with just the fundamental noise sources.
Figure 1.14: Fundamental sources of noise in advanced gravitational wave detectors. Reproduced from [190]

Figure 1.14 introduces the set of noises that are inherent for the particular detector design. The purpose of introducing these noise curves is firstly to describe the ultimate goal of the control of parametric instability. Secondly, the motivation for detector design choices that ultimately lead to parametric instability can be highlighted.

The ultimate goal of parametric instability control is to improve the sensitivity of advanced gravitational wave detectors. As such any control scheme must not introduce or increase noise contributions to the main interferometer output. We do not want to add another line to Figure 1.13 due to a noise contribution from parametric instability control.

In the following mini-sections the noise contributions that are relevant for parametric instability are described in reference to Figure 1.14.

a) Quantum noise

We examined in the previous section how shot noise limited prototype gravitational wave detector sensitivity. Advanced detector configurations are designed to roughly achieve the free mass *standard quantum limit* (SQL) - this is the limit dictated by Heisenberg’s uncertainty relation where the contributions from from shot
noise and radiation pressure noise are equal. Rough estimates of these quantities can be made by examining a simple Michelson interferometer (following Saulson’s derivation [224]) where all the light leaving the interferometer is in one fringe 

\[ P = P_0 \times \cos^2 \pi/\lambda_0(L_x - L_y), \]

where \( \lambda_0 \) is the wavelength of light. Coherent lasers produce photons with a Poisson distribution. As such the root mean squared fluctuation when counting \( N \) photons is \( \sqrt{N} \). The number of photons in a measurement of power \( P \) over time \( t \) is \( N = \lambda_0/(2\pi\hbar)Pt \), where \( \hbar \) is the Plank constant. The best sensitivity in this case will be achieved when \( P = P_0/2 \). For analysis of detector sensitivity the noise needs to be expressed as a spectral density \( S_{sn} \). The shot noise is given by [224]

\[ S_{sn}(f) = \frac{hc\lambda_0}{2\pi P}. \]  

(1.11)

Here \( c \) is the speed of light. Note that \( S_{sn} \) is frequency independent.

As already indicated, the radiation pressure fluctuations disturb the mirror. Radiation pressure exerts a force \( 2P_0/c \) which again has Poisson statistical fluctuations that along with the mirror dynamics result in a radiation pressure noise displacement spectral density \( S_{rp} \):

\[ S_{rp}(f) = \frac{1}{M^2 f^4} \frac{\hbar P}{2\pi^3 c\lambda_0}. \]  

(1.12)

Here \( M \) is the mass of the test mass. Note that \( S_{rp}(f) \) have a \( f^{-4} \) frequency dependence which is imposed by the response of the suspended mirrors.

The optimum power for a gravitational wave detector may be found by equating \( S_{sn} \) and \( S_{rp} \):

\[ P_{opt} = c\lambda\pi Mf^2 \]  

(1.13)

Selecting a reasonable mass \( M = 10 \) kg requiring optimum sensitivity at 100 Hz and a laser wavelength of 1 \( \mu \)m we find the optimum power of \( P_{opt} = 10^6 \) W. This is a very simplified analysis, it is clear in Figure 1.14 that the quantum noise at high frequency is not flat as would be expected by Equation 1.11. This is because the cavity storage time has not been taken into account. This simple analysis does however

![Figure 1.15: Diagram of the quantum noise limited displacement spectral density for a Michelson interferometer showing high medium and low power quantum noise limited sensitivity and the free mass standard quantum limit.](image)
provide an order of magnitude estimate of the power level required to achieve the free mass standard quantum limit (SQL) which is defined as the minimum noise that can be achieved at any power. The displacement noise spectral density of the SQL, $S_{\text{sql}}^x$, is displayed in Figure 1.15 and given by Equation 1.14.

$$S_{\text{sql}}^x = \frac{\hbar}{M\pi^2 f^2} \quad (1.14)$$

The large power required to reach the SQL at 100Hz leads inevitably to some complex and initially unexpected optomechanical interactions in advanced gravitational wave detector configurations, including the main subject of this thesis, parametric instability.

b) Substrate thermal noise

The fluctuation dissipation theorem states that where motion results in dissipation a reverse process exists where thermal fluctuations result in motion. As such any lossy system will move due to statistical temperature fluctuations. Gillespie and Raab analysed this thermal noise contribution from the test mass substrate with an eigenmode expansion [129] using the equipartition theorem. This is a reasonable approximation for materials with homogeneous loss angle, for more complex objects the approached derived by Levin [180] presented in the following section must be used. Gillespie and Raab show the displacement spectral density $S_{\text{sub}}^x$ contribution from the $m^{\text{th}}$ normal mode to be

$$S_{\text{sub}}^x(\omega) = \frac{4k_B T}{\mu_m \omega} \left[ \frac{\omega_m^2 \phi_m(\omega)}{(\omega^2 - \omega_m^2) + \omega_m^4 \phi_m(\omega)^2} \right]. \quad (1.15)$$

Here $\mu_m$ and $\omega_m$ are the mode effective mass and radial frequency of mode $m$, $k_B$ is the Boltzmann constant, $T$ is the temperature, $\phi_m(\omega)$ is the energy loss per cycle at radial frequency $\omega$. $\phi_m(\omega_m) = 1/Q_m$ considering a single mode $m$. From examining Equation 1.15 we find that frequencies far from resonance have a lower thermal noise contribution if the Q factors of modes closest to the gravitational wave detection band are high. The test mass eigenmodes of advanced gravitational wave detectors have frequencies higher than the gravitational wave detection band so the Q factors of the lowest eigenfrequencies of the test mass are most important. To demonstrate the principle Figure 1.16 displays the noise contribution calculated with Equation 1.15 for the first 10 Advanced LIGO test mass eigenfrequencies assuming they all have an effective mass of 10 kg and for two assumed Q factors $10^6$ and $10^7$. 

28
Figure 1.16: Advanced LIGO test mass substrate thermal noise contribution assuming two different Q factors $10^6$ and $10^7$ for all modes. Thermal noise in the 0-4 kHz gravitational wave detector band is clearly decreased with larger Q factors.

In Figure 1.16 we see that increasing the Q factor increases the peak thermal noise while it decreases the noise outside the eigenmode linewidth. We are interested in frequencies from 30 Hz to 4 kHz. Ideally we therefore would choose the largest Q factor mirror possible. However as we will see in Section 3.2, susceptibility to parametric instability scales linearly with test mass eigenmode Q factor, which means that reducing the losses, to reduce thermal noise, increases the likelihood and severity of parametric instability.

c) Coating Thermal Noise

The coating Brownian thermal noise results from the acoustic losses of the multiple $\text{SiO}_2$, $\text{Ti}_2\text{O}_5$ dielectric layers that provide the high reflectivity coating on the test masses. Again it arises in accordance with the fluctuation dissipation theorem. For a Gaussian profile laser beam sensing the position of a coated mirror the spectral noise contribution derived by Levin [180] and is given by Harry et al [141] as

$$S_{\text{coat}}^x(f) = 2k_B T \phi_{\text{eff}} \frac{1 - \sigma}{\pi^{3/2} f_w E}.$$  \hspace{1cm} (1.16)

Here $S_{\text{coat}}^x(f)$ is the displacement noise power spectral density, $k_B$ is the Boltzmann constant, $T$ is the temperature, $\sigma$ and $E$ are the Poisson ratio and Young’s modulus of the substrate material, $\phi_{\text{eff}}$ is the effective loss angle of the mirror, $w$ is the half width of the Gaussian beam on the mirror. The equation is valid far from resonance. Levin [180] describes how in the case where the surface being sensed is composed of a material with a larger loss angle the contribution to the noise spectral density...
scales with $1/w^2$ rather than $1/w$ as in Equation 1.16. So even though the high reflectivity coating deposited on the mirrors do not significantly reduce the Q factor of the entire mirror, they significantly increase the displacement noise, particularly for small Gaussian beams. There is therefore a requirement to make the beam spots on the mirror as large as possible. This is achieved by designing high g factor cavities which will be explained in Chapter 2 Section 2.3.1. Unfortunately the use of high g factor cavities generally increases susceptibility to parametric instability, but in a complex manner that will be introduced in Chapter 3, Section 3.2.6.

\textit{d) Seismic noise}

Vibration isolation from seismic noise is a critical component of gravitational wave detectors. Here I will not discuss details, but will discuss a few issues relevant to this thesis. Isolation systems generally are composed of active and passive components that represented by a cascade of mass-spring elements with control elements. While seismic noise does not have a direct impact on parametric instability we will see in Chapter 6 Section 6.3 that low frequency residual motion of the optics can alter the very sensitive parametric instability interactions. In some cases residual motion can dramatically reduce the expected severity of instabilities. The test mass residual motion at low frequency generally results from finite bandwidth control loops and residual motion of low frequency suspension resonant modes. As different detectors use very different suspension systems there may be significant variability between sites. However generally with either passive of active isolation systems we expect excess low frequencies motion for the following reasons.

Generally passive type suspension systems use a low frequency pendulum resonance as a mechanical low pass filter. However such a filter has positive gain on resonance which would amplify ground motion. Control systems are often used to lower the residual motion on resonance.

Active suspensions generally use a combination of feed forward and feedback. As such good inertial sensors are required for feedback and good seismometers for feed forward of ground vibration. Poor low frequency sensitivity and cross coupling of tilt and horizontal degrees of freedom make it difficult to achieve large isolation factors at low frequency [94].

\textbf{1.2.4 Gingin High Optical Power Test Facility}

\textit{Gingin} is the site of a gravitational wave detector test facility called the High Optical Power Test Facility (HOPTF). Much of the work described in this thesis took place at the HOPTF, where we learnt to create and control parametric instability. For this
reason I here give some details of the facility and the optical cavity in which the first
demonstration of parametric instability in a suspended, high power optical cavity was
achieved (Chapter 6, Section 6.3).

The HOPTF has been used for testing the UWA compact seismic vibration isolation systems and for investigating high optical power effects in gravitational wave detectors such as thermal effects and optomechanical effects. The site has a vacuum envelop for an 80 m Michelson interferometer. However during my candidature it was set up as two ∼80 m linear optical cavities to suit experimental objectives. These two experiments called the East Arm and South Arm experiments can be seen in Figure 1.17 as viewed from the top of the leaning tower of Gingin⁴. The following mini-sections summarise the key technical features of the HOPTF.

![Figure 1.17: AIGO, site of the Gingin high optical power test facility, the south arm and east arm experiments are marked. Other buildings visible are; a science museum called the Gravity Discovery Center, a public observatory, the Zadco robotic telescope, a geodesic gallery and an accommodation building](image)

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a) Isolation and Suspension Systems

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⁴The leaning tower of Gingin is a scale model of the leaning tower in Pisa Italy, where Galileo’s fabled experiments took place
The vibration isolation system at the Gingin HOPTF is called the *UWA compact vibration isolation system*. It is a multistage system with some similarity to the VIRGO super attenuator [76, 222]. It uses similar isolator resonant mode frequencies and a similar number of stages, each stage is based on innovative technologies that include Euler springs, La Coste springs, self-damped pendulums and Roberts Linkages [101, 260, 34]. The mirror suspension system uses a unique modular niobium cantilever system that has demonstrated very low acoustic losses [119]. In Figure 1.18 a picture and simplified diagram of the isolator with dominant resonant frequencies marked. Each stage has active control loops (not shown) to damp resonances.

![Figure 1.18: Left panel, an image of the UWA compact vibration isolation system. Right panel, schematic of the isolation system with resonant frequencies indicated.](image)

We will see in Section 6.3 how residual motion in some of the suspension resonant modes suppress parametric instability.

*b) Optical Cavities*

The experimental objectives at the HOPTF 2012 to 2016 were to achieve the conditions where parametric instability could be directly observed. To suit this objective two ~80 m optical cavities with high reflectivity mirrors, with g factors close to the limit of geometric stability were constructed.
The *East Arm Cavity* was set up specifically to study parametric instability. This required achieving conditions comparable to those of Advanced LIGO, but in the sub-scale format possible at the HOPTF. The two mirrors are supported by the suspension systems described in mini-section a) above. The 80 m hydrocarbon free vacuum envelope is maintained at $10^{-7}$ Torr.

Figure 1.19: Optical layout of the Gingin HOPTF east arm cavity. An Non-Planar Ring Oscillator (NPRO) seed laser pumps a commercial (Nufern) 50 W fiber amplifier. The laser is locked to the cavity using Pound-Drever-Hall locking. A $CO_2$ laser is used to tune the radius of curvature of the input test mass.

Figure 1.19 shows a schematic of the optical layout. A 500 mW Non-Planar Ring Oscillator (NPRO) provides a stable seed laser. A commercial 50 W Nufern Fiber laser amplifier is pumped by the NPRO. When required this beam is then passed through a pre-mode cleaner (Not shown) to spatially filter the beam and reduce phase and intensity noise. This beam is delivered to the $L = 74$ m cavity at the input test mass (ITM) as a 11.5 mm beam. A cartoon of the cavity is shown in Figure 1.20 with parameters labelled.

The radius of curvature (RoC) of the ITM is $R_{ITM} = 37.4$ m and the end test mass (ETM) $R_{ETM} = 37.3$ m. This ‘cold’ RoC should produce a cavity waist of $w_0 = 1.1$ mm approximately in the center of the cavity and the beam size on the ITM and ETM is $w_{ITM} \approx w_{ETM} \approx 11.5$ mm. However the small optical absorption of the mirror’s coatings result in some of the circulating beam being absorbed, heating the mirror. Through thermal expansion the RoC changes. The tuning rates for both absorption from the circulating laser and intentional deformation induced by heating
with a CO₂ laser are reported in Table 1.1. These were estimated from the direct measurement of the change in frequency of the first order transverse mode resonant in the cavity, this process is described in Chapter 2 Section 2.3.2. Pound Drever Hall locking that will be described in Section 2.3 is used to maintain the cavity on optical resonance. Reflected signals and transmission signals are available for interrogation of the cavity.

The vibration isolation system has a real time control system that enables damping of resonant modes and maintains the vibration isolation system in its linear range. It also provides controls to enable the user to slowly adjust mirror positions and alignments.

An extensive characterisation of the East Arm cavity was performed by Fang, Blair and Zhao [118]. An interesting effect observed in this cavity was degeneracy between the fundamental and very high order transverse optical modes (10th – 20th order). It is thought that these degeneracies cause up to 30% reduction in the cavity finesse at specific mirror RoC [118]. The cavity parameters measured in these investigations are given in Table 1.1:
Table 1.1: Measured east arm cavity parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>TM Diam.</td>
<td>100 mm</td>
<td>Specified test mass (TM) diameter</td>
</tr>
<tr>
<td>TM Wid.</td>
<td>50 mm</td>
<td>Specified TM thickness</td>
</tr>
<tr>
<td>$R_{ETM}$</td>
<td>37.3 m</td>
<td>Specified end TM (ETM) radius of curvature</td>
</tr>
<tr>
<td>$R_{ITM}$</td>
<td>37.4 m</td>
<td>Specified input TM (ITM) radius of curvature</td>
</tr>
<tr>
<td>ETM T</td>
<td>20 ppm</td>
<td>Specified ETM transmission</td>
</tr>
<tr>
<td>ITM T</td>
<td>200 ppm</td>
<td>Specified ITM transmission</td>
</tr>
<tr>
<td>Mass</td>
<td>880 g</td>
<td>Specified TM mass (density 2200 kg/m$^3$)</td>
</tr>
<tr>
<td>L</td>
<td>73.929 m</td>
<td>Measured cavity length</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>14500±300</td>
<td>Measured cavity finesse</td>
</tr>
<tr>
<td>g-factor</td>
<td>0.98-0.90</td>
<td>Measured cavity g factor $g_l \approx [-0.995, -0.93], g_E \approx -0.991$</td>
</tr>
<tr>
<td>$\frac{\partial R_{ITM}}{\partial P_{CO2}}$</td>
<td>$\sim1$ m/W</td>
<td>Change in ITM radius of curvature with CO2 laser power</td>
</tr>
<tr>
<td>$\frac{\partial R}{\partial P_{cav}}$</td>
<td>$\sim0.015$ m/kW</td>
<td>Change in TM radius of curvature cavity with change in contained power (due to coating absorption)</td>
</tr>
</tbody>
</table>

A simple technology that has been very useful at the HOPTF is the “Donger” - a small solenoid driven hammer. Dongers are installed on both test masses of the East arm to excite resonant modes of the test masses. Both birefringent (crossed polariser) readout and surface deformation readouts described by Fang et al [120] were used to measure acoustic mode frequencies and Q factors. This was combined with finite element modelling to identify, resonant modes of the test masses.
Figure 1.21: Gingin east arm test mass eigenmode shape generated with Ansys [23]. Frequency and Q factor measurements shown were made on the input test mass.

Figure 1.21 shows a selection of resonant modes and Q factors for the ITM. Investigating mode Q factors, Fang et al [120] found the contributions to the loss angle from the suspension system and substrate. The modular, replaceable suspension system contributed less than 10% of the substrate loss at 100 Hz. This was an important result as the high Q factors achieved confirmed that the east arm is a suitable cavity for the study of parametric instability control. Also this investigation led to an improved
finite element model such that the mode shape of parametrically unstable modes at frequencies of at 152 kHz presented in Chapter 6 Section 6.3 could be identified with some confidence.

The *South arm cavity* at the Gingin HOPTF was used in experiment reported in Chapter 4 and many of the experiments reviewed in Chapter 3, Section 3.3 and Chapter 6 Section 6.3. The cavity was first set up to observe thermal effecting in high finesse, high power optical cavities. The cavity is half symmetric cavity on single stage suspensions.

![Image: South Arm 77m half symmetric cavity](image)

**Figure 1.22**: South Arm 77 m half symmetric cavity, labelled parameters are $R_{ETM}$ the radius of curvature of the end test mass (ETM) and $w_0$, $w_{ETM}$ the beam size of the beam waist (minimum size) and beam size on the ETM respectively.

The south arm cavity has sapphire mirrors with flat ITM shown in Figure 1.22 and an ETM RoC of $R_{ETM} = 720$ m. The g factor tuning is achieved by means of a fused silica compensation plate wound with 6 turns of Nichrome wire (not shown). Finesse of 1400 have been achieved.

The above optical cavities have played a critical role in experimentally verifying parametric instability theory and testing techniques for parametric instability avoidance and suppression. In Chapter 5 and Chapter 6 Section 6.2 we will see how these techniques were successfully applied at Advanced LIGO resulting in the first detection of gravitational waves.

### 1.3 Direct Detection of Gravitational Waves and the Future of Gravitational Wave Astronomy

14 September 2015 marked the end of the drama-filled quest to detect gravitational waves. There is one last historical drama to relate. This drama had large personal and
psychological implications for many people in the gravitational wave community. In 2010 there had been an “event” that had sparked the gravitational wave community into action. The event nicknamed the *Big Dog* event was a signal that appeared in the gravitational wave detector channels of initial LIGO and VIRGO. The article written by Jonah Kanner and Alan Weinstein [155] about the Big Dog event could easily have been written about the 2015 event. They talk about the meticulous months of work required to verify they data. They relate the difficulties in getting 700 scientist to agree on the wording of a paper. I can imagine the despair and frustration when an envelope was opened in “Arcadia in 2011 that told us all that the Big Dog was a Big Fake” [155]. It had been a *blind injection*. A gravitational wave signal had intentionally injected into the instrument to test of the whole process end to end. A practice run.

Publication of the paper in February 2016 [13] must have been a huge relief for those who had been through the ordeal of the *Big Dog* event. It was also a time of huge excitement. At the Livingston site, the event had first been talked about with skepticism. “Too perfect”, “too soon” were common opinions. However you could see the glint of excitement in peoples eyes, and people got to work. To do lists were created and divided up amongst scientists. A few key people had to manage the enormous flood of information, sorting the important from the mundane.

Eventually when the list of possible scenarios to explain the event was reduced to just three: a real gravitational wave, a lightning strike in Burkina Fasso or a malicious hack of Advanced LIGO electronics, the excitement became palpable. I was finally convinced when Matthew Evans gave a presentation at the face to face commissioning meeting in November on the work he had been doing to verify that our gravitational wave signal was not a malicious hack.

In Advanced LIGO’s first observing run the coalescence of two black hole was definitively detected, twice! There was a third event that was more likely to be a gravitational wave from a third pair of black holes [182]. These are the first observations of binary black hole systems. They are the first observations of black holes in the mass range from 20-100 $M_\odot$. They also significantly increase the number of black holes with known masses as can be seen in Figure 1.23.
With the observation of black hole mergers in this mass range we can constrain models for black hole formation. In the follow-up paper on the implications of GW150914 [72] it is claimed this event must have occurred in a low metallicity environment and requires weak solar wind in the progenitor stars that formed to two black holes. Prior to the observation it was claimed by Lee and Hong [153] that Advanced detectors would first detect binary black holes. This prediction was based on N-body simulations of nuclear star clusters (regions of high star density in galaxies). Models for the environments in which binaries form may be constrained with future measured statistics of mass ratio of binary systems and black hole spin measurements.

The signal to noise ratio of the first gravitational wave event observed was remarkable [10]. In Figure 1.24 a plot of the signals measured at the LIGO Hanford site and the LIGO Livingston site are shown. The difference in arrival time of the two signals is 7 ms. Less than the propagation time (10 ms) for a gravitational wave traveling at the speed of light. The difference in arrival time allows triangulation of the source location. A localization [237] (that used all available measured parameters) of the
gravitational wave source to a region of $\sim 600\,\text{degrees}^2$ in the southern hemisphere sky. The first part of the plots (0.25-0.4 s) in Figure 1.24 show the *in-spiral*, when the two black holes orbital distance is rapidly shrinking as they dissipate energy to gravitational waves. Where the signal reaches a maximum amplitude around 0.42 s is the *merger*, highly non-linear spacetime deformations result as the black holes coalesce. Finally the *ringdown* period ($> 0.42$ s) is visible as a damped high frequency signal, this is the ringdown of the normal modes of the resulting black hole. This ringdown phase is like a drum that has been struck, the sound dissipates as it loses energy to sound waves, however in this case the vibrating black hole’s energy is lost to gravitational waves.
Figure 1.24: Gravitational wave signal of the event GW150914 from the LIGO Livingston and Hanford detectors. The top panels show signals from the detectors filtered with 35-350 Hz bandpass filters, also notch filters remove narrow noise features visible in Figure 1.14 page 26. The second panel down shows the numerical relativity mode of the in-spiral projected into the 35-350 Hz detector bandwidth. The shaded regions show 90% confidence intervals for signal reconstructions, this dark shaded region is based on a binary black hole model and the light shaded region is based on a sin-gaussian decomposition of the waveform. The first panels from the top show the residual noise after the model from the second panel is subtracted from the data in the first panel. This residual is indistinguishable from detector noise. Finally the bottom panels shows the signals as spectrograms. The colour represents the normalised amplitude of the signal as a function of frequency in the vertical axis and time in the horizontal axis. Image reproduced from [13]

As the ringdown has a much lower strain amplitude it was not expected that the first gravitational wave detection would reveal the feature. With all three pieces of information estimates of all the parameters that define the binary coalescence can be made [74].

Rough estimates can be made reading off properties of the in-spiral from Figure 1.24. From the in-spiral frequency $f = 2\pi \omega_g$ and rate $\dot{f}$ of the gravitational wave
signal, the *chirp mass* $M_{\text{chirp}}$ can be determined with Equation 1.17:

$$M_{\text{chirp}} = \left(\frac{M_1 M_2}{M_1 + M_2}\right)^{3/5} = \frac{\hat{c}^3}{G} \left(\frac{5}{9} \pi^{-8/3} f^{-11/3} \dot{f} \right)^{3/5} \tag{1.17}$$

The chirp mass is approximately $30 M_\odot$. Assuming equal masses ($M_1 = M_2$) the total mass of the system is $69 M_\odot$. The distance between the center of mass of these two black holes and the speed can be calculated by the frequency of the last stable orbit $\sim 150$ Hz. This implies the black holes were orbiting 75 times per second. Using Kepler’s third law $d_{\text{orbit}} = (\frac{M_1 + M_2}{2\pi \omega_{\text{orbit}}^2})^{1/3}$ the separation between is only $\sim 350$ km. By inference the black holes are moving at half the speed of light.

The frequency of the quasinormal mode in the ringdown phase is $\sim 250$ Hz. The mass of the final black hole can be estimated from the quasinormal mode frequency. In [74] a fit to a full numerical model was used to estimate the mass of the initial black holes as $29 M_\odot$ and $36 M_\odot$ and the final black hole as $62 M_\odot$. Approximately $2.4 M_\odot$ radiated as gravitational waves in one fifth of a second (while $\approx 3 M_\odot$ radiated in the entire in-spiral that likely lasted billions of years).

The distance to the binary system can be calculated from the maximum strain amplitude of the signal by solving Equation 1.1 on page 7 for $d$. This rough estimate of $\sim 700$ MPc for a binary system where the orbit plane is facing earth. More detailed estimates with both detector strain amplitudes reveal the orbit inclined by between 35 and 135 degrees and the likely luminosity distance of $410$ MPc.

For the precise estimates of these parameters reported in [74] the waveform is fitted to a full general relativistic model.

The second event labelled GW151226 [73] in Figure 1.23 was the coalescence of one 14.2$M_\odot$ black hole and one 7.5$M_\odot$ black hole. The smaller black holes have higher frequency signals, the quasinormal mode frequency is 450 Hz. Smaller systems produce smaller gravitational wave strain. The system was a similar distance (450 MPc) to GW150914. However the peak gravitational wave strain of $3.4 \times 10^{-22}$, which is a factor of three smaller that GW150914.

Observing an electromagnetic or neutrino counterpart to gravitational wave signals could possibly explain some of the mysteries of astronomy. Current models of back hole mergers do not predict any electromagnetic counterpart. Efforts were made none the less to observe the region of sky thought to have produced these mergers [237, 24]. No significant correlations were observed, constraining both the optical and neutrino emissions from such mergers. With Advanced VIRGO joining the advanced detector network in 2017, the angular uncertainty in the localisation of
the event will be reduced. In the event of a bright merger the systems set up for fast follow up and multi messenger astronomy will be invaluable.

With the detection of gravitational waves from a class of black holes that were not know to exist we can be assured that in the future many more gravitational waves of this kind will be detected. The event rate of this type of binary was estimated in [12] to be \(2-600 \text{ Gpc}^{-3}\text{yr}^{-3}\). With the prediction that between 7 and 20 events will be detected in Advanced LIGO’s second observation run and between 30 and 70 in Advanced LIGO’s third observation run. This dramatic increase in expected observation rates is expected with the enhancements to the detectors required to achieve the planned Advanced LIGO sensitivity [11]. One of the major enhancements that is expected to increase detector sensitivity is to increase the optical power. We will see in Chapters 5 and 6 that increased optical power will require control of parametric instability.

In the future more gravitational wave detectors will come online. More gravitational wave detectors will allow better angular resolution. Particularly the addition of southern hemisphere detectors. New detectors also allow new improvements to be incorporated to improve detector sensitivity. In the next section we explore the proposals for the next generations of gravitational wave detectors.

1.4 Next Generation Gravitational Wave Detectors

In this section we review what the future likely holds for gravitational wave astronomy. Ground based detectors will likely be high optical power, large interferometric devices. These devices will have significant optomechanical interactions. We review space detectors, focusing on the recent LISA proposal [88]. These detectors will open one new window to a new spectrum of gravitational waves. Optomechanical interactions in these devices are unlikely due to low optical power. Finally we review pulsar timing arrays which promise to open a third window, the nano Hertz spectrum of gravitational waves.

1.4.1 Third Generation Ground Based Detectors

Third generation ground based detectors will take the lessons learnt from building the first generation and advanced detector configurations to construct detectors that have an order of magnitude improvement in strain sensitivity over the advanced detector network.
a) Einstein Telescope

The Einstein Telescope (ET) is a European proposal for an underground gravitational wave detector with the goal to achieve a factor of 10 improvement on the advanced detector configuration’s sensitivity, the target frequency range is from 10 Hz to 10 kHz [223]. The proposed detector is actually six interferometers collocated in the same triangular underground system of shafts as shown in Figure 1.25.

Figure 1.25: Einstein Telescope planned configuration of 6 detectors in 3 underground shafts. Copied from [14]

ET will use two detectors in each triangle to achieve the desired frequency range. This is known as a xylophone arrangement - a term originally coined for bar detectors, where different detectors are constructed for different frequency ranges. One ET detector would be a cryogenic low frequency detector using resonant signal recycling to achieve high sensitivity between 5-35Hz [14]. The other detector would be a high frequency detector. The sensitivity of the composite system might look something like Figure 1.26.
Within the ET proposal [14] several proposed configurations are discussed. The D configuration (with sensitivity shown in Figure 1.26) is representative of all configurations from the perspective of parametric instability control. The high frequency detector plans to use a 3 MW beam circulating in 10 km arm cavities. This large optical power will be resonant between two 200 kg, 62 cm diameter fused silica mirrors. The ET proposal [14] considers parametric instability. They found that parametric gains 1.5 times as high as Advanced LIGO at design optical power are likely and there are likely to be seven times as many unstable modes, based on an analysis of the optical and acoustic mode densities.\footnote{This is considering the lower optical mode density achieved using high order Laguerre Gaussian beams to create large spot sizes on the mirrors with low diffraction losses}

\textit{b) Asia and Australia Gravitational Wave Detector}

[53] describes an 8 km interferometer based on a scaled-up Advanced LIGO configuration. This is suggested as a first stage in development of new detectors in Australia and China. Estimates [154] show that these detectors could give major improvements in the angular resolution of the world-wide gravitational wave detector network. In addition the improved sensitivity would increase the event rate of events like those already observed to between $3 \times 10^3$ and $10^4$ events per year [154].

KAGRA which has already been discussed in Section 1.2.2 also proposes using features generally considered as third generation technology. Such features include

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\textbf{Figure 1.26:} Einstein Telescope proposed sensitivity for the D configuration. The ET-D-LF is a low frequency cryogenic detector operating at relatively low power ($\sim 10 \text{ kW}$) The ET-D-HF is the high frequency detector requiring 3 MW circulating optical power in the arm cavities. The red trace is the combined sensitivity. Copied from [6]
being underground to reduce seismic noise and cryogenics to reduce thermal noise.

c) Next Generation Gravitational Wave Detectors in the USA

The proposed third generation gravitational wave detector in the United States has been called the Cosmic Explorer. The aim is to create a detector that will be sensitive to the entire cosmos for 50 solar mass binary coalescences. The designs are preliminary but generally broken into the two design concepts; long and simple or short and complex.

Longer detectors are more sensitive. In [102] Dwyer reports that a 40 km detector can achieve an order of magnitude improvement over the Advanced LIGO design sensitivity. The sole addition to technology that is currently in use at Advanced LIGO is quantum squeezing discussed in Section 2.5.3.

There are many technologies that have been proposed and even tested that could improve the sensitivity of detectors at their current scale. Many of these techniques are exceedingly complex. Miller [202] compares several of these promising technologies including; frequency dependent squeezing to reduce quantum noise, coating improvements that will reduce coating thermal noise, increasing the mirror mass to reduce radiation pressure noise and changing the suspensions to reduce suspension thermal noise (See figure 1.14). Up to a factor of two improvement in the strain sensitivity is reported with each improvement.

1.4.2 Space Detectors

All ground based detectors have to date been limited to a fairly narrow range of frequencies. The upper limit has been set by what is thought to be the highest frequency sources we might reasonably expect to detect - binary neutron stars. A coalescing pair of neutron stars 1.4 solar masses each will produce a black hole of \( \approx 2.7 \) solar masses having a maximum chirp frequency of 700 Hz and a ringdown frequency of \( \sim 5 \) kHz. The lower frequency limit of ground based detectors is set by technical limitations of seismic isolation to \( \sim 10 \) Hz. Broadband isolation of seismic noise below 10 Hz has not yet been achieved at the level required for gravitational wave detection.

Space detectors do not suffer from seismic noise, in fact they are more limited in the high frequency part of the spectrum. Any attempt to reduce shot noise by increasing the laser power would result in an untenable radiation pressure force. The eLISA proposal specified 0.6 W of transmitted light. This would produce a radiation pressure force of 2 nN, which is not insignificant when compared to the Field Emission
Electric Thruster capability of $30\mu N$ and the sunlight radiation pressure force of $25\mu N$ [106].

The first proposals for a space based gravitational wave detector were made by Faller and Bender [20] in 1981. The proposal considered $\approx 10^6$ and $\approx 10^8$ km baseline interferometers stationed at or around Lagrangian points. The main goal proposed was to detect known galactic binaries such as the cataclysmic variable binary star AM Canum Venaticorum, a cataclysmic dwarf nova binary WZ Sagittae and a close white dwarf with a companion Iota Bootis, these systems have expected gravitational wave frequencies of 1.9 mHz, 0.4 mHz, and 0.09 mHz respectively and are 100-1000 light years from earth.

In the 1980’s the LAGOS project based on these ideas was not funded. However many of the concepts have been adopted by later proposals. The proposed transmission apertures have reduced in diameter as laser technologies improved allowing for higher transmission power.

The LISA (Laser Interferometer Space Antenna) proposal [38] in the year 2000 was to be a joint National Aeronautics and Space Administration (NASA) and European Space Agency (ESA) mission. The proposal fixed the space antenna geometry to 3 identical spacecraft in orbit about the sun, trailing the earth by 20 degrees. They would each orbit the sun in ellipses that result in the formation of 3 spacecraft revolving around each other, separated by 5 million kilometers, once per year. To achieve this the plane defined by the three spacecraft is inclined 60 degrees to the plane of earths orbit about the sun as can be seen in Figure 1.27.

\footnote{With the recent success of LISA pathfinder it appears cold gas thrusters may be used in place of Field Emission Electric Thrusters [88]}
An artist’s impression of an individual LISA spacecraft is shown in the inset in Figure 1.27. The spacecraft transmits and receive laser beams to and from the two other LISA spacecraft craft to make a triple interferometer.

The types of signals we might detect with the LISA are shown on the projected sensitivity curve in Figure 1.28 reproduced from the 2017 LISA proposal [88]. LISA signals will likely consist of many compact binaries in the galaxy, stellar mass binaries, massive black hole binaries and extreme mass ratio in-spirals from cosmological distances and a confusion of background signals.
Figure 1.28: The Laser Interferometer Space Antenna proposed sensitivity plotted as strain amplitude against frequency, with expected amplitudes of various signals. MBHB traces show the expected strain amplitude of 3 equal mass black hole binaries, with total intrinsic mass $10^7$, $10^8$ and $10^7 M_\odot$ at a redshift of $z = 3$. Blue * mark known galactic compact binaries on a background of expected galactic binaries (Gal. Bin.) which would be detected with a signal to noise ratio greater than 7. The EMRI traces show the expected strain amplitude of extreme mass ratio in spirals of stellar mass black holes $10^{-60} M_\odot$ plunging into massive black holes of $10^5-10^5 M_\odot$ at a redshit of $z = 1.2$. The Advanced LIGO detection GW150914 is projected into the LISA sensitivity range along with range of other LIGO-type black hole binaries (BHB). Finally the millions of other binaries create a background gravitational wave noise in the LISA sensitivity spectrum shown as the grey region. It will not be possible for LISA to resolve these objects. Image reproduced from [88].

The LISA project has had many incarnations due to funding constraints. The latest proposal [88] in response to the ESA call for concept proposals for L3 space mission to “study the gravitational Universe” [3] was submitted January 2017. The projected launch date in this proposal is in the year 2030. This proposal alters the constellation size, telescope size relative to the 2000 LISA proposal. They also propose using cold gas thrusters after successful demonstration on the LISA Pathfinder mission.

In 2016 the LISA pathfinder (LISAPF) mission gave the LISA project a huge boost.
with the successful demonstration of less than a femto-g of gravitational acceleration noise between 0.7 and 30 mHz. This noise performance was demonstrated with a test mass cocooned in a spacecraft located at the L1 Lagrangian point. This demonstrates the core technology of LISA, the drag free performance determined by measurement of a freely floating test mass (a metal cube shielded within the center of the spacecraft) that is sensed using interferometry. After a month of operation LISAPF had almost demonstrated design requirements. Residual acceleration noise in the LISA frequency range was likely due to gas damping from residual gas on the spacecraft.

Most recent reports [88] have shown that this noise has reduced with time and LISAPF has surpassed design requirements as demonstrated in Figure 1.29 reproduced from the ESA proposal [88]. With this promising work it appears that space based gravitational wave detectors will in the future, broaden our gravitational wave horizon and spectrum from the audio band into sub-audio frequencies. The monsters of the universe (giant black holes) will reveal their mysteries in these grave tones, serenading us from the entirety of space and time to which we have access.

1.4.3 Pulsar Timing Arrays

There is now a large collection of known millisecond pulsars. These millisecond pulsars are what were referred to before as the clocks of the milky way galaxy. Some of these pulsars are used by the Parkes Pulsar Timing Array [152], European Pulsar Timing Array [32] and the North American Nanohertz Observatory for Gravitational Waves [25] to search for gravitational waves. When a gravitational wave passes through the earth it moves the earth relative to the distant pulsars. The signature of gravitational waves would be the correlated apparent change in the pulsar timing residuals caused by the motion of the earth relative to each pulsar.

For many years it has been recognised that the unique time signatures of pulsars make them attractive markers for galactic navigation. The suns location relative to
certain known pulsars was sent on a plaque carried by NASA’s Pioneer 10 and 11 probes is shown in Figure 1.30. These probes were launched in the early 1970s.

![Figure 1.30: Pioneer Plaque giving earth location relative to pulsars. Copied from [208]](image)

Pioneer 10 made last contact in 2003, 12.2 billion km from earth. If some alien race find our probe and recognise the pulsars, they could locate us in the galaxy. In this sense maybe pulsars should be referred to as the galaxy’s GPS system rather than merely the galaxy’s clocks.

In Figure 1.30 the Earth shown at the apex of the lines. If we consider the Earth perturbed by a gravitational wave, each pulsar clock signal will be perturbed according to the wave polarisation and direction relative to the line joining the Earth and the pulsar.

The key problem in pulsar timing is the intrinsic timing noise of pulsars. For unknown reasons many pulsars have poor timing stability. Just a few pulsars exhibit timing residuals $\sim 100$ ns, which is required for significant gravitational wave searches.

Results to date have provided significant limits on the models for massive black hole growth in the universe. The frequency range that pulsar timing arrays can address is in the nano Hertz range. At such low frequencies signals take years to accumulate, particularly with relatively few stable millisecond pulsars. However new telescopes such as FAST and the SKA offer promise of many more pulsar discoveries,
which will allow better gravitational wave strain estimates, just like getting signals from more GPS satellites improves GPS localisation accuracy.

The most promising sources of gravitational waves in the nano Hertz frequency band are from coalescence of super-massive black holes ($\sim 10^5 - 10^9 M_\odot$) and stochastic background signals from the big bag and the confusion of many other black hole events.

### 1.5 Summary

In this Chapter I have introduced gravitational wave starting from Laplace’s recognition that the finite speed of gravity would cause dissipation, I have reviewed the theory of gravitational waves and the history of the many valiant efforts that have been made to detect them. I have reviewed the concepts of interferometric gravitational wave detectors and I have introduced the Gingin High Optical Power Test Facility where much of this thesis work took place and which led me to be able to participate in the momentous first detection of gravitational waves. I then briefly reviewed the proposals for the future searches for gravitational waves.

In this chapter I have shown that current and future ground based gravitational wave detectors use high optical power that results in optomechanical interactions. In the next chapter I shall review the optomechanics of gravitational wave detectors.
Chapter 2

An Introduction to the Optomechanics in Gravitational Wave Detectors

Optomechanics is the core concept behind gravitational wave detection. In this chapter I review optomechanics, first from a very simple and historical viewpoint, then the details of the optical and mechanical systems in gravitational wave detectors are described. This is followed by a review of several examples of optomechanical interactions that are relevant to gravitational wave detectors.

2.1 Overview

When light interacts with a mechanical system there is an intrinsic optomechanical coupling due to the radiation pressure forces exerted by the light. If any light is reflected off the mechanical system, any motion of the mechanical system produces changes in phase of the reflected light.

From everyday experience radiation pressure is a strange concept. Tanning on the beach you do not feel the pressure of the suns rays. When you switch on the headlights, your car does not slow down. In 1862 Maxwell predicted that there would be a radiation pressure force $F$ equal to the power of light $P$ divided by the speed of light $c$. It is interesting to look at the magnitude of this force to understand why we don’t experience radiation pressure in our everyday lives. Car headlights emit $P \sim 100 \text{W}$, producing a force of $\sim 0.3 \mu \text{N}$, less than a billionth the force the car engine can apply.

One must be careful when it appears that radiation pressure forces interact with everyday objects. Crookes’ radiometer [82] is an evacuated bulb with vanes supported
on a needle bearing. As shown in Figure 2.1 one side of each vane is black, the other silver. It spins with the silvered surface of the vanes preceding the black surface when sunlight falls on the vanes.

Crookes’ radiometer was a point of controversy from its invention in 1873 till Osborne Reynolds correctly described it’s thermo-dynamics in 1879. Many believed it to be driven by the radiation pressure James Maxwell had, at that time, recently predicted. There was some historical physics drama when Maxwell beat Reynolds to publication with his review [194] of Reynolds’ work that Maxwell had refereed. The explanation of both men was that the residual gas (vacuum systems weren’t what they are today) warms when light is absorbed by the black surfaces making a pressure differential to the cooler silvered surface where minimal light is absorbed. This pressure differential moves gas past the vane from the warmer surface to the cooler surface - driving the vanes in reaction, a thermodynamic system. If it was an optomechanical system, where radiation pressure drove the vanes, the radiometer would spin the other way as the silvered surfaces reflect photons, receiving twice the momentum \((2P/c\) Newtons) of the black surfaces that absorb them \((P/c\) Newtons).

To make interesting optomechanical systems, the force of light has to be made significant compared to the dissipative forces in the mechanics. Consider a one dimensional optomechanical system where light shines on a movable mirror, with dynamics governed by spring constant \(k\) (and some unspecified damping) as in Figure 2.2. We see that there is a radiation pressure force \(2P/c\) that moves the mirror a distance \(dz\). In the steady state the restoring force from the spring constant \(k\) will balance the radiation pressure force \(F\).

Work is done against the spring. Where did the energy come from? The photons reflected off the mirrors lose some energy. The dynamic compression of the spring causes a Doppler shift and the photons consequently lose energy. The gravitational wave interaction with an optical cavity can also be considered in terms of the Doppler
friction. The frequency shift of the photon arises from the Doppler effect due to the motion of the mirror as it relaxes to its new position. This concept was first introduced by Einstein in 1909 as radiation friction [104]. It was extensively studied by Ma et al in relation to the energy interaction between gravitational waves and interferometers [183].

In the quantum world of molecular interactions the Doppler shift discussed above is called Raman scattering. With molecular interactions only energy transitions between quantized molecular states are allowed, leading to distinct spectral characteristics of the scattered light. With large macroscopic mirrors such as those in gravitational wave detectors the allowed energy can generally be considered continuous allowing the use of classical approximations. However in cryogenic mechanical systems consisting of cooled micro-mirrors, the quantum nature of the mechanical oscillator can become significant [239].

One way to make the radiation pressure force have a significant effect is to make the mechanic system weak, like the first optical levitation experiments by Ashkin [26] in 2006. In this levitation experiment micrometer sized particles weighing just picograms are suspended against gravity in an optical resonator with \( \sim 1 \) W of laser power.

If the laser power is high enough and the mechanical losses are low enough radiation pressure becomes significant at the macroscopic scale. This is how most of the interactions with kg scale mirrors described in Chapters 3-8 become significant. These interactions do not require enormously powerful lasers. Resonance in an optical cavity is used to increase the power of the optical field inside the cavity. This
creates a measurable static radiation pressure force on the mirrors in high power optical cavities such as Advanced LIGO’s arm cavities where hundreds of kilowatts of laser power accumulate. The force, typically $\sim$ milliNewtons, is sufficient to cause macroscopic deflection of the mirror pendulums.

An optical cavity can produce a feedback system where mechanical motion can influence the optical field in the cavity and the optical field in the cavity can in turn influence the mechanical degrees of freedom through radiation pressure. In Figure 2.3 a one dimensional cavity is shown to demonstrate this principle. In this model we see

$$F = 2P/c$$

the familiar radiation pressure effect. However the optical cavity will only be resonant if it is an integer ($N$) number of wavelengths ($\lambda_0$) long, i.e. $L + dz = N\lambda_0$. Assume the cavity is resonant at length $L$ then $2L = N\lambda_0$. Consider an experiment that begins with almost no light resonant in the cavity such that radiation pressure term can be considered zero. As the laser power increases radiation pressure acts to push the mirrors apart. However this results in less cavity resonance and less radiation pressure force. All sorts of scenarios can ensue ranging from chaotic systems [188], to potentially the most rigid objects on earth [59].

Interactions are very dependent on the properties of the mechanical system. For example the interaction between light and a mechanical system can be increased when the mechanical system has small dissipative losses. In this way small radiation pressure forces exerted in phase with mechanical resonant motion can result is large resonant amplitudes.
In this Chapter I will first study mechanical resonators in Section 2.2, focusing on the eigenmodes of a cylinder which are extensively studied in this thesis. In Section 2.3 the theory of optical cavities is reviewed. These theories are combined to investigate optomechanical systems in Section 2.4. While in Section 2.5 a range of interesting optomechanical interactions related to gravitational wave detectors are explored.

## 2.2 Mechanical Resonators

The mechanical oscillator in an opomechanical system can be modelled as an idealised damped mass-spring oscillator as illustrated in Figure 2.4.

![Figure 2.4](image)

Figure 2.4: One dimensional damped mass spring mechanical resonator model, $F$ is an applied force, $dz$ is the resulting movement of object with mass $m$ that interacts with an idealized infinitely rigid restraint via a spring with constant $k$ and dissipative damper $C$.

This model is relevant to many real world systems; test masses suspended as pendulums, test mass internal eigenmodes and torsional oscillators. All mechanical resonators of interest here have a mass $M$ that satisfies $\omega^2 = k/M$ however in many cases the mass is an effective mass $\mu_m$ which measures the mass that contributes to the motion measured at a specific point or in a specific manner. The spring parameter $k$ in the mechanical resonator may often be determined from the mechanical properties, but may be varied by the application of feedback forces or by radiation pressure.

This thesis is concerned with two oscillator types: pendula and internal resonant modes of gravitational wave detector test masses. Both these harmonic systems can be modelled as one dimensional oscillators. A pendulums eigenfrequency is: $\omega_m = \sqrt{g/L'}$, where $L'$ is the pendulum length, $g$ is the gravitational acceleration on earth.

Test masses in gravitational wave detectors are effectively solid cylinders. The eigenmodes of a cylinder have many possible eigenfrequencies associated with the degrees of freedom of the cylinder. Each eigenmode can, to a very good approximation, be treated as an independent harmonic oscillator.
The damping coefficient $C$ in general can be complex defining both the amplitude and phase of the dissipative damping force. In this thesis viscous damping is often assumed where the dissipative force is in phase with the mode velocity. Viscous damping can be defined by either the quality factor, exponential decay time or linewidth. The Exponential decay time constant $\tau$ is the time it takes a resonator amplitude to damp to $1/e$ its original amplitude, here I use the convention where $\tau$ is negative for damping and $\tau$ is positive for unstable exponential growth time constant. The Quality factor $Q$ is a dimensionless parameter given by $\pi$ times the number of cycles that occur before the amplitude has reached $1/e$ of its original amplitude. The Linewidth $\gamma$ used in this thesis is given by the full width at half maximum of the spectral response. To see how these are related we examine the equation of motion and the time domain and frequency domain response of the system. The equation of motion is given by:

$$m\ddot{z} + \frac{2m}{\tau} \dot{z} + kz = F$$

(2.1)

Here $F$ is an external forcing term. The solutions to Equation 2.1 in the time domain and in the frequency domain give the responses shown in Figure 2.5. The left panel shows the amplitude response of the resonator to a step function at $t = 0$ applied as forcing term $F$. In the right panel the frequency response is shown. It shows the steady state amplitude ratio and phase difference between the resonator amplitude and a sinusoidal forcing term $F$ at radial frequency $\omega$. These figures graphically demonstrate the relationship between the three equivalent descriptors of damping:

$$Q = \frac{\omega_m}{\gamma} = -\frac{\tau\omega_m}{2}$$

(2.2)
In this thesis and in many other applications of optomechanics we are interested in the interaction of optical cavities with spring mounted mirrors at their ends. Our primary interest is usually in the longitudinal motion of the mirror. However we will see that torsional mechanical modes can lead to unwanted interactions and that test mass eigenmodes with complex surface profiles can interact with complex optical modes.

Different resonators can interact with light in different ways. Pendula can have motion in the direction of the optical axis of the cavity - longitudinal motion - or can be torsional, rotating a mirror, changing the cavity geometry. Test mass eigenmodes can have a range of interactions. Figure 2.6 shows some examples of surface deformation due to test mass eigenmodes. The mode in Figure 2.6 (a) approximates longitudinal motion, (b-c) approximate torsional motion while (d-f) show extremely complex surface profiles. We will see in Chapter 3 that these can couple to particular optical cavity modes.

Figure 2.6: Surface deformation perpendicular to the page of 6 eigenmodes of Advanced LIGO test masses, red represents a deformation out of the page a blue a deformation into the page.

2.2.1 Test Mass Elastic Eigenmodes

The test mass elastic eigenmodes are the mechanical resonances of the test mass. Generally these resonances are at sonic to ultrasonic frequencies so are sometimes referred to as acoustic modes to distinguish them from optical modes (100 THz frequencies).
The test mass approximates a cylinder. The eigen-frequencies of a cylindrical body were derived by Chree in 1886 [66] and summarised in the context of gravitational wave detectors by Strigin [234]

\[
\frac{\omega}{2\pi} = \frac{\beta}{2\pi}(2n/\rho)^{1/2}.
\] (2.3)

Here \(\omega/2\pi\) is the resonant frequency, \(\beta\) is a spatial parameter that has certain allowed values, \(\rho\) is the density and \(n = E/(\sigma + 1)\) where \(E\) is the Young’s modulus and \(\sigma\) is the Poisson ratio.

Due to rotational symmetry the displacement of the eigenmodes is best described in cylindrical coordinates \(\vec{u}=(u_r,u_z,u_\phi)\). The displacement vector for the eigenmodes is described by Equations 2.4 to 2.6.

\[
\begin{align*}
    u_r &= -A_m \cos \omega_m t \sin \beta z J_1(\beta r) \\
    u_z &= A_m \cos \omega_m t \cos \beta z J_0(\beta r) \\
    u_\phi &= 0
\end{align*}
\] (2.4) (2.5) (2.6)

under the conditions,

\[
\begin{align*}
\beta R &= a_i, \beta H = b_j \\
    a_i \exists J_1(a_i) &= 0 \\
    b_j &= \pi + j2\pi
\end{align*}
\]

Each elastic eigenmode is defined by a deformation vector \(\vec{u}\) that scales with the mode amplitude \(A_m\) and oscillates with the mode frequency \(\omega_m\). Due to rotational symmetry there are no resonances with dependence on the rotational degree of freedom. The displacement along the optic (z) axis is generally of most interest. The z displacement along the radial axis and depth axis are depicted in Figure 2.7 for 6 of the low order modes for a cylinder close to the dimensions of a LIGO test mass.

A test mass may be considered as an approximately cylindrical body. However, as studied by Strigin [234] and further investigated in Chapter 5 and Chapter 8, the attachment of suspension structures modify the cylindrical symmetry resulting in significant differences in eigenmodes. Specifically, rotational symmetry is broken destroying the degeneracy between vertically and horizontally oriented modes.

To model test masses more precisely analytical solutions cannot be used or are very complicated. Finite element modelling is generally used throughout the thesis as it allows the calculation of eigen-modes of specific test mass designs, including suspension structures and also under specific thermal conditions.
Finite element modelling allows the surface deformation, the deformation of the high reflectively surface, to be calculated such as those presented in Figure 2.6. This is an integral step in analysing the strength of various optomechanical interactions.

Another required quantity from the finite element models is an estimate of the effective mass. As mentioned in Section 2.2 the effective mass is defines the mass of a resonant mode as seen by the system that interacts with it. The effective mass is calculated in COMSOL [77] is:

\[
\mu_m = M \times \frac{\int u_{z,m} dV}{\sum_{j=1}^{\inf} \int u_{z,j} dV},
\]

where \( M \) is the mass of the object, \( u_{z,m} \) is the displacement field in a particular direction (\( z \) in this case) for mode \( m \), \( \int dV \) is the integral over the objects volume and the sum as over all eigenmodes. The COMSOL effective mass does not take into account the interacting system and hence it is of limited use. Throughout this thesis
the problem of effective mass is avoided by incorporating the effective mass into a
term called the effective mass scaled overlap factor generally abbreviated to overlap
factor. The overlap factor encompasses the interaction strength and effective mass
of the test mass eigenmode. These overlap factors can be numerically integrated
with the interacting pressure distribution $f_\perp$, the eigenmode surface displacement
distribution in the direction of the force $u_\perp$ and the volume displacement distribution
$\vec{u}$

$$
\Lambda = \frac{V \left( \int_S f_\perp(r_\perp) u_\perp \, dr_\perp \right)^2}{\int_S |f_0|^2 \, dr_\perp \int_S |f_\perp|^2 \, \vec{u}^2 \, dV} \tag{2.8}
$$

Here $\int_S d\vec{r}_\perp$ denotes integration over the interacting surface and $\int_V dV$ over the entire
volume.

In optomechanics, the interacting field $f_\perp$ is produced by the optical fields in an
optical cavity. In the following section the properties of these fields are reviewed.

### 2.3 Optical Cavities

Imagine you shine a 1W laser at a 99% reflective mirror, you observe a reflected
beam with intensity 990mW. Someone places (possibly far away) a second identical
mirror to reflect the 10mW transmitted beam. Suddenly the reflection off the 99%
reflective mirror vanishes. To investigate, you look at the beam between the mirrors
and note bright scattered light off dust particles. This is the magic of optical cavities
(of course in reality it is beautiful physics rather than magic). To examine how this
works, consider a model with the parameters shown in Figure 2.8.

