1 Components of variance from two calculations of alpha

**Components of variance of scales with a bi-factor subscale structure from two calculations of $\alpha$**

**Abstract**

Since Cronbach’s (1951) elaboration of $\alpha$ from its introduction by Guttman (1945), this coefficient has become ubiquitous in characterising assessment instruments in education, psychology and other social sciences. Also ubiquitous are caveats on the calculation and interpretation of this coefficient. This paper summarises a recent contribution (Andrich, 2015) on the use of coefficient $\alpha$ which complements these many caveats. It shows that in the presence of a simple bi-factor structure of a scale where unique components of variance are homogeneous in magnitude, three components of variance and the common latent common correlation among the subscales, can be calculated from the ratio of two calculations of $\alpha$, one at the level of the items, the other at the level of the subscales. It was suggested that with these two ready calculations and their interpretation, and the reporting of all four indices in the analysis of scales with a subscale structure, would reduce the miss-interpretation of this coefficient. An illustrative example of the application of the calculations is also shown.

**Key words:** bi-factor structure, coefficient alpha, components of variance, subscales, dimensionality
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Components of variance of scales with a bi-factor subscale structure from two calculations of \( \alpha \)

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Because the many pitfalls in interpreting \( \alpha \) as a reliability index, and ways to overcome these pitfalls have been recommended in the literature (e.g., Cortina, 1993; Davenport, Davison, Liou & Love, 2015; Green, Lissitz, & Mulaik, 1977; Komaroff, 1997; McDonald, 1978; Rae, 2006; Raykov, 1998; Schmitt, 1996; Sijtsma, 2009, 2015; Van Zyl, Neudecker, & Nel, 2000; Zinbarg, Revelle, Yovel, & Li, 2005), this paper does not canvass the various caveats in the application and interpretation of coefficient \( \alpha \). Instead, it focuses immediately on a subscale structure which has specific effects on the calculation of \( \alpha \). The presence of subscales suggests a simple bi-factor (Reise, Moore & Haviland, 2010; Reise, Morizot & Hays, 2007; Green & Yang, 2015). In addition, because the proofs of the results are provided in Andrich (2015), primarily summary results are presented and illustrated, and the main derivations that are shown, are presented in the Appendix.

Subscale structure

Many scales in education and the social sciences which are constructed to be used in a sense which is unidimensional, that is, for persons to be summarised by a single score, are nevertheless composed of identifiable subscales. They are composed of subscales because multiple items are used to assess each identifiable aspect of a scale. Therefore, technically, the scale is not unidimensional and the degree of non-unidimensionality depends on the relationships between items within and among the subscales. The explicit presence of subscales implies a bi-factor structure. The results in Andrich (2015), and summarized here, formalize and take advantage of the feature that \( \alpha \) calculated at the level of the entire set of items is greater than when it is calculated at the level of the subscales (e.g., Marais & Andrich, 2008; Rae, 2006; Zenisky, Hambleton, & Sireci, 2002).

Formalizing a subscale structure of a scale

The formalization of a subscale structure is facilitated by beginning with an item level resolution of the observed score of classical test theory (CTT) into a true and error score. Let the observed score of person \( n \) on item \( i \), \( i = 1,2,\ldots,I \), be \( x_{ni} \), \( x_{ni} = 0,1,2,\ldots m_i \), where \( m_i \) is the maximum score of item \( i \) and let

\[
x_{ni} = r_n + e_{ni}
\]  

(1)
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where \( \tau_n \) is the value on the variable \( \tau \) common to the responses of person \( n \) to all \( I \) items and \( \epsilon_{ni} \) is the error component of this response to item \( i \). Because the error is taken to be uncorrelated with the true score, \( \text{COV}[\tau, \epsilon] = 0 \), and taken to be homogeneous across persons, \( V[\epsilon_i] = \sigma^2 = \sigma^2_\epsilon \),

\[
V[x_i] = \sigma^2 + \sigma^2_\epsilon.
\] (2)

Let \( y_n \) be the sum of the scores of person \( n \) on the \( I \) items of a scale. Then

\[
y_n = \sum_{i=1}^{I} x_{ni} = \sum_{i=1}^{I} (\tau_{ni} + \epsilon_{ni}) = I \tau_n + \sum_{i=1}^{I} \epsilon_{ni}.
\] (3)

