Combinatorially Regular Euler Polytopes

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Abstract

The main aim of this thesis is to classify the combinatorially regular Euler incidence polytopes. The classification is completed except for a few exceptional cases, where the subgroup structure of certain related Coxeter groups is not sufficiently well known. The topological structure of the objects is not considered.

The regular three dimensional (spherical) polytopes have been known since antiquity, and the higher dimensional ones since late last century. In recent times, various authors have attempted to abstract the concept of regular polytopes into purely combinatorial (non-geometric) settings. Currently, the most widely studied (but not the only) such abstraction is Egon Schulte’s regular incidence polytopes, where a polytope is regular if and only if its automorphism group acts transitively on its set of flags. The key difference between this work and other work on incidence polytopes is that instead of defining regularity in the above terms, a combinatorial definition is used – an object being called combinatorially regular if its local structure remains the same all over the object (specifically, if corresponding sections are isomorphic). It is shown that any polytope that is regular in Schulte’s sense is also regular in the combinatorial sense. This means that combinatorial regularity is a weaker condition that the earlier one, making the results obtained correspondingly stronger. The Eulerian condition is a similar condition to that satisfied by the classical geometric polytopes, and allows us to obtain an overview of the combinatorially regular incidence polytopes. While it is true that some authors have examined combinatorial ‘polytopes’ that are ‘less regular’ than usual, a literature search reveals that objects as ‘weakly’ regular as those examined in the thesis have usually been ignored. In particular, no-one has attempted a classification such as the one given.

The thesis contains a brief overview of some known results about geometric polytopes, and then a number of combinatorial results about the incidence polytopes are stated and proved. Some examples are given of particular polytopes, and then a strong link is established between the combinatorially regular incidence polytopes, and the theory of Coxeter groups. The main theorem of the thesis is a statement of this link, which is a classification of what might be termed ‘locally spherical’ combinatorially regular incidence polytopes, the classification being in terms of certain subgroups of certain Coxeter groups. Although nowhere is the classification explicitly restricted to such ‘locally spherical’ polytopes, this result becomes the lynchpin of the classification theorems of the next-to-last chapter, which attempt to describe exactly what combinatorially regular Euler incidence polytopes exist with certain Schlāflis symbols. Except for a few particular cases, the combinatorially regular Euler incidence polytopes are completely described.
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CHAPTER 1

How to Read This Thesis

"Begin at the beginning", the king said gravely, "and go on till you come to the end; then stop."
Lewis Carroll, 1865

While the words of the king may well be sound advice to a young girl struggling to recount her adventures in a wondrous and confusing land, they are rarely expedient when tackling a mathematical work. Hence a description of the contents of this thesis is provided, which should allow the reader to choose whichever style of reading is best suited to his or her purpose.

1.1 Aim of the Work

The aim of this work is simply stated — it is to classify the class of combinatorially regular Euler incidence polytopes. The d-incidence polytopes are an abstraction of the d-dimensional classical geometric polytopes, and were first introduced by Schulte, in his Dissertation [47] at the University of Dortmund in Germany. His work was subsequently published in three papers ([17], [48] and [49]), and has since gained a strong degree of acceptance. See §3.1 for more details.

Schulte was interested in those incidence polytopes which satisfy a rather strong regularity condition, and most research into these objects has expanded on that theme. His definition of regularity requires that the object possess a kind of global symmetry, and this symmetry greatly facilitates their study. In this thesis, we introduce the concept of "combinatorial" regularity. This definition requires that on a local level, the polytope "looks" the same all over, but says nothing about its global structure. The term "combinatorially" regular was chosen for two reasons. Firstly, its definition is purely combinatorial in nature (not requiring reference to any external algebraic structures constructed from the polytope), and secondly because it is analogous to the definitions of regularity used in other combinatorial contexts (for example in graph theory). The two definitions of regularity are discussed in §3.4. This is the first time weaker regularity than the standard has been studied in depth.

Combinatorial regularity is a strictly weaker condition than regularity, in that all regular incidence polytopes are combinatorially regular, but not vice versa.

1 But not all — see for example [52].
2 See Theorem 3.4.3 and the notes following it.
CHAPTER 1: How to Read This Thesis

The implications of this for our study are twofold. First of all, it makes any results about them significantly stronger (and so more interesting, having broader application). Secondly, it makes such results significantly harder to obtain. Considering that the classification of even the regular polytopes is far from complete, an attempt to classify the merely combinatorially regular polytopes would be far too ambitious here! Hence, we restrict the class of combinatorially regular polytopes to be examined, by applying a certain Eulerian condition (described more fully in §3.2). This condition has the advantages that it is arguably a “natural” condition, and that by applying it we may gain something of an overview of the field of combinatorially regular polytopes. Further, it is not so restrictive that no interesting results arise.

As was stated above, the aim of the thesis is to classify the combinatorially regular Euler incidence polytopes. It can be said that this aim has been largely achieved3. Chapter 7 contains the following three theorems.

Theorem 7.1.3: If d is even, there exists a one to one correspondence between isomorphism classes of indecomposable combinatorially regular Euler d- incidence polytopes and d-CS pairs.

Theorem 7.1.4: If d is odd, there is a one to one correspondence between isomorphism classes of indecomposable combinatorially regular Euler d- incidence polytopes and finite Coxeter systems (with d generators) whose graph is a path.

Theorem 7.1.5: There is a one to one correspondence between combinatorially regular Euler incidence polytopes, and finite sequences of indecomposable combinatorially regular Euler incidence polytopes.

A ‘Coxeter system’ (defined more fully in §5.1.1) is a certain kind of group, along with a particular set of generators for it, and a d-CS pair (defined in §7.1) is a certain type of Coxeter system, along with a conjugacy class of (so-called) ‘sparse’ subgroups of it.

These theorems thrust the study of combinatorially regular Euler polytopes firmly into the study of Coxeter groups and their subgroups. The interesting thing, however, is that these results could only be derived after the bulk of the classification had been completed. Below, there is a brief outline of the thesis, which explains how this unusual state of affairs came about.

1.2 Thesis Outline

The main story starts in Chapter 3, in fact, since the second chapter is devoted to an exposition of the (no less interesting) geometric polytopes. This second chapter opens with a few technical results pertaining to the geometric polytopes, and closes with a statement of the classification of the regular ones.

Chapter 3 then begins with the definition of an incidence polytope, along with some brief notes about what is already known about these objects. Moving on to Section 3.2, we meet the Eulerian condition that was mentioned above, and discover what it means for a polytope to be ‘Euler’. The attack begins in earnest in Section 3.3.

In this section, a number of combinatorial results are proven. Some of these are simple, such as isomorphism, and some more deep, such as indecomposability. Still others are merely technical, included for pragmatic reasons. All contribute, in some way or other, to the final goal. In Section 3.4, the two definitions of regularity are given, and the concept of the Schläfi Symbol of a combinatorially regular polytope is explained. It is shown how combinatorial regularity and the Schläfi Symbol relate to the ideas and results of §3.3, so that all of these concepts may be used together later on. The last section of this chapter establishes a link between the incidence polytopes and the more traditional geometric ones, showing that every geometric polytope has a ‘combinatorial counterpart’ – an Euler incidence polytope that corresponds to it in a natural way.

Chapter 4 digresses from the general study of combinatorially regular Euler incidence polytopes, and develops numerous examples of these objects. Some of the examples (such as the cubes and simplices) turn out to be isomorphic to combinatorial counterparts of geometric polytopes, whereas others (such as the halfcubes) do not. The examples are intended to aid the understanding of the classification: besides providing names for a few of the polytopes that arise, they hopefully give the reader some reasonably concrete pictures of what would otherwise be quite abstract mathematical structures.

In the fifth chapter, Coxeter groups are first encountered. A cursory overview of Coxeter group theory is given in §5.1.1, and §5.1.2 gives some additional results about those specific Coxeter groups that prove the most useful in this work. In §5.1.3, we see developed the first explicit link between our polytopes and the Coxeter groups, and gain a taste of the power of group theoretic methods.

Section 5.2 turns away, once again, from the general study of combinatorially regular Euler polytopes, and spends its pages analysing two further examples of polytopes. These are the ‘universal’ polytope constructed from a Coxeter group, and the ‘quotient’ polytope, constructed also from a subgroup of the Coxeter group. In particular, this section examines quotients constructed from

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3 Although, in some cases the classification is not terribly explicit.
certain particular subgroups called 'sparse' subgroups. These quotients turn out to have certain nice properties, in particular that their 'facets' and 'vertex figures' are universal (see Corollary 5.2.24). This property also enables us to calculate certain numerical results about these quotients (such as to determine exactly when they are Euler).

Section 5.3 is perhaps the key that unlocks the whole classification. Here it is shown that any incidence polytope is a quotient, and more particularly, there is a theorem (paraphrased below) about quotients by sparse subgroups.

Theorem 5.3.4: (Paraphrased) A partially ordered set is a combinatorially regular incidence polytope with universal facets and vertex figures if and only if it is a quotient of a universal polytope by a sparse subgroup of a related Coaster group.

Why should this be the key, one might ask, when it only concerns those very special polytopes whose facets and vertex figures are universal? The answer lies in Chapter 6.

The foundation has been laid. The 'Classifications' chapter builds on this foundation in a systematic manner. Sections 6.1 and 6.2 analyse in depth certain Schlafli Symbols, classifying for all d the combinatorially regular Euler d-polytopes with those Schlafli Symbols. It is shown in all cases considered, that the polytopes are of the form of the above theorem.

In Section 6.3.1, the reader is reminded that the 1- and 2-polytopes were classified in §3.3.1. Noting then that all 2-polytopes are universal, it is realised that all combinatorially regular 3-polytopes are covered by the theorem. An analysis of the Eulerian condition reveals then that all the indecomposable combinatorially regular Euler 3-incidence polytopes are in fact universal. It follows that the theorem also covers the combinatorially regular Euler 4-polytopes, but this time, it turns out that not all are universal.

Even so, we can use the new-found knowledge of the 4-polytopes to gain a solid grip on the 5 dimensional case, since the facets and vertex figures of a 5-polytope must be 4-polytopes. In fact, it is shown in §6.3.4 that if an indecomposable combinatorially regular 5-polytope has either facets or vertex figures that are not universal, then it fails to be Euler, and further, there are only a few possibilities for the Schlafli Symbol of such a polytope. An inductive argument using some of the results of §3.4 reveals exactly what Schlafli Symbols

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4 We should note here that by their definition, the facets and vertex figures of a d-polytope are (d - 1)-polytopes

5 Or almost. In actual fact, the case analysed in §6.2 proves somewhat difficult to cover explicitly in full detail.

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A d-polytope with d ≥ 5 can have, and they turn out to be exactly the Schlafli Symbols that were covered in §6.1 and §6.2. This completes the classification. Chapter 7 sums up the results of the thesis, drawing attention once again to its more interesting features, and pointing out some possible directions for future research.

1.3 Knowledge Required and Other Warnings

This thesis assumes little by way of previous knowledge, although at least a cursory understanding of group theory, modular arithmetic, vector space and perhaps module theory, and some general concepts of abstract algebra will probably be necessary. Most of the deeper concepts are at least defined, but prior knowledge of them will be beneficial. There are also an abundance of concepts relating specifically to combinatorial polytopes. One could not expect a general reader to have prior knowledge of these! For this reason, and because they are so central to the thesis, these concepts are always carefully defined when introduced, and occasionally examples are provided to illustrate them.

Even the reader already familiar with the concepts would find it beneficial to work through these examples, as they are often referred to later in the work, for example to underpin a proof. A classic example is that of the n-cycles, which are defined and proved to be 2-polytopes in §3.2, are shown to encompass all 2-polytopes in Theorem 3.3.10, and then in §3.4.3 become an integral part of the definition of the Schlafli Symbol. All words defined in the thesis are listed in the index, so that the reader need never founder in the face of an unfamiliar term.

A note about the numbering system. Each of the chapters is divided into sections, and many of the sections are divided into subsections. The sections are numbered sequentially within their chapters, and the subsections within their sections, so that '§5.1.3' refers to the 3rd subsection of the 1st section of the 5th chapter. Then, 'Lemma 3.4.15' refers to the 12th lemma or theorem or corollary of Section 3.4. Likewise, Figure 2.4.1 is the 1st figure of §2.4, and Table 3.6.3 is the 6th table in §6.3. The only exceptions to this numbering scheme are the tables in Appendix A. Since this appendix is not divided into sections, references to its tables are of the form 'Table A.1' and so forth.

Some parts of the thesis will be quite light reading, others more difficult. It is hoped that overall, it will be an enjoyable and rewarding experience.
CHAPTER 2
Geometric Polytopes

Mathematicians have long regarded it as demeaning to work on problems related to elementary geometry in two or three dimensions, in spite of the fact that it is precisely this sort of mathematics which is of practical value.

Branko Grünbaum & G. C. Shepherd, 1985

2.1 Introductory Concepts

There are two equivalent ways found in the literature of defining a geometric polytope (convex polytope), namely as the convex hull of a finite set of points in \( \mathbb{R}^d \), or as a bounded nonempty intersection of closed half-spaces. I will give these definitions more formally later on, along with a proof that they are, in fact, equivalent, but first, I will explain some of the concepts needed to understand them. The reader interested in a more thorough treatment has a wide range of works to choose from, for example any of [22], [29] or [6].

Given any finite subset \( \{v_1, \ldots, v_k\} \) of \( \mathbb{R}^d \), we say \( w \in \mathbb{R}^d \) is a linear combination of the \( v_i \) if there exist real constants \( \lambda_1, \ldots, \lambda_k \) such that \( w = \lambda_1 v_1 + \cdots + \lambda_k v_k \). If the \( \lambda_i \) sum to 1, then \( w \) is also called an affine combination of the \( v_i \). If, in addition, the \( \lambda_i \) are all non-negative, then \( w \) is also called a convex combination. A set is said to be linear (respectively, affine or convex) if every linear (respectively, affine or convex) combination of elements in the set is also in the set. Note that a nonempty linear set is just a vector subspace of \( \mathbb{R}^d \). Although it is not obvious from the definition, it is not hard to prove that a nonempty affine set is a translate or additive coset of a vector subspace. Note also that the usual definition of a convex set only requires that convex combinations of pairs of its elements be in the set, but it can be shown that this definition is equivalent to the one given above [29, Thm 2.15]. Next, we define the linear span (linear hull) of an arbitrary set \( S \). Again, there are three types. The linear span (linear hull) of \( S \) is the set of all linear combinations of elements of \( S \), or equivalently, the intersection of all linear sets containing \( S \). It is denoted \( \text{lin} S \). We similarly define the affine span (hull) and the convex hull (span) of \( S \), denoting them \( \text{aff} S \) and \( \text{conv} S \). The equivalence of the two definitions given for a convex hull of a set is shown in [29, Thm 2.22]. The proof there goes through with only slight modification for the affine span (as, in fact, Lay states) and also for the linear span.

We know from vector space theory that the dimension of a linear set is. It will be useful to have a definition for the dimension of an arbitrary set. This is done as follows. Since an affine set is a translate of a unique linear set \(^6\) (that is, for every affine set \( A \), there exists a unique linear set \( L \) such that \( A = x + L = \{x + t : t \in L\} \) for some vector \( x \)), we can define the dimension of an affine set to be the dimension of the linear set of which it is a translate. Now, the dimension of an arbitrary set \( S \) can be defined as follows: the dimension of \( S \) is the dimension of its affine hull.

We are now in a position to examine some relations concerning particular affine subsets of \( \mathbb{R}^d \), namely those whose dimension is \( d - 1 \), called hyperplanes. One way of describing hyperplanes is in terms of linear functionals. Otherwise known as linear forms, which are linear maps from \( \mathbb{R}^d \) to the real line. If \( f \) is a nonzero linear functional, and \( \alpha \) is a real constant, then the set \( H = \{x \in \mathbb{R}^d : f(x) = \alpha\} \) will be a hyperplane, denoted by \( \{f : \alpha\} \). Similarly, if \( H \) is any hyperplane, then there is a linear functional \( f \) and a real constant \( \alpha \) such that \( \{x \in \mathbb{R}^d : f(x) = \alpha\} = H \). This \( f \) will be unique up to ‘scaling’ in the sense that if \( \{f : \lambda \alpha\} = \{g : \beta \} \) for two nonzero linear functionals \( f \) and \( g \), then there exists a nonzero constant \( \lambda \) such that \( f = \lambda g \) and \( \alpha = \lambda \beta \) [29, Thm 3.2] and [29, Thm 3.3]. Another useful fact is that for every linear form \( f \), there exists a vector \( y \) such that for all \( x \in \mathbb{R}^d \), \( f(x) = \langle x, y \rangle \), the inner product or dot product of \( x \) and \( y \).

Some further definitions — any hyperplane \( H = \{f : \alpha\} \) determines two closed half-spaces, \( \{x \in \mathbb{R}^d : f(x) \geq \alpha\} \) and \( \{x \in \mathbb{R}^d : f(x) \leq \alpha\} \) and two open half-spaces, \( \{x \in \mathbb{R}^d : f(x) > \alpha\} \) and \( \{x \in \mathbb{R}^d : f(x) < \alpha\} \) [29, Defn 4.8]. The hyperplane is said to bound a set \( A \) if \( A \) is a subset of one of the closed half-spaces, and to separate \( A \) and \( B \) if it bounds both, one in each half-space. [29, Defn 4.9 and 4.2]. The hyperplane strictly separates \( A \) and \( B \), in addition, both \( A \cap H \) and \( B \cap H \) are empty, that is, if \( A \) and \( B \) are contained within the open half-spaces, with \( A \) in one and \( B \) in the other [29, Defn 4.3]. Note that since the two closed half-spaces are not disjoint, it is possible to have a hyperplane separating two sets which are not disjoint. As an extreme example, any hyperplane separates itself from itself!

A hyperplane \( H \) is said to support a set \( S \) at \( x \) if \( H \) bounds \( S \), and \( x \in H \cap S \) [29, Defn 5.1]. Lay proves an interesting theorem about convex sets using this definition, namely that for every point \( x \) on the boundary of a closed convex set \( S \) there is at least one hyperplane supporting \( S \) at \( x \). A subset \( F \) of a compact (that is, closed and bounded \(^6\)) convex set \( S \) is called a face of \( S \) if either \( F \) is the empty set, \( F = S \), or there is a hyperplane \( H \) which supports \( S \) such that \( H \cap S = F \). The set \( S \) itself, and the empty set, are called improper faces, and all other faces are called proper faces. For example, the proper faces of the unit disc in \( \mathbb{R}^2 \) are just the points on the boundary. More relevant to our topic, the

\(^6\) See [29, Thm 2.13] and [29, Defn 2.7]

\(^7\) See [29, Thm 5.2]

\(^8\) Strictly speaking, not all closed bounded sets are compact, except in finite dimensional spaces, which we will be dealing with exclusively.
proper faces of a square in \( \mathbb{R}^2 \) are the four corner points and the four sides. If a proper face has dimension \( i \), it is called an \( i \)-face, so the unit disc has only 0-faces, whereas the square has both 0-faces and 1-faces, even though both are 2-dimensional. If \( S \) is a \( d \)-dimensional compact convex set, its 0-faces are called vertices, its 1-faces edges, and its \((d - 1)\)-faces facets (29, Defn 20.1). The set of vertices of a convex set \( S \) is denoted \( \text{vert}S \). A note of caution: in this thesis, the term ‘face’ will at times be used to mean ‘proper face’. Now sufficient tools have been introduced to be able to understand the two common definitions of a convex polytope, here called a geometric polytope.

2.2 Definitions of a Polytope

Definition: (1) A geometric \((d-)\)polytope in \( \mathbb{R}^n \) is the \((d\)-dimensional) convex hull of a finite set.

Definition: (2) A geometric \((d-)\)polytope in \( \mathbb{R}^n \) is the \((d\)-dimensional) bounded nonempty intersection of a finite number of closed half-spaces.

Theorem 2.2.1: The two definitions given above are equivalent.

Proof: See either [26, Thms 20.8 and 20.9], or [6, Thm 9.2].

Note that a 3-polytope is usually called a polyhedron, and a 2-polytope a polygon. Some examples of geometric polytopes (there are many others) are the cube in \( \mathbb{R}^2 \) (a 3-polytope) which is the convex hull of the set of its eight corner points, or any line segment in \( \mathbb{R}^n \) (1-polytopes), each of which is the convex hull of its two endpoints. Another important example is the \( d \)-simplex, which is the convex hull of any set of \( d + 1 \) points which do not all lie on the same \( d \)-dimensional affine set. (The 2-simplex is just a triangle, and the 3-simplex is just a tetrahedron.) Although in this thesis, the term geometric polytope will be used to refer to an object satisfying the above definitions, there are a number of related, and equally interesting concepts found in the literature, often also called polytopes (see for example [23]). Thus, when reading a text on polytopes, it is always a good idea to check exactly what the author in question means by the term. The book ‘Proofs and Relativations’ by Imre Lakatos [28], although primarily a book about how mathematics in general is done, amplifies this point very well. In this thesis, the word ‘polytope’ will be used for two different concepts\(^9\), namely for geometric polytopes, and incidence polytopes (which will be defined in the next chapter), but where there is any possible confusion, I will be careful to distinguish between them.