We assume perfect mirrors that have power reflectivity $R = 99\%$ and 1\% trans-
mission. When light is first incident on mirror M1 in Figure 2.8, 99\% is reflected
($E_7$) and 1\% is transmitted ($E_2$). It goes on to reflect off M2 where is loses 1\% in
transmission ($E_4$). At mirror M1 the beam ($E_6$) is just under 1\%. Depending on the
phase of the reflected portion of $E_6$, $E_2'$, this light will either interfere constructively
$E_2 + E_2' > E_2$ or interfere destructively $E_2 + E_2' < E_2$ with the new light entering
the cavity. If the cavity length is an integer number of half wavelengths, the light
will experience maximal constructive interference. In this case the cavity is optically
resonant with the particular wavelength of light. Every round trip adds another 1\%
of $E_1$. The light is accumulated until the 1\% added every round trip is equal to the
Figure 2.8: Model optical cavity composed of two mirrors M1 and M2. Both mirrors have the same reflectivity R and the distance between the mirrors is L. $E_1$ to $E_7$ described the field amplitude, direction and phase at various locations and the primed field amplitudes are used in descriptions in the text.

round trip loss, 2% through M1 and M2. As the light accumulates 1% of $E_6$ is transmitted through M1 as $E'_7$ which destructively interferes with $E_7$, thereby conserving optical power.

If the losses of the cavity, in this case just $(1 - R)$ transmitted through M2, are equal to the input transmission $(1 - R)$ the cavity is said to be *impedance matched*, in this case when the cavity reaches a steady state $E'_7$ transmitted through M1 perfectly cancels $E_7$. No light is reflected. The intensity inside the cavity is now increased to $1/((1 - R)$ while the transmitted beam $E_4$ has the same power as the input beam $E_1$. This concept is used throughout interferometric gravitational wave detectors.

The relation between cavity length and field amplitude can be derived from the system of equations:

\[ E_2 = \sqrt{(1 - R)} E_1 e^{-i\omega_0 t} + \sqrt{R} E_6 \left( t - 2 \frac{L}{c} \right), \quad (2.9) \]
\[ E_6 = \sqrt{R} E_2 \left( t - 2 \frac{L}{c} \right), \quad (2.10) \]
\[ E_7 = \sqrt{R} E_1 - \sqrt{(1 - R)} E_6, \quad (2.11) \]
\[ E_4 = \sqrt{(1 - R)} E'_2 \left( t - \frac{L}{c} \right). \quad (2.12) \]

From Equations 2.9 to 2.12 some interesting properties can be found. Considering $E_1$ as input and $E_4$ as output the cavity transfer function can be examined as a function of length $L$ or frequency of light $\omega_0$ over a range of mirror reflectivity as shown.
in Figure 2.9. When the cavity length changes by half of the wavelength of light (Figure 2.9 top axis $2L = \lambda$) the transfer function repeats, going from one resonant peak to the next. These resonant peaks are known as different axial or *longitudinal modes* of the cavity, indexed by the letter $l$. As can be seen in Figure 2.9, the change in frequency from one longitudinal mode $\omega_l$ to reach the next longitudinal mode $\omega_{l+1}$ is called the *free spectral range* and is related only to the cavity length.

$$\omega_{fsr} = \frac{\pi c}{L}$$  \hspace{1cm} (2.13)

Here $c$ is the speed of light. The FWHM linewidth of the resonance $\gamma_0$ is related to length and the cavity losses. In the idealized case with no excess cavity loss it depends only on the reflectivity of the mirrors:

$$\gamma_0 = \frac{(1 - R)^2}{2L} \frac{\omega_{fsr}}{2\pi}$$  \hspace{1cm} (2.14)

There is a strong resemblance between the optical cavity resonance and the mechanical resonator. Definitions of Q factors, linewidth and time constants are the same in both cases. The *finesse* of an optical cavity describes how many round trips ($2L$) light makes in the cavity.

$$\mathcal{F} = \frac{\omega_{fsr}}{\gamma_0} \approx \frac{2\pi}{(1 - R)}$$  \hspace{1cm} (2.15)

In practice the creation of optical cavities can be challenging. The linewidth of Figure 2.9 can be expressed in length units:

$$\gamma_0 \approx \frac{\lambda_0}{\mathcal{F}}$$
For a moderately high finesse of $10^4$ the linewidth is $10^{-4}\lambda$. The mirror positions have to be maintained to $\lambda/10^4$ precision to maintain resonance. Acoustic and seismic vibration of the mirrors and thermal drift in laser wavelength generally means control systems are required to maintain resonance on a particular longitudinal mode. One of the most common control techniques used is called *Pound Drever Hall locking* (PDH) [99, 48]. The phase of the light incident on the cavity ($E_1$) is modulated, producing modulation sidebands which do not resonate in the cavity, together with some *carrier light* that will resonate in the cavity. The ratio of sideband to carrier amplitude is determined by the *modulation depth* which is related to the maximum frequency deviation for sinusoidal modulation. In reflection off the cavity ($E_7$) the modulation sidebands beat with the reflected carrier\(^1\) to produce a signal at the modulation frequency that is proportional to the deviation of the laser frequency from resonance. Demodulating the error signal and filtering out high frequency components provides a control signal that can be used to either control the laser wavelength so that it follows the cavity length or it can be used to control the cavity length so that the cavity length follows the laser wavelength. The wavelength control scheme was used for the Gingin experiments in Chapters 4, 6 and 7. While Advanced LIGO, Virgo and Kagra use or will use [123, 27, 15] a combination of schemes.

### 2.3.1 Cavity Geometry

A three dimensional optical resonator can no longer be described simply by the distance between two plane mirrors. Such a configuration in three dimensions allows photons to bounce back and forth at many different angles, each angle experiences a different cavity length. Diffraction from any finite aperture results in light inevitably escaping a plane-parallel cavity configuration. We might imagine a sphere would make an ideal cavity that traps all the photons in the cavity. However such a configuration results in the same instability as two flat mirrors. Because diffraction allows a spread in photon trajectories, photons can “walk off” sideways in the sphere resulting in photons filling all possible positions and directions in the sphere.

There is a stability condition for optical cavities that relates to the length and radius of curvature of the mirrors. The stability condition is described in terms of the cavity *g factor*, which for stability is required to have a value in the range: $0 < g < 1$. This g factor of a cavity is the product of the g factors for each end mirror:

$$\text{g} = g_1g_2, \quad g_i = 1 - \frac{L}{R_i}, \quad (i = 1, 2). \quad (2.16)$$

\(^1\)There will be some reflected carrier assuming the cavity is not perfectly impedance matched
Here $L$ is the cavity length and $R_i$ is the radius of curvature of mirror $i$. Figure 2.10 shows a range of cavity configurations, the stable regions are highlighted blue. Several cavity configurations are depicted for the red dots shown in the g factor space. Symmetric cavities, where both mirrors have the same radius of curvature, fall on the dashed red line, the configurations at the limit of stability in this case are the two configurations discussed above, the plane-parallel and the concentric cavities. The configurations used throughout this thesis are listed here:

1. Gingin south arm - between hemispherical and plane parallel $g = 0.92$
2. Gingin east arm - symmetric, approaching concentric $g = 0.98$
3. Advanced LIGO arm cavity - ~ symmetric, approaching concentric $g = 0.81$

In the following section we study the resonances that are supported in optical cavities and examine cavity geometric stability from another perspective.

2.3.2 Transverse Electromagnetic Modes

The modes of optical cavities must always satisfy the requirement that the cavity length is an integer number of half wavelengths. If the mirrors have spherical shape, the simplest mode that satisfies this condition is a mode with a Gaussian beam profile. However a whole family of transverse modes can also exist in which the optical mode...
has a more complex structure. These modes with different spatial profiles are called *higher order transverse electromagnetic modes* (HOTEM).

If the HOTEM is resonant at frequency $\omega_0$ the mode is *degenerate* with the fundamental mode in the cavity [212]. Two modes with different spatial profiles resonating at the same time was explored by Fang et al [115] in Gingin optical cavities. Normally cavities are designed to avoid degeneracy. The frequencies of HOTEMs are generally not the same as the fundamental mode because the phase velocity or *Gouy phase* [130] of the HOTEM is different to the fundamental mode. The Gouy phase is the phase difference between a focused Gaussian beam and a plane wave of the same frequency.

HOTEM frequencies are defined by the longitudinal mode they belong to $l$ and two parameters $p$ and $q$ that define their spatial structure.

$$\omega_{lpq} = \omega_0 + l\omega_{fsr} + (p + q)\omega_{fsr}\frac{\phi_{rtg}}{\pi}. \quad (2.17)$$

Here $\omega_{lpq}$ is the radial frequency of the HOTEM, $\omega_0$ is the radial frequency of the fundamental mode, $\omega_{fsr}$ is the free spectral range and $\phi_{rtg}$ is the round trip difference in Gouy phase between the HOTEM and the fundamental mode. The Gouy phase changes by $\pi/2$ from the infinite far field to the beam waist following $\phi_{qouy} = \arctan z/z_R$ as a beam propagates a distance $z$ from the beam waist, where $z_R$ is the Rayleigh distance. The cavity geometry determines what proportion of the Rayleigh distance is encompassed by the cavity and hence the round trip Gouy phase:

$$\phi_{rtg} = \arccos \pm \sqrt{g} \quad (2.18)$$

Here the $\pm$ depends on the cavity geometry, $-$ for cavities in the lower left quadrant of Figure 2.10 and $+$ for cavities in the upper right quadrant.

Examining Equations 2.17 and 2.18 we find the limits on cavity stability (defined in Section 2.3.1 where the $g$ factor was equal to one), are cavity geometries where all the TEM resonate at the same frequency (ie they are degenerate).

Under the assumption of infinite mirrors in a cavity with stable geometry, analytic solutions to Maxwell’s equations define the allowed field distributions for cavities with boundary conditions of spherical mirrors. The solutions in Cartesian co-ordinates are known as Hermite-Gaussian (HG) modes while the solutions in polar co-ordinates define Laguerre-Gaussian (LG) modes. The solutions are given in Equations 2.19 and
2.20:

\[
HG_{pq}(x, y, z) = E_0 \frac{w_0}{w(z)} H_p(\sqrt{\frac{2}{w(z)}}) H_q(\sqrt{\frac{2}{w(z)}})
\times e^{-\frac{x^2+y^2}{w(z)^2}} \exp -i[kz - (1+p+q) \arctan \frac{z}{z_R} + \frac{k(x^2+y^2)}{2R(z)}] \tag{2.19}
\]

\[
LG_{pq}(\rho, \phi) = 2 \sqrt{(p/2)!} \frac{1}{(1+\delta p)\pi(p/2+q)!} w(z) \left( \cos q\phi \right)
\times \left[ \sqrt{2\rho} \right]^q L_{p/2}^q \left( \frac{2\rho^2}{w(z)^2} \right) e^{-\rho^2/w(z)^2}
\times \exp i\left[ \frac{kp^2}{2R} - (2p/2 + q + 1) \arctan \frac{z}{z_R} \right] \tag{2.20}
\]

Here \( H_n \) signify Hermite polynomials, \( L_{p/2}^q \) signify the generalised Laguerre polynomials, \( w(z) \) is the beam size as a function of displacement along the optic axis \( z \).

The convention of only permitting even values of \( p \) in LG modes allows the mode order to be calculated simply as \( n = p + q \). The LG and HG functions are two independent orthogonal basis sets. Either basis set can be used to describe any beam geometry as a linear combination of the infinite basis set. As such any HG mode can be expressed as a linear combination of LG modes and vice versa [169].

Any resonant mode in a cavity with non-ideal mirrors can be described as an infinite sum of basis elements of either HG or LG basis sets. However single LG basis elements are better approximations of the modes that resonate in rotationally symmetric cavities that approach ideal cavities. The HG basis elements are better approximations of modes that resonate in cavities without rotational symmetry, that approach ideal cavities. The symmetries of these modes can be seen in the various intensity profiles depicted in Figure 2.11. In this figure the asymmetry in the HG modes would be associated with asymmetry between the vertical and horizontal orientations.

Consider now how to design a cavity where the HOTEM are not degenerate. To do this we are not interested in the longitudinal order of modes with varying \( l \) value, but only in the mode spacing between adjacent modes. This mode spacing is a function of transverse mode order \( n \). The frequency of the HOTEM can be calculated by putting Equation 2.18 into Equation 2.17. The frequency of the HOTEM of order \( n = p + q \) relative to the fundamental mode \( \omega_0 \) in the infinite mirror approximation is,

\[
\Delta \omega_n = \omega_0 - \omega_n = \frac{c}{L} (p + q) \arccos \left( \pm \sqrt{\frac{1 - \frac{L}{R_1}(1 - \frac{L}{R_2})}{(1 - \frac{L}{R_1})}} \right) \tag{2.21}
\]
Equation 2.21 defines the mode spacing as a function of the two mirror radii of curvature $R_1$ and $R_2$. As before the ± depends on the cavity geometry, − for cavities in the concentric $g$ quadrant of Figure 2.10 and + for cavities in the plane-parallel quadrant.

Equation 2.21 is used extensively in Chapter 5 to estimate the transverse mode frequencies and rate of change with thermal aberration in the optic. It is used in Chapter 6 to estimate the ring heater tuning used during parametric instability damping experiments and for estimating the transverse mode modulation depth for suppressing instability. Equation 2.21 is also used in Chapter 8 to estimate the HOTEM frequencies and the rate these frequencies are tuned by thermal heat power on the test mass.
We must be somewhat careful when applying Equations 2.19, 2.20 and 2.21 in Advanced LIGO arm cavities as the cavity g factor is relatively large \((g_1g_2 = 0.81)\) so the beams are large (a \(\sim\)10 cm diameter contains \(\sim\)97% of the power) and the mirrors are not so large (34 cm diameter). Barriga et al [35] did a detailed study of diffraction losses in LIGO arm cavities and the expected effect on HOTEM frequencies, shape and linewidth. Particularly Barriga et al found that \(\omega_n\) is not the same for all combinations of \(p\) and \(q\). Generally where this becomes important in this thesis \(\omega_{pq}\) is used to distinguish particular mode frequencies.

Due to the deviation in measurement from the nominal mode spacing of \(\omega_n\) of Equation 2.21, comparisons are made throughout this thesis to numerical estimates of the HOTEM frequencies, mode shapes, linewidths and tuning rates for the specific cavity being considered.

### 2.3.3 Transverse Modes Equivalence to Beam Position and Alignment

It is possible to decompose any beam into a linear combination of a set of basis functions such as one of the HG or LG basis sets from the previous section. One decomposition of particular interest for this thesis is the decomposition of beam tilt or beam position change applied at the beam waist, to an otherwise perfect Gaussian beam [22]. Both of these can be represented to first order by the addition of a single Hermite Gaussian mode either \(HG_{01}\) or \(HG_{10}\) depending whether the tilt or misalignment is in the vertical or horizontal direction. The first order approximation is valid when the beam position misalignment \(\partial x\) or \(\partial y\) is small relative to the waist size \(w_0\) \((\partial x/w_0 \ll 1, \partial y/w_0 \ll 1)\) and when the change in angle \(\partial \theta\) is small relative to the divergence angle \(\theta_0\), \((\partial \theta/\theta_0 \ll 1)\). This gives rise to Equations 2.22 and 2.23:

\[
E_{out} = E_{00} + \frac{\partial x}{w_0} E_{01} \tag{2.22}
\]

Here \(E_{out}\) is the resulting beam when the beam \(E_{00}\) undergoes a change in position of \(\partial x\). The resulting beam is modified by the addition of small proportion of first order mode \(E_{01}\) which is proportional to the ratio of the change in position \(\partial x\) and beam waist size \(w_0\). The approximation is valid when \(\partial x/w_0 \ll 1\).

\[
E_{out} = E_{00} + j\frac{\pi w_0}{\lambda} \partial \theta E_{01} \tag{2.23}
\]
For angular misalignment Equation 2.23 states that the resulting beam $E_{out}$ with angular misalignment $\partial \theta$ relative to the cavity optic axis is composed of the fundamental mode $E_{00}$ with additional first order mode $E_{01}$ with amplitude proportional to the angular misalignment $\partial \theta$. This approximation is valid when $\partial \theta \pi w_0 / \lambda \ll 1$.

Conversely the resulting change in beam position and alignment can be calculated from the transverse mode power in the beam as used by Kwee et al [177] for characterising laser noise.

This modal decomposition can also give useful information on longitudinal degrees of freedom. In this case changes in waist position along the optic axis and changes in waist size can be equated to the addition of second order transverse modes [21]. This equivalence was used by Miller and Evans [203] in their proposal to use second order modes for cavity length control and by Kwee et al [177] for their “beam quality and pointing measurement breadboard” used to characterise the Advanced LIGO laser systems at the Albert Einstein Institute in Hanover.

In Chapter 4 the principle of Equation 2.23 is successfully used to generate higher order modes of a specific frequency. These high order modes resonate in the cavity creating a radiation pressure beat note that drives the eigenmodes of the mirror. Later in Chapter 4 the general idea that the phase distortion can generate any higher order modes is suggested as a means of measuring the parametric gain (a parameter useful in assessing a systems susceptibility to parametric instability) when the system is below the parametric instability threshold. In Chapter 8, the beam position transverse mode equivalence of Equation 2.22 is used extensively to explain the unexpected coupling of test mass eigenmode signals throughout the interferometer.

### 2.3.4 Filtering Properties

As with any resonator an optical cavity filters light passing through it. This filtering is due to photons being stored in the cavity for a lifetime $\tau_c = FL/c$. The filtering effect on amplitude noise or length noise is a single pole low-pass filter with a cavity pole that can be expressed in terms of the cavities frequency response:

$$\gamma_\omega = \frac{\omega_{fsr}}{2F}$$  \hspace{1cm} (2.24)

Or as a displacement linewidth:

$$\gamma_d = \frac{\lambda}{2F},  \hspace{1cm} (2.25)$$

Here the interpretation is that a length disturbance of a mirror in the cavity produces an optical sideband that is attenuated if it is outside the cavity linewidth.
The spatial filtering properties of a cavity are somewhat more complicated. Gener-
ally a high finesse optical cavity is a very good spatial filter as most HOTEM will not
resonate because of the extra Guoy phase. The exceptional case is that of degeneracy
discussed above.

To determine HOTEM attenuation for a particular cavity, the spatial profile of the
pump beam must be decomposed into the cavity eigenmodes. The spectral density
of each eigenmode in the pump beam may be determined. Then the attenuation of
each of these basis elements may be determined from the cavity transfer function
for this eigenmode. For a simple cavity this can be defined by the frequency and
cavity finesse of this eigenmode. For example, consider a TEM$_{01}$ mode at frequency
$2\pi\omega_{01} = 5.1$ kHz incident on an impedance matched cavity with a $2\pi\omega_{01} = 5$ kHz
mode spacing and linewidth of $\gamma = 200$ Hz. Half the TEM$_{01}$ mode will be reflected
and half will be transmitted.

This method is used in Chapter 8 Section 8.1.2 to determine the coupling of
transverse modes through the interferometer. In this case coupled optical resonators
are treated as independent and the HG mode decomposition is assumed.

### 2.3.5 Mode Matching and Measuring Transverse Optical Modes

Mode matching is the term used to describe the spatial manipulation of a laser beam
from one optical system, such as a laser, to enable the maximum optical power transfer
to the next optical system, such as an optical cavity. We are normally interested
in matching to the fundamental mode. In this case we require a maximum power
transfer to the fundamental mode of the cavity and a minimum coupling of higher
order transverse modes.

We can determine the coupling coefficient from one mode with normalised mode
distribution $E_{p'q'}$ to any other mode with normalised mode distribution $E_{pq}$ by inte-
grating the mode optical overlap factor given in Equation 2.26,

$$\int_S |E_{p'q'} * E_{pq}| d\vec{r}_\perp \quad (2.26)$$

Here $\int_S d\vec{r}_\perp$ is the integral over the surface where this coupling can occur.

A similar overlap integral is used to estimate the sensing function when a super-
position of fundamental and HOTEM are incident on a photodiode, this photodiode
overlap factor is given by:

$$\int_{PD} E_{00}^* E_{pq} + E_{00} E_{pq}^* d\vec{r} \quad (2.27)$$
Here $\oint_{PD} d\vec{r}$ is the integral over the surface area of the photodiode.

The overlap factor of Equation 2.26 is used in Chapter 8 Section 8.1 to estimate the optical overlap factor (coupling) of different transverse modes between optical systems in the advanced LIGO interferometer. Equation 2.27 is used to estimate the expected amplitude and phase of the these superpositions of modes as sensed on a quadrant photodiode throughout this thesis. In this case contributions to each quadrant must be calculated separately replacing $\oint_{PD}$ with $\oint_{Q(1-4)}$.

### Figure 2.12: The quadrant photodetector overlap at frequency $\Delta \omega_{p+q}$ for various low order Hermite-Gaussian and Laguerre-Gaussian modes (same selection as Figure 2.11), red indicates a positive field amplitude while blue indicates a negative field amplitude (a small addition to large Gaussian mode field amplitude).

Generally in this thesis the relative phase is of interest as this allows spatial information of $E_{pq}$ to be ascertained. As such care must be taken to maintain the geometric phase relationship between $E_{00}$ and $E_{pq}$ in numerical simulations. Examples of the $E_{00} * E_{pq}$ are shown in Figure 2.12 with a QPD superimposed.
2.4 Optomechanics

We can now describe the interactions between a mechanical resonator from Section 2.2 and an optical cavity from Section 2.3.

Consider the one dimensional model of Figure 2.3 in which the optical cavity is tuned to resonate. A PDH control system compensates for the static radiation pressure which acts to lengthen the cavity. Compensation is achieved by applying a feedback force to the mirrors which maintain cavity length such that it resonates. This is known as locking the cavity. It is possible to lock a cavity off resonance by introducing an offset voltage into the PDH locking loop. This condition is known as cavity detuning.

In the scenario depicted in Figure 2.13, the linewidth of the pump laser at \( \omega_0 \) is much less than the linewidth of the cavity. In Figure 2.13 the potential field amplitude in the cavity is plotted as a function of the pump frequency (lower axis) in grey. One particular pump frequency is highlighted with a red arrow showing a cavity detuning of \( \omega_0 - \omega_c \). This detuning can equivalently be achieved by maintaining the pump frequency and changing the cavity length as depicted on the top axis. In this case an increase of the cavity length by \( \delta z \) results in a cavity detuning of \( \omega_0 - \omega_c \). When the pump field \( \omega_0 \) is lower in frequency than the cavity resonance \( \omega_c \) it is commonly called red detuned. When \( \omega_0 > \omega_c \) its called blue detuned. In the blue detuned case, the derivative of optical power with respect to length is negative. We know the radiation pressure on the mirror is proportional to optical power and if there is a derivative of force with respect to length \( z \) it gives rise to an effective spring constant. This is known as the
optical spring constant $k_{os}$:

$$k_{os} \propto \frac{\partial F}{\partial z} \propto \frac{\partial E_z^2}{\partial z}$$

The blue detuned cavity results in a positive effective optical spring constant $k_{os}$, while in the red detuned case the system acquires a negative optical spring constant.

If we ignore the PDH control for the moment, this means that the optical power in the cavity changes the resonant frequency of the movable mirror. The equation of motion is:

$$m\ddot{x} + \frac{2m}{\tau} \dot{x} + (k + k_{os})x = F,$$

resulting in a resonant frequency $\omega'_m = \sqrt{\frac{k+k_{os}}{m}}$. This red detuning reduces the resonant frequency, while blue detuning increases the resonant frequency. However this is not the only effect of detuning a cavity.

Let's again examine the one dimensional model of Figure 2.3. The mechanical resonator modulates the phase of light in the cavity at a frequency $\omega_m$ and creates two sidebands $\omega_0 \pm \omega_m$. If the modulation frequency is less than the linewidth of the optical mode $\omega_m < \gamma_0$ then both sidebands are supported by the cavity resonant mode, but they may have different amplitudes as shown in Figure 2.14. Another special case is the one where the modulation frequency is close to an integer multiple of the free spectral range $\omega_m \approx N\omega_{fsr}$ (see Section 2.4.1). If the modulation frequency is large compared to the cavity linewidth then the scattered field is not optically resonant and the optomechanical interaction is weak. First we will consider the case where the scattering sidebands are within the linewidth of the cavity mode. This is known as an unresolved sideband interaction.

Take the case where the modulation frequency is similar to the linewidth of the optical mode $\omega_m \approx \gamma_0$. The length modulation at $\omega_m$ produces sidebands at $\omega_0 \pm \omega_m$. If we assume the cavity is tuned perfectly on resonance then $\omega_c = \omega_0$. By symmetry both sidebands resonate equally in the cavity. Each sideband beats with the pump frequency and creates a radiation pressure force on the mirror. But by symmetry these beat note have opposite phase. Thus the radiation pressure forces from each sideband cancel.

If we again detune the cavity by introducing an offset voltage into the PDH locking loop, the modulation sidebands are no longer symmetric and one radiation pressure force will dominate over the other.

In Figure 2.14 the right panel shows a blue detuned cavity. In this case the lower sideband dominates. The beat note of the lower sideband $\omega_0 - \omega_m$ with the carrier produces a radiation pressure force that has a phase advance relative to the
mechanical motion, exciting further motion. This process results in heating the mechanical mode. The heating optomechanical interaction (that we saw earlier produce a positive optical spring) produces negative damping which decreases the linewidth \( \gamma_{\text{eff}} = \gamma - \gamma_{os} \) and increases the modes effective time constant \( \tau \). Conversely when the cavity is red detuned, the upper sideband is larger. It produces a radiation pressure force that is phase retarded relative to the mechanical motion, reducing the mechanical oscillator motion. This process results in cooling. The cooling optomechanical interaction produces positive damping which increases the linewidth \( \gamma_{\text{eff}} = \gamma + \gamma_{os} \) and decreases the modes effective time constant \( \tau \).

Table 2.1: Relation between detuning, optical spring and optical damping

<table>
<thead>
<tr>
<th>Tuning</th>
<th>Optical spring ( k_{os} )</th>
<th>Optical damping ( \gamma_{os} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>On resonance</td>
<td>( k_{os} = 0 )</td>
<td>( \gamma_{os} = 0 )</td>
</tr>
<tr>
<td>Red detuning</td>
<td>( k_{os}^{-ve} )</td>
<td>( \gamma_{os}^{+ve} )</td>
</tr>
<tr>
<td>Blue detuning</td>
<td>( k_{os}^{+ve} )</td>
<td>( \gamma_{os}^{-ve} )</td>
</tr>
</tbody>
</table>

These three types of unresolved sideband interactions are summarised in Table 2.1. Because there is no new mode involved in these interactions they are referred to as two mode interactions, involving one optical mode and one mechanical mode. It is convenient to think of the two cases as quantum scattering of photons and phonons. The two processes are illustrated in Figure 2.15. In the left panel a carrier photon \( \omega_0 \) scatters with a phonon to produce a lower frequency photon. Clearly by this process an incoming photon must inject a phonon into the mechanical resonator to conserve energy. This is conventionally called a Stokes process. The process associated with the upper sideband consists of an incoming photon gaining energy from the mechanical
resonator. Thus a phonon must be absorbed from the mechanical resonator. This process is called the anti-Stokes process.

\[ \omega_0 - \omega_m = \omega_s \]

(a) Stokes - Heating

\[ \omega_0 + \omega_m = \omega_a \]

(b) Anti-Stokes - Cooling

Figure 2.15: Optomechanical interactions as quantum scattering processes. (a) A TEM\(_{00}\) photon (\(\omega_0\)) undergoes pair creation through the Stokes process to produce a phonon \(\omega_m\) in the mechanical resonator and a TEM\(_s\) (Stokes) photon at a lower frequency \(\omega_s\) within the Lorentzian frequency envelope and Gaussian mode shape of the TEM\(_{00}\). (b) The equivalent anti-Stokes process cools the mechanical mode by emitting a TEM\(_a\) (anti-Stokes) photon

Braginsky et al showed that the two phenomena discussed above are common to all optomechanical systems [245], this can be neatly wrapped up in the phrase. *Positive rigidity and negative damping* or *Negative rigidity and positive damping.*

### 2.4.1 Multimode Interactions

Now consider what happens if the mechanical mode sidebands are resonant in the next longitudinal mode or for that matter any other transverse mode. If the mechanical mode frequency is greater than the linewidth of the fundamental mode the interaction is known as a *resolved sideband* interaction. In long optical cavities sidebands are produced by the mechanical resonance that can potentially be resonant in several different transverse electromagnetic modes (TEM) depending on the mechanical mode frequency \(\omega_m\) as depicted in Figure 2.16. Each case presented in the figure is a *three mode interaction*, it is an interaction between the fundamental mode at \(\omega_0\) a mechanical mode with frequency \(\omega_m\) and a higher order optical mode with frequency \(\omega_n \approx \omega_0 - \omega_m\). In Figure 2.16 only the Stokes sidebands are shown because they are the ones that give rise to parametric instability. Also because the spectral distribution of TEMs is asymmetric in optical cavities it is rare for stokes and anti-Stokes processes to occur simultaneously for a single mechanical resonance.
Figure 2.16: The three-mode interaction at $\omega_{m1}$ is with 3rd order modes, possible radiation pressure distributions are displayed to the right of the peak. At $\omega_{m2}$ the interaction is with the fundamental mode (longitudinal radiation pressure force). At $\omega_{m3}$ the dominant interaction will be with the second order modes with ideal potential radiation pressure distribution on the right of the peak.

Now consider that the sidebands generated by the mechanical mode will only be resonant if two conditions are satisfied. Firstly the frequency of the mechanical mode must be close to the mode spacing from the TEM$_{00}$ fundamental mode to the higher order TEM$_{pq}$. Secondly the scattered light must have a spatial distribution that allows it to couple to the TEM. Equally the TEM must be able to couple to the mechanical resonator. This can only happen if the radiation pressure force of the beat note of the two optical modes is able to interact with the mechanical oscillator.

As an example consider a three dimensional system where the mechanical vibration is in the longitudinal direction. By inspection of the mode shapes only even order Hermite Gaussian modes and Laguere-Gaussian modes with $q = 0$ can have radiation pressure distributions that act with a finite longitudinal component. For all other optical modes, the average radiation pressure distribution is zero over the entire surface of the mirror. To visually assess this refer to Figure 2.12. Remember that the beating is between the high power carrier ($\omega_0$) and the low power TEM. The radiation pressure consists of a large DC component from the pump beam in the cavity plus a small AC term due to the beat between the fundamental and high order TEM. Under these circumstances we interpret the red colours as positive force and blue colours as negative force. Clearly only a few of the modes such as $HG_{02}$, $HG_{20}$ and $LG_{02}$ have a finite longitudinal component.

The above analysis is quantified by the *mass scaled overlap factor* introduced
in Equation 3.16 page 99 and examined in detail in Chapter 3 Section 3.2.7. This parameter partly determines the susceptibility to three mode interactions:

As mentioned earlier the asymmetry of the cavity higher order TEM around the fundamental mode (Figure 2.16) results in a very low probability of symmetric sidebands creating radiation pressure forces that cancel, such as happens with a perfectly tuned unresolved sideband interaction. The only way this can occur in an unresolved sideband interaction is if the TEM frequency is perfectly tuned $\omega_m \approx \omega_{fsr}/2$, or overlap parameters for two TEM are by chance identical.

2.4.2 Stability and Chaos

In early investigations into the stability of Fabry Pérot cavities it was realised that if the storage time of the cavity became large with respect to the period of oscillation of mechanical motion multistability or even chaos could ensue [95]. Other investigations by Tourrenc indicated that the arm cavities of gravitational wave detectors would have enhanced stability [243].

However Soimeno [229] considered transverse motions and showed that tilt instabilities can result if a threshold power is surpassed.

In the following section we will briefly review a selection of observed, predicted and potential optomechanical interactions in gravitational wave detectors, to give an indication of the range of phenomena.

2.5 Examples of Optomechanics in Gravitational Wave Detectors

The hundreds of kilowatts of optical power planned in gravitational wave detectors produce radiation pressure effects, but exactly how these will manifest is an ongoing study. In this section I briefly review a selection of optomechanical effects in gravitational wave detectors, to give the reader a sense of the range of optomechanical interactions in gravitational wave detectors. An angular instability that makes angular control of mirrors in high power cavities challenging is reviewed in Section 2.5.1. The role of the signal recycling cavity of dual recycled interferometers is reviewed in Section 2.5.2. Small amounts of power resonant in this cavity allow dramatic changes in the peak sensitivity and bandwidth of a gravitational wave detector. In Section 2.5.3 a proposal for an optomechanical device designed to achieve broadband sensitivity improvement is reviewed.
2.5.1 Sidles Sigg Instability

In 2006 Sidles and Sigg [228] analysed the torsional stiffness of an optical cavity. This is a torsional optical spring effect that couples the mirrors of the cavity. Two types of instability are shown in Figure 2.17 that relate to the type of beam motion in the cavity. The symmetric degree of freedom is associated with a rotation of the beam about the beam waist, while the antisymmetric degree of freedom is related to a translation of the beam waist.

![Figure 2.17](image)

Sidles and Sigg calculated the optical spring constant coupling the mirrors based on a purely geometric argument much like the optical spring description in Section 2.4.

The two eigenvalues of the torsional stiffness matrix are given in Equations 2.29 and 2.30

\[ k_a = \mp \frac{LP}{c} \frac{4w^4}{w_0^4}, \]
\[ k_b = \pm \frac{LP}{c}, \]

Here \( w \) is the beam size on the optic and \( w_0 \) is the beam waist size, the \( \mp \) and \( \pm \) relate the cavity geometry. For near-concentric cavities \( k_a \) is positive and \( k_b \)
is negative, while for near plane-parallel the signs are reversed. For any cavity we therefore expect there to be two torsional couplings, one inducing positive rigidity and negative damping and one inducing negative rigidity and positive damping between the mirrors.

The effect of the instability was demonstrated at initial LIGO showing very good agreement between simulation and theory [151]. The angular control systems were not adversely affected by the small changes in spring constant and damping coefficient.

In Advanced LIGO the torsional stiffness matrix eigenvalues estimated by Sidles and Sigg are very large compared to the stiffness of the final pendulum stage. $k_{\text{pendulum}} = 6 \text{N.m}$, $k_a = 301 \text{N.m}$ and $k_b = -11.5 \text{N.m}$. The large positive spring constant $k_a$ coupling the mirrors will greatly increase the frequency of torsional modes of the individual mirrors and reduce the damping coefficient. The smaller negative spring constant is still larger than the pendulum response. Therefore at high power the torsional mode of the pendulum would not be stable without control. A control system will be required to achieve angular stability in both cases. Also angular controls must be modified as optical power is increased in gravitational wave detector optical cavities to maintain stability.

In the next subsection the manner by which detector sensitivity can be altered by optomechanically induced changes in the dynamics of mirrors is reviewed.

### 2.5.2 Signal Recycling Cavity Tuning

The interferometer is ideal for gravitational wave detection. The beam splitter naturally separates the pump light and the signal sidebands we want to measure. In the absence of signals the ideal interferometer has no light exiting the dark port. In the presence of signal, the dark port contains only the signal sidebands.

We saw above in our discussion of the simple optomechanical cavity that changes in the dynamics - heating cooling and optical springs - was associated with the relative size of the sidebands. In the interferometer the sidebands can be manipulated by detuning or by changing the linewidth of the cavity itself, which changes the resonant gain of the sidebands.

In 1997 Braginsky proposed a novel detector topology that was built on the optical spring principle. By detuning a cavity the mechanical impedance of the cavity could be tuned [59]. This detector topology was known as an optical bar detector due to its similarity to high Q resonant bar detectors discussed in Section 1.2.1.

In 2001 Buonanno and Chen [64] described how large optical powers in advanced gravitational wave detectors with detuned signal recycling cavities would couple the
mirror motion of the two mirrors of the Fabry Pérot cavities with massive optical springs, converting multiple individual mirror resonances into one single optomechanical resonance. They also demonstrated how changing the tuning condition of the signal recycling cavity (SRC) resulted in an optical spring that could improve a detector sensitivity past the free mass standard quantum limit (SQL) introduced in Section 1.2.3.

Lazebny et al [166] showed how narrow band improvements in strain sensitivity, more than 10 times better than the SQL can be achieved with a detuned SRC. They also demonstrated how increasing the SRC finesse can be used to increase the peak sensitivity at the cost of reduced bandwidth. Figure 2.18, from Lazebney’s paper, shows the bandwidth to peak sensitivity trade-off for various SRC detuning scenarios. The linewidth \( \gamma_t \) of the differential degree of freedom is partly determined by the reflectivity of the signal recycling mirror. Detuning the signal recycling cavity changes the optomechanics which leads to the sensitivity enhancement shown.

![Figure 2.18](image)

Figure 2.18: Left panel - Projected sensitivity of a gravitational wave detector with various signal recycling cavity tuning scenarios solid line \( \gamma_t/2\pi \times 100 = 0.1 \), dashed line \( \gamma_t/2\pi \times 100 = 0.01 \) and dotted line \( \gamma_t/2\pi \times 100 = 0.001 \), straight line represents the free mass standard quantum limit. Right panel shows the effective impedance for the same detuning scenarios, straight line represents the impedance of a free mass. Copied from [166]

Lazebny et al showed that the improved sensitivity is associated with optical rigidity that changes the mechanical impedance of the detector. In the right panel of Figure 2.18 the effective mechanical impedance of the detector is plotted in relation to the impedance of a free mass (straight line). In the left panel the quantum noise limited sensitivity is plotted for the same detuning scenarios. In essence changing the signal recycling tuning has modified the dynamics of the system, thereby creating a modified quantum limited sensitivity with improved performance over the free mass quantum limited sensitivity over a narrow frequency range.
2.5.3 Optomechanically Induced Transparency - a 12cm Squeezing Angle Rotation Filter Cavity

In this subsection I examine shot noise and radiation pressure noise from a different perspective. Gravitational waves induce differential motion in the arms of gravitational wave detectors, this differential motion is measured using laser light. To increase the signal, a strong optical field is required. The interferometer output is then locked to a dark fringe to reduce shot noise associated with the pump field that does not contain gravitational wave signal. In this configuration, quantum vacuum noise will couple into the interferometer through the output dark port and beat with the strong optical field inside the interferometer. The amplitude fluctuations of the vacuum noise will create quantum radiation pressure noise that makes the suspended test masses move randomly, while the phase fluctuations of the vacuum noise will create shot noise on the photodetector that would otherwise ideally only sense the signal sidebands.

It has been demonstrated [145] that quantum fluctuations of light may be squeezed to either: reduce amplitude noise and increase phase noise, or reduce phase noise and increase amplitude noise\(^2\). However, it is impossible to squeeze both amplitude and phase at the same time because that violates Heisenberg’s uncertainty principle. In the left panel of Figure 2.19 quantum vacuum fluctuations are shown as a grey region in the amplitude, phase space diagram. A phase noise squeezed vacuum state is then shown in red. In the right panel the effect of squeezing on the quantum noise limited sensitivity of a gravitational wave detector is displayed.

The signal recycling cavity detuning described in Section 2.5.2 also produces correlations between amplitude noise and phase noise within the detector that result in squeezing. However Lazebny [166] showed that this ponderomotive squeezing effect is very small for the configurations he examined. Normally the term *squeezing*, in the field of gravitational wave detection, refers to the technique of squeezing the vacuum fluctuations entering the interferometer through the instruments dark port. Phase noise squeezing in this fashion was demonstrated at GEO600 and LIGO [246, 2] to be effective at improving detector high frequency sensitivity.

The demonstrated sensitivity improvements have been achieved by squeezing the phase noise entering the interferometer dark port. We saw in Chapter 1 Section 1.2.3 that the high frequency sensitivity of detectors is limited by shot noise. Squeezing phase noise reduces shot noise increasing the detector sensitivity if the low frequency

\(^2\)Or fluctuations may be squeezed at any particular angle in amplitude phase space
sensitivity is limited by non-quantum noise sources. A cartoon demonstration of the improved shot noise limited sensitivity is shown in red in the right panel of Figure 2.19.

We also saw in Chapter 1 that radiation pressure noise dominates the quantum noise limited sensitivity in the low frequency band of gravitational wave detectors because of the frequency dependent mechanical impedance of free masses. As detector using more optical power the radiation pressure noise is expected to limit detector sensitivity. In Figure 2.19 it is clear that by reducing shot noise at high frequency the phase radiation pressure noise at low frequency got worse. To reduce radiation pressure noise the vacuum fluctuations entering the interferometer dark port must be squeezed in amplitude. Amplitude noise squeezing results in the blue curve in the right panel of Figure 2.19. Kimble et al [168] proposed to squeeze the vacuum state in a frequency dependent manner; in particular, squeeze the amplitude noise at low frequency and squeeze the phase noise at high frequency. For optimum sensitivity the squeezing angle should rotate with frequencies between the low and high frequency regions. In principle, it is simple to create frequency dependent squeezing; a phase squeezed vacuum, like the one produced by Heidmann [145], passed through a filter cavity will become a frequency dependent squeezed source. The midpoint between low and high frequency is determined by the cavity linewidth. This technique has been demonstrated with large cavity linewidths $\sim 10^7$ Hz [68].
To match the advanced detector configuration the linewidth of the filter cavity must be \( \sim 100 \text{Hz} \). The cavity must also have extremely low losses so as not to introduce vacuum fluctuations that would disturb the squeezed vacuum state. To achieve these requirements, filter cavities need to be large (\( \sim 10\text{-}1000 \text{m} \)) and have very low round trip losses (\( \sim 1\text{-}100 \text{ppm} \)). Evans et al [111] proposed a realistic 16 meter filter cavity design for an Advanced LIGO upgrade and also demonstrated frequency dependent squeezing experimentally.

Ma et al [184] proposed that an optomechanical cavity just \( \sim 10 \text{cm} \) long could achieve the required squeezed angle rotation using a technique called \textit{optomechanically induced transparency}. The optomechanical interaction couples the optical and mechanical resonators together to create an optomechanical resonator with tunable linewidth. The optomechanical linewidth can be tuned in a region between the optical cavity linewidth (generally \( \gg 1 \text{kHz} \)) and the mechanical resonator linewidth (generally \( \ll 1 \text{Hz} \)). With a moderately high Q factor mechanical resonator, the linewidth requirements for filter cavities are easily achieved. The technique was demonstrated by Qin in 2014 [218]. However it is noted that this was a classical demonstration of squeezed angle rotation. Generally thermal noise from the mechanical resonator would have to be greatly reduced to create a practicable filter cavity. This is because the thermally driven vibration scatters part of the strong optical pump field into the squeezed vacuum field reducing the squeezing level. Investigations into techniques to reduce thermal noise in these filter cavities are ongoing.

There are many more interesting aspects of optomechanics in gravitational wave detectors that are beyond the scope of this review. Applications include quantum entanglement [185], ground state cooling [71], white light signal recycling [217] and optical dilution [211]. An entire text-book could be written on the subject, see Aspelmeyer’s [29] 65 page “Cavity Optomechanics” for detailed further reading. Hopefully this section has given an indication of the ways in which optomechanics affects gravitational wave detectors and is being used in the design of future detectors.

### 2.6 Summary

In this chapter we have reviewed the theory of optomechanical interactions. Energy can be transferred efficiently between optical and mechanical systems when coupling is enhanced by resonance. One optical resonance can be coupled to one mechanical resonance in unresolved sideband interactions. Alternatively energy may be transferred from one resonant optical mode to another in resolved sideband schemes. These
interactions naturally result in radiation pressure acting to change the damping coefficient and spring constant of the mechanical resonances. Blue detuned cavities, where the dominant mechanical mode sideband has a lower frequency than the pump frequency, result in reduced damping coefficients. Red detuned cavities increase the damping coefficient which decreases the mechanical mode amplitude and exponential decay time constant.

These radiation pressure effects also change the mechanical resonant frequency through the optical spring effect. In blue detuned systems where there is a decrease in the damping coefficient, the mechanical mode frequency increases. In red detuned systems where there is an increase in the damping coefficient or cooling, the mechanical mode frequency decreases. This can easily be remembered by the phrase, *positive spring - negative damping* and *negative spring - positive damping*.

We have also explored how different transverse optical modes have different interactions with optics and how some superpositions of modes are identical in nature to changes in beam position and angle. We have explored how different optical modes may be measured as a beat note with the fundamental mode on photodiodes and how the expected amplitude and phase of signals on multi-element photodiodes may be calculated.

Finally we have explored a few examples of optomechanics in advanced gravitational wave detectors and how some of these effects may be used in the future to improve detector sensitivity.

Now we will go on to review the theory and history of parametric instability, which is the main subject of this thesis.
Chapter 3

Three Mode Parametric Instability

3.1 Introduction

In Chapter 1 we showed that advanced laser interferometer gravitational wave detectors such as Advanced LIGO [142] and Advanced Virgo [5] aim to achieve a strain sensitivity \( h \sim 10^{-24}/\sqrt{Hz} \) at \( \sim 100 Hz \). To achieve this sensitivity without recourse to quantum squeezing, it is necessary to balance the shot noise and the radiation pressure noise. To achieve this balance at 100 Hz requires the optical power in the arm cavities to approach 1 MW.

With such enormous optical power, radiation pressure forces significantly alter the behaviour of suspended optical cavities as we saw in Chapter 2. Control systems compensate these affects to maintain correct and stable position and alignment. In this chapter we show that a radiation pressure instability can excite elastic resonant modes of advanced gravitational wave detector mirrors.

In Chapter 2 we also showed how optomechanical interactions manifest in various forms. Unresolved sideband configurations, resolved sideband configurations, heating configurations and cooling configurations. Three mode parametric instability is a resolved sideband heating configuration. It involves an optical field resonant in the optical cavity we call the pump field, an elastic eigenmode resonant in a mirror and a resonant higher order transverse electromagnetic mode (TEM). These three modes are shown in Figure 3.1 labelled a-c.

This TEM is defined by its order numbers \( p \) and \( q \) (TEM\(_{pq}\)). Here, the convention of defining Laguerre Gaussian modes as \( LG_{p,q} \) where \( p \) may only be even is continued such that \( n = p + q \) represents the mode order (while Hermite Gaussian modes \( HG_{p,q} \) \( p \) may be any whole number). Different TEM modes have different spatial profiles as discussed in Chapter 2 Section 2.3.2. In Figure 3.1 we see a cartoon example of the spatial profiles of three modes responsible for a parametric instability.
Figure 3.1: Cartoon of three modes that interact creating parametric instability. Two mirrors are shown, a weak pump beam (red) enters from the left. The pump field is resonant in the cavity (red). The field cross section is inset and labelled (a). There is an elastic eigenmode resonant in the right mirror labelled (b). The surface deformation from this mechanical resonance scatters light into the $TEM_{10}$ that is also resonant in the cavity (orange). The field cross section is inset and labelled (c).

The cavity configuration of advanced gravitational wave detectors is near concentric to maximise the beam sizes on the optic. This is done to minimise coating thermal noise contributions to the interferometers noise budget discussed in Chapter 1. Near concentric cavities have a $g$ factor approaching 1. The mode spacing approaches one free spectral range. In this situation the relative frequencies of the fundamental optical mode, TEM modes and an example test mass resonant mode’s sidebands are depicted in Figure 3.2. Peaks in the figure are colored by longitudinal mode number. It is interesting to observe that the $\omega_m$ sidebands fall onto TEM that do not belong to the same longitudinal mode group as the carrier.

In Section 3.2 we will review the theory of parametric instability starting with Braginsky’s prediction [60] in 2001 that parametric instability would affect proposed advanced gravitational wave detector configurations. Since 2001 several different interpretations and derivations have been made that provide insight into parametric instabilities in gravitational wave detectors. A review of these interpretations along with experimental investigations makes up the remainder of Section 3.2.

Before parametric instability had been observed, techniques for controlling parametric instability were proposed and many were also experimentally investigated, showing their effectiveness. In Section 3.3 we review the proposals for controlling parametric instability and some of these experiments.
Figure 3.2: Optical mode structure of an advanced gravitational wave detector cavity, showing the frequency of a mechanical mode with potential for large three mode interactions. The fundamental mode frequency $\omega_0$ is marked as frequency 0, $\omega_{fsr} \approx 2\pi \times 37.5$ kHz is the free spectral range. $\Delta \omega_1 \approx 2\pi \times 5$ kHz is the mode spacing between the fundamental and first order, $\Delta \omega'_1 = \omega_{fsr} - \Delta \omega_1$ is the mode spacing the fundamental mode with the same longitudinal order, $\omega_m \approx 2\pi \times 15$ kHz is the radial frequency of an acoustic mode making sidebands on the red fundamental mode where there is a third order transverse mode resonance.

3.2 Historical Review of Parametric Instability Theory

3.2.1 Parametric Gain

Braginsky’s approach was to solve the Lagrangian of a Fabry Pérot resonator supporting two optical modes TEM$_{00}$ and TEM$_{pq}$ with amplitude $E_{00}$ and $E_{pq}$, radial frequencies $\omega_{00}$ and $\omega_{pq}$ and full width half maximum (FWHM) linewidth $\gamma_{00}$ and $\gamma_{pq}$ respectively. If we use the infinite mirror approximation, modes of order $n = p + q$ have the same frequency and loss so $\omega_n \approx \omega_{pq}$ and $\gamma_n \approx \gamma_{pq}$. A mechanical resonance characterised by its frequency $\omega_m$ and its FWHM linewidth $\gamma_m$ is supported in one of the mirrors.

By solving the Lagrangian at the frequency of the three mode resonance - where $\omega_{00} - \omega_{pq} = \omega_m$ a stability requirement was derived.

$$\frac{4PQ_nQ_mB^2_{m,n}}{ML_c\omega_m^2} \leq 1 \quad (3.1)$$

Here $L$ is the cavity length $P$ is the optical power in the fundamental mode of the
cavity, $B^2_{m,n}$\(^1\) is a spatial overlap parameter discussed in detail in Section 3.2.7, $c$ is the speed of light, $M$ is the mirrors mass and $Q_n = \omega_n / \gamma_n$ and $Q_m = \omega_m / \gamma_m$ are the Q factors of the TEM\(_{pq}\) and mechanical mode respectively.

For the condition of equality in Equation 3.1 the optomechanical interaction imparts energy to the mechanical oscillator at the same rate as the mechanical oscillator dissipates energy. If the left term was equal to two, energy would be imparted to the mechanical system at twice its dissipation rate. The time evolution in this case would be an exponential ring up with a time constant equal to (but negative of) the ring down time constant of the mechanical system (Polyakov [214] et al explore time evolution in more detail). In this sense the term to the right is the parametric gain $R_m$ of mechanical mode $m$. It is the maximal parametric gain $R_{\text{max}}$ as it is only valid for the three mode resonance condition $\omega_{00} - \omega_{pq} = \omega_m$. In any other situation the parametric gain will always be smaller.

$$R_{\text{max}} = \frac{4PQ_nQ_m}{MLc\omega^2_m} B^2_{m,n}$$

(3.2)

When examining the resonant case it is instructive to consider the interaction in the quantum picture that was illustrated in the Feynman diagram in Chapter 2, Section 2.15 page 77. The stokes interaction results in the mechanical mode gaining energy, it is being heated. The high occupation number of the carrier with radial frequency $\omega_{00}$ ensures that the 3-mode interaction (represented by a three quanta vertex in a Feynman diagram) is dominated by photon-phonon pair creation. A carrier photon decays coherently into a transverse mode photon and a phonon. The phonons contribute to the occupation number of the acoustic mode, and if the rate of this process exceeds the thermal relaxation rate of the mode, the acoustic mode occupation number will increase with time. The anti-Stokes interaction indicates there should be an associated 3 mode cooling process that will be discussed shortly.

Braginsky et al. [60] also considered the parametric gain in the condition where the resonance condition is not perfectly satisfied. In Equation 3.3 the two scattering sidebands, the Stokes and anti-Stokes, generated by the test mass eigenmode, are represented in the expression for the parametric gain $R_m$ for an acoustic mode $m$.

$$R_m = \frac{4PQ_m}{MLc\omega^2_m} \left( \frac{Q_s B^2_{m,s}}{1 + (\Delta \omega_s / \gamma_s)^2} - \frac{Q_a B^2_{m,a}}{1 + (\Delta \omega_a / \gamma_a)^2} \right)$$

(3.3)

Here subscripts $s$ and $a$ denote the Stokes and anti-Stokes modes. $\gamma_s$ and $\gamma_a$ represent the linewidth of these optical mode and $Q_s$ and $Q_a$ represent the Q factors. The

\(^1\)Notation is adjusted from Braginsky’s notation for consistency throughout the thesis.
term $\Delta \omega_s$ and $\Delta \omega_a$ are the difference in frequency between the mechanical mode frequency and the optical beat note frequency $\Delta \omega_{s/a} = \omega_m - (\omega_0 \mp \omega_{ns/na})$. The two optical mode frequencies, $\omega_{ns/na}$ are determined from the optical cavity mode structure shown in Figure 3.2 on page 89. The negative sign on the right term in the brackets indicates that the anti-Stokes interaction reduces the parametric gain while the Stokes interaction increases the parametric gain. This raises the question: Will the anti-Stokes optical sideband cool the mechanical eigenmode as fast of the Stokes sideband heats the eigenmode to stabilise the system.

In 2002 Braginsky published a followup paper [61] that addressed the concern that the Stokes and anti-Stokes optical modes acting on the same test mass eigenmode effectively cancel. The asymmetry of the optical sideband structure shown in Figure 3.2 means that even if the stokes and anti-Stokes modes both resonate they will have different optical mode shape. These different shapes result in different overlap integrals and hence different interaction strength. An example is shown in Figure 3.3. In this example a 32 kHz acoustic mode creates two sidebands. There is a significant difference in optical Q factor (resonant amplification) and mode shape. Hence, differences in overlap integrals to any of the $HG_{pq}$ and $LG_{pq}$ mode shapes shown. Only the $HG_{10}$ has a large overlap integral.

![Figure 3.3: The three-mode interaction, acoustic mode with radial frequency $\omega_m$, with surface deformation shown next to $\omega_0$ peak, creates two sidebands. The optical modes have different resonant gain, and significantly different possible mode shapes. Hence interaction strengths will be significantly different. (This model is based on a 32 kHz acoustic mode in an Advanced LIGO arm cavity)](image-url)
As we do not expect symmetric heating and cooling we expect to find some mechanical modes that are heated by the Stokes sidebands and some mechanical modes that are cooled by the anti-Stokes sidebands. We expect to find very few where the Stokes and anti-Stokes interactions cancel.

Braginsky’s 2002 paper also expands the analysis from a Fabry Pérot cavity to a power recycled Michelson interferometer. This analysis produces three results depending on the tuning condition of the recycling cavity.