Further, let

\[
t_n = I \tau_n; \ e_n = \sum_{i=1}^{I} \epsilon_{ni},
\] (4)

giving the standard CTT true score equation

\[
y_n = t_n + e_n,
\] (5)

where \( t_n \) is the scale true score and \( e_n \) is the error score of person \( n \). In addition,

\[
\sigma^2_t = V[I \tau] = I^2 V[\tau] = I^2 \sigma^2_\tau,
\]
\[
\sigma^2_\epsilon = V[\epsilon] = IV[\epsilon] = I \sigma^2_\epsilon.
\] (6)

Note that the variance of the true score \( t \) is a quadratic function of the number of items, \((I^2)\sigma^2_\tau\), while the error variance is only a linear function \((I)\sigma^2_\epsilon\). This ensures that as the number of items increases the variance of the true scores increases at a greater rate than the error variance, resulting in successive items adding precision to the assessment.

The bi-factor structure is formalized at the item level. Elaborating Eq. (1), the observed value \( x_{ni} \) is resolved according to

\[
x_{ni} = \tau_{ni} + \nu_{ni} + \epsilon_{ni},
\] (7)

where \( x_{ni} \), \( \epsilon_{ni} \) retain the same meaning as in Eq. (1), \( \nu_{ni} \) is the value of the unique aspect \( \nu \) of the response of person \( n \) to item \( i \), and \( c_i \) is the weight for item \( i \). With a subscale structure, suppose the weights of all items within subscale \( s \) are identical, that is \( c_i = c_s \) for
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all items within each subscale \( s, s = 1, 2, \ldots, S \), and let \( \nu_{ns} \) be the value of the unique aspect \( \nu \) for subscale \( s \). Then

\[
x_{nis} = \tau_n + c_s \nu_{ns} + \epsilon_{nis},
\]

where \( x_{nis}, \epsilon_{nis} \) are respectively the observed and error scores of person \( n \) to any item \( i \) on subscale \( s \).

In a bi-factor structure it is assumed that covariances between the true scores and error scores, and between all pairs of subscales, are all zero, that is, that

\[
\text{COV}[\nu_s, \tau] = \text{COV}[\nu_s, \nu_r] = \text{COV}[\nu_s, \epsilon_s] = 0
\]

for all subscales. Again it is assumed that the item error variances are homogeneous, that is that \( \sigma^2_{as} = \sigma^2_{e} \). Therefore

\[
V[x_{nis}] = \sigma^2_{\tau} + c_s^2 \sigma^2_{ss} + \sigma^2_{e},
\]

where \( c_s^2 \sigma^2_{ss} \) is then the unique variance of subscale \( c_s^2 \).

A simplifying assumption is that the unique variances of items within subscales are also homogeneous across subscales, that is, \( \sigma^2_{ss} = \sigma^2_{v} \). With \( c_s \) variable, this assumption is not particularly restrictive. However, the assumption that the unique weight of each item in each subscale is the same within and among subscales is also made, that is \( c_s = c; s = 1, 2, \ldots, S \). The consequence is that it assumes that the magnitude of the unique variances of the items within and among all subscales are the same, or at least homogeneous. This is a stronger assumption than simply that \( \sigma^2_{ss} = \sigma^2_{v} \). However, with the constraint of an a-priori subscale structure within the constraints elaborated in the last section, this assumption is not implausible. In addition, as summarized below, it facilitates the calculation of the common true variance, the unique variance and the error variance from just two calculations of \( \alpha \) and provides a rapid overview of the properties of the scale.

Without loss of generality, and because of the weight \( c_s \) that appears with the unique aspect \( \nu \) of each subscale, we can let the variances of the common variable \( \tau \) and the unique aspect \( \nu \) be the same, that is

\[
\sigma^2_{\tau} = \sigma^2_{v},
\]

for all subscales \( s \). Then \( c_s^2 \) is the proportion of the unique variance \( \sigma^2_{v} \) of an item in subscale \( s \) relative to the total common variance \( \sigma^2_{\tau} \) of that item.
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In the first instance and for simplicity of exposition, let the number of items \( K_s \) in each scale be the same, \( K_s = K, s = 1,2,\ldots,S \), giving the total number of items \( I = SK \). This assumption can be relaxed.

**The components of variance**

Suppose the total score of person \( n \) on subscale \( s \) is \( y_{ns} \).