2.3 Properties of Polytopes

2.3.1 Faces:

The definition of the faces of a (geometric) polytope was given in \( \S 2.1 \). Here, I state several known results about them.

Theorem 2.3.1: If \( F \) is a face of a polytope \( P \), and \( G \) is a face of \( F \), then \( G \) is a face of \( P \).

Proof: See [6, Thm 5.2].

This result is also stated in [29, Thm 20.12], where only an outline of a proof is given. Lay refers the reader to [22] for further details.

Theorem 2.3.2: If \( F \) and \( G \) are faces of a polytope \( P \), then \( F \cap G \) is a face of \( G \).

Proof: This result is derived from [29, Thm 20.11].

Corollary 2.3.3: If \( \{F_1, \ldots, F_k\} \) is a set of faces of a polytope \( P \), then \( \bigcap \{F_i\} \) is a face of \( P \).

Proof: This follows from the previous two theorems.

This theorem is also shown in [29, Thm 20.6]. Bredstedt (6|8) neither proves it, nor even states it as a theorem, but regards it as an obvious property of the faces of a closed convex set (see [6, p30]).

Theorem 2.3.4: All the faces of a polytope \( P \) are polytopes.

Proof: See [6, Cor 7.3], [6, Cor 8.4], or [29, Thm 20.5].

Theorem 2.3.5: Any \( d \)-polytope \( P \) has facets, and hence also \( j \)-faces for all \( 0 \leq j < d \).

Proof: [6, Thm 8.3] shows that a polytope has facets, or \((d-1)\)-faces. If \( P \) has a \((k+1)\)-face \( A \), then \( A \) is a \((k+1)\)-polytope (Theorem 2.3.4), has facets, that is, \( k \)-faces. Theorem 2.3.1 then tells us that these \( k \)-faces are also \( k \)-faces of \( P \). Thus, by induction, \( P \) has \( j \)-faces for all \( 0 \leq j \leq d - 1 \).

After having shown that there actually exist faces, it would be nice to know that they don’t swarm like locusts all over our polytope.
CHAPTER 2: Geometric Polytopes

Theorem 2.3.6: The number of faces of a geometric polytope \( P \) is finite.

**Proof:** This is shown in [29, Thm 20.5]. Brandsted actually proves it twice, once for each of the two definitions of a geometric polytope (this is before he has shown the two to be equivalent). See [6, Cor 7.4] and [6, Cor 8.5].

Note that for each of the (finite number of) facets of a geometric polytope \( P \), the supporting hyperplane of that facet (which is just its affine span) defines two halfspaces, one containing \( P \) as a subset, the other not.

Theorem 2.3.7: \( P \) is the intersection of all the halfspaces containing \( P \) defined by supporting hyperplanes (that is, the affine hulls) of facets of \( P \).

**Proof:** This is shown in [29, Thm 20.8], although it is stated there as a slightly weaker result. It could also be derived as a corollary from [6, Thm 8.2].

Theorem 2.3.8: \( P \) is the convex span of the set of its vertices.

**Proof:** See [29, Thm 20.4] or [6, Thm 7.2].

Now I state a very well-known result about geometric polytopes, the famous Euler’s relation, which goes as follows.

Theorem 2.3.9: Let \( P \) be a geometric \( d \)-polytope, and let \( f_i(P) \) be the number of its \( i \)-faces. Then \( f_0(P) - f_1(P) + \cdots + (-1)^{d+1}f_{d-1}(P) = 1 - (-1)^d \).

**Proof:** For a full proof, see [22, §8.1–§8.2] or [6, Thm 16.1]. Since this is an important result, I will give a proof outline here. I will be following the proof given in [6].

The proof is done by induction on the dimension of the polytope \( P \). It is clear that the formula works when \( P \) is a 0- or a 1-polytope, and it is easy to show that it works when \( P \) is a 2-polytope. It is then assumed that the dimension of \( P \) is \( d \geq 3 \) and that the theorem holds for all polytopes whose dimension is less than \( d \). Then, a linear form \( f \) (not related to the function \( f_i \)) is chosen such that \( x, y \neq f(x_i) \neq f(y_i) \) for vertices \( x_i \) and \( y_i \) of \( P \), and the vertices are labelled in such a way that \( i < j \) \( \Rightarrow f(x_i) < f(x_j) \) (i.e., \( i \in \{1, \ldots, m\} \)), and then a set of \( 2m - 1 \) distinct, parallel hyperplanes \( H_k \) are chosen so that the \( H_k \) with odd \( k \) pass through the vertices of \( P \), and for all \( k \), \( H_k \) is the only \( H_k \) strictly separating \( H_{k+1} - 1 \) from \( H_{k+2} - 1 \). Letting \( P_k = H_k \cap P \) gives a sequence \( P_1, \ldots, P_{2m-1} \), where \( P_1 = \{x_1\} \), \( P_{2m-1} = \{x_m\} \), and all the other \( P_k \) are \((d-1)\)-polytopes. Thus, Euler’s relation holds for each, giving

\[
\sum_{j=2}^{2m-1} (-1)^j f_j(P) = 0, \quad k = 1, \ldots, 2m - 1,
\]

whence it is deduced

\[
\sum_{k=1}^{2m-1} (-1)^{k+1} \sum_{j=2}^{k} (-1)^j f_j(P_k) = 0,
\]

that is,

\[
\sum_{j=2}^{2m-1} (-1)^{j+1} \sum_{k=1}^{2m-1} (-1)^k f_j(P_k) = 0.
\]

Then, it is shown that

\[
\sum_{k=1}^{2m-1} (-1)^k f_j(P_k) = \begin{cases} -1, & j = -1; \\ f_i(P) - f_{i-1}(P), & j = 0; \\ f_{i+1}(P), & j = 1, \ldots, d - 1. \end{cases}
\]

And if we combine the last two results above, we obtain Euler’s relation as required.

2.3.2 Combinatorial Equivalence:

Euler’s Formula is a theorem about the combinatorial properties of a polytope. Much research into polytopes is along these lines, being less concerned with the relative positions in space of the various faces of a polytope, but more with what kinds of faces there are, and how they are interconnected. We make the following definition:

**Definition:** If \( P \) and \( Q \) are (geometric) polytopes, they are said to be isomorphic or combinatorially equivalent or of the same combinatorial type if there is a bijection between the set of faces of \( P \) and that of \( Q \) which preserves inclusion.

Another useful definition is the following: \( P \) and \( Q \) are said to be dual to one another if there is a bijection between the sets of their faces which reverses inclusion. The word ‘dual’ is also used as a noun: We say that \( Q \) is a dual of \( P \) (and vice versa). Note that if \( P \) and \( Q \) are of the same combinatorial type, their duals will be as well; in particular, any two duals of \( P \) will be isomorphic. If \( P \) is dual to itself, it is said to be self-dual.

It is then a natural question to ask: what are the possible combinatorial types of geometric \( d \)-polytopes? Indeed this very question has doubtless had its role in sparking interest in some of the abstractions to be looked at in §3.1. Grünbaum (in [22, Thm 5.5.2]) states:
CHAPTER 2: Geometric Polytopes

Section 2.3: Properties of Polytopes

Theorem 2.3.10: The enumeration problem for d-polytopes is solvable. That is, there exists an algorithm for the determination of all the different combinatorial types of d-polytopes with k vertices.

The proof of this theorem relies on some rather heavy tools of mathematical logic\(^{10}\), and does not yield an algorithm that is practical for large values of k.

So far, no classification of the d-polytopes (for d ≥ 4) has been made. In fact even if we aim for the more modest goal of classifying the possible values of the vector \((f_0(P), \ldots, f_{d-1}(P))\), we still have a problem which for d ≥ 4 has not yet been solved. Only some partial results are known.

The situation for d = 3 is much brighter, but before we begin to explore this area, we need some elementary concepts relating to graph theory. Firstly, the graph of a polytope is just the graph defined on its vertices where we say two vertices are adjacent if there is an edge of P containing both. Also we need to know the definitions of some properties of graphs – namely that a graph is planar if (roughly) it can be drawn on a plane in such a way that the edges only intersect at the vertices. Also, a graph is k-connected if between any two vertices, there are (at least) k disjoint paths. So any graph will be 0-connected.

If we construct a graph with k vertices by placing an edge between every pair of them (the complete graph), the graph obtained will be (k – 1)-connected, the (k – 1) disjoint paths between vertices v and v’ being v, \{v, v’\}, v’, and v, \{v, w\}, w, \{w, v’\}, v’ for each vertex w not equal to v or v’.

In what he called the Fundamental Theorem of Convex Types, Ernst Steinitz proved

Theorem 2.3.11: A graph is isomorphic to the graph of a 3-polytope if and only if it is planar and 3-connected.

Proof: See [54] or [22, Thm 13.1.1].

The higher dimensional equivalents of this result do not hold, although for a long while it was assumed that it did, in the sense that theorems about the higher dimensional equivalents of planar 3-connected graphs were automatically treated as theorems about higher dimensional polytopes as well. It wasn’t until 1967 that a counterexample to this assumption was found\(^{11}\).

\(^{10}\) Interestingly, the proof does not go through if instead of considering polytopes in \(\mathbb{R}^d\), we consider polytopes in \(\mathbb{Q}^d\) – and [22, 15.5] gives an example of a polytope which is not ‘rationally realisable’, that is, a polytope \(P\) such that there is no polytope isomorphic to \(P\) whose vertices all have rational coordinates.

\(^{11}\) See [29].

Many useful facts about 3-polytopes are known, such as the possible values of the vector \((f_0(P), f_1(P), f_2(P))\) (see [22, Thm 10.3.2]), and some information about the number of combinatorial types with various numbers of vertices and facets\(^{12}\). The reader interested in a concise general summary of what is currently known in the area of combinatorial types of d-polytopes (including \(d ≥ 4\)) should consult [4].

The 2-polytopes (or polygons) are familiar objects, and their combinatorial types have a very simple classification. Given n, all n-gons are combinatorially equivalent. Thus, for each \(n ≥ 3\), there exists exactly one combinatorial type of n-gon. Clearly, there is only one combinatorial type of 1-polytope.

Even though little is known in general about d-polytopes for \(d ≥ 4\), much more can be said about specific subclasses of polytopes. Later (in §2.4 and §2.5) we shall examine just such a class: the regular polytope.

2.3.3 Duals:

Recall that two geometric polytopes \(P\) and \(Q\) are said to be dual to one another if there is a structure-reversing bijection from the faces of one to those of the other, and that under those circumstances, \(Q\) is said to be a dual of \(P\) (and vice versa). We call the bijection in question a dual map between \(P\) and \(Q\).

Brendsted tells us [6, Thm 10.2] that for any polytope \(P\), there will exist polytopes dual to it, and in fact if \(P\) is a d-polytope, its duals will be also [6, Thm 10.3]. Both [6] and [29] give particular examples of dual polytopes using the notion of the polar set, defined below.

Definition: For any subset \(M\) of \(\mathbb{R}^d\), the polar set \(M^*\) of \(M\) is the set \(M^* = \{y \in \mathbb{R}^d : \forall x \in M, x^T y ≤ 1\} = \{y \in \mathbb{R}^d : \sup_{x \in M} x^T y ≤ 1\} \).

Brendsted [6, Thm 6.1] shows that if \(M\) is bounded, then the origin is an interior point of \(M^*\). Then, in [6, Thm 6.2], he shows that for any subset \(M\) of \(\mathbb{R}^d\), \(M^{**}\) is the smallest closed convex set containing both \(M\) and the origin. From those, he deduces the corollary that if \(C\) is any compact (that is, closed and bounded) convex set with the origin as an interior point, then \(C^*\) is likewise a compact convex set with the origin as an interior point, and in addition \(C^{**} = \overline{C}\).

A geometric polytope is a compact convex set, being the convex hull of a finite (hence compact) set. Using this, it can be shown that for any polytope \(P\) with 0 an interior point of \(P\), \(P^*\) is a polytope also [6, Thm 9.1]. Another useful fact is that if \(F\) is a face of \(P\), we can define a face \(F^\Delta\) of \(P^*\) quite simply, via \(F^\Delta = \{y \in P^* : \forall x \in F, x^T y = 1\}\) (see [6, Thm 6.6] or [29, Lem 23.9]).

\(^{12}\) See [3].
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It can be shown that the map \( \psi : F \mapsto F^\alpha \) is a dual map between \( P \) and \( P^* \), and therefore we conclude that \( P \) and \( P^* \) are dual polytopes. Also, \( \psi^2 = 1 \) in the sense that \( (P^\alpha)^\alpha = P \) for all faces \( F \) of \( P \) (see [6, Cor. 6.8]). Note that even if \( 0 \) is not an interior point of \( P \), we can find a dual to \( P \) by looking at the polar of the set \( P - a = \{ x - a : x \in P \} \), where \( a \) is an interior point of \( P \).

One reason for introducing the dual at this point is that it makes it a little easier to understand the relevance of the concept of the vertex figure, to be introduced in the next section.

2.3.4 Vertex Figures:

Any facet \( F \) of a polytope \( P \) has the property that any face of \( P \) which is contained in \( F \) is also a face of \( F \). In that sense the facet \( F \) contains all the 'information' there is about the faces of \( P \) surrounding it. This is not true of the vertexes, however. Any vertex is just a single point, and therefore contains no information whatsoever about the faces it is contained in. The vertex figure is defined to address this problem.

Unfortunately, there are a number of definitions of the term 'vertex figure' in the literature. The definition we will use follows that given in [6, p68] and [22, p49]. However, we will also be interested in the concept as given in [12] and [29] (which is not well-defined in general, but only in certain special cases). For this concept, we will use the rather cumbersome term 'special vertex figure'. Fortunately, we won't need to use it much.

Definition: Let \( P \) be a \( d \)-polytope, and let \( x_0 \) be a vertex of \( P \). Let \( H \) be a hyperplane which strictly separates \( x_0 \) from \( (\text{vert}(P)) \setminus \{ x_0 \} \), that is, from the rest of the vertexes of \( P \). Then the \( (d - 1) \)-polytope \( P \cap H \) is called a vertex figure of \( P \) at \( x_0 \). Note that there will in general be many different vertex figures for a given polytope \( P \) and vertex \( x_0 \). However [6, Cor 11.3] shows that they will all be combinatorially equivalent. Yemelichev, in [62, p96], calls these objects vertex sections.

Theorem 11.5 of [6] is an important one - it helps to understand a little how the vertex figure provides information about the faces surrounding the vertex in question. I state it as a theorem here.

Theorem 2.3.12: If \( P^* \) is a dual of a polytope \( P \), \( \psi \) is an inclusion reversing bijection from the faces of \( P \) to those of \( P^* \), and \( Q \) is a vertex figure of \( P \) at a vertex \( q \), then \( \psi(Q) \) is a polytope dual to \( Q \).

That is, the vertex figures of \( P \) are dual to the facets of the dual of \( P \). Thus:

Theorem 2.3.13: Given \( P \), \( q \) and \( Q \) as defined in the previous theorem, there is an inclusion-preserving one to one correspondence between faces of \( Q \) and faces of \( P \) which properly contain \( q \).

Proof: Let \( P^* \) be a dual of \( P \), let \( \psi \) be a dual map between \( P \) and \( P^* \), and let \( \phi \) be a dual map between \( q \phi \) and \( Q \) as per Theorem 2.3.2 (above). We prove that the required map is \( \psi \phi \). Now if \( F_1 \) and \( F_2 \) are faces of \( P \) with \( q \subset F_1, F_2 \), we note that \( F_1 \psi, F_2 \psi \subset \phi \psi \) (by choice of \( \psi \)), and so \( F_1 \psi \) and \( F_2 \psi \) are faces of \( \phi \psi \). But then \( F_1 \psi \phi \) and \( F_2 \psi \phi \) must be faces of \( Q \), by choice of \( \phi \), so \( \psi \phi \) is a map from the faces of \( P \) containing \( q \) to the faces of \( Q \). Since both \( \psi \) and \( \phi \) are bijections, so is \( \psi \phi \), and finally, if \( F_1 \subset F_2 \), then \( F_2 \psi \subset F_1 \psi \), so \( F_1 \psi \phi \subset F_2 \psi \phi \) as required.

It can also be shown that

Theorem 2.3.14: For any polytope \( P \), the facets of a vertex figure of \( P \) at a vertex \( x \) are just vertex figures at \( x \) of the facets of \( P \) which contain \( x \).

In the next section, we will be interested in one particular vertex figure at each vertex \( x \), namely the one given by the convex span of the midpoints of all the edges of \( P \) containing \( x \), whenever this is actually a vertex figure (which will occur when the affine span of these points is a hyperplane). This is the vertex figure we will call the special vertex figure (as mentioned earlier). Note that a 2-polytope will always have a special vertex figure at every vertex.

2.4 Regular Polytopes

The regular polytopes, because of their symmetry, have an aesthetic value which is partly what makes them the best known (amongst the general population) of all the various subclasses of geometric polytopes. The reader will probably recognise the regular 3-polytopes, or polyhedra, which are displayed in Figure 2.4.1.

Let us examine the mathematical definition of a regular polytope and some related concepts, before running through their classification.
2.4.1 The Definition:

Two subsets $S_1$ and $S_2$ of $\mathbb{R}^n$ are said to be congruent if there is a distance preserving map $T$ with $T(S_1) = S_2$. Such a map is then called a congruence. It can be shown that all congruences are invertible affine transformations. The symmetry group of a polytope $P$ is the set of all congruences which fix $P$. We define a regular polytope to be a 2-polytope all of whose facets (that is, edges) are congruent, and all of whose special vertex figures are congruent. Thus, for example, a square is a regular polygon. We then define a $d$-polytope to be regular if $d \leq 1$, or if it is a regular polygon, or if $d \geq 3$ and all its facets are regular and at each vertex, the special vertex figure exists and is regular (cf [12, p123]).

Cumbrous though it may seem, this is the standard definition of regularity for a geometric polytope. Other equivalent definitions have been used, for example based on whether or not the polytope possesses a certain degree of symmetry. In fact, it is analogous the latter kind of definition which are often used to define regularity for combinatorial abstractions of the geometric polytopes (see §3.1 and §3.4).

Now given a regular polytope $P$, there exist polytopes of the same combinatorial type which are not themselves regular. Thus regularity is more than a purely combinatorial property of the polytope. Nonetheless, we do want to divide the set of all geometric polytopes into some kind of isomorphism classes, or we will be forced to concede that there exist an uncountably infinite number of regular polytopes in any dimension. A suitable division is by "similarity": we say two polytopes are similar if there exists some nonzero $\lambda \in \mathbb{R}$ such that $P$ and $\lambda Q$ are congruent.

Note that if two polytopes are similar, they are also combinatorially equivalent. It will turn out that for regular polytopes, the converse is also true, but this will not be proved directly in this thesis, but rather, will be noted by perusing the classification of all (similarity classes of) geometric polytopes, given in §2.5. We close this section with two further results, which will not be proven in detail here.

**Theorem 2.4.1:** Every regular polytope has a regular dual.

**Proof:** It can be shown that if we choose the origin of $\mathbb{R}^n$ to be at the centre of the regular polytope $P$, then the polar of $P$ will be regular. In fact, the vertices of the polar lie on lines which pass through the origin, and through the centres of the facets of $P$.

**Theorem 2.4.2:** If $P$ is a regular geometric polytope, all its facets are congruent, and all its special vertex figures are congruent.

**Proof:** This may be derived from [32, Prop 2.1].

2.4.2 Schl"afi Symbols:

The Schl"afi Symbol is named for the Swiss mathematician Ludwig Schl"afi (see §2.5.3 and [12, p142]), and is defined for any regular geometric polytope $P$. We speak of the Schl"afi Symbol of $P$.

Basically, it is defined inductively – the regular $p$-gon being given the Schl"afi Symbol $\{p\}$, and a regular polyhedron being given the Schl"afi Symbol $\{p\}$ if its facets are $p$-gons and its (special) vertex figures $q$-gons. Since for any polytope $P$, the facets of a vertex figure of $P$ at a vertex $x$ are just vertex figures at $x$ of the facets of $P$ which contain $x$, or more succinctly, the "[facets] of the [special] vertex figures are the [special] vertex figures of [ facets]" ([12, p129]), we can conclude that if a regular 4-polytope has facets whose Schl"afi Symbols are $\{p\}$, its special vertex figures must have Schl"afi Symbol $\{p\}$ for some $p$, and so we can define the Schl"afi Symbol of this 4-polytope to be $\{p\}$. Similarly for higher dimensions, a regular $d$-polytope has Schl"afi Symbol $\{p_1\ldots p_{d-1}\}$ if its facets have Schl"afi Symbol $\{p_1\ldots p_{d-2}\}$ and its special vertex figures $\{p_1\ldots p_{d-1}\}$.

The concept of the Schl"afi Symbol of a (combinatorially) regular incidence polytope (to be given in §3.4.4) will mirror this one.
2.5 The Classification of Regular Polytopes

The regular geometrical polytopes have been completely classified. Possibly the best text available on the subject is [12].