1. The ‘rare’ case, where the high order transverse mode is close to resonance \((\omega_{00} - \omega_{pq} - \omega_m \ll \gamma_{pr})\) in the signal or power recycling cavities:

\[
\frac{4PQ\omega_m^2}{M L c \omega_m^2} B_{m,n}^2 \times \left(1 + \frac{\gamma_n}{2\gamma_{pr}}\right) > 1,
\]

where the \(\gamma_{pr}\) is the are the spectral full width half maximum bandwidth of the coupled arm cavities and power recycling cavities.

2. The ‘likely’ case where the high order transverse mode is far from resonance \((\omega_{00} - \omega_{pq} - \omega_m \gg \gamma_{pr}\gamma_n)\) in the recycling cavities:

\[
\frac{4PQ\omega_m^2}{M L c \omega_m^2} B_{m,n}^2 \times \left(1 + \frac{1}{2(1 + \frac{(\omega_{00} - \omega_{pq} - \omega_m)^2}{\gamma_n^2})}\right) > 1.
\]

3. The general case valid under the assumption \(\gamma_m \ll \gamma_{pr} \ll \gamma_n\) will not be presented in this thesis as the purpose of these descriptions is to highlight descriptions in points 1 and 2. With advanced detector configurations where the recycling cavities linewidth is small \(\delta_{pr} \sim 1\) Hz there is a small chance of a dramatic increase in the parametric gain, occurring when the recycling cavity is resonant to the higher order transverse optical mode.

Zhao et al [269] did a careful simulation analysis of parametric instability in Advanced LIGO in 2005. Finite element simulations of the test masses and analytic models of the optical cavities were used to predict that Advanced LIGO would inevitably suffer from parametric instability even considering a range of possible mirror radius of curvature that could be achieved using thermal tuning systems that will be discussed in Section 3.2.6.

### 3.2.2 A Note on Two Mode Instabilities

Two mode instabilities occur in a very similar manner to three mode parametric instability. In this case a scattered field is resonant within the pump mode linewidth, ie it is an unresolved sideband instability. These instabilities were identified by Kippenberg
et al [171] in 2005 in micro-toroidal resonators. Several mechanical resonances were excited by Brillouin scattering of the optical field in a whispering gallery mode of the toroid. In such two mode interactions mode shape of the scattered field is nominally identical to the pump field. Experimental results presented by Corbitt [79] show that in a suspended free space optical cavity such two mode instabilities only achieve high spatial overlap with drumhead type modes. Two mode instabilities are not relevant to advanced gravitational wave detectors as the cavity linewidth is \( \sim 100 \text{Hz} \) and the first test mass resonances are around 5 kHz.

### 3.2.3 Feedback Loop Model

In 2009, Evans et al [110] generalised the formalism presented by Braginsky et al [60, 61], Gurkovsky et al [140] and Strigin et al [234] to include interactions with all the optical fields in an interferometer. The result is very similar to previous derivations, however the optical interaction that was hitherto restricted to two optical modes is now represented by the sum of optical modes \( n \) interacting with the mechanical mode \( m \).

\[
R_m = \frac{4\pi Q_m P}{M \omega_m^2 c \lambda_0} \sum_{n=1}^{\infty} \text{Re}[G_n] B_{m,n}^2
\]  

(3.6)

Here \( G_n \) is the transfer function for an optical field \( n \) leaving the test mass surface to the field incident on that same surface.

A powerful conceptual model was also presented in pictorial form by Evans et al, the parametric instability feedback loop. In the diagram in Figure 3.4, thermal noise leads to resonant mode in the mirror having some finite amplitude. A portion of the TEM\(_{00}\) field resonant in the optical cavity scatters off the resonant mode in the mirror producing sidebands \( \omega_m \) from \( \omega_{00} \) and a portion of that scattered light is resonant as a TEM\(_{01}\) field in the cavity at a frequency \( \omega_{00} - \omega_m \). The TEM\(_{01}\) and TEM\(_{00}\) fields beat in the cavity to produce a radiation pressure field at frequency \( \omega_m \). If the scattering process was a stokes process the radiation pressure force from this beating will be in phase with the acoustic mode producing excitation. While if it was anti-Stokes scattering into TEM\(_{01}\) at \( \omega_{00} + \omega_m \) the radiation pressure force is out of phase producing radiation pressure cooling of the resonant mode in the mirror. We can consider what happens when the three mode resonance is not satisfied in this model. In that case the sideband at \( \omega_{00} \pm \omega_m \) does not coincide with the peak of the TEM\(_{01}\) resonance. The amplitude response is reduced and some phase is introduced. In this situation we would expect a lower parametric gain and a
small optical spring component, increasing or decreasing the optomechanical resonant frequency depending on which side of the TEM\textsubscript{01}.

### 3.2.4 Quantization of Three Mode Parametric Interactions

Miao investigated three mode interactions through the lens of a quantised Hamiltonian [197]. This study gives some useful insights for the experimentalist. We can think of parametric instability affecting the decay rate $\gamma_m$ and thermal occupation number $\bar{n}_\text{th}$ of the mechanical resonator, in the classical derivation these are affected by three mode interactions as in Equation 3.7:

$$
\gamma'_m \approx (1 - R_m) \gamma_m \quad \text{and} \quad \bar{n}'_\text{th} = \frac{\bar{n}_\text{th}}{1 - R_m}
$$

Here the primes (’) indicate resulting quantities with parametric gain $R_m$.

Miao defines a coupling constant

$$
G_0 \equiv \sqrt{\frac{B_{m,n}^2 \bar{\omega}_m \omega_n}{M \omega_m L^2}}
$$

Where notation has been converted for consistency.
Miao goes on to demonstrate that the decay rate and frequency will be affected by three mode interactions as:

\[
\gamma'_m = \gamma_m + \frac{4G_0^2 \bar{a}^2 \omega_m \gamma_n (\omega_n - \omega_o)}{[(\omega_m - (\omega_n - \omega_o))^2 - \gamma_n^2] [(\omega_m + (\omega_n - \omega_o))^2 + \gamma_n^2]}, \tag{3.9}
\]

\[
\omega'_m = \omega_m + \frac{G_0^2 \bar{a}^2 (\omega_m^2 - (\omega_n - \omega_o)^2 - \gamma_n^2)}{[(\omega_m - (\omega_n - \omega_o))^2 + \gamma_n^2] [(\omega_m + (\omega_n - \omega_o))^2 + \gamma_n^2]} \tag{3.10}
\]

Here \( \bar{a} = \sqrt{2P/\gamma_0 \hbar \omega_0} \) is a measure of the optical power in the fundamental mode of the cavity. On resonance, where \( \omega_n - \omega_0 = \omega_m \), Equations 3.9 and 3.10 simplify to:

\[
\omega'_m \approx \omega_m - \frac{G_0^2 \bar{a}^2}{4\omega_m} \quad \text{and} \quad \gamma'_m \approx \gamma_m + \frac{G_0^2 \bar{a}^2}{\gamma_n}. \tag{3.11}
\]

These equations are useful for calculating the expected mechanical mode frequency shift with parametric gain. The classical equation for parametric gain (Equation 3.3) derived by Braginsky can be retrieved from Equation 3.11 by defining \( R_m = (\gamma_m - \gamma'_m)/\gamma_m \). In contrast the thermal occupation number can no longer be arbitrarily small as allowed by Equation 3.7, now it has minimum set by vacuum fluctuations \( \bar{n}_{\text{quant}} \) in Equation 3.12.

\[
\bar{n}'_{\text{th}} = \bar{n}_{\text{th}} \left( 1 - R_m \right) + \bar{n}_{\text{quant}} \tag{3.12}
\]

Indicating that where three mode interactions are used to cool a mode, the mode thermal occupation will be limited to a minimum of

\[
\bar{n}_{\text{quant}} \approx \left( \frac{\gamma_n}{2\omega_m} \right)^2. \tag{3.13}
\]

In general thermal fluctuations are always larger than vacuum fluctuations so this has little effect on problematic parametric instabilities in gravitational wave detectors. But the process gives us valuable insights that are used in Chapter 7 and Chapter 8.

### 3.2.5 Parametric Gain Dependence on Optical Power

The linear dependence of parametric gain on TEM\(_{00}\) power in Equations 3.2 to 3.6 indicates that any optomechanical system will become parametrically unstable if the power can be increased enough. The power at which a system becomes unstable is the threshold power for instability, assuming all other aspects of the system stationary. Tomes and Carmon [242] demonstrated this threshold power effect in the first observations of three mode parametric instability in 2009. In these experiments a spherical optical resonator supporting whispering gallery modes much like Kippenberg’s [171] 2 mode experiments was used. The instability was due to three mode interactions.
between two optical whispering gallery modes and one vibrational whispering gallery mode.

In section 3.2.1 the relation between parametric gain and the time evolution of the acoustic mode amplitude was discussed. This relation was demonstrated by Chen et al [69] in the first observation of parametric instability in a free space optical cavity. In these experiments a nano-gram membrane resonator acted as the coupling mirror between two cavities shown in Figure 3.5(a). This configuration provided easy tuning from heating to cooling simply by changing the membrane position $z_o$.

The right panel of Figure 3.5(e) shows two sets of data. We are interested in the blue circles, they show the ring up time constant on the vertical axis as a function of input power on the horizontal axis. The data fits the parametric instability model well. Instability occurs at powers greater than the threshold power which is marked by the grey line.

The relation between parametric gain and optical power,

$$\tau_m = \frac{2Q_m}{\omega_m(R_m - 1)}$$

was used in the first observation of parametric instability at Advanced LIGO in 2014 [113] to estimate the Q factor $Q_m$. In Figure 3.6 the effect changing the optical power (green trace) has on exponential time constant of the mode amplitude (blue trace) can be clearly seen.

Demonstrating linearity between parametric gain and optical power provides a convincing argument that the observed instability is a radiation pressure instability.
Figure 3.6: First parametric instability at Advanced LIGO. Optical power is shown in green. When the power is reduced the growth of the mechanical eigenmode is slowed. When it is further reduced the time constant changes sign and the eigenmode reduces in amplitude. Copied from [113].
### 3.2.6 Transverse Optical Mode Thermal Tuning

Braginsky’s seminal work [60] proposed that parametric instabilities could be avoided by modifying the mirror surface profile. Modifying the surface profile can have a variety of effects (see Section 3.3.7). Here we restrict the changes in mirror profile to simply changes in radius of curvature. As we showed in Chapter 2 Section 2.3.2 changes in radius of curvature affect the mode spacing of higher order transverse optical modes. The relation between mode spacing of a TEM$_{pq}$ and mirror radii of curvature is reproduced here for convenience.

$$\Delta \omega = \omega_{00} - \omega_{n} = \frac{c}{L}(m + n) \cos^{-1}(\pm \sqrt{(1 - \frac{L}{R_1})(1 - \frac{L}{R_2})})$$  \hspace{1cm} (3.15)

Here $R_1$ and $R_2$ are the radii of curvature of the end mirrors of the cavity, $p$ and $q$ are integers describing the order of the optical mode. The ± sign depends on the cavity configuration.

Equation 3.15 assumes spherical mirrors. To determine the mode spacing for mirrors with arbitrary surface profiles an approximation is used in this thesis. The approximate mode spacing is determined by the average radius of curvature, averaged over the area of the beam spot on the mirror. This approximation is used in Chapter 5 and Section 6.3.

By changing the radius of curvature of the mirror the frequency of the TEM$_n$ mode can be changed to move away from the triple resonant condition $\omega_{00} - \omega_{pq} = \omega_m$.

In 2005 Zhao et al [269] proposed that heat loads applied to testmasses would be adequate for tuning $\omega_n$ away from the triple resonance in Advanced LIGO optics. A heat load changes surface profile through thermal expansion or the effective curvature experienced in transmission of an optic through the thermo-optic effect. This scheme became known as thermal tuning and thermal detuning of parametric instability.

Thermal tuning for parametric instability control was extensively investigated with a 80 m high power cavity [265, 92]. In these experiments ring heaters and compensation plates were used to tune three mode interactions. In 2012 Susmithan et al [235] performed experiments showing thermal tuning with a CO$_2$ laser is effective at changing the mirror radius of curvature. CO$_2$ laser thermal tuning is relatively fast compared to ring heater thermal tuning. In the left panel of Figure 3.7 the experimental layout shows how the CO$_2$ laser power was applied to the mirror. Sensing with a Hartmann wavefront sensor showed the heat induced wavefront distortion. The beam profiler measured the change in beam size of a beam transmitted from the cavity. The radius of curvature of the optic was then derived from the beam size. The QPD
measured the parametric amplification of a test mass acoustic mode. (Note: this is a combination of mechanical mode amplification and optical mode amplification)

Figure 3.7: (Left panel) Experimental layout showing 77 m cavity, CO$_2$ heating, beam profile, Hartmann wavefront sensor and quadrant photodetector for measuring the signals from the three mode interactions. (Right panel) The parametric gain amplification of the resonant mode thermal noise amplitude as a function of CO$_2$ laser power, (inset) time evolution of the radius of curvature when CO$_2$ laser power is changed at point 'b'.

In the right panel of Figure 3.7 the observed TEM$_{01}$ amplitude is plotted as a function of CO$_2$ heating power. The three mode resonant condition is satisfied when the CO$_2$ heating power was $\approx 0.62$ W. The method of thermal tuning is now fully confirmed. It has been used frequently at both Advanced LIGO and the Gingin High Optical Power Test Facility [156] to tune three mode interactions. At the Advanced LIGO detectors, ring heaters near the test masses are used to tune the radius of curvature of the test mass mirrors. This will be investigated in detail in Chapter 5.

3.2.7 Overlap Factor

The overlap parameter $B_{m,n}^2$ groups spatial information of the optical and mechanical modes in Braginsky’s derivation. We define $\vec{u}$ to be the displacement field of the mirror eigenmode, $u_z$ is the surface deformation in the direction along the optic axis ($z$) of the high reflectivity surface, $E_0$ and $E_n$ are the optical fields distributions over the mirror’s surface of the fundamental and $n^{th}$ optical mode respectively,

$$B_{m,n}^2 = \frac{V \left( \oint_S E_0(r^-_\perp) E_n(r^-_\perp) u_z(r^-_\perp) \, dr^-_\perp \right)^2}{\oint_S |E_0|^2 \, dr^-_\perp \, \oint_S |E_n|^2 \, dr^-_\perp \oint_V |\vec{u}|^2 \, dV}.$$  

(3.16)

Here $\oint_S \, dr^-_\perp$ denotes integration over the mirrors surface and $\oint_V \, dV$ over the mirrors volume.
The numerator of Equation 3.16 will be large if the field distribution $E_n$ and the mechanical mode surface deformation $u_z$ have a similar shape. As such a visual analysis gives a good indication of the overlap integral. In Figure 3.8 four optical modes are displayed as the radiation pressure field $E_{00} \ast E_{pq}^*$ in the left column and 12 test mass eigenmodes to the right, the eigenmodes are selected so there is one with a relatively large overlap parameter for each optical mode. Before reading the caption, the reader may like to inspect the mode shapes and try to identify acoustic mode shapes in the right 3 columns that have a high spatial overlap with optical modes shown in the left column.

Figure 3.8: Visual analysis of overlap factors, on the left column the radiation pressure distribution is plotted $(E_{00} \ast E_{pq}^*)$ for various modes (Ideal $HG_{30}$, simulated $n = 3$, ideal $HG_{22}$ and ideal $HG_{53}$) to the right the surface deformation of various test mass eigenmodes are plotted. The largest overlap integral for each row top to bottom is column 2, 3, 2 and 1

We see from Equation 3.16 that the surface deformation does not completely de-
scribe the overlap parameter. Modes with small surface displacements relative to the average volume displacement field will have a smaller overlap parameter. So computation of the overlap parameter is required. Computing the overlap parameter is straightforward using analytical (Section 3.2.6) or simulated optical mode shapes (see [35] for a good discussion comparing techniques for simulating optical mode shapes) and finite element simulation of the displacement field of the test mass eigenmodes. However as shown in Chapter 5 there are many subtleties in the calculation of the overlap that are not covered in the literature.

Observations of resonant enhancement of the scattered field are evidence that there is overlap between the optical modes and the surface deformation. In 2008 Zhao et al [270] demonstrated this resonance of the scattered field for the mode shown in Figure 3.9(a). The overlap parameter can be ‘visually integrated’ in the vertical profiles of the TEM $HG_{01}$ field and the product of the TEM $00$ and test mass eigenmode or ‘acoustic’ mode surface profile shown in Figure 3.9(b).

Figure 3.9: (a) the surface deformation of the 158kHz test mass eigenmode observed the interact with the TEM$_{01}$ optical mode. (b) The vertical profile of the TEM$_{01}$ optical mode and the produce of the eigenmode (acoustic mode) and TEM$_{00}$ mode, Copied with permission from [270]

In Chapter 4 a low power characterisation of parametric instability experiment is studied. Analysis includes an experimental estimate of the overlap factor that is within a factor of two of the predicted maximum value.

Several experimental observations of three mode interactions [49, 158] have noted that the parametric interaction strength is beam position dependent. To some degree this is expected from the change in the overlap parameter as the beam moves relative to the test mass. For example, the overlap factor of the acoustic mode in row 1 column 4 with the ideal optical mode shown in column 1 row 1 Figure 3.8 is zero
when the optical mode is perfectly aligned with the acoustic mode. However the overlap parameter grows quickly with any misalignment of the beam in the right or left direction.

Heinert studied this relation between beam position and parametric gain in simulations of advanced LIGO in 2011 [146]. He demonstrated that deviation from perfect central alignment of \( \sim 1 \text{ cm} \) is likely to cause more parametrically unstable modes. In Section 6.3 we will see that many parameters change when the beam position changes. We will show that oscillation in beam position can even be used to suppress instability.

### 3.2.8 Parametric Instability Saturation

In the first demonstration of free space three mode parametric instability Chen et al [69] showed parametric instability excites acoustic modes to amplitudes where they cease to grow, their amplitudes saturates. In Figure 3.10 the measured amplitude of

![Figure 3.10: First parametric instability in free space cavity showing saturation. Point are measured data, colours represent 3 different power levels, solid lines are the modelling results, Figure reproduced with permission from [69].](image)

the optical beat signal is plotted against time (points) for three different optical power levels in three different colours. From time zero the exponential ring-up of parametric instability can be observed as an exponential increase in the beating signal. The time constant of this ring-up is inversely related to power. The exponential ring-up then saturates after a time also inversely related to power. The saturation amplitude is very similar for all the traces reported.
Polyakov and Vyatchanin [214] had predicted this saturation effect in 2007. In 2014 Danilishin et al [87] explained how scattering from the surface deformation due to the test mass eigenmode will eventually reduce the power in the fundamental mode to the threshold power for instability. Danilishin’s modelling results are reproduced in Figure 3.11.

![Diagram](image)

Figure 3.11: Danilishin’s model for parametric instability saturation in a Fabry Pérot cavity. Mode shapes are depicted top center. Feynman diagram top right. Spectral cartoon of the modes $\Omega_m \equiv \omega_m$ is the mechanical mode frequency, $\omega_S = \omega_{pq}$ is the stokes mode frequency and $\omega_0$ is the fundamental mode frequency. In the center the time evolution of each mode is shown for the non-linear model. Figure reproduced with permission from [87].

In Figure 3.11 (top left) a model Fabry Pérot cavity is depicted. The mode structure for the model is shown in the cavity, in cross section to the right of the cavity in a Feynman diagram (top right) and in frequency space along the bottom of the figure. The amplitudes of the different modes are plotted as a function of time through the center of this image. The mechanical mode and the stokes mode amplitudes grow while the fundamental mode amplitude reduces to the threshold power. When the fundamental mode reaches the threshold power for instability the mode amplitude stops growing as demonstrated in Chen et al’s experiment.

The condition for instability changes as the mode amplitude changes when the saturation phenomena is considered. Danilishin et al. expressed this as a new stability criterion:

$$ R_{NL} = \frac{R_{max}}{(1 + \frac{\gamma_{pq}}{\gamma_0 |x_m|^2})^2 + \left(\frac{1}{\gamma_{pq}} \Delta \omega_0\right)^2} \leq 1. \quad (3.17) $$

In the extreme case the power in the transverse mode could reach threshold power resulting in a cascade of multiple instabilities. This phenomena has been modeled in
two mode instabilities by Marquardt [188] naming it a multi-stability phenomenon.

To date non-linear affects have not been observed at Advanced LIGO. The cavity control systems go out of range, resulting in loss of interferometer control, before the saturation can stabilise the cavity [113].

### 3.2.9 Parametric Instability in Coupled Cavities

After Braginsky [61] extended parametric instability to power recycled Michelson interferometer. Strigin et al.[234] extended the theory to a dual recycling interferometer detector and showed that the multi-cavity coupling could reduce the effective line-width to a sub-Hz range. Again if the high order cavity mode involved in parametric instability is resonant in both the arm cavities and the recycling cavity, extremely high three mode parametric gain could occur. However such instabilities are very unlikely.

Detailed analysis of a dual recycling interferometer with realistic test masses by Gras et al.[136] showed that the highest gain could reach \( \sim 1000 \) corresponding to acoustic ring-up times \( \sim \) seconds. Gras et al’s simulations beautifully show the relation between the potential parametric gain and the tuning conditions for various parts of the interferometer. An example simulation shown in Figure 3.12 depicts the parametric gain for a particular mode as a function of test mass radii of curvature. In the top panel the two arm cavities are adjusted in a symmetric manner. Note the appearance of a double peak with detuning of the signal recycling cavity. While in the lower panel asymmetry is introduced and only one cavity’s radius of curvature is adjusted resulting in a signal resonant peak. The signal recycling tuning \( \theta \) moves the resonant peak in test mass radius of curvature coordinates and changes the ‘linewidth’ in radius of curvature units.

There are too many variables and too many uncertainties to produce exact estimates of parametric instability in dual recycled Michelson interferometers. Hence, statistical approaches have been used to estimate the likelihood of parametric instability. Generally these approaches use numerical optical simulation packages such as Finesse [9], Optickle [112], Oscar [93] or FOOP [261] to estimate the parametric gain. Statistical information is attained by running Monte Carlo simulations over the range of likely cavity configurations. Such a simulation was used by Matthew Evans in 2012 to predict that at 125W injection there is a 20% chance of no parametric instability at Advanced LIGO. For conditions like Advanced LIGO’s Observation Run 1 where the injected optical power was 25W this statistical approach predicted a 20% chance of at least one parametrically unstable mode as shown in Figure 3.13.
As we will see in Chapter 5 we now know that there was one parametrically unstable mode at 25 W injected optical power (\(\sim 100 \text{kW} \) contained optical power in the arm cavity). Was the instability at Advanced LIGO a statistical anomaly? Or will there be many more instabilities than predicted? It is somewhat more complicated.

These simulations, undertaken before parametric instability was observed, did not take into account the dynamics of the cavity mirrors heating and cooling due to absorption of the high reflectivity coatings. These dynamics will be explored in Chapter 8 Sections 8.2 and 8.3.2. These investigations show that the detector tuning for parametric instability changes as the mirrors warm. Observations have now shown that the detector is exposed to many more unstable modes, however these unstable modes will be unstable for a period as the mirrors warm, then they will be stable when the mirrors reach thermal equilibrium, they are transient instabilities. These
transient instabilities provide a strong motivation to reduce the thermal transients in the mirrors.

3.3 Review of Proposed Mitigation Strategies

3.3.1 Suppression of Parametric Instability

From Equation 3.6 it can be seen that many of the design requirements for advanced interferometers such as high power for lower shot noise and high Q materials for low thermal noise tend to increase parametric gain. It also indicates how parametric instability can be controlled by the following strategies:

- Reduce the circulating optical power
- Avoid the three mode resonance condition
- Change the spatial overlap
- Reduce the mechanical mode Q factor
- Reduce the finesse of the higher order optical mode
The key is to control the instability without affecting core design requirements of the gravitational wave detector [249]. So a strategy such as reducing the optical power is not a useful control strategy as it will result in increased shot noise. Lowering the Q of the test mass by choosing a material with a larger loss angle is also not a good strategy as it will result in increased thermal noise.

In the following sections parametric instability control schemes are reviewed.

### 3.3.2 Passive Dampers

Because the threshold for instability is proportional to the acoustic Q factor $Q_m$ of the test masses, adding damping is a logical approach. Ring damper’s were proposed by Gras et al [133] in 2009. The proposal involves adding a material with high mechanical loss to the circumference of the test mass. The Q factor of eigenmodes of the test mass would be reduced. Modeling demonstrated that a rubber ring would be effective and that gold coatings that have been proposed to improve the thermal and electrical conductivity of the test mass will reduce parametric gains a little. However Gras et al’s simulations also show that ring dampers may introduce unacceptable levels of thermal noise.

The thermal noise increases due to the increase in the mechanical loss of surface deformation of the test mass in the gravitational wave detector bandwidth 10-4000 Hz. So a passive damper would ideally have a large mechanical loss at the high frequency of acoustic mode (5-90 kHz) and low mechanical loss at the low gravitational wave frequencies 0.03-4 kHz.

Acoustic mode dampers are an electro-mechical low-pass filter designed by Gras et al [138, 114] for this purpose. They are composed of a piece of piezoelectric material, a reaction mass and a shunt resistor across the piezo. This forms a mechanical resonator that is attached to the test mass. The capacitance of the piezoelectric material along with the shunt resistance results a resonant electro-mechanical damper.

Experiments by Gras et al [134] showed that this method has potential, reducing Q factors by 1-2 orders of magnitude. However high mechanical loss of the piezoelectric and epoxy materials used in these tests result in unacceptable increases in Advanced LIGO thermal noise [134]. Investigations into better materials are ongoing.

### 3.3.3 Mechanical Feedback Damping

A feedback system that senses the mode amplitude and applies a mechanical feedback force to the test mass can also be used to effectively reduces the Q factor of the
eigenmode. Electrostatic drives can apply mechanical feedback to the test masses at Advanced LIGO. In this section we consider electrostatic damping of parametric instability.

Electrostatic damping was proposed by Ju et al [157] in 2009 and simulation and experiments were performed by Miller et al [204] in 2011 that largely confirmed it would be effective.

Its operation is dependent on the magnitude of the overlap integral achievable between the actuator and the thousands of potentially unstable acoustic modes. Because the overlap integral depends strongly on relative locations of the electrostatic combs it is difficult to be certain that it will have sufficient overlap for all potentially unstable modes.

Demonstration of damping parametric instability with the electrostatic drives is the subject of Chapter 6 Section 6.2 with details of this method described therein.

### 3.3.4 Optical Feedback

Optical feedback can suppresses instability by suppressing the build up of the transverse optical mode associated with the instability. This is achieved by injecting the higher order transverse optical mode with the opposite phase the the higher order transverse optical resonant in the cavity. Interference results nulling the circulating high order transverse mode. This technique was first proposed by Zhang et al [264] in 2010 and was tested in 2010 by Fan et al [117] in an 80 m suspended optical cavity in the configuration shown in Figure 3.14.

At the time of Fan’s experiment [117] parametric instability had not been demonstrated, so an excitation was applied with a capacitive actuator. Optical feedback was demonstrated by suppressing the higher order transverse optical mode’s amplitude.

In Chapter 4 a different modulation scheme is used to the same effect, in this case to constructively interfere with the resonant higher order transverse optical mode. With the resonant enhancement of this mode radiation pressure excitation of a test mass eigenmode was demonstrated. Zhao [271] proposed that this parametric instability damping strategy could be incorporated into advanced gravitational wave detector designs with minimal additional hardware. This method involves detecting the onset of instability, generating an interference beam and injecting it into the optical cavities. This sounds simple, but is complicated by the need to generate several transverse mode shapes of precise frequency with minimal injected noise.
3.3.5 Radiation Pressure Damping

Shortly after predicting parametric instability Braginsky et al [62] proposed that radiation pressure could be used to damp parametric instabilities. Any beam of light can provide a radiation pressure force on a test mass. However the radiation pressure has to dissipate energy from the acoustic mode at the rate the three mode interaction is imparting energy to the acoustic mode of the test mass.

The Photon Calibrator, or PCal, laser has been suggested as a possible candidate. While the PCal laser may be powerful enough at present the beam geometry is not suitable. The PCal design has two beams of equal power, separated vertically, incident on the test mass. There is currently no means of applying amplitude modulation to the two beams independently. In this geometry interaction with the test mass eigenmodes at 15.5 kHz and 15 kHz will be weak.

A radiation pressure damping system with 4 carefully placed beams (to avoid nodes in modes surface deformation) that can be independently modulated with a peak to peak force of $\sim 10\text{nN}$ may be effective. Such a system would require one $\sim 10\text{W}$ laser source per test mass. An advantage of such a scheme is that beam positions can be adjusted easily to achieve optimum damping for a particular set of parametrically unstable modes. This parametric instability damping scheme should be further investigated.
3.3.6 Substrate Selection

The density of resonant modes is dependent on the speed of sound in the material. By choosing a material that has a suitable speed of sound the eigenmode density at frequencies that are likely to suffer from parametric instability can be minimised. This region is where the transverse optical modes beat note frequency is likely to cause instability. Zhao et al [269] used simulation to compare the severity of instability in Advanced LIGO with two different substrate materials - fused silica and sapphire. The comparison of number of unstable modes and maximum parametric gain over a range of transverse optical mode tuning frequencies are shown in Figure 3.15. Over

![Figure 3.15: The maximum parametric gain for an Advanced LIGO arm cavity as a function of the mirror radius of curvature. (a) Sappire optics, (b) fused silica optics.](image)

the tuning range displayed sapphire test masses result in far fewer unstable modes due to the lower mode density. This means its possible to tune to $\Delta \omega_{pq}$ further from the mechanical resonance $\omega_m$ resulting in lower parametric gains. It was shown by Zhao et al that using sapphire test masses and appropriate tuning a parametrically stable could be achieved. In Figure 3.16 The number of unstable modes is plotted as a function of mirror radius of curvature. Regions around 2090 and 2125 m are parametrically stable.

These studies were followed up with a report by Ju et al [159] in 2006 that proposed using sapphire test masses at the end test masses of the Michelson interferometer arms and fused silica test masses as input mirrors to the arm Fabry Pérot cavities. One of the problems with sapphire is that its thermal conductivity is very low. This means that large thermal gradients form when heat loads are applied and any corrective heat load, such as those proposed in Section 3.2.6 to control the mirror radius of curvature, take a long time to stabilise. Ju’s simulations showed that by using the proposed design the severity of parametric instability could be reduced while thermal
compensation could be performed reasonably fast with the fused silica input test masses.

### 3.3.7 Reducing the Q factor of Higher Order Transverse Optical Modes

**Surface Profiles**

The parametric gain would be reduced if the resonant enhancement, via the TEM$_{pq}$ mode, of scattered field was reduced. In a sense this is how thermal tuning of Section 3.2.6 works by moving the TEM$_{pq}$ resonant frequency away from $\omega_{00} - \omega_m$. Another way to approach the problem is to reduce the finesse of the TEM$_{pq}$.

One method to reduce the finesse of the higher order modes is to modify the surface profiles of the mirrors. In 2016 Matsko et al [192] showed through simulation that surface profiles could be designed that increase the round trip loss of higher order modes, maintaining the critical round-trip loss of the fundamental mode below 3 ppm.

Example surface profile used in their study are shown in Figure 3.17.

Such mirror profiles affect the stability of the interferometer when misalignment is considered. There are several other aspects that also would require detailed studies before any serious proposal for implementing such a system. In addition manufacturing such mirror profiles to the tolerances required for advanced gravitational wave detectors would be challenging.
Figure 3.17: Surface profiles that increase the round trip loss of high order transverse optical modes. Copied from [192]
Filter Cavities Another suggestion that has been raised numerous times [199, 137, 96, 139] is that filter cavities could be used at or in place of the end test masses to remove the higher order optical modes from the arm cavities. The problem with such proposals is the difficulty in frequency and mode matching many higher order modes to the filter cavity.

Green et al [139] propose using a filter cavity (or extraction cavity) in transmission of the arm cavities or in reflection of the OMC and IMC. This cavity has a mode cleaner (inherent to the IMC and OMC) as end mirror such that the high order transverse modes have a high resonant gain in the cavity while the fundamental mode has low gain due to the high transmission of the mode cleaner. The cavity must be completely degenerate such that all mode resonate. Also mode shape must be preserved in this degenerate cavity. The design of a cavity with these requirements appears challenging. Simulation results that assume the preservation of mode shape show reduction in the number of parametrically unstable modes.

The discussion with Miao, De Salvo and Gras have been about a cavity composed of the ETM and another mirror to make a composite end mirror like a Khalili cavity [167]. This Khalili filter cavity must achieve high reflectivity of the fundamental mode in the cavity and high transmission of the high order modes. A standard Khalili cavity will achieve this to some degree. To get resonant extraction of the high order transverse modes a mode structure in the Khalili filter cavity approaching that of the arm cavity is required. The idea is not practical due to the extreme length or extreme g factors required of such a cavity.

3.4 Summary

In this section we have reviewed the theory of parametric instability and have explored many experiments that prepared the gravitational wave community for parametric instability in advanced gravitational wave detectors. In the next chapter we apply this theory in a 80m cavity to show the radiation pressure excitation of acoustic modes and in doing so determine the parametric gain of the interaction.
Chapter 4

Low Power Characterisation of Parametric Instability

4.1 Introduction

Parametric instability is a demonstrated threat to advanced detectors as discussed in the previous chapter. The effect of instability is the exponential increase in the amplitude of mechanical resonances of test masses, with associated increase in the conversion of the fundamental to high order optical modes. The instability could continue until the fundamental mode is depleted sufficiently such that the system reaches equilibrium, however more commonly it would be expected to cause a loss of cavity locking and detector dysfunction. Several control mechanisms were reviewed in the previous chapter and various aspects of these control mechanisms have been tested.

In this chapter the first demonstration of radiation pressure driving of ultrasonic acoustic modes via pairs of optical modes in gravitational wave detector type optical cavities is described. The chapter is based on the paper “Radiation pressure excitation of test mass ultrasonic modes via three mode opto-acoustic interactions in a suspended Fabry-Pérot cavity” [49] by Carl Blair, S. Susmithan, Chunnong Zhao, Qi Fang, Li Ju and David Blair. Small changes have been made to make it more relevant to this thesis.

In the experiment presented in this chapter, 0.4W of TEM\(_{01}\) mode and 1kW of fundamental TEM\(_{00}\) mode were circulated inside the cavity. A 181.6kHz excitation was observed with amplitude 5 \(\times\) 10\(^{-13}\) m. The results presented in this chapter verify the radiation pressure excitation term in the parametric instability feedback model [60] shown in Figure 4.1. The interaction parametric gain was (3.8 ± 0.5) \(\times\) 10\(^{-3}\) and the mass-ratio scaled opto-acoustic overlap factor was 2.7 ± 0.4.
The ability to estimate such low parametric gains may be useful in predicting the parametric instability threshold power in gravitational wave detectors. Also the radiation pressure control of the acoustic modes is a key aspect of the optical control of parametric instability discussed in Chapter 3 Section 3.3.4. The experiment presented in the chapter is another demonstration of the optical actuation principle.

4.2 Method

As explained in Chapter 3 Section 3.2.1 if the parametric gain \( R_m \) exceeds unity for a particular acoustic mode \( m \), the mode will grow exponentially. The experiment described here is part of a research program aiming to determine the best way of controlling instability.

Three-mode parametric instability (PI) can be broken down into components in the feedback loop model described in Chapter 3 Section 3.2.3. As we saw in Sections 3.2.6 and 3.2.7 two of the three components that make up this feedback loop, as labelled in Figure 4.1, have been experimentally verified:

a) Thermal tuning: It was predicted in 2005 that PI can be tuned through thermally induced radius of curvature (RoC) variations in the cavity mirrors [269]. This tuning of the resonant interaction was demonstrated in 2008 [198].

b) Acoustic generation of transverse optical modes: The acoustic generation of a specific resonant transverse mode has amplitude determined by the overlap factor \( a \) number that determines the spatial overlap between the acoustic and optical modes. This was reported by Zhao et al [270]. In 2011 three mode interactions were used to detect thermally excited acoustic modes, and these interactions were shown to enable high sensitivity spectroscopy of the thermally excited acoustic mode spectrum [268].

The third component closes the feedback loop (labelled (c) in Figure 4.1). It is the radiation pressure excitation of the acoustic modes via beating between the high power carrier optical mode and the transverse optical mode.

This chapter reports the first observation of such radiation pressure excitation in a long optical cavity designed as a sub-scale version of a gravitational wave detector cavity. The specific interaction being studied is the three mode interaction (3MI) between a TEM\(_{00}\) mode, a TEM\(_{01}\) mode and a mirror acoustic mode at 181.6kHz. The frequency of the TEM\(_{01}\) mode is tuned about 181.6kHz above the TEM\(_{00}\) using a CO\(_2\) laser to thermally deform the mirror so as to accomplish RoC tuning. The system is shown schematically in Figure 4.2.
Both TEM\textsubscript{00} and TEM\textsubscript{01} modes are injected into the cavity. The beating between these two signals is detected at the measurement point, and provides the radiation pressure field on the mirror. The radiation pressure field and thermal noise both excite acoustic mode resonances of the mirror. When the injected TEM\textsubscript{01} mode is switched off the exponentially decaying beat signal demonstrates the radiation pressure excitation.

Because the cavity configuration is near flat-flat only the anti-Stokes process satisfies the resonance condition. The stored optical power in the cavity was about 1 kW. Spontaneous parametric instability cannot be achieved under these conditions. However by artificially increasing the transverse mode power we can mimic the conditions of instability and measure the excitation of the acoustic mode by observing its free ring down.

This mechanism would result in instability if it was a stokes interaction and the parametric gain $R_m$ was greater than unity \cite{60}. In our experiment the parametric gain $|R_m| << 1$. To mimic the high gain regime we artificially generate TEM\textsubscript{01} power. Our results are dominated by the inserted TEM\textsubscript{01} power. In this situation the parametric gain has little effect on the results, although in a higher power experiment the negative parametric gain associated with the anti-Stokes process would lead to suppression of the driving term.
4.3 Experimental Setup

The experimental layout is shown in Figure 4.2. The Nd:YAG laser with a wavelength of 1064 nm is frequency locked to the 77 meter suspended cavity. The laser beam first passes through a pre-mode cleaner (PMC) to create a pure TEM\(_{00}\) mode. The beam from the PMC is reflected by a piezo actuated mirror.

![Experimental Layout](image)

Figure 4.2: Experimental Layout, showing laser, pre-mode cleaner, mechanical excitation of TEM\(_{01}\) using a piezo driven mirror, the PDH locking loop, the optical cavity with end mirror radius of curvature tuned using CO\(_2\) laser heating and readout of the transmitted beam using differential degrees of freedom x and y from a quadrant photo detector (QPD).

The actuator is driven at a frequency matched to the mechanical resonant mode (shown as a surface deformation on the bottom left mirror in Figure 4.1) of the end test mass (ETM) mirror to generate TEM\(_{01}\) mode modulation side-bands. The TEM\(_{00}\) mode carrier with TEM\(_{01}\) mode side-bands are injected into the 80 meter suspended cavity. The CO\(_2\) laser power is adjusted to tune the RoC of the ETM until the cavity mode frequency gap between TEM\(_{01}\) and TEM\(_{00}\) matches the ETM acoustic mode frequency. This causes the TEM\(_{00}\) carrier and a single TEM\(_{01}\) side-band to resonate simultaneously inside the cavity.

The beat frequency between the TEM\(_{00}\) and TEM\(_{01}\) is precisely tuned to the resonant frequency of the acoustic mode of interest of the ETM, at 181.6 kHz. The transmitted beam is monitored using the quadrant photo-detector (QPD). When the TEM\(_{01}\) is being injected, the QPD mainly measures the amplitude of the transmitted TEM\(_{01}\) mode. After the driving is stopped, the QPD measures the TEM\(_{01}\) mode amplitude generated by the acoustic mode scattering, and therefore the amplitude of
the acoustic mode. The measured ring-down curve of the signal after the excitation is switched off proves the existence of the radiation pressure excitation of the acoustic mode.

4.4 Experimental Results

Let $E_{00}$ and $E_{01}$ represent the amplitude of the circulating TEM$_{00}$ mode and TEM$_{01}$ respectively, the radiation pressure force excitation drives the acoustic mode amplitude to $x_m$ that scatters a part of the TEM$_{00}$ mode into TEM$_{01}$ mode with amplitude of $E^S_{01}$. If the power transmission of the ETM is $T_c$, the QPD detection efficiency is $g$, then the output of the QPD at the acoustic mode frequency and the DC component are:

$$V_{drv} = 2g T_c |E_{00}(E_{01} + E^S_{01})|$$  \hspace{1cm} (4.1)

and

$$V_{DC} = g T_c |E_{00}|^2.$$  \hspace{1cm} (4.2)

Immediately after turning off the driving ($E_{01} = 0$), the output of the QPD at the acoustic mode frequency is

$$V_{rd} = 2g T_c |E_{00} E^S_{01}|.$$  \hspace{1cm} (4.3)

Here $E^S_{01}$ is proportional to the acoustic mode amplitude $x_m$ and should ring down with the acoustic mode dissipation rate. In the case of the current experiment, the parametric gain $R_m$ is much smaller than unity. Hence the TEM$_{01}$ created by scattering should be much smaller than the driving field, i.e: $E^S_{01} << E_{01}$. The parametric gain in the feedback loop model is the open loop gain. If we break the loop at the TEM$_{01}$ injection point and measure $E^S_{01}$ relative to $E_{01}$, then the parametric gain is given by

$$|R_m| = 2g T_c \left| \frac{E_{00} E^S_{01}}{E_{01}} \right| \approx \left| \frac{V_{rd}}{V_{drv}} \right|.$$  \hspace{1cm} (4.4)

By measuring the ring down time we obtain the Q-factor of the acoustic mode. Note that when there is no injection of the driving field there are two possible acoustic mode driving terms, the thermal excitation will drive the mode amplitude as it approaches its thermally excited amplitude and the parametric interaction (in this case anti-stokes process) could act to damp the ring-down signal. However for this experiment the low value of $R_m$ and the signal amplitude achieved being much greater than the thermal level mean that these effects are small.

Figure 4.3 shows the output of the QPD vertical differential quadrant readout. Initially, the injection of the driving TEM$_{01}$ mode field transmitted through the cavity,
beats with the TEM\textsubscript{00} mode field and creates a large amplitude output at 181.6 kHz. A spectrum analyzer is used to down-convert the signal frequency to 5 Hz for easy detection, with amplitude of 64 ± 0.5 mV.

In the top panel of Figure 4.3, the driving applied to the piezo was turned off at 0.109 seconds. The QPD measurement of the ring-down can then be seen when the vertical axis scale is expanded, in the bottom panel of Figure 4.3. In this panel the reference time (the moment the driving was turned off) has been set to zero. The measurement presented here could be affected by the cavity ring down time and by electromagnetic transients. For this reason the first 0.035 seconds that do not correlate well with an exponential ring down are not used in the analysis.

We can clearly see the acoustic mode ring-down curve after turning off the driving. The fitted curve shows a ring-down time of 0.5 ± 0.1 seconds, corresponding to a
mechanical Q-factor of $Q_m \approx (3 \pm 1) \times 10^5$. The ring-down amplitude at the time the driving was turned off was $V_{rd}(0) = 0.24 \pm 0.03 \text{mV}$.

From these results the parametric gain can be calculated to be $R_m = (3.8 \pm 0.5) \times 10^{-3}$ using Equation 4.4. Equation 3.3 can then used to determine the mass scaled overlap factor $2.7 \pm 0.4$. Here the effective mass ratio $\approx 13$ has been derived from ANSYS simulations. The parameters used in this analysis are listed in Table 4.1.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_m$</td>
<td>$(3 \pm 1) \times 10^5$</td>
<td>Q factor of the 181.6kHz mode</td>
</tr>
<tr>
<td>$Q_{01}^S$</td>
<td>$2 \times 10^{11}$</td>
<td>Q factor of the TEM$_{01}$ optical mode</td>
</tr>
<tr>
<td>$P$</td>
<td>1kW</td>
<td>Power contained in cavity</td>
</tr>
<tr>
<td>$M$</td>
<td>5.6 kg</td>
<td>Mass of the ETM mirror</td>
</tr>
<tr>
<td>$L$</td>
<td>77 m</td>
<td>Length of cavity</td>
</tr>
<tr>
<td>$\omega_m$</td>
<td>$2\pi \times 181.6 \times 10^3 \text{rad/s}$</td>
<td>Angular frequency of acoustic mode</td>
</tr>
</tbody>
</table>

The overlap factor derived experimentally is about 51% of the expected value considering perfect alignment of the optical and acoustic modes used in this experiment. In multiple experiments we observed that the ability to excite the acoustic mode was very sensitive to cavity alignment. This was expected because the overlap factor is strongly modulated by lateral displacement of the optical modes relative to the test mass acoustic modes as investigated by Heinert et al [146]. This also points to a simple method of reducing the parametric gain of a single mode near the threshold of instability, by finely adjusting the spot positions on the mirror test masses. This sensitivity in relation to the Advanced LIGO optical configuration is studied in the next chapter Section 5.4.

### 4.5 Summary

By injecting a high order mode simultaneously with the fundamental mode into an optical cavity, we have verified the radiation pressure driving term in the theory of three mode parametric instability. We have demonstrated a simple method for measuring the parametric gain and the overlap factor of three mode interactions at relatively low optical power and very low parametric gain. The technique can be used in long baseline interferometer gravitational wave detectors to identify and characterize potentially unstable acoustic modes in situ when the detector is operating at low optical power. This can allow the design of appropriate control schemes that
will mitigate instabilities that will occur when sensitivity demands require operation at high optical power.
Chapter 5

Characterization of Parametric Instability at LIGO

5.1 Introduction

The observation of parametric instability in the Advanced LIGO interferometers resulted in the need to experimentally verify the predictions of the parameters that govern parametric instability. This verification is required to assess the effectiveness of strategies for the suppression of parametric instability in Advanced LIGO and future high power gravitational wave detectors. In this chapter, the tools developed to measure these parameters are presented. A single cavity model is used to develop a method for optimal tuning of the optical mode spacing. This model is also used to make predictions for Advanced LIGO at design power. The predictions indicate a maximum parametric gain of 56 with 800 kW of laser power in the arm cavities assuming the worst case scenario. In the best case scenario, under the assumption that the cavity geometry can be maintained with appropriate thermal compensation, the maximum parametric gain still exceeds the threshold for instability by a factor of 2.4.

This chapter is an expansion of an article published in a special edition of Science China, Physics, Mechanics & Astronomy "The next detectors for gravitational wave astronomy" Section 5 titled "Three mode parametric instability and their control for advanced gravitational wave detectors" by Carl Blair.

This chapter is organized as follows: The background is presented in Section 5.1. In Section 5.2, the behavior of the acoustic modes in the interferometer test masses is explored. In Section 5.3, the theoretical and measured behavior of optical modes in advanced LIGO arm cavities is studied with the introduction of the single cavity model. Then in Section 5.4, the spatial overlap is considered, these results are
combined into a predicted parametric gain with experimental verification from all observed cases of parametric instability prior to Advanced LIGO Observation Run 1 in Section 5.5. Finally the results are summarized and prospects for the future are considered in Section 5.6.

5.1.1 Previous work

It takes extreme technology, like long baseline laser interferometers, to enter the regime where parametric instability can occur: 40 kg mirrors with ultra-low acoustic losses, 4 km long optical cavities, and very high optical power of the order of hundreds of kilowatts produce conditions susceptible to parametric instability at Advanced LIGO. Numerous experiments presented in Chapter 3 document improvements in our understanding of the physics of parametric instability and also highlight difficulties in experimentally achieving free running instability in large suspended optical cavities. In Fang Qi’s Thesis [115] the design and characterisation of a cavity designed for research and observation of parametric instability is presented. Many of the methods used by Fang et al [115] in conjunction with many of the methods presented in Chapter 3 are developed in this chapter for an analysis of the Advanced LIGO arm cavities.

In November 2014 parametric instability became a reality for operations of the Advanced LIGO detectors. This was shortly after the observation of parametric instability at the Gingin facility [272] that will be presented in Section 6.3. Instability was observed [113] at the Advanced LIGO Livingston facility operating in a commissioning phase prior to Observation Run 1. The instability appeared as an exponential ring-up of a resonant mode of a test mass at a frequency $\approx 15.5$ kHz. The optical power contained in the arm cavities was 5% the optical power required to reach Advanced LIGO’s full design sensitivity. The interferometer’s control systems became unstable when the amplitude of the acoustic mode exceeded its quiescent amplitude by $\sim 4.5 \pm 1$ orders of magnitude, very close to predictions made in by Ju et al [157] in 2009. Parametric gains up to 2.2 were observed with an operating stored optical power in the arm cavities of up to 100 kW.

Exact predictions for parametric instability have been difficult, predominantly due to the sensitivity of the higher order optical mode frequency to mirror radius of curvature, but also due to large uncertainties in parameters like the Q factors of the acoustic modes and difficulty in exact modelling of the test mass resonant modes [249] due to various uncertainties such as the elastic moduli of the test mass material. Previously, statistical approaches have been used to predict the severity of parametric instability by the metric of the number of unstable acoustic modes and the
maximum parametric gain [136] or probability of instability as presented in Chapter 3 Section 3.2.9. To date these statistical approaches have been accurate to the order of magnitude level.

To some extent parametric instability control is hampered by lack of knowledge of which acoustic and optical modes are likely to be unstable, since this depends very strongly on unpredictable details such as thermal deformations and alignment variations. It would be very useful to have a means of diagnosing and predicting parametric instability before it occurs such as the low power parametric gain estimates presented in Chapter 4. Such low power estimates at LIGO Livingston prior to Observation Run 1 would have required a very accurate knowledge of the cavity and test mass parameters.

In this chapter measurements from the LIGO Livingston Y arm are used to improve the parametric instability model such that predictions of future parametric instabilities will be more certain. The improved model is compared with observations of parametric instability at LIGO Livingston in 2015.

### 5.2 Acoustic Modes of the Test Masses

The frequency, quality factor and surface deformation of acoustic modes are key parameters in estimating the strength of three mode interactions. Additionally there are dynamic effects due to the test mass heating and cooling that result in variations in mode frequencies. In this section a method for identifying test mass resonant modes is presented and compared with a finite element model. By fine tuning the material properties, the model is improved to match the observed resonant frequencies. The correlation between acoustic mode frequency and temperature is introduced and this property is then used as a method for identifying with which test mass an observed resonance is associated. Finally various methods for measuring the quality factor of the elastic modes are compared.

#### 5.2.1 Acoustic Mode Frequency and Quality Factor

In this subsection the behaviour of test mass eigenfrequencies is examined with reference to analytic solutions for eigenmodes of cylindrical bodies. The relation between Q factor and loss angle is established and issues that could affect measurement of the true test mass acoustic mode Q factor are also discussed.
5.2.1.1 Test mass acoustic mode frequency and temperature

As mentioned in section 2.2.1, the eigenfrequencies of a simple cylindrical body are a function of the cylinder radius to length ratio $\beta$, as well as the density $\rho$, the Young’s modulus $E$ and the Poisson ratio $\sigma$ of the material, $\omega_m = \beta \sqrt{E/\rho/(1+\sigma)}$ (Equation 2.3 on page 60).

It is apparent that eigenfrequencies will change with any change in the test mass shape or elastic constants. As both of these are dependent on temperature we expect eigenfrequencies to be temperature dependent.

For Advanced LIGO and Virgo detectors, the chosen material of the mirror test masses is fused silica. At LIGO it is Heraeus SURPRASIL 3001 [148]. The material specification states the Young’s modulus $E = 70$ GPa, Poisson ratio $\sigma = 0.17$ [148] and thermal expansion coefficient $5.1 \times 10^{-7}$ m/K.

The Young’s modulus of fused silica has an interesting property. Rather than continually decreasing with temperature it has an inversion from $-200$ C° to 1000 C°. Around 17 C° Spinner [231, 232] reported fused silica $\partial E/\partial T = 10.2$ GPa/K and $\partial \sigma/\partial T = 5.5 \times 10^{-5}$/K. Measurements of the suspension fused silica fibers at LIGO yielded $\partial E/\partial T = 12.2 \pm 0.4$ GPa/K [36].

Using Equation 2.3 and the above parameters, we can estimate the fractional frequency change $\frac{\partial \omega}{\omega \partial T}$ due to a uniform change in test mass temperature $\partial T$. The fractional change in frequency due to Young’s modulus is proportional to half the fractional change in Young’s modulus with temperature $\frac{\partial \omega}{\omega \partial T} = \frac{\partial E}{2E \partial T} = 7.7 \times 10^{-5}$. The Poisson ratio contribution to the fraction frequency change is $-\frac{1}{2(1+\sigma)} \frac{\partial \sigma}{\partial T}$, $\approx -3 \times 10^{-5}$. The thermal expansion of the test mass contributes in two ways. (1) Through the change in density $-\frac{\partial \rho}{\rho \partial T} \approx -10^{-7}$. (2) Through the change in the linear dimension through the parameter $\beta - \frac{\partial \beta}{\beta \partial T} \approx 10^{-7}$.

From these results we expect the change in elastic moduli with a uniform change in temperature to dominate by two orders of magnitude over the effect of thermal expansion. Consequently in this chapter thermal expansion is often ignored when calculating thermal dependence of eigenfrequencies.

If the temperature change is not uniform across the test mass, the situation is more complex. The effect of non-uniform temperature distributions on test mass eigenfrequencies will be explore in Chapter 7 Section 7.4.
5.2.1.2 Test mass resonant mode quality factors

The quality factors \( Q \) of acoustic modes are determined by the dissipation of the material or connected objects. Many efforts have been made to model the dissipation in the test masses \([213, 135, 115]\). Generally this can be done by summing the loss angle induced by the various loss mechanisms of the system.

\[
Q_m = \frac{1}{\phi_{\text{total}}}, \quad \phi_{\text{total}} = \phi'_{\text{substrate}} + \phi'_{\text{coating}} + \phi'_{\text{geometry}} + \phi'_{\text{suspension}}.
\]  

(5.1)

Here \( \phi_{\text{total}} \) is the loss angle of the system and \( \phi'_{\text{substrate}}, \phi'_{\text{coating}}, \phi'_{\text{geometry}}, \phi'_{\text{suspension}} \) are the loss angles induced in the system by the substrate, coatings, geometry (including surface finish) and the method of suspension. Considering only fundamental substrate losses we would expect a Q factor of \( \sim 10^8 \) \([213, 83]\). Measurements [113] of LIGO test masses presented in Section 5.2.4 have found Q factors of \( \sim 10^7 \).