Then the total score of person \( n \) on the scale is given simply by

\[
y_n = \sum_{s=1}^{S} y_{ns} = \sum_{s=1}^{S} \sum_{i=1}^{K_s} x_{nis}.
\]  

(11)

The Appendix shows that the variance of \( y \) is given by

\[
V(y) = \sigma_t^2 + \sigma_u^2 + \sigma_e^2,
\]  

(12)

where

\[
\sigma_t^2 = S^2K^2 \sigma_r^2 = I^2 \sigma_r^2
\]  

(13)

remains the common true score variance (across all subscales),

\[
\sigma_u^2 = SK^2c^2 \sigma_v^2
\]  

(14)

is the total unique variance of the subscales (the sum of the unique variances among the subscales), and

\[
\sigma_e^2 = SK \sigma_e^2
\]  

(15)

is the total error variance of the scale. Note that by analogy to the effect of increasing the number of items, it follows from Eq. (13) that each additional subscale adds quadratically \( (S^2(K^2 \sigma_r^2)) \) to the true common variance \( \sigma_t^2 \) while from Eq. (14) it adds only linearly \( (S(K^2c^2 \sigma_v^2)) \) to the unique variance \( \sigma_u^2 \), ensuring that, with the addition of each subscale, the common variance increases relative to the unique variance. From Eq. (15), it is evident that the error increases, as in the case without a subscale structure, linearly with the number of items per subscale and with the number of subscales, that is linearly as a function of the total number of items.
The correlation between items from different subscales

It is shown readily (Andrich, 2015) that the correlation $\rho_{sr}$ between any pair of subscales $s$ and $r$ is the same as the correlation between a pair of items from the two subscales, and is given by

$$\rho_{su} = \frac{\sigma_i^2}{\sqrt{\sigma_i^2 + c_i^2 \sigma_{is}^2}} \sqrt{\sigma_c^2 + c_i^2 \sigma_{sr}^2}.$$  \hspace{1cm} (16)

Applying (i) the condition that the unique variances within subscales are homogeneous across subscales, that is, $\sigma_{is}^2 = \sigma_{ir}^2 = \sigma_{is}^2$, (ii) from Eq. (10) that $\sigma_i^2 = \sigma_{ic}^2$, and (iii) that the item weights are the same for all items within subscales and across subscales, that is $c_r = c_s$ for all $s, r$ gives,

$$\rho_{sr} = \frac{1}{1 + c_s^2} \text{ for all } s, r.$$  \hspace{1cm} (17)

In the case of just two subscales, $\rho_{sr}$ is identical to the correlation between two subscales corrected for attenuation because of error using estimates of the reliability of each subscale. Given a single value of $c_s$ across all pairs of subscales, $\rho_{sr}$ is a summary correlation between all pairs of subscales. Clearly, if $c_s = c_r = 0$, $\rho_{sr} = 1$.

$\alpha$ and the components of variance with a subscale structure

The two calculations of $\alpha$ are made from the standard formula (Cronbach, 1951). Calculated at the level of discrete items, ignoring the subscale structure, $\alpha$ is given by

$$\alpha = \frac{I}{I-1} \frac{V[\sum_{i=1}^{I} x_{ni}] - \sum_{i=1}^{I} V[x_i]}{V[y]}.$$  \hspace{1cm} (18)

Calculated at the level of the subscales, the items within each subscale are summed to give the subscale score $y_{ns} = \sum_{i=1}^{K} x_{nis}$. Then using the same structure as Eq. (18), $\alpha$ is given by

$$\alpha = \frac{S}{S-1} \frac{V[\sum_{s=1}^{S} y_{ns}] - \sum_{s=1}^{S} V[y_s]}{V[y]}.$$  \hspace{1cm} (19)
Table 1 shows the resultant ratio of components of variance relative to the total variance from each of these two calculations of $\alpha$. As a frame of reference, the components of variance are shown also for calculations of both Eqs. (18) and (19) when $c_i = 0$ and $\rho_{sr} = 1$, that is when the scale is unidimensional but when the calculations are carried out both at the item and subscale levels. The notations for these two calculations are respectively $\alpha$ for the standard case at the item level, and $\alpha_0$ at the subscale level. These values are shown in the first row of Table 1. It is evident that the two calculations estimate the same ratio of variances, the proportion of the common true variance relative to the total variance. In this case, with all assumptions of CTT met entirely, $\alpha$ and $\alpha_0$ provide an estimate of the internal consistency reliability. In practice, and even if the assumption that $\rho_{sr} = 1$ is met entirely, that is the scale is unidimensional, the two calculations might show slight differences because of capitalization of any chance relationships between items within subscales. However, if $\alpha \approx \alpha_0$, then there is little evidence of multidimensionality in the hypothesized subscale structure. On the other hand, if they are substantially different, it is the first simple indication that some multidimensionality is present.