2.5.1 The 1- and 2-polytopes:

Line segments are regular, and are all similar. These are the 1-polytopes. More interesting are the regular 2-polytopes, or regular polygons. As Coxeter ([12, p.1]) points out 'everyon is acquainted with some of the regular polygons', which have all sides equal, and all angles equal. For any number of sides \( p \), there exists a regular polygon, although for large \( p \), they begin to look very much like circles. For example, it has been shown that it is "possible" to construct a regular 65537-gon using only a ruler and a compass. However, if one were to do so, making each edge 1cm long, the polygon would need to be over 200m across, and the radii of the incircle and the circumcircle would differ by less than a quarter of a micrometre – approximately the wavelength of ultraviolet light!

You would need an ultraviolet camera to distinguish between this polygon and a circle – not to mention a very sharp pencil to draw it! The convex hull of the points \( \{ (\cos \frac{2k\pi}{p}, \sin \frac{2k\pi}{p}) : k \in \mathbb{Z} \} \) is a regular \( p \)-gon.

2.5.2 The 3-polytopes:

The only five regular geometric 3-polytopes (regular polyhedra) are the five so-called platonic solids. Information about them is given in Tables 2.5.1 and 2.5.2 below. In Table 2.5.1, the columns headed \( V \), \( E \) and \( F \) give (respectively) the numbers of vertices, edges and faces for each of the regular polyhedra. Table 2.5.2 shows how each such polyhedron may be formed as a convex hull of a finite set. In this table, \( \phi \) represents the number \( \sqrt{3+2} \), and \( \sigma \) is the permutation \( (1,2,3) \), so \( e_{1\sigma} = e_2 \), \( e_{2\sigma} = e_3 \) and \( e_{3\sigma} = e_1 \).\n
<table>
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</tbody>
</table>

The credit for the first constructions of these figures does not go to man – each of them occurs naturally in one form or another, although not all of these occurances are visible to the naked eye. The tetrahedron, cube, and octahedron all occur as crystals. These by no means exhaust the numbers of possible forms of crystals\(^\text{13}\), of which there are 48. Neither the regular icosahedron nor the regular dodecahedron are amongst them, although one of the forms, called the pyritohedron (named for the group of minerals of which it is typical) is combinatorially equivalent to the regular dodecahedron, having twelve pentagonal faces, three per vertex. In the microscopic world, there exist certain species of plankton whose skeletons are shaped like various regular polyhedra, some examples being Circoporus Octahedrus, Circogonia Icosahedra, Lithocbeschus Geometricus and Circorhexogena Dodecachorda, of the order Radiolaria in the class Sarcodina. The shapes of these creatures should be obvious from their names, but the curious reader can find sketches of two of them in [2, p6]. A more recent discovery is of a series of new types of Carbon molecules, known as the fullerences\(^\text{14}\) for an easy to read exposition of this discovery. Although the most easily produced of these (C_{60}) looks more or less spherical, some of the larger varieties (such as C_{70}, C_{80} and C_{90}) are hypothesised to take on the form of slightly rounded icosahedra, a few nanometres across.

The root of the human discovery of these shapes, particularly of the simpler ones, is probably impossible to trace. Some authors\(^\text{15}\) credit Pythagoras (550BC) with being familiar with them, whereas others indicate that he may only have been familiar only with the tetrahedron, cube, and dodecahedron, crediting the discovery of the other two to Thaetetus (an Athenian), who in any case gave a mathematical description of all five\(^\text{16}\). Coxeter (see [12, §1.9]) credits Plato (400BC) with having made models of them, and mentions that one of the earlier Pythagoreans used all five in a correspondence between the

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\(^{13}\) See for example [53, p212].

\(^{14}\) For example [43].

\(^{15}\) [56], [18, book XIII].
polyhedra and the nature of the universe as it was then perceived. It is from Plato's name that the term "Platonic Solids" is derived.

At the other end of the Aegean sea was another civilization which is little heard of nowadays, the Etruscans. There is a possibility that these people predated the Greeks in their awareness of at least some of the regular polyhedra, as evidenced by the discovery near Padua (in Northern Italy) in the late 1800's of a dodecahedron made of soapstone, and dating from about 500BC. It may be argued, however, that the construction of this form was inspired by the pyritohedron mentioned earlier, as pyrite minerals are relatively abundant in that part of the world. This being so, the Etruscans may have had no mathematical understanding of the regular figures at all. In any case, the Greeks are usually credited with being the first human discoverers.

2.5.3 The 4-polytopes:

There are exactly six regular geometrical 4-polytopes. This fact was first discovered by the Swiss mathematician Ludwig Schlaffi, who also characterised the regular polytopes in all higher dimensions. His efforts were first published in full in 1900, six years posthumously, although parts of it were published in 1855 and 1858. Interestingly, between 1880 and 1900, Schlaffi's results were rediscovered independently by at least nine other mathematicians (see [12, pp.143-144] for more details).

For a description of these results, I refer you once again to [12], which being more modern, is probably more accessible. Descriptions of the six regular geometric 4-polytopes may be found in the Table 2.5.3 (below). V, E, F and S denote the numbers of vertices, edges, 2-faces, and facets, for each of the regular 4-polytopes.

The 4-Simplex, although a four dimensional object, is actually most easily expressed as a convex hull of a set of points in $\mathbb{R}^4$, namely $\conv\{e_i : 1 \leq i < 5\}$. The cube can be defined as $\conv\{\pm 1, \pm 1, \pm 1\} \subseteq \mathbb{R}^4$, and the Cross as $\conv\{\pm e_1, \pm e_2, \pm e_3, \pm e_4\}$. The 4-Cube and the 4-Cross, dual to one another, may be regarded as the four dimensional analogues of the three dimensional cube and the octahedron respectively (mentioned in the last section), whereas the 4-Simplex may be regarded as the analogue of the tetrahedron.

An example of the 24-cell is $\conv\{\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm (\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$. It is self-dual. Note that its vertex set may be divided into two parts -- one part corresponding to the vertices of a cross, and the other part corresponding to the vertices of a cube.

An example of the 600-cell is the convex hull of those points obtained by applying even permutations to the coordinates of $(0, \pm 1/2, \pm 1/2, \pm 1/2)$ and $(\pm 1, \pm 1, \pm 1, \pm 1/2)$, where once again $\phi = \sqrt{3}/2$. The 120-cell has 600 vertices, it would therefore be tedious to list them all. An example of it would be the polar set of the 600-cell given above. Alternatively, one could find the vertices of the 120-cell by taking the means of all sets of four "mutually adjacent" vertices of the 600-cell. This is because the facets of the 600-cell are tetrahedra and any set of four mutually adjacent vertices yields a facet, whose centre will be a vertex of a 120-cell (in fact, a scaled version of the polar). A simple test to see if two vertices are adjacent are if the angle subtended by them from the centre of the 600-cell is $30^\circ$. Given the above set of vertices of the 600-cell, this simply means that any pair of adjacent vertices $x_1$ and $x_2$ satisfies $x_1 \cdot x_2 = \frac{3}{5}$.  

2.5.4 From Then On:

If $d \geq 5$, there are exactly three similarity classes of regular geometric $d$-polytopes. They are the (geometric) $d$-Simplex, $d$-Cube, and $d$-Cross. Details are given in tables 2.5.4 and 2.5.5.

As mentioned in the last section, these results were first discovered by Schlaffi and were published in [46]. Note that each of these examples also occurs for

<table>
<thead>
<tr>
<th>Name</th>
<th>S.Sym.</th>
<th>V</th>
<th>E</th>
<th>F</th>
<th>S</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Simplex</td>
<td>(3)(3)</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>Self</td>
</tr>
<tr>
<td>4-Cube</td>
<td>(4)(3)</td>
<td>16</td>
<td>24</td>
<td>32</td>
<td>8</td>
<td>4-Cross</td>
</tr>
<tr>
<td>4-Cross</td>
<td>(3)(4)</td>
<td>8</td>
<td>32</td>
<td>24</td>
<td>16</td>
<td>4-Cube</td>
</tr>
<tr>
<td>24-cell</td>
<td>(4)(3)</td>
<td>24</td>
<td>96</td>
<td>96</td>
<td>24</td>
<td>Self</td>
</tr>
<tr>
<td>120-cell</td>
<td>(5)(3)</td>
<td>600</td>
<td>1200</td>
<td>720</td>
<td>120</td>
<td>600-cell</td>
</tr>
<tr>
<td>600-cell</td>
<td>(3)(5)</td>
<td>120</td>
<td>720</td>
<td>1200</td>
<td>600</td>
<td>120-cell</td>
</tr>
</tbody>
</table>
A number of things may be noted from the overall classification. For example, if two regular polytopes have the same Schl"{a}fli Symbol, they are combinatorially equivalent. This turns out not to be the case for combinatorially regular incidence polytopes. Also note that any two regular geometric polytopes are combinatorially equivalent, they will also be similar, although only the converse is true for general geometric polytopes. Nonetheless, we still need to use similarity classes instead of combinatorial equivalence classes to exclude some non-regular polytopes (such as the pachyderm mentioned earlier) from the set of regular polytopes.

This having been done, combinatorial objects may be found that satisfy the new axioms, but for which there does not exist a corresponding geometrical object (or at least not in the geometry which inspired the abstraction). Instead of being regarded as a failure on the part of the new definition to accurately capture the old concept, this makes the new objects even more interesting to study, particularly if the definition has a certain "aesthetic" quality all its own.

One early effort in this direction is Branko Gr"{u}nbaum's *polystromata* (See [24]). The concept of a polystron is an extremely broad one -- any partially ordered set with a single least element is a polystron. Gr"{u}nbaum describes how results in a number of different branches of mathematics may be phrased in terms of his objects. He also looks at particular examples of polystromata which possess certain symmetry properties. Little other work has been done specifically on
these objects since then, except for a paper (99) by Colbourn and Weiss, which gives a census of all the known regular 3-polyhedral in a particular type. Numerous other papers, however, mention them as examples of other objects. (See for example [13]).

An abstraction which seems to have gained considerable acceptance in the mathematical community is Egon Schulte's incidence complexes. These were originally described in his 1980 thesis [47], which was subsequently published in three separate articles, [17], [48] and [49]. His definition is as follows.

Definition: A d-incidence complex is a partially ordered set \( \mathcal{X} \), satisfying the properties 11 to 14 below.

(11) The set \( \mathcal{X} \) has a unique minimal element \( \emptyset \), and a unique maximal element \( K \).

Each totally ordered subset of \( \mathcal{X} \) with exactly \( t + 1 \) elements is called a chain of length \( t \). We call the maximal chains the flags of \( \mathcal{X} \). Then, the second requirement is as follows.

(12) Every chain of \( \mathcal{X} \) is contained in a flag of \( \mathcal{X} \). Also, there is some positive integer \( d \) such that every flag has length \( d + 1 \).  

Lemma 3.1.1: If \( \mathcal{X} \) satisfies 11 and 12 above, then for all \( a \in \mathcal{X} \), \( \emptyset \leq a \leq K \).

Proof: Let \( a_0 \) be an element of \( \mathcal{X} \) such that \( a_0 \notin K \). In particular, \( a_1 \neq K \), so there exists an element \( a_1 \) of \( \mathcal{X} \) which is strictly greater than \( a_0 \), that is, \( a_0 < a_1 \). But if, now, \( a_1 \leq K \), then this tells us that \( a_0 < K \) (by the transitivity of the partial order), which contradicts our assumption that \( a_0 \notin K \).

So \( a_1 \notin K \) either. Hence there is an element \( a_2 \) of \( \mathcal{X} \) satisfying \( a_1 < a_2 \), and \( a_2 \notin K \). This process may be continued as long as required to construct a chain \( (a_0, a_1, \ldots, a_n) \) of elements of \( \mathcal{X} \), with \( n \geq d + 2 \). This chain must be contained in some flag of \( \mathcal{X} \), and this flag will have \( d + 2 \) elements (since flags have length \( d + 1 \)). Hence the chain has at most \( d + 2 \) elements, leading us to conclude that \( m + 1 \leq d + 2 \). This is a contradiction. Hence our initial assumption, that there exists an \( a_0 \) with \( a_0 \notin K \), must have been mistaken. A similar argument shows that there cannot exist an element \( b_0 \) of \( \mathcal{X} \) with \( \emptyset \leq b_0 \).

So \( \emptyset \) and \( K \) are not only maximal and minimal, but also a maximum and a minimum. For any \( f, g \in \mathcal{X} \) with \( f \leq g \), we define the section \( (f, g) \) to be

\[(f, g) = \{ h : h \in \mathcal{X} \text{ and } f \leq h \leq g \}. \]

Theorem 3.1.2: Let \( \mathcal{X} \) be a poset satisfying 11 and 12 above. Then any section of \( \mathcal{X} \) also satisfies 11 and 12.

Proof: This is stated without proof in [17]. Consider \( F = (f, g) = \{ h : f \leq h \leq g \} \), a section of \( \mathcal{X} \). Now \( f \) and \( g \) will serve as the minimal and maximal elements \( K \) and \( 1 \) is bounded by \( 1 \). Now, let \( C \in \mathcal{X} \). Then \( \mathbb{C} \cup (f, g) \) will be a chain of \( \mathcal{X} \), and hence will be contained in a flag \( F \) of \( \mathcal{X} \). We show that \( F \cap \mathcal{X} \) is a maximal chain of \( \mathcal{X} \). To do this, let \( G \) be a chain of \( \mathcal{X} \) with \( F \cap \mathcal{X} \subseteq G \), and let \( t \in G \). Now for all \( u \in \{ t \} \cup F \), we have either \( u = t \) or \( u \in F \cap \mathcal{X} \), (whence either \( u \geq t \) or \( u \leq t \)), or \( u \in F \cap \mathcal{X} \), then \( u \in G \), so either \( u \leq t \) or \( u \geq t \), since \( G \) is totally ordered. Thus \( F \cup \{ t \} \) is a totally ordered subset of \( \mathcal{X} \), whence (since \( F \) is a flag) it must be that \( F \cup \{ t \} = F \), so \( t \in F \). Thus \( G \subseteq F \cap \mathcal{X} \). It follows that \( F \cap \mathcal{X} \) is a maximal chain of \( \mathcal{X} \), so \( C \) is indeed contained in a chain, as required.

To show that all the flags of \( \mathcal{X} \) have the same size, we first choose two flags \( C_1 \) and \( C_2 \) of \( \mathcal{X} \), and if they are not the same size, let \( C_1 \) be the larger of the two. Then, we construct a flag \( F \) containing \( C_1 \) (which can be done, since \( C_1 \) is a chain in \( \mathcal{X} \)). Note that \( f, g \in C_1 \), so \( f, g \notin F \). Then, the set \( G = (F \cap C_1, C_2) \) is a chain of \( \mathcal{X} \), since for all \( u \in G \), either \( u \in C_1 \) or \( u \in F \), or (without loss) \( u \in F \cap C_1 \) and \( u \in C_2 \). If \( t \in F \cap C_1 \), then it must be that \( t > g \geq u \) or \( t < f \leq u \). Otherwise, \( m \) must be an element of \( \mathcal{X} \), whence \( F \cap \mathcal{X} \) is a chain of \( \mathcal{X} \) containing \( C_1 \). In all these cases, we have either \( t \geq u \) or \( t \leq u \), so indeed \( G = (F \cap C_1, C_2) \) is a chain of \( \mathcal{X} \). In this fact set \( m \) is a maximal chain, for if \( H \) is any chain of \( \mathcal{X} \) containing \( C_1 \), then \( t \in (H \cap C_1) \cup C_2 \) which will be a chain of \( \mathcal{X} \) containing \( (C_1 \cup C_2) \). Now \( F \cap C_1 \) as we have seen, contains no element of \( \mathcal{X} \). Thus \( C_1 \cup C_2 = (F \cap C_1) \cup C_2 \), and \( C_1 \cup C_2 = (F \cap C_1) \cup C_2 \), which equals \( F \) since \( C_1 \subseteq F \). Thus \( t \in (H \cap C_1) \cup C_2 \), which is a chain of \( \mathcal{X} \) containing \( F \). Since \( F \) was a flag of \( \mathcal{X} \), we conclude that \( t \in C \). So if \( t \in C \), then either \( t \in C \), or \( t \in (H \cap C_1) \), then \( t \in C_1 \), whence \( t \in F \cap C_1 \), whence \( t \in C_1 \). It follows that \( H \subseteq F \) and \( G = (F \cap C_1) \cup C_2 \), and the same size, since \( \mathcal{X} \) satisfies 12. But \( C_1 \) is a flag of \( \mathcal{X} \), and \( C_2 \) is a flag of \( \mathcal{X} \), and it may be deduced that \( |C_1| \leq |C_2| \), that is, \( |C_1| \leq |C_2| \), which is a contradiction unless \( C_1 \) and \( C_2 \) have the same size. This completes the proof.

The properties 11 and 12 allow us to assign a dimension to the sections of \( \mathcal{X} \) (including \( \mathcal{X} \) itself), and also to its elements. For any poset \( \mathcal{X} \) satisfying 11 and 12 above, we define its dimension to be

\[ \dim \mathcal{X} = d, \]
where \(d + 1\) is the length of the length of the flags of \(\mathcal{S}\) (that is, the flags of \(\mathcal{S}\) have \(d + 1\) elements, including the maximal and minimal elements of \(\mathcal{S}\)). Thus, since sections of such a poset also satisfy 11 and 12, for any \(f, g \in \mathcal{S}\) with \(f \leq g\), the expression \(\dim(f, g)\) is well defined. An \(i\)-dimensional section shall be called an \(i\)-section of \(\mathcal{S}\). We define the dimension of an element \(f\) of \(\mathcal{S}\) via

\[
\dim f = \dim(\emptyset, f),
\]

where \(\emptyset\) is the minimal element of \(\mathcal{S}\). Note that \(\dim \emptyset = -1\) and if \(K\) is the maximal element of \(\mathcal{S}\), then \(\dim K = d\). It may be shown that a flag \(F\) of \(\mathcal{S}\) contains exactly one element of each dimension \(j\) with \(-1 < j \leq d\). There is a danger that some ambiguity may arise in our notation if \(F\) is an element of more than one complex (for example, an element of \(\mathcal{S}\) and also of a section \(\mathcal{S}^i\) of \(\mathcal{S}\)). The dimension function \(\dim f = \dim(\emptyset, f)\) would give different values in such situations, the ambiguity shall be removed by use of such notation as \(\dim \mathcal{S} f\) or \(\dim \mathcal{S}^i f\). We also have the following result.

**Theorem 3.1.3:** For \(x\) and \(y\), elements of \(\mathcal{S}\), we have \(\dim(x, y) = \dim y - \dim x - 1\).

**Proof:** (Outline.) Let \(\mathcal{S} = (x, y)\), and let \(\dim x = i\) and \(\dim y = j\). Let \(F\) be a flag containing \(x\) of \((0, y)\), so \(F\) has \(j + 2\) elements. Then \(F \cap \mathcal{S}\) will be a flag of \((0, y)\), and so has \(i + 2\) elements. Then, since \(F\) is equal to the disjoint union \((F \cap \mathcal{S}) \cup (F \cap \mathcal{S}^i)\), we have \(|F| = |F \cap \mathcal{S}| + |F \cap \mathcal{S}^i|\), whence \(j + 2 = (\dim \mathcal{S} + 2) + (i + 2 - 1)\), which may be rearranged to obtain the required result.

If \(\dim \mathcal{S}(\emptyset, f) = t\), then we call \((\emptyset, f)\) an \(i\)-face of \(\mathcal{S}\).

As there is a one to one correspondence between the elements of \(\mathcal{S}\) and the faces of \(\mathcal{S}\), taking \(f\) to \((\emptyset, f)\), some authors like to use the term i-faces to refer to the elements of \(\mathcal{S}\) of dimension \(t\). The 0-faces (and the elements of \(\mathcal{S}\), also to the elements of \(\mathcal{S}\)) of dimension 0 of \(\mathcal{S}\) will be called the vertices of \(\mathcal{S}\), and the 1-faces will be called the edges. Also, the \((d - 1)\)-faces of \(\mathcal{S}\) will be called the facets of \(\mathcal{S}\).

Further, we define the edges or cofaces of \(\mathcal{S}\) to be those sections of the form \((f, K)\). Those cofaces \((f, K)\) where \(f\) is a vertex shall be called vertex cofaces (or vertex edges). We similarly define edge edges (or edge cofaces). Finally, two sections \((f, g)\) and \((f', g')\) shall be said to be corresponding if \(\dim f = \dim f'\) and \(\dim g = \dim g'\).

The third property an incidence polytope must satisfy relates to the connectivity of the whole structure. To explain this property, still more definitions are needed.
Theorem 3.1.4: Any section of an incidence complex will itself be an incidence complex, and any section of an incidence polytope will itself be an incidence polytope.

Proof: We have seen (Theorem 3.1.2) that a section satisfies 11 and 12 and it is straightforward to show that it satisfies 13. We also could use Theorem 3.1.3 to show that any section $\mathcal{S} = (f, g)$ of $\mathcal{X}$ satisfies $\dim x = \dim y - 1$ for all $x, y \in \mathcal{X}$, so that if $x$ and $y$ are elements of $\mathcal{X}$ with $\dim x + 1 = \dim y - 1 = i$, then $x$ and $y$ are elements of $\mathcal{X}$ with $\dim x + 1 = \dim y - 1 = i + i$ for all $x, y \in \mathcal{X}$, and satisfying 14 also, making it an incidence complex. If $\mathcal{X}$ is a polytope, that is, if all the $k_i$ are equal to 2, then $\mathcal{X}$ is a polytope also.