The Q factor of the resonant mode could be determined by either the exponential decay time constant \( \tau_m \) of the mode amplitude or the linewidth \( \gamma_m \) of the mode frequency spectrum peak.

To measure the exponential decay time the mode must be excited above the noise level. Assuming that the mode amplitude is excited to a certain level, the mode amplitude should follow the exponential decay \( Ae^{t/\tau_m}e^{-i\omega_m t} \) after the excitation ceases, with \( A \) the mode amplitude at an initial time \( t = 0 \), \( \omega_m \) the radial frequency of the mode. If the measurement is contaminated by additive noise like thermal noise a constant term \( C \) may be added \((ae^{t/\tau_m} + C)e^{-i\omega_m t}\) to represent the noise.

To measure the mode linewidth, a spectrum of the mode amplitude must be recorded and the full width at half maximum (FWHM) of the observed resonant peak \( \gamma_m \) provides the linewidth estimate. The resonant peak must be well resolved that the FWHM \( \gamma_m >> f_s/N = 1/t_w \), where \( f_s \) is the sample frequency, \( N \) is number of samples used, \( t_w \) is the measurement time. Caution must be taken in such measurements because linewidth measurements can be corrupted by any change in mode frequency over the duration of the measurement.

With one of these measurements the Q factor can be estimate,

\[
Q_m = \frac{\omega_m}{\gamma_m} = \frac{\tau_m \omega_m}{2},
\]  

(5.2)

where \( Q_m \) is the quality factor of mode \( m \), \( \omega_m/2\pi \) is its frequency, \( \gamma_m \) is the FWHM linewidth and \( \tau_m \) is exponential decay time constant.

One complication in measuring Q factors in situ at LIGO is that all methods for measuring the Q factor rely on there being optical power in the cavity. The measured
Q is the effective Q factor, which changes with parametric gain given by the following equation
\[ Q_{\text{eff}} = Q_m \frac{1}{1 - R_m}. \]  
(5.3)

Here \( Q_{\text{eff}} \) is the effective Q factor of a mode with an intrinsic Q factor of \( Q_m \) influenced by a three mode interaction with parametric gain \( R_m \) (described in Section 3.2.1).

This makes it difficult to determine the intrinsic Q factor. For example, consider a mode that is parametrically unstable (\( R_m > 1 \)) with 100 kW of optical power in the arm cavity. If a ringdown of this mode is measured with 10 kW of optical power in the arm cavity (\( R_m > 0.1 \)), the effective Q factor \( Q_{\text{eff}} \) will be more than 11% larger than the intrinsic Q factor \( Q_m \). If the gain \( R_m \) approaches unity a measurement of the effective Q factor \( Q_{\text{eff}} \) will deviate dramatically from the intrinsic \( Q_m \).

To solve this problem the effective Q factor can be measured several times with different optical power. As the parametric gain \( R_m \) is proportional to power, Equation 5.3 can be used to extrapolate to zero power and hence determine the intrinsic Q factor \( Q_m \).

This is equivalent to a method presented by Evans [113] where the time constant of the first observed parametrically unstable modes at Advanced LIGO were measured at two different operating powers, one below threshold power and one above threshold power then \( Q_m \) was interpolated. Equation 5.3 still applies with \( R_m > 1 \). But the meaningful parameter would be the effective time constant \( \tau_{\text{eff}} \)
\[ \tau_{\text{eff}} = \frac{2Q_m}{\omega_m(1 - R_m)}. \]  
(5.4)

### 5.2.2 Measurement and Methods for Acoustic Mode Identification

In this section we study how the theory of resonant modes presented in Section 5.2.1 can be applied to measure material properties of the test mass. We study how measurements of the test mass eigenmodes can be made, and we examine how other useful information can be obtained from these measurements.

#### 5.2.2.1 Measurements of acoustic modes using Advanced LIGO

It has been experimentally observed that acoustic modes of test masses can be monitored in various outputs of the interferometer. In normal conditions the acoustic modes of the test masses, assumed to be at the thermal noise level, are rather quiet.
However, with three mode parametric interaction, some of the modes will be excited ($R_m > 1$) or even in stable conditions the modes signal can be amplified. The three mode interaction mechanism for sensing acoustic modes is described below, while other possible coupling mechanisms are discussed in Chapter 8 Section 8.1. Figure 5.1 is a schematic showing the sensors where modes are visible with a reasonable signal to noise ratio in parametrically stable configurations.

![Figure 5.1: Interferometer sensors that are sensitive to resonant modes in the four test masses. The most important sensors are the transmission quadrant photo-detectors (QPD) of the end test masses ETMX and ETMY followed by the photo-detectors located after the output mode cleaner (OMCPDs), the anti-symmetric port (AS) QPDs and OMC reflection (REFL) QPDs. Other elements shown in this diagram are; the input test masses (ITMX and ITMY), power recycling mirror (PRM), signal recycling mirror (SRM) and laser beams in red.](image)

When the test mass resonant modes have large amplitudes (due, for example, to parametric instability) the signals couple into many photodiodes within the interferometer. Most notably when the mode amplitudes are larger that a factor of $\sim 10^3$ above quiescent levels, they are down-converted through aliasing\(^1\) into the detection

\(^1\)Aliasing is a down-conversion in frequency that arises when a signal is sampled at less than twice the signal frequency.
band of the main interferometer gravitational wave sensing channel. This is where parametric instability was first detected [113].

The aliased signal in the main interferometer output is significantly attenuated by anti-aliasing 10 kHz low pass filters. The baseband signal has a quite remarkable signal to noise ratio between 10 and $10^3$ as can be seen in Figure 5.2. This figure shows a particular snapshot in time of several different signals which may be used to measure the test mass resonant modes. These signals include the OMCPDs, with the highest signals to noise ratio, quadrants of the arm transmission QPDs and OMC AS QPDs.

Figure 5.2: Amplitude spectral density plot for comparison of signal to noise ratio of different sensors of the 15540Hz group of modes at LIGO Livingston. The output mode cleaner photodetector signal (black trace) resolves all four modes at 15516, 15525, 15536 and 15539Hz. These peaks have modulation side-bands presumably from test mass motion. Some output mode cleaner alignment quadrant photodetector (QPD) quadrants are displayed in bright colours. Some arm transmission QPD quadrants signals are displayed in dark colours. In these only the 15539Hz mode is resolved.

Throughout the chapter and later in this thesis we will see that signals on some photodiodes are not always visible. Sometimes they appear and disappear from a spectrum in minutes to hours. The strength of some coupling mechanisms depends on the optical gain of high order transverse modes in the cavity.

To help to understand this relation the following example is introduced. Imagine an acoustic mode is exciting the first order transverse electromagnetic mode. Let’s assume this mode approximates a first order Hermite Gaussian mode ($HG_{01}$). A combination of the $HG_{01}$ and fundamental mode are measured by a QPD in transmission

---

2A baseband signal is the original frequency range of a signal before it is converted, modulated or aliased into a different frequency range
of the arm cavity. In this case quadrants of transmission QPDs can be combined to be effective at sensing the beat signal of the fundamental and $HG_{01}$ modes as a beat signal at the original mechanical mode frequency $\omega_m$. The beat signal appears like a fluctuation in beam position on the QPD as described in Chapter 2 Section 2.3.3. The signal amplitude observed on the QPD will be dependent on the quantity of $HG_{01}$ in the cavity. The quantity of $HG_{01}$ in the cavity is dependent on the acoustic mode amplitude and the degree to which the $HG_{01}$ is resonant - the optical gain of the $HG_{01}$ mode at this frequency. This is very closely related to the optical gain term in parametric instability relation. While the parametric gain is proportional to $(E_{00} \ast E_{01})^2$ the optical readout gain term is proportional to $(E_{00} \ast E_{01})$.

Sensing via the high order transverse optical mode may not be common to all photodiodes. In Chapter 8 Section 8.1 possible coupling scenarios are compared, in one scenario modes couple directly to a length degree of freedom, ie the optical sideband produced from the mechanical excitation has a fundamental mode shape. In this case the mode amplitude and signal to noise ratio will not be dependent on optical gain of the high order transverse optical mode. If this is the case then the ratio of the amplitude measured on two different sensors may change.

Another useful feature of the acoustic modes measured through the interferometer that can be identified in Figure 5.2 is that there are 4 modes. One mode for each test mass. These four modes are all the same eigenmode. Eigenfrequencies are generally separated by up to 0.25% of their resonant frequency, that is 40 Hz at 15.5 kHz.

In this section a lot of analysis is done just to determine to which test mass a particular mode belongs. One might ask why not use a sensor that can measure each test masses resonant mode individually, like the optical levers. The answer is that sensors like the optical levers are orders of magnitude too insensitive to measure acoustic modes at quiescent levels.

On account of their large signal to noise ratio the OMCPDs are chosen for the majority of investigations described in Sections 5.2.3. However the coupling mechanism is not well understood, this will be discussed in Chapter 8 Section 8.1. The mechanism producing the signal on the arm transmission QPDs is well understood as described previously. As such these signals are used to gain information in two ways. (1) The amplitude of the optical field in a particular arm cavity can be ascertained. (2) As introduced in Chapter 2 Section 2.3.5 the spatial information about the optical field can be ascertained from the phase of the signals on individual quadrants of the QPDs as shown in Figure 5.10 on page 143.
5.2.2.2 Frequency correlation for identification of acoustic modes

The information of which mode belongs to which test mass is of importance to parametric instability control. We can use the following two facts to identify a modes test mass. 1) acoustic modes’ frequencies change with temperature as described in Section 5.2.1.1. 2) test masses are spatially separated such that the changes in temperature of different test mass are not correlated. To identify acoustic modes that belong to the same test mass we must just find correlated fractional frequency shift $\frac{\partial \omega_m}{\omega_m \partial T}$.

Temperature fluctuations arising from normal interferometer operation induce small changes in frequency. This makes it hard to identify frequency changes due to ambient changes in temperature, and hence it is hard to correlate these changes and identify modes belonging to the same test mass. A method is devised whereby the ring heater on each test mass is used to induce larger temperature fluctuations. These changes in temperature and associated changes in eigenfrequency are used to identify to which test mass an observed resonant mode belongs.

![Acoustic Mode frequency against time](image)

Figure 5.3: An example measurement of the response of various acoustic mode frequencies (measured in transmission of the arm cavities through the three mode interaction) to steps in ring heater power. The ETM ring heaters were changed three times, up time=0, down at $\sim 3000$ seconds and up again at $\sim 11000$ seconds. The modes can be seen to fall into two groups those that change frequency with changes in ring heater power and those that change frequency solely due to heating from absorption of the contained laser power in the test mass coatings.
Figure 5.3 shows an example where this method was first used to identify modes by test mass. The power of both ETM ring heaters was changed three times: up at the beginning, down at \( \sim 3000 \) seconds and up again at \( \sim 11000 \) seconds. The relative change in frequency of the five observed acoustic modes vary in two groups. Figure 5.3 shows modes 15004 Hz and 15538 Hz are affected by the ring heater power change. Modes 14980 Hz, 15058 Hz and 15527 Hz are unaffected by the ring heater, only changing due to heating from the coating absorption of the main cavity laser and changes in ambient temperature. Therefore modes 15004 Hz and 15538 Hz belong to the ETMs, while the others belong to the ITMs.

In this case the measurements were taken from the Y arm transmission QPD signals. Not all modes are measurable at all times due to the marginal and varying signal to noise ratio. A consistent method for identifying acoustic modes should therefore use the OMCPCs as the signal to noise ratio is better and many more modes may be resolved. By changing the power applied by the ring heater of each test mass sequentially a mode’s test mass can be identified by the time when the mode frequency started to change. The correlation between test masses then can be used to verify marginal measurements.

The changes in frequency induced by the ringheater are small so many samples are required to resolve them \( \Delta f_{\text{meas}} \approx 2t_w/N \), where \( \Delta f_{\text{meas}} \) is the minimum resolvable change in frequency, \( t_w \) is the sample period and \( N \) the number of samples. The ring heater has a slow effect on the test mass eigenfrequencies \( \Delta f_{RH}/f_m = \zeta \Delta P_{RH} \Delta t_{RH} \), where \( \zeta \) is the ring heater tuning rate coefficient measured to be \( \approx 6 \times 10^{-6} \) Hz/(Hz.W.hours), \( \Delta f_{RH}/f_m \) is the relative change in frequency of mode \( m \), \( \Delta P_{RH} \) is the change in ring heater power and \( \Delta t_{RH} \) is the time between subsequent frequency measurements. There is therefore a relation between the change in ring heater power used and the time between sequential changes in ring heater power:

\[
\zeta \Delta P_{RH} \Delta t_{RH} f_m > \frac{t_w}{N} \quad (5.5)
\]

A 1 W change in ring heater power with half hour intervals between successive test masses change in ring heater power easily resolves different test masses.

This method is used in Section 5.2.3 to identify a large subset of the modes between 5 kHz and 30 kHz.
5.2.2.3 Acoustic mode relative amplitude

Three mode interactions in a dual recycled interferometer are extremely complex. It is useful for the experimentalist to develop simplified conceptual models for understanding these interactions. Useful information can be obtained by looking at the relative amplitude of transverse electromagnetic mode (TEM) in various parts of the interferometer. Here a simplified conceptual model is used to explain how the relative amplitude of an observed TEM\(_{pq}\) can be used to indicate the location of the test mass eigenmode that is generating the TEM\(_{pq}\).

This study was motivated by the observation that a test mass eigenmode was always observed to have a larger signal in one arm transmission QPD relative to the other arm transmission QPD. While other eigenmodes displayed the opposite relative amplitudes relation. An example of differences in relative amplitude will be seen in Figure 5.10 on page 143.

Consider a test mass eigenmode resonant in a Y arm end test mass (ETMY) as in Figure 5.4(a). The test mass eigenmode scatters light \(E_{\text{scat}}\) into the TEM\(_{pq}\) that is resonant in the Y arm optical cavity. We would like to know the proportion of the TEM\(_{pq}\) field exiting the interferometer from ETMX relative to ETMY.

First we consider the case where the arm cavities are symmetric, both resonant to TEM\(_{pq}\). A single cavity approximation of the recycling cavities can be used. In Figure 5.4(b) the rare case of both recycling cavities being anti resonant to TEM\(_{pq}\) is shown, in this case the cancellation of the field in the recycling cavities results in almost no transmission of the TEM\(_{pq}\) field to the X arm.

In Figure 5.4(c) the case where one recycling cavity is on resonance for the TEM\(_{pq}\) field and the other cavity is anti-resonant, the round trip loss of the recycling cavity is 75% resulting in half the TEM\(_{pq}\) leaving the X arm when compared to the field leaving the Y arm. A similar argument can be made for all recycling cavity detuning. The expectation being that the ratio of the field leaving ETMX to the field leaving ETMY be between zero and one.

The final scenario in Figure 5.4(d) depicts the case where both recycling cavities are resonant for the TEM\(_{pq}\) mode. In this rare case the single mirror approximation of the recycling cavities is a 100% transmissive mirror. The power in the the X arm and Y arm will be the same.

This simplified analysis leads to the conclusion that for symmetric arm cavities we would expect the signal amplitude to be biggest in transmission of the arm with the resonant test mass. The proportion by which they differ is related to the resonant condition of the TEM\(_{pq}\) in the recycling cavities.
Figure 5.4: Model for estimating the relative high order optical mode (TEM$_{pq}$) amplitude in either arm cavity. (a) is the interferometer configuration. (b) is the model where both recycling cavities are anti resonant to TEM$_{pq}$, the single mirror approximation is an infinitely reflective mirror replacing the combination of input mirrors I1 and I2. (c) models one recycling cavity resonant for TEM$_{pq}$ and one anti-resonant, the single mirror equivalent is 100% transmission + 50% loss from the beam splitter. While (d) is the model where both recycling cavities are on resonance, in this case the single mirror equivalent is 100% transmissive.
If we consider the case where there is asymmetry between the arm cavities, there is a condition where this rule of thumb can be violated. If the TEM\(_{pq}\) field is far from resonance in the Y arm cavity where it is generated and the recycling cavities are close to resonance and the X arm cavity is resonant to the TEM\(_{pq}\) then it is possible for the field exiting the X arm to be larger than the field exiting the Y arm.

5.2.2.4 Finite element model for test mass resonant modes

There are several limitations in using an analytic description of acoustic modes based on Equation 2.3, such as the inability to create an analytic model with complex features such as the flat sections and ears on the mirrors depicted in Figure 5.5 that provide suspension points. These features break rotational symmetry introducing new non-degenerate degrees of freedom. A simple cylindrical model therefore underestimates the number of modes. In this section finite element modelling (FEM) will be used to estimate the test mass eigenfrequencies, see Strigin [234] for a comparison between FEM and the analytic expression of Equation 2.3. More importantly, the finite element analysis can assist in identifying the mode shapes of the resonant peaks. This is important in the estimation of the three mode parametric interaction gain of the system.

A COMSOL [77] model was built and compared to measurements of the LIGO Livingston test masses. The COMSOL model is based on the LIGO Livingston end test mass Y (ETMY). The initial parameters for the model are either those measured prior to installation [4] or they are values from specifications [148, 43, 187]. These are summarised in Table 5.1 and Figure 5.5.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter</td>
<td>340.13mm</td>
<td>Diameter as measured by LIGO</td>
</tr>
<tr>
<td>Depth</td>
<td>199.59mm</td>
<td>Depth as measured at LIGO</td>
</tr>
<tr>
<td>Mass</td>
<td>39564g</td>
<td>Mass as measured at LIGO (density 2203kg/m(^3))</td>
</tr>
<tr>
<td>Wedge</td>
<td>0.07deg</td>
<td>Optic wedge as measured at LIGO</td>
</tr>
<tr>
<td>E</td>
<td>70GPa</td>
<td>Young’s modulus as specified by manufacturer</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.17</td>
<td>Poisson as specified by manufacturer</td>
</tr>
</tbody>
</table>
In a preliminary COMSOL eigenfrequency analysis, the elastic moduli were roughly adjusted to produce a model that attained agreement to modes that appeared in groups of four resonant peaks, such as those depicted in Figure 5.12 on page 146. As mentioned before, each resonant peak belongs to one of the four test masses, while the four peaks have the same eigenmode shape. The elastic moduli in the COMSOL model may be refined in this manner, but only to the point that the residual between measurement and simulation frequency approaches the spread in the 4 measured eigenfrequencies (from the four test masses) for a particular eigenmode, this is $\sim 0.25\%$ of the mode frequency as previously mentioned.

The estimates of the elastic moduli are then further optimised. The frequency shift method described in Section 5.2.2.2 is used to determine a set of resonant modes that belong to a single test mass. With this set of measured eigenfrequencies belonging to a single test mass, the elastic moduli are optimised to the resolution of the simulation. Assuming experimental parameters such as temperature distribution are properly taken into account.

In the Livingston study only the geometry of the particular test mass was used in the COMSOL model, while in the later studies at Hanford the simulated temperature distribution from the ring heater and absorbed cavity power is also applied in the COMSOL simulation, the improvement is marginal.

5.2.3 Experimental Identification of Test Mass Resonant Modes

In this section the mode identification results are presented. The optimised elastic moduli are presented and the improvement in the simulated eigenfrequencies estimates
with the optimised elastic moduli is shown. The parametrically unstable modes are identified by mode shape and test mass. Then for verification, the relative amplitude of the signals measured at the arm transmission ports are compared and the phases on the QPD quadrants are investigated.

5.2.3.1 Mode frequency shift due to thermal load

Using the ring heater on the test masses, mode frequency shift was measured. Measurements of the OMC transmission were taken over several hours. Two examples of measurements of mode frequency shift are given in Figures 5.6 and 5.7. In Figure 5.6 only ETMX and ETMY ring heaters’ power was changed for the LIGO Livingston investigation. The changes in thermal load resulted in changes in eigenfrequencies clearly identifying the ETMX and ETMY modes. The results of the Hanford investigation shown in Figure 5.7 demonstrate the complete method in one graph. The ETM ring heaters were reduced in power by 1 W and ITM ring heaters were increased in power by 1 W to maintain similar cavity geometry. A half hour interval between each change in power easily resolved the different test masses.

![Figure 5.6: Example measurement of mode frequency shift for test mass identification of Livingston end test mass (ETM) modes. The ETMX ring heater power was increased at zero hours, then the ETMY ring heater power was reduced at 2 hours. Two mode groups are compared, solid lines represent the 6.05 kHz mode group while points represent the 15.53 kHz mode group that is used as a reference measurement. In both groups the ETMX and ETMY modes are clearly identified.](image)
Figure 5.7: Example measurement of the complete method for test mass identification at Hanford using the frequency shift method. Four modes from the 9.87kHz group are plotted with the reference 15.2kHz group. The sequential changes in ring heater power can be clearly seen identifying each mode, the order that the ring heater power was changed was ETMX, ETMY, ITMY then ITMX.

These figures only identify modes from one mode group. They are examples from the entire investigation. The full Livingston result is attached as Appendix B. It demonstrates the confident identification of 26 modes between 5kHz and 18kHz and the identification of several other modes by test mass but not mode shape.

5.2.3.2 Mode identification through finite element modelling

A COMSOL model of the Livingston ETMY is built with dimensions measured prior to installation. By adjusting the rather poorly known Young’s modulus and Poisson ratio, the simulated mode frequencies can be tuned. Through an iterative approach the residual between measurement and simulation was reduced. The values obtained for the elastic moduli are $E = 72.7$ and $\sigma = 0.164$. The final model parameters are given in Table 5.2.
Table 5.2: Parameters for the tuned COMSOL model

<table>
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</tr>
<tr>
<td>Wedge</td>
<td>0.07deg</td>
<td>Optic wedge as measured at LIGO</td>
</tr>
<tr>
<td>E</td>
<td>72.7GPa</td>
<td>Young’s modulus from fitting observed mode frequencies (17°C)</td>
</tr>
<tr>
<td>σ</td>
<td>0.164</td>
<td>Poisson ratio from fitting observed mode frequencies (17°C)</td>
</tr>
<tr>
<td>$\frac{\partial E}{\partial T}$</td>
<td>11.5MPa/°C</td>
<td>Change in Young’s modulus with Temperature from Spinner</td>
</tr>
<tr>
<td>$\frac{\partial \sigma}{\partial T}$</td>
<td>$5.5 \times 10^{-5}$ /°C</td>
<td>Change in Poisson ratio with Temperature from Spinner</td>
</tr>
</tbody>
</table>
The improvement in mode frequency estimate from the optimised model is shown in Figure 5.8 and Table 5.3. The mesh for the optimisation was chosen such that further increases in mesh density produced less than 4 Hz improvement in acoustic mode frequency estimates. The optimization was performed until the root mean squared (rms) error in the mode frequency estimate reached 4 Hz. The Livingston mode identification investigation is described in detail in Appendix B, where observed resonant modes are identified by the test mass they belong to and a complete list of the modes identified is compiled.

Figure 5.8: Improvement in the estimated frequency from material property optimisation
Table 5.3: Improvement in mode frequency estimate after tuning the Young’s modulus and Poisson ratio

<table>
<thead>
<tr>
<th>#</th>
<th>Mode</th>
<th>Measured $f$ (Hz)</th>
<th>Simulation Specification (Hz)</th>
<th>Simulation Tuned (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Butterfly</td>
<td>5945</td>
<td>5815</td>
<td>5941</td>
</tr>
<tr>
<td>2</td>
<td>Butterfly</td>
<td>6049</td>
<td>5919</td>
<td>6047</td>
</tr>
<tr>
<td>3</td>
<td>Drumhead</td>
<td>8153</td>
<td>8014</td>
<td>8153</td>
</tr>
<tr>
<td>4</td>
<td>Cylinder</td>
<td>8313</td>
<td>8135</td>
<td>8312</td>
</tr>
<tr>
<td>5</td>
<td>Cylinder</td>
<td>9100</td>
<td>8918</td>
<td>9098</td>
</tr>
<tr>
<td>6</td>
<td>Cylinder</td>
<td>9334</td>
<td>9150</td>
<td>9335</td>
</tr>
<tr>
<td>7</td>
<td>Drumhead</td>
<td>9833</td>
<td>9634</td>
<td>9834</td>
</tr>
<tr>
<td>8</td>
<td>Drumhead O2v</td>
<td>9885</td>
<td>9685</td>
<td>9886</td>
</tr>
<tr>
<td>9</td>
<td>Drumhead O2h</td>
<td>10216</td>
<td>9989</td>
<td>10206</td>
</tr>
<tr>
<td>10</td>
<td>Butterfly O1.5</td>
<td>10425</td>
<td>10204</td>
<td>10425</td>
</tr>
<tr>
<td>11</td>
<td>Butterfly</td>
<td>12638</td>
<td>12374</td>
<td>12637</td>
</tr>
<tr>
<td>12</td>
<td>Drumhead</td>
<td>13010</td>
<td>12786</td>
<td>13012</td>
</tr>
<tr>
<td>13</td>
<td>Drumhead O2</td>
<td>13074</td>
<td>12830</td>
<td>13069</td>
</tr>
<tr>
<td>14</td>
<td>Butterfly</td>
<td>13206</td>
<td>12937</td>
<td>13206</td>
</tr>
<tr>
<td>15</td>
<td>Butterfly</td>
<td>13272</td>
<td>13004</td>
<td>13274</td>
</tr>
<tr>
<td>16</td>
<td>Butterfly O2</td>
<td>14468</td>
<td>14152</td>
<td>14461</td>
</tr>
<tr>
<td>17</td>
<td>Drumhead O2v</td>
<td>15014</td>
<td>14719</td>
<td>15018</td>
</tr>
<tr>
<td>18</td>
<td>Drumhead O2h</td>
<td>15081</td>
<td>14784</td>
<td>15085</td>
</tr>
<tr>
<td>19</td>
<td>Drumhead</td>
<td>15220</td>
<td>14919</td>
<td>15222</td>
</tr>
<tr>
<td>20</td>
<td>Drumhead O2v</td>
<td>15538</td>
<td>15224</td>
<td>15537</td>
</tr>
<tr>
<td>21</td>
<td>Drumhead O2h</td>
<td>15624</td>
<td>15309</td>
<td>15622</td>
</tr>
<tr>
<td>22</td>
<td>Butterfly O2</td>
<td>16272</td>
<td>15921</td>
<td>16269</td>
</tr>
<tr>
<td>23</td>
<td>Butterfly O2</td>
<td>16917</td>
<td>16565</td>
<td>16911</td>
</tr>
<tr>
<td>24</td>
<td>Butterfly O2</td>
<td>17026</td>
<td>16683</td>
<td>17028</td>
</tr>
<tr>
<td>25</td>
<td>Butterfly O2.5</td>
<td>17907</td>
<td>17558</td>
<td>17914</td>
</tr>
</tbody>
</table>
By referring to Table 5.3 and to the results in Appendix B we can say that
the two modes that became parametrically unstable at Livingston in 2015, prior to
Observation Run 1 are those depicted in Figure 5.9.

Figure 5.9: The simulated (COMSOL) surface deformation perpendicular to the sur-
face of the two acoustic modes responsible for parametric instability in Advanced
LIGO. red colour indicates a deformation out of the page while blue colours represent
a deformation into the page (a) is the 15004 Hz acoustic mode and (b) is the 15538 Hz
acoustic mode, insets are total surface deformation for the 3D volume

These instabilities were an ETMX mode of the form in Figure 5.9a at 15004 Hz and
an ETMY mode of the form in figure 5.9b at 15538 Hz. We will see in the following
section that they both interact with a third order optical mode that resembles $HG_{03}$
or $LG_{11}$ (oriented vertically).

5.2.3.3 Quadrant photodetector amplitude and phase

The experimental mode identification is confirmed in two ways:

1.) By using the relative amplitudes of the arm transmission signals shown in
Figure 5.10(a). In this figure the ETMY resonance at 15538 Hz is compared in the
pitch degree of freedom on 4 arm transmission QPDs. Both ETMY QPD signals
are larger by between two and five times the ETMX signals. This indicates that the
signal comes from the Y arm.
Figure 5.10: (a) Relative amplitude of the arm transmission signals for the Y arm 15538 Hz mode. (b) Relative phase of 4 quadrants of Y arm transmission for the 15538 Hz mode and X arm transmission for the 15004 Hz mode.

2.) By using the observed phase of the signals on the individual quadrants of the arm transmission photo-detectors. Figure 5.10(b) shows the QPD phases measured with respect to the OMCPD phase for the 15538 Hz mode and the 15004 Hz mode. To explain how this is useful in mode identification we need three pieces of information from the analysis of the optical mode.

i) We need to know what optical mode shapes are likely to produce the QPD
phases measurements in Figure 5.10(b).

ii) We can restrict analysis to the optical mode order (frequency) that would allow interaction with with acoustic modes at $\approx 15\text{kHz}$.

iii) We need to know what acoustic mode shapes will interact, or have a high overlap factor, with the optical mode identified.

We will see in the subsequent sections that that optical modes at 15kHz must be dominated by third order optical modes.

By comparing the quadrant phases to third order optical modes in Figure 5.18 on page 156 or Chapter 2 Figure 2.12 on page 73 it can be established that the optical mode shape most similar to the acoustic mode shape are the $HG_{30}$ or $LG_{21}$ modes.

A visual assessment of these optical mode shapes and the acoustic mode shapes shown in Figure 5.9 on page 142 indicate that the the overlap factor will likely be high for this mode. Particularly it rules out high overlap factors for neighbouring acoustic modes that do not have vertical orientation in surface deformation.

These methods indicate mode shape and the arm within which the instability arises that are consistent with the identification made with the the acoustic mode frequency shift method in Section 5.2.3.2.

5.2.4 Quality Factor Measurements

In Section 5.2.1 the expected Q factor of the Advanced LIGO test masses and methods for measuring the Q factor were described. In this section the application of three methods are considered 1.) Ringdown measurement (following some excitation). 2.) Estimate by extrapolation from a series of optical power and time constant measurements. 3.) Estimate from a linewidth measurement of the observed resonance.

5.2.4.1 Ringdown method

As described in Section 5.2.1.2 the the intrinsic mode Q factor measurement may not be straight forward at Advanced LIGO as the three mode interaction modifies the effective Q factor of the acoustic modes.

For modes that have large parametric Again, the three mode interactions are always strong and the effect on the measured effective Q factor ($Q_{eff}$) is large. The relation between effective Q factor and power is described by Equation 5.3 where the parametric gain $R_m$ is proportional to power. A series of measurements of the effective Q factor can be taken at different optical power such that a fit to the data may be used to estimate the natural Q factor by extrapolating the an optical power of zero.
Measuring the Q factor with very low cavity optical power using electrostatic drive as a means of controlled excitation would avoid the complication of influence of the parametric gain. However, the electrostatic drive control channels for parametric instability were installed weeks before the LIGO observation run 1 and experiment time could not be dedicated to measuring the Q of acoustic modes. The only means of excitation was parametric instability itself. Using parametric instability to measure the Q factor is the subject of the next subsection.

The technique of electrostatic excitation and ringdown measurement was later used extensively at Hanford in 2016. Some results are presented in Chapter 8.

### 5.2.4.2 Quality factor from parametric gain optical power dependence

Several measurements of the time constant $\tau_{\text{eff}}$ of a parametrically unstable mode with different optical power in the optical cavity can be used to estimate the Q factor of test mass eigenmodes. This is essentially the same theory as ringdown measurements except that in this case parametric instability provides the excitation.

Noting the proportionality between the parametric gain and the optical power in the cavity ($R_m = A \times P$ where $A$ is a constant), Equation 5.4 on page 127 can be written

$$A \times P = 2Q_m/\left(\omega_m \tau_{\text{eff}}\right) + 1. \quad (5.6)$$

Here $A = \frac{M c L \omega_m^2 (1 + \Delta \omega/\delta o)}{Q_m Q_\Omega B_{m,n}}$ with parameters defined in Equation 3.3 is assumed constant. Using several measurements of ring-up or ringdown time constant $\tau_{\text{eff}}$ and cavity optical power $P$ we can solve for $Q_m$ and the constant parameter $A$.

In Figure 5.11 the optical power is plotted against the effective time constant to show an example of such a fitting.

![Figure 5.11: The fit used to estimate the Q of the acoustic mode in the 15004Hz mode with the measured data as the inset. The estimated Q is $6 \times 10^6$.](image-url)
This method has been used to estimate the Q factor of the two modes responsible for instability. The 15538 Hz mode Q factor estimate was $(12 \pm 4) \times 10^6$ while the estimate for the 15004 Hz mode was $(6 \pm 3) \times 10^6$.

In this chapter these Q factors are used to calculate the parametric gain of these modes given an observed effective time constant.

### 5.2.4.3 Linewidth method

The Q factor can be estimated from the linewidth. The output of the output mode cleaner (OMC) is remarkably sensitive to the acoustic modes as shown in Figure 5.12.

![Figure 5.12: An example measurement of the OMC transmission depicting a group of 4 acoustic modes associated with the 4 test masses, associated with the 15.53 kHz acoustic mode depicted in figure 5.9 b](image)

From Figure 5.12 we see that there is ample signal to noise ratio to measure the acoustic mode linewidth. However there is a problem in measuring linewidths of the order 1 mHz. Long stretches of data are required and as can be seen in Figure 5.3 the acoustic mode frequency changes with temperature. As $\sim 1000$ sec are required to resolve 1 mHz, this frequency change results in smearing of the acoustic mode peak as measured in a standard amplitude spectral density (ASD) measurement. This prevents direct measurements of the Q by this method. It was found that the minimum linewidth was recorded with $\sim 2000$ sec of data. These linewidth measurements give a lower bound on the Q-factor of $7 \pm 2 \times 10^6$ for both the ETMX 15004 Hz mode and the ETMY 15538 Hz mode. However in general Q factor estimates ranging from 1-7 million are obtained depending of the thermal stability of the optic at the time of measurement.

One solution to the problem of the mode frequency shifting is to use the knowledge of the behavior of the resonant mode frequency to de-smear the peak in the ASD.
Frequency tracking is used to estimate the evolution of the mode frequency over half an hour with a linear fit.

\[ d(t)' = \sin\left(\omega_m + \frac{\partial \omega_m}{\partial t}t\right) \]  

(5.7)

The data \(d(t)\) is filtered with a 2 Hz wide filter centered on the mode of interest and multiplied by a simulated version \(d(t)'\) defined by a linear sweep in frequency given in Equation 5.7 with a 1 Hz offset frequency defines a new function \(e(t)\) given by:

\[
e(t) = d(t) \times d(t)'
\approx \sin((\omega_m + \frac{\partial \omega_m}{\partial t}t)t) \times \sin((\omega_m + (\frac{\partial \omega_m}{\partial t} + 1)t)t)
= \frac{1}{2} \cos(t) - \cos((2\omega_m + (2\frac{\partial \omega_m}{\partial t} + 1)t)t),
\]  

(5.8)

When \(e(t)\) is low pass filtered it becomes a new function \(f(t)\) given by:

\[
f(t) = \frac{1}{2} \cos(t)
\]  

(5.9)

The function \(f(t)\) preserves the linewidth of \(d(t)\) removing only the smearing effect due to the linear component of the mode frequency shift.

Figure 5.13: Left - The data is multiplied by a linear frequency sweep with a 1 Hz frequency offset to produce down-conversion to 1 Hz. This removes the linear component of the change in frequency. From the measured linewidth the estimated \(Q\) is \(5.6 \times 10^6\). Right - The same data down-converted with \(\sin((\bar{\omega}_m + 1)t)\). From the measured linewidth the estimated \(Q\) is \(3.5 \times 10^6\)

Figure 5.13 shows a case for which the de-smearing increased the \(Q\) factor estimate. the \(Q\) estimate increased from a \(Q\) of 3 million to 6 million.

In general de-smearing was found not to give substantial improvements on the estimated lower bound for \(Q_m\). When there is a large linear change in frequency this method gives a substantial increase in the \(Q\) estimate, but when the frequency change is small, little or no change in \(Q\) estimate is observed. This is presumably due to the
inadequacy of the linear chirp model. The complicated side-band structure indicates that frequency up-conversion of optic motion or other low frequency noise influences the spectrum and probably the linewidth measurement.

A better understanding of this side-band structure will be required before linewidth measurement using this method gives accurate Q estimates.
5.3 Optical Gain of Advanced LIGO Arm Cavities

As studied in Chapter 2, the optical gain of a single cavity as a function of detuning in length or frequency is a Lorentz function centered around the fundamental mode with many other Lorentz functions superimposed. These other peaks in the optical gain function are resonances of the higher order optical modes.

![Simulated cavity scan of LIGO Livingston Y Arm Cavity. The blue line represents the power response of the cavity over one wavelength length detuning (ie one free spectral range). The simulated optical mode shapes are overlaid above their respective peaks.](image)

The frequency of these peaks are determined by the radius of curvature (RoC) of the mirrors and the length of the cavity as described in Section 5.3.1.1. The linewidth is determined from the losses in the cavity. The situation gets more complicated when we consider coupled cavities [136, 110]. However from Figure 7 in [113] and investigations in [136] the single cavity model is shown to be a reasonable approximation of a coupled cavity. Additionally we can expect that dynamic effects, explored in Chapter 6 Section 6.3, will average out the very high parametric gains associated with optical resonances in the recycling cavities. The analysis in this section is restricted to a single cavity approximation. In this section, measurements of the optical mode spacing are compared with theory and simulation. First I shall review the essential theory.

5.3.1 Review of Cavity Theory for Optical Gain Measurements

In this section I present a review of specific topics that will be addressed in experimental investigations reported in Sections 5.3.2 and 5.3.3.
5.3.1.1 Transverse electromagnetic mode spacing

Parametrically unstable acoustic modes are driven by radiation pressure of the beat note between the fundamental TEM\(_{00}\) and the higher order optical mode TEM\(_{pq}\). The radiation pressure will only be significant if the TEM\(_{pq}\) amplitude in the cavity is large. As only a small fraction is scattered from TEM\(_{00}\) to TEM\(_{pq}\) by the acoustic mode, the TEM\(_{pq}\) must be resonant in the cavity to attain a large amplitude. The frequencies of the transverse modes of a cavity were introduced in Chapter 3 Section 3.2.6. It is convenient to introduce a term \(\Delta f_{pq}\) which is the mode spacing between the fundamental TEM\(_{00}\) and a high order mode TEM\(_{pq}\). This mode spacing defines the frequencies where TEM\(_{pq}\) will resonate most. This frequency is determined by the mirror RoC and the cavity length \(L\) represented here as the free spectral range \(f_{fsr} = c/2L\), where \(c\) is the speed of light.

\[
\Delta f_{pq} = \frac{f_{fsr}(p + q)}{\pi} \cos^{-1}(\sqrt{g_1 g_2}), \tag{5.10}
\]

where \(g_{1,2} = (1 - L/R_{1,2})\), \(R_1\) and \(R_2\) are the RoC the mirrors. \(p\) and \(q\) are the order of the TEM\(_{pq}\), in the Hermite-Gaussian basis HG\(_{p,q}\) or in the Laguerre-Gauss basis LG\(_{p/2,q}\). These basis functions represent the ideal modes of a cavity with infinite radius mirrors (no clipping losses). A comparison between ideal modes and simulated mode is studied by Barriga [35]. He shows how diffraction losses affect the frequency, linewidth, gain and mode shape. I will note a few important points Barriga raises:

1. The change in mode frequency up to a few kHz from ideal mode frequencies significantly alters which mechanical modes one expects to become unstable.
2. The change in mode shape affects the overlap integral.
3. Not all ideal HG or LG modes are supported. An unsupported ideal mode injected in simulation results in the resonance of a supported mode shape of the same order, the resonant mode may have a significantly different overlap to the acoustic mode.
4. Clipping losses reduce the optical gain of high order modes.
5. Clipping losses from the flats (Figure 5.5) break the degeneracy from rotational symmetry resulting in a perfectly aligned cavity having differences in gain and frequency between horizontally and vertically oriented optical modes.

Advanced LIGO fiducial error mirror maps applied in simulation also have an effect on symmetry breaking as can be seen from the simulation that will be presented in Figures 5.17, 5.14 and investigations in Section 5.4. The mirror maps create a preferential mode alignment at a certain frequency as well as frequency differences between some HG\(_{pq}\) and HG\(_{qp}\) modes.
5.3.1.2 Linewidth of an optical cavity

The cavity finesse $F$ is determined by transmission coefficients $T_i$ and $T_e$ for the ITM and ETM and the absorption, scattering and diffraction losses $L_a, L_s$ and $L_d$ of the cavity. Finesse is related to the cavity linewidth by Equation 5.11.

$$F \equiv \frac{f_{fsr}}{\gamma} = \frac{2\pi}{T_i + T_e + L_a + L_s + L_d}$$  (5.11)

Where $f_{fsr}$ is the free spectral range and $\gamma$ is the full width at half maximum of the Lorenz peak.

The losses in Equation 5.11 are dependent on the order of the mode. The losses of the high order optical mode are larger than those of the fundamental mode primarily due to diffraction losses - where light escapes the mirror due to its finite size, Barriga shows how this significantly affects the optical linewidth [35]. The diffraction losses are larger as there is more optical power close to the border of the mirror, these diffraction losses may be approximated by the clipping loss approximation - where the proportion of the ideal transverse mode shape falling outside the mirror’s aperture is treated as the proportional loss. This clipping loss approximation is shown in D’Ambrosio’s [86] Figure 3. However this is not the only mechanism that will increase the losses of higher order modes, transmission, scattering and absorption losses may increase, as the coating degrades towards the edge of the mirror.

The linewidth is a parameter that is changed significantly when considering coupled cavities. A resonance of a coupled cavity has a higher effective finesse when compared to a single cavity. Simulations in [136] show that the linewidth of resonances in the LIGO coupled cavities can be one hundredth the single cavity approximation under specific tuning conditions.

Such narrow features in the optical spectrum are likely to be affected by dynamic detuning resulting from residual motion at low frequency as examined in Chapter 6 Section 6.3. This detuning results from small changes in beam position on the optic and small figure errors in the mirrors. This leads to changes in the effective radius of curvature experienced by the beam and hence changes in the high order optical mode spacing. Time averaging these changes in mode spacing will result in smearing of the narrow linewidth features, much like the acoustic linewidths are smeared by changes in acoustic mode frequency.

Due to the preceding two arguments the linewidth is treated as an unconstrained parameter in this chapter. In Section 5.3.3 measurements of the optical linewidth are presented.
5.3.1.3 Thermal transients in cavity geometry

The high order optical mode frequency is very sensitive to the RoC of the optics which itself is sensitive to the thermal state of the optics. The large laser power build-up in the cavity directly heats the mirrors primarily through coating absorption and this results in a thermal deformation [267], this deformation was described analytically by Hello and Vinet [147]. The finite element analysis package COMSOL has also been used to simulate the thermal transient of the test mass. As yet there are large discrepancies between simulation and the measured thermal state of the optic as can be seen from the measurements of the acoustic mode frequency and measurements of the LG20 mode made during early testing on the interferometer [8]. Work done to address this discrepancy [250, 219] is further discussed in Chapter 7 Section 7.3. For the purposes of this chapter an extremely simplified model of the thermal transient is used, it is defined by a half hour linear change in RoC followed by a constant 22 μDiopter per W of absorbed laser power. In Figure 5.15 the simplified model is compare to a COMSOL [77] Heat Transfer and Thermal Expansion simulation estimate of the change in RoC.

Figure 5.15: Change in the mirror radius of curvature due to the thermal transient from self heating from 0.1W absorbed optical power in the coating. The blue curve represents the simulated defocus generated in COMSOL, while the red curve represents the simplified model, a linear piece wise model was chosen for simplicity and to keep the required history of cavity optical power data to a minimum.

Changes in ring heater power are also used extensively in this chapter. Heating the ring heater makes a temperature gradient that results in a change in RoC that has the opposite sign (ie compensates) to RoC due to self heating. Again an extremely
simplified model of the thermal transient is used. The simplified ring heater model is defined by a 1.5 hour linear change in RoC followed by a constant. In Figure 5.16 the simplified model is compare to the COMSOL simulated response in the panel on the left, while on the right Brooks’ measurement is compared to a simulated response.

![Figure 5.16: Comparison of ring heater defocus measurement (blue) COMSOL simulation (green) and simplified model (red). Right - Simplified ring heater defocus model consists of a linear piece wise model to fit a 1W change in ring heater power. This was chosen for simplicity and to keep the history of ring heater power data required to a minimum. Right panel a 13.3W change in ring heater power defocus measured using the Hartman wavefront sensor. Right Panel copied from Brooks[8]](image)

In the right panel of Figure 5.16 the model fits the measured data relatively well in the first 1.5 hours. This is consistent with Wang et al’s [250] investigations studied in Section 7.3, where the discrepancy in the thermal model was attributed radiative coupling to elements surrounding the optic. This radiative coupling affects the optic temperature very slowly.

The simplified model is compared to the COMSOL simulation in left panel of Figure 5.16. The simplified model rate of change in RoC after a change in the ring heater power is $3.0 \pm 0.2 \text{m/Wh}$ for 1.5 hours. Followed by a constant $1.1 \mu\text{Diopter}$ per W of ring heater power.

Intentional changes in ring heater power are used to provide an approximately linear sweep in the high order optical mode frequency. This is used in Section 5.3.3 to estimate the high order optical mode spacing and linewidth.

### 5.3.1.4 Oscar simulation

The mode shape, frequency and linewidth are modified by the finite size of the mirrors [35]. The fast Fourier transform based optic simulation package Oscar [93] is used to simulate the LIGO arm cavities to compare simulated results with analytic results.
The optical simulation uses the values of Table 5.4. Mirror maps measured prior to installation are applied in the simulation.

Figure 5.17: Optical modes shapes of an ideal cavity (red) compared to modes shapes from an Oscar simulation (blue) of finite mirrors with figure error mirror maps applied.
5.3.1.5 Quadrant photodetector study of transverse mode orientation

Information about the spatial distribution and orientation of the optical modes can be attained by comparing the phase of the four quadrants of the QPD in transmission of the arm cavities as described in Chapter 2 Section 2.3.5.

The expected amplitude and relative phase can be calculated by integrating the beat note over the quadrant area

\[
\iint_Q E_{00} \ast E_{pq} d\vec{r}_\perp.
\]

Here \(\iint_Q d\vec{r}_\perp\) in the integral over the quadrant surface, \(E_{00}\) is the field distribution of the fundamental mode and \(E_{pq}\) is the field distribution of the transverse electromagnetic mode of order \(n = p + q\).

In Figure 5.18 various ideal Hermite Gaussian (HG\(_{pq}\)) and Laguerre-Gauss (LG\(_{pq}\)) modes are displayed as the transverse electro-magnetic mode (TEM\(_{pq}\)) power distribution in black and white. The expected phase of the signals on the individual quadrants of the QPDs are displayed in colour next the the mode shape. The simulated phase in this figure is against an arbitrary reference.
5.3.2 Analysis of the LIGO Arm Cavity

5.3.2.1 Expected transverse mode spacing

For the 4km long arm cavities of advanced LIGO the free spectral range is \( f_{fsr} = 37.5 \text{kHz} \). The parameters that govern optical mode spacing and linewidth as measured prior to installation are shown in Table 5.4 [4].
Table 5.4: LIGO Livingston Y Arm Cavity Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>4000 m</td>
<td>Cavity length</td>
</tr>
<tr>
<td>D&lt;sub&gt;ETM&lt;/sub&gt;</td>
<td>340.13 mm</td>
<td>End test mass (ETM) diameter as measured by LIGO</td>
</tr>
<tr>
<td>R&lt;sub&gt;ETM&lt;/sub&gt;</td>
<td>2242.4 m</td>
<td>Radius of curvature of the LLO Y arm ETM measured over 164 mm diameter</td>
</tr>
<tr>
<td>L&lt;sub&gt;i,ETM&lt;/sub&gt;</td>
<td>3.6 ppm</td>
<td>HR surface transmission of the LLO Y arm ETM</td>
</tr>
<tr>
<td>L&lt;sub&gt;a,ETM&lt;/sub&gt;</td>
<td>1.8 ppm</td>
<td>HR surface absorption of the LLO Y arm ETM</td>
</tr>
<tr>
<td>L&lt;sub&gt;s,ETM&lt;/sub&gt;</td>
<td>9 ppm</td>
<td>HR surface scatter of the LLO Y arm ETM</td>
</tr>
<tr>
<td>D&lt;sub&gt;ITM&lt;/sub&gt;</td>
<td>339.92 mm</td>
<td>Input test mass (ITM) diameter as measured by LIGO</td>
</tr>
<tr>
<td>R&lt;sub&gt;ITM&lt;/sub&gt;</td>
<td>1940.7 m</td>
<td>Radius of curvature of the LLO Y arm ITM measured over 164 mm diameter</td>
</tr>
<tr>
<td>L&lt;sub&gt;i,ITM&lt;/sub&gt;</td>
<td>1.48%</td>
<td>HR surface transmission of the LLO Y arm ITM</td>
</tr>
<tr>
<td>L&lt;sub&gt;a,ITM&lt;/sub&gt;</td>
<td>0.3 ppm</td>
<td>HR surface absorption of the LLO Y arm ITM</td>
</tr>
<tr>
<td>L&lt;sub&gt;s,ITM&lt;/sub&gt;</td>
<td>14 ppm</td>
<td>HR surface scatter of the LLO Y arm ITM</td>
</tr>
</tbody>
</table>

Using RoC and Length from Table 5.4 and Equation 5.10 results in the estimate of the transverse mode spacing given in Table 5.5.

Table 5.5: Calculated single arm mode spacing from measured RoC

<table>
<thead>
<tr>
<th>ARM</th>
<th>TEM&lt;sub&gt;01&lt;/sub&gt;</th>
<th>TEM&lt;sub&gt;02&lt;/sub&gt;</th>
<th>TEM&lt;sub&gt;03&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>XARM</td>
<td>4972</td>
<td>9943</td>
<td>14915</td>
</tr>
<tr>
<td>YARM</td>
<td>5045</td>
<td>10091</td>
<td>15137</td>
</tr>
</tbody>
</table>

The region sampled to estimate the radius of curvature is 160.4 mm which is large relative to the beam diameter. The full mirror maps are used to estimate the average RoC over the diameter of the spot sizes on the mirrors. The estimated beam size is 108 mm on the ITM and 126 mm on the ETM using the 1/e<sup>2</sup> convention. For the LIGO Livingston Y arm the RoC of the central 108 mm region of the ITM is 2.4 m longer at R<sub>i</sub> = 1943.1 m and the ETM is 4.5 m longer at R<sub>e</sub> = 2246.9 m. The theoretical mode spacing would therefore be 5137 Hz.

Finally this can be compared with an Oscar simulation that includes the full mirror map for each mirror and clipping losses that assume the mirror reflective coating covers the full mirror front surface.
Table 5.6: Simulated single arm mode spacing using Oscar and full mirror maps

<table>
<thead>
<tr>
<th>ARM</th>
<th>HG01</th>
<th>HG02</th>
<th>HG03</th>
<th>TEM04*</th>
<th>TEM05*</th>
<th>TEM06*</th>
<th>Uncer</th>
</tr>
</thead>
<tbody>
<tr>
<td>XARM</td>
<td>4950</td>
<td>10125</td>
<td>15375</td>
<td>20775</td>
<td>26250</td>
<td>32138</td>
<td>40Hz</td>
</tr>
<tr>
<td>YARM</td>
<td>5025</td>
<td>10237</td>
<td>15487</td>
<td>20850</td>
<td>26362</td>
<td>32063</td>
<td>40Hz</td>
</tr>
</tbody>
</table>

* indicates the mode shape is not recognizable in Oscar simulation.

5.3.2.2 Expected transverse mode linewidth

The finesse of the arm cavity can be estimated as 448 from Equation 5.11 together with the data presented in Table 5.4, from which a linewidth of 83.7Hz is derived. It is known that there are excess losses in the cavity. The linewidth of the fundamental mode in the cavity has been measured several times [84] and is significantly position dependent [178]. Using a nominal central alignment the finesse was measured as $F = 417 \pm 3$ and thus the linewidth $\gamma = f_{fsr}/F = 89.9 \pm 0.6\ Hz$. The high order optical mode linewidth nominally is governed by the same absorption, transmission and scattering loses. The increase in the diffraction losses from the increased size of the TEM$_{03}$ mode are small. If we use the clipping loss approximation the expected increase in clipping losses is a few ppm. This estimate neglects the increase in losses from coating degradation towards the edge of the mirror, the expected increase in finesse of the coupled cavity and dynamic effects. In Section 5.3.3 when fitting the optical gain curve, the initial condition for the fit is the linewidth of the fundamental mode. However the possibility of a wide range of optical linewidths was considered.

5.3.3 Measurements of the Optical Mode Spacing and Linewidth

In this section the high order optical mode spacing and linewidth is estimated. Measurements of the parametric gain over the first two hours of a ring heater transient are used. The models presented in Figures 5.15 and 5.16 are used to restrict the analysis to a finite history and convert time into an estimated change in mode spacing.

It is assumed that the only parameter changing for the duration of the transient is the mode spacing. The location of the peak of the Lorentz curve can be used to estimate the mode spacing and the linewidth represents the linewidth of the high order optical mode.
5.3.3.1 X Arm

Figure 5.19: The parametric gain of the 15004Hz ETM X mode as a function of estimated mode frequency shift after a transient in the ring heater in blue with a Lorentz fit in red.

In Figure 5.19 measurements of the mode amplitude time constant have been used to estimate the parametric gain as a function of time. Time is converted into an estimate of the change in mode spacing from the history of the ring heater and contained power using the models of Figures 5.15 and 5.16. It can be seen that a Lorentzian fit approximates that parametric gain as a function of mode spacing, the peak of the Lorentz fit is 28 Hz to the left of zero, with an optical linewidth of 65 Hz. Based on this measurement the X arm HOOM spacing with 100 kW contained power and a ring heater operating at 1.1 W can be estimated as $15032^{+20}_{-5}$ Hz and the peak parametric gain may be estimated as $1.7 \pm 1$. This measurement was made when the ring heater was in a faulty state, but provides the most convincing linewidth estimate yet obtained.