The second row of Table 1 shows the ratio of variances when $c_i > 0$, $\rho_{sr} < 1$, that is there is a subscale structure technically violating unidimensionality. The respective calculations at the item and subscale levels are notated $\alpha_c$ and $\alpha_s$. It is evident from the second row of the Table that $\alpha_c$ is inflated relative to the case when $c_i = 0$ and $\rho_{sr} = 1$. It is a well-known result that shows that $\alpha$ is not an index of unidimensionality – it is inflated by lack of unidimensionality. It is also evident from the second row of Table 1 that calculated at the subscale structure level, $\alpha_s$ is the proportion of the common true variance relative to the total variance:

$$\alpha_s = \frac{\sigma_y^2}{\sigma_i^2 + \sigma_u^2 + \sigma_e^2}.$$  (20)

Clearly, if $c > 0$, then $\alpha_s < \alpha_c$, which reflects the well-known result, noted earlier, that if there is a subscale structure, $\alpha$ calculated at the subscale level has a smaller value than when calculated at the item level, which is inflated by the lack of unidimensionality. In summing the scores of items within subscales, the unique variance of the subscales is
absorbed into the variance of the summed scores. This is the relationship that is formalized with simplifying assumptions, and exploited to estimate a range of components variance.

**Table 1. Conditions under which $\alpha$ is calculated and its values**

<table>
<thead>
<tr>
<th>Not taking account of the subscale structure</th>
<th>Taking account of the subscale structure</th>
<th>Effect on $\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>standard case: $c_s = 0$, $\rho_{sr} = 1$</td>
<td>$\alpha_0 = \frac{\sigma_r^2}{\sigma_y^2}$, $\sigma_u^2 = 0$</td>
<td>$\alpha = \alpha_0$.</td>
</tr>
<tr>
<td>$c_s &gt; 0$, $\rho_{sr} &lt; 1$</td>
<td>$\alpha_s = \frac{\sigma_r^2 + \sigma_u^2 S(K-1)/(SK-1)}{\sigma_y^2}$</td>
<td>$\alpha_s &gt; 0$.</td>
</tr>
<tr>
<td>$\sigma_y^2 = \sigma_r^2 + \sigma_u^2 + \sigma_e^2$</td>
<td>$\alpha_c = \frac{\sigma_r^2}{\sigma_y^2}$</td>
<td>$\alpha_c &gt; \alpha_s$</td>
</tr>
</tbody>
</table>

**Estimating $c$**

With the assumption $\sigma_r^2 = \sigma_v^2$, $c_s = c_v$ for all $s$, $r$ and simplifying (Andrich, 2105) gives

$$\alpha_c / \alpha_s = 1 + c_s^2 (K-1)/(SK-1),$$

from which

$$c_s^2 = \left( \frac{SK-1}{K-1} \right) \left( \frac{\alpha_c}{\alpha_s} - 1 \right).$$

(22)

Table 1 shows that if $c > 0$ then $\alpha_c > \alpha_s$ and $\alpha_c / \alpha_s > 1$. Therefore, because $SK-1 > K-1$, Eq. (22) implies that if $\alpha_c > \alpha_S$, then $c^2 > 0$, as required. If $S = 1$ then $\alpha_S$ cannot be calculated but in that case there are no subscales. Without loss of generality, $c = +\sqrt{c^2}$ is taken implying that the items are correlated positively.

The requirement that each subscale has the same number of items can be relaxed. Though easy to implement, the somewhat cumbersome expression is shown in the Appendix.