These four axioms, as has been mentioned, are intended to reflect certain properties of the set of faces of a geometric polytope. In §3.5, we will take a closer look at the link between the geometric polytopes and the incidence polytopes, in particular showing that the set of faces of a polytope do in fact satisfy these axioms. The question of whether or not these are the most "natural" combinatorial axioms one could choose, is probably impossible to answer.

The incidence polytopes have, however, attracted significant interest particularly the case of "regular" incidence polytopes, where a regular incidence polytope $\mathcal{X}$ is one whose automorphism group is flag-transitive, that is, for any two flags $F$ and $G$, there exists a structure preserving bijection from $\mathcal{X}$ to itself (an automorphism) which maps $F$ to $G$. More on this in §3.4.

Numerous examples of these objects have been found. Most papers dealing with the problem of finding such examples involve the use of a group theoretical construction given in [49], where a one-to-one correspondence is established between regular incidence polytopes and certain groups, specifically, the class of those groups which can occur as automorphism groups of the polytopes. Given the group, the polytope is uniquely identified, and certain of its combinatorial properties may be discerned. Authors who have taken advantage of this construction include McMullen and Schulte [34, 36, 37], Monson [40] and Monson and Weiss [59, 60]. Certain other questions have also been addressed, such as Weiss [59, 60].

For example, James Oxley in [41], his 1975 Master's thesis, introduced what he called combinatorial polyhedra and polyhedral configurations. The latter can be shown (after some work) to be equivalent to 3-incidence polytopes, with a regular 3-incidence polytope being a "flag-transitive" polyhedral configuration. Oxley's work might therefore be quite useful to those studying Schulte's more widely known objects. Oxley, in his thesis, proved a number of results, for example that there exists one-to-one correspondence between flag-transitive polyhedral configurations and their symmetry groups. This result corresponds to the group theoretic characterisation of regular incidence polytopes mentioned earlier.
3.2 Euler Polytopes

Recall the difference between an incidence complex and an incidence polytope. For a d-incidence complex, we have for any i, if f < g satisfy \( \dim f + 1 = i = \dim g - 1 \), then \( |\{h : f < h < g\}| = k_i \), where the cardinals \( k_i \) depend only on \( i \), and not on \( f \) and \( g \). For the complex to be a polytope, each \( k_i \) is forced to be equal to 2.

We could rephrase this, in terms of the so-called Euler’s condition. A complex \( X \) satisfies Euler’s condition if and only if

\[
\sum_{j=1}^{\dim X} (-1)^{j+1} |X_j| = 0,
\]

where \( X_j = \{h \in X : \dim X h = j\} \). Note that this may be re-written as

\[
\sum_{k \in X} (-1)^{\dim X k + 1} = 0,
\]

and only has meaning if the \( X_j \) are all finite.

Note that if \( \mathcal{Y} = \{f, g\} \) is a 1-section of a complex, then \( \mathcal{Y} = \{h : f \leq h \leq g\} = \{f, g\} \cup \{h : f < h < g\} = \{f, g, h_1, h_2, \ldots, h_k\} \). This yields \( \mathcal{Y}_1 = \{f, g\} \), \( \mathcal{Y}_0 = \{h_1, \ldots, h_k\} \), and \( \mathcal{Y}_1 = \{g\} \). Thus \( |\mathcal{Y}_1| - |\mathcal{Y}_0| = 1 - k_1 + 1 = 2 - k_1 \) if \( k_1 \) is finite). \( \mathcal{Y} \) will therefore satisfy Euler’s condition if and only if \( k_1 = 2 \).

So an incidence complex is a polytope if and only if all 1-sections satisfy Euler’s condition. We might then call an incidence polytope a 1-Euler incidence complex. We make the following definitions.

Definition: A d-incidence complex shall be called k-Euler if for all \( 0 \leq j \leq k \), every j-section satisfies Euler’s condition. Evidently, if \( l \leq k \) and a complex \( X \) is k-Euler, it will also be l-Euler. All the sections of a k-Euler polytope are also k-Euler. A d-Euler d-incidence complex shall be called an Euler incidence complex (or, more simply, an Euler complex or polytope). Complexes of this type are the main focus of this work. It will also be useful to consider the class of \( (d - 1) \)-Euler d-incidence complexes, which shall be described as sub-Euler. We will be focusing mainly on Euler and sub-Euler polytopes. The reason for narrowing the focus in this manner is so that the problem of studying the combinatorially regular incidence polytopes does not become too large a task for this introductory work.

Discussions of k-Euler incidence polytopes are not found elsewhere in the literature.

Now let us look at some simple examples.

Theorem 3.2.1: The set \( X = \{0, K\} \) with \( 0 < K \) is an Euler polytope of dimension 0.

Proof: This is a matter of checking the axioms 11 to 14 one by one, and noting that \( X \) itself satisfies Euler’s condition, with \( |X_1| - |X_0| = 1 - 1 = 0 \).

Theorem 3.2.2: The set \( \{0, h_1, h_2, \ldots, h_k, K\} \) with \( 0 < h_i \) and \( h_i < K \) for each \( i \), and \( h_i \) and \( h_j \) not comparable when \( i \neq j \) (so neither \( h_i \leq h_j \) nor \( h_j \leq h_i \)), is a 1-incidence complex, and is an Euler 1-polytope if \( k = 2 \).

Proof: This is again a matter of checking the axioms one by one, and if \( k = 2 \), checking Euler’s formula for \( X \) itself, and for its 0-sections.

Definition: If \( n \geq 2 \), we make the following definition — let \( C = \{1\} \) be a cyclic group of order \( n \). (So if \( n \in \mathbb{Z}^+ \) then \( C \cong \mathbb{Z}_n \) and if \( n = \infty \) then \( C \cong \mathbb{Z} \)). Then, let \( X = \{0, K\} \cup \{t_n : x : x \in C\} \), and define the partial order as follows. For all \( x \in C \), let \( 0 < t_x, t_{x+1}, K \), let \( t_x < t_{x+1} \). \( t_x < t_{x+1} \), and let \( T_x < K \). Then \( X \) is called an n-cycle. Figure 3.2.1 depicts a 7-cycle, and should give an idea of the structure of the more general case.

![Figure 3.2.1](image-url)
The following theorem is useful.

Theorem 3.3.2.3: The n-cycle is an Euler polytope for any finite \( n \geq 2 \), and is a sub-Euler polytope if \( n = \infty \).

Proof: It is trivial to check 11 and easy to check 12. If \( y - x = k \cdot 1 \) for some \( k \in \mathbb{Z}^+ \), then \( t_x, t_{x+1}, t_{x+2}, \ldots, t_{y-1}, t_y \) will be a sequence connecting \( t_x \) and \( t_y \). It is just as easy to demonstrate sequences connecting any other pair of elements of the n-cycle, so the n-cycle is weakly connected. Since all its other sections have dimension less than 2, they are likewise weakly connected, so the n-cycle satisfies 13. Furthermore, it is not hard to check that the sets of the form \( \{z: t_x < z < T_x\} \) and \( \{z: 0 < z < T_x\} \) both have exactly two elements, and so \( \mathcal{X} \) satisfies 14, making it a complex, indeed a polytope. Being a polytope, it is 1-Euler, that is, sub-Euler. If \( n \) is finite, then the polytope itself also satisfies Euler’s condition: the elements with dimension 0 are just the n elements \( t_x \), and those with dimension 1 are the \( T_x \), so

\[
(-1)^{1} \mathcal{X}_{-1} + (-1)^{0} \mathcal{X}_0 + (-1)^{-1} \mathcal{X}_1 + (-1)^{1} \mathcal{X}_1 = -1 + n - n + 1 = 0.
\]

It is therefore 2-Euler, that is, Euler.

3.3 Properties of Euler Polytopes

3.3.1 Isomorphism:

We define two posets \( \mathcal{X} \) and \( \mathcal{Z} \) to be isomorphic, and write \( \mathcal{X} \cong \mathcal{Z} \) if there exists a map \( \psi \) from \( \mathcal{X} \) to \( \mathcal{Z} \) which is one to one, onto, and preserves the partial order (that is, \( x \leq y \) if and only if \( \psi(x) \leq \psi(y) \)). We call such a \( \psi \) an isomorphism from \( \mathcal{X} \) to \( \mathcal{Z} \). Any such map will have an inverse, and this inverse will also be an isomorphism. In fact, the relation of isomorphism is an equivalence relation on the class of all posets. The standard practice of regarding isomorphic objects as essentially the "same" object will be followed during this classification, so classification results will be of the form "up to isomorphism, the only complexes satisfying X are \( \mathcal{X}_1 \), \( \mathcal{X}_2 \), \( \mathcal{X}_3 \), \ldots", meaning that any complex satisfying \( \mathcal{X} \) is isomorphic to one of the \( \mathcal{X}_i \). If a map between two posets is structure preserving and one to one, but not necessarily onto, it is called a monomorphism.

Isomorphism is of course a 'standard' concept, used in many branches of mathematics. It would therefore be justifiable to skim over this section, only giving partial proofs, and so on. I have gone into detail, in the hope that the reader unfamiliar with incidence polytopes may gain a deeper understanding of them by working through the proofs.

The first "trivial" result that we shall prove in detail is that if \( \mathcal{X} \) and \( \mathcal{Z} \) are isomorphic, and \( \mathcal{X} \) is a polytope, then \( \mathcal{Z} \) is also a polytope. For the next few lemmas, let \( \mathcal{X} \) and \( \mathcal{Z} \) be isomorphic posets, and let \( \psi \) be an isomorphism from \( \mathcal{X} \) to \( \mathcal{Z} \).

Lemma 3.3.1.1: If \( \mathcal{X} \) satisfies 11 and 12, so does \( \mathcal{Z} \). In fact, \( \dim \mathcal{X} = \dim \mathcal{Z} \).

Proof: Let \( \mathcal{X} \) and \( \mathcal{Z} \) be the unique minimal and maximal elements of \( \mathcal{X} \) respectively, and let \( b \) be an arbitrary element of \( \mathcal{Z} \). Then, there exists a unique \( a \in \mathcal{X} \) such that \( \psi(a) = b \). By Lemma 3.1.1, \( \mathcal{X} \leq a \leq \mathcal{Z} \), and so \( (\mathcal{X})_{\psi} \leq b \leq (\mathcal{Z})_{\psi} \). Consider \( (\mathcal{X})_{\psi} \). Since there exists no element \( b \) of \( \mathcal{Z} \) with \( b < ((\mathcal{X})_{\psi}) \), it follows that \( (\mathcal{X})_{\psi} \) is minimal in \( \mathcal{Z} \). But no other element \( z \) of \( \mathcal{Z} \) can be minimal, since \( x \neq ((\mathcal{X})_{\psi}) \), then \( x < z \), so \( (\mathcal{X})_{\psi} < z \). Thus \( (\mathcal{X})_{\psi} \) is a unique minimal element of \( \mathcal{Z} \). Similarly, \( \mathcal{Z} \) is a unique maximal element of \( \mathcal{Z} \). Denote \( (\mathcal{X})_{\psi} \) by \( \mathcal{X}_{\psi} \) and \( (\mathcal{Z})_{\psi} \) by \( \mathcal{Z}_{\psi} \).

Now, let \( C \) be a chain of \( \mathcal{Z} \) and let \( C' = \{c_{\psi}^{-1} \subseteq C \} \). Now \( c_{\psi}^{-1} \leq c_{\psi} \) if and only if \( c \leq d \), so \( C' \) will be totally ordered by virtue of the fact that \( C \) is a chain. Then, \( C' \) will be contained in a flag \( F' \) of \( \mathcal{Z} \). Let \( F = \{f' \subseteq F' \} \). Certainly, \( C \subseteq F \), since if \( c \in C \), then \( c_{\psi}^{-1} \subseteq C' \) also, giving \( c_{\psi}^{-1} \subseteq F \). It remains to be shown that \( F \) is a flag of \( \mathcal{X} \). Note first that it will be a totally ordered subset. Secondly, if \( F \) is a chain of \( \mathcal{Z} \) with \( F \subseteq C \), then \( C' = G_{\psi}^{-1} \) is a chain of \( \mathcal{X} \) with \( F' \subseteq C' \), whence \( F' \subseteq C' \) (since \( F \) is a flag of \( \mathcal{Z} \)). Thus \( F = G_{\psi} \), so \( F \) is a maximal chain of \( \mathcal{Z} \). We have shown that any chain \( C \) of \( \mathcal{Z} \) is contained in a flag. Now let \( F \) be any flag of \( \mathcal{Z} \), and let \( \dim \mathcal{X} = d \). Then \( F' = \{f_{\psi}^{-1} \subseteq F \} \) will be a flag of \( \mathcal{X} \). But then \( F \) has \( d+2 \) elements, showing that \( |F| = d+2 \) also, and so the dimension of \( \mathcal{Z} \) is \( d \). This completes the proof.

Lemma 3.3.2.2: Let \( \mathcal{X} = (f, g) \) be a section of \( \mathcal{X} \). Then \( \{\psi: s \in \mathcal{X}\} \) is a section of \( \mathcal{Z} \), equal in fact to \( (f_{\psi}, g_{\psi}) \) and isomorphic to \( \mathcal{X} \).

Proof: Since \( f_{\psi} \leq s_{\psi} \) and \( g_{\psi} \leq s_{\psi} \) if and only if \( f \leq s \) and \( s \leq g \), the set \( \{\psi: s \in \mathcal{X}\} \) equals \( (f_{\psi}, g_{\psi}) \). Let \( \psi \) be restricted to \( \mathcal{Z} \). This map will be one to one and structure preserving, since \( \psi \) is, and certainly maps \( \mathcal{X} \) onto \( \mathcal{Z} \). Thus \( \mathcal{X} \) and \( \mathcal{Z} \) are isomorphic.

A corollary to this is that \( \dim \mathcal{X} = \dim \mathcal{Z} \) for any \( f \in \mathcal{X} \), and hence that for any section \( \mathcal{X} \) of \( \mathcal{X} \), if \( \mathcal{Z} = \mathcal{X} \), then

\[
\sum_{x \in \mathcal{X}} (-1)^{\mathcal{X}(x)+1} = \sum_{x \in \mathcal{X}} (-1)^{\mathcal{Z}(x)^{+}+1}.
\]

From this we can deduce the following.

Corollary 3.3.3.3: If \( \mathcal{X} \) satisfies 11 and 12 and is isomorphic to \( \mathcal{Z} \), then \( \mathcal{Z} \) is \( k \)-Euler if and only if \( \mathcal{X} \) is also \( k \)-Euler.

From now, let \( \mathcal{X} \) and \( \mathcal{Z} \) be isomorphic posets satisfying 11 and 12, with \( \psi \) defined as before. We continue to work through the incidence complex axioms.
Lemma 3.3.4: If \( \mathcal{K} \) satisfies 13, then so does \( \mathcal{L} \).

**Proof:** Let \( \mathcal{S} \) be a section of \( \mathcal{L} \), equal to \((t,T)\), say. Then \( \mathcal{S}' = (s \psi^{-1} : s \in \mathcal{S}') \) is a section of \( \mathcal{K} \) (Lemma 3.3.2). Let \( f, g \in \mathcal{S} \). Then \( f \psi^{-1}, g \psi^{-1} \in \mathcal{S}' \). But \( \mathcal{S}' \) is strongly connected, so \( \mathcal{S}' \) is weakly connected. Thus there exists a sequence of elements \((f \psi^{-1} \leq h'_1, \ldots, h'_n \leq g \psi^{-1}) \) of \( \mathcal{S}' \), none equal to either \( f \psi^{-1} \) or \( g \psi^{-1} \), such that for each \( i \), either \( h'_i \leq h'_{i+1} \) or \( h'_{i+1} \leq h'_i \). Letting \( h'_i \psi = h_i \), we deduce that \((f \leq h_1, h_2, \ldots, h_n \leq g) \) satisfies \( h_i \leq h'_{i+1} \) or \( h_{i+1} \leq h_i \) for each \( i \), and that none of the \( h_i \) are equal to either \( t \) or \( T \). Thus \( \mathcal{S} \) is weakly connected, and since \( \mathcal{S} \) was an arbitrary section of \( \mathcal{L} \), \( \mathcal{L} \) is strongly connected.

Finally, we have the following lemma.

**Lemma 3.3.5:** Let \( \mathcal{K} \) be such that for each \( i \) with \( 0 \leq i \leq d - 1 \), there exists some \( k_i \geq 2 \) such that for any two elements \( f' \) and \( g' \) of \( \mathcal{K} \) with \( f' < g' \) and \( \dim f' + 1 = i = \dim g' - 1 \), the set \( \{ h' : f' < h' < g' \} \) has \( k_i \) elements. Then, for each such \( i \), given any \( f, g \in \mathcal{L} \) with \( f < g \) and \( \dim f + 1 = i = \dim g - 1 \), the set \( \{ h : f < h < g \} \) has \( k_i \) elements also. That is, if \( \mathcal{K} \) satisfies 14, then so does \( \mathcal{L} \), with the same constants \( k_i \).

**Proof:** Let \( f, g \in \mathcal{L} \) be as stated. Then \( f \psi^{-1} < g \psi^{-1} \) and \( \dim \mathcal{K} f \psi^{-1} + 1 = i = \dim \mathcal{K} g \psi^{-1} - 1 \) (see the note following Lemma 3.3.2). Hence there exist exactly \( k_i \) elements \( h' \) of \( \mathcal{K} \) with \( f \psi^{-1} < h' < g \psi^{-1} \). But these elements will be in one to one correspondence with the elements \( h \) of \( \mathcal{L} \) satisfying \( f < h < g \). Thus \( \{ h : f < h < g \} \) has \( k_i \) elements, as required.

So, given any two posets \( \mathcal{K} \) and \( \mathcal{L} \), we have the following theorem.

**Theorem 3.3.6:** If \( \mathcal{K} \cong \mathcal{L} \), then \( \mathcal{K} \) is a \((k\text{-Euler})\) complex, if and only if \( \mathcal{L} \) is likewise a \((k\text{-Euler})\) complex.

**Proof:** This follows easily from the above results.

This is, of course, to be expected.

Now that we have developed the concept of isomorphism, we can prove our first classification results.

**Theorem 3.3.7:** Up to isomorphism, there is only one 0-complex \( \mathcal{K} \), and this has only two elements.

**Proof:** We encountered a 0-complex \( \mathcal{K} = (\emptyset, K) \) in Theorem 3.2.1. If \( \mathcal{L} \) is a 0-complex, let \( \psi : \mathcal{L} \to \mathcal{K} \) satisfy \( \emptyset_{\mathcal{L}} = \emptyset_{\mathcal{K}} \) and \( K_{\psi} = L \), where \( \emptyset_{\mathcal{L}} \) and \( L \) are (respectively) the minimal and maximal elements of \( \mathcal{L} \). It is easily shown that \( \psi \) is an isomorphism from \( \mathcal{K} \) to \( \mathcal{L} \), thus any 0-complex is isomorphic to \( \mathcal{K} \).

This result has an immediate corollary.

**Corollary 3.3.8:** Any incidence complex \( \mathcal{K} \) is 0-Euler.

**Proof:** The above theorem shows that any 0-section of \( \mathcal{K} \) is a 0-complex, and thus (by Theorem 3.2.1) is Euler, and so satisfies Euler's condition.

Now we move on to higher dimensions.

**Theorem 3.3.9:** Any 1-complex has at least 2 vertices. Also, for each (not necessarily finite) \( k \geq 2 \), there is exactly one 1-complex with \( k \) vertices.

**Proof:** Let \( \mathcal{L} \) be a 1-complex, and let \( \emptyset_{\mathcal{L}} \) and \( L \) be its minimal and maximal elements. Since \( \dim \emptyset_{\mathcal{L}} = 0 \), the set \( \{ z : \emptyset_{\mathcal{L}} < z < L \} \) has \( \ell \) elements, for some \( \ell \geq 2 \) (by property 14). All these elements will have dimension 0, for if \( \dim g = -1 \) then \( g = \emptyset_{\mathcal{L}} \), and if \( \dim g \geq 1 \) but \( g \neq L \), then there exist proper chains of \( \mathcal{L} \), specifically flags of \( (\emptyset_{\mathcal{L}}, g) \), with too many elements. Thus \( \mathcal{L} \) consists of \( \emptyset_{\mathcal{L}} \) and \( L \), and its \( \ell \) elements. Let \( \ell = k \), and call the vertices \( h'_{1}, h'_{2}, \ldots, h'_{\ell} \). Then it is easy to show that \( \psi \) defined by \( (\emptyset_{\mathcal{L}})_{\psi} = \emptyset_{\mathcal{K}}, (L)_{\psi} = K, \) and \( (h')_{\psi} = h_{i} \) is an isomorphism between \( \mathcal{L} \) and the complex \( \mathcal{K} \) described in Theorem 3.2.2.