Here an analysis of the ring heater fault is presented. Based on the measurements of the ring heater electrical impedance the ring heater fault was diagnosed as two short circuits as shown in Figure 5.20. This fault was fixed with modification to the drive electronics that held these points on the ring heater at a common potential confirming the diagnosis. The ring heater drive electronics and the faulty ring heater were simulated with LTSPICE [108] to obtain the power dissipated in each segment of the ring heater shown in the left panel of Figure 5.20. Then this power dissipation was applied in a COMSOL model. The resulting surface deformation is displayed in the right panel of Figure 5.20.
Figure 5.20: Left - The ring heater segment impedance derived from measurements at V1,V2,V3,V4 and ground. Right - The simulated surface deformation for the faulty ring heater with 0.68 W dissipated in section A, 2 W dissipated in section B, 0.45 W dissipated in section C and 0.57 W dissipated in section D as derived from a LTSPICE simulation of the faulty ring heater and electronics.

An Oscar simulation with the COMSOL simulated surface deformation of Figure 5.20 estimates the faulty ring heater would shift the third order mode approximately 100 Hz relative to the case where the ring heater is operating correctly. The simulation also predicts $\approx 100 \text{ Hz}$ split between the TEM$_{03}$ and TEM$_{30}$.

After the ring heater was repaired, instability was induced with 3 W of ring heater power as can be seen in Figure 5.21. In this figure the ring heater power, mode amplitude and parametric gain is plotted as a function of time.
Figure 5.21: A step from 1W to 3W in the ETMX ring heater was used to induce instability in the 15004 Hz mode, the red trace shows the ring heater power the green trace the mode amplitude and the blue trace the parametric gain with Lorentzian fit in dashed blue.

From Figure 5.21 it appears that the transient passes a maximum in the parametric gain one hour after the change in ring heater power. From this the estimated mode spacing with 100 kW contained power and a ring heater operating at 1.1 W can be estimated as 15165 Hz. There is almost no information regarding the optical mode linewidth in this ring heater sweep. Assuming the point of maximum parametric gain was passed the maximum parametric gain estimate of 1.3 is reliable.

5.3.3.2 Y Arm

The sweep for the Y arm is shown in Figure 5.22. Again it does not sufficiently constrain the mode linewidth. However for a range of optical mode linewidths from 1-90 Hz the estimated mode spacing with 100 kW contained power and a ring heater operating at 1.1 W is contained in the range 15418 ± 5 Hz.
The maximum parametric gain is also not well constrained, given a range of possible linewidths from 10-90 Hz least squares fitting predicts a maximum parametric gain of 7.

5.4 Overlap Parameter

The mass scaled overlap parameter $B_{m,n}^2$ that determines parametric coupling is reproduced here from Chapter 3:

$$B_{m,n}^2 = \frac{V \left( \oint_S |E_0(r_{\perp})E_n(r_{\perp})u_z\,dr_{\perp}\right)^2}{\oint_S |E_0|^2 \oint_S |E_n|^2 \oint_V |\vec{u}|^2 dV},$$

(5.13)

where $E_0$ and $E_n$ are the optical fields’ distributions over the mirror’s surfaces shown as their product in Figure 5.23, $\vec{u}_m$ the $n^{th}$ eigenmode deformation vector and $u_z$ is its surface deformation in the direction of the optic axis shown in Figure 5.17, $\int d\vec{r}$ are integrals over the surface and $\int d\vec{V}$ integrals over the mirror volume.

The calculated overlap parameter for ideal Hermite Gaussian modes (infinite mirrors) is 0.11 for the 15538 Hz mode. Using the Oscar simulated beam shapes the overlap factor is 0.0782 for the 15538 Hz mode and 0.0321 for the 15004 Hz mode.

5.4.0.1 Beam de-centering

The overlap factor will change if the beams are not centered on the optic. In this section it is assumed that the de-centering is small such that changes in mode shape and frequency due to the modified boundary conditions can be neglected. Under this assumption the overlap factor can be calculated by the convolution of the optical modes shapes product (the product of the two shapes in Figure 5.23) with the mode surface deformation.
During the period prior to Observation Run 1, the beam position was estimated from the actuator gains that minimized the coupling between test mass angular degrees of freedom and length degrees of freedom. This process was performed sporadically and estimated spot positions were mostly within 1 cm of the center of the optic.

Here the normalised overlap factor is calculated, by calculating the numerator of Equation 5.13 and dividing by the maximum overlap. This calculation is performed over the expected range of beam positions on the optic and is displayed as a function of beam position in Figure 5.24. The distribution of the optical modes is shown in Figure 5.23 while the spatial distribution of the acoustic mode surface deformation in the direction of the optic axis were shown in Figure 5.9 on page 142.

Figure 5.23: The spatial profile of the optical modes generated from Oscar simulations that are used for the overlap calculations left the TEM$_{00}$ mode and on the right the TEM$_{03}$ mode

Figure 5.24: The normalised convolution of the optical beat-note and the mirror surface deformation parallel to the cavity optic axis for the 15004 Hz mode (left) and 15538 Hz mode (right)
The normalized mass scaled overlap factor displayed as a function of position in Figure 5.24 shows that during normal interferometer operation with less than 10 mm variation in spot position the maximum reduction in the overlap factor is approximately 3.5% for the 15004 Hz mode and 5% for the 15538 Hz mode. These variations are small compared to the unexplained changes in parametric gain. It will be seen in Section 5.5 that the uncertainty due to beam de-centering is small relative to other uncertainties in the parametric gain estimate.

5.4.0.2 Beam rotation and optical mode preferred alignment

When taking measurements of the individual quadrant phase, like those presented in Figure 5.10(b), it was noted that there appears to be rotation of the optical mode on the arm transmission QPDs, particularly when the ring heater malfunctioned. Changing the ring heater power in this case appeared to produce a rotation of the signals on the QPD.

The idea presented in this section is that there may be a preferential angle in TEM modes. The effect of the rotation of optical modes on parametric instability was considered by Gras et al [132] in their analysis of the use of flat-top beams in gravitational wave detector cavities. The angle is likely to be associated with cavity alignment combined with astigmatism in the optics.

In the case of the faulty ring heater at LIGO Livingston the expected heat distribution applied in a COMSOL simulation produces the surface deformation shown in Figure 5.20. This surface deformation produces some astigmatism at a specific angle. The surface deformation is applied in an Oscar simulation (with tilt removed). The field distribution of the resulting 3rd order mode peak is compared to the case where the ring heater is operating normally in Figure 5.25.
A rotation of the TEM$_{03}$ optical mode of 12 degrees can be seen between the two panels on Figure 5.25. Also the shape of the mode changes, in the right panel representing the simulation with a normal ring heater thermal deformation the mode resembles a LG$_{11}$ mode, while in the left panel simulation with the faulty ring heater resembles a HG$_{03}$ mode. The faulty ring heater simulation also results in an increase in the difference in frequency between the TEM$_{03}$ and TEM$_{30}$ modes to about 100 Hz.

This would at first appear to be contrary to the results of [35] where it is discussed that the flats of the mirrors break the symmetry thereby causing differences between the HG$_{30}$ and HG$_{03}$ modes. But OSCAR simulations of the Y arm cavity show that applying the LIGO mirror maps to a perfectly aligned cavity produces the dominant source of symmetry breaking. If a misalignment of 1$\mu$deg is applied to one mirror in the cavity, misalignment becomes the dominant source of symmetry breaking of the third order modes. This means that the preferential alignment of optical modes may be defined by figure errors in the reflective surface or by misalignment. A simulation of the overlap parameter as a function of rotation between the optical mode and the acoustic mode was undertaken. In Figure 5.26 the overlap factor of the TEM$_{03}$ mode is shown through 180 degrees of rotation. The form of the curve is roughly sinusoidal for both the 15004 Hz mode and the 15538 Hz mode.
Figure 5.26: Simulation of the parametric instability overlap factor as a function of beam rotation to the ETMY 15538 Hz mode and the ETMX15004 Hz mode

The rotation angle produced in the simulation of the faulty ring heater was 12 degrees, herein this is considered the worst case optical mode rotation. In Section 5.5 this possible beam rotation is included in the uncertainty budget in Figures 5.27 and 5.28. Under this worst case rotation there is a 20% reduction in the overlap parameter.

5.5 Parametric Gain

Now we can estimate the parametric gain. The estimates of the mode spacing are derived from a 2 hour history of the cavity power and the ring heater power applied to the models in Figure 5.15 and 5.16. The maximum parametric gain, optical linewidth (65 Hz in all cases) and zero detuning frequency from Section 5.3.3 and uncertainties applied from the uncertainty in Q factor (Section 5.2.4.2) and overlap factor (Section 5.4). Equation 3.6 can now be used to derive the expected parametric gain for the simplified single cavity approximation. In Figures 5.27 and 5.28 the estimate of the parametric gain is compared to the observed parametric gain for all cases where the modes in one cavity were unstable from November 2014 to August 2015 and several points showing where the a particular mode was parametrically stable. When large changes in ring heater power were applied there was one observation of a 15015 Hz instability, this instability is assumed to have the same properties as its ETMX cousin at 15004 Hz. Uncertainties in detuning are derived from a 20% uncertainty from the models for the transient in RoC due to cavity power and ring heater changes. Parametric gain uncertainties arise from uncertainties in the overlap factor, the Q factor estimate and the power estimate.
Figure 5.27: Comparison of the measured parametric gain (scaled to 100 kW) as a function of the estimated de-tuning ($\Delta \omega$) for X arm instability at 15536 Hz. The two single arm models overlaid reflect the faulty ring heater case red and repaired ring heater case in blue. All the observed instabilities happened in the faulty state. Short vertical error bars are due to overlap factor (Section 5.4) the remaining error bar includes the remaining optical gain uncertainty (Section 5.3). The horizontal error bars are due to the simple model used to estimate the mode spacing and the experimental error in mode spacing (Section 5.3).

Figure 5.28: Comparison of the measured parametric gain (scaled to 100 kW) as a function of the estimated de-tuning ($\Delta \omega$) for all Y arm instabilities. The green points are the 15015 Hz mode and the blue points are the 15538 Hz mode. The single arm model is overlaid. Short vertical error bars are due to overlap factor (Section 5.4) the remaining error bar includes the remaining optical gain uncertainty (Section 5.3). The horizontal error bars are due to the simple model used to estimate the mode spacing and the experimental error in mode spacing (Section 5.3). The large uncertainty in Q factor (Section 5.2.4) must be considered when comparing the green points to the blue points.
The data of Figures 5.27 and 5.28 follow the trend of the expected curve. However there are large uncertainties and many outliers. The observations of the 15538 Hz mode seem consistent with a maximum parametric gain of 7 and a maximum parametric gain of 3 for the 15004 Hz mode. The 15015 Hz mode likely has a maximum parametric gain greater than 7 (or a Q factor less than $6 \times 10^6$).

Figure 5.28 also shows that the adjustments in the ring heater setting prior to Observation Run 1 have achieved close to an optimum location for the third order transverse mode. In the thermal equilibrium state the parametric gain of unstable acoustic modes are symmetrically distributed about the nominal optical mode spacing.

5.6 Summary

We have reviewed the physics of parametric instability in relation to Advanced LIGO arm cavities and considered many subtle effects observed in real interferometers that influence parametric instability. In general the observations largely confirm predictions made in 2005 [269], but subtle effects are shown to be significant to obtaining a detailed understanding. Studies on the Advanced LIGO interferometer at Livingston have shown that observed parametric instabilities are caused by 3rd order arm cavity TEM and two different test mass eigenmodes.

A model and method for determining the acoustic mode frequencies in the Advanced LIGO test masses has been demonstrated. The method refines the estimates of the test mass elastic moduli, here we report a Youngs modulus of 72.7GPa and Poisson ratio of 0.164 at 17°C. The mode identification method produces confident identification of 25 eigenmodes in all four test masses. Improvements in high frequency sensing examined in discussion Section 8.4 will be required to identify more test mass eigenmodes.

The simplified single cavity model of the optical transfer function can be used to successfully steer the TEM away from parametrically unstable acoustic modes. Experimental results show the TEM$_{03}$ may be tuned to a region between the 15.0 kHz acoustic modes and the 15.54 kHz modes that is parametrically stable at Observation Run 1 operating optical power. There is a region from 15.3 kHz to 15.5 kHz with no acoustic modes. At Observation Run 1 operating power the former region allows a wide tuning range of the TEM$_{03}$. However much of this tuning range is required as the thermal transient from the absorbed optical power in the mirror coatings results in a sweep in frequency of the TEM$_{03}$.
The parametric gain was compared to the estimated TEM frequency using a further simplification on the single cavity model, there are large uncertainties and many outliers in this comparison.

The spatial overlap parameter was examined. The spatial overlap parameter for ideal Hermite Gauss modes is 40% larger than the calculation using the Oscar simulated mode shape. It is shown that spatial overlap change with beam de-centering is of minimal concern when considering avoidance of parametric instability for Advanced LIGO. It is shown that the change in spatial overlap parameter with beam rotation can have a significant affect on the parametric gain, particularly in the case examined where there was significant astigmatism.

Measuring the Q factors of modes that are currently parametrically stable will be essential to determine at what optical power these modes will become unstable, this will enable estimates of the maximal optical power that thermal tuning alone can accommodate. Refinements of measurements of the optical transfer function such as results presented in Section 5.3.3 and comparison against a dual recycled Michelson interferometer model that includes mirror dynamics will also be invaluable for diagnosis of parametric instability.
Chapter 6

Demonstrations of the Control of Parametric Instability

6.1 Introduction

Parametric instability was observed at the Gingin High Optical Power Facility [273] shortly before parametric instability started affecting Advanced LIGO [113]. In both cases strategies for suppressing instability were demonstrated. This chapter is based on two published papers; “First Demonstration of Electrostatic Damping of Parametric Instability at Advanced LIGO” by Carl Blair, Slawek Gras, Richard Abbott et al [50] and “Parametric Instability in Long Optical Cavities and Suppression by Dynamic Transverse Mode Frequency Modulation ” by Chunmeng Zhao, Li Ju, Qi Fang, Carl Blair, Jiayi Qin and David Blair [272]. In this chapter I have amended notation to be consistent with this thesis and made minor modifications to make the content more relevant to this thesis.

This chapter provides the details on the two parametric instability suppression techniques that have been demonstrated. Section 6.2 discusses the implementation and testing of a feedback control system to suppress parametric instabilities at Advanced LIGO. Section 6.3 explains how unforeseen dynamics of suspended optical cavities led to the realisation that these dynamics produce an inherent suppression mechanism involving the dynamic detuning of the optical modes. This effect was discovered when parametric instability was first observed at the Gingin High Optical Power Facility (hereafter referred to as Gingin).

The role of this mechanism was also studied in relation to Advanced LIGO. This work showed that it will likely have little effect on instabilities where the resonance is dominated by optical power build-up in the arm cavities. However where power
buildup in the recycling cavities has been predicted [110] to result in the largest para-
metric gains, the dynamic detuning affect may well tranquilize the instability close to
the level expected from coupled arm cavities. The possibility of using these dynamics
for intentional parametric instability suppression is also explored. In Section 6.4 the
results from the parametric instability suppression experiments are summarised.
6.2 First Demonstration of Electrostatic Damping of Parametric Instability

In this section the control of a parametrically unstable mode by actively damping a 15.54 kHz resonant mode of an Advanced LIGO test mass using electrostatic force actuators is demonstrated. In Section 6.2.1 the setting in which these experiments were performed is described. Then the physics of parametric instability (PI) relevant to these experiments is discussed in Section 6.2.2. The status of PI avoidance via the control of the mirror radius of curvature is discussed in Section 6.2.3. The Advanced LIGO electrostatic drive system is introduced and the physics behind its interaction with the test mass modes is explained in Section 6.2.4. The configuration used for this experiment is summarised 6.2.5; successful damping observations are reported 6.2.6; and the implications for high power operation of Advanced LIGO are discussed in Section 6.2.7.

6.2.1 Introduction

Three mode parametric instability (PI) was first observed in 2009 in solid state micro-cavities [242], in 2012 it was observed in a free space cavity [69] at UWA, then in 2014 in an 80 m cavity [272] and soon afterwards during the commissioning of Advanced LIGO [113]. Left uncontrolled, PI at Advanced LIGO results in the optical cavity control systems becoming unstable on time scales of minutes to hours [113].

The first detection of gravitational waves was made by two Advanced LIGO laser interferometer gravitational wave detectors with about 100 kW of circulating power in their arm cavities [13]. To achieve this power level required suppression of PI through thermal tuning of the higher-order mode eigen-frequency [267] that was introduced in Section 3.2.6 and is introduced in the context of this experiment in Section 6.2.3. This tuning allowed the optical power to be increased in Advanced LIGO from about 5% to 12% of the design power of 800 kW, sufficient to attain a strain sensitivity of $10^{-23} \text{Hz}^{-\frac{1}{2}}$ at 100 Hz.

At design power it will not be possible to avoid instabilities using thermal tuning alone for two reasons. First the parametric gain is proportional to optical power and second the acoustic mode density is so high that thermal detuning for one acoustic mode brings other modes into resonance [269, 113].

Several methods are likely to be useful for controlling PI as discussed in Section 3.3. At LIGO active thermal tuning will minimize the effects of thermal transients [63, 116, 219] and maintain operation near the parametric gain minimum. In
the future, acoustic mode dampers attached to the test masses [134] could damp acoustic modes. Active damping of acoustic modes may also be used to suppress instabilities by applying feedback forces to the test masses. One possible active damping scheme in which electrostatic forces are used to actuate on test mass eigen-modes is the subject of this section.

6.2.2 Parametric Instability Model for this Experiment

The equation for parametric gain $R_m$, derived by Braginsky [60] and generalised by Ju [159] is reproduced here for clarity in the form presented by Evans [110]

$$R_m = \frac{8\pi Q_m P}{M \omega_m^2 c \lambda_0} \sum_{n=1}^{\infty} \text{Re}[G_n] B_{m,n}^2.$$  (6.1)

Here $Q_m$ is the quality factor (Q) of the mechanical mode $m$, $P$ is the power in the fundamental optical mode of the cavity, $M$ is the mass of the test mass, $c$ is the speed of light, $\lambda_0$ is the wavelength of light, $\omega_m$ is the mechanical mode angular frequency, $G_n$ is the transfer function for an optical field leaving the test mass surface to the field incident on that same surface and $B_{m,n}$ is the spatial overlap between the optical beat note pressure distribution and the mechanical mode surface deformation.

The simplified model used in this chapter is that of a single cavity and a single optical mode. For a simulation analysis including arms and recycling cavities see [136, 110]. In Section 6.3 the dynamics of the transverse optical modes are studied, these dynamics blur narrow linewidth optical resonances possibly making the recycling cavities less relevant. In the simplified case we consider the TEM$_{03}$ mode as it dominates the optical interaction with the acoustic mode investigated here. Equation 6.2 defines corresponding optical transfer function

$$\text{Re}[G_{03}] = \frac{c}{L \pi \gamma_{03} (1 + \Delta \omega^2 / \gamma_{03}^2)}.$$  (6.2)

Here $\gamma_{03}$ is the half-width at half maximum of the TEM$_{03}$ optical mode frequency distribution, $L$ is the length of the cavity, $\Delta \omega$ is the spacing in frequency between the mechanical mode $\omega_m$ and the beat note of the fundamental and TEM$_{03}$ optical modes. In general the parametric gain changes the time constant of the mechanical mode as in Equation 6.3.

$$\tau_{pi} = \tau_m / (1 - R_m)$$  (6.3)

Here $\tau_m$ is the natural time constant of the mechanical mode and $\tau_{pi}$ is the time constant of the mode influenced by the optomechanical interaction. If the parametric gain $R_m$ exceeds unity the mode becomes unstable as $\tau_{pi}$ becomes positive.

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6.2.3 Thermal Tuning

Thermal tuning was used to control PI in Advanced LIGO’s Observation Run 1 and was integral to the experiment described here, so the situation is re-examined in detail below. Thermal tuning is achieved using radiative ring heaters that surround the barrel of each test mass without physical contact (see Figure 6.2 in Section 6.2.4). Applying power to the ring heater decreases the radius of curvature (RoC) of the mirrors. This changes the cavity g factor and tunes the mode spacing between the fundamental \((TEM_{00})\) and higher order transverse electromagnetic \((TEM_{pq})\) modes in the cavity, thereby tuning the parametric gain by changing \(\Delta \omega\) in Equation 6.2.

Figure 6.1: The relative location of the optical and mechanical modes during Advanced LIGO Observation run 1. Mechanical modes measured in transmission of the output mode cleaner shown in blue with mode surface deformation generated from FEM modeling overlay-ed. These modes appear in groups of four, one for each test mass. They have linewidth \(\sim 1mHz\). The optical transfer function for the \(TEM_{03}\) optical cavity mode for a simplified single cavity is shown in solid red and with the ring heater turned off dashed red. The shape of the \(TEM_{03}\) mode simulated with OSCAR [93] is inset below the peak.

Figure 6.1 shows five groups of mechanical modes (A-E) and the optical transfer function of Equation 6.2 as a function of \(\omega_m\) for the \(TEM_{03}\) mode. The ring heater tuning used during Advanced LIGOs first observing run [70] is shown in bold red. In the figure acoustic modes associated with the four test masses appear as groups of blue peaks labelled A-E at frequencies from 15kHz to 15.5kHz. The mode shapes of the acoustic modes and the \(TEM_{03}\) modes are illustrated. Without thermal tuning, the optical gain curve in Figure 6.1 shifts to higher frequency, decreasing the frequency spacing \(\Delta \omega\) with mechanical mode group E as depicted in dashed red. This leads to
the instability of this group of modes. (Note: In Section 5.2.1 the mirror acoustic mode frequencies were demonstrated to be only weakly tuned by heater power, due to the elastic moduli of fused silica having a low temperature dependence).

If the ring heater power is increased inducing approximately 5 m change in RoC, optical transfer function peak in Figure 6.1 moves left about 400 Hz, decreasing the value $\Delta \omega$ for mode group A. This has resulted in instability of two of the modes in mode group A. The mode groups C and D are stable as the second and fourth order optical modes that might be excited by these modes are far from resonance. Mode group B is also stable at the circulating optical power used in this experiment, presumably due to either lower quality factor $Q_m$ or lower optical gain $G_{30}$ of the TEM$_{30}$ mode as investigated in [35]. Extrapolating from Equation 6.2 and the observed parametric gain, increasing the interferometer power by a factor of 3 results in no stable region. Mode group A at 15.00 kHz and group E at 15.54 kHz will be unstable simultaneously.

### 6.2.4 Electrostatic Control

Electrostatic control of PI was proposed [157] and studied in the context of the LIGO electrostatic control combs by Miller et al [204]. Here we report studies of electrostatic feedback damping for the group E modes at 15.54 kHz.

The main purpose of the electrostatic drive (ESD) is to provide longitudinal actuation on the test masses for lock acquisition [205] and hold the arm cavities on resonance. It creates a force between the test masses and their counterpart reaction masses through the interaction of the fused silica test masses with the electric fields generated by a comb of gold conductors that are deposited on the reaction mass. The physical locations of these components are depicted in Figure 6.2. The structure of the gold comb is shown in Figure 6.3 along with the force density on the test mass.

The force applied to the test mass $F_{ESD}$ is dominated by the dipole attraction of the test mass dielectric to the electric field between the electrodes of the gold comb. $F_{app,m}$ is the fraction $b_m$ of this force that couples to the acoustic mode:

$$F_{app,m} = b_m F_{ESD,Q} = b_m \alpha_Q \times \frac{1}{2}(V_{bias} - V_Q)^2$$

(6.4)

Here $\alpha_Q$ is the force coefficient for a single quadrant resulting in a force $F_{ESD,Q}$, while $V_{bias}$ and $V_Q$ are the voltages of the ESD electrodes defined in Figure 6.3. The overlap $b_m$ between the ESD force distribution $f_{ESD,Q}$ and the displacement $\hat{u}_m$ of the surface
for a particular acoustic mode $m$ can be approximated as a surface integral derived by Miller [204]:

$$b_m \approx \left| \int \int_S \overline{f}_{ESD,Q} \cdot (\overline{u}_m \cdot \hat{z}) \, dS \right| \quad (6.5)$$

If a feedback system is created that senses the mode amplitude and provides a viscous damping force using the ESD, the resulting time constant of the mode $\tau_{esd}$ is given by:

$$\tau_{esd} = \left( \frac{1}{\tau_m} + \frac{K_m}{2\mu_m} \right)^{-1} \quad (6.6)$$

Here $K_m$ is the gain applied between the velocity measurement and the ESD actuation force on a mode with time constant $\tau_m$ and effective mass $\mu_m$. Reducing the effective time constant lowers the effective parametric gain:

$$R_{eff} = R_m \times \frac{\tau_{esd}}{\tau_m} \quad (6.7)$$

The force required $F_{req}$ to reduce a parametric gain $R_m$ to an effective parametric gain $R_{eff}$ when the mode amplitude is the thermally excited amplitude was used by Miller [204] to predict the forces required from the ESD for damping PI:

$$F_{req} = \frac{x_m \mu_m \omega_m^2}{b_m} \left( \frac{R_m - R_{eff}}{Q_m R_{eff}} \right) \quad (6.8)$$
Here $x_m$ is the acoustic mode amplitude. At the thermally excited amplitude $x_m = \sqrt{k_B T/\mu_m \omega_m^2}$, where $k_B$ is the Boltzmann constant and $T$ is the test mass temperature.

Having described the forces in the feedback loop we will now describe the experimental configuration.

### 6.2.5 Feedback Loop - The Experiment Configuration

Figure 6.4 shows the damping feedback loop implemented on the end test mass of the Y-arm (ETMY). The error signal used for mode damping is constructed from a quadrant photodiode (QPD) that receives light transmitted by ETMY. By suitably combining QPD elements, we measure the beat signal between the cavity $TEM_{00}$ mode and the $TEM_{03}$ mode that is being excited by the 15,538 Hz ETMY acoustic mode. This signal is bandpass filtered with a central frequency of 15,538 Hz and bandwidth of 10 Hz. The signal is then phase shifted to produce a control signal that is 90 degrees out of phase with the mode amplitude (velocity damping). A damping force is applied, with adjustable gain, to two quadrants of the ETMY electrostatic actuator.
Figure 6.4: A simplified schematic of advanced LIGO showing key components for damping PI in ETMY. Components shown include input and end test masses (ITM/ETM), beam-splitter (BS), power and signal recycling mirrors (PRM/SRM), the laser source (LS), the output mode cleaner (OMC), the OMC transmission photodiode (OMCPD) and the arm transmission quadrant photodiode (QPDX/Y). While 4 reaction masses exist, only the Y end reaction mass is shown (ERMY) with key components of the damping loop. These components generate a differential signal from the vertical orientation of QPDY, filter the signal with a 10 Hz wide band pass filter centered on the 15,538 Hz mode, apply gain $K_m'$ and phase $\phi$ set in the digital control system and then differentially drive the upper right Q1 and lower left Q3 quadrants of the ESD.
6.2.6 Results

PI stabilization via active damping was demonstrated by first inducing the ETMY 15,538 Hz to become parametrically unstable. This was achieved by turning off the ring heater tuning, so that the TEM$_{03}$ mode optical gain curve better overlapped this acoustic mode, as discussed in Section 6.2.3 and shown in Figure 6.1. When the mode amplitude became significantly elevated in the QPD signal, the damping loop was closed with a control gain to achieve a clear damping of the mode amplitude and a control phase optimised to within ±15 degrees of viscous damping. The mode amplitude was monitored using the photodiode at the main output of the interferometer (labelled OMCPD in Figure 6.4), as it was found to provide a higher signal-to-noise ratio than the QPD.

![Figure 6.5: Damping of parametric instability. Upper panel, the 15,538 Hz ETMY mode is unstable ringing up with a time constant of 182±9 sec and estimated parametric gain of $R_m = 2.4$. Then at 0 sec control gain is applied resulting in an exponential decay with a time constant of 23 ± 1 sec and effective parametric gain $R_{eff,m} = 0.18$. Lower panel, the control force over the same period.](image)

The results are shown in Figure 6.5, which plots the mode amplitude during the unstable ring-up phase with time constant $\tau_{pi}$ 182 sec, followed by the ringdown time constant $\tau_{eff}$ due to optical gain and damping of -23 sec. From the ring-up we estimate...
the parametric gain to be $2.4 \pm 0.8$ from Equation 6.3. With the damping applied:

$$R_{\text{eff}} = \frac{R_m \tau_{\text{eff}}}{\tau_m + R_m \tau_{\text{eff}}}$$

the effective parametric gain is reduced to a stable value of $R_{\text{eff}} = 0.18 \pm 0.06$. The uncertainty is primarily due to the uncertainty in the estimate of $\tau_m$ which was obtained by the method described in Section 5.2.4.2, fitting the Q factor from several measurements of the parametric gain at different powers.

At the onset of active damping (time $t = 0$ in Figure 6.5), the feedback control signal produces an estimated force of $F_{\text{ESD}} = 0.62 \text{nN rms}$ (at 15,538 Hz). As the mode amplitude decreased the control force dropped to a steady state value of 0.03 nN rms. Over a 20 minute period in this damped state, the peak control force was 0.11 nN peak.

Table 6.1 summarises control and cavity parameters.

### 6.2.7 Discussion

The force required to damp the 15,538 Hz mode when Advanced LIGO reaches design power can be determined from the ESD force used to achieve the observed parametric gain suppression presented here, combined with the expected parametric gain when operated at high power:

$$\frac{F_{\text{req}}}{F_{\text{ESD}}} = \frac{R_{\text{eff}}}{R_{\text{req}}} \frac{R_{\text{max}} - R_{\text{req}}}{R_{\text{req}} - R_{\text{m}} - R_{\text{eff}}}$$  (6.10)

The maximum parametric gain $R_{\text{max}}$ where $\Delta \omega = 0$ is calculated using Equation 6.2. For the 15,538 Hz mode the de-tuning is $\Delta \omega \approx 50 \text{ Hz}$ with zero ring heater
power, so $R_{\text{max}} \approx 7$ for the power level of these experiments. This is consistent with results presented in Chapter 5. At full design power the maximum gain will be $R_{\text{max}} \approx 56$. To obtain a quantitative result, we set a requirement for damping such that the effective parametric gain of unstable acoustic modes after damping be $R_{\text{req}} = 0.1$.

Using Equation 6.10, the measurements of $R_m$ and $R_{\text{eff}}$, the maximum force required to maintain the damped state at high power is $F_{\text{ESD}} = 1.5 \, \text{nN rms}$. Prior to this investigation Miller predicted [204] that a control force of approximately 10 nN rms would be required to maintain this mode at the thermally excited level.

The PI control system must cope with elevated mode amplitudes as the PI mode may build up before PI control can be engaged. There is therefore a requirement for some control range or safety factor, defined as the available voltage divided by drive voltage in damped state, such that the control system will not saturate if the mode amplitude is a multiple of the safety factor times the damped state amplitude. The average ESD drive voltage $V_{Q1} = -V_{Q3}$ over the duration that the mode was in the damped state was 0.42 mV rms. However during this time $V_Q$ reached a maximum value of $\pm 1.4 \, \text{mV peak out of a } \pm 20 \, \text{V control range}$. This leads to a safety factor of more than 10,000. At high power the safety factor will be reduced by the required force ratio according to Equation 6.10. This results in an expected safety factor of 310.

As the laser power is increased, additional modes are likely to become unstable. The parametric gain of these modes should be less than the gain of mode group E provided the optical transfer function in these experiments is maintained. However these modes may also have lower spatial overlap $b_m$ with the ESD. Miller’s simulation [204] show some modes in the 30-90 kHz range will require up to 30 times the control force $F_{\text{ESD}}$ required to damp the group E modes. Even in this situation the PI safety factor is approximately 10.

Coupling of PI control forces presented here to noise in the main interferometer output was insignificant. A detailed investigation will be required when commissioning the complete parametric instability control system. This will be examined in more detail in Chapter 8 Section 8.3.4.
6.3 Transverse Optical Mode Frequency Modulation for Parametric Instability Suppression

In this section the first observation of three mode parametric instability in a long suspended optical cavity is presented. Introduced in Chapter 1 Section 1.2.4, the Gingin High Optical Power facility east arm cavity used in these experiments is designed to be comparable to those of advanced gravitational wave detectors by maintaining the mirror mass to optical power ratio \(266\) and using seismic isolation \(101\). The cavity length was 74 m, the test masses are 0.9 kg each and the optical power was \(\sim 30\) kW.

Our results show that the previous modelling assumption that transverse optical modes are stable in frequency except for frequency drifts on a thermal deformation time scale is unlikely to be valid for suspended mass optical cavities. It is demonstrated that mirror figure errors cause a dependence of transverse mode offset frequency on spot position. Combined with low frequency residual motion of suspended mirrors, this leads to transverse mode frequency modulation which suppresses the effective parametric gain.

6.3.1 Introduction

Three mode parametric interactions are extremely sensitive to test mass mirror parameters. This extreme sensitivity was emphasised by Ju et al.\([160]\) who showed that mirror figure errors - radius of curvature changes corresponding to wave-front deformations of \(10^{-6}\)\(\lambda\) could easily be observed by monitoring three mode interactions in Advanced interferometers. This extreme sensitivity will be studied in the context of gaining information about the interferometer in Chapter 7. In this section we see how this extreme sensitivity resulted in the surprising result that the Gingin High Optical Power facility’s 74 m cavity is predominantly stable to three mode parametric instability when containing \(\sim 30\) kW of optical power.

Previous modelling has ignored two practical aspects of suspended mass interferometers: Firstly that mirrors have figure errors \(\sim 1\) nm, and second that the laser spot position on the mirrors fluctuates due to residual low frequency seismic motion. The presence of figure errors means that the average radius of curvature of the region of the mirror intercepted by the laser beam depends on the beam location. This radius of curvature determines the transverse mode offset frequency \(\omega_{00} - \omega_{pq}\), here \(p\) and \(q\) are integers that describe the transverse mode order. Because low frequency fluctuations of the spot position causes the laser spot to intercept different regions of the mirror...
surface, it follows that there will be *Transverse electromagnetic mode frequency modulation* - modulation of the optical transverse mode offset frequency. If this frequency modulation amplitude exceeds the transverse mode optical linewidth, the parametric gain will be time dependent. This can create a situation where parametric instability does not have time to develop because it is only on-resonance intermittently, and for too short a time for instability to grow to problematic levels.

In this section we will show that the above phenomenon is likely to reduce the average parametric gain in some cases, thereby significantly reducing the risk of instability. Results were confirmed by modelling and by measurements on a 74 m optical cavity at the Gingin High Optical Power Facility. Recognition of this frequency modulation suppression mechanism also leads to methods by which suppression can be enhanced either by modulated thermal actuation or spot position dithering at frequencies below the gravitational wave sensitivity band.

In Section 6.3.2 we summarise the theory of parametric instability suppression with Transverse electromagnetic mode (TEM) frequency modulation and present modelling results showing how individual unstable modes can be suppressed by seismically induced TEM frequency modulation. In Section 6.3.3 we use Advanced LIGO test mass mirror metrology data to estimate the frequency modulation expected for small spot position motions in Advanced LIGO. In Section 6.3.4 we present results obtained at the Gingin High Optical Power Facility of both the observation of parametric instability and the frequency modulation that greatly reduces the risk and severity of instability. We discuss the results, and their implications for parametric instability suppression using this technique at Advanced LIGO. Thermal actuation modelling results are used to estimate the suppression factors achievable.

### 6.3.2 Theory of Transverse Mode Frequency Modulation

As already emphasised three mode opto-acoustic interactions occur when the frequency difference between an optical cavity pump mode at frequency $\omega_{00}$ and a transverse mode at frequency $\omega_{pq}$ is appropriately tuned to the frequency of an acoustic mode at frequency $\omega_{m}$. This three mode interaction resonance is defined by $\Delta \omega = (\omega_{00} - \omega_{pq}) - \omega_{m} = 0$. The parametric gain $R_{m}$ characterises the ratio of optical energy input compared to mirror mechanical mode losses. If $R_{m} > 1$, the system is unstable, and the acoustic mode will grow exponentially until either non-linearities cause saturation [87], or else the cavity loses lock.

In this section we are concerned only with small amplitude excitation and ignore non-linearities. The magnitude of $R_{m}$ depends on cavity input power, on acoustic
and optical mode losses, and on the spatial overlap between the relevant modes. For any pair of acoustic and optical modes, the gain $R_m$ can be expressed as [60]

$$R_m = \frac{P B_{m,n}^2 \omega_{pq}}{M \omega_m L^2 \gamma_m \gamma_{00} \gamma_{pq}} \frac{1}{1 + \left(\frac{\Delta \omega}{\gamma_{pq}}\right)^2}, \quad (\gamma_m \ll \gamma_{pq}) \quad (6.11)$$

Here $P$ is the input power to the cavity, $\gamma_{00}$, $\gamma_{pq}$, and $\gamma_m$ are the linewidths of the two optical modes and the acoustic mode the test mass respectively, $M$ is the effective mass of the test mass, $L$ is the length of the cavity. The factor $B_{m,n}^2$ is the mode shape overlap integral described in Section 3.2.7 which describes how well the acoustic mode shape matches the optical mode shape. The optical mode gap $\omega_{00} - \omega_{pq}$ is a function of the radius of curvature (RoC) of the mirrors of the optical cavity as shown in Section 3.2.6. The equation for the optical mode gap is reproduced here for convenience.

$$\omega_{00} - \omega_{pq} = \frac{c}{L} (p + q) \cos^{-1} \left( \pm \sqrt{(1 - \frac{L}{R_1})(1 - \frac{L}{R_2})} \right) \quad (6.12)$$

where $R_1$ and $R_2$ are the radii of curvature of the end mirrors of the cavity. The sign depends on the cavity configuration. Equation 6.12 assumes perfect spherical mirrors, but we assume that in the case where mirror figure errors are present the mode gap is defined by the average RoC at the laser spot position, averaged over the effective spot size.

Equation 6.11 considers only the Stokes process where parametric amplification or instability processes occur due to a single high order optical mode. Here we focus particularly on the case where dynamic detuning causes $\omega_{00} - \omega_{pq}$ and hence $\Delta \omega$ to be time dependent. We consider the case of harmonic detuning given by

$$\Delta \omega(t) = \Delta \omega_0 \cos(\omega_d t), \quad (6.13)$$

where $\omega_d$ is a dynamic tuning frequency. In suspended mass interferometers the test mass mirrors are supported by low frequency pendula which isolate against vibration. The test mass positions are controlled by feedback, but finite residual motion is inevitable because of the requirement that the test masses be inertial within the gravitational wave signal band.

Thus test masses can be expected to have significant motion at pendulum normal mode frequencies 0.11 Hz. This gives rise to a modulation in the laser spot position. If the mirrors are imperfect, the mirror RoC (averaged over the laser spot size) will vary smoothly with spot position. In this case, modulation in spot position can modulate the transverse mode offset frequency, causing time-dependent detuning fluctuations.
Spot position motion will also modulate the modal overlap parameter. However, for millimeter-scale motions, the overlap parameter modulation is small compared with the effect of detuning and is ignored in the following analysis.

Assuming that $\Delta \omega$ changes according to Equation 6.13, the parametric gain is given by

$$R(t) = \frac{R_{\text{max}}}{(1 + 2a \cos \omega_d t)^2},$$  \hspace{1cm} (6.14)$$

where $a = \Delta \omega_0 / \gamma_{pq}$ is the normalised frequency detuning modulation amplitude. Equation 6.14 allows estimation of the effects of modulation on the growth of parametric instability. As discussed above, modelling has shown that the characteristic ring-up time scale for parametric instability in a detector similar to Advanced LIGO is likely to be $\sim 10^2$ s [87]. Since $\omega_d$ is fast compared with these ring-up times, one would expect to observe modulated signal growth.

Figure 6.6: Acoustic mode amplitude ring-up curves for various detuning amplitudes. Here we assume maximum gain $R_{\text{max}}=6$, an acoustic mode ringdown time $\tau_m=6$ and a dynamic modulation frequency of 0.1 Hz. For comparison, cases for on resonance ($a=0$) with $R_{\text{max}}=6$ and $R_{\text{max}}=1.45$ are also plotted.

Figure 6.6 shows examples of possible acoustic mode ring-up curves. We assume parameters comparable to those of the experiment reported in this section: $f_d = \omega_d/2\pi = 0.1$ Hz, $R_{\text{max}} = 6$, and a normalised detuning amplitude $a = 1, 2, 4$ and $10$. It is sufficient to choose a typical acoustic mode decay time $\tau_m$ without the need to specify the acoustic mode frequency. We chose to use $\tau_m = 6$ s, corresponding
to acoustic quality factor $Q_m = 10^6$ and $4 \times 10^6$ for frequencies 50 kHz and 200 kHz respectively. Results are compared with acoustic mode ring-up curves in the absence of dynamic detuning ($a = 0$) for $R = 6$. In the case of $a = 4$ the ring-up slope is equivalent to that of a system with $a = 0$ and $R = 1.45$ as indicated in Figure 6.7. This represents a gain suppression factor $\approx 4$. Clearly in all cases, frequency modulation suppresses the effective parametric gain as determined by the average slope of the ring-up curves. For $a = 10$ we see that instability is replaced by a modulated acoustic mode amplitude. While not harmonic the evolution of the acoustic mode amplitude is stable in time. The equivalent parametric gain $R_a$ in the presence of harmonic dynamic detuning is given by:

$$R_a = \frac{R_{\text{max}}}{\sqrt{1 + 4a^2}}.$$  \hspace{1cm} (6.15)

The suppression of effective parametric gain as a function of modulation amplitude is shown in Figure 6.7. Parametric gain can be suppressed by an order of magnitude for $a = 10$. In Section 6.3.3 we will show that such suppression can occur naturally as a result of mirror imperfections.

The mechanism discussed above occurs because the dynamic detuning modulation frequency is fast compared with the acoustic mode ring-up time scale. The observed acoustic mode amplitude modulation occurs at double the dynamic detuning fre-
frequency $\omega_d$ assuming the static detuning is small. While the effective parametric gain is independent of $\omega_d$, the peak to peak acoustic mode amplitude within one cycle is inversely dependent on $\omega_d$. Figure 6.8 shows some examples for three different dynamic detuning frequencies. The amplitude modulation waveform is highly non-linear since it is due to a Lorentzian modulation acting on the exponent.

Figure 6.8: As the dynamic detuning frequency is increased, the acoustic mode amplitude excursions are reduced. The effective parametric gain is unaltered. Here three detuning frequencies 1 Hz, 0.1 Hz and 0.01 Hz are shown. The detuning amplitude is fixed at $a=2.5$ ($R_{max}=6$ and $\tau=6$).

Figure 6.8 it shows that if the detuning frequency is too slow, the acoustic mode amplitude could grow to a very large value within half a detuning period. It is possible to define a lower limit for the dynamic detuning frequency $f_{d-lim}$ to prevent the acoustic mode amplitude excursion from exceeding $\beta$ times its original value within one cycle. Figure 6.9 shows three curves showing the lower limit of the dynamic detuning frequency as a function of detuning amplitude $a$, for two values of $R_{max}$ and two values of the acoustic amplitude excursion limit $\beta$. For example if $R_{max} = 6$, and $\tau = 6 \text{s}$, and a requirement $\beta = 5$, then the dynamic detuning frequency is limited within the range 0.2-0.6 Hz assuming detuning modulation amplitudes $a$ is between 2 and 8 optical linewidths. We see that larger detuning frequencies or larger detuning amplitudes both act to reduce amplitude excursions. This defines the parameter space for suppressing parametric instability by the dynamic detuning mechanism.
Figure 6.9: Dynamic detuning frequency limit to prevent amplitude excursions exceeding a predetermined value $\beta$. For example: for $R_{\text{max}}=10$ and $\tau=6$, with an amplitude growth requirement of $\beta=2$, the minimum dynamic detuning frequency is limited within the range 0.6 Hz and 0.2 Hz for detuning amplitude $a$ in the range of 3 to 8.

Next we consider the application of the above theory to the 74 m cavity at Gingin and to Advanced LIGO optical cavities.

6.3.3 Frequency Modulation by Mirror Figure Errors in Advanced LIGO Test Masses

The test mass mirrors in Advanced LIGO have radii of curvature of approximately 2 km. Figure errors imply that the effective RoC depends on the spot position. For example, using Equation 6.12, with Advanced LIGO arm cavities, if the end test mass (ETM) RoC changes 1 m from nominal value of 2242 m (corresponding to a sagittal change within a beam diameter $\approx 0.3$ nm), the TEM$_{10}$ cavity mode frequency will change by 13 Hz.

The residual motion of the test masses in interferometer arm cavities cause high order mode frequency modulation. Residual angular motion creates beam residual motion on the test mass surface of a few millimeters [122]. Depending on the test mass figure errors, this residual motion causes dynamic detuning of the cavity high order mode frequency at the frequencies of test mass pitch and yaw motion.

To estimate the Advanced LIGO arm cavity high-order mode frequency changes as a function of the residual test mass angular motion, we used measured test mass
surface data [44, 45, 46, 47] in interferometer simulation code (OSCAR [93] and FOPG [261]) to simulate cavity transverse mode detuning. In the simulation, we fixed the input test mass (ITM) and modeled ETM misalignment at various angles from zero to 0.25 microradians. Figure 6.10(a) shows representation of the input data in the form of a cross section of the surface across of a portion of the test mass diameter. This figure shows how the figure errors increase with distance from the center of the test mass.

Figure 6.10: (a) Typical Advanced LIGO test mass figure errors (compared to a perfect sphere of RoC of 2242 m) showing deformations over a portion of the mirror diameter. (b) FFT code model for frequency offset as a function of test mass angular motion.
Figure 6.10(b) shows the calculated data for real two dimensional surface profiles. It shows that the cavity mode spacing increases roughly quadratically with ETM misalignment angle. Note that 0.1 microradians corresponds to $\approx 2$ mm beam position displacement on the test mass in a typical Advanced LIGO arm cavity. We extended the simulation to the test mass rotation corresponding to a spot displacement $\approx 6$ mm. We note that in reality, both test masses move independently of each other, thereby creating somewhat larger detuning amplitudes.

Cavity high-order mode-frequency modulation can also be artificially created by applying modulated heating to the test mass. We used ANSYS FEM simulation software [23] to simulate the transient thermal deformation of the test mass surface under sinusoidal heating power. Figure 6.11 shows the maximum thermal deformation when a 0.1 Hz modulated heating beam of 50 mm radius with 2 W peak-to-peak power amplitude is applied to the test mass front surface. Simulation using FFT code [93] indicates that this deformation corresponds to a cavity mode spacing frequency change of $\approx 40$ Hz in Advanced LIGO arm cavities.

![Figure 6.11: The maximum thermal deformation when 0.1 Hz sinusoidal heating power of amplitude 2W is applied to the front surface of the test mass. The peak deformation amplitude in nm is shown on the vertical axis and in colour for clarity.](image_url)

The above results indicate that passive detuning frequency modulation in Advanced LIGO would be expected to be $\sim$ few Hz for the TEM$_{10}$ transverse mode, which is much smaller than the arm cavity linewidth and does not have a significant
effect on the parametric gain. However, this could be increased to $\approx 40$ Hz modulation using CO$_2$ laser heating. It is important to note that the highest predicted parametric gains in Advanced LIGO are for modes up to fourth order [110].

Equation 6.12 implies that detuning scales with mode order. Thus, the above estimates would correspond to at least 4 times larger modulation ($\approx 160$ Hz) for fourth-order instabilities. The Advanced LIGO arm cavity half-linewidth is $\approx 40$ Hz. Thus, the parametric gain associated with arm cavity optical modes would be suppressed by a factor of approximately 3, and for low order cavity modes the above modulations would be negligible.

Gras and Evans [110, 136] show that the highest parametric gain instabilities are associated with modes that are resonant in the power recycling cavity. For these modes the coupled cavity linewidth is about 0.3 Hz [136] and the normalised dynamic detuning amplitude can exceed 50 times of the coupled cavity linewidth. Thus, it is most likely that intrinsic passive detuning in Advanced LIGO will lead to a parametric gain suppression factor $>100$ for those modes resonant inside the recycling cavity. A detailed simulation to explore how the cavity high order mode frequency modulation affects the broad spectrum of parametric instability in Advanced LIGO is beyond the scope of this investigation.

Experimental observations of parametric instability at the Gingin High Optical Power Facility presented in the next section largely confirm the above theory.

### 6.3.4 High Optical Power Cavity Observations

We studied three-mode parametric instabilities at the Gingin high optical power facility [265] described in Section 1.2.4. The experimental setup is shown in Figure 6.12. A 74 m-long optical cavity with fused silica test masses is suspended from high-performance vibration isolators [33, 101] by a modular four-wire test mass suspension system developed at University of Western Australia. The test masses are installed in two large vacuum chambers connected by a 400 mm vacuum pipe. The system was assembled in clean room conditions and uses a hydrocarbon free vacuum system to enable high optical power densities to be achieved.

Both test masses are 50 mm in diameter and 50 mm thick, with mass $\approx 0.8$ kg. The nominal RoC of the two test masses is 37.5 and 37.4 m, respectively. The test masses have a very sparse mode spectrum compared to Advanced LIGO test masses, so that three-mode parametric interactions need to be tuned to specific candidate acoustic modes. This is achieved by using a power-stabilized CO$_2$ laser to thermally tune the ITM RoC to create three-mode tuning for the specific candidate acoustic
Figure 6.12: Schematic diagram of the experimental setup: The laser light from a seed laser is amplified by a 50W fiber laser amplifier. The high optical power laser beam is injected into the 74 m-long optical cavity. The seed laser is frequency locked to the long cavity using pound-drever-hall locking. The cavity transmitted beam is detected by a quadrant photodiode (QPD). The differential signal from the QPD measures the beating between the cavity fundamental mode and the first order mode.

The measured cavity finesse is $14500 \pm 300$. The light source is a 50W fiber laser amplifier fed by a 400mW NdYAG NPRO seed laser. The seed laser is frequency locked to the long cavity using pound-drever-hall locking $[48, 99]$. The cavity transmission is detected by a quadrant photodiode (QPD). The differential output of the QPD measures the beating between the cavity fundamental mode and the first order mode while the sum of the QPD output measures the total cavity transmitted power. A spectrum analyzer (Agilent 89410A) and a PC are used to analyze and record the signal.

Using ANSYS simulations we first analysed the test mass acoustic mode structure and frequencies, for details see $[115]$. These ANSYS simulations are performed in much the same manner as the COMSOL simulations for Advanced LIGO test masses presented in Section 5.2.2.4, page 135.

We then identified one particular acoustic mode that has good overlap with the cavity first order mode and minimum vibration amplitude at the suspension point to minimize the mechanical loss introduced by the suspension. Our target mode, with simulation frequency 150.49kHz, is in the range for easy CO$_2$ laser thermal tuning.
The mode amplitude distribution on the test mass surface is shown in Figure 6.13. The overlap factor taking into account the mode effective mass is 16. The measured mode frequency is $\approx 150.2\text{kHz}$ (depending on the temperature). The measured mechanical Q factor using the ringdown method is $\approx 3.4 \times 10^6$. Three mode interaction conditions are achieved by tuning the TEM$_{00}$ and TEM$_{10}$ mode spacing close to 150.2kHz using CO$_2$ laser thermal tuning.

Measurement of the tuning is relatively easy because residual laser noise gives rise to a small amount of TEM$_{10}$ mode power inside the cavity which beats with the TEM$_{00}$ at the QPD, allowing the TEM$_{10}$ offset frequency to be monitored as a beat note. This provides a means for monitoring the mode spacing by measuring the cavity transmitted power on the QPD where the two modes are mixed. The mode spacing was observed to fluctuate with a typical peak-to-peak amplitude of a few kHz. To confirm that these fluctuations were associated with beam spot position on the test masses, we recorded the cavity mode spacing and the beam position simultaneously for the ITM.

Figure 6.14 shows the mode spacing as a function of the beam position on the ITM in the horizontal direction. The beam position was determined by recording the video of the CCD camera. The video was analyzed to determine the beam position relative to the test mass diameter. In the horizontal orientation there is a linear correlation between mode spacing and beam position. The solid line in Figure 6.14 is a linear least squares fit to the measurement data. The significant scatter is due to the
The correlation between horizontal spot position of the ITM alone and the transverse mode frequency indicates that the frequency detuning is caused by the spot position change. The spread of the data is due to the laser beam position also changing on the ETM. The horizontal axis gives the spot position relative to the center of the test mass where +ve is right as viewed from within the cavity. The line represents a linear least squares fit to the measured data.

The fact that we recorded only the ITM beam position while the ETM beam position is varying by a similar amount. The effect is more difficult to measure in the other axis because the suspensions introduce much smaller vertical beam position fluctuations. However, the single-axis correlation is sufficient to confirm our conjecture that mirror figure errors translate into dynamic detuning.

We do not have precise metrology of our test mass mirror profiles. However, the observed fluctuations are consistent with the mirror figure error specification of 1 nm. It is interesting to note that in principle simultaneous measurement of spot position on both test masses and transverse mode frequency offset could be used to allow precise metrology of the test masses.

When the cavity is correctly tuned, the three-mode interaction occurs, and the signal at the QPD is dominated by the beating between TEM$_{00}$ and TEM$_{10}$ modes at the acoustic mode frequency. The signal is proportional to the acoustic mode amplitude, the TEM$_{00}$ mode power and the TEM$_{10}$ mode detuning. The signal is normally most easily observed by mixing the acoustic frequency with a local oscillator, combined with a low pass filter, so as to reduce the signal frequency to $<$10 Hz.

As discussed above residual motion causes cavity detuning. The residual mo-
tion amplitude depends on environmental noise, which excites the suspension normal
modes. Most of the time we observe dynamic detuning with a frequency amplitude
of 15 kHz. Even under these circumstances the acoustic mode signal at frequency
≈ 150.2 kHz can normally be clearly observed. Wind forces on the laboratory, micro-
seismic activity and human activity all contribute to degrading the residual motion.
During periods of low seismic noise, residual motion is reduced. For periods of approx-
imately 30 seconds the detuning amplitudes can be less than a few cavity linewidths.
In these short periods of time conditions are suitable for observing three-mode para-
metric instability.

To observe the signature of parametric instability we increased the cavity circu-
lating power to ≈ 30 kW. For periods of time up to 30 seconds, when the dynamic
detuning is low, the acoustic signal can be observed ringing up with time, as shown
in Figure 6.15. In this case the acoustic signal frequency was down converted to
0.91 Hz as discussed above. Observations under optimally tuned low noise conditions
show the acoustic signal growing for ~ 10 seconds.