From the estimate of $c_s^2$ and calculated values of $\alpha_c$, $\alpha_s$ and $\sigma_y^2$, the proportions and actual values of the true and unique variances can be estimated. Thus first from $\alpha_s = \sigma_r^2 / \sigma_y^2$,
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\[ \sigma_i^2 = \sigma_y^2 \alpha_s. \] (23)

However, \( \sigma_i^2 = S^2 K^2 \sigma_v^2 \), and \( \sigma_u^2 = SK^2 \sigma_v^2 \); hence

\[ \sigma_u^2 = SK^2 c^2 \sigma_v^2 = S^2 K^2 c^2 \sigma_v^2 / S = c^2 \sigma_i^2 / S. \] (24)

Finally from Eq. (12)

\[ \sigma_e^2 = \sigma_y^2 - \sigma_i^2 - \sigma_u^2. \] (25)

From estimates of \( \sigma_i^2, \sigma_u^2, \sigma_v^2 \), their respective proportions relative to the total variance \( \sigma_y^2 \) are readily calculated. In addition, the proportion of the true variance relative to the sum of true and unique variances, notated \( A \), is given by

\[ A = \frac{\sigma_i^2}{(\sigma_i^2 + \sigma_u^2)}, \] (26)

and the proportion of the sum of the true common and unique variance relative to the total variance, is given by \( \omega \) (Green & Yang, 2015)

\[ \omega = \frac{\sigma_i^2 + \sigma_u^2}{\sigma_y^2}. \] (27)

**A ready interpretation of \( \alpha_c \), \( \alpha_s \) and \( \alpha_c / \alpha_s \)**

To provide a relatively straightforward interpretation of \( \alpha_c, \alpha_s \) and \( \alpha_c / \alpha_s \), an approximation for the ratio \( S[K - 1]/[SK - 1] \) can be used when the number of items per subscale is relatively large. It is shown in the Appendix that as the number of items becomes larger, the ratio \( S[K - 1]/[SK - 1] \) approaches 1, though it is always a little less than 1; for all practical purposes it is 1. For example, if there are 15 items per subscale and there are two subscales, the value of \( S[K - 1]/[SK - 1] = 0.97 \) which is a little less than 1. Substituting 1 for this ratio simplifies the interpretation of the relevant ratios of variances considerably. Thus from \( \alpha_c = \frac{\sigma_i^2 + \sigma_u^2}{S(K - 1)/(SK - 1)} \) \( \sigma_y^2 \) in Table 1, and substituting \( S(K - 1)/(SK - 1) = 1 \), gives the much simpler expression

\[ \alpha_c < \frac{(\sigma_i^2 + \sigma_u^2)}{\sigma_y^2} \equiv \frac{(\sigma_i^2 + \sigma_u^2)}{\sigma_y^2} \cdot \] (28)

Eq. (29) indicates that \( \alpha_c \) is the lower bound of the proportion of the sum of the common true and unique variances relative to the total, and can generally be interpreted as its equivalent.
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From Table 1, it is immediately clear that $\alpha_s$ is the proportion of the true common variance relative to the total variance. Next, denoting the ratio $\alpha_s / \alpha_c$ by $\alpha_A$, it can be shown readily that substituting the expressions for $\alpha_s$, $\alpha_c$ from Table 1 gives

$$\alpha_A = \frac{\alpha_s}{\alpha_c} = \frac{\sigma_i^2}{\sigma_i^2 + \sigma_u^2 S(K - 1)/(SK - 1)}.$$  \hspace{1cm} (29)

Again, substituting $S(K - 1)/(SK - 1)=1$, gives

$$\alpha_A = \alpha_s / \alpha_c > \frac{\sigma_i^2}{(\sigma_i^2 + \sigma_u^2)} \equiv \frac{\sigma_i^2}{(\sigma_i^2 + \sigma_u^2)}.$$  \hspace{1cm} (30)

Thus $\alpha_A$ is the upper bound of the proportion of common true variance relative to the sum of the common true and unique variances and can be interpreted as its equivalent: that is, $\alpha_A$ is the proportion of common true variance relative to the variance that is not error variance.

**Example**

The data used for illustrative purpose comes from Australian Scholastic Aptitude test (ASAT) which is a 100 item multiple choice test constructed to cover four areas of scholastic achievement – Mathematics and Science; Humanities and Social Science. It was generally used at Year 12 level to assess students who are applying to enter universities in Australia. The number of students for whom data were made available is a random sample of 1000 students. However, 13 of the students did not have complete responses and they were left out of the analysis. The remaining sample consisted of 490 girls and 497 boys. For the purpose of this paper, the items for Mathematics and Science are taken to form one subscale, and those for Humanities and Social Science are taken to form the second subscale. The number of items in each of these subscales is 50.