Since a polytope must be 1-Euler, a 1-polytope must be Euler, and so satisfy Euler's condition. Thus \(-1 + k_0 - 1 = 0\), forcing \( k_0 \) to equal 2. Thus there is only one 1-polytope up to isomorphism, this having two vertices.

The next theorem shows us that all 2-polytopes are in fact n-cycles.\(^{28}\)

\(^{28}\) The n-cycles were defined in §3.2.
Theorem 3.3.10: Let $\mathcal{L}$ be a 2-incidence polytope. Then $\mathcal{L}$ is isomorphic to an $n$-cycle for some $n \geq 2$.

Proof: Note that $\mathcal{L} = \mathcal{L}_{-1} \cup \mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2$, where $\mathcal{L}_1$ is the set of all 2-dimensional elements of $\mathcal{L}$. Then, we construct a map $\phi_0$ from $\mathcal{L}$ to $\mathcal{L}_0$, and another $\phi_1$ from $\mathcal{L}$ to $\mathcal{L}_1$ as follows. Choose an arbitrary element $x \in \mathcal{L}_2$ and $y \in \mathcal{L}_1$ such that $x < y$. Let $\phi_0(y) = x$ and $\phi_0(x) = y$. Then, for each $\phi_1(k)$ and $\phi_0(k)$ (for $k \geq 0$), we construct $\phi_1(k+1)$ and $\phi_0(k+1)$ as follows: since $\mathcal{L}$ is a polytope, there are exactly two elements $T$ of $\mathcal{L}_2$ satisfying $\phi_1(k) < T < \mathcal{L}_2$ (where $\mathcal{L}_2$ is the maximal element of $\mathcal{L}$). One of these is $\phi_1(k)$. Let the other be $\phi_1(k+1)$. Then, there must be two elements $T_1, T_2$ of $\mathcal{L}_2$ such that $\phi_1(k) < T_1, T_2 < \phi_1(k+1)$. One of these is $\phi_1(k)$, let the other be $\phi_1(k+1)$. Thus $\phi_0(m)$ and $\phi_1(m)$ are defined for all $m \geq 0$. They may similarly be defined for $m < 0$ in a consistent way, hence $\phi_0(m)$ and $\phi_1(m)$ are defined for all $m \in \mathbb{Z}$. That these maps are onto may be shown from the fact that polytopes are connected. Note that $\phi_0(k) < \phi_1(k), \phi_1(k+1)$ for all $k$, that is, that $\phi_0(k), \phi_0(k+1) < \phi_1(k)$. Now if $\phi_0$ is one to one, then $\phi_1$ is also, for if $\phi_0(i) = \phi_0(j)$, then $\phi_0(i), \phi_0(i-1) = \phi_0(k)$, $\phi_0(k-1)$. If $k \neq i$, then $k = i - 1$ and $k - 1 = i$, which is a contradiction. In this case, it may be deduced from this that if either of $\phi_0$ and $\phi_1$ is one to one, then both must be. In this case, we construct a map $\psi$ from the cycle $(\{k, K\} \cup \{T, T_1, T_2 : \in \mathbb{Z}\}$ to $\mathcal{L}_2$ via $(\{k, K\} \psi) = T, T_1, T_2 \psi = \phi_1(k)$, and then $t_1 \psi = \phi_0(x)$ and $T_2 \psi = \phi_1(x)$. It is then not difficult to show that this map is an isomorphism between $\mathcal{L}$ and $\mathcal{L}_2$.

If $\phi_0$ is not one to one, however, let $l > k$ be such that $\phi_0(k) = \phi_0(l)$. Then either $\phi_1(k) = \phi_0(k), \phi_1(k+1) = \phi_1(l)$, and in fact, if $\phi_0(k) = \phi_0(l)$, then $\phi_0(k+1) = \phi_0(l-1)$, and in fact, if $\phi_0(k) = \phi_0(l)$, then $\phi_0(k+1) = \phi_0(l-1)$. By careful choice of $l$, we obtain a contradiction -- either $\phi_0(k) = \phi_0(k+1)$ or $\phi_0(k) = \phi_0(l-1)$. Thus instead $\phi_0(k) = \phi_0(l)$; from this it may be shown that $\phi_0(k+1) = \phi_0(l)$. Then, if $\phi_0(k) = \phi_0(l)$, then $\phi_0(k+1) = \phi_0(l+1)$ for all $l \in \mathbb{Z}$, whence there exists a positive integer $i$ such that $i \neq k$. Then $i \neq k$, and $i \neq k$. Hence $i$ is a multiple of $n$, and in fact $\phi_0(k) = \phi_0(l)$ if and only if $k$ and $l$ differ by a multiple of $n$ (and similar results for $\phi_1$). Now, let $\mathcal{L}_2$ be another 2-cycle $\{T, T_1, T_2 : \in \mathbb{Z}\}$, where $C = \{l : (l)\}$ is a cyclic group of order $n$. We construct the map $\psi$ from $\mathcal{L}_2$ to $\mathcal{L}_2$ via $(\{k, K\} \psi) = T, T_1, T_2 \psi = \phi_1(j)$, and if $x = j$, let $(x_1 \psi = \phi_0(j)$ and $(x_2 \psi = \phi_1(j))$. Using the facts about $\phi_0$ and $\phi_1$ that we have derived, it is not hard to show that $\psi$ is a well-defined map (so if $i \neq k$, then $\phi_0(k) = \phi_0(l)$ and $\phi_1(k) = \phi_1(l)$), and then $i$ is a multiple of $n$. Then $\mathcal{L}_2$ is isomorphic to $\mathcal{L}_2$, and so $\mathcal{L}_2$ is an $n$-cycle, as required. This completes the proof.

Corollary 3.3.11: Let $\mathcal{L}$ be a finite 2-color set. If $\mathcal{L}$ is 1-Euler, then it is $\mathcal{L}_2$.

Proof: If it is 1-Euler, it is a polytope, and hence an $n$-cycle. But if it is finite, it cannot be an $n$-cycle. Thus, by Theorem 3.3.13, it is 1-Euler.

Corollary 3.3.12: Let $\mathcal{L}_2$ be a 2-color set of all of whose proper sections are Eulerian. If $\mathcal{L}_2$ is 1-Euler, then it is 2-Euler.

Proof: In fact, for this corollary, we need only the 2-sections to be finite. For then, any 2-section will be 1-Euler, if by virtue of the fact that it is, and hence it satisfies the conditions of the previous corollary. Thus, all 2-sections of $\mathcal{L}_2$ are Eulerian, and so $\mathcal{L}_2$ is Eulerian.

3.3.2 Duals:

Just as we have duals for geometric polytopes, so we have them for incidence complexes. In fact, their formulation in the latter case is a lot simpler. We define the dual of a poset $\mathcal{L} = (\mathcal{L}, \leq)$ to be the poset $\mathcal{L}^* = (\mathcal{L}, \geq)$. More generally, we say that $\mathcal{L}^*$ is dual to $\mathcal{L}$ (and is a dual of $\mathcal{L}$) if there is a structure reversing bijection (called a dual map) from $\mathcal{L}$ to $\mathcal{L}^*$ (that is, there is a bijection $\psi$ which satisfies $f < \psi$ if and only if $\psi < g$). If $\mathcal{L}$ is dual to itself, we call it self-dual. The following lemmas are easy to prove.

Lemma 3.3.13: A dual of a dual of $\mathcal{L}$ is isomorphic to $\mathcal{L}$.

Proof: The composition of the dual maps will yield an isomorphism.

Lemma 3.3.14: The sections of a dual of $\mathcal{L}$ are dual to the sections of $\mathcal{L}$.

Proof: If $\psi$ is the map from $\mathcal{L}$ to a dual $\mathcal{L}^*$ of $\mathcal{L}$, then it is easily seen that $\psi$ restricted to $\{f, g\}$ is a structure reversing bijection from this section of $\mathcal{L}$ to the section $\{\psi f, \psi g\}^*$.

Lemma 3.3.15: If $\mathcal{L}$ satisfies $\mathcal{L}_1$ and $\mathcal{L}_2$, then so does any dual of $\mathcal{L}$. Also, $\dim \mathcal{L}_1 = \dim \mathcal{L}_2$.

Proof: If $\varnothing$ and $K$ are the minimal and maximal elements respectively of $\mathcal{L}$, then $\varnothing^*$ and $K^*$ are the minimal and maximal elements (respectively) of $\mathcal{L}^*$. Also, if $C$ is any chain of $\mathcal{L}$, then $C_\varnothing^*$ will be a totally ordered subset of $\mathcal{L}^*$. Secondly, if $C^*$ is not a maximal totally ordered subset of $\mathcal{L}$, then $C^*$ must be contained in some other chain $C_\varnothing^*$. Then $C_\varnothing^*$ will be a chain of $\mathcal{L}^*$, so if $C$ is a flag of $\mathcal{L}^*$, then $C_\varnothing^*$ is a flag of $\mathcal{L}^*$. We deduce that all flags of $\mathcal{L}^*$ have the same length as those of $\mathcal{L}$, and thus in particular that $\mathcal{L}$ and $\mathcal{L}^*$ have the
same dimension. We have still not proved that all chains of $\mathcal{X}^*$ are contained in flags. Let $C$ be any chain of $\mathcal{X}^*$. Then $C \psi^{-1}$ is a chain of $\mathcal{X}$, and so is contained in a flag $F$. But then $C = C \psi^{-1} \phi$ must be contained in the flag $F \phi$ of $\mathcal{X}^*$. This completes the proof.

Let $\dim \mathcal{X} = d$, and let $f \in \mathcal{X}$. Then we have $\dim \psi = \dim(\mathcal{X}^*, \mathcal{X} \psi) = \dim(\mathcal{X}, \mathcal{X} \psi) = \dim(f, \mathcal{X})$ (Lemma 3.3.15, applied to the poset $(f, \mathcal{X})$) which by Theorem 3.1.3 equals $d - (\dim f + 1) = d - \dim f - 1$. Thus we have the following.

Lemma 3.3.16: $\mathcal{X}$ satisfies Euler’s formula if and only if $\mathcal{X}^*$ does.

Proof: Note that

$$\sum_{f \psi \in \mathcal{X}^*} (-1)^{\dim f + 1} = \sum_{f \in \mathcal{X}} (-1)^{\dim f} - \sum_{f \psi \in \mathcal{X}^*} (-1)^{\dim f - 1}$$

which will equal zero if and only if $\mathcal{X}$ satisfies Euler’s formula.

Corollary 3.3.17: $\mathcal{X}$ is k-Euler if and only if $\mathcal{X}^*$ is k-Euler.

Proof: The $j$-sections of $\mathcal{X}^*$ will satisfy Euler’s formula if and only if their duals, the $j$-sections of $\mathcal{X}$, do so.

Theorem 3.3.18: If $\mathcal{X}$ is an incidence polytope, then so is $\mathcal{X}^*$.

Proof: Let $\psi$ be a structure-reversing bijection from $\mathcal{X}$ to $\mathcal{X}^*$. We have already seen that if $\mathcal{X}$ satisfies I1 and I2, then so will $\mathcal{X}^*$. Let $\mathcal{X}^*$ be any section of $\mathcal{X}^*$. Then $\mathcal{X} = \mathcal{X}^* \psi^{-1}$ will be a section of $\mathcal{X}$, and will be weakly connected since $\mathcal{X}$ is strongly connected. If $h_0, \ldots, h_m$ is a sequence demonstrating this, then $h_0 \psi, \ldots, h_m \psi$ will be a sequence demonstrating that $\mathcal{X}^*$ is also weakly connected. Hence $\mathcal{X}^*$ is strongly connected, as required for property I3. Finally, let $f, g \psi \in \mathcal{X}^*$ be such that $f \psi < g \psi$, and such that dim $f + 1 = i = \dim g - 1$. Then $f$ and $g$, elements of $\mathcal{X}$, will satisfy $f > g$ and $(d - \dim f - 1) + 1 = i = (d - \dim g - 1) - 1$, that is, $\dim g + 1 = d - i - 1 = \dim f - 1$. Thus the set $\{h : g < h < f\}$ has $k_{i-1}$ elements, and hence will be the set $\{h \psi : f \psi < h \psi < g \psi\}$, independently of $f \psi$ and $g \psi$ (since $\psi$ is a bijection). Therefore $\mathcal{X}^*$ satisfies I4 also, and is complex. If $\mathcal{X}$ is a polytope, that is, is 1-Euler, then by Corollary 3.3.17, $\mathcal{X}^*$ is also.

3.3.3 (i,j)-Connectivity:

In this subsection, we will see a result which is similar to weak connectivity. It is more of a technical result, useful later on, than one which is interesting in its own right.

Property 13 tells us that a complex must be strongly connected; that is, all its sections must be weakly connected. A poset $\mathcal{P}$ (satisfying I1 and I2) is weakly connected if its dimension is 0 or 1 or if for any $f, g \in \mathcal{P} \setminus \{\emptyset, \mathcal{P}\}$, there exists a sequence $h_1, \ldots, h_m$ of elements of $\mathcal{P} \setminus \{\emptyset, \mathcal{P}\}$ such that $f \neq h_1 \neq h_2 \neq \cdots \neq h_{m-1} > h_m \leq g$. Since any complex is a section of itself, any complex is weakly connected, as well as strongly. The main theorem of this subsection shows that the sequence of $h_i$ may be chosen in a rather special way. We prove a lemma that leads to the main result.

Lemma 3.3.19: If $x, y \in \mathcal{X}$ with dim $x = \dim y = i \neq -1$, and if $i < j < d$, then there exists a sequence $x = a_0, \ldots, a_m = y$ such that for all $k$, if $k$ is even, dim $a_1 \leq i$ and $a_k < a_{k-1}, a_{k+1}$, but if $k$ is odd, dim $a_1 \geq j$ and $a_k < a_{k-1}, a_{k+1}$.

Proof: Let $x \leq h_1 > h_2 > \cdots > h_{m-1} > h_m \leq y$ be a sequence such as given by the fact that $\mathcal{X}$ is weakly connected (see Figure 3.3.1).
Now, given this sequence, we can ensure that for all odd \( k \), we have \( \dim h_k \geq j \). For if any such \( h_k \) satisfies \( \dim h_k < j \), choose \( h' \in (h_1, K) \) to satisfy \( \dim_{\mathcal{X}} h'_k = j \). Then, since \( h_{k-1}, h_{k+1} < h_k < h'_k \), we can simply replace \( h_k \) with \( h'_k \) to obtain a sequence of the required form (see Figure 3.3.3). We similarly ensure that \( \dim_{\mathcal{X}} h_k \leq i \) for all even \( k \), as required.

Now, since \( \dim x = \dim y = i \), we may use the previous lemma to find a sequence \( x = h_0 < g_0 > h_1 < \ldots < g_{n-1} > h_n = y \) where for all \( k, \dim g_k \geq j \) and \( \dim h_k \leq i \). I claim that this sequence can be transformed to one of the required form. This may be shown by means of another induction, this time on \( t \), the number of \( g_k \) and \( h_k \) such that \( \dim g_k > j \) or \( \dim h_k < i \).

If \( t = 0 \) there is nothing to prove, because the sequence is already in the form required to complete the proof. Now assume that \( t > 1 \), and that any sequence with \( t \leq s - 1 \) may be transformed as required. We have two possibilities. Either there exists some \( k \) with \( \dim g_k > j \), or there doesn’t.

Assume first that there does exist such a \( k \), and consider the sequence \( \mathcal{S} = (0, g_k) \). Now \( \dim \mathcal{S} > j \), so, since \( j > i > -1 \), we have \( d - 1 \geq \dim \mathcal{S} \geq 2 \). Note also that \( h_k, h_{k+1} \in \mathcal{S} \). Now, let us choose \( x' \in \mathcal{S} \) such that \( \dim x' = i \), and \( h_k \leq x' \), and also choose \( y' \in \mathcal{S} \) such that \( \dim y' = i \) and \( h_{k+1} \leq y' \). Then we can apply \((i, j)\)-connectivity to the complex \( \mathcal{S} \) (recall that we are assuming, by the first inductive hypothesis, that the theorem holds for complexes whose dimension is less than \( d \)). Thus there exists a sequence \( x'' = c_0 < d_0 < c_1 < \ldots < d_{m-1} < e_m = y' \) where \( \dim_{\mathcal{S}} c_i = i \) and \( \dim_{\mathcal{S}} d_i = j \) for each \( i \). Now \( \dim_{\mathcal{S}} a = \dim_{\mathcal{X}} a \) for all \( a \in \mathcal{S} \). Thus, if we insert \( c_0 < d_0 < c_1 < \ldots < d_{m-1} < e_m \) in place of \( g_k \), and re-label all the elements of the sequence, we obtain a new sequence

\[
x = h_0 < g'_0 > h'_1 < \ldots < g'_{n-1} > h'_n = y
\]

where for all \( k, \dim g'_k \geq j \) and \( \dim h'_k \leq i \), but with the number of \( g'_k \) and \( h'_k \) with \( \dim g'_k > j \) or \( \dim h'_k < i \) being only \( s - 1 \). But according to our (second) inductive hypothesis, this new sequence may be transformed into one of the required form. We have almost completed the proof, except that we have yet to consider the possibility that there is (in the sequence with \( t = s \)) no \( g_k \) with \( \dim g_k > j \). In this case, there must exist some \( h_k \) with \( \dim h_k < i \). Then, a similar argument (applying \((i - \dim h_{k-1} - 1, j - \dim h_{k-1} - 1)\)-connectivity to \((h_k, K)\)) yields the required reduction in \( t \) for the induction to proceed. This completes the proof.

### 3.3.4 Subpolytopes

Often, when studying some algebraic object, one is interested also in the ‘subobjects’ - subsets of the object which are also ‘objects’ in their own right - examples being subgroups of groups, subspaces of vector spaces, submodules of modules, and so forth. It is possible to likewise define subpolytopes, which is exactly what we do here. The topic will not be explored in great depth here, the main result of this subsection being a useful result about the non-existence of certain subpolytopes.
Definition: Let $\mathcal{X}$ be a $d$-incidence complex. A subset $\mathcal{L}$ of $\mathcal{X}$ is called a subcomplex of $\mathcal{X}$ if $\mathcal{L}$ remains a complex when the partial order is restricted to it. If $\mathcal{L}$ is a polytope, we call it a subpolytope of $\mathcal{X}$. If $\mathcal{L}$ has dimension $k$, we call it a $k$-subcomplex (or $k$-subpolytope) of $\mathcal{X}$.

Let $\mathcal{X}$ be any complex. Then certainly $\mathcal{X}$ is a subcomplex of itself. In fact, all the sections of $\mathcal{X}$ are subcomplexes of $\mathcal{X}$. Usually (but not always), $\mathcal{X}$ also has multitudes of $k$-subcomplexes not to any of its $k$-sections and its future of $\mathcal{X}$ (as it happens). From now on, let $\mathcal{X}$ be a $d$-incidence polytope, and let $\mathcal{L}$ be a subcomplex of $\mathcal{X}$. To avoid ambiguity, we denote the section $(x,y)$ of $\mathcal{X}$ by $(x,y)_{\mathcal{X}}$. It is easy to show that for $x,y \in \mathcal{L}$, we have $(x,y)_{\mathcal{X}} = (x,y)_{\mathcal{X}} \cap \mathcal{L}$.

Lemma 3.3.21: Let $\mathcal{L}$ be a $d$-subcomplex of a $d$-complex $\mathcal{X}$. Then the minimal and maximal elements of $\mathcal{L}$ are those of $\mathcal{X}$.

Proof: Let $F$ be a flag of $\mathcal{L}$. Certainly $0_{\mathcal{X}}, L \subseteq \mathcal{L}$. However, $F$ will also be a shell of $\mathcal{X}$, and will be $0_{\mathcal{X}} \cup \{0_{\mathcal{X}}\}$. This latter will be contained in a flag of $\mathcal{X}$ in the same way, it must be that $\mathcal{X}$ and $\mathcal{L}$ have the same dimension $d$, so we must be that $\mathcal{L}$. Therefore, since $\mathcal{X}$ and $\mathcal{L}$ have the same dimension $d$, we must be that $\mathcal{L}$. Since $\mathcal{L}$ and $\mathcal{X}$ have the same dimension $d$, we must be that $\mathcal{L}$. Applying the same lemma to $\mathcal{X}$ yields $\mathcal{L} \subseteq \mathcal{X}$, and so in particular, $\mathcal{X} \subseteq \mathcal{L}$. Hence $\mathcal{L} \subseteq \mathcal{X}$. A similar argument shows that $0_{\mathcal{X}} = 0_{\mathcal{L}}$.

From now on, denote the (common) minimal element of $\mathcal{X}$ and $\mathcal{L}$ by $0_{\mathcal{X}}$ and $0_{\mathcal{L}}$, and the maximal element by $K$. Before we proceed with the next few lemmas, let us note that for any subcomplex $\mathcal{L}$ of $\mathcal{X}$, if $x,y \in \mathcal{L}$, then $\dim((x,y)_{\mathcal{L}})$.

Lemma 3.3.22: For all $x \in \mathcal{L}$, we have $\dim_{\mathcal{L}} x = \dim_{\mathcal{X}} x$.