The amplitude growth is modulated in a more complex manner than the single
modulation frequency model used in Section 6.3.2, due to the presence of several low-
frequency modulation frequencies associated with the angular motion of both test
masses. Beating also occurs, due to the fact that the two test masses have closely
spaced suspension normal modes. The beating causes the detuning amplitude to
vary periodically over time scales ~ 30 seconds. The effective parametric gain based
on observed ring-ups during times of minimum detuning amplitude such as shown in
Figure 6.15 is $R_a \approx 1.45$.

In Figure 6.15 we fit to a double frequency of a single 0.15 Hz suspension mode
to model the dynamic detuning. This gives a reasonable fit to the data, an exact fit
is not possible due to the stochastic nature of the seismic excitation of the normal
modes.
Figure 6.15: The QPD differential output signal at test mass acoustic mode frequency (150.28 kHz). The signal was down-converted to 0.91 Hz by mixing with an local oscillator signal (green points). The signal amplitude ring-up with time is clear. A fit using 2 modulation frequencies is shown for comparison (blue) and the envelope function of this fit is shown in red.

6.4 Summary of Demonstrated Parametric Instability Suppression Techniques

In this chapter I have reported the first two parametric instability suppression techniques to be demonstrated in free space optical cavities.

At LIGO Livingston we demonstrated the first case of electrostatic control of parametric instability. An unstable acoustic mode at 15,538 Hz with a parametric gain of $2.4 \pm 0.8$ was successfully damped to a gain of $0.18 \pm 0.06$, using electrostatic control forces. The damping force required to keep the mode in the damped state was 0.03 nN rms. FEM simulations predicted that the electrostatic drive would need to apply approximately six times this control force to maintain the mode amplitude at the thermally excited level. When Advanced LIGO is operating at high power it is estimated that damping the 15.54 kHz mode group to an effective parametric gain of 0.1 will result in a safety factor $\approx 310$. It is predicted that unstable modes that are most difficult to damp may be controlled as long as amplitudes can be maintained below the safety factor of 10 times their quiescent amplitude.

At the Gingin High Optical Power Facility we created conditions in which three-mode parametric instability can occur in a suspended high-power optical cavity de-
signed to mimic conditions comparable to those in advanced gravitational wave detectors. We observed time-dependent growth of a 150.2 kHz acoustic mode, consistent with a new model of parametric instability for suspended mass optical cavities. The gain in this parametric instability regime is lower than previously expected and modulated by low-frequency residual motion of the suspended cavity. Results are consistent with the new model for instability in which transverse mode frequency fluctuations act to reduce the time averaged parametric gain through dynamic detuning which itself is caused by residual motion in the presence of nm-level mirror figure errors.

Simulations of Advanced LIGO optical cavities indicate that the same phenomenon will act to reduce the risk of parametric instability for the highest parametric gain modes. Mirror imperfections have beneficial effects in this regard. Simulations also indicate that simple methods for reducing parametric gain by thermal modulation of test masses or by low-frequency dithering of the beam positions on the test masses may be effective. Further studies on full-scale detectors to quantify the dynamic detuning and linewidths of transverse modes are needed to quantify these effects.

In this chapter we have studied the suppression of parametric instability, the troublesome feature now confirmed to manifest in Advanced Gravitational wave detectors. However we have also seen the extreme sensitivity of three mode interactions to mirror imperfection. Previously, in Chapter 5, we saw how the sensitive three mode read out of the acoustic modes could be used to learn valuable information about the test masses. In the following chapter will explore how this extreme sensitivity can be used to turn a troublesome feature into a beneficial feature.
Chapter 7

Three Mode Interactions - Precision Monitoring Tools in Gravitational Wave detectors

7.1 Introduction

The extreme sensitivity of three mode interactions to various cavity degrees of freedom noted throughout this thesis indicates that very sensitive tools could be created. In this chapter we explore the potential of such tools. The chapter is based on three papers. “Three Mode Interactions as a Precision Monitoring Tool for Advanced Laser Interferometers” [158] by Ju Li, ... Carl Blair, et al. uses a collection of results from the Gingin High Optical Power facility to argue that three mode interaction monitoring will be a useful tool in gravitational wave detectors (Section 7.2). Section 7.3 is based on the paper “Thermal modelling of Advanced LIGO test masses” [250] by Haoyu Wang, Carl Blair et al. This paper describes experiments that use three mode interaction readout of acoustic modes as test mass thermometers. This precise thermometer allows an improved thermal model of the test mass to be created. Finally Section 7.4 is based on a paper in preparation preliminarily entitled, “Measuring Temperature Distribution Inside Advanced LIGO Test Masses from Elastic Eigen-Frequencies” [51] by Carl Blair and Yuri Levin. This paper extends the test mass thermometer concept. Estimates of the thermal profile within the test mass are made from measurements of many acoustic modes.

In Chapter 3 we reviewed the extensive studies of three mode parametric interactions [270] in an 80 m optical cavity at the Gingin high optical power facility [156], hereafter called Gingin. By thermally changing the radius of curvature of a cavity test mass mirror, using either a thermal compensation plate [92] or a CO₂ laser [236],
Parametric interaction signals have been observed with high signal to noise ratio. The monitoring of three mode interactions (3MI) for the purpose of parametric instability suppression has already been proposed and demonstrated. This is the method of optical feedback control [117, 49]. It requires the monitoring of an acoustically excited high order optical mode, externally generating an out-of-phase optical signal, and re-injecting it to the cavity, so that interference suppresses the offending mode.

Parametric instability has become a topic of intense interest in the gravitational wave community as it threatens to limit detector sensitivity. 3MI monitoring techniques are being applied in advanced gravitational wave detectors for the purpose of avoiding parametric instability. This is by no means the only potential application.

There are several factors that give 3MI monitoring an ability to diagnose multiple degrees of freedom in an interferometer including radii of curvature, inhomogeneous thermal distortion and optical alignments. Below we summarise six ways that 3MI can be used in advanced gravitational wave detectors.

a) Transverse mode metrology. Each acoustic mode signal is the result of an opto-acoustic overlap between an acoustic mode and one or more high order transverse cavity modes. The overlap depends strongly on the high order mode position and relative orientation on the test mass surface, so transverse mode positions and orientations can be estimated.

b) Parametric instability predictions. Because parametric gain depends linearly on input laser power, monitoring 3MI gain at low power can allow prediction of parametric instability at a higher power.

c) Transverse mode monitoring of thermal inhomogeneities. The high order transverse modes sample larger and different areas of the test mass compared with the main TEM$_{00}$ beam. Hence 3MI monitoring can in principle detect thermally induced inhomogeneous radius of curvature variations.

d) Test mass mode identification and thermal state. Each acoustic mode signal is associated with an individual test mass. The acoustic mode frequency dependence on temperature, $\partial \omega_m / \omega_m \approx 10^{-4} K^{-1}$, enables individual modes to be identified simply by observing the tuning when the thermal environment is altered (eg by warming the vacuum envelope of the test mass suspension tank). In principle the thermal state of a test mass could be monitored by this means as explored in Section 7.3 and Section 7.4. Or by comparing 3MI gains of several acoustic modes as discussed in Chapter 8 Section 8.5. This may provide a better error signal for controlling the test mass thermal state.
e) Control parameter estimation. As we show below, \( \approx 700 \) acoustic modes should be able to be monitored simultaneously. These signals are not all independent, because many have common associated transverse modes. However since many acoustic mode shapes can be inferred based on finite element modeling, unique solutions for important interferometer control parameters may be obtainable.

f) Dual recycled interferometer metrology. Parametric gains also depend on other interferometer components such as the beam-splitter, the power and signal recycling mirrors and power recycling cavity compensation plates, which all affect the transverse mode resonance. These components represent additional degrees of freedom that might be able to be measured through multiple 3MI monitoring.

To turn the above concepts into useful tools depends on the aspects of these signals studied in Chapter 5 such as the signal to noise ratio, the stability of measurements and quality of finite element models. Also 3MI monitoring will only be useful if unique solutions can be found. This chapter explores these possibilities and identifies future research directions, some of which will be discussed in Chapter 8.

This is not the first proposal or use of optomechanical interaction with test mass eigenmodes in gravitational wave detectors. In 2012 Fricke [121] demonstrated that the lowest order drumhead modes could be used as differential arm length modulation signals so that the output mode cleaner cavity could be optimised for transmission of the arm cavities fundamental mode. According to Fricke’s description this is a two mode interaction monitoring scheme.

In Section 7.2 we will show that 3MI monitoring is clearly a useful tool for single cavities, by reviewing several experiments at Gingin. In the subsequent two sections particular implementations of 3MI monitoring at Advanced LIGO are studied. In Section 7.3 we describe the use of 3MIs as test mass thermometers. Analysis of this data enabled the refinement of thermal models of Advanced LIGO test masses. While in Section 7.4 this concept is extended: changes in the thermal distribution within the test mass are estimated from changes in test mass eigenfrequencies.

### 7.2 Three Mode Interaction Monitoring at Gingin

In this section three mode interaction (3MI) monitoring of numerous mostly low gain acoustic modes is demonstrated and two ways in which this monitoring can be useful are proposed.

Firstly 3MI can be used to obtain advance warning of parametric instabilities that are likely to occur at higher optical power. This can allow detector commissioners time
to design specific control strategies for specific predicted instabilities. Secondly, the
method can provide error signals which might be useful in controlling alignments and
inhomogeneous temperature gradients that may not otherwise be easily measured.

The proposed method relies on the fact that all acoustic modes are thermally ex-
cited and very well vibration isolated. For this reason the $kT$ mean thermal energy of
each mode can be treated as a calibration signal. Provided mode energy is integrated
over several relaxation times (typically integrating for $10^{-3}$ seconds) it provides a
calibration signal with precision that increases with integration time. If the paramet-
ric gain approaches unity the calibration must be corrected by the parametric mode
amplification which acts to increase the mode temperature by a factor $1/(1 - R_m)$.

The sensitivity of 3MI monitoring arises from the sensitivity of parametric inter-
action gain to thermally induced test mass radius of curvature (RoC) changes, and
test mass alignment. For example, in a single arm cavity, the gain of some interac-
tions changes by up to $100\%$ for a few centimeters change in the $\approx 2000m$ RoC of the
advanced LIGO mirrors, or just a few mm for Gingin mirrors. These changes may be
created by very small changes in mirror heat load.

Advanced LIGO and Virgo already plan an extensive program of wave-front mon-
itoring using Hartmann sensors to detect thermal distortions of test masses [63]. The
3MI monitoring proposed here does not replace the use of Hartmann sensors, nor
other sensors such as beam spot imaging, RF modulation wavefront sensors [123] or
optical levers. At a minimum it can provide early experimental prediction of paramet-
ric instability, but it has the possibility of providing many more benefits as discussed
further below.

This section is based on a variety of results, Subsection 7.2.1 is a review of simula-
tions initiated by Zhao [269] and Gras [136]. Subsection 7.2.2 summarises experimen-
tal results from the south arm cavity, these experiments were mostly performed by Su-
smithan, Zhao, Fan and Fang [274, 268, 236, 235], while Subsection 7.2.3 summarises
results from experimental investigations on the East arm cavity, these experiments
were mostly performed by Qi, Blair and Zhao [272, 115, 119].

\subsection{Theory of Three Mode Interaction Monitoring of Optical Modes}

Figure 7.1(a) shows a snapshot of acoustic modes of a test mass in a 4 km cavity with
a particular RoC (which defines the transverse mode spectrum). The figure shows
the 3MI gain as a function of frequency. Higher gain modes tend to cluster around
the transverse mode offset frequencies. Those points with $R_m > 1$ would normally
need to be suppressed to below unity for stable operation of the interferometer. If the RoC changes or if the laser spot moves on the mirror, the points representing the parametric gain move up or down such as in the examples discussed below.

Typically, for a fused silica test mass for advanced detectors, we expect about 700 modes per test mass for $R_m > 10^{-3}$ as shown in Figure 7.1(b). We choose $R_m \approx 10^{-3}$ as a threshold because modes have easily been observed at this level in experimental studies at Gingin.

Figure 7.2 shows a selection of 3MI gains as a function of RoC for a typical advanced interferometer model. It can be seen that some modes change their gain by several orders of magnitude over a change in RoC of a few meters and some show asymptotic behavior in which the gain can change by 100% in a change in RoC of a few cm. Amongst the large number of modes there are always a few with such high sensitivity to RoC.

It is useful to calculate the magnitude of the wave-front distortions corresponding to the observed RoC sensitivity discussed above. A change in RoC translates to a fractional change in wave-front according to the relation given in Equation 7.1.

$$\Delta d = \frac{r^2}{2R_1^2} \Delta R_1,$$

where $R_1$ is the mirror RoC, and $r$ is the effective radius of the test mass and $d$ is the depth of the mirror deformation. Assuming $\Delta R_1 \approx 2$ mm can be resolved, $\Delta d \approx 2 \times 10^{-13}$ m. For $\lambda = 1064$ nm radiation, this means that 3MI has the capability
of monitoring wavefront changes $\approx 2 \times 10^{-7}\lambda$. In this sense 3MI offers monitoring at an unprecedented precision.

### 7.2.2 Sensitivity to CO$_2$ Laser Tuning of Transverse Mode Frequency

This subsection describes how three-mode gain for single cavity interactions can be monitored by observing beat signals in the transmitted or reflected light due to thermal excitation of acoustic modes. It is a review of several previously published results in the Gingin South Arm [274, 268, 236, 235]. In addition it is described how 3MI signals can be used at low optical power to predict parametric instabilities that could occur at higher power, and how at any power, the observed mode amplitudes can be used to control the interferometer operating point against slow environmental perturbations. We summarize data on an 80 m cavity that demonstrates these effects.

3MI have been investigated in both of the $\approx 80\text{ m}$ optical cavities at the Gingin facility that was introduced in Chapter 1 Section 1.2.4. The systems are rather simple. A cavity with high g factor is locked$^1$ using about 2 W of TEM$_{00}$ injected 1064 nm light. A quadrant photodetector monitors the transmitted beam. A CO$_2$ laser is used to apply variable heating to the centre of the end test mass, thereby creating a temperature gradient that deforms the mirror.

$^1$lock - bringing a cavity into its controlled operational state, in this case on resonance
The south arm of the Gingin facility is a cavity with sapphire test masses while the east arm cavity is composed of fused silica test masses. Both cavities have allowed observation of 3MIs. Because the test masses in both cavities are relatively small, the acoustic mode density is low and instead of seeing a large number of modes simultaneously, modes are observed only when the CO$_2$ laser power has correctly tuned the transverse mode frequency to allow a particular mode to be observed, as we saw in Chapter 4.

Modes with parametric gain $\sim 10^{-3}$ are relatively easily observed. Normally monitoring is done using a quadrant photodetector which is particularly sensitive to the TEM$_{01}$ and TEM$_{10}$ modes as described in Chapter 2 Section 2.3.5. Simulations predict that parametric interactions will be strong for optical modes up to order nine. As such generally it would be advantageous to use at least a 9-element photodetector with the ability to make linear combinations of channels to achieve maximum sensitivity to a particular optical mode shape.

In the south arm sapphire test mass cavity, Zhao et al [267] first observed 3MI interactions as described in Chapter 3 Section 3.2.7. The parametric gain $R_m \approx 0.01$ was measured when the RoC of the end test mass was appropriately thermally tuned [268]. The thermal peak had an amplitude of a few parts in $10^{-15}$ m (corresponding to $kT$ of energy), and the noise floor was about $10^{-17}$ m$/\sqrt{\text{Hz}}$. From this observation we can predict from Equation 3.1 on page 89 that if the input laser power was increased from 2 W to 200 W the mode would reach the instability threshold, with $R\approx 1$.

In Chapter 3 Section 3.2.6 it was described how changing the RoC of the test masses changes the transverse mode spacing. The sapphire test masses in the south arm cavity were tuned in such a manner using CO$_2$ laser heating [235]. When the optical beat frequency matched the acoustic mode frequency a thermal peak was observed in the arm transmission QPDs. The observed thermal peak as a function of heating power is shown in Figure 7.3. A few hundred milliwatts change in heating power was sufficient to sweep across the entire resonant peak. Typically the 3MI gain changes by a factor 3 as the heating power changes by $\approx 100$ mW.

Another observation was that the position and size of the CO$_2$ laser spot significantly alters the 3MI signal. In this case it is assumed that inhomogeneous changes in the shape of the mirror are being observed, which distort both the transverse mode shape and the mode frequency. However changes in overlap factor also contribute significantly due to the small Gingin test masses.
7.2.3 Tuning of Transverse Mode Frequency from Absorption of Circulating Beam

In this subsection experiments that relate to the temperature of test masses in the Gingin East Arm cavity are summarised. The results from these experiments demonstrate the high sensitivity of 3MI to the thermal state of the optic. This sensitivity is demonstrated as both; the sensitivity of the high order transverse optical mode frequency to thermal distorting of the test mass RoC and as the sensitivity of eigenfrequencies of the test mass to the test mass temperature. The east arm cavity, introduced in Chapter 1 Section 1.2.4, was designed specifically to study 3MI. Fang Qi and I [115] worked extensively characterising this cavity and doing the first 3MI experiments on the East Arm cavity that culminated the first observations of parametric instability in a long optical cavity reported in Chapter 6 Section 6.3. The cavity is a near concentric cavity similar to an Advanced LIGO arm cavity, with fused silica test masses 100 mm diameter × 50 mm thickness in a cavity of length 73.9 m, and radii of curvature of 37.4 m and 37.5 m. The measured cavity finesse is 14500 ± 300, and the cavity g factor is ≈ 0.98, leading to a transverse mode spacing (calculated with Equation 2.21 Chapter 2 page 68) of around ≈ 100 kHz. This means that test mass acoustic modes at 100 kHz can scatter the cavity fundamental mode.
into the cavity first order mode while acoustic modes near 200 kHz will scatter the fundamental mode into the second order mode, etc.

Much like the results from the sapphire test masses thermal tuning using \( \sim 1 \text{ W} \) CO\(_2\) laser heating can decrease the g factor to 0.95, increasing the mode spacing to about \( \approx 150 \text{ kHz} \).

Without the CO\(_2\) laser, increasing the cavity power from 500 W to 5 kW in the fused silica east arm cavity resulted in tuning of 3MIs similar to the CO\(_2\) laser tuning result in the south arm described above. The carrier laser heating causes self-induced thermal lensing, thereby shifting the TEM\(_{01}\) mode frequency which determines the frequency at which acoustic modes are resonant. At low power we can observe a 94 kHz acoustic mode, but as the cavity power increases this ceases to be detectable, while other modes appear at \( \approx 105 \text{ kHz} \) and then 112 kHz, as shown in Figure 7.4.

![Figure 7.4: Three acoustic modes of the fused silica test masses with Q-factor \( \approx 4 \times 10^4 \), observed by 3MIs. In this case the cavity transverse mode is tuned by optical absorption in the test masses, causing different acoustic modes to be resonant.](image)

Without CO\(_2\) laser heating and at a cavity power level of \( \approx 3 \text{ kW} \), it was observed that a few millimeters change in position of the laser spot on a test mass caused easily detectable changes in the 3MI signal demonstrating a high sensitivity of the 3MI signal to laser beam alignments. This sensitivity is presumably either due to changes in overlap parameter discussed in Chapter 5 Section 5.4 or change in effective RoC discussed in Chapter 6 Section 6.3.2. These effects are summarised.

1. Changes in beam position can cause large reductions in the overlap parameter of Equation 3.16 page 99, so we may expect some modes to fall below the noise floor while different acoustic modes become visible.
2. The mode spacing and hence 3MI signal change with beam spot position on the test mass due to figure errors in the mirror changing the effective RoC experienced by the beam as it moves across the test mass. Changes in RoC change the g factor and mode spacing as described in Chapter 6 Section 6.3.4. These figure errors are partly manufacturing tolerance but may also be caused by inhomogeneous optical absorption.

Thus it is possible to use relative mode amplitudes to diagnose changes of the optical cavity conditions such as g factor, spot position and mirror figure errors.

Since the acoustic modes have moderately high Q-factors, the acoustic mode frequency is easily measured. We have observed strong thermal tuning of the acoustic mode frequencies in our fused silica test masses due to the temperature dependence of Youngs Modulus. As discussed in Chapter 5 Section 5.2.1.1 the frequency tuning coefficient \( \frac{\partial f}{(f \partial T)} \) in fused silica is anomalous and positive [232], and has a magnitude \( \approx 10^{-4} \) per degree. The acoustic mode frequencies increase with time due to the increasing average temperature of the test mass as it is warmed from absorption of the circulating optical power. Correlating the frequency change of eigenmodes with CO\(_2\) laser heating provides a means of identifying the test mass associated with a particular acoustic mode in a very similar manner to eigenmode and test mass identification discussed in detail in Chapter 5 Section 5.2.3 where ring heaters and ambient temperature fluctuations created eigenfrequency fluctuations.

7.2.4 Summary of Gingin Studies

In this section 3MIs normally present in large scale gravitational wave detector arm cavities are demonstrated to have a sensitivity that would allow them to be used as precision monitoring tools sensitive to small variations in mirror radii of curvature, in beam spot positions and in mirror temperatures. The high sensitivity of acoustic mode readout to small changes in test mass absorbed power has been demonstrated by both modelling and observations in two optical cavities. The method could be useful for providing ancillary data to enable stable operation of high optical power interferometers in the presence of slow thermal fluctuations.

Experimental results have been presented that demonstrate the ability for low power measurements to predict modes likely to become unstable at high optical power in a single arm cavity. Both alignment and thermal perturbations in a single cavity are easily observed. In a more complex advanced interferometer the parametric gain depends on the Guoy phase associated with the entire coupled cavity configuration. The idealised modelling presented (that neglects the thermal compensation
masses and optical inhomogeneities) shows that the method carries over to full dual recycling interferometers, but the large number of degrees of freedom means that disentangling the data will be a complex task. Some preliminary attempts to diagonalise cavity mode spacing and test mass temperature degrees of freedom were presented in Chapter 5, more detailed studies of test mass temperature will be presented in the remaining section of this chapter. Subsequently in Chapter 8 Sections 8.5 and 8.5 plans for further 3MI monitoring will be discussed.

Assuming that the noise level for sensing gain fluctuations $\Delta R_m/R_m \approx 10\%$, the most steeply varying modes in our simulations allow RoC control to about 1 ppm, corresponding to about 2 mm RoC precision, which corresponds to optical path length control $\approx 2 \times 10^{-13}$ m and wave front errors $\approx 2 \times 10^{-7}$. The multiple channels of data that are predicted to be available in cavities with large test masses correspond to hundreds of acoustic modes which sample different test masses, their thermal distortion and their alignment. We suggest that monitoring these channels can allow improved control of high power interferometers, helping to predict and prevent parametric instability, and also possibly helping to minimize glitches that can occur due to the interaction of varying diameter beams with loss points on the mirrors.

However it is important to note that complete diagonalization (to allow extraction of individual error parameters in an advanced interferometer) would require a sufficient number of independent modes. This point is uncertain, so it is not clear how many degrees of freedom could actually be controlled. Clearly experimental testing of these ideas is required. Because there are so many acoustic mode channels there will be a need to determine which channels to use and how to combine them to create useful tools for separate purposes such as individual test mass alignment and thermal distortion probes.

In the following sections we will explore two detailed investigations into three mode monitoring of the Advanced LIGO test masses.
7.3 Three Mode Interaction for Monitoring of Test Mass Temperature

As we have seen in the preceding section three mode interactions (3MI) create high sensitivity transducer for test mass elastic eigenmodes. In Advanced LIGO, several test mass eigenmodes are visible when parametrically stable in the arm transmission 3MI readout. In the preceding section we saw that important information may be obtained from 3MI monitoring.

This section focuses on the thermal dependence of the test mass eigenfrequencies. This property allows 3MI monitoring of test mass eigenmodes to be used as a test mass thermometer. These thermal probes are then used to improve the thermal model of the test mass allowing better prediction of the long term thermal behaviour.

Prior to this investigation a technique to continuously monitor test mass temperatures developed at VIRGO [216] had been applied at initial LIGO. Short thermal transients of test mass resonant mode frequencies were used to estimate the absorption coefficients of the coatings [174].

In developing strategies for damping parametric instability in Section 6.2 knowledge of the total change in mode frequency was required. Measurements were taken of the cavity absorption heating coefficient and ring heater heating coefficients and compared to a finite element model of the test mass. We observed that the thermal transient over long periods does not fit well with the simple model that had been assumed to that point. Haoyu Wang then did an extensive finite element modelling investigation to explore the parameter space that could be responsible for this discrepancy.

In this section a more complex model is developed to explain the long term thermal behavior of the test masses. The model provides a coating absorption estimate of 1.5±0.2 ppm and estimates that 0.8±0.5 ppm of the circulating light is scattered on to the ring heater.

7.3.1 Motivation for Improving the Thermal Model

The benefits of having a good thermal model of the test mass are far broader than just parametric instability suppression.

The thermal transient in the test masses impacts on the performance of the interferometer in three ways:

1. Thermal lensing [233, 91] is caused by a change of refractive index via the thermo-optic effect. This induces a change in the optical path length within optical
components resulting in aberration in the power recycling and signal recycling cavities. This effect is mitigated by laser heating an optic known as the compensation plate [267].

2. Deformations of high-reflectivity coatings of test masses are caused by thermal expansion. The first order deformation is a change of the radius of curvature (ROC) of the mirror [147], which impacts on mode matching [221]. A ring heater surrounding the mirror is used to compensate for this effect [219].

3. A change in the tuning conditions for parametric instabilities [60, 110] is caused by two factors. First, the change in ROC mentioned above will shift the frequency of the transverse optical modes (TEM) resonant in the cavity, changing the mode spacing between the fundamental mode and TEM modes. Second, the Young’s modulus of mirror substrate has a small positive thermal dependence which results in an increase in the mechanical mode frequencies as the mirror warms. Parametric instabilities are most severe when the mechanical mode frequency equals the frequency spacing between the fundamental mode and TEM modes, this condition will be altered by the thermal transient.

Scattering from the test mass coating has also been a major concern [220], as any scattered light that returns into the main beam introduces phase noise, reducing detector sensitivity. As we will see the temperature of the test mass is also sensitive to surrounding objects. In conjunction with a temperature sensor in a component called the ring heater the scattered light can be estimated from this thermal model.

Increases in test mass coating absorption are indicative of degradation or contamination of the test mass high reflectivity coatings. Monitoring of the test mass thermal transient with verification against this model could be used to indicate to advanced gravitational wave detector operators the state of degradation or contamination.

All these issues need to be addressed in order to minimise their impact on detector sensitivity. The power of the circulating beam in arm cavities at Advanced LIGO during Observation Run 1 (2015) was 100 kW and thermal effects were managed with various mitigation strategies [63]. However, these effects will become more severe in future observation runs when the detectors will employ higher laser power to reach design sensitivity. Many of the current strategies will therefore require further attention.

To aid the design and development of such strategies, a good thermal model of the test masses is required. Here the thermal model includes optical absorption, thermal conductivity and surface emissivity in addition to thermal coupling to the surroundings.
7.3.2 Elastic Eigenfrequency Temperature Probes

In Section 5.2.1.1 the thermal dependence of fused silica elastic moduli was shown to result in a test mass eigenfrequency temperature dependence. The very high Q factor of the eigenmodes allows very accurate measurements of the eigenfrequencies. As such the test mass eigenfrequencies can be used as very accurate test mass thermometers. We saw eigenfrequencies depend on the mirror dimensions and on two material properties that have a temperature dependence: the Young’s modulus and the Poisson ratio. The eigen-frequencies of a cylinder can be expressed analytically as described in Section 2.2.1

\[ \omega_m = \beta_m \sqrt{\frac{E}{\rho(1+\nu)}}, \]  

(7.2)

where \( \beta_m \) is a parameter encompassing several aspects of cylinder dimensions and has restricted values, \( E \) is the Young’s Modulus, \( \nu \) is the Poisson ratio and \( \rho \) is the density of the material.

In Section 5.2.1.1 it was shown and in next section it will be shown in more detail that changes in eigenfrequencies due to the thermal expansion of the substrate are negligible when compared to the change due to the temperature dependence of \( E \) and \( \nu \). We also note that the change due to the Young’s modulus dominates over the change due to the Poisson ratio, as discussed in Section 5.2. Most materials soften with increasing temperature, but the Young’s modulus of fused silica increases in the \((-200, 1100)\) °C interval [232, 231] and the rate of change at 17 °C is 11.5 GPa/K, so eigen-frequencies increase as the test mass becomes warmer. Under the assumption of homogeneous changes in temperature all eigenmodes change at the same relative rate. Inhomogeneous changes in temperature will be studied in Section 7.4.

Mechanical mode frequencies of test masses are coupled to the cavities’ transmission ports when arm cavities are locked - controlled to a state where their fundamental optical mode is resonant. Prior to Observation Run 1 sensing noise limited sensitivity to test mass resonant modes in the 3MI readout. The limiting parametric gain for detection of these modes is well above the prediction of \( R_m > 10^{-3} \) from Section 7.2. Even so when there is sufficient signal amplitude and interferometer operating times are of adequate duration the full thermal transient in the test mass resonant frequencies may be observed.

We experimentally track the frequency of the 15.5kHz mode on the ETMY as read out through a photodiode at the cavity’s output port [54, 50]. We focus on this mode because it has significant sensing gain and is known for producing the first
observed parametric instability [113]. Figure 7.5 shows the mode’s frequency shift over a 36 hour lock on 8th October 2015 (blue curve). The figure also shows the ambient temperature data as measured by a sensor inside the vacuum tank. The green curve represents the raw sensor data and the red curve represents the data after a single pole low pass filter (LPF) is applied with a time constant of 7.2 hours determined by our finite element model. We can see about 25 hours into the lock that the frequency change is completely determined by the ambient temperature fluctuation.

Figure 7.5: Frequency shift of the 15540 Hz eigenmode and ambient temperature fluctuation during a 36 hour lock of LIGO Livingston’s ETMY.

7.3.3 Heat Transfer Model of Advanced LIGO Test Masses

7.3.3.1 Thermal coupling

We use COMSOL Multiphysics’s [77] Heat Transfer module to build a model of the test mass. To improve over previous attempts to model the thermal transient of the mirror we include the radiative heat transfer between the mirror and elements in proximity of the mirror.

Figure 7.6 depicts different mechanisms by which heat is transferred in the system. When the arm cavity is locked ≈100 kW of laser power is incident on the test masses. The absorption of the high reflectivity coatings, in the order of 1 ppm [4], result in a portion of the light being absorbed and converted into heat. In addition, scattered light from the beam will illuminate elements surrounding the mirror, part of which will be absorbed and converted into heat by those elements.

Some of this absorbed heat will then radiatively couple to surrounding elements introducing longer time constants to the test mass thermal transient. The surrounding elements used in this model include the reaction mass and the ring heater. All other
Figure 7.6: A heat transfer model between end test mass (ETM), reaction mass (RM), Ring heater (RH), ambient isothermal bath and an extra term encompassing complex structures surrounding the test mass that could not be modeled in detail. Red arrows represent the energy from the intra-cavity beam or scattering thereof. Green arrows represent the radiative heat couplings between different elements and black arrows radiative heat couplings with the ambient isothermal bath.

nearby objects are simplified to a single empirical element which we call the ‘extra term’.

Finally, all elements will exchange heat with the ambient via radiation. The ambient temperature is recorded by a sensor in the vacuum tank and inserted into the model. The amount of scattered light hitting the RM as well as the radiative coupling between the RM and the RH are negligible and have been omitted from Figure 7.6 for simplicity.

7.3.3.2 Ambient temperature and estimates of circulating laser absorption

We can assume that each of heat transfer mechanisms discussed in the previous section has a linear effect on the eigenfrequencies of the test mass. This constitutes a good approximation since each process only changes the overall temperature of the test mass by approximately $0.1^\circ C$. The mechanical mode frequency of the mirror can thus be expressed as:

$$f(t) = f_{\text{laser}}(t) + f_{\text{surroundings}}(t) + f_{\text{ambient}}(t)$$  \hspace{1cm} (7.3)

where each term on the right hand side corresponds to contributions from the laser impinging on the ETM, the radiative heat transfer with surroundings (RH, RM and extra term), and the radiative heat transfer with the ambient respectively.
We examine mode frequency data from 5 locks where the cavity was in a clean thermal state. We define a clean state as state where light was not circulating in the cavity for at least 30 hours prior to the lock. This allows for the test mass, that has a thermal time constant in the order of 10 hours to come into thermal equilibrium with surrounding elements. When lock is achieved, the circulating laser power steps from 0 to 100 kW in $\sim 100$ sec and then fluctuates within 5% of 100 kW.

By looking only at the first three hours of each lock we rule out slow radiation terms and concentrate only on $f_{\text{laser}}(t)$. For each lock, we remove the influence of the ambient temperature fluctuation. This is done by low-pass filtering the ambient temperature data (like in Figure 7.5) and then removing the estimated contribution $f_{\text{ambient}}$ from the measured eigenfrequency. The corrected data estimates the mode frequency change due only to the laser power. This data is then scaled by the inverse of the cavity transmitted power for each lock. By doing this the corrected mechanical mode frequency shifts should be equivalent for all locks given that the coating absorption of the test mass doesn’t change.

7.3.3.3 Finite element model

The Advanced LIGO ETM is a cylinder with a 170 mm radius and 200 mm depth made of Heraeus Suprasil 3001 fused silica. When the 100 kW laser beam impinges on it, the high-reflectivity coating of the ETM will absorb some energy and convert it into heat. In the model, a heat source boundary condition with a Gaussian profile is used to simulate the beam spot on the mirror surface. The intensity of a Gaussian beam with beam radius $w$ and power $P_0$ at a distance $r$ from the beam axis is

$$I(r) = \frac{2P_0}{\pi w^2} \exp\left(-\frac{2r^2}{w^2}\right) \text{ W/m}^2,$$

where $w = 6.2$ cm and $P_0 = 100$ kW for the ETM. The energy absorbed by the coating is given by $\alpha_{\text{coat}} I(r)$ where $\alpha_{\text{coat}}$ is the absorption coefficient of the coating. For a laser power of 100 kW, a coating absorption of 1 ppm corresponds to a total absorbed energy of 0.1 W. The high reflectivity and low transmission of the ETM coating result in a negligible bulk absorption.

The reaction mass (RM) is another fused silica cylinder with the same radius as the ETM, but a depth of only 130 mm. It’s located coaxially 5 mm away from the rear surface of the ETM. Its main use is for longitudinal actuation of the test mass via electrostatic force.
The ring heater (RH) consists of a glass ring surrounding the ETM. The ring is wrapped with Nichrome wire and a U shaped aluminium shield surrounds it. The inner surface of the aluminium shield is gold-plated for reflecting infrared. The purpose of the ring heater is to create a thermal gradient to compensate the change in RoC induced by the laser heating on the surface of the optic. During Observation Run 1 the power fed to the RH was small and fixed, changing the overall ETM temperature by less than 1°C. Since the ETM-RH system was in thermal equilibrium we don’t take this gradient into account, as it has negligible effect on ETM’s temperature fluctuations. The RH is, however, subject to light scattered by the ITM and ETM. The thermal sensor inside the RH shield registers a quick rise in temperature in the first five hours of each lock.

The short term thermal behaviour of the system is well understood and easy to predict reliably. The reliability of long-term models, however, depends on how accurately test mass surroundings are modelled. Simulating all surrounding elements in detail quickly becomes an impossible task as the computation time rises exponentially with each new radiative surface. To approximate effects of all elements in the vicinity of the test mass we introduce an extra term, an aluminium annulus that surrounds the ETM. This extra term couples to the ETM radiatively and also receives scattered light from the cavity. Material properties of fused silica and aluminium used in the model are listed in Table 7.1.

![3D view of the geometry of the ETM, RH, RM and extra term as modeled in COMSOL (225° slice).](image)

The only input of the model is the laser power step function that takes values of 0 and 100 kW when the cavity is locked. There are 5 free parameters to tune: the energy absorbed by the ETM coating directly from the circulating beam, $\alpha_{coat}$, the scattered energy absorbed by the ring heater, $\alpha_{RH}$, and by the extra term, $\alpha_{ext}$, the
Table 7.1: Material properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Fused silica</th>
<th>Aluminium</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>2203</td>
<td>2700</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Young’s modulus at 16°C</td>
<td>$E$</td>
<td>73×10$^9$</td>
<td>–</td>
<td>Pa</td>
</tr>
<tr>
<td>dE/dT at 16°C</td>
<td>$dE/dT$</td>
<td>11.5×10$^6$</td>
<td>–</td>
<td>Pa/K</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>$\nu$</td>
<td>0.17</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>$k$</td>
<td>1.38</td>
<td>160</td>
<td>W/(m·K)</td>
</tr>
<tr>
<td>Thermal expansion coefficient</td>
<td>$\alpha$</td>
<td>5.5×10$^{-7}$</td>
<td>–</td>
<td>1/K</td>
</tr>
<tr>
<td>Heat capacity</td>
<td>$C_p$</td>
<td>740</td>
<td>900</td>
<td>J/(kg·K)</td>
</tr>
<tr>
<td>Relative permittivity</td>
<td>$\epsilon$</td>
<td>3.8</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>Surface emissivity</td>
<td>$\epsilon_e$</td>
<td>0.93</td>
<td>0.1</td>
<td>–</td>
</tr>
</tbody>
</table>

thickness of the extra term, $d_{\text{ext}}$, and its position, $p_{\text{ext}}$. As $\alpha_{\text{ext}}$, $d_{\text{ext}}$ and $p_{\text{ext}}$ have a complex association, we just find an empirical value for each of them. Outputs of the model are the frequency shift of the 15540 Hz mechanical mode and the ring heater temperature change as a function of time. The output data is compared with the corrected measured data. Figure 7.8 provides input/output diagram for this model.

Figure 7.8: Inputs, outputs and free parameters of the model (described in the text)

7.3.4 Tuning the Model

The FEM model calculates eigenfrequencies of the ETM over time. These results are then compared to measured data. Tuning the model required four steps:

1. The ambient contribution to the test mass temperature is removed from the measured data using the model displayed in Figure 7.5.

2. The laser heating term is estimated from the first three hours of the lock where the dominant term is coating absorption. This gives an estimate of the coating absorption coefficient, $\alpha_{\text{coat}}$.

3. The ring heater’s scattered light heating term is also estimated from the first three hours of the lock. The temperature recorded by the ring heater sensor is used to estimate the amount of scattered light absorbed by this element $\alpha_{\text{RH}}$.
4. Finally parameters of the extra heating term are obtained. Setting $\alpha_{\text{coat}}$ and $\alpha_{RH}$ from the previous two steps the extra term parameters are chosen to minimise the residual between the measured data and the long-term simulation of the test mass, including all heat transfer effects depicted in Figure 7.6. The parameters adjusted are dimensions, scattered light absorption and the proximity of the ring to the test mass.

The first point was described in section 7.3.2. The following three points are described in detail in the following subsections.

7.3.4.1 Coating absorption measurement

Acoustic mode frequency data from the first three hours of several locks was used to estimate the energy absorbed by the Livingston ETMY coating. We chose 5 lock periods from April 12, July 26, October 27, November 18 and December 21, 2015. These lock stretches were chosen because in each case there was no light in arm cavities and no anomalies in the laboratory temperature for at least 30 hours prior, justifying the assumption that the test mass was in thermal equilibrium with its surroundings at the start of the lock. The raw data from these locks is shown in the left plot of Figure 7.9. The right plot of Figure 7.9 has the ambient temperature

Figure 7.9: Mode frequency shift data of first 3 hours. The raw data shows a large disagreement due to parametric differences between locks. After the data is calibrated, the trends are in agreement with modelled coating absorptions of 1.3 to 1.7 ppm.

effect removed and a lesser effect of scaling for small differences in laser power in the arm cavities is accounted for. The simulated (Black) curves display the effect of coating absorption only for 130 and 170 mW of laser power absorbed. Slow radiative effects start affecting the test mass temperature and hence eigenfrequency after about 1.5 hours. All 5 locks display a very similar trend but don’t exactly overlap, there
are many possible explanations such as the simple ambient temperature to test mass
temperature transfer function or unaccounted for heating terms such as in vacuum
electronics.

The simulated curves correspond to coating absorption coefficients $\alpha_{\text{coat}}$ between
1.3 and 1.7 ppm. This is much larger than the absorption measured prior to installation of 0.3 ppm [4]. This may indicate contamination.

For comparison another method that has been used to estimate the ETMY absorp-
tion is introduced. Measuring the thermal lensing with the Hartmann wavefront
sensor [63], the change in optical path length from the thermo-optic effect has been
used to estimate the coating absorption of the ETM in the Y arm, between 1.3 to 2.2
ppm.

7.3.4.2 Scattered light

The scattered light absorbed by the ring heater completely dominates the initial
thermal transient of this element. In our model this is described by a heat source
on the cavity facing surface of the ring heater shield. The data from the ring heater
temperature sensor is compared to the model absorption estimates in Figure 7.10. We
obtain estimates for $\alpha_{\text{RH}}$ of 35 to 50 ppb of the contained power in the cavity. The
absorption of aluminium at 1064nm varies greatly between 4% and 12% depending
on the surface roughness and the thickness of the oxide layer [39]. As a result, the
estimated scattered light incident on the ring heater could range from 0.3 to 1.25 ppm
of the 100 kW beam. This is less than the current total scattering loss of the Y-arm
cavity, which is about 20 ppm [262].

![Figure 7.10: Ring heater temperature data of first 3 hours. The corrected data agrees with the models for 35 ppb and 50 ppb of ring heater absorption.](image)
7.3.4.3 Long-term simulation

In this section acoustic mode frequency data from April 12, 2015, spanning 31 hours, is used to tune the parameters of the extra term. Our long-term model includes the ETM, RM, RH and the extra term depicted in Figure 7.6. The coating absorption is set to the medium value of the estimated range which is 1.5 ppm, as derived in Section 7.3.4.1. The scattered light absorbed by the ring heater is chosen to be 35 ppb of the circulating laser power, because this value agrees better with the April data, as derived in section 7.3.4.2. Finally the ambient heating term is generated using the method displayed in Figure 7.5.

![Tuning the model for data on April 12](image)

Figure 7.11: Long-term comparison of the April data (black curves) with both models (red curves). The linearised contributions from the various components can be examined separately.

In Figure 7.11 the contributions to the model are displayed as a linear decomposition based on Equation 7.3. The top panel shows a green dashed line representing the coating absorption component of the 15.5 kHz mode frequency change. The ring heater scattered light heating component is shown dashed blue, it turns out to be very small. The red dashed line is the ambient temperature contribution to the mode frequency change. The sum of these three contributions determines the simple model
shown in solid blue. In the improved model, the parameters of the extra term ($d_{\text{ext}}$, $p_{\text{ext}}$ and $\alpha_{\text{ext}}$) were adjusted based on visual comparison and the resulting error was reduced to less than 5%.

The bottom panel of Figure 7.11 shows the ring heater measured temperature data (black curve) and the component heating terms from our model are shown. The green dashed line is the ring heater scattered light absorption term, the blue dashed line is the radiation influence from the ETM and the dashed red is the ambient contribution. The solid blue and red lines are the sum of component terms corresponding to results of the simple model and improved model respectively.

We can see in both sub-plots that the simple model deviates from the data after about 2 hours. The slow effect indicates radiative coupling, resulting in a maximum error of 40% in frequency change. The extra term geometry parameters, $p_{\text{ext}}$ and $d_{\text{ext}}$, were chosen to match the time constant of the residual based on visual comparison. The extra term absorbed power parameter, $\alpha_{\text{ext}}$, was subsequently tuned to minimise the residual between the model output and measured April data. Also $\alpha_{\text{ext}}$ was limited to be consistent with the range of small angle scattering loss from ITMY ($<10$ ppm) which should dominate heating of structures surrounding the test mass.
7.3.5 Results

Figure 7.12 shows results of applying the model developed with the April data set to a data-set starting December 21, 2015. The first lock lasted 17 hours and was followed by 3 locks each of about 10 hours. The black curve in Figure 7.12 is the raw data and the red curve is the prediction from the model developed in the previous section.

![Testing the model for data on December 21](image)

Figure 7.12: Testing the tuned model with the December lock data. The black line is the raw data, the red curve is the improved model, and the orange line in the lower plot represents the ambient contribution to the ring heater temperature.

There is good agreement between the model and the measured data in the first lock period, where the deviation between model and experiment is only 3%. However, there is significant deviation in subsequent locks. This discrepancy is also observed in the long-term ring heater temperature comparison. The ambient temperature sensor does not show such a long-term trend. As shown in the lower panel of Figure 7.12, the measured ring heater temperature (black curve) at the beginning of the 4th lock was lower than at the start of the 1st lock, from the ambient temperature sensor we expect similar ambient thermal conditions. The ring heater data and acoustic mode frequency data indicate a decrease in ambient temperature. There are many possible explanations for such a discrepancy such as a lock dependent heating terms other than the laser or another heating term, an inadequate model of the ambient temperature transfer function to the test mass or possibly corrupt data. There is
some suspicion that the temperature sensor we were using may have been faulty as it was associated with increased electronic noise noted by Matinov et al in Reference [189] Figure 14. Data stretches from long locks starting from a clean thermal state are rare, and further tests of this model will require data from Advanced LIGO’s second observation run.

Despite the deviation between model and experiment observed in Figure 7.12, we demonstrate a factor of 6 improvement in the error in the temperature estimate over the first 17 hour lock. We expect an improvement in the model accuracy given reliable ambient temperature data and following investigations of long-term features of the transfer function from ambient temperature to test mass temperature.

7.3.6 Summary and Further Work

Mechanical mode frequencies measured through three mode interactions can be used as very accurate thermometers. These accurate thermometers can then be used to infer thermal properties of the test masses. We have provided an estimate of LIGO Livingstons ETMY coating absorption of $1.5 \pm 0.2$ ppm. The scattered light incident on the ring heater is estimated to be $0.8 \pm 0.5$ ppm.

We have described a thermal model that reduces the error in long-term estimates of the test mass temperature by a factor of 6. The accuracy of the thermal model of the test mass was improved with the addition of an extra term to account for radiative contributions from objects in the vicinity of the test mass. The model was tuned to a data set from April, matching the ETMY measured temperature shift within 5%. The model was then tested on a data set from December. The agreement is good over the first 17 hour lock but deviates significantly in subsequent locks. We expect to improve the model further when reliable ambient temperature sensors become available.

These measurements on the average test mass temperature assume homogeneous temperature variations whereas in fact there are thermal gradients in the test mass. In the following section the effect of these gradients in temperature are explored and an exciting new proposition is made.
7.4 Estimating Thermal Profile Test Mass Eigen-Frequencies

In the investigations in the previous section it was observed that a single test mass’s resonant mechanical modes do not all change by the same relative frequency as would be expected for a uniform change in temperature applied to a test mass whose mode frequencies obey Equation 7.2. This observation leads to an exciting proposition. Can the changes in a set of mode frequencies reveal the thermal profile in the test mass? Discussion with Yuri Levin led to his development of a formalism to transform test mass temperature distributions into test mass eigenfrequency vectors and vice versa. Levin’s formalism and investigations into useful tools developed from this formalism are encompassed in a paper in preparation entitled Measuring Temperature Distribution Inside Advanced LIGO Test Masses from Elastic Eigen-Frequencies [51] by Carl Blair and Yuri Levin are presented in this section.

7.4.1 Motivation to Measure the Test Mass Thermal Profile

Knowledge of the test mass thermal profile is critical to the adaptive optics of the dual recycled Michelson interferometer that is the core of advanced gravitational wave detectors. This is because the thermal state of the test mass influences several aspects of the instrument including the two listed here:

1. Thermal actuation is used to control the surface profile of the test masses for tuning the arm cavity transverse mode frequencies.
2. Thermal actuation is used to control the mode matching between the arm cavities and the recycling cavities.

Therefore it is useful for LIGO scientists and engineers to be able to monitor the 3-dimensional temperature field inside each test mass.

The current monitoring system is a Hartmann wavefront sensor [63] that measures the wavefront distortion experienced by a probe beam that passes through the test mass and reaction mass / compensation plate twice. From this the radius of curvature of the test mass and single pass wave-front distortion are inferred. From these the transverse mode spacing estimate and corner mode matching are inferred respectively. In this section we propose that an auxiliary sensor of the test mass thermal profile can be achieved by monitoring the frequencies of multiple resonant modes of the test masses. These auxiliary sensors may be used in conjunction with the Hartmann wavefront sensor to improve current measurements or to measure new degrees of freedom.
The modes also play a major role as a driver of parametric instability that needs to be well-controlled at high circulating power. The formalism developed here also gives an improved understanding of the behaviour of test mass eigenfrequencies as the test mass changes its thermal state during thermal transients caused by optical power build up in the interferometer or changes in thermal actuators that control adaptive optics in Advanced LIGO.

Some effort has already been spent designing and implementing a system that monitors the small changes in mode frequencies as demonstrated in the previous section. In what follows we argue that it is in principle straightforward to use these measurements to also monitor in real time the 3-dimensional temperature distribution inside the test mass and, by extension, the thermal distortion of the mirror surfaces.

The section is arranged as follows. In Subsection 7.4.2 we develop the mathematical formalism for computing the changes in mode frequencies and for solving the inverse problem - estimating the change in temperature field from a change in a set of eigenfrequencies. In Subsection 7.4.3 we present a COMSOL model that is used to test the formalism. In Subsection 7.4.4 the forwards problem of frequency estimation from a thermal profile is tested against a COMSOL eigenfrequency analysis. In Subsection 7.4.5 the problem of symmetry in the test mass is highlighted. In Subsection 7.4.6 the inverse problem is demonstrated, calculating a simulated thermal profile from changes in eigenfrequency generated in a COMSOL eigenfrequency analysis. It is shown how averaging can be used to avoid symmetry problems. Finally in Subsection 7.4.7 we show a concrete example estimating the change in thermal distribution of an Advanced LIGO test mass and comparing it to COMSOL Heat Transfer simulation of the physical system.

### 7.4.2 General formalism

#### 7.4.2.1 The preamble: linearity and the inverse problem

The changes in the mode frequencies $\delta \omega_i$ ($\omega_i$ is used in place of $\omega_m$ to represent a set of eigen-frequencies) are linear functions of the changes in the temperature inside the mirror, $\delta T(r)$ (herein for more readable equations bold-faced letters will be used in place of $\vec{r}$ notation to denote three-dimensional vectors). Mathematically this can be expressed as follows:

$$\delta \omega_i = \int \rho(r) f_i(r) \delta T(r) d^3 r,$$

(7.5)
where \( \rho(r) \) is the density, and functions \( f_i(r) \) are formfactors that will be discussed in the next subsection. It is convenient to introduce an inner product between functions,

\[
\langle f, g \rangle \equiv \int \rho(r)f(r)g(r)d^3r
\]  

(7.6)

and similarly between vector fields:

\[
\langle a, b \rangle \equiv \int \rho(r)a(r) \cdot b(r)d^3r.
\]  

(7.7)

The factor \( \rho(r) \) ensures that the integral is restricted to the mirror volume, and as will be seen below, is useful for expressing orthogonality relations between the mirror mode displacements. Equation 7.5 can be written simply as

\[
\delta \omega_i = \langle f_i, \delta T \rangle.
\]  

(7.8)

To solve the inverse problem, we expand \( \delta T(r) \) in a series:

\[
\delta T(r) = p_i f_i(r).
\]  

(7.9)

Here and below we use the Einstein convention, where the summation of repeated indices is assumed. In general, one expects the functions \( f_i(r) \) to be linearly independent. However in some cases where a high degree of symmetry is present, it may turn out that this is not so. In such situation, one needs to restrict the series above to a linearly independent subset of functions spanning the whole function space, so that the expansion is unique.

Substituting the expansion above into Equation 7.8 results in a matrix equation

\[
\delta \omega_i = C_{ij} p_j,
\]  

(7.10)

where

\[
C_{ij} \equiv \langle f_i, f_j \rangle.
\]  

(7.11)

One therefore has

\[
\delta T(r) = C_{ij}^{-1} \delta \omega_i f_j(r),
\]  

(7.12)

where \( C_{ij}^{-1} \) are the elements of \( C^{-1} \). Since one monitors only a finite number \( N \) of the normal modes, in practice one should restrict \( C_{ij} \) to be the \( N \)-dimensional square matrix.

In this subsection we demonstrated that once the formfactors \( f_i(r) \) are known, the computation of the temperature field inside the mirror is straightforward. We now use elasticity theory to compute these formfactors.
7.4.2.2 Computation of the formfactors $f_i(r)$

Consider a Lagrangian displacement $\xi(r,t)$ of the mirror from its position of rest. In the linear approximation, the elasto-dynamic equations of motion are

$$\rho(r) \frac{\partial^2 \xi}{\partial t^2} = \hat{L}(\xi)$$

(7.13)

where $\hat{L}$ is the operator representing the elastic restoring force and given by

$$\hat{L}(\xi)(m) = \frac{\partial \sigma_{mn}}{\partial x_n} = \frac{\partial [c_{mnkl} \epsilon_{kl}]}{\partial x_n},$$

(7.14)

where

$$\epsilon_{kl} = (\xi_{k,l} + \xi_{l,k})/2$$

(7.15)

is the shear tensor, $c_{mnkl}$ is the elasticity tensor, and

$$\sigma_{mn} = c_{mnkl} \epsilon_{kl}$$

(7.16)

is the elastic stress tensor. A normal mode with angular frequency $\omega_i$ is characterised by the wavefunction $\xi^{(i)}(r)$ that satisfies the following eigen-equation:

$$\hat{L}(\xi^{(i)}) = -\omega_i^2 \rho(r) \xi^{(i)}$$

(7.17)

Importantly, the normal modes satisfy orthogonality relation

$$\langle \xi^{(i)}, \xi^{(j)} \rangle = \langle \xi^{(i)}, \xi^{(i)} \rangle \delta_{ij}.$$  

(7.18)

Consider now a perturbation $\delta \hat{L}$ due to the change in temperature of the mirror:

$$\delta \hat{L}(\xi)(m) = \frac{\partial}{\partial x_n} \left[ \delta T(r) \frac{\partial [c_{mnkl} \epsilon_{kl}]}{\partial T} \right].$$

(7.19)

Here we take into account the fact that the elasticity tensor is temperature-dependent. Strictly speaking, there is another contribution to the change in $\hat{L}$ that is due to thermal expansion of the mirror. However, the thermal expansion coefficient of the Advanced LIGO mirror substrate is very small. Numerically, it is about $0.003 \times d \log E/dT$, where $E$ is the Young modulus of the fused silica mirror substrate. Therefore we can safely neglect the thermal expansion effect as sub-dominant.