The area of Mathematics and Science are clearly substantively different from the area Humanities and Social Science. In addition, the items assessing these areas were not focussed on a specific Year 12 curriculum. However, because the students were preparing for examinations in these specific disciplines for entrance into universities in Western Australia or in other parts of Australia, students were expected to have studied across the areas of Mathematics and Science and Humanities and Social Science. As a result, it was expected that the scores on these tests would have a positive correlation.
Table 2 shows the results of the analysis of ASAT according to the above bi-factor structure and formulae, which were produced directly by the software RUMM2030 (Andrich, Sheridan & Luo, 2015). The values of $\alpha_c$, $\alpha_S$, and $\alpha_A$, $c$ and $\rho$ are shown in the second row, and the interpretation of these values of $\alpha$ as proportions of the relative variances when the approximation that $S[K-1]/[SK-1]=1$ when K is large is used are shown in the fourth row. It is evident that the values are virtually identical and there would be no misinterpretation if one set of values rather than the other were used. Table 3 shows the values of each of the components of variance relative to the total variance calculated using Eqs. 13 - 15.

Table 2. Three calculations of $\alpha$ and the approximations of proportions of variance together with the estimated true correlation between two subscales of the ASAT

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_c$</th>
<th>$\alpha_S$</th>
<th>$\alpha_A$</th>
<th>c</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{\sigma_t^2 + \sigma_u^2 S(K-1)}{(SK-1)}$</td>
<td>$\frac{\sigma_t^2}{\sigma_y^2}$</td>
<td>$\frac{\sigma_t^2}{\sigma_y^2}$</td>
<td>$\frac{\sigma_t^2}{\sigma_y^2}$</td>
<td>$\frac{\sigma_t^2}{\sigma_y^2}$ + $\frac{\sigma_u^2}{\sigma_y^2}$</td>
</tr>
<tr>
<td></td>
<td>0.924</td>
<td>0.737</td>
<td>0.798</td>
<td>0.716</td>
<td>0.661</td>
</tr>
</tbody>
</table>

Table 3. Calculations of variance components from two subscales of the ASAT and proportions relative to the total variance

<table>
<thead>
<tr>
<th>Components of Variance</th>
<th>Total</th>
<th>Common true</th>
<th>Unique</th>
<th>Sum true and unique</th>
<th>Error variance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>259.928</td>
<td>191.285</td>
<td>48.981</td>
<td>240.266</td>
<td>19.262</td>
</tr>
<tr>
<td>Proportion</td>
<td>1.000</td>
<td>0.737</td>
<td>0.189</td>
<td>0.924</td>
<td>0.074</td>
</tr>
</tbody>
</table>
The results in Tables 2 and 3 are readily calculated, and provide a more comprehensive understanding of the ASAT than either of the two values of $\alpha$ could provide on its own. In addition, in reporting these values, including $\alpha_s = 0.737$ and the latent correlation of $\rho = 0.673$, there is little likelihood of making the mistake that a high value of $\alpha_c$, 0.924, reflects a highly unidimensional scale. On the other hand, it is also clear that the proportion of error variance, 0.074, is relatively small and perhaps the total score can be used in test equating, which was the purpose of the test.

**Some caveats**

Although not often stressed, one of the caveats concerning the application of $\alpha$ in general, and in the presence of subscales in particular, is that it rests on the assumption of continuous measured variables. This is evident from Eq. (5) in which both the true score $t_n$ and the error score $e_n$ are real numbers. However, in general, the observed score, which is postulated to be the sum of these, is generally an integer. For the approximation to hold, it is necessary that the number of useful score points on the integer scale is reasonably large, of the order of 30 points or more, and in general no less. In addition, it is most important that there are no floor or ceiling effects in the data. In the case of say 30 dichotomous items or 10 polytomous items each with a maximum score of 3, the observed scores should range between approximately 5 and 25. In particular, distributions skewed artificially because of floor or ceiling effects renders the calculation of $\alpha$ essentially meaningless.

In the specific application of subscales described in this paper, it is necessary to have an a-priori definition of the subscales. This will occur when a scale is constructed deliberately to have multiple items assess different, well defined aspects of the scale. In this case, and in order for the different aspects to be assessed with equal precision, it is likely that the different aspects will have similar numbers of items. Although the formulae above do not require exactly the same numbers of items in subscales, the assumption of homogeneous variances among subscales implies that there should be similar numbers of items in each of the subscales.