Proof: Consider the cofaces at $x$, of both $\mathcal{X}$ and $\mathcal{L}$. We know that $\dim((x,K)_{\mathcal{L}}) \leq \dim((x,K)_{\mathcal{L}})$.

Theorem 3.3.24: Let $\mathcal{X}$ be a $d$-incidence polytope and let $\mathcal{L}$ be a $d$-subcomplex of $\mathcal{X}$. Then $\mathcal{L} \cong \mathcal{X}$.

Proof: We prove this by induction on $d$. If $d = 0$, then Theorem 3.3.3 tells us that $\mathcal{X}$ and $\mathcal{L}$ are isomorphic and finite. Thus $|\mathcal{X}| = |\mathcal{L}|$, and so $\mathcal{X} = \mathcal{L}$.

By induction, let $\mathcal{L} \subseteq \mathcal{X}$ be a $d$-incidence polytope, and let $\mathcal{L}$ be a subcomplex of $\mathcal{X}$. To avoid ambiguity, we denote the section $(x,y)$ of $\mathcal{X}$ by $(x,y)_{\mathcal{X}}$. It is easy to show that for $x,y \in \mathcal{L}$, we have $(x,y)_{\mathcal{X}} = (x,y)_{\mathcal{X}} \cap \mathcal{L}$. The formula $\dim((x,K)_{\mathcal{L}}) = \dim((x,K)_{\mathcal{L}}) - \dim((x,y)_{\mathcal{L}}) - 1 = d - 1 - \dim((x,y)_{\mathcal{L}})$ holds. Thus it must be that $\dim_{\mathcal{L}} x \geq \dim_{\mathcal{X}} x$. By virtue of the fact that $\dim((0,x)_{\mathcal{L}}) \leq \dim((0,x)_{\mathcal{L}})$, we deduce that $\dim_{\mathcal{L}} x \leq \dim_{\mathcal{X}} x$, and we have equality.

Lemma 3.3.23: For all $x \in \mathcal{L}$, we have $\dim((x,y)_{\mathcal{L}}) = \dim((x,y)_{\mathcal{L}})$.

Proof: For $\mathcal{L}$ equal to either $\mathcal{X}$ or $\mathcal{L}$, we have $\dim((x,y)_{\mathcal{L}}) = \dim((x,y)_{\mathcal{L}})$.

Note that this result has a corollary.

Corollary 3.3.25: Let $\psi$ be a monomorphism from a $d$-incidence complex $\mathcal{L}$ to a $d$-incidence polytope $\mathcal{X}$. Then $\psi$ is an isomorphism also.

Proof: The image of $\psi$ is a $d$-subcomplex (isomorphic to $\mathcal{L}$) of $\mathcal{X}$. Hence this image is equal to $\mathcal{X}$, and so $\psi$ is onto, whence $\mathcal{X} \cong \mathcal{L}$.
3.3.5 Decomposable Complexes:

Decomposability will prove to be a powerful tool in the attempt to classify the (combinatorially regular) Euler polytopes, as it greatly reduces the number of different cases which need to be considered. In order to gain a solid understanding of this concept, we need first to define the composition of two complexes.

Let \( X \) and \( L \) be \( k \)- and \( l \)-incidence complexes respectively, and let \( X' \) and \( L' \) be disjoint posets such that \( X \cong X' \oplus \Pi \) and \( L \cong L' \oplus \Pi \). Then let \( M = X' \ crowds L' \ominus \{K', \emptyset_x \} \) be given the partial order

\[
(x,y) \in M \quad \text{if and only if} \quad \begin{cases} x \in X', \quad x \leq X \ y, \quad \text{and} \quad x \leq X' \ y, \quad \text{or} \\ x \in X', \quad x \leq X' \ y. \end{cases}
\]

We call \( M \) a composition of \( X \) and \( L \), and denote it by \( M = X \oplus L \). In a sense, we have identified \( K \) with the whole of \( L \), and \( \emptyset \) with the whole of \( X \).

Note that if \( i = 0 \), then \( X \oplus L \cong X \), and if \( k = 0 \), then \( X \oplus L \cong L \).

Lemma 3.3.26: If \( X \) and \( L \) are posets satisfying I1 and I2 with \( \dim X = k \) and \( \dim L = l \), then \( X \oplus L \) also satisfies I1 and I2, and \( \dim (X \oplus L) = k + l \).

Proof: \( X = X \oplus L \) has \( L' \) as its unique maximal element, and \( \emptyset_x \) as its unique minimal element, and so satisfies I1. Now, let \( C \) be any chain of \( X \), and write it in the form \( C_{X'} \cup C_{X'} \), where \( C_{X'} = C \cap X \) and \( C_{X'} = C \cap L \). Then \( C_{X'} \) is contained in a flag \( F_{X'} \) of \( X' \), and \( C_{X'} \) is a flag \( F_{X'} \) of \( L' \). Now \( F = F_{X'} \cup F_{X'} \cup \emptyset_x \) will certainly be a totally ordered subset of \( X \), and in addition \( C \subseteq F \). We show that \( C \) is a maximal totally ordered subset by letting \( G \) be a chain of \( X \) with \( C \subseteq G \), and letting \( x \in G \). Then \( F \cup \{x\} \) is totally ordered. If \( x \in X \), then \( F_{X'} \cup \{x\} \) must also be totally ordered, and so \( x \in F_{X'} \). But then either \( x = X' \) (which is impossible since \( x \in X \)), or \( x \in F_{X'} \ominus \{K'\} \), which would imply \( x \in F \). Similarly, if \( x \in L \) we deduce that \( x \in F \). Thus \( F \) is a maximal totally ordered subset of \( X \) containing \( C \), and thus \( C \) is contained in a flag of \( X \). Now consider the case where \( C \) is a flag of \( X \). Then we may write \( C \) in the form \( C = C_{X'} \cup C_{X'} \), where \( C_{X'} \ominus \{K'\} \) is a flag of \( X' \) and \( C_{X'} \cup \emptyset_x \) is a flag of \( X' \). It follows then that \([C] = [C_{X'}] + [C_{X'}] = (k + 2 - 1) + (l + 2 - 1) = (k + l) + 2 \). Thus the flags of \( M \) have length \( k + l + 1 \), giving \( \dim M = k + l \).

Having shown this lemma, we note a few properties of the composition which will help show that \( X \oplus L \) is a complex when \( X \) and \( L \) are. For example, if \( x, y \in M \), with \( x < y \), then

\[
(x,y) \in M \quad \text{if and only if} \quad \begin{cases} x, y \in X' \quad \text{or} \\ x, y \in L' \quad \text{or} \\ (x, y) \in \emptyset_x. \end{cases}
\]

Also, we can find some useful results about the dimensions of various elements of \( M \).

Lemma 3.3.27: If \( x \in X \ominus \{K'\} \), then \( x, y \in X' \). Also, if \( y \in X' \ominus \{K'\} \), then \( x, y \in X' \). Also, if \( y \in L' \ominus \{K'\} \), then \( x, y \in L' \).

Proof: Since \( \emptyset_x \cup \emptyset_x \in X' \ominus \{K'\} \), it must be that they have the same dimension, so \( \dim X' = \dim X' \cup \emptyset_x \) as required. Now if \( y \) is as given in the statement of the lemma, then \( \dim (X' \cup \emptyset_x, y) = \dim (X' \cup \emptyset_x, K') + \dim (\emptyset_x, y) = k + \dim (\emptyset_x, y) \). Thus \( \dim X' = k + \dim X' \).}

Corollary 3.3.28: If \( x, y \in X \) satisfy \( x \leq y \), where \( X \) is either \( X' \) or \( L' \), then \( x, y \in X \). Also, \( x, y \in X \).}

Proof: This follows from the above lemma, using the relation \( \dim (x, y) \leq \dim y - \dim x - 1 \) (Theorem 3.1.3).

Note that if \( x \in X' \ominus \{K'\} \), then \( \dim (x, y) = \dim x + 1 \).

Theorem 3.3.29: If \( X \) and \( L \) are complexes, so is \( X \oplus L \).

Proof: We have already seen that \( X \oplus L \) will satisfy I1 and I2. To show I3, let \( (x, y) \) be a section of \( M \). If \( x, y \in X' \ominus \{K'\} \), then \( (x, y) \) is weakly connected, by virtue of the fact that \( (x, y) \) is. Consider the case when \( x \in X' \ominus \{K'\} \) and \( y \in L' \ominus \{K'\} \), so \( (x, y) \) is weakly connected. If \( (x, y) \) is weakly connected by virtue of the fact that \( (x, y) \) is weakly connected. If \( (x, y) \) is weakly connected by virtue of the fact that \( (x, y) \) is weakly connected.

Note that if \( x \in X' \ominus \{K'\} \), then \( \dim (x, y) = \dim (x, y) - 1 \).

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Lemma 3.3.30: If \( X \) and \( \mathcal{L} \) satisfy Euler's condition, then so does \( X' = X \circ \mathcal{L} \).

Proof: Let \( x \in X \). If \( x \in X' \), then \((-1)^{\dim X' + 1} = \sum_{x \in X'} (-1)^{\dim x'} x' \). But if \( x \notin X' \), then \((-1)^{\dim X' + 1} = \sum_{x \in X'} (-1)^{\dim x'} x' \). Thus

\[
\sum_{x \in X} (-1)^{\dim x} x \equiv \sum_{x \in X'} (-1)^{\dim x'} x' = \sum_{x \in X'} (-1)^{\dim x'} x'.
\]

as required.

Theorem 3.3.31: If \( X \) and \( \mathcal{L} \) are j-Euler, then so is \( X' = X \circ \mathcal{L} \).

Proof: Any i-section \( \mathcal{S} \) of \( X \) (with \( i \leq j \)) will either be an i-section of \( X' \), and i-section of \( \mathcal{L} \), or a composition of an i-section of \( X' \) with an i-section of \( \mathcal{L} \), where \( i' + i'' = i \). In the former two cases, the section will satisfy Euler's condition by the virtue of the fact that \( i \leq j \) and so i-sections of \( X' \) and \( \mathcal{L} \) satisfy Euler's condition. In the latter case, note that \( i' + i'' \leq i \leq j \), and so \( \mathcal{S} \) is a composition of two complexes which satisfy Euler's condition. Then the previous lemma tells us that in this case also, \( \mathcal{S} \) satisfies Euler's condition. Therefore, all i-sections of \( X' \) with \( i \leq j \) satisfy Euler's condition.

In particular, if \( X \) and \( \mathcal{L} \) were polytopes (that is, 1-Euler), then \( X \circ \mathcal{L} \) would also be a polytope.

Theorem 3.3.32: If \( X \) and \( \mathcal{L} \) are Euler, so is \( X' = X \circ \mathcal{L} \).

Proof: The proof of this theorem is very similar to that of the last one.

This latter theorem is not simply a special case of the earlier one. For example, let \( X \) be a 2-complex and let \( \mathcal{L} \) be 3-complex, and let both be Euler. This means that \( X \) is 2-Euler, and \( \mathcal{L} \) is 3-Euler (and hence 2-Euler also). So Theorem 3.3.31 tells us that \( X \circ \mathcal{L} \) is 2-Euler. However, Theorem 3.3.32 tells us that it is in fact 5-Euler, being an Euler 5-complex.

Having defined the composition \( X \circ \mathcal{L} \) of two complexes, and discovered that \( X \circ \mathcal{L} \) is a complex also, we make the following definitions. A complex \( X \) is decomposable if it is isomorphic to \( X \circ \mathcal{L} \) for some \( X \) and \( \mathcal{L} \) of dimension greater than zero. In this case, \( X \) and \( \mathcal{L} \) are called composition factors of \( X \). If \( X \) is not decomposable, it is said to be indecomposable.

We note (without proof) a few simple results.
and $d < \nu y$, for then we have $x < \nu y$, $c < \nu y$, and $d \leq \nu y$, as required. Therefore, if $z < \nu y$, then $x < \nu y$, and show that $z < \nu y$ also. Again, we have a number of cases. Either (i) $x, y \in \mathcal{L}'$, and so $x < \nu y$, (ii) $x, y \in \mathcal{X}'$, and so $x < \nu y$, or (iii) $x \in \mathcal{X}$ and $y \in \mathcal{L}'$. In each of these three cases, $x < \nu y$ as required. The fourth case (iv) $x \in \mathcal{Y}$ and $y \in \mathcal{X}$ cannot occur, for this would require $x > a$ and $y < b$, giving $\dim x > k - 1$ and $\dim y < k$, and so $\dim x > \dim y$, which would contradict the supposition that $x < \nu y$. Thus the relations $< \nu$ and $< \nu$ are identical, as required.

Now we can prove the following lemma and theorem.

**Lemma 3.3.37:** If $\mathcal{X}'$ and $\mathcal{L}'$ are $d'$-complexes, and $\mathcal{X}''$ and $\mathcal{L}''$ are $d''$-complexes, with $\mathcal{X} = \mathcal{X}' \circ \mathcal{X}''$ isomorphic to $\mathcal{L} = \mathcal{L}' \circ \mathcal{L}''$, then $\mathcal{X} = \mathcal{X}' \circ \mathcal{X}''$ and $\mathcal{L} = \mathcal{L}' \circ \mathcal{L}''$.

**Proof:** Let $\phi$ be the isomorphism from $\mathcal{X}$ to $\mathcal{L}$. Now for any $y \in \mathcal{X}$ with dimension $d'$, we have $\mathcal{X}'$ isomorphic to $(\mathcal{X}', y)$. But the latter is isomorphic (by Theorem 3.3.2) to $(\mathcal{X}, \phi, y)$, which is equal to $(\mathcal{X}, \phi, y)$, where $\phi$ is an element of $\mathcal{L}$ with dimension the same as that of $y$. Thus $\dim x \phi = d'$. But this means that $(\mathcal{X}, \phi, y) \cong \mathcal{L}$, so $\mathcal{X} \cong \mathcal{L}$ as required. A similar argument shows that $\mathcal{X}'' \cong \mathcal{L}''$.

The next theorem is similar in form to the Jordan-Hölder theorem for groups. For this reason, we shall call it by the same name.

**Theorem 3.3.38:** (Jordan-Hölder) If $\mathcal{X}', \mathcal{X}''$, ..., $\mathcal{X}^{(n)}$ and $\mathcal{L}', \mathcal{L}''$, ..., $\mathcal{L}^{(n)}$ are indecomposable complexes with nonzero dimension, such that the complex $\mathcal{X} = \mathcal{X}' \circ ... \circ \mathcal{X}^{(n)}$ is isomorphic to the complex $\mathcal{L} = \mathcal{L}' \circ ... \circ \mathcal{L}^{(n)}$, then $m = n$ and for all $i$ we have $\mathcal{X}^{(i)} \cong \mathcal{L}^{(i)}$.

**Proof:** Assume without loss that $m \leq n$. The proof will be by induction on $m$. If $m = n = 1$ then there is nothing to prove. However, if $m = 1$ and $n > 1$, then this would contradict the supposition that $\mathcal{X}'$ is indecomposable.

Assume now that $m = p$ and that the theorem holds whenever $m < p$. Write $\mathcal{X}$ as $\mathcal{X} \circ \mathcal{X}^{(n)} \circ ... \circ \mathcal{X}^{(n)}$, and $\mathcal{L}$ as $\mathcal{L} \circ \mathcal{L}^{(n)} \circ ... \mathcal{L}^{(n)}$. Let $\mathcal{X}'$ be a $k$-complex, and $\mathcal{L}'$ be an l-complex. I claim that $k = l$. For if $k \neq l$, consider first the case $k < l$. Then let $a, b \in \mathcal{L}'$, with $\dim a = k - 1$ and $\dim b = k$. If $\phi$ is an isomorphism from $\mathcal{X}$ to $\mathcal{L}$, then we have $\dim x \phi^{-1} = \dim x a = \dim x b = k - 1$. Similarly, $\dim x b^{-1} = k$. Thus $a \phi^{-1} \in \mathcal{X}'$, but $b \phi^{-1} \in \mathcal{X}'$. It follows that $a \phi^{-1} < b \phi^{-1}$, so $a < b$. But since $a$ and $b$ were arbitrary elements of $\mathcal{L}'$ with these dimensions, we can apply Theorem 3.3.36 to conclude that $\mathcal{L}'$ is decomposable, which contradicts our supposition. If on the other hand $k > l$, a similar argument shows that $\mathcal{X}'$ is decomposable, again a contradiction.

Thus $k = l$. But then, the previous lemma implies that $\mathcal{X}' \cong \mathcal{L}'$, and $\mathcal{X}^{(n)} \circ ... \circ \mathcal{X}^{(n)} \cong \mathcal{L}^{(n)} \circ ... \circ \mathcal{L}^{(n)}$. Then, the inductive hypothesis may be applied to the latter, yielding $n - 1 = p - 1 = m - 1$, and also $\mathcal{X} \cong \mathcal{L} \circ ... \circ \mathcal{L}$, as required.

So any complex has a unique decomposition into nontrivial indecomposable composition factors. More importantly, this theorem tells us that there is a one to one correspondence between (isomorphism classes of) incidence complexes, and finite sequences of (isomorphism classes of) indecomposable incidence complexes of dimension greater than zero. Similar statements could be made about incidence polytopes, j-Euler complexes, or Euler complexes. or even (as we shall see) combinatorially regular incidence complexes. Thus, to classify any of these objects, it suffices to classify the indecomposable specimens.

### 3.4 Regular Incidence Polytopes

In this section, we shall contrast definitions of two different types of regularity for incidence polytopes. Firstly, there will be that used by Schulte and later authors, and secondly, there will be a weaker definition more in line with those used in other areas of combinatorics.

#### 3.4.1 Automorphism Groups:

Recall that an isomorphism from $\mathcal{X}$ to $\mathcal{L}$ is a map which is one to one, onto, and structure-preserving (see §3.3.1). An automorphism of a complex $\mathcal{X}$ is an isomorphism from $\mathcal{X}$ to $\mathcal{X}$. Note that certainly there exist automorphisms – the identity map is an example. Also, any two may be combined via composition of maps to form another automorphism. In fact, the set of all automorphisms of a complex $\mathcal{X}$ forms a group under composition, called the automorphism group of $\mathcal{X}$, and denoted $\text{Aut}(\mathcal{X})$.

The group $\text{Aut}(\mathcal{X})$ therefore acts on the set $\mathcal{X}$ via the group action $g \ast a = a$ for any $g \in \mathcal{X}$ and $a \in \text{Aut}(\mathcal{X})$. In fact, it will be more useful to regard $\text{Aut}(\mathcal{X})$ as acting on the set of flags $\mathcal{F}(\mathcal{X})$ of $\mathcal{X}$, via the action $\mathcal{G} \ast \{f : f \in \mathcal{F}\}$, for all $\mathcal{F} \in \mathcal{F}(\mathcal{X})$. Now the automorphisms of $\mathcal{X}$, when regarded as acting on the set of flags of $\mathcal{X}$, have an important property, which we define below. Given a flag $F$ of $\mathcal{X}$, let $(F)_i$ be the element of $F$ with dimension $i$. $\text{Sym}(\mathcal{F}(\mathcal{X}))$ denotes the set of all permutations of $\mathcal{F}(\mathcal{X})$.

**Definition:** An element $a$ of $\text{Sym}(\mathcal{F}(\mathcal{X}))$ has consistent projections if for any two flags $F$ and $G$ of $\mathcal{X}$ with $(F)_i = (G)_i$ for some $i$, we also have $(F^{(i)})_i = (G^{(i)})_i$. An automorphism, when regarded as an element of $\text{Sym}(\mathcal{F}(\mathcal{X}))$ will have consistent projections, for if $F$ and $G$ are two flags with $(F)_i = (G)_i$, then...
(F^n)_i} is equal to (F)_i, α by definition, and so is in turn equal to (G)_i, which equals (G^n)_i.

In general, there exist permutations of Sym(F(X)) with consistent projections which are not automorphisms. However, these permutations can only exist if X satisfies certain properties which lift out of the class of complexes being examined. In particular, we have the following two lemmas.

Let X be an incidence complex. For any i, let k_i = \{h \in \{f < h < g\}\}, where f and g are elements of X with \text{dim} f + 1 = i = \text{dim} g - 1.

Lemma 3.4.1: Let α ∈ Sym(F(X)) have consistent projections. If all the k_i are finite, then α^{-1} also has consistent projections.