The first order perturbation theory for Equation 7.17 gives

$$\delta \hat{L} \left( \xi^{(i)} \right) + \hat{L} \left( \delta \xi^{(i)} \right) = -\omega_i^2 \rho(r) \delta \xi^{(i)} - 2 \omega_i \delta \omega_i \rho(r) \xi^{(i)}.$$  

(7.20)
Following the standard procedure, we now impose the constraint
\[
\langle \xi^{(i)}, \delta \xi^{(i)} \rangle = 0. \tag{7.21}
\]
This can always be achieved by multiplying \( \xi^{(i)} + \delta \xi^{(i)} \) by an appropriate scaling factor. We now multiply Equation 7.20 by \( \xi^{(i)} \), and integrate over the volume. Using the orthogonality relations Equations 7.18 and 7.21, and the self-adjointness of \( \frac{1}{\rho(\mathbf{r})} \hat{L} \), we get
\[
\langle \xi^{(i)}, \frac{1}{\rho(\mathbf{r})} \delta \hat{L} \left( \xi^{(i)} \right) \rangle = -2\omega_i \delta \omega_i \langle \xi^{(i)}, \xi^{(i)} \rangle. \tag{7.22}
\]
Therefore, the change of the mode’s angular frequency is given by
\[
\delta \omega_i = -\frac{1}{2\omega_i} \frac{\langle \xi^{(i)}, \frac{1}{\rho(\mathbf{r})} \delta \hat{L} \left( \xi^{(i)} \right) \rangle}{\langle \xi^{(i)}, \xi^{(i)} \rangle}. \tag{7.23}
\]

We are now ready to determine the formfactor \( f_i(\mathbf{r}) \). To achieve this, we write down the numerator of the above equation explicitly as an integral over volume:
\[
\langle \xi^{(i)}, \frac{1}{\rho(\mathbf{r})} \delta \hat{L} \left( \xi^{(i)} \right) \rangle = \int \xi^{(i)} \delta \hat{L} \left( \xi^{(i)} \right) \ d^3 \mathbf{r} = \int \xi^{(i)} \left( \frac{\partial}{\partial x_n} \left( \delta T(\mathbf{r}) \frac{\partial c_{mnkl}^{(i)}}{\partial T} \epsilon_{kl}^{(i)}(\mathbf{r}) \right) \right) \ d^3 \mathbf{r} = -\int \delta T(\mathbf{r}) \frac{\partial c_{mnkl}}{\partial T} \epsilon_{mn}^{(i)} \epsilon_{kl}^{(i)}(\mathbf{r}) \ d^3 \mathbf{r}.
\]
The last step is obtained by integrating by parts, using Gauss’ theorem, recalling that \( \sigma_{mn} = \delta \sigma_{mn} = 0 \) at the surface of the test mass, and using the symmetry of the elasticity tensor with respect to the indices \( m \) and \( n \) (the latter insures that the stress tensor is symmetric). From this expression, we conclude that the formfactor is given by
\[
f_i(\mathbf{r}) = \frac{1}{N_i \rho(\mathbf{r})} \frac{\partial c_{mnkl}(\mathbf{r})}{\partial T} \epsilon_{mn}^{(i)}(\mathbf{r}) \epsilon_{kl}^{(i)}(\mathbf{r}), \tag{7.24}
\]
where the normalization factor is given by
\[
N_i = 2\omega_i \int \rho(\mathbf{r}) |\xi^{(i)}(\mathbf{r})|^2 \ d^3 \mathbf{r} = \frac{4W_i}{\omega_i}, \tag{7.25}
\]
where \( W_i \) is the total energy of the mode. It is worth noting that
\[
c_{mnkl} \epsilon_{mn} \epsilon_{kl} = 2U(\mathbf{r}) \tag{7.26}
\]
where \( U(\mathbf{r}) \) is the energy density of elastic deformation. For an isotropic medium such as fused silica glass,
\[ c_{mnkl} \epsilon_{mn} \epsilon_{kl} = 2U(\mathbf{r}) = W (\epsilon_{ll})^2 + 2G \epsilon_{ik}^s \epsilon_{ik}^s, \quad (7.27) \]

where \( E \) is the Young modulus, \( G \) is the shear modulus, and \( \epsilon_{ik}^s \) is the incompressible part of the shear,

\[ \epsilon_{ik}^s = \epsilon_{ik} - \frac{1}{3} \epsilon_{ll} \delta_{ik}. \quad (7.28) \]

A simple way of rewriting the formfactor in Equation 7.24, that may be handy in the context of using materials engineering packages like COMSOL or ANSYS, is as follows:

\[ f_i(\mathbf{r}) = \frac{\omega_i}{2W_i \rho(\mathbf{r})} \left[ \frac{\partial U^{(i)}(\mathbf{r})}{\partial T} \right] \xi^{(i)}, \quad (7.29) \]

where the notation implies that the partial derivative with respect to temperature is evaluated with the mode displacement \( \xi^{(i)}(\mathbf{r}) \) being held constant. This completes our computation of the formfactors.
7.4.3 COMSOL Model to Test Formfactor Solution

To test the formfactor solution of Equation 7.29 a COMSOL [77] model was built very similar to that used in Section 5.2.2.4. This model is based on the Advanced LIGO Livingston Y end test mass. Model parameters are given in Table 7.2 and and the model geometry is displayed in Figure 7.13.

Table 7.2: Parameters for the COMSOL Models

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter</td>
<td>340.13mm</td>
<td>Diameter</td>
</tr>
<tr>
<td>Depth</td>
<td>199.59mm</td>
<td>Depth</td>
</tr>
<tr>
<td>$\rho$</td>
<td>2203kg/m$^3$</td>
<td>Density (mass 39564g)</td>
</tr>
<tr>
<td>Wedge</td>
<td>0.07deg</td>
<td>Optic wedge</td>
</tr>
<tr>
<td>$E$</td>
<td>72.7GPa</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.164</td>
<td>Poisson ratio</td>
</tr>
<tr>
<td>$\partial E/\partial T$</td>
<td>11.5 MPa</td>
<td>Thermal dependence of Young’s modulus</td>
</tr>
<tr>
<td>$\partial \sigma/\partial T$</td>
<td>$1.55 \times 10^{-5}$</td>
<td>Thermal dependence of Poisson ratio</td>
</tr>
</tbody>
</table>

Figure 7.13: The geometry used for the COMSOL simulation

COMSOL is used to perform a steady state thermal analysis and an eigenfrequency analysis. Using just the eigenfrequency analysis the formfactors are calculated using Equation 7.29 within COMSOL In this calculation the total strain energy can be used in place of $W_i$ as the amplitude can be assumed to be at its maximum for the deformation calculated. Here the kinetic contribution to the total energy is zero. The 3D formfactors are then exported to Matlab [191] for further processing. Formfactors and total mode displacement are shown in Figure 7.14 for a selection of modes.
In following sections the following additional parameters are calculated within the COMSOL model:

1. The analytic frequency shift from Equation 7.5, where \( T(r) \) either is produced by a COMSOL Heat transfer and Thermal expansion analysis or an arbitrary thermal distribution \( Z(z) \), \( R(r) \) and \( \Phi(\phi) \) in cylindrical co-ordinates may be applied.

2. The diagonal elements of the conversion matrix \( C_{ij} \) as in Equation 7.11. This calculation is used to verify the scaling parameters between integration performed in Matlab and integration performed in COMSOL. The result is not perfect indicating a difference in method or meshing as can be seen in Figure 7.15.
7.4.4 Testing the Forwards Model

An arbitrary temperature distribution $R = J_1(r)$, $Z = z$ and $\Phi = \sin(\phi)$ above room temperature is applied to the test mass in the COMSOL simulation and the eigen-frequency analysis is run to produce a set of eigenfrequencies. The eigenfrequencies produced without the thermal distribution (i.e. just at room temperature) are subtracted to produce the COMSOL estimate of $\Delta \omega_i$. In conjunction Equation 7.8 is applied in COMSOL to calculate the expected change in eigen-frequency for that same temperature distribution. The comparison of the analytically calculated result and the simulation result are shown in Figure 7.16 showing very good agreement.

It is thought that numerical errors in the COMSOL simulation are likely to cause discrepancies of $\sim 0.02 \text{mHz}$ visible in the inset image in Figure 7.16.
Figure 7.16: Comparison between the frequency shift predicted by a COMSOL eigen-frequency simulation and the frequency shift predicted by the analytic expression for 224 eigen-modes influenced by an arbitrary thermal disturbance, inset zoomed in to 0.1 mHz

As we saw in Section 5.2.2 mode identification presents an continuing problem for parametric instability control. The forwards problem presented here presents a new tool for mode identification. By applying a known thermal transient to the test mass the transient in mode frequencies can be calculated. By comparing the calculated mode frequencies with the measured mode frequencies, mode identification can be confirmed.

7.4.5 Thoughts on Symmetry

There are two main symmetries in Advanced LIGO test masses: The symmetry front to back and the symmetry right to left. There is also rotational symmetry but this is broken by the flats and ears as can be seen in Figure 7.13. The result of this symmetry being broken can be observed as a change in eigen-frequency of $\sim 0.5\%$ between the vertical and horizontal modes as can be seen in Figure 6.1. The symmetry up-down should be broken by the ear placement below the optic axis. However simulation does not show mode splitting for this symmetry. This will be discussed further in Section 8.2.2.

The symmetries in the test mass result in symmetries in the eigenmodes. Eigen-modes that are symmetric have the same frequency, which is called degenerate eigen-
modes. The modes can no longer be distinguished. The modes will be resolved into two modes when an asymmetry is introduced that is large enough to split the mode frequencies so they are resolved.

The symmetry of the eigenmodes means that information can not be obtained about thermal distributions that break that symmetry. Eigenfrequencies will be affected equally by the portion on one 'side' of the symmetry as the other. This will be true unless the asymmetric thermal distribution is strong enough to produce the mode splitting associated with breaking that particular symmetry. Mode splitting is explored in Section 8.2.2 where it is shown that thermal asymmetry does not produce mode splitting with thermal gradients up to $10\times$ those expected in Advanced LIGO test masses.

Resigned to the fact that this information is lost, we are left to explore what information can be extracted. In Figure 7.17 225 eigenfrequencies are compared between 2 simulations.

Figure 7.17: The figure shows a comparison between two COMSOL simulations. The horizontal axis represents the mode frequency shifts resulting from a Gaussian heat profile applied to the right of the optic axis while the vertical axis represents the mode frequency shifts when the heat profile is applied to the left of the optic axis. The expected outcome for identical thermal distributions is also plotted. This data illustrates the degeneracy problem which causes exactly the same eigenfrequency shift from two different (but symmetric) thermal profiles.

In one simulation the temperature distribution has a Gaussian profile centered at 10 cm to the right of optic axis, with a uniform depth profile. In the other, the Gaussian profile is centered 10 cm to the left of optic axis.
It is clear that these are indistinguishable. Therefore there is no way that the inverse calculation with one set of eigenfrequencies will produce the two thermal profiles inset in Figure 7.17 that would be expected from such a calculation. If it is possible to average over the symmetry then this will represent the average temperature profile. In the case represented here the input thermal distribution averaged left-right will both produce the same thermal distribution. This will roughly produce half the original Gaussian thermal profile 10 cm to either side of the optic axis.

7.4.6 Testing the Inverse Calculation of the Thermal Profile

To check that the shift in eigen-frequency can predict a change in thermal profile in the Advanced LIGO test masses, a deviation in the thermal distribution is applied to the COMSOL model, a Zernike polynomial $Z_2^1$ is used with a uniform distribution through the depth of the optic. The simulated change in frequency of the first 225 modes is then used to estimate the Zernike polynomial, the comparison between the Zernike polynomial and the estimate from Equation 7.12 are displayed in Figure 7.18a-d.

The result is unimpressive, there is very little resemblance between the thermal profiles. It must be highlighted here that symmetry in the test mass, discussed in the previous section, results in degeneracy in the eigen-mode frequency shifts. This degeneracy results in the loss of spatial information.
Figure 7.18: Comparing the input temperature field to the temperature field in the test mass estimated from the change in frequency of 225 eigenmodes. (a) Top left, the input temperature field, uniform in depth. (b) Top right, the estimated front (high reflectivity) surface temperature field. (c) Bottom left, the estimated temperature field in the center of the test mass. (d) Bottom right, the estimated back (anti reflective) surface temperature field.

To test whether we are recovering the thermal profile the thermal profile is averaged in Figures 7.19. In b.) the average is over the depth of the optic while in c.) the average is over depth and rotational angle leaving only the radial position.

In Figure 7.19 the resemblance between the surface profile of the depth average is apparent. When distilled by averaging to just the radial component it is clear that the inverse problem is providing an estimate of the thermal profile. However it is also clear that with the eigenmode decomposition, 225 modes do not provide enough spatial components to accurately describe a quadratic radial temperature distribution.

Also it is clear that the symmetries in the optic result is the drastic loss of information. It is presumed that the depth symmetry causes the profiles in Figure 7.18 to have large deviations from the expected thermal profile. It is then presumed that the right-left symmetry and up down symmetry cause Figure 7.19 (b) not to resemble (a). If these assumptions are correct the maximum information available is the average
Figure 7.19: Figure comparing averaged temperature distribution estimates from simulated change in frequency of 225 eigen-frequencies. (a) Top left, the input temperature field, uniform in depth. (b) Top right, the estimated temperature field averaged though the depth of the optic. (c) Bottom left, averaged through depth and rotation of the optic. (d) Bottom right, the average radial profile estimated from change in eigenfrequencies compared to the expected radial profile, the fit is good but is limited by the fact that the highest order modes only have a forth order radial component quadrant temperature profile averaged in depth.

However there are some other interesting observations that can be made from these figures. In both Figures 7.18 and 7.19 it is clear that the estimated thermal distribution is symmetric right to left, but not symmetry up to down. This is due to the ears breaking the vertical symmetry, it can be seen in Figure 7.13 that the ears are placed under the optic axis (so that the suspension point is very close to the optic axis). So though spatial information about the temperature profile has not been recovered in this symmetry it is clearly not perfectly symmetric. This symmetry breaking is examined in detail from another perspective in Chapter 8 Section 8.2.2.
7.4.7 Thermal Distribution Estimate in Advanced LIGO Test Masses

Data from LIGO Livingston Y end test mass is used in this example. A set of 26 mode frequencies given in the Appendix A were recorded over a 1 hour period. Mode frequencies were chosen from ETMY modes that have previously been identified by the method in Section 5.2.3 [54]. They had to have a reasonable signal to noise ratio during the measurement period used in this experiment (0520-0637 UCT 18 Aug 2015). Some rows of the conversion matrix $C$ are close to being linearly dependent, so the eigenfrequencies that were associated with rows that reduced a singular value of $C$ to below $10^{-3}$ were removed.

The period used for this example is characterised by: The front surface heating due to the power absorbed from the main cavity laser which had been applied continuously for 2 hours prior to the first measurement and more or less continuously for 10 hours prior to that. The first measurement was immediately preceded by a drop in ring heater power of 1W. The second measurement was taken very shortly after a 3W increase in ring heater power.

![Figure 7.20: A preliminary comparison between the temperature distribution predicted by COMSOL (left) and the temperature distribution estimated from the change in 26 eigenfrequencies (right). The temperature distributions are averaged over the depth of the optic. The smooth thermal profile on the right was expected due to the limited number of eigenmodes used. However the underestimate of the temperature change was not expected](image)

A 24h history of the contained cavity power and ring heater power were used to estimate the thermal profile of the test mass using a COMSOL Heat Transfer simulation of the test mass, considering only radiative cooling. Preliminary results from the COMSOL simulation and the temperature distribution estimated from the
change in 26 eigenfrequencies are compared in Figure 7.20. Visible radial streaking in the COMSOL thermal profile estimate is thought to be due to the coarse mesh used to speed up the simulation.

In this preliminary comparison the temperature distribution is averaged over the depth of the optic, this removes the front to back symmetry problem. There is a factor of 4 excess in the temperature distribution estimated from the mode frequency shift. The measurement also indicates excess heating on the flats that is not observed in the simulation. It is not clear why this estimate does not suffer from the rotational symmetry problem from the previous section. The restricted number of modes available result in missing spatial resolution components smoothing the thermal profile in this case. This may be ameliorated as changes in Advanced LIGO hardware now allow measurement of more than 100 modes per test mass. These changes will be further discussed in Section 8.4.

7.4.8 Discussion

In this Section we presented a formalism for computing the three-dimensional temperature distribution inside LIGO mirror test masses, by monitoring the frequency drifts of a multitude of the test mass’ vibrational modes. Equations 7.24, 7.25 and 7.29 can be used for practical evaluation of the formfactors $f_i(r)$ that connect the temperature changes inside the test masses with the eigenfrequency changes of their modes. Once the formfactors are computed, Equations 7.11 and 7.12 can be used to invert the problem and work out the temperature field from the measured changes in temperature. King et al. [170] give a computationally efficient algorithm for how to connect the temperature field to thermal deformations of the mirror surfaces. The calculation of the formfactors and mode frequency shift have been tested with a COMSOL simulation of the LIGO test mass showing very good agreement between the analytic expression and a COMSOL eigen-frequency analysis. Inverting the problem has been tested. Problems with symmetry are highlighted. These problems can be circumvented by averaging over the symmetry to give good results. However problems with the low spatial frequencies of modes even when considering a relatively large number of modes have been highlighted. Finally two measurements of 26 eigen-frequencies of LIGO Livingston ETM Y test mass are used to compute the change in thermal distribution in the test mass. This result is compared to a COMSOL simulation of the period. The COMSOL simulation uses a 24 h history of ring heating power and cavity contained power as input, to estimate the thermal profile in the test mass at
the times the eigenfrequencies were measured. The very preliminary result is inconclusive, low spatial frequencies mean the thermal profile is smoothed and the scale appears to be in error by a factor of 4. Further investigation will be required.

It is possible that a selection of modes could be used to target a particular thermal distribution with better accuracy than the complete 3D reconstruction of the thermal profile. However it is hard to see how the problems with symmetry in the test mass can be resolved and hence such a system will be limited in its ability to distinguish a thermal distribution that has right-left or front-back asymmetry.

7.5 Summary

In this chapter we have explored three mode interaction monitoring of gravitational wave detector degrees of freedom. In Section 7.2 a variety of three mode interaction monitoring programs are proposed: (1) Early warning of parametric instability, (2) Estimates of the optical gain, (3) Beam position estimates could be made from the sensitivity of optical gain to beam position or even beam size estimates, (4) Mirror temperatures can be inferred and finally, (5) Mirror radii of curvature can be inferred.

Data showing how an early warning of parametric instability can be achieved is presented as are estimates of the optical gain and the inferred mirror radii of curvature. These three mode interaction monitoring tools have been implemented on the ∼80m optical cavities at the Gingin High Optical Power facility.

We then studied in detail the use of eigenfrequencies of the test mass to monitor the temperature of the test mass. In Section 7.3 this monitoring was used to improve the thermal model of the test mass. The proposed thermal model has been tested showing a factor of 6 improvement in the temperature estimate after 19 hours.

Then we studied the possibility of using a set of eigenfrequencies to measure the 3D temperature profile in the test mass. A formalism is presented that allows the direct translation of changes in test mass eigenfrequency to changes in temperature and vice versa. This model is then tested again finite element simulation. Results are promising, however the symmetry of the test mass means that much spatial information is lost and the high frequency of eigenmodes with very high spatial frequencies means that it is unlikely that very high spatial resolution temperature maps will be possible. Even so a preliminary measurement on Advanced LIGO test masses indicates that useful spatial information may be obtained however further investigation is required.
These are just the first of the possible tools that could be developed using three mode interaction monitoring. We saw in chapter 5 when characterising the Advanced LIGO cavities for parametric instability that we required estimates of the transverse electromagnetic mode spacing. We saw that observations of parametric instability could be used to confirmed these estimates. We can envisage these concepts being extended to real time monitoring of the optical gain of the interferometer. However in practice it is hard to design a robust system and confirm its predictions, this will be explored further in the discussion Section 8.5. In the remaining sections of Chapter 8 we will discuss recent developments with respect to parametric instability at LIGO Hanford.
Chapter 8

Preliminary Investigations and Future Work

Experimentation on optomechanical three mode interactions and parametric instability in gravitational wave detector optical cavities has only recently become possible. This means that there is a wealth of interesting effects to be investigated many of which were beyond the scope of this thesis. In this chapter further studies that could improve on the investigations presented in this thesis are discussed. The chapter has large and varied content, below is a summary.

Interferometer Measurements of Test Mass Resonant Modes

The LIGO instruments use multiple photodetectors for sensing different degrees of freedom. Several of these photodetectors are sensitive to eigenmodes of the test masses. In Section 8.1 the photodetector signal channels used for the experimental studies reported in Chapter 5 and Sections 6.2, 7.3 and 7.4 are compared. This highlights the fact that our understanding of the coupling mechanism between the acoustic modes in the mirrors and the observed signals is still incomplete. With better understanding these signals could have valuable applications such as beam centering on the test masses or output mode cleaner alignment in addition to parametric instability control.

New Parametrically Unstable modes at LIGO Hanford

During the last six months of my PhD candidacy I spent two months at LIGO Hanford helping with the effort to achieve 150 kW circulating optical power for Advanced LIGO’s second observation run. Section 8.2 details how this increase in optical power resulted in more instabilities.

First this section discussed the probably mode shape of the $\sim 47.5$ kHz instabilities.

Second a phenomena of mode splitting was observed. The investigation into asymmetries that may cause this mode splitting indicate the test mass ears are responsible.
Electrostatic Damping of Parametric Instability with Thermal Transients

The increase optical power used at Hanford necessitated the upgrade of the parametric instability damping system described in Chapter 6 Section 6.2. The difficulties encountered damping parametric instability with dynamic affects due to thermal transients in the optics from the coating absorption of the circulating beam is discussed in Section 8.3.

Upgrades for High Frequency Sensing and Actuation on Parametric Instabilities

In Section 8.4 the changes made to electrostatic damping to enable continuous interferometer operation with \(\sim 150\, \text{kW}\) optical power in the arm cavities are presented. This includes a discussion of the limitations of the system implemented and further required work.

Three Mode Interaction Monitoring of the Optical Transfer Function

In Chapter 7 the use of three mode interaction monitoring of the optical gain of transverse electromagnetic modes was discussed. In Section 8.5 the difficulty of such monitoring schemes and methods to improve on them are discussed.

8.1 Interferometer Measurements of Test Mass Resonant Modes

In Chapters 5 and 6 the many photodiodes available for measuring the acoustic modes in the mirrors were discussed. Figure 8.1 shows ten photodetectors that all show significant acoustic mode signals. All of them have the potential to provide different information about the interferometer. To make use of this information the mechanism by which the mechanical vibration of the mirror couples to the photodetector must be understood. In this section the possible coupling mechanisms and the potential uses of such signals are examined.

8.1.1 Coupling Mechanisms

As explained in Chapter 5 the coupling of the acoustic mode to the arm transmission is via three mode interactions. The fundamental cavity optical mode is scattered by the acoustic mode in the mirror into a higher order transverse optical mode that is resonantly enhanced by the cavity. This is evident from the fact that acoustic mode signals are only observed when there is significant resonant enhancement, typically requiring a parametric gain \(|R| >\sim 0.1\).

It might normally be assumed that the high order transverse mode directly couples acoustic modes signals throughout the interferometer. If the transverse mode coupled
the acoustic mode signal directly to a particular sensor we would expect a mode amplitude to reflect the gain of that particular mode. In Chapter 6 Figure 6.1 on page 174 we see mode groups C and D have large amplitudes. These modes have low overlap to third order optical modes and therefore low parametric gain. It is surprising that they are so large. In Chapter 5 Section 5.2.3.2 60% of the test mass acoustic modes at LLO, from 6-18kHz were identified. There appears to be little relation between the mode amplitude and the mode overlap to the relevant higher order transverse mode. Below I will present three possible mechanisms by which the signal can appear on the output mode cleaner photodiodes (which are the main interferometer output sensors).

Before discussing these three specific mechanisms I will discuss experiments with the OMC reflection photodiodes. These are discussed first because they prove the presence of the transverse modes at the output mode clearer input, and thereby emphasise why the signal seen at the main output is surprisingly strong. The discussion refers to Figure 8.1.

Figure 8.1: Interferometer sensors that are sensitive to resonant modes in the four test masses. The most important sensors are the transmission quadrant photo-detectors (QPD) of the end test masses ETMX and ETMY followed by the photo-detectors located after the output mode cleaner (OMCPDs), the anti-symmetric port (AS) QPDs and OMC reflection (REFL) QPDs. Other elements shown in this diagram are; the input test masses (ITMX and ITMY), power recycling mirror (PRM), signal recycling mirror (SRM) and laser beams in red

Some of the $\text{TEM}_{pq}$ power resonant in the arm cavities is transmitted through
the corner cavity composed of the input test masses (ITM)s, power recycling mirror (PRM) and signal recycling mirror (SRM). We saw in Chapter 5 Section 5.2.3.2 how this results in the signal coupling from one arm to the other. It is therefore inevitable that some of the TEM$_{pq}$ will be incident on the OMC. This would appear to be the most straightforward path by which the signal arrives at the OMC reflection (REFL) quadrant photo-detectors (QPD) and at the anti-symmetric port QPDs.

To see a measurable signal on these QPDs the TEM$_{pq}$ must beat with the fundamental mode. To confirm that the TEM$_{pq}$ is incident on the OMC REFL QPDs the phase of the signal observed on these QPDs can be compared when the OMC is locked and when it is not locked. When the OMC is not locked OMC REFL QPD receives mostly fundamental mode. However when the OMC is locked, the fundamental mode is strongly suppressed by the OMC transmission. The reflected signal contains all the spatial modes and frequency content not transmitted by the OMC. This is commonly referred to as junk light. The Livingston OMC is slightly over coupled with an input transmission of 7600ppm and a finesse of 405 [173], so there is a small ($\approx 2\%$) amount of fundamental mode reflected with the opposite phase. When the OMC is locked, the TEM$_{pq}$ beat note phase was observed to change by 180 degrees. This phase change is just what is expected for a TEM$_{pq}$ mode incident on the over-coupled OMC.

According to this analysis, the TEM$_{pq}$ mode is reflected by the OMC. Hence we would expect very low TEM$_{pq}$ power to appear at the OMCPD. Therefore we would expect any three mode interaction signals to be very small. Given that in reality PI shows up strongly in the OMCPD, there must be other mechanisms at work to make the signal visible. If we can properly understand how these PI signals occur, we should be able to optimise their signal to noise ratio to make them much more useful, and more easily suppressed.

As mentioned above there are three possible ways the acoustically modulated signal can be transmitted from the mirror to the OMCPD. Two involve three mode interactions and the third is an off resonance two mode interaction.

1. OMC Input Misalignment

If the alignment or mode matching to the OMC is poor, incident TEM$_{pq}$ may be converted to TEM$_{00}$ resonant in the OMC (The OMC linewidth is $\approx 680$ kHz $\gg$ acoustic mode frequencies [173]). This coupling ma be expressed as a function of the beam position of the incident beam on the output mode cleaner.

$$E_{OMCPD} = a(r^\perp)E_{pq}$$  (8.1)
Here $E_{OMCPD}$ is the field amplitude of the TEM$_{00}$ in the output mode cleaner at the frequency of the incident TEM$_{pq}$, $a(\vec{r}_\perp)$ is the coupling between the incident TEM$_{pq}$ and the OMC TEM$_{00}$ as a function of beam position and pointing and $E_{pq}$ is the field amplitude of the incident TEM$_{pq}$. As the frequency of the incident TEM$_{pq}$ mode is small compared to the linewidth of the OMC (680 kHz) the OMC gain at this frequency is ignored.

![Figure 8.2: Left, layout showing misalignment circled green. Right TEM$_{03}$ coupling to OMC fundamental mode $a(\vec{r}_\perp)$, in this case $E_{pq}$ is assumed to be $HG_{03}$ and only beam position is changed in $\vec{r}_\perp$.](image)

For the Advanced LIGO design and assuming a third order Hermite Gaussian mode $E_{30}$ the overlap parameter is calculated for a range in incident beam positions. This coupling function is displayed in Figure 8.2.

As it is hard to read specific values a range of values are read off the figure. For a TEM$_{03}$ mode displaced vertically 50 µm $a(\vec{r}_\perp) = 0.002$, displaced 0.2 mm $a(\vec{r}_\perp) = 0.03$ and displaced 0.5 mm $a(\vec{r}_\perp) = 0.07$.

2. **Misalignment from the OMC output to the OMCPDs**

In this case the frequency of the TEM$_{pq}$ incident on the OMC input is far from the resonance of the TEM$_{pq}$ in the OMC, so is significantly attenuated by $c$ as described in Chapter 2 Section 2.3.4. The portion that is transmitted beats with the fundamental mode. If the photodiode is large or the beam is perfectly aligned, this beat signal is zero. However any misalignment that causes beam clipping will result in power fluctuations on the photodiode at the beat frequency. In this case:

$$E_{OMCPD} = b(\vec{r}_\perp)cE_{pq} \quad (8.2)$$

where $b(\vec{r}_\perp)$ is the TEM$_{pq}$ coupling from the OMC output to the photodiode as a function of beam position on the photodiode and $c$ is the TEM$_{pq}$ coupling function.
through the OMC. The OMC PD is a 3 mm diameter InGaAs diode while the beam diameter \((1/e^2)\) is 1.04 mm \([172]\). From these numbers the function \(b(\vec{r}_\perp)\) can be calculated for the TEM\(_{03}\) as \(\oint_{PD} E_{00} \ast E_{03} d(\vec{r}_\perp)\) which is plotted in Figure 8.3. Again specific values are read off the figure. For a vertical beam displacement of 50 \(\mu\)m the coupling function is \(b(\vec{r}_\perp) = 10^{-5}\), for 100 \(\mu\)m, \(b(\vec{r}_\perp) = 10^{-4}\), for \(b(\vec{r}_\perp) = 250 \mu\)m - \(10^{-3}\) and for 1 mm \(b(\vec{r}_\perp) = 10^{-2}\). The coupling \(c\) of the TEM\(_{03}\) mode through the OMC at 15 kHz can be calculated from the OMC mode spacing of 58.1 MHz and the cavity pole at 300 kHz to be \(-50\) dB.

In the first case (point 1 above), a thermally excited mode with an rms amplitude of \(10^{-15} \text{ m}/\sqrt{\text{Hz}}\) would scatter light a proportion \(B_{m,3}\) into a resonant TEM\(_{03}\) which is about 0.03 for the 15 kHz mode (Chapter 5 Section 5.4). This mode is then assumed to be resonant in the arm cavity resulting in a transferal coefficient of 1 (assumes the same resonant enhancement of TEM\(_{pq}\) and TEM\(_{00}\)) and then \(a(\vec{r})\) determines the coupling to the OMC fundamental mode output. If we assume a 0.2 mm offset we would expect \(10^{-18} \text{ m}/\sqrt{\text{Hz}}\) equivalent displacement to be observed in the OMC signal. About the same level as shot noise.

If this mechanism is correct, we would expect to see change in the amplitude of a subset of TEM\(_{pq}\) signals on the OMCPDs when the OMC alignment drifts. This has not been observed. However a careful experiment to directly test for this correlation has not been performed.

The second case (point 2) appears unlikely due to the massive attenuation of the TEM\(_{pq}\) when it does not coincide in frequency with the OMC’s TEM\(_{pq}\) which is \(\approx\)
\((p+q) \times 58\) MHz. Being so far from resonance results in a severe reduction in the TEM\(_{pq}\) noise at the OMC output [176].

3. Non-transverse mode coupling

The other possibility is that the the TEM\(_{pq}\) mode is not responsible for coupling the signal from the test masses to the OMC. It is possible that the resonant modes are seen as length fluctuations in the arm cavities. This is equivalent to the creation of a TEM\(_{00}\) sideband at \(\omega_m\). At the beam splitter these phase changes become differential arm length signals. In the case of a drum-head type mode such as the 15220 Hz mode in Figure 8.5 (a) it is clear that there is a length fluctuation, which gives rise to high overlap to the TEM\(_{00}\). However for the 15 kHz mode responsible for the parametric instability at LIGO, it can be seen from Figure 8.5(b) and (c) that the beam must be off-center in the vertical axis to produce high overlap to the TEM\(_{00}\).

Figure 8.4: The exaggerated deformation of two eigen-modes demonstrating the change in cavity length degree of freedom for the 15.2 kHz mode (TEM\(_{00}\) sidebands) (a). In (b) a 15 kHz eigen-mode has no coupling to length while in (c) when the beam is off center there is coupling to the cavity length.

The modulation of the acoustic mode scatters light into many optical modes at
frequency $\omega_m$

$$E_{arm} = \frac{2\pi i}{\lambda_0} \sum_{n=0}^{\infty} A_m E_{00} G_n B_{m,n}. \quad (8.3)$$

Figure 8.5 shows the proportion overlap to the fundamental $E_{00}$ mode shape. This can be computed using the overlap integral $B_{m,n}$ of Equation 8.3 and is plotted below for the TEM$_{03}$ as a function of beam decentering.

Figure 8.5: Test mass eigenmode surface deformation coupling to the TEM$_{00}$ mode shape sideband at radial frequency $\omega_m$ as a function of vertical beam position

In Figure 8.5 the overlap parameter between the Eigenmode surface deformation and the TEM$_{00}$ is plotted as a function of vertical beam position on the test mass.

The optical transfer function of the arm cavity $G_{00}$ significantly attenuates the signal. The corner frequency of the cavity is $\sim 100$ Hz, so TEM$_{00}$ at 15.5 kHz will be attenuated by 43 dB. So for a 10 mm beam displacement the minimum attenuation expected for the 15.5 kHz modes is 93 dB. So a thermally excited mode with an rms amplitude of $10^{-15}$ m/$\sqrt{\text{Hz}}$ would be observed with an amplitude of $\sim 10^{-20}$ m/$\sqrt{\text{Hz}}$. For reference shot noise at 15 kHz should be $\sim 10^{-18}$ m/$\sqrt{\text{Hz}}$.

The above hypothesis may be experimentally investigated by looking for a correlation between beam position on the optic and signal amplitude of modes like the one depicted in Figure 8.5(b). This was attempted at LIGO Livingston on the 30th October 2015. The test masses were driven with low frequency oscillations in pitch and yaw - (dithering) so that the beam position could be tracked on the optics. This sensitivity arises from the same mechanism proposed here. As the beam moves from the center of rotation of the optic, optic rotation gets converted into cavity length,
then beam position is calculated with the actuation geometry. While the optic angles were being dithered the OMC transmission signal was recorded for the 15538 Hz mode.

The beam vertical position was plotted against the amplitude of the mode seen in transmission of the OMC in Figure 8.6. We would expect a linear correlation between beam position on the optic and amplitude of the signal with a y intercept of zero if the TEM$_{00}$ sideband provided the means of coupling the signal to the OMC transmission.

![Figure 8.6: Correlation between the ratio of the OMC and arm transmission amplitudes for the 15538 Hz ETMY mode and the vertical beam position on ETMY](image)

The correlation is poor for the two 1 hour segments where OMC transmission signals were collected.

Without more data with better signal to noise ratio in both the beam position estimate and the mode amplitudes no conclusion can be drawn from this result. However if we could believe the fit, the non-zero intercept would indicate the OMC transmission signal is not zero when the beam is perfectly aligned. This implies there is another coupling mechanism at play or the beam position from the low frequency dither lines is incorrect.

The most likely coupling mechanism presented above is point (1), where the high order transverse mode resonant in the arm cavities couples to the fundamental mode of the OMC with misalignment or mode mismatch. Experimentally testing this hypothesis by modulating the beam position or mode matching on the OMC input would be very valuable. As the estimate presented here can not described the observed coupling.
8.1.2 Potential Application of Signals

The acoustic modes of the test masses may provide a wealth of dither signals. Dither signals are intentionally injected signals used routinely to make measurements such as the beam position estimates from the previous section. As mentioned in Chapter 7, the first drumhead modes were used by Fricke et al [121] to modulate the arm cavity fundamental mode to optimise the output mode cleaner mode matching. These acoustic mode dither signals could also be used for centering the beams on the test masses. The reference position of such beam centering would be the center of the acoustic mode.

Each acoustic mode has a different shape, so it is possible to draw more information than just beam position. One can envisage that a pair of rotationally symmetric drum modes with different average displacement over the beam areas could give a measure of the beam radius and hence cavity geometry. There are many more possibilities that are not directly related to parametric instability.

Such measurements may require that the mode amplitude be maintained to ensure a good signal to noise ratio for the measurement. The feedback loop damping system presented in Chapter 6 Section 6.2 has been used with an amplitude set-point - an offset in the zero-point of a control loop, for the purpose of maintaining the mode amplitude.

Additionally there is discrepancy between Gingin experiments that report high signal to noise ratio measurements of acoustic modes in arm transmission signals for parametric gains of $R < 10^{-3}$, while at Advanced LIGO the sensitivity limit appears to be around $R > 10^{-1}$. Having high sensitivity to three mode interactions will improve parametric instability control and the prospects for three mode interaction monitoring. Recent investigation at Gingin have shown a high sensitivity to beam position on the arm transmission photo-detectors, further investigations at LIGO are suggested.

8.2 New Parametrically Unstable modes at LIGO Hanford

At the end of Advanced LIGO’s Observation Run 1 it was decided that Livingston should put effort into understanding the unexplained noise in the gravitational wave signal channel spectrum between 30-100 Hz while Hanford would increase optical power [70]. This two pronged approach minimised the duplication of effort to reach
these goals. Both efforts have been successful. Observation Run 2 began on the 30 November 2016. Livingston achieved approximately 25% increase in sensitivity as measured by the average binary neutron star coalescence sensitivity range and Hanford operated with an optical power close to twice that during Observation Run 1.

Prior to increasing the optical power at Hanford it was apparent from the investigations in Chapter 5 that either the thermal transient in the high order optical mode spacing must be reduced or active damping of parametric instability must be implemented. In the commissioning phase at Hanford, prior to Observation Run 2, the optical power was increased. The two unstable mode groups A and E in Chapter 6 Section 6.2 were both unstable. However not at the same time. The thermal transient in the mirror radius of curvature first caused group A modes to be unstable for a period, typically half an hour. Then these modes became stable and mode group E became unstable. When the ring heater power was increased to reduce the severity of the group E mode instability new instabilities appeared. This section provides details on transient parametric instability observations and some explanation of the extra resonances observed. The next two sections explain the commissioning of electrostatic damping of parametric instability at Hanford.

8.2.1 New Unstable Modes

When the optical power was increased the ring heater power was also increased from 1W to 2W to maintain the same approximate high order optical mode spacing. This was applied as 1.5W on the ETMs and 0.5W to the ITMs to maintain mode matching to the recycling cavities. Poor mode matching of the arm cavities and the signal recycling cavity results in contrast defect - where light beams from each arm cavity do not overlap perfectly when recombining at the beam splitter. This results in excess light in the SRC.

When the ring heater power is increased the 15kHz group of modes are unstable when the interferometer is locked. However, this is only a transient phenomena. When the mirrors warm, the optical mode spacing increases and the 15kHz modes quickly become stable as can be seen from the small rise in amplitude of many of the signals in in the first pane of Figure 8.7.

However two hours after optical power was contained in the arm cavities the 15.54kHz modes became unstable. In the case presented in Figure 8.7 electrostatic damping was not effective. Then the offending modes grew to an amplitude where the interferometer control systems could no longer maintain their set-points. This
Figure 8.7: This is a series of figures from the LIGO Summary Pages [186]. The amplitude of the 15 kHz, 15.5 kHz modes are plotted as a function of time in hours in the first two panels. In the third panel a spectrogram of the new 18 kHz acoustic modes showing increase in amplitude of several modes at the beginning of lock and power being distributed into neighboring frequencies.
occurred at 5 pm local time, and the commissioners departed. Twelve hours later the instrument was operational again and a similar behaviour was observed, in which the 15 kHz mode amplitudes increase at the beginning of the lock.

In the bottom panel of Figure 8.7 a transient instability was observed in a new eigen-mode or modes of the test mass. The new modes are aliased\(^1\) from 47.5 kHz to their observed frequency of approximately 18 kHz. This can be confirmed by the mode identification technique presented in Chapter 5 Section 5.2.3. Using this technique the mode frequency is compared to modes known to belong to particular test masses. In Figure 8.8 the two modes that caused instability (18045 and 18056) and one other mode (18065) clearly change in frequency in the opposite direction to the remaining modes.

![Figure 8.8: The relative change in frequency of some 18 kHz modes with reference to know 15.2 kHz modes labelled by test mass. The ETMY ring heater power was reduced at 1.75 hours. The trend of the cyan, magenta and red dots and red crosses all change direction (though red crosses and magenta dots have a much shallower trend). Cyan (18039 Hz) and red (18057 Hz) are aliased ETMY modes. The green (18063 Hz) are an aliased ETMX mode and the magenta (18002 Hz) a baseband ETMY mode.](image)

This anti-correlation is indicative that the mode is aliased. The sample rate of the Control and Data Systems (CDS) [58] at Advanced LIGO is \(2^{16}\) Sa/sec. If aliased from the first superNyquist band the modes native frequencies are, \(2^{15} + (2^{15} - \omega_o / 2\pi) = 47.48\) kHz and \(47.492\) kHz (here \(\omega_o\) is the observed radial frequency, see subsequent

\(^1\)Aliasing is the down-conversion in frequency that happens when a signal with a frequency greater than the Nyquist frequency is sampled. Aliasing makes signals from two different frequencies indistinguishable
Equation 8.4). The relative frequency shift of these modes is plotted with reference to the 15200 Hz modes identified by test mass in Figure 8.9.

![Figure 8.9: Frequency evolution of the 47.5 kHz modes showing correlation with ETMX and ETMY 15.2 kHz modes](image)

Figure 8.9 indicates that both modes that have caused instability around 47.5 kHz are ETMY modes. Their super-Nyquist frequencies are 47,498 Hz and 47,480 Hz. COMSOL models do not predict such close mode spacing at these frequencies, this will be examined in the next section.

Modes at these frequencies must be driven by optical modes of second order associated with the second free spectral range \( 1/2\pi(\omega_{\text{far}} + 2 \times \omega_{\text{hoom}}) \approx 47.5 \ kHz \). We expect optical mode shapes like TEM\(_{20}\), TEM\(_{02}\) or TEM\(_{11}\). The COMSOL models do not predict modes that have obvious overlap with these mode shapes at 47.5 kHz. The closest mode is at frequency 47.936 Hz shown in Figure 8.10 and is shown with a range of neighbouring modes in Figure 8.11 in the next section. The mode identification must be improved to give confidence in mode identification of frequencies greater than 18 kHz, this will be studied in Section 8.4.

It is also very important to correctly identify the optical mode shape. One approach is to investigate the phase of the signals on the QPDs as described in Chapter 2 Section 2.3.5. This gives indicative information about the optical mode shape.

Figure 8.10 provides evidence that the acoustic modes are scattering light into a TEM\(_{11}\) optical mode.

The ring-down technique described in Section 5.2.4.1 was used to measure the Q factor of the 47,480 Hz mode to be 5.4 million. Parametric gains as high as \( R_m = 3 \) were observed.
In the preceding section two unstable modes were presented separated by only 17 Hz. COMSOL simulations presented in Section 5.2.2.4 predict that modes around 47 kHz are spaced on average by 30 Hz. The chance of two neighboring modes being parametrically unstable seems low considering the mode shapes of the 36 modes in Figure 8.11.
There is only one mode in Figure 8.11 that appears to have high overlap to the TEM$_{11}$ optical mode, this is the mode circled in red. However in Figure 8.10 we saw that there were two unstable ETMY modes around 47.5 kHz. This on its own would not be a convincing argument to say there are extra modes as there have been no measurements to verify the COMSOL model in the vicinity of these frequencies. However this is not the first time extra modes have appeared.

First, in the parametric instability damping experiments at Livingston extra modes were observed. In an attempt to measure the Q factor of the 15081 Hz ETMY mode, a 15085 Hz resonance was rung up that must also be in ETMY. It had a low Q factor for a test mass eigen-mode. Second, when Hanford commenced operating with higher optical power it became clear that there were at least twice as many resonances as expected for some eigen-modes. Some of the clearer examples are shown in Figures 8.12, 8.13 and 8.14.

The Mode Identification technique of Section 5.2.3 confirmed that the frequency evolution of these modes correlates with known test mass eigen-modes.
Figure 8.12: Amplitude Spectral Density of various mode groups that clearly display extra modes in the 15070 Hz mode group. Where these modes have been identified they are labelled above the peak.

Figure 8.13: Amplitude Spectral Density of the 15600 Hz mode group clearly display extra resonances but resonances are not all identified. Where these modes have been identified they are labelled above the peak (ETMX+ is beating with another mode in this case).

Several avenues have been studied in an attempt to explain these resonances. The following paragraphs contain a brief description of each attempt and each outcome.
8.2.2.1 High coupling to violin modes

The first measurement of the 18085 Hz mode linewidth indicated the extra mode’s Q factor was a lot lower $10^4 - 10^5$ which suggested this resonance might have a different origin such as violin modes. The violin modes are the string modes of the fused silica fibers that support the test mass. Their fundamental frequency is 500 Hz. However harmonics are visible in the interferometer output out to 3 kHz. At 15 kHz we might expect something like the 33-35$^{th}$ harmonic to be resonant (The violin mode ‘harmonics’ are slightly an-harmonic due to a taper in the fiber at the attachment point [230]). This explanation may be satisfactory to explain one-off resonances like the one seen at Livingston but it is very unlikely to explain all the modes seen in Figures 8.12, 8.13 and 8.14.

8.2.2.2 Geometrical asymmetry

Generally any symmetric shape will have degenerate eigen-modes. Degeneracy meaning that several modes with different shape/orientation have the same frequency and are therefore not distinguishable. Changing the geometric symmetry will break the degeneracy leading to splitting of modes with different shape/orientation in that symmetry. This can clearly be seen with the LIGO test masses. The LIGO test masses are close to being cylinders. The flats and ears however break the rotational symmetry and this can be seen in the mode structure as a separation in frequency of the
vertically and horizontally orientated modes of about 0.5% of the mode frequency. This can be seen in Figure 6.1 on page 174 where mode group A at 15kHz is the rotated version of mode group B at 15.07kHz.

There are several symmetries in the test mass, specifically these are up-down, left-right, and front-back symmetries, as can be seen in Figure 8.15. However there are very few degenerate mode frequencies due to these symmetries. Particularly there are no degenerate modes less than 16kHz.

Figure 8.15: Symmetries of the Advanced LIGO test mass, highlighting points at which symmetries are broken. (left) Front to back symmetry broken by the RoC and wedge in the optic. (center) Up to down symmetry is broken by the ears and wedge in the optic. (right) Left to right symmetry is nominally perfect.

Figure 8.15 shows the front-back symmetry is broken by the wedge in the optical faces, as well as the RoC of the front surface. These are very small geometric asymmetries that have negligible effect on mode frequencies.

The up-down symmetry is also broken by the wedge. In addition the ear placement below the axis of rotation of the cylinder breaks this symmetry.

The left-right symmetry should only be broken by manufacturing tolerances which are very tight compared to the other geometric asymmetries presented in the previous paragraphs.

COMSOL was not able to identify any extra modes associated with the design geometry. To investigate whether the closely spaced modes were somehow not being resolved by COMSOL the asymmetries were amplified in simulation.

1. The ear placement tolerance was amplified several 1000-fold by placing one ear 2cm from its nominal position and/or rotating it 10 degrees to break the left-right symmetry. There were no new modes observed in the COMSOL simulation. Mode frequencies around 15kHz changed by $\sim$1Hz.

2. The wedge angle was increased 10-fold (0.75 degrees) and RoC decreased 100-fold (20m). COMSOL simulations did not produce any new 15.7kHz or 15.6kHz
modes. Mode frequencies changed by $\sim 10$ Hz.

3. The ears were made 5-8 times heavier. This had a significant effect on the horizontally oriented modes like the 15070 Hz group, changing the frequency considerably and also creating 'extra' modes. It also created an extra 15 kHz vertically oriented mode but not the 15.54 kHz mode, shifting its frequency marginally to 15.52 kHz.

Figure 8.16: Surface deformation and total deformation of modes that split when heavy ears are used in the COMSOL simulation. Top Row - The ears density is increased to $8 \times 2203 \text{kg/m}^3$ the 15600 Hz group split into a 15457 Hz and 15523 Hz modes. Second Row - The ears density is increased to $5 \times 2203 \text{kg/m}^3$ the 15070 Hz group split into a 14726 Hz and 14957 Hz modes. Third Row - The ears density is increased to $8 \times 2203 \text{kg/m}^3$ the 15000 Hz group split into a 14993 Hz and 15023 Hz modes.

The results of the heavy ears simulation are presented in Figure 8.16. The surface deformation of the extra modes show differences in up-down symmetry, while left-right symmetries are maintained (front-back symmetry is also maintained but this is not clear in these images). The high amplitude of the ear deformation that can be seen in the 3D deformation images is common to all extra modes. This indicates the fiber suspensions (not modelled here) might be relevant in these simulations.  

This investigation is preliminary and needs a more careful investigation. If this explanation was correct, one would expect a geometric asymmetry that is identical

\footnotesize{Tracking the 'heavy ear' extra modes through a sweep in ear density it has since become clear that these modes decrease in frequency significantly with ear mass indicating they may be ear modes that have just by chance ended up near the test mass modes.}
for all test masses to result in a measured split frequency that is also identical for different test masses. This also is not observed in experimental observations.

### 8.2.2.3 Thermal asymmetry

Another possible cause of asymmetry is the thermal asymmetry caused by the ring heater heating the outer portion of the test mass at the back and the cavity beam heating the central portion of the front of the test mass as shown in Figure 8.17. Again using COMSOL simulations thermal loads that produce 100 times the thermal gradient expected in the optic were examined. No extra modes were observed.

![Figure 8.17: Thermal front-back asymmetry in the test masses with colour showing test mass temperature. 0.2 W heating from main laser absorption is visible on the left surface and the ring heater heating a ring near the right end of the test mass with 1.6 W is visible](image)

### 8.3 Electrostatic Damping of Parametric Instability with Thermal Transients

There are large thermal transients in the Advanced LIGO interferometer test masses. These transients will get larger as the stored optical power is increased to achieve design sensitivity. These transients make damping parametric instability challenging. In this section the challenges encountered will be discussed along with solutions where applicable.
8.3.1 Acoustic Mode Frequency Transient

As the test masses warm, their eigen-frequencies increase as demonstrated in Sections 5.2.3.1 and 7.3. In Section 6.2 damping was demonstrated with a 10 Hz wide band pass filter. This filter linewidth was chosen such that it separates most eigen-modes. However there are some modes such as the 15541 ETMX and 15542ETMY separated by roughly 1 Hz. To separate these modes using filtering requires $\sim 1$ Hz wide filters. An 8th order Butter-worth 1 Hz wide band-pass filter bode plot shown in Figure 8.18, gives 10 dB rejection of a mode 1 Hz away.

![Figure 8.18: Eighth order Butterworth filter bode plot](image)

The rate of change of phase with frequency of this filter is greater than 120 deg/Hz. This is untenable when we consider the result from Section 7.3 were it was shown that the mode frequency shift is 0.2 Hz/100 kW of cavity power. With changes in ambient temperature in the LIGO facilities are included, the changes in eigen-frequencies increase to 1 Hz.

From many hours damping parametric instability through thermal transients it is apparent that the bandpass filter is not the only cause of the change in the damping phase. It has been observed that when large damping forces are applied, the damping itself can change the eigen-frequency by several Hz. This is expected if the damping phase does not result in perfect viscous damping, because the proportional gain component of the feedback loop acts as a modified spring constant which changes the resonant frequency.

Thermal transients also change the radius of curvature of the mirrors and hence the optical transfer function changes. We would expect a 180 degree change in the optical transfer function phase as we go from one side of the TEM$_{pq}$ resonance to the other. This provides ever increasing motivation to reduce the thermal transient in the RoC of the arm cavity mirrors. This will be explored further in Section 8.3.2.
Some modes have been observed to cross in frequency as one test mass cools and another heats. This gives rise to a beating signal which makes the active damping ineffective. It would therefore be desirable to make the filters as narrow as possible so that duration of the beat signal can be minimised. This is possible using phase locked loops (PLL)s to track the eigen-frequencies. Arbitrarily narrow filters may be used as long as they result in a stable phase locked loop. The problem of phase changes associated with changes in mode frequency relative to the filter would then be eliminated.

Several approaches have been studied with the intent of controlling parametrically unstable modes that cross in frequency. Ed Daw and his group have developed a tool called i-wave [89]. It is a dynamic infinite impulse response (IIR) band-pass filter with some similarities to a PLL. I-wave has been demonstrated to track crossing modes.

Equivalently a conventional digital PLL has been implemented in CDS and investigated. Appropriate filtering on the PLL error signal is required. From preliminary investigations the large beat in the error signal from the approaching modes results in an extended period where the error signal is less effective and the PLL often jumps modes. A smarter system that monitors multiple modes from the same test mass and uses the correlation in eigen-frequencies in the test mass may provide a solution to crossing mode frequencies with a conventional PLL.

Another possible solution is to find error signals that better discriminate between test masses. There are QPD’s in transmission of each arm. As we saw in Section 5.2.2.3 the mode amplitude is largest in the arm that contains the resonant test mass. By suitably combining the QPD signals, with the appropriate relative phase, a signal that maximises a resonance in one arm and minimises the resonance in the other arm is possible.

Similarly there are two QPDs in transmission of each arm. The two QPDs measure different guoy phases of the cavity. Generally one would expect that ITM resonant motion would have a different amplitude ratio in the QPD’s to the amplitude ratio of ETM resonant motion. It may be possible to isolate ITM and ETM modes in much the same manner as auto-alignment systems [115] with appropriate combinations of the two QPDs signals. However the readout at the arm transmission is readout of TEM\textsubscript{pq} and we would expect the TEM\textsubscript{pq} generated by the ITM to be identical to the ETM so the method is unlikely to be possible.
8.3.2 Optical Mode Transient

The optical mode transient caused by changes in each mirror RoC as they warm, causes changes in the parametric gain and creates transient parametric instability as explored in Section 6.2.3 and 8.2. Test mass modes become unstable when an interacting optical mode sweeps in frequency past the acoustic mode and then become stable again as the optical modes moves farther from the acoustic mode resonant frequency. The change in the HOOM spacing due the the thermal deformation of the front surfaces of the mirrors results in many more unstable modes than predicted by simulations for a static interferometer. As the optical power in the interferometer increases the thermal transient increases meaning that the number of unstable test mass modes will increase dramatically. These modes will not all be unstable simultaneously, however from a practical perspective it is still problematic as each unstable mode needs a tuned damping control system.