Finally, the approximations above work best when there are few subscales and relatively large items per subscale, as in the example. However, even in the case where this is not the case, then a difference between $\alpha_c$ and $\alpha_s$, and an estimate of $c^2$ which is greater than 0,
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will indicate a subscale structure, and then if required, this can be studied with more detailed methods.
Summary

It is well known that coefficient $\alpha$ can be misinterpreted, particularly with respect to the assessment of dimensionality. However, because it is easy to calculate and because it gives an index of reliability under ideal conditions, it is reported routinely in assessing assessment scales in education and other social sciences, with different degrees of misinterpretation. In the presence of subscales where multiple items assess different aspects of a scale, misinterpretations are even more likely. Under some reasonable simplifying assumptions, this paper shows that relevant components of variance and a summary correlation between pairs of subscales can be calculated from just two calculations of $\alpha$, one at the level of the items and the other at the level of the subscales. It is suggested that at least the following four values be reported: (i) $\alpha_c$ which can be interpreted as approximately the proportion of the sum of true common and subscale unique variances relative to the total variance, (ii) $\alpha_s$ which is the proportion of the common true score variance relative to the total variance, (iii) $\alpha_A$ which can be interpreted as the proportion of the true common variance relative to the sum of the true and unique variance; and (iv) the correlation $\rho$, which is automatically corrected for attenuation for error and is the average correlation among subscales. With a report of all four indices, there is a reduced likelihood for misinterpretation and a more comprehensive understanding of the properties of a scale. Although these indices can be calculated by other means, it seems that in applied settings where the properties of scales are assessed, they are rarely reported. It is suggested that because these values can be calculated easily based on two calculations of $\alpha$, each of which is readily interpreted, they are more likely to be reported in the assessment of the properties of a scale.
Appendix

Components of variance from the total score

From Eqs. (8) and (11),

\[ y_n = \sum_{s=1}^{S} y_{ns} = \sum_{s=1}^{S} \sum_{i=1}^{K} x_{nis} = \sum_{s=1}^{S} \sum_{i=1}^{K} (\tau_n + c_s \nu_{ns} + \varepsilon_{nis}) \]

\[ = SK \tau_n + K \sum_{s=1}^{S} c_s \nu_{ns} + \sum_{s=1}^{S} \sum_{i=1}^{K} \varepsilon_{nis} ; \]

\[ V[y] = V[\sum_{s=1}^{S} y_s] = V[\sum_{s=1}^{S} \sum_{i=1}^{K} x_{is}] = S^2 K^2 \sigma^2_i + SK^2 c^2_s \sigma^2_{\nu} + SK \sigma^2_{\varepsilon} \]

\[ = \sigma^2_i + \sigma^2_{\nu} + \sigma^2_{\varepsilon} . \]

Variable number of items per subscale

Let \( k_s \) be the number of items in subscale \( s \). Then it can be shown that (Andrich, 2015)

\[ c^2_s = \frac{\left( \frac{\alpha_c}{\alpha_s} \right) \left( \sum_{s=1}^{S} k_s - 1 \right) \left( \sum_{s=1}^{S} \sum_{i=1}^{K} k_{is} \right) - \left( \sum_{s=1}^{S} \sum_{i=1}^{K} k_{is} \right) - \left( \sum_{s=1}^{S} k_s (k_s - 1) \right)}{\sum_{s=1}^{S} k_s (k_s - 1)} . \]

In addition, the terms \( S(K-1) \), \((SK - 1)\) for \( \alpha_c \) in Table 1 become respectively

\[ S(K-1) = \sum_{s=1}^{S} (K_s - 1) , \quad (SK - 1) = (\sum_{s=1}^{S} K_s) - 1 , \quad \text{and} \quad S^2 K^2 = (\sum_{s=1}^{S} K_s)^2 . \]

Limit of the ratio \( S[K-1]/[SK-1] \) with increasing number of items

The ratio \( S[K-1]/[SK-1] \) appears in a number of equations and has a simple limit as the number of items per subscale increases. Thus

\[ \lim_{K \to \infty} S[K-1]/[SK-1] = \lim_{K \to \infty} [S(1-1/K)/(S-1/K)] \]

\[ = S / S = 1 . \]

Because \( S[K-1] < [SK-1] \), the above limit approaches 1 from below.
Components of variance from two calculations of alpha

References


Zinbarg, R. E., Revelle, W., Yovel, I., & Li, W. (2005). Cronbach’s $\alpha$, Revelle’s $\beta$, and McDonald’s $\omega_H$: their relations with each other and two alternative conceptualizations of reliability. *Psychometrika*, 70, 123-133.