Proof: Let X and α be as given, and let F and G be flags of X such that (F)_i = (G)_i for some i. Now - applying the fact that a complex is flag connected, and that ((F)_i, K) is a complex, we can find a sequence of flags F \cap (F_i)_j, K = H_j \cup \ldots \cup H_m = G \cap (F_i)_j, K \cap (F_i)_j, K such that for each j, H_j and H'_j differ by exactly one element. Similarly, we choose a sequence of flags (F_i)_j \cap F = H''_m \cup \ldots \cup H''_j = G \cap (F_i)_j \cap (F_i)_j, such that for each j, H''_j and H'''_j differ by exactly one element. Note that for any j' and j'', it is the case that H''_j \cap H''_j is a flag of X, and in fact (H''_j \cap H''_j)_j = (F)_j. In fact, we can form a sequence of such flags via
\[ H_j = \begin{cases} \overset{\bigcup H''_m}{H''_1 \cup H''_m} & \text{for } j = 1 \ldots m, \text{ or} \\ \overset{\bigcup H''_n}{H''_m \cup H''_n} & \text{for } j = m \ldots n. \end{cases} \]

Note then that (H_j)_i = (H''_j)_i, (H'_j)_i = (H'''_j)_i, and (F_j)_i = (F''_j)_i, for each j. Now fix j, and let A be the set \{A \in F(X) : (A)_i = (H''_j)_i, for all i' ≠ j\}. That is, A is the set of flags of X which differ from H''_j by at most an element of dimension i'. Similarly, let B be the set \{B \in F(X) : (B)_i = (H''_j)_i, for all i' ≠ j\}. Note that |A| = |B| = k_j, by property 14. Since α has consistent projections, if (A)_i = (H''_j)_i, then (A''_i)_i = (H''_j)_i. Therefore, if A \in A, then A'' \in B, so A'' \subseteq B. But α is a permutation of the set of flags of X, and so A'' has the same size as A, which in turn has the same size (k_j) as B, since k_j is finite, we may deduce that A'' = B. However, H'_{j+1} \subseteq B. It follows that H''_j \subseteq A, and so for any j, the flags H''_j and H'''_j differ by at most one element, whose dimension is i_j, which is not equal to i. That is, (H''_j)_j = (H'''_j)_j for each j, and so also (H''_j)_j = (H'''_j)_j. Therefore, since (F_j) \cap (F'_j) \cup (F_j) \cap (F''_j) = F and (similarly) H''_j = G, we deduce that (F''_j)_i = (G''_j)_i, and so α^{-1} has consistent projections, as required.

Section 3.4: Regular Incidence Polytopes

We next use this fact to prove that for such X, the automorphisms are characterised by those permutations of F(X) with consistent projections. This fact will be extremely useful when we come to considering how this and other groups can act on our incidence polytopes.

Lemma 3.4.2: Let X be an incidence complex where all the k_i are finite, and let α be any permutation of F(X) with consistent projections. Then α induces an automorphism of X.

Proof: Given such an α, we define α', a permutation of X, as follows. Let x ∈ X, with \text{dim} x = i, and choose F ∈ F(X) such that (F)_i = x. Then, define xα' = (F')_i. This will not depend on the choice of F, because α has consistent projections. Now α' will have consistent projections, so also induces a map from X to X. This map (call it (α')') will serve as an inverse for α', and hence α' is one to one and onto. Finally, α' preserves the structure of our complex. Let y, z ∈ X with z < y. Then, if \text{dim} x = i and \text{dim} y = j, there exists a flag F with \{x, y\} ⊆ F. But then, xα'yα' = (xα')(yα') = yα', so xα' < yα' if and only if x < y. This completes the proof that α' is an automorphism.

Note that this proof required the k_i to all be finite. If in fact, some are infinite, then the result does not hold, and it is possible to give examples of complexes where not all permutations with consistent projections induce automorphisms.

Example: Let X = \{0, K\} \cup \{a_n, b_n : n \in \mathbb{Z}\}, with the partial order given by 0 ≤ x ≤ K for all x ∈ X, and an < bn if and only if either m > 0 or m + n is even. This will be a 2-incidence complex with k_0 = k_∞ = ∞. The set of flags is F(X) = \{\{0, a_n, b_n, K\} : m > 0 or m + n is even\}. We define the permutation α on F(X) via
\[ \{0, a_m, b_n, K\}^α = \begin{cases} \{0, a_{m+1}, b_n, K\} & \text{if } m > 0, \\ \{0, a_m, b_{n+2}, K\} & \text{if } m < 0. \end{cases} \]

It is not hard to check that α is a well-defined permutation and that α has consistent projections. However, α^{-1} does not have consistent projections. For example, let F = \{0, a_1, b_1, K\} and G = \{0, a_1, b_2, K\}, so (F)_1 = (G)_1. Then F''_1 = \{0, a_0, b_2, K\} and G''_1 = \{0, a_1, b_1, K\}, so that (F''_1)_1 = a_0, as distinct from (G''_1)_1 which equals a_1. Thus α cannot induce an automorphism of X, since α^{-1} does not.
3.4.2 Regular Incidence Complexes:

A d-incidence complex is said to be regular if its automorphism group acts transitively on its set of flags. This is the definition of regularity used by most authors working on incidence complexes. However, more in line with the definitions of regularity used in other combinatorial contexts (for example, in graph theory), we say a complex is combinatorially regular if for any two corresponding sections \((x, y)\) and \((x', y')\) (that is, sections such that \(d x = d x'\) and \(d y = d y')\), we have \((x, y) \cong (x', y')\). Note that if \(\mathcal{P}\) is combinatorially regular, then any section of \(\mathcal{P}\) is also combinatorially regular. To minimise the chance of confusion, complexes whose automorphism groups act transitively on the set of flags shall also be called flag regular. Note that the term "combinatorially regular" is used in [32] for the concept of (flag) regularity.

The flag regular complexes have been examined in some detail by other authors (see §3.1.1 for references). It is not our purpose here to duplicate their work, but rather, to obtain some information about those complexes which are combinatorially regular. First, we shall note that combinatorial regularity is a weaker definition than flag regularity.

**Theorem 3.4.3:** If a complex \(\mathcal{X}\) is flag regular, it is also combinatorially regular.

**Proof:** Let \((x, y)\) and \((x', y')\) be corresponding sections of \(\mathcal{X}\). Then, let \(F\) and \(F'\) be flags of \(\mathcal{X}\) such that \(x, y \in F\) and \(x', y' \in F'\). Since \(\mathcal{X}\) is (flag) regular, there exists an automorphism \(\phi\) of \(\mathcal{X}\) such that \(F = F'\), and so \(x = x'\) and \(y = y'\). But \(\phi\) is an automorphism from \(\mathcal{X}\) to \(\mathcal{X}\), so by Theorem 3.3.2, we have \((x, y) \cong (x, y)\), which is required.

There exist numerous examples of complexes which are combinatorially regular without being regular, so the combinatorial definition is strictly weaker. That this is so may be seen (for example) from Theorems 6.3.9 and 6.3.10 and the associated Tables 6.3.3 and 6.3.4, or from Theorem B.2.3 (in Appendix B, which is associated with §6.3). Let us now examine these two kinds of regularity more closely.

**Theorem 3.4.4:** For \(d \leq 1\), all 1-complexes are flag regular.

**Proof:** For \(d = 0\), there is only one 1-complex (see Theorem 3.3.7), and it has only one flag. Hence the automorphism group cannot fail to act transitively on the flags, and so the complex is regular. Let \(d = 1\), and let \(\{0, h_1, h_2, \ldots, h_k, K\}\) be a 1-complex (all 1-complexes are of this form, by Theorem 3.3.9). Then any permutation of the \(h_i\) will be an automorphism. Now any flag is of the form \(\{0, h_i, K\}\) for some \(i\); if we want to find an automorphism taking a flag \(\{0, h_i, K\}\) to the flag \(\{0, h_j, K\}\), it suffices to find a permutation taking \(h_i\) to \(h_j\). The permutation transposing \(h_i\) and \(h_j\) will do. Thus the automorphism group is flag transitive, and the complex is regular.

**Theorem 3.4.5:** All 2-complexes are combinatorially regular.

**Proof:** Let \((x, y)\) and \((x', y')\) be corresponding sections of a 2-complex \(\mathcal{X}\). If \(d x = d x'\) and \(d y = d y'\), then \((x, y) \cong (x', y')\), so they are isomorphic. Also, if \(d x = d x'\) and \(d y = d y'\), then \((x, y) \cong (x', y')\) since all 0-complexes are isomorphic. But if \(d x = d x'\) and \(d y = d y'\), then \((x, y) \cong (x', y') \cup \{h_1, \ldots, h_i\}\) and \((x', y') \cup \{h_{i+1}, \ldots, h_k\}\) (where \(i\) is given by \(d + 1 = i + d y - 1\)). Then, the map \(\psi\) defined by \(x \psi = x', y \psi = y'\), and \(h \psi = h\) is an isomorphism between these two sections. Hence any two corresponding sections are isomorphic, and \(\mathcal{X}\) is combinatorially regular.

It may further be shown that if the 2-complex is a polytope, then it is in fact flag regular. This result will be proved here, as it is not needed for what follows, and is much easier to prove with the machinery to be developed in Chapter 5. Let us now examine these two kinds of regularity more closely.

**Theorem 3.4.6:** If \(\mathcal{X}\) is combinatorially (or flag) regular, and \(\mathcal{X} \cong \mathcal{L}\), then \(\mathcal{L}\) is also combinatorially (or flag) regular.

**Proof:** Let \(\psi\) be an isomorphism from \(\mathcal{X}\) to \(\mathcal{L}\), and let \(\mathcal{P}\) and \(\mathcal{P}'\) be corresponding sections of \(\mathcal{L}\). Then \((\mathcal{P}')\psi^{-1}\) and \((\mathcal{P}')\psi^{-1}\) will be corresponding sections of \(\mathcal{X}\), and hence will be isomorphic if \(\mathcal{X}\) is combinatorially regular. Likewise, if \(F\) and \(G\) are flags of \(\mathcal{L}\), then \(F\psi^{-1}\) and \(G\psi^{-1}\) are flags of \(\mathcal{X}\), so if \(\mathcal{X}\) is flag regular, there is an automorphism \(\alpha\) of \(\mathcal{X}\) taking \(F\psi^{-1}\) to \(G\psi^{-1}\). Then \(\psi^{-1}\alpha\psi\) is an automorphism of \(\mathcal{L}\) taking \(F\psi^{-1}\) to \(G\psi^{-1}\).

**Theorem 3.4.7:** If \(\mathcal{X}\) is combinatorially (or flag) regular, and \(\mathcal{X}'\) is dual to \(\mathcal{X}\), then \(\mathcal{X}'\) is also combinatorially (or flag) regular.

**Proof:** The proof of this theorem is almost exactly the same as that of the previous one, with \(\psi\) being a structure-reversing bijection instead of a structure-preserving one.

This latter theorem will cut down the work required almost by half. The next one yields an even greater saving.
Theorem 3.4.8: If \( \mathcal{N} \) and \( \mathcal{L} \) are combinatorially regular, then so is their composition \( \mathcal{N} = \mathcal{N} \circ \mathcal{L} \).

**Proof:** Let \( \dim \mathcal{N} = k \) and \( \dim \mathcal{L} = l \), and let \( \mathcal{N} = (\mathcal{X} \cup \mathcal{L}) \circ (K', \mathcal{V}) \), where \( \mathcal{N} \cong \mathcal{X} \) and \( \mathcal{L} \cong \mathcal{L}' \). Also, let \( z \) and \( z' \) be any elements of \( \mathcal{N} \) which satisfy \( \dim_{\mathcal{N}} z = \dim_{\mathcal{N}} z' \). If \( x, x' \in \mathcal{X} \), then it immediately follows that \( \dim_{\mathcal{X}} x = \dim_{\mathcal{X}} x' \). If, however, \( x, x' \in \mathcal{L}' \), then we deduce that \( k + \dim_{\mathcal{N}} x = k + \dim_{\mathcal{N}} x' \), so \( \dim_{\mathcal{N}} x = \dim_{\mathcal{N}} x' \). Thus if \( (z, y) \) and \( (z', y') \) are corresponding sections of \( \mathcal{N} \) or \( \mathcal{L} \), then either they are corresponding sections of \( \mathcal{X} \) or \( \mathcal{L} \) (and so are isomorphic), or else they may be written \((z, y) = (x, K') \circ (\mathcal{V}, y) \) and \((z', y') = (x', K') \circ (\mathcal{V}, y') \), where \((x, K')\) and \((x', K')\) are corresponding sections of \( \mathcal{X} \), as are \((\mathcal{V}, y)\) and \((\mathcal{V}, y')\) of \( \mathcal{L} \). We may conclude that \((x, K') \cong (z, y) \) and \((\mathcal{V}, y) \cong (z', y') \), and hence (by Theorem 3.3.35) that \((x, y) \cong (z', y') \) as required.

Theorem 3.4.9: If \( \mathcal{X} \) and \( \mathcal{L} \) are flag regular, then so is \( \mathcal{X} \circ \mathcal{L} \).

**Proof:** Let \( \mathcal{X}' \) and \( \mathcal{L}' \) be as before, and let \( F \) and \( G \) be flags of \( \mathcal{N} = \mathcal{X} \circ \mathcal{L} \). Then \( F \) may be written \((F_{\mathcal{X}} \cup F_{\mathcal{L}}) \circ (K', \mathcal{V}) \), where \( F_{\mathcal{X}} \) is a flag of \( \mathcal{X} \) and \( F_{\mathcal{L}} \) a flag of \( \mathcal{L} \). We similarly can construct \( G_{\mathcal{X}} \) and \( G_{\mathcal{L}} \). Then \( \mathcal{X} ' \cong \mathcal{X} \) and \( \mathcal{L} ' \cong \mathcal{L} \) are regular, there exists an automorphism \( \alpha ' \) of \( \mathcal{X} ' \) which maps \( F_{\mathcal{X}} \) to \( G_{\mathcal{X}} \), and an automorphism \( \alpha '' \) of \( \mathcal{L} ' \) which maps \( F_{\mathcal{L}} \) to \( G_{\mathcal{L}} \). From \( \alpha ' \) and \( \alpha '' \) we may define \( \alpha \), a map from \( \mathcal{N} \) to itself, as follows. Let \( a \in \mathcal{N} \). Then \( \alpha a = \alpha a' \) if \( a \in \mathcal{X} \), and \( \alpha a = \alpha a'' \) if \( a \in \mathcal{L} \). It is not hard to check that \( \alpha \) is well defined, one to one, onto, and preserves the partial order, and also that it maps \( F \) to \( G \). Thus for any two flags of \( \mathcal{N} \), there is an automorphism mapping one to the other, as required for \( \mathcal{N} = \mathcal{X} \circ \mathcal{L} \) to be regular.

In the next subsection, we shall examine the concept of the Schl"afli Symbol of a regular incidence polytope. Later on, the classification of the regular Euler polytopes will be done in terms of the Schl"afli Symbol, meaning that theorems will be obtained of the form "If \( \mathcal{N} \) is a combinatorially regular Euler polytope with Schl"afli Symbol ..., then \( \mathcal{N} \) is isomorphic to one of ...". The concepts of duality and decomposability will be very helpful because (as we shall see) they bear very simple relationships with the Schl"afli Symbol.

3.4.3 Schl"afli Symbols:

**Definition:** Given a combinatorially regular d-incidence polytope \( \mathcal{X} \), with \( d \geq 1 \), we define the Schl"afli Symbol of \( \mathcal{X} \) to be \( \{q_1 \ldots q_d \} \), if whenever \( \dim x + 2 = i = \dim y - 1 \), the section \((x, y)\) is a q_i-cycle. So, for example, the Schl"afli Symbol of a 1-polytope is \( \{\} \), and that of an n-cycle is \( \{n\} \). Below are a few simple results resulting from the definition.
The next theorem shows that there is a simple relationship between Schl"afli Symbols and decomposability. Recall that we are intending to classify the combinatorially regular Euler polytopes in terms of their Schl"afli Symbols. Hence, if we are to use the results about decomposability, it becomes very important to find such a relationship.

**Theorem 3.4.14:** Let \( \mathcal{N} \) be a combinatorially regular \((k+1)\)-incidence polytope, where \( k, l \geq 1 \). Then \( \mathcal{N} = \mathcal{K} \circ \mathcal{L} \) for some combinatorially regular \( k \)- and \( l \)-incidence polytopes if and only if the \( k \)-th entry in the Schl"afli Symbol of \( \mathcal{N} \) is 2.

**Proof:** First, we prove \((\Rightarrow)\). This is done simply by checking the definition of the Schl"afli Symbol. Let \( x, y \in \mathcal{N} \) satisfy \( \text{dim}_x x + 2 = k = \text{dim}_y y - 1 \). Then \( x \in \mathcal{K} \) and \( y \in \mathcal{L} \), so \( (x, y) = (x, K) \circ (y, y) \). It follows, since \( \text{dim}_x x = k - 2 \) and \( \text{dim}_y y = (k + 1) - k = 1 \), that we deduce both \((x, K)\) and \((y, y)\) are 1-polytopes, and so satisfy \([x, K] = [y, y] = 4\). Thus \([x, y]\) is 4 + 4 - \([K, y] = 8 = 2\). But there exists some \( n \) for which \((x, y)\) is an \( n \)-cycle, whence \([x, y]\) = \(2n + 2\). Thus we have \(2n + 2 = 6\), so that \((x, y)\) is a 2-cycle, and hence also that the \(k\)-th entry in the Schl"afli Symbol of \(\mathcal{N}\) is 2.

Proving the converse \((\Leftarrow)\) is somewhat harder. Denote by \(\mathcal{A}\) the set of elements of \(\mathcal{N}\) with dimension \(i\), and let \(b \in \mathcal{A}_{k-1}\). This proof proceeds in three stages, each stage applying \((i, j)\)-connectivity (Theorem 3.3.20) for some \(i \) and \(j\). First, fix \(a \in \mathcal{A}_{k-2}\), such that \(a < b\). Then, for any \(c, d \in \mathcal{A}\) with \(\text{dim}_c c = k \) and \(\text{dim}_d d = k + 1 \) and \(c < d\), we have \(\{a, c\} \) is a 1-polytope, and \(\{a, d\}\) is a 2-cycle. Hence the sets \(\{b \in \mathcal{A}_{k-1} : a < b < c\}\) and \(\{b \in \mathcal{A}_{k-1} : a < b < d\}\) are equal, the former being a subset of the latter, and both having exactly two elements (see Figure 3.4.1). Now, let \(c^* \in \mathcal{A} \cap \mathcal{A}_{k-2}\), and consider \(\{b \in \mathcal{A}_{k-1} : a < b < c^*\}\). This tells us that there exists a sequence \(c = c_0, c_1, c_2, \ldots, c_m = c^*\) of elements of \(\mathcal{N}\), such that for each \(i\), \(m\), we have \(\text{dim}_c c_i = \text{dim}_c c_{i+1} = k \) and \(\text{dim}_d d_i = k + 1\). From this, we may deduce that for each \(i\), the sets \(\{b \in \mathcal{A}_{k-1} : a < b < c_i\}\), \(\{b \in \mathcal{A}_{k-1} : a < b < c_{i+1}\}\) are all equal. It follows that for all \(c, c^* \in \mathcal{A} \cap \mathcal{A}_{k-2}\), we have \(\{b \in \mathcal{A}_{k-1} : a < b < c\}\) = \(\{b \in \mathcal{A}_{k-1} : a < b < c^*\}\) (see Figure 3.4.2).

![Figure 3.4.1](image1)

![Figure 3.4.2](image2)

This set has two elements, one of them \(b^*\). Call the other \(b^*\). Now \((b, N) \cap \mathcal{A}_{k-2}\) = \{\(c^* \in \mathcal{A}_{k-2} : b < c\)\}. But if \(c^*\) is an element of this set, then \(b^* < c\) also, and so \(c^* \in \mathcal{A} \cap \mathcal{A}_{k-2}\). Thus \((b, N) \cap \mathcal{A}_{k-2} \subseteq \mathcal{A} \cap \mathcal{A}_{k-2}\). By symmetry, these sets are in fact equal (see Figure 3.4.3). We have deduced that given any \(a \in \mathcal{A}_{k-2}\) and \(c \in \mathcal{A}_{k-2}\) with \(a < e\), if \(b\) and \(b^*\) satisfy \(a < b \) and \(b^* < c\), we have \(\{c^* \in \mathcal{A}_{k-2} : b < c\}\) = \(\{c^* \in \mathcal{A}_{k-2} : b^* < c\}\). We can extend this result as follows. Let \(b, b^*\) be any two elements of \((a, N) \cap \mathcal{A}_{k-2}\). Applying \(0,1\)-connectivity, this time on the complex \((a, N)\), we discover that there is a sequence \(b = b_0, b_1, \ldots, b_m = b^*\) such that for any \(i\) we have \(c_i \in \mathcal{A}_{k-2}\) with \(a < c_i\) and \(b_i, b_{i+1}\) satisfy \(a < b_i < b_{i+1}\). Thus, for any \(i\), the sets \(\{c_i \in \mathcal{A}_{k-2} : b < c\}\) and \(\{c_i \in \mathcal{A}_{k-2} : b < c\}\) are equal. It follows that given \(a \in \mathcal{A}_{k-2}\), we have \(\{c \in \mathcal{A}_{k-2} : b < c\}\) = \(\{c \in \mathcal{A}_{k-2} : b^* < c\}\) for any \(b, b^*\) in \((a, N) \cap \mathcal{A}_{k-2}\) (see Figure 3.4.4).