There is another effect that was touched on briefly in Section 8.3 - the phase component of optical transfer function. As an optical mode frequency passes an acoustic mode resonant frequency the phase of the scattered light resonant in the cavity changes by 180 degrees. This means that the phase of the signal on the photodiodes will also experience a change of phase relative to the eigenmode phase. Hence this phase gets added to the control loop phase. In Figure 8.19, this single cavity idealisation is used to show the expected optical transfer function $|G_{03}|$ for optical mode spacing $\omega_{00} - \omega_{03}$ in the range thermal states expected when Advanced LIGO operates at its design optical power of 800kW.

In the figure it is clear that the changes in transverse optical modes spacing currently experienced, $\sim$100 Hz, cause severe changes in damping phases. A control system that is marginally stable, damping a parametrically unstable mode in one thermal state, may be exciting it in another thermal state. This may explain the unexplained changes in phase described in the previous section.

The complexity and amount of infrastructure required for electrostatic damping hundreds of unstable modes with loop damping phase changing through the thermal transient may make this approach unattractive. For this reason in Section 8.3.3 methods for reducing the thermal transient are discussed.

In Section 8.4 the current infrastructure for damping up to 32 unstable modes is described.
8.3.3 Reducing Thermal Transients

The thermal transient in the optics not only makes damping parametric instability difficult, it also affects the mode matching of various cavities in the interferometer and *wave front sensing* signals that are used for interferometer alignment.

For parametric instability active compensation with the ring heater may be used to reduce the thermal transient in the RoC. However the ring heater response is slow. It takes almost 15 min for a change in ring heater power to significantly affect the optic RoC. This means that any feedback system would not be able to keep up with the thermal transient when the optical power is first incident on the optics. A scheme could be devised where the ring heater changes in advance of locking the interferometer.

There is an additional issue for parametric instability. If the scheme from the previous paragraph worked the ring heater would maintain the RoC but would be unable to maintain the test mass eigen-frequencies at the same time. To achieve an approximately stable RoC the ring heater must apply significant additional heating that results in more than twice the change in the temperature of the optic compared to the thermal transient without RoC correction. To avoid this problem another approach has been explored.

Preheating the test masses is a more elegant approach. The aim of such an approach is to always maintain the 'hot' state. This may be achieved using a CO$_2$ laser in the geometry described in Sections 4.3 and 6.3.3. For the purpose described
here the system must be designed to heat the test mass with the same heating profile and heat load as the circulating beam in the cavity applies when the interferometer is locked. CO$_2$ lasers must then apply this heat load to the test masses whenever the cavity unlocks, maintaining the heat load and distribution on the test mass.

When the interferometer is being brought into its locked state, optical power is increased in various optical cavities. This increase in optical power generally follows a step series like function, but it is somewhat arbitrary - depending on environmental conditions. As this power is increased the CO$_2$ laser power can be decreased with a feedback system from the optical power transmitted from the relevant arm to maintain a constant heat load. With no change in the heat load or distribution on the test mass there should be no thermal transient in either the RoC of the optic or the eigen-frequencies of the optic.

8.3.4 Parametric Instability Induced Noise in Gravitational Wave Detection Band

It is important to consider the possibility that parametric instability could cause noise in the gravitational wave frequency bands. Two mechanisms are explored in this subsection.

1. Actuation forces that are applied to damp parametric instability may be down-converted through mechanisms such as the quadratic term in Equation 6.4 resulting in noise in the gravitational wave frequency band.

2. Beating of optical modes could provide a means of down-conversion and transmission of signals to the output mode cleaner where they may be coupled to the fundamental mode output.

A preliminary noise investigation was performed when doing the experiments reported in Section 6.2. A test mass eigenmode was driven to a large amplitude. At this large amplitude the control gain was was applied as described in Section 6.2. It was increased until the control signal reached a defined proportion of the control range. At about 1/10th the control range, low frequency noise appeared in the main interferometer output. At 1/3rd of the control range noise with an one divided by frequency squared characteristic appears. Assuming a linear projection this noise is a factor of 100 below the noise floor in the damping experiments presented in Section 6.2 when the mode was in the damped state. It must be noted that this preliminary noise investigation was performed before installation of the new electronics presented in Section 8.4. New 10-80 kHz bandpass analogue filters filter out low frequency signals, possibly reducing the coupling to low frequency force noises.
A second noise process was observed when two test mass eigenmode’s amplitudes were elevated and being actively controlled, and possibly saturating. This was observed at LIGO Hanford with the electronics described in Section 8.4. Modes at 15542 Hz and 15009 Hz were elevated \( \sim 10,000 \) times quiescent levels. The resulting beat note between the two modes appeared in the main interferometer output at 532 Hz. It was observed to have a functional form of approximately 

\[
x_{532} = \frac{1}{4} \sqrt{x_{15542} \times x_{15009}}
\]

as shown in Figure 8.21. Two possibilities are suggested for the noise coupling described in Figure 8.21. The ESD drive signal could be down-converted to a low frequency force noise, as observed at LIGO Livingston in Figure 8.20 or the optical modes are beating to produce down-conversion. Either way such coupling is problematic. It can produce signals in the gravitational wave output.

A detailed investigation to determine the means by which signals couple to the baseband gravitational wave detector output is required. With a clear understanding of the mechanism the relation between mode amplitudes, control gains and frequency can be used to do noise projections to ensure parametric instability is not corrupting gravitational wave data.

### 8.4 Upgrades for High Frequency Sensing and Actuation on Parametric Instabilities

Prior to commissioning the parametric instability damping system at Hanford, all digitised signals passed through \( \sim 10 \text{kHz} \) low-pass anti-aliasing (AA) filters and all
Figure 8.21: Two test mass eigenmodes 15542 Hz and 15009 Hz have amplitudes ~10,000 times quiescent levels, ESD feedback control is on, possibly saturating. The beat note at 532 Hz is ~ one order of magnitude above the noise floor. This plot show that the produce of the 15542 Hz mode amplitude and the 15009 Hz amplitude is approximately proportional to the amplitude of the 532 Hz noise.

actuation signals coming from the digital system passed though ~10 kHz low-pass anti-imaging (AI) filters. The AA filter prevents high frequency content contaminating the sampled data. The AI filter converts the 'sample and hold' square waveform produced by the digital to analogue conversion into a smooth waveform by removing the unwanted high frequency content.

The signals used for parametric instability measurement and actuation are the first signals at Advanced LIGO to use the digital to analogue conversion (DAC) and analogue to digital conversion (ADC) sample rates of 65536 Sa/Sec. The AA and AI filters significantly attenuated parametric instability signals. These filters were replaced with 10-90 kHz band-pass filters for the parametric instability sensing and actuation signals.

8.4.1 New Electronics

Using a 65536 Sa/sec sampling system for signals up to 90 kHz is unconventional. As previously mentioned, the Nyquist frequency criterion states that signals greater than the Nyquist frequency (32,769 Hz) will be aliased. Aliasing results in a down-conversion in frequency. This is shown graphically in Figure 8.22. The aliasing results
in super-Nyquist frequencies $F_{sn}$ being down-converted in frequency to a base-band frequency $f_{bb}$ by:

$$f_{bb} = f_s - (f_{sn} \mod f_s), \forall \left\lfloor \frac{f_{sn}}{f_s} \right\rfloor \in \mathbb{O}$$

$$f_{bb} = (f_{sn} \mod f_s), \forall \left\lfloor \frac{f_{sn}}{f_s} \right\rfloor \in \mathbb{E}$$

(8.4)

Here $f_s$ is the sampled frequency, $\mathbb{O}$ are odd integers and $\mathbb{E}$ even integers. Normally this process corrupts information in the base band frequencies $f_{bb}$. However for parametric instability signals we are only interested in the frequency content within the linewidth of the acoustic mode. As acoustic mode linewidth is very small there is little chance that super-Nyquist acoustic modes alias to the the same frequency as the baseband modes.

However as we saw in Sections 7.3 and 7.4 the thermal transient results in these modes changing frequency. Baseband modes increase in frequency with temperature while super-Nyquist modes between 32768Hz and 65536Hz will be reducing in base-band frequency with temperature at a rate twice that of base-band modes on average. This significantly increases the risk that modes from the base-band will cross the baseband frequencies of super-Nyquist modes. This is shown graphically in Figure 8.23.

The main purpose of the new electronics was to be more sensitive to parametrically
unstable modes and to have more actuation strength from the electrostatic drive (ESD) at parametric instability frequencies. The high frequency sensing electronics was installed on a new signal path for a single QPD (QPD B) in transmission of both arm cavities and also on the OMCPDs. The arm transmission QPDs are generally used in parametric instability control loops while the OMCPDs are generally used for monitoring. The dramatic increase in the sensitivity to high frequency modes can be seen in the section on mode identification to 90 kHz, Section 8.4.4, Figure 8.4.4.

The other half of the story is high frequency actuation. The same filters were also installed on a separate ESD actuation path installed for damping parametric instability. The principle of aliasing in ADC has an equivalent principle called imaging where signals are up-converted by the digital to analogue conversion using a sample and hold technique. Imaging is a perfect mirror of aliasing, such that a mode aliased from $f_{sn}$ to $f_{bb}$ when applied to the DAC will be imaged to produce a signal at $f_{sn}$, in addition to signals at $f_{bb}$, $f_{s} + f_{bb}$, $2f_{s} - f_{bb}$, etc. These additional signals result in the same increased likelihood that modes will overlap during a thermal transient. Crossing modes are less problematic when actuating as it is very unlikely that the actuation pattern for a parametrically unstable mode will have high spatial overlap ($b_m$ from Section 6.2) with the other mode. Even if there was high overlap the modes would have to coincide in frequency for a long time for the mode amplitude to increase by a factor of $10^4$ where it starts becoming a problem for operation of the interferometer.
It must be noted here that both analogue to digital conversion and digital to analogue conversion have transfer functions inherent to the processes. With sample and hold type conversion a zero order hold model can be used to show this sampling function is a sinc function. However generally any sampling method will have a zero in its transfer function at the sampling frequency.

Figure 8.24: Analogue to digital conversion and digital to analogue conversion response function, with anti-imaging/aliasing filter superimposed.

The red curve in Figure 8.24 shows the sampling transfer function. Zero response at multiples of the sampling frequency (65536 Hz) highlight the fact that this system will have very little sensitivity to modes in the vicinity of the sample frequency.

Equivalently the sampling response function becomes the dominant feature in the actuation transfer function as can be seen in the measured ESD actuation in Figure 8.25. There is therefore very little actuation voltage applied to the ESD at these frequencies.

Figure 8.25: Actuation transfer function from control counts to Volts on four quadrants of the ETMY electrostatic drive, compared to the model transfer function
The low sensitivity and small actuation voltages available at frequencies close to the sample frequency make potential parametric instabilities in this frequency region a high risk to interferometer operation. If parametric instability arose in this region transverse optical mode tuning with the ring heaters or reducing the optical power would be required to maintain stability. Either of these options may have consequences that limit the detector sensitivity. This region in the subject of the following section.

8.4.2 Parametric Instability Dark Region

The sampling function of a system that utilises aliasing to monitor super-Nyquist frequencies has zero amplitude response at the sampling frequency. There is a region between $\sim 60 \text{kHz}$ and $\sim 70 \text{kHz}$ where the monitoring system presented here has poor sensitivity to parametric instability and a region from $\sim 65-66 \text{kHz}$ where it has no sensitivity to parametric instability. This means that if there was parametric instability between 65 and 66 kHz the instrument may unlock before any signal is registered by the digital system.

Regular Analogue monitoring of this band was performed with a SR785 spectrum analyser connected to a single quadrant of the Y arm transmission QPDs at Hanford to ensure there were no unstable modes in this region prior to Advanced LIGO’s Observation Run 2. There were some modes that appeared to behave like test mass eigen-modes as indicated in Figure 8.26.

![Figure 8.26: Peaks that behave like test mass eigen-modes in the region with no sensitivity](image)

Figure 8.26: Peaks that behave like test mass eigen-modes in the region with no sensitivity
These peaks were only visible in the spectrum when there was high optical power in the arm cavities. Their amplitudes rose and fell like other eigen-modes and the amount the frequency changed during the periods they were monitored was in the order of 1 Hz consistent with other eigen-frequencies. These signals were not visible in the sampled data. However none had amplitude excursions greater than a factor of 100. The region appears parametrically stable for Advanced LIGO’s Observation Run 2 configuration. Ongoing monitoring of this region will be essential when interferometer configuration changes take place.

In the event of parametric instability in this region one solution would be to use analogue down-conversion of sensing signals and up-conversion of control signals with a common analogue oscillator. Filtering the QPD signals with a 61-69 kHz analogue bandpass filter then multiplying by a 61 kHz oscillator would produce a 0-8 kHz signal that may be sampled in a regular 16 kSa/sec CDS channel.

8.4.3 Parametric Instability Damping configuration

There are several other aspects of damping parametric instability that were addressed when commissioning the electrostatic damping system. Two examples follow. (1) How to get signals from the arm transmission QPDs to the ITM ESDs for damping ITM modes. (2) How to efficiently implement tens of phase lock loops tracking parametrically unstable modes.

The configuration that was developed uses digital heterodyne down-conversion of parts of the spectrum where parametrically unstable modes are present to (a) efficiently send information from point to point using a smaller bandwidth than the full 65 kSa/sec sample rate and (b) allow the phase locked loops to operate at a slower clock cycle greatly reducing the amount of computation time required.

A schematic of the damping configuration is displayed in Figure 8.26. (For the complete flow diagram please see Appendix C)
Figure 8.27: Signal flow diagram of parametric instability damping system for LIGO Hanford Observation Run 2. Grey blocks are real time loops running on a particular computer, green blocks are electronic components. Only components for the X arm are displayed. They are replicated for the Y arm. Highlighted numbers are explained in the text.

In Figure 8.27 the parametric instability damping signal can be traced from the QPD, labelled (1), through the whitening\(^3\)(2) and AA filters (3). A cartoon spectrum is shown (4) indicating the location and amplitude of several modes including superyquist modes. After sampling by ADC1 (5) the spectrum of the sampled signal is shown (6) with colour coding maintained. In the h1susex real time computing loop (7) the signal is split into 8 × 1 kHz bands with heterodyne down-conversion. These signals are multiplexed (8) onto a 16 kSa/sec communication channel and sent from the end station to the interferometer corner 4 km away (9).

\(^3\)Whitening filters optionally apply between -60 and 15 dB gain and up to 6 orders of high pass filtering
At the corner station the signals are de-multiplexed (10) in h1omc.mdl real time loop. Signals from the two end stations are passed to the h1susprocpi.mdl real time loop for processing into control signals (11). The output mode cleaner transmission signals are also available from h1omcpi.mdl even though they are generally only used for monitoring (12).

In the h1susprocpi.mdl loop (11) linear combinations of error signals can be chosen. These signals are then band-pass filtered. Peak tracking with a phase locked loop or iWave can be applied. Finally linear combinations of modes can be added to the appropriate communications channel. This communications channel selects the test mass and frequency band for damping.

ITM channels are sent directly (13) to the ITM which is a simplified version of the ETM signal flow path. ETM channels are sent back to h1omc.mdl to again be multiplexed (14) to send down the arm (15). They are de-multiplexed at the end station in h1pemex.mdl (16) and up-converted to the same frequency they were sampled at in h1susetmxpi.mdl (17). A spectrum showing the same set of modes is shown (18). In reality only a subset of modes detected by the ETMX QPD will be sent to the ETMX ESD as many of the signals sensed at ETMX are ITMX modes and ETMY and ITMY modes are also visible in the ETMX QPD. The DAC (19) creates high frequency images, up-converting modes to their native super-Nyquist frequencies (20). Note this makes a large number of redundant signals. The output of the DAC is then sent through the 10-80kHz band-pass anti-imaging filter (21) and low noise ESD driver (22) before being applied to two quadrants of the ESD (23). Two quadrants were chosen for historical reasons. In the future four quadrants may be used to increase the available actuation force on certain eigen-modes.

8.4.4 Mode Identification to 90 kHz

The new high frequency sensing and actuation electronics described in Section 8.4 greatly increased the number of modes visible in the OMC transmission spectrum. In Figure 8.28 a sense of the increase in number of modes can be ascertained. This measurement was taken with the whitening filter gain adjusted to provide approximately the same dynamic range for the 15kHz modes.

Several small mode identification data sets have been used to identify some subsets of visible modes. Observations of modes up to 55kHz have been made. Mode identification of all visible modes 5kHz to at least 48kHz will be performed when possible as this will greatly increase our confidence in the 47.5kHz test mass eigen-mode shape.
Figure 8.28: The new high frequency electronics shown in blue is sensitive to many more resonant peaks in the right half of the plot.

8.5 Three Mode Interaction Monitoring of the Optical Transfer Function

As discussed in Chapter 5 very little information is currently obtained from the interferometer about the optical transfer function. During my time at Livingston and Hanford I tried several methods for measuring aspects of the optical transfer function.

Once a mode is parametrically unstable it provides a very good measure of the optical gain of the ensemble of optical modes that interact with it, at the frequency of the unstable acoustic mode assuming the other parameters in the parametric gain formula are known. When electrostatic damping is used to stabilise parametric instability the open loop gain of the damping control loop provides a measure of the optical gain. And the open loop phase may be used to provide a relative measure of the optical transfer function phase.

Each eigen-mode samples a particular frequency of the optical transfer function. It would be desirable to obtain information from more eigen-modes to obtain more information about the transfer function. Obtaining information from eigen-modes that are parametrically stable would be desirable. This is similar to the problem described in Chapter 4.

Using the participation theorem of John Waterston [252] thermal calibration [144] of acoustic modes may be performed by assuming that the acoustic mode is in ther-
mal equilibrium at some point. At Advanced LIGO where parametric gain can significantly alter mode amplitudes the start of a lock stretch is the best approximation of thermal equilibrium. The assumption of thermal equilibrium has been debated because there are ‘strong’ electrostatic forces that catch the test masses when locking [275]. These forces could excite the test mass eigen-modes during the lock acquisition process. These lock acquisition signals are filtered by low pass filters so there so the power at test mass eigenfrequencies should be attenuated.

The amplitude of an eigenmode will change when acted upon by a three mode interaction radiation pressure forces. This was expressed by Evans [110] as,

$$\frac{\Delta A_m}{A_m} = \frac{4\pi Q_m P}{M \omega_m^2 c \lambda_0} \sum_{n=0}^{\infty} G_n B_{m,n}^2.$$  (8.5)

Using the same simplification as in Chapter 6, Equation 6.2 where we assume that the interaction is with only one optical mode.

$$\frac{\Delta A_m}{A_m} = \frac{4Q_m P}{ML \omega_m^2 \lambda_0 \gamma (1 + \Delta \omega_{m,n}^2 / \gamma^2)} B_{m,n}^2$$  (8.6)

We assume the the original acoustic mode is in thermal equilibrium and that all terms other than $\Delta \omega_{m,n}$ are constant in equation 8.6. $\Delta \omega_{m,n}$ can be expressed as a function of $\Delta A_m$.

$$\Delta \omega_{m,n}^2 = \left(\frac{4Q_m P}{ML \omega_m^2 \lambda_0 \gamma} \frac{A_m B_{m,n}^2}{\Delta A_m} - 1\right) \gamma^2$$  (8.7)

This equation has two solutions for a single measurement of $\Delta A_m$, however if two modes that interact with the same optical mode are available a single solution can be obtained.

It must be clarified which amplitude we are talking about. When the parametric gain changes, two things are changing. The Q factor of the mode increases by $(1/(1 - R_m))$ derived from Equation 6.3 in Section 6.2.2, that produces an increase in the modes absolute amplitude but does not increase the rms amplitude of the mode. Energy is also imparted to the mode producing an increase in the rms amplitude of the mode by the factor $1/(1 - R_m)$ as presented in Section 7.2. When a mode’s amplitude is small, as is generally the case for parametrically stable modes, the only way to measure them is to take the ASD of a long time segment. As explored in Section 5.2.4 to date there is no scheme to resolve the acoustic mode linewidth. Measurements that do not resolve the linewidth provide estimates of the rms amplitude. Information on the absolute amplitude is lost.
We must also be clear where the information is being measured. The QPDs in transmission of the arm measure the transverse mode amplitude. This is resonantly enhanced by the readout gain of Equation 8.5.

\[ G_{ro} = \sqrt{R_m} \tag{8.8} \]

The point here is to measure parametrically stable modes, at current operating power very few modes are visible in the arm transmission QPDs as only some modes have sufficient readout gain. Using the OMC transmission provides a larger signal to noise ratio, but depending on the coupling mechanism (section 8.1.2) the observed amplitude may be proportional to the mode amplitude, or to the mode amplitude multiplied by the readout gain.

This scheme could be extended by identifying more modes and their interacting optical mode. With a clear understanding of the OMCPD signals many modes are potentially available to produce a more complete estimate of the optical transfer function of the advanced LIGO cavities. This should be performed in conjunction with simulation of dual recycled Michelson interferometers to produce improvements on the single arm cavity simplification used here.

### 8.6 Chapter Summary

In this chapter the main issues in commissioning a parametric instability control scheme have been discussed. Reducing thermal transients will greatly simplify parametric instability control. Electronics designed specifically for parametric instability have been described and there is a discussion of the control scheme design for Advanced LIGO’s Second Observation Run.

The need for acoustic mode identification to 90kHz has been emphasised. This will improve our understanding of the new acoustic modes presented in this chapter. But more importantly will prepare the commissioning team for future increases in power where more acoustic modes are sure to become parametrically unstable. Measurements of acoustic mode quality factors will also greatly improve our ability to predict when modes will become unstable. A combination of acoustic mode identification and Q factor measurements will allow better predictions of parametric instability. Such predictions will guide the way to Advanced LIGO high power operation where electrostatic damping may be adequate or acoustic mode dampers described in Section 3.3.2 may be required to control parametric instability.
As discussed in Section 8.3 thermal transients in the optics are one of the major difficulties in damping parametric instability. Here a system using CO$_2$ lasers is describe that will nominally give zero thermal transient in either test mass eigenfrequency or optical transfer function. This system needs to be tested.

There is also a wealth of information available from three mode interactions. This was the subject of Chapter 7 but is further expanded in this discussion chapter to include potential applications arising from a better understanding of the interferometer.

I give two examples. (1) In Section 8.1 we saw how a better understand the acoustic modes coupling mechanism to the output mode cleaner could provide a wealth of information about the interferometer such as alignment signals and beam size signals. (2) Signals from the output mode cleaner can also potentially give a method for real time monitoring of the interferometer optical transfer function for different transverse electromagnetic modes as discussed in Section 8.5.
Chapter 9

Conclusion

Parametric instability is a fundamental problem in gravitational wave detectors that use high optical power to achieve optimum quantum noise limited strain sensitivity at 100 Hz. In this thesis I describe the control of parametric instability at Advanced LIGO. In Advanced LIGO’s first Observation run (2015-2016) thermal detuning was used to avoid the three mode resonance that causes parametric instability. This detuning allowed the optical power to be increase from $\sim 5\%$ (40 kW) to $\sim 12\%$ (100 kW) the design operating optical power. At this optical power the two Advanced LIGO instruments detected the first two gravitational waves ever detected.

The characterisation of parametric instability presented in Chapter 5 demonstrates that thermal tuning will not work when intra-cavity power reaches 300 kW. To prepare for higher optical power, electrostatic control of parametric instability was demonstrated, this demonstration is reported in Chapter 6. This technique was demonstrated with the interferometer tuned for instability with 100 kW intra-cavity power. Effective damping was demonstrated with negligible noise introduced to the gravitational wave channel. By extrapolation from the electrostatic damping experimental results it was estimated that this technique will be able to control parametric instability at design operating power of 800 kW. In Advanced LIGO’s second observation run (2016-2017) electrostatic damping has become an integral part of operating the Hanford instrument with 200 kW intra-cavity power. Electrostatic damping was required in the second observation run because of large thermal effects.

Thermal effects in the test masses create large changes in parametric gain. These effects are studied in Chapter 5. In Chapter 8 the challenges thermal effects create for the electrostatic damping technique are discussed. In Chapter 7 two monitoring systems that rely on three mode interactions are shown to produce information about the thermal state of the test mass. Such information may be useful for diagnosing
systems such as the $CO_2$ laser preheating system discussed in Chapter 8 that could ideally eliminate thermal transients.

Electrostatic actuation on acoustic modes uncovered extra unidentified resonances that also interacted with actuation forces. In Chapter 8 it is shown that these extra modes are likely to be associated with a vertical asymmetry caused by the suspension structures attached to the test mass. Better understanding of these extra modes will allow an electrostatic damping system to be designed that is compatible with these extra modes.

Electrostatic damping is dependent on the quality of acoustic mode signals. The arm transmission signals are well understood and provide reasonable quality signals. In chapter 8 it is highlighted that the quality of the arm transmission signals at Advanced LIGO are not as good as equivalent signals at the Gingin High Optical Power Test Facility. This discrepancy requires further investigation. At Advanced LIGO acoustic mode signals in the transmission of the output mode cleaner have high signal to noise ratio. In Chapter 8 preliminary investigations into these acoustic mode signals reveal a likely mechanism for coupling the mechanical motion to the output mode cleaner transmission. However rough estimates of coupling are discrepant with measurements. Experiments to determine the mechanism and quantify the coupling are suggested.

Experimental results from the Gingin facility that provide a method of low power characterisation of parametric instability are presented in Chapter 4. This type of characterisation should be performed at Advanced LIGO when time permits to prepare for future increases in optical power. Such a characterisation requires resonant modes of the test mass to be identified by eigenmode and test mass. The mode identification presented in Chapter 5 can be used to identify acoustic mode from 5-90 kHz as discussed in Chapter 8.

The experimental investigations were preceded in Chapters 1, 2 and 3 by a basic introduction to gravitational waves, optomechanics and parametric instability in gravitational wave detectors. These chapters were designed to provide an introduction to the key experimental concepts, particularly to engage an audience such as new students entering the field. The possibility of using three mode interaction monitoring for a variety of exciting applications is discussed in Chapter 7. With so many potential investigations I hope that new students may be inspired to take on the challenge of untangling the acoustic mode degrees of freedom so that some of these monitoring schemes such as three mode monitoring of the optical transfer function discussed in Chapter 8 may be realized.
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Appendices
Appendix A

Thermoelastic Formfactors of Advanced LIGO Test Masses

This appendix provides a list of images of the formfactors that are used for converting changes in thermal profiles to changes in testmass elastic eigenfrequencies and vice versa. The set of formfactors presented here are the 26 formfactors used for the estimate of the thermal profile in the Advanced LIGO test mass on the 18 August 2015 as presented in Section 7.4.7.

The list provides the reader with the ability to visually evaluate the inherent limit to the spatial resolution that can be achieved with this set of 26 eigenmodes.

Figure A.1: Images of the two highest frequency formfactors for Advanced LIGO test masses used to estimate the thermal profile in Section 7.4.7. Mode frequencies are labelled to the right of the formfactor image.
Figure A.2: Set of images of the formfactors for Advanced LIGO test masses used to estimate the thermal profile in Section 7.4.7. Mode frequencies are labelled to the right of the formfactor image.
Figure A.3: Set of images of the formfactors for Advanced LIGO test masses used to estimate the thermal profile in Section 7.4.7. Mode frequencies are labelled to the right of the formfactor image.
Figure A.4: Set of images of the formfactors for Advanced LIGO test masses used to estimate the thermal profile in Section 7.4.7. Mode frequencies are labelled to the right of the formfactor image.
Appendix B

LIGO Testmass Eigenmode Identification

This LIGO report on test mass elastic mode identification from thermal dependence of eigen-frequencies. This is published in the internal LIGO Document Control Center (DCC).
Identification of Test Mass Resonant Modes

Blair C.

Distribution of this draft:

LIGO Scientific Collaboration
Summary

• A comparison between a COMSOL model of the test masses and measured resonances is made and a selection of mode shapes are presented

• A method for identifying which test mass a resonances appears in is described and results presented

• Using this mode identification of LLO ETMY modes the Young’s modulus and Poisson ratio of a COMSOL model are tuned to minimize the rms residual between the model and measurement

• The fitted Young’s modulus is 72.7Gpa and the Poisson ratio 0.164

1 Introduction

Parametric instability [1] has been affecting the operation off the Advanced LIGO detectors since November 2014 [2]. Instability is minimized by tuning the radius of curvature of the mirrors with ring heaters. Electrostatic drives acting on the acoustic modes are used in a feedback loop to suppress instability when the ring heaters are not sufficient. To function correctly the resonant modes must be identified and their behaviour must be understood. It is therefore desirable that the acoustic mode shape, frequency and location are known. This document specifies a method for identifying the majority of mode 5-18 kHz then some modes up to 40 kHz. Then the identification of LLO ETMY modes from 5-18kHz are used to fine tune the estimate of the Young’s modulus to 72.7GPa and Poisson ratio to 0.164.

2 Method

A COMSOL model of the test mass is used to estimate the mode frequency. The 3 dimensional model parameters are those measured pre-installation tabulated here [3]. Simplifications used in this model are that the wedge and radius of curvature are not included and that the ears and suspension fibers are not included. Also Young’s modulus and Poisson ratio have large uncertainties so significant offsets in frequency are expected. The COMSOL ’very fine’ mesh is used with ≈ 45,000 domain elements, the estimated error for this mesh depends on the complexity of the mode shape. Mode 5-10kHz error is 0.1Hz, 10-20kHz - 20Hz, 20-30kHz - 200Hz.

Other physical parameters required are listed in the following table.

<table>
<thead>
<tr>
<th>Model Parameters</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus @291K</td>
<td>$E$</td>
<td>$73 \times 10^9 Pa$</td>
</tr>
<tr>
<td>Poisson Ratio @291K</td>
<td>$\nu$</td>
<td>0.17</td>
</tr>
<tr>
<td>density</td>
<td>$\rho$</td>
<td>$2203 kg/m^3$</td>
</tr>
</tbody>
</table>
The test mass resonant modes are visible at various ports of the instrument, see figure 2. The signal to noise ratio varies significantly. The signal with the highest signal to noise ratio is the signals from the diodes in transmission of the output mode cleaner (OMC DC PD’s).

However signal from QPD’s provide more information about things like mode orientation, which was invaluable when first identifying the modes responsible for parametric instability. We also have a good understanding of the mechanism by which they are sensed. For this reason they were the first signals used to demonstrate electrostatic damping of parametric instability [4].

However the OMC DC PD’s generally have the best signal to noise ratio. For the purposes of this document discussion will be restricted to the OMC DC PD’s as we wish to identify as many modes as possible.
Figure 2: The sensors that are useful for sensing the test mass resonant modes. The arm transmis-

sion quadrant photo detectors ETMX and ETMY, QPDAs and QPDBs, the output mode cleaner re-

flection (OMC REFL) is junk light but has a similar signal to noise ratio as the arms transmission QPDs indicating some carrier in the junk light. The output mode cleaner (OMC) alignment sensing QPD’s (OMC AS) mechanism is understood, low power results in about the same SNR as the arm transmission. The photo-diodes in transmission of the OMC (OMC DC PDs) have about 30 times the signal to noise ratio of the arm transmission

3 Mode Shape

To identify the modes a measurement of the acoustic modes is compared against the model. Many of the acoustic modes are visible in transmission of the output mode cleaner. While the mechanism by which they arrive is not well understood, the frequency of these modes has been compared to the frequency of the modes measured in transmission of the arms and the agreement good. To measure modes up to 32kHz a signal sampled at 64KSa/sec must be used. There is no such stored signal so the test point for the photo diode channels must be used at LLO these are IOP_LSC0_MADC0_TP.CH12 and 13. Either DTT or gwpy may be used to grab live data from the test point. An 8 second psd is sufficient to identify many modes, however it may be desirable the use longer PSDs to improve the SNR.

Modes come in groups of 4, one for each test mass. From the sample data in figure 3 the modes can be lined up and compared to the COMSOL model. A selection of modes is shown overlayed in the figure from such an operation. The full list of modes is attached using the LLO ETMY model. A selection of modes is shown below with this simple model.
Figure 3: A spectrum from the IOP_LSC0_MADC0_TP_CH13 test point with a selection of mode surface profiles from COMSOL overlayed.

<table>
<thead>
<tr>
<th>Mode Shape</th>
<th>Simulated Frequency</th>
<th>Measured Frequency</th>
<th>COMSOL surface deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Butterfly</td>
<td>5932</td>
<td>5942 ± 6</td>
<td></td>
</tr>
<tr>
<td>First Drumhead</td>
<td>8182</td>
<td>8156 ± 3</td>
<td></td>
</tr>
<tr>
<td>First Drumhead of Second order, vertical</td>
<td>9843</td>
<td>9815 ± 16</td>
<td></td>
</tr>
<tr>
<td>First Drumhead of Second order, horizontal</td>
<td>9896</td>
<td>9873 ± 12</td>
<td></td>
</tr>
<tr>
<td>Second Drumhead</td>
<td>13031</td>
<td>13010 ± 14</td>
<td></td>
</tr>
</tbody>
</table>
This gives some confidence in the COMSOL model. In the next section a method to identify the test mass of a particular resonant mode observed in the OMC transmission will be examined. This will also provide more insight into which signals are associated with test mass resonances.

## 4 Test Mass Identification

There is a temperature dependence of the Young’s modulus of fused silica [5, 6]. From table 4 we see this dependence is small.

<table>
<thead>
<tr>
<th>Model Parameters</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus @291K</td>
<td>$E$</td>
<td>$73 \times 10^9 Pa$</td>
</tr>
<tr>
<td>Poisson Ratio @291K</td>
<td>$\nu$</td>
<td>0.17</td>
</tr>
<tr>
<td>$E$ thermal dependence @291K</td>
<td>$dE/dT$</td>
<td>$11.5 \times 10^6 Pa/K$</td>
</tr>
<tr>
<td>$\nu$ thermal dependence @291K</td>
<td>$d\nu/dT$</td>
<td>$5 \times 10^{-5}$</td>
</tr>
<tr>
<td>Coefficient of Thermal expansion</td>
<td>$\alpha$</td>
<td>$0.55 \times 10^{-6} 1/K$</td>
</tr>
<tr>
<td>Heat Capacity</td>
<td>$C_p$</td>
<td>703 J/kg/K</td>
</tr>
<tr>
<td>Thermal Conductivity</td>
<td>$k$</td>
<td>1.38 W/m/K</td>
</tr>
<tr>
<td>Emissivity</td>
<td>$\epsilon$</td>
<td>0.9</td>
</tr>
</tbody>
</table>

The mode frequency is related to the Young’s modulus by [7]:

$$\omega_m = \sqrt{2\beta} \sqrt{\frac{E}{2\rho(1+\nu)}}, \quad (1)$$

where $\beta$ is a term related to the dimension of the cylinder, $E$ is the Young’s modulus, $\rho$ is the density and $\nu$ is the Poisson ratio.

The change in frequency of fused silica is dominated by this change in Young’s modulus, while the changes in dimension due to thermal expansion and change in Poisson ratio due to its thermal dependence may be ignored.

The small thermal dependence of the young’s modulus is enough to make very accurate estimates of the test mass temperature [8] from the acoustic mode frequency. As such the modes of the same test mass may be identified from the correlated relative change in frequency of the modes $df/f$. It is easier to induce a change in the temperature in the test masses using the ring heater. In this way a distinctive change can be observed.

By stepping the ring heater changes in mode frequency are easily identified as demonstrated in LLO alog18002.
Figure 4: The relative change in mode frequency when the ETM ring heaters were stepped up at the start of lock. Then stepped down after 40 min.

An opportunistic set of data on the 18th August 2015 was used to identify ETM modes at LLO. The historic ring heater power is depicted in figure 5. The OMC data was recorded in half hour sections starting at hour 5 in this figure. From this point the ETMX is heating while ETMY is cooling.

A few examples of modes that are identified are shown in figure 6.
Figure 5: The Ring heater setting in the build up to measurements shown in the following figure

Figure 6: Top to bottom: Second butterfly mode at 6040Hz, A Differential butterfly mode at 13195Hz and the second order drum-head mode at 15010Hz compared to the 15530Hz group of modes

From these plots it is clear that modes 6052Hz, 13200Hz, 15004Hz and 15536Hz are ETMX modes, while modes 6049Hz, 13206Hz, 15013Hz and 15538Hz are from ETMY. Such a method was used to identify all modes visible in this lock stretch. This list is in "LLOETMmodes.csv"
5 Tuning COMSOL Model Parameters

In this section model elastic parameters are adjusted from the manufactures specified values to get agreement between model and experiment.

The COMSOL model is based on the LIGO Livingston ETMY, in this case the wedge and ears are also included, the RoC of the mirror is not included.

By using the test mass identification of Section 4 the ETMY modes were identified. These mode frequencies are compared to the COMSOL finite element simulation in the first column of Table 1.

By adjusting the Young’s modulus and Poisson ratio the modes frequencies can be tuned, reducing the residual between measurement and simulation. The Values obtained are $E = 72.7$ and $\sigma = 0.164$ A comparison of the mode frequencies prior to and after tuning the elastic parameters is shown in Table 1. The mesh for the optimisation was chosen such that further increases in mesh density produced less than 4Hz improvement in acoustic mode frequency estimates. The optimization was performed until the rms error in the mode frequency estimate reached 4Hz.

From Table 1 we see that there is a significant improvement comparing the material specification $E=70$GPa and $\sigma=0.17$ with the tuned model were $E=72.7$GPa and $\sigma=0.164$. The largest deviation is reduced from 350 Hz to 10 Hz.
### Table 1: Tuning the Young’s modulus and Poison ratio

<table>
<thead>
<tr>
<th>Mode</th>
<th>Measured $f$ (Hz)</th>
<th>Simulation Specification (Hz)</th>
<th>Simulation Tuned (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Butterfly</td>
<td>5945</td>
<td>5815</td>
<td>5941</td>
</tr>
<tr>
<td>Butterfly</td>
<td>6049</td>
<td>5919</td>
<td>6047</td>
</tr>
<tr>
<td>Drumhead</td>
<td>8153</td>
<td>8014</td>
<td>8153</td>
</tr>
<tr>
<td>Cylinder</td>
<td>8313</td>
<td>8135</td>
<td>8312</td>
</tr>
<tr>
<td>Cylinder</td>
<td>9100</td>
<td>8918</td>
<td>9098</td>
</tr>
<tr>
<td>Cylinder</td>
<td>9334</td>
<td>9150</td>
<td>9335</td>
</tr>
<tr>
<td>Drumhead</td>
<td>9833</td>
<td>9634</td>
<td>9834</td>
</tr>
<tr>
<td>Drumhead O2v</td>
<td>9885</td>
<td>9685</td>
<td>9886</td>
</tr>
<tr>
<td>Drumhead O2h</td>
<td>10216</td>
<td>9989</td>
<td>10206</td>
</tr>
<tr>
<td>Butterfly O1.5</td>
<td>10425</td>
<td>10204</td>
<td>10425</td>
</tr>
<tr>
<td>Butterfly</td>
<td>12638</td>
<td>12374</td>
<td>12637</td>
</tr>
<tr>
<td>Drumhead</td>
<td>13010</td>
<td>12786</td>
<td>13012</td>
</tr>
<tr>
<td>Drumhead O2</td>
<td>13074</td>
<td>12830</td>
<td>13069</td>
</tr>
<tr>
<td>Butterfly</td>
<td>13206</td>
<td>12937</td>
<td>13206</td>
</tr>
<tr>
<td>Butterfly</td>
<td>13272</td>
<td>13004</td>
<td>13274</td>
</tr>
<tr>
<td>Butterfly O2</td>
<td>14468</td>
<td>14152</td>
<td>14461</td>
</tr>
<tr>
<td>Drumhead O2v</td>
<td>15014</td>
<td>14719</td>
<td>15018</td>
</tr>
<tr>
<td>Drumhead O2h</td>
<td>15081</td>
<td>14784</td>
<td>15085</td>
</tr>
<tr>
<td>Drumhead</td>
<td>15220</td>
<td>14919</td>
<td>15222</td>
</tr>
<tr>
<td>Drumhead O2v</td>
<td>15538</td>
<td>15224</td>
<td>15537</td>
</tr>
<tr>
<td>Drumhead O2h</td>
<td>15624</td>
<td>15309</td>
<td>15622</td>
</tr>
<tr>
<td>Butterfly O2</td>
<td>16272</td>
<td>15921</td>
<td>16269</td>
</tr>
<tr>
<td>Butterfly O2</td>
<td>16917</td>
<td>16565</td>
<td>16911</td>
</tr>
<tr>
<td>Butterfly O2</td>
<td>17026</td>
<td>16683</td>
<td>17028</td>
</tr>
<tr>
<td>Butterfly O2.5</td>
<td>17907</td>
<td>17558</td>
<td>17914</td>
</tr>
</tbody>
</table>

### 6 Method for Acoustic Mode Test Mass Identification

The following is a suggested method for determining test mass mode locations.

1. With the interferometer locked start capturing data from one of the OMC DP PDs at 65kSa/Sec

2. Step one ring heater by 0.1W - 0.2W (top and bottom at the ‘same’ time

3. After 30 minutes return this ring heater to its original state and step the next ring heater. Repeat until all 4 ring heaters have been modulated

4. Use a script such as that in appendix A to identify the frequency shift of the mode you are interested in

Note1: ITM ring heaters have never been stepped. So no guarantee this will work.

Note2, it may be advisable to watch the acoustic mode amplitudes in case stepping the ring heater results in parametric instability (a DTT session with the same OMC DC DP signal is a good way to watch).
7 Appendix A

A simple script to track a single acoustic mode frequency. To use:

1. Save data with gwpy. “gwdv_record -o DCPD64k_20150922_1918_3600.hdf 1800 L1:IOP-LSC0_MADC0_TP_CH12”. I have found half hour to one hour segments are manageable.

2. Set files names such that the script can read them.

3. First look at the 15530Hz group of modes.

4. Plot figure 2 with the range of frequencies (f9 - f10) that you are interested. (15500 - 15550)

5. Find the specific peaks in figure 2 you are interested in and set bounds f1-f8 to differentiate each peak.

6. Run the entire script - modes are compared to the 15530Hz group of modes. Before you save this group of modes there will be an error.

7. Go back to step 4 and run the next group of modes you are interested in.

```matlab
clear;
filetimes={'0210','0339','0417','0437','0520'};
j=1;
for j = 1:max(size(filetimes))
    tstr=filetimes(j);
    file_name = char(strcat('DCPD64k_20150818_',tstr,'_1800.hdf'));
    metainfo = hdf5info(file_name);
    chnName = metainfo.GroupHierarchy.Datasets(1).Name(1);
    data = hdf5read(metainfo.GroupHierarchy.Datasets(1));
    tt = startTime + [0:1/sampleRate:(length(data)-1)/sampleRate]';
    eval(char(strcat('startTime_',tstr,'=startTime')));

    sampfft_DCPD=2^23;
    Fs=65536;
    ovl = 0.7; % ovl = 1 -> independant amplitude estimates
    Fs_DCPD=65536;
    L=floor((max(size(data))-sampfft_DCPD)/sampfft_DCPD/ovl);
    i=1;
    H_DCPD=spectrum.welch('hann',sampfft_DCPD,0);
    amp_DC_PD=[];
    ind_DC_PD=[];
    f9=18200; f10=18250;
    f1=18235; f2=18237;
    f3=18238; f4=18240;
```
for i = 1:L
    dCPDspec = psd(h_DCPD.data(round(1+sampfft_DCPD*ovl*i):round(sampfft_DCPD+sampfft_DCPSpec.data)*2));
    subplot(2,2,1)
    plot(dCPDspec.Frequencies([round(f1/Fs*sizedata(1)):round(f2/Fs*sizedata(1)-1)]),
         sqrt(dCPDspec.data([round(f1/Fs*sizedata(1)):round(f2/Fs*sizedata(1)-1)])),'b'
    subplot(2,2,2)
    plot(dCPDspec.Frequencies([round(f3/Fs*sizedata(1)):round(f4/Fs*sizedata(1)-1)]),
         sqrt(dCPDspec.data([round(f3/Fs*sizedata(1)):round(f4/Fs*sizedata(1)-1)])),'b'
    subplot(2,2,3)
    plot(dCPDspec.Frequencies([round(f5/Fs*sizedata(1)):round(f6/Fs*sizedata(1)-1)]),
         sqrt(dCPDspec.data([round(f5/Fs*sizedata(1)):round(f6/Fs*sizedata(1)-1)])),'b'
    subplot(2,2,4)
    plot(dCPDspec.Frequencies([round(f7/Fs*sizedata(1)):round(f8/Fs*sizedata(1)-1)]),
         sqrt(dCPDspec.data([round(f7/Fs*sizedata(1)):round(f8/Fs*sizedata(1)-1)])),'b'
    [amp_DC_PD(1,i), ind_DC_PD(1,i)] = max(sqrt(dCPDspec.data([round(f1/Fs*sizedata(1)):
        round(f2/Fs*sizedata(1)-1)])))
    [amp_DC_PD(2,i), ind_DC_PD(2,i)] = max(sqrt(dCPDspec.data([round(f3/Fs*sizedata(1)):
        round(f4/Fs*sizedata(1)-1)])))
    [amp_DC_PD(3,i), ind_DC_PD(3,i)] = max(sqrt(dCPDspec.data([round(f5/Fs*sizedata(1)):
        round(f6/Fs*sizedata(1)-1)])))
    [amp_DC_PD(4,i), ind_DC_PD(4,i)] = max(sqrt(dCPDspec.data([round(f7/Fs*sizedata(1)):
        round(f8/Fs*sizedata(1)-1)])))
end;

% figure(5)
% hold on
% plot(dfOnf(1),'.');
% plot(dfOnf(2),'rx');
% plot(dfOnf(4),'go');
% hold off;

eval(['dfOnf1_',char(tstr),' = ',char(dfOnf1(1,:))',']);
end;

% figure(6)
% hold on
% plot(dfOnf(2),'.');
% plot(dfOnf(3),'rx');
% plot(dfOnf(4),'go');
% hold off;

eval(['dfOnf2_',char(tstr),' = ',char(dfOnf2(1,:))',']);
end;

% figure(7)
% hold on
% plot(dfOnf(3),'.');
% plot(dfOnf(4),'go');
% hold off;

eval(['dfOnf3_',char(tstr),' = ',char(dfOnf3(1,:))',']);
end;

% figure(8)
% hold on
% plot(dfOnf(4),'.');
% plot(dfOnf(2),'rx');
% plot(dfOnf(3),'go');
% hold off;

eval(['dfOnf4_',char(tstr),' = ',char(dfOnf4(1,:))',']);
end;

timeVec = [[1:L]*sampfft_DCPD/Fs*ovl1],[[1:L]*sampfft_DCPD/Fs*ovl1+(startTime_0339-startTime_0339)];
eval(char(strcat(['ModeMatrix',num2str((f3+f4)/2),',',[dfOnf2_0210,dfOnf2_0339,dfOnf2_0417,dfOnf2_0437,dfOnf2_0520]-dfOnf2_0210(1)))]);
eval(char(strcat(['ModeMatrix',num2str((f5+f6)/2),',',[dfOnf3_0210,dfOnf3_0339,dfOnf3_0417,dfOnf3_0437,dfOnf3_0520]-dfOnf3_0210(1))]);
eval(char(strcat(['ModeMatrix',num2str((f7+f8)/2),',',[dfOnf4_0210,dfOnf4_0339,dfOnf4_0417,dfOnf4_0437,dfOnf4_0520]-dfOnf4_0210(1))]);
temp2 = [dfOnf2_0210, dfOnf2_0339, dfOnf2_0417, dfOnf2_0437, dfOnf2_0520] - dfOnf2_0210(1);  
temp3 = [dfOnf3_0210, dfOnf3_0339, dfOnf3_0417, dfOnf3_0437, dfOnf3_0520] - dfOnf3_0210(1);  
temp4 = [dfOnf4_0210, dfOnf4_0339, dfOnf4_0417, dfOnf4_0437, dfOnf4_0520] - dfOnf4_0210(1);  

load('ModeMatrix15530.mat');  
figure(5)  
plot(timevec, dfOnf(1));  
hold on;  
plot(timevec, dfOnf(2), '.');  
plot(timevec, dfOnf(3), 'rx');  
plot(timevec, dfOnf(4), 'go');  
hold off;  
plot(timeVec([1:14, 29:74])/3600, temp1(:,[2:15, 30:end])', 'v')  
hold on;  
plot(timeVec([1:14, 29:74])/3600, temp2(:,[2:15, 30:end])', 'xg')  
plot(timeVec([1:14, 29:74])/3600, temp3(:,[2:15, 30:end])', 'og')  
plot(timeVec([1:14, 29:74])/3600, temp4(:,[2:15, 30:end])', 'r')  
plot(timeVec([1:14, 29:75])/3600, ModeMatrix15530(:,[2:15, 30:end])', '-', 'LineWidth', 2)  
hold off  
set(gca, 'LineWidth', 2, 'Yscale', 'linear', 'box', 'off', 'ycolor', 'black', 'xcolor', 'black',  
'Xlim', [0, 3.5], 'Ylim', [-0.1, 0.3]*10^-4, 'fontname', 'courier',  
'fontweight', 'bold');  
xstring = [('f{\textbf{Time (hours)}}')];  
ystring = '\bf{Mode Rel Frequency Shift ($\frac{\partial f}{f}$)}';  
% tstring = '\bf{Relative Frequency Shift of 8310HZ Modes}';  
% hleg = legend([[\bf{0}, num2str(round((f1+f2)/2))]];[\bf{0}, num2str(round((f3+f4)/2))]);[0  
% hleg = legend([(num2str(round((f1+f2)/2))));(num2str(round((f3+f4)/2))]);(num2str(round((f5+f6+f7+f8)/8))));  
xlabel(xstring, 'interpreter', 'latex', 'fontsize', 14)  
ylabel(ystring, 'interpreter', 'latex', 'fontsize', 14)  
% title(tstring, 'interpreter', 'latex', 'fontsize', 14)  
set(hleg, 'interpreter', 'latex', 'fontsize', 12)  
filestring = char(strcat('ModeMatrix', num2str(round((f1+f2+f3+f4+f5+f6+f7+f8)/8))));  
save(filestring, filestring)


Appendix C

Architecture for Active Damping of Parametric Instability at LIGO Hanford Observation Run 2

LIGO DCC signal flow diagram for active damping of parametric instability.
Signal flow diagram for active damping of parametric instabilities O2
Mode 1

Mode 32

NOTE: Mode numbering as of 25 July uses modes 1-8 for ITMX 9-16 for ITMY
17-24 for ETMX and 25-32 for ETMY

Modes required are
Mode 1 14985Hz, fa, DOF, ITMX
Mode 2 15522Hz, fa, DOF, ITMX
Mode 9 14980Hz, fa, DOF, ITMY
Mode 10 15515Hz, fa, DOF, ITMY
Mode 17 15008Hz, fa, DOF, ETMX
Mode 18 15541Hz, fa, DOF, ETMX
Mode 25 15509Hz, fa, DOF, ETMY
Mode 26 15542Hz, fa, DOF, ETMY
Mode 27 18040Hz, fb, DOFI, ETMY
Mode 28 18056Hz, fb, DOFI, ETMY

Model h1susproci.mdl is currently running on h1oa0, when a cpu becomes available on h1sc0 it will be moved there.
Heterodyne down-conversion, 8 conversions(a-h) for each photodiode error signal covering 8 x 1kHz bands, each band has independent sensing DOF.

Note1: Degrees of freedom (DOF) of QPD signals in use 25 July 2016 are DOFf= pitch (1.5k and 3.5k), DOFI=pringle (for 4k) and DOFm=seg 1 (for unknown models)

Note2: Conversion frequencies in use 25 July 2016 are (required) fs=300kHz, f0=178kHz (for monitoring) f1=1380kHz, fs=153kHz, f=16800kHz, fs=21600kHz, f=25800kHz

Heterodyne up-conversion, 8 conversions(a-h) for each test mass control signal covering 8 x 1kHz bands, each band has independent actuation DOF.

Note3: ITM DOF in use 25 July 2016 (required) DOFpersent, DOF=pringle (pitching) DOFm=UL, DOFm=LR quadrant ETM DOF in use are (required) DOFm=UL, DOF=UR, DOFm=UL, DOF=LR (tracking) DOFm=UL, DOF=UR

Heterodyne up-conversion and down-conversion performed in 64kSa/sec RCG models h1susitmpi.mdl running on h1susb23, h1susetmxpi.mdl on h1susex, h1susetmpi.mdl on h1susex and h1omcpi.mdl on h1sc0.
Appendix D
Publication List

Short Author List Publications


Carl Blair. "The next detectors for gravitational wave astronomy” Section 5

"Three mode parametric instabilities and their control for advanced gravitational wave detectors” [54], Science China, Physics, Mechanics & Astronomy, 58 120405 (2015), (Ref. Chap. 5).


Q. Fang, C. Blair, C. Zhao, L. Ju, Degeneracy losses in a 7/4 Suspended Fabry-Pérot Cavity, paper in preparation, 2017

Q. Fang, C. Blair, C. Zhao, L. Ju, D. Blair, Mechanical loss in a Suspended Test Mass with Modular Suspension System, paper in preparation, 2017

LIGO Collaboration, Instrument Science Author List and Long Author List Publications


LIGO Scientific Collaboration, Virgo Collaboration, Characterization of transient noise in Advanced LIGO relevant to gravitational wave signal GW150914, Classical and Quantum Gravity


LIGO Scientific Collaboration, Virgo Collaboration, First low frequency all-sky search for continuous gravitational wave signals, Phys. Rev. D 93, 042007, February 2016


LIGO Scientific Collaboration, Advanced LIGO, Classical and Quantum Gravity, Volume 32, Number 7, April 2015

LIGO Scientific Collaboration, Virgo Collaboration, A directed search for gravitational waves from Scorpius X-1 with initial LIGO, Phys. Rev. D 91, 062008, March 2015

M. Adier, ..., C. Blair, ..., et al, Concepts and research for future detectors, General Relativity and Gravitation, 46:1749, August 2014,