![Figure 3.4.3](image3)

![Figure 3.4.4](image4)

We now appeal \((i, j)\)-connectivity a third and last time – this time applying \((k-2, k-1)\)-connectivity to \(a\), so that for any \(b, b^* \in \mathcal{N}_{k-2}\), there exists a sequence \(b = b_0, b_1, \ldots, b_m = b^*\) such that for each \(i\) we have \(a \in \mathcal{A}_{k-2}\) and \(b_i, b_{i+1}\) = \(\{a, N\} \cap \mathcal{A}_{k-2}\). From the results of the previous paragraph, it follows, for any \(i\), that \(\{c \in \mathcal{A}_{k-2} : b_i < c\}\) = \(\{c \in \mathcal{A}_{k-2} : b < c\}\) and hence that \(\{c \in \mathcal{A}_{k-2} : b^* < c\}\) = \(\{c \in \mathcal{A}_{k-2} : b^* < c\}\) (see Figure 3.4.5).

![Figure 3.4.5](image5)
CHAPTER 3: Combinatorial Polytopes

The above theorem tells us that when classifying combinatorially regular polytopes, we can ignore those whose Schl"{a}fi Symbols contain a 2, since these are decomposable, and may (by Theorem 3.3.38) be written in a unique way as a composition of indecomposable combinatorially regular polytopes.

3.4.4 Number Crunching:

This subsection contains a simple lemma which, although not terribly deep, will prove quite useful later. It relates to the relative sizes of the sets $\mathcal{X}_i$ for a combinatorially regular complex $\mathcal{X}$.

Lemma 3.4.15: Let $\mathcal{X}$ be a combinatorially regular $d$-incidence complex. Then for $i < j$, we have

$$|(a_i, K) \cap \mathcal{X}_j| \cdot |\mathcal{X}_i| = \sum (a_j) \cdot |\mathcal{X}_i| \cdot |\mathcal{X}_j|,$$

where $a_i$ and $a_j$ are (arbitrary) elements of $\mathcal{X}_i$ and $\mathcal{X}_j$, respectively.

Proof: Let $a_i \in \mathcal{X}_i$ and $a_j \in \mathcal{X}_j$, and consider the set $\mathcal{F} = \{(x, y) : x \in \mathcal{X}_i, y \in \mathcal{X}_j, x < y\}$. The number of elements of this set will be given by

$$|\mathcal{F}| = \sum_{x \in \mathcal{X}_i} |\{(y, x) : y \in \mathcal{X}_j, x < y\}| = \sum_{x \in \mathcal{X}_i} |\mathcal{X}_j| - \sum_{x \in \mathcal{X}_i} |(x, K) \cap \mathcal{X}_j|.$$

But since $\mathcal{X}$ is combinatorially regular, $(x, K)$ is isomorphic to $(a_i, K)$ for each $x$, from which we can deduce after a little work that for each $x_i, |(x, K) \cap \mathcal{X}_j| = |(a_i, K) \cap \mathcal{X}_j|$. Thus we have $|\mathcal{F}| = \sum_{x \in \mathcal{X}_i} |(a_i, K) \cap \mathcal{X}_j| = \sum_{x \in \mathcal{X}_i} |(x, K) \cap \mathcal{X}_j|$. By a similar argument, $|\mathcal{F}| = \sum_{x \in \mathcal{X}_i} |(a_i, K) \cap \mathcal{X}_j|$, and so these two products are equal, as required. The result may be shown to hold even if $|\mathcal{F}|$ is infinite.

This result is chiefly useful for the following reason. The Schl"{a}fi Symbol of a combinatorially regular polytope tells us the Schl"{a}fi Symbols of its facets and vertex figures. If the class of polytopes with the latter Schl"{a}fi Symbols has been completely described, we can deduce from this description the possible values of the $|(a_i, K) \cap \mathcal{X}_j|$ and $|(a_i, K) \cap \mathcal{X}_j|$. In this sense, the Schl"{a}fi Symbol of a polytope $\mathcal{X}$ can tell us a great deal about the relative sizes of the $\mathcal{X}_i$. If, in addition, we know that $\mathcal{X}$ is Euler, we can combine this information with Euler’s condition. This can yield some very strong results, at times even showing that no combinatorially regular Euler polytope with a given Schl"{a}fi Symbol can exist.

3.5 The Geometric Link

Combinatorial geometry, particularly the study of incidence polytopes, is inspired by classical geometry. It follows that the concepts studied in the two fields ought to bear some resemblance to one another.

In the definition of an incidence polytope, the elements of the partially ordered set are intended to represent the faces of a (perhaps hypothetical) geometric polytope, and the partial order the relation of inclusion between those faces.

In this section, we clarify the link between geometric polytopes and the combinatorial objects, and hence see that it is reasonable to call them polytopes.

This will be done by defining, for any geometric polytope, an incidence polytope that corresponds to it in a natural way.

Definition: Let $P$ be a geometric polytope. Then, let $\mathcal{P}$ be the set of faces of $P$, including $P$ itself, and the empty set $\emptyset$. We define the combinatorial counterpart $\mathcal{P}$ of $P$ to be $\mathcal{P}$ equipped with the partial order $\subset$. In the literature, this is often called the face lattice of $P$. We shall unfortunately be forced to avoid this term, as the word ‘lattice’ is used elsewhere for a quite different concept.

We proceed to show that the combinatorial counterpart is an incidence polytope. The first step is to lift a theorem from Brundsted that will be of great assistance.

Theorem 3.5.1: Let $\mathcal{P}$ be the combinatorial counterpart of a geometric polytope $P$, and let $x$ and $y$ be faces of $P$ with $x \subseteq y$. Then there exists some geometric polytope $Q$ with $\dim Q = \dim y - \dim x - 1$ such that the partially ordered set $(x, y) = \{z : z \subseteq x \subseteq y\}$ is isomorphic to the combinatorial counterpart $\mathcal{Q}$ of $Q$, and if $\phi$ is an isomorphism from $(x, y)$ to $\mathcal{Q}$, then for all $z \in (x, y)$, we have $\dim \phi(z) = \dim z - 1 - \dim x$.

Proof: See [6, Thm 11.4].

Lemma 3.5.2: If $P$ is a geometrical polytope, then $\mathcal{P}$ satisfies $11$ and $12$.

Proof: (See also [32, Prop 3.1].) Certainly, $\emptyset$ and $P$ will (respectively) be unique minimal and maximal elements of $\mathcal{P}$. Now since $\mathcal{P}$ is a finite set (by Theorem 2.3.6), any chain must be contained in a maximal chain. Now $F$ is a flag, and $x, y \in F$ with $x \subseteq y$, then $x = \emptyset$ or $y$ is a proper face of $y$ (by Theorem 2.3.2). It follows that $\dim x \neq \dim y$, so $F$ has at most one element of each dimension $i \in \{-1, 0, \ldots, d\}$. Next, note that $\emptyset, P \in F$, and let $i$ be the least element of $\{-1, 0, \ldots, d\}$ such that $F$ has no element of dimension $i$. Let $x, y \in F$ be such that $\dim x = i - 1$, and $\dim y = j$ where $j > i$ is such that there is no element $z$ of $F$ with $\dim z < \dim x < \dim y$. Theorem 3.5.1
then tells us that there exists a geometric polytope $Q$ such that the partially ordered set $\langle x, y \rangle = \{ x \in P : x \leq z \leq y \}$ is isomorphic to the combinatorial counterpart $\mathcal{D}$ of $Q$, and furthermore, if $\phi$ is any isomorphism from $(x, y)$ to $\mathcal{D}$, then for all $z \in (x, y)$, $\dim \phi(z)$ is equal to $\dim x - 1 - \dim z$. Now $Q$, being a polytope, has vertices, so there exists $z \in (x, y)$ such that $\dim \phi(x) = 0$, that is, $\dim z = 0 + 1 + (i - 1) = i$. Then, we can obtain a contradiction by noting that the totally ordered set $F \cup \{ x \}$ properly contains $F$. It follows that any flag of $\mathcal{P}$ has exactly one element of possible dimension $i \in \{ -1, 0, \ldots, d \}$; that is, $F$ has $d + 2$ elements, as required.

Note then that the poset $\mathcal{P}$ has the same "dimension" as the geometric polytope $P$. This is despite the fact that the definition of the dimension of a poset satisfying $I_1$ and $I_2$ bears no obvious similarity to the definition of dimension of a subset of $R^n$. Note further that we could show that for any $i$-face $x$ of $P$, we have $\dim_{\mathcal{P}} x = i$, by noting that $\emptyset, x$ is just the set of faces (that is, the combinatorial counterpart) of the polytope $x$.

Before we move on, let us recall to mind the definition of the graph of a polytope. This, as mentioned in §2.3.2, is the graph defined on the vertices (0-faces) of the polytope, with two vertices being regarded as adjacent if and only if the polytope has a 1-face (an edge) containing them both. Even though the vertices of the polytope are the vertices of the graph, and the edges of the graph are (in a sense) represented by the edges of the polytope, one should always be careful to discern the context in which a term is used. This will be especially so when we come to the next lemma, and begin to talk about strongly and weakly connected posets, and connected graphs, all in one breath.

Recall that a strongly connected poset is one such that all its sections are weakly connected, and a weakly connected poset is one such that for any two of its elements $f$ and $g$ (except the minimal and maximal elements), there exists a sequence $f = h_1, h_2, \ldots, h_n = g$ such that for each $i$, either $h_i \leq h_{i+1}$, or $h_i \geq h_{i+1}$, and none of the $h_i$ are the maximal or minimal elements of the poset. However, a connected graph is one such that for any two vertices $x$ and $y$, there exists a sequence $x = v_1, e_1, v_2, e_2, \ldots, v_{n-1}, e_{n-1}, v_n = y$ of vertices $v_i$ and edges $e_i$ of the graph such that $v_i, v_{i+1} \in e_i$ for each $i$. The definitions look similar but are saying somewhat different things. We will be using both concepts in the proof of the next result.

**Lemma 3.5.3.** Let $\mathcal{P}$ be the combinatorial counterpart of a geometric polytope $P$. Then $\mathcal{P}$ satisfies $I_3$, that is, it is strongly connected.

**Proof:** (Also shown in [32, Prop 3.3].) To show that $\mathcal{P}$ is strongly connected, we need to show that any section of $\mathcal{P}$ is weakly connected. Now Theorem 3.5.1 tells us that any section of $\mathcal{P}$ is in fact isomorphic to a combinatorial counterpart. It suffices, then, to show that the combinatorial counterpart $\mathcal{D}$ of an arbitrary geometric polytope $Q$ is weakly connected – but this follows from the fact that the graph of $Q$ is connected (see [6, Thm 15.5]), since for any proper faces $f, g \in \mathcal{D}$, a path in the graph of $Q$ which connects a vertex of $f$ to one of $g$ may be expanded, if $\dim Q \geq 2$, to yield a sequence of the required form of proper faces of $\mathcal{D}$.

Finally, we have the following.

**Lemma 3.5.4.** If $P$ is a polytope, and $x$ and $y$ are faces of $P$ with $x \subseteq y$ and $\dim x + 1 = \dim y - 1$, then the set $\{ z : z \subset x \subset y \}$ has exactly two elements.

**Proof:** (Also in [32, Prop 3.2(1)].) By Theorem 3.5.1, the section $(x, y)$ will be isomorphic to the combinatorial counterpart $\mathcal{D}$ of some polytope $Q$. By the same theorem, this $Q$ will have dimension $\dim y - \dim x - 1$, that is, 1. The only convex subsets which have dimension 1 are in fact line segments, and these have four faces each – the empty set, the two endpoints, and the segment itself. Thus the set $(x, y) = \{ z : z \subset x \subset y \}$ has four elements, whence the set $\{ z : z \subset x \subset y \}$ has the required two.

**Theorem 3.5.5.** The combinatorial counterpart $\mathcal{P}$ of $P$ is an incidence polytope.

**Proof:** See Lemmas 3.5.2, 3.5.3 and 3.5.4.

I state without proof two simple results.

**Theorem 3.5.6.** If $P$ and $Q$ are geometric polytopes of the same combinatorial type then their combinatorial counterparts are isomorphic.

**Theorem 3.5.7.** If $P$ and $Q$ are geometric polytopes, and $P$ is a dual of $Q$, then the combinatorial counterpart of $P$ is a dual of the combinatorial counterpart of $Q$.

Another important theorem is the following.
Theorem 3.5.8: If a regular geometric $d$-polytope, its combinatorial counterpart ${\mathcal{P}}$ is flag regular, and hence also combinatorially regular.

Proof: The simplest way to prove this is to appeal to [32, Prop 2.1], where it is shown that a polytope is regular if and only if its symmetry group is transitive on the set of flags of the combinatorial counterpart, and to note that any symmetry of $P$ induces an automorphism of ${\mathcal{P}}$. This shows that ${\mathcal{P}}$ is in fact flag regular. It is therefore combinatorially regular by Theorem 3.4.3. ■

Any geometric polytope has a combinatorial counterpart. The converse to this does not hold, however; there exist incidence polytopes (even Euler incidence polytopes) with no 'geometric counterpart'. The simplest examples of such incidence polytopes are decomposable.

Theorem 3.5.9: If $P$ is a geometric polytope, its combinatorial counterpart ${\mathcal{P}}$ is indecomposable.

Proof: By contradiction. Assume ${\mathcal{P}}$ is decomposable, and let $A$ be a facet of $P$. Note that $A \subseteq P$. Since ${\mathcal{P}}$ is decomposable, it follows that $x \subseteq A$ for all elements $x$ of ${\mathcal{P}}$, so in fact $\text{vert} P \subseteq \text{vert} A$. Thus $\text{conv}(\text{vert}P) \subseteq \text{conv}(\text{vert}A)$, from which [29, Thm 20.4] informs us that $P \subseteq A$. This is a contradiction, since it would imply that the $(d - 1)$-dimensional set $P$ is equal to the $(d - 1)$-dimensional set $A$.

Nevertheless, even indecomposability is no guarantee for an incidence polytope to be a combinatorial counterpart to some geometric polytope, as the halfcubes of §4.2 will demonstrate.

32 See §2.4.1
33 Recall the definitions of vert $P$ and conv(vert $P$) §2.1 - vert $P$ is the set of vertices of $P$, and conv(vert $P$) is the convex hull of vert $P$.

CHAPTER 4

Some Examples

We have enough good precepts, but few good examples.
(Nous avons d'assez bons préceptes, mais pas de bons maîtres)

Vauvenargues, 1746

In this chapter we shall see numerous examples of combinatorially regular incidence polytopes. The examples fall into two categories. There are the general classes (that is, constructions that yield $d$-polytopes for infinitely many $d \in \mathbb{Z}^+$), and there are the sporadic examples ($d$-polytopes for specific (small) $d$ which cannot be readily generalised to higher dimensions).

All of the examples will occur in the classification results of Chapter 6. In many cases, notably the simplices, the cubes and halfcubes, and the crosses and halfcrosses, the examples given here turn out to be the only combinatorially regular Euler incidence polytopes that exist with those Schlöfli Symbols. However, no attempt will be made in this chapter to prove any such uniqueness results.

The descriptions given here are reasonably neat constructions based on other well-known abstract algebraic objects. They are by no means the only neat way of describing these polytopes. In fact, for some polytopes, this thesis gives as many as three or even four different constructions – for example the combinatorial simplices described here in the first section here can also be constructed as the combinatorial counterparts of the geometric simplices (see §3.5 and Theorem 4.1.1), or (as we shall see in Chapters 5 and 6), as so-called universal polytopes constructed from certain Coxeter groups.

As has been stated, nowhere in this chapter is any attempt made to prove any uniqueness theorems. Hence, in one sense, this chapter makes no progress towards the classification. However, it is hoped that it will give the reader a better intuitive grasp of some of the polytopes that do exist, making it easier for him or her to understand the classification and the concepts underlying it. It also provides a set of useful pegs on which to hang some of the results of Chapter 6.
4.1 Combinatorial Simplices

The combinatorial simplices, as has been mentioned, turn out to be combinatorial counterparts of the geometric simplices. They may be defined as follows. Let $R$ be a set with $d+1$ elements ($d$ is finite), and let $\mathcal{R}$ be the power set of $R$, that is, the set of all subsets of $R$.

Definition: $\mathcal{R}$, equipped with the partial order $\subseteq$, shall be called a (combinatorial) $d$-simplex.

Clearly, the structure of $\mathcal{R}$ depends only on the size of the set $R$ chosen, and not on $R$ itself — if $R$ and $R'$ have the same size, then any bijection from $R$ to $R'$ will induce a natural isomorphism from $\mathcal{R}$ to $\mathcal{R}'$. We deduce that for any given $d$, all $d$-simplices are isomorphic.

Now $\mathcal{R}$ will certainly be a partially ordered set. We may show immediately that it is isomorphic to the combinatorial counterpart of the geometric simplex.

Theorem 4.1.1: The combinatorial counterpart of a geometric $d$-simplex is isomorphic to a combinatorial $d$-simplex.

Proof: Let $S$ be a geometric $d$-simplex, and let $R$ be the set of its vertices. Note that $|R| = d+1$ (from Table 2.5.5), and let $\mathcal{S}$ be the combinatorial $d$-simplex based on $R$. Furthermore, let $\mathcal{S}$ be the set of faces of $S$, equipped with the inclusion relation, so $\mathcal{S}$ is the combinatorial counterpart of $S$. Then, we define a map $\psi: \mathcal{S} \rightarrow \mathcal{R}$ as follows. For any face $s$ of $S$, let $s \psi$ be the vertex set of $s$. Now $\psi$ is a well-defined map. This is so, because by Theorem 2.3.1, any vertex (0-face) of $s$ will also be a 0-dimensional face of $S$ (that is, a vertex of $S$), so $s \psi = \text{vert} \subseteq \text{vert} S$. Also, by Theorems 2.3.4 and 2.3.8, we conclude that $\text{conv}(s \psi) = s$, so if $s \psi = t \psi$, then $s = t$, making $\psi$ one to one. $\psi$ also preserves the partial order, for if $s$ and $t$ are faces of $S$ with $s \subseteq t$, then $s$ is a face of $t$ (by Theorem 2.3.2), and so any vertex of $s$ is also a vertex of $t$ (by Theorem 2.3.1), so $s \psi \subseteq t \psi$. Conversely, let $s$ and $t$ be such that $s \psi \subseteq t \psi$. Now $s = \text{conv}(s \psi)$ is the intersection of all convex sets containing $s \psi$ (by definition of the convex hull). But $t = \text{conv}(t \psi)$ is both convex, and contains $t \psi$ and hence also $s \psi$. Thus $\text{conv}(s \psi) \subseteq \text{conv}(t \psi)$, yielding $s \psi \subseteq t \psi$. Thus $\psi$ is a structure-preserving monomorphism from the $d$-incidence polytope $\mathcal{S}$ to the $d$-incidence polytope $\mathcal{R}$, and so by Corollary 3.2.25, it is an isomorphism also.

It is known that a combinatorial counterpart of a regular geometric polytope is in fact a flag regular incidence polytope (see Theorems 3.5.5 and 3.5.8). We therefore obtain immediately the following.

Corollary 4.1.2: The $d$-simplex $\mathcal{R}$ is a combinatorially regular incidence polytope.

It will be useful to note in particular:

Corollary 4.1.3: The $d$-simplex is strongly connected.

The next result, about the sections of a simplex, will make it easy to show a number of other results which become useful later.

Theorem 4.1.4: Let $X \subseteq Y \subseteq R$ satisfy $|X| = i$ and $|Y| = j$, with $0 \leq i < j \leq d+1$. Then the section $\langle X, Y \rangle$ of $\mathcal{R}$ is isomorphic to a combinatorial $(j-i-1)$-simplex.

Proof: The set $Y \setminus X$ has $j-i$ elements. Let $\mathcal{S}'$ be the $(j-i-1)$-simplex defined from it. Let $\psi$ be the map from $(X,Y)$ to $\mathcal{S}'$, defined by $Z \psi = Z \setminus X$. We shall do this proof in an unusual order. First, we prove for $Z', Z'' \subseteq (X,Y)$ that $Z \subseteq Z'$ if and only if $Z \setminus X \subseteq Z' \setminus X$. Assume first that $Z \subseteq Z'$. Then if $z \in Z \setminus X$, we have $z \in Z$, so $z \in Z'$. But since also $z \notin X$, it follows that $z \in Z' \setminus X$, so $Z \setminus X \subseteq Z' \setminus X$. To show the converse, assume that $Z' \setminus X \subseteq Z \setminus X$. Then, if $z \in Z'$, either $z \in Z$ (since $X \subseteq Z'$), or $z \in Z \setminus X$, in which case $z \notin Z'$, since $Z' \setminus X \subseteq Z \setminus X \subseteq Z'$. So in either case we have $z \in Z'$, and so $Z \subseteq Z'$ as required, that is, $Z \subseteq Z'$ if and only if $Z \setminus X \subseteq Z' \setminus X$, as claimed. Now, note that for any sets $A$ and $B$, we have $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. Thus, for any $Z', Z'' \subseteq (X,Y)$, the statement $Z \subseteq Z'$ is equivalent to the statement $Z \subseteq Z'$ and $Z' \subseteq Z$, which is true if and only if $Z \setminus X \subseteq Z' \setminus X$ and $Z' \setminus X \subseteq Z \setminus X$, that is, $Z \setminus X = Z' \setminus X$. So $Z = Z'$ if and only if $Z \setminus X = Z' \setminus X$. Noting also that $Z \in (X,Y)$ implies $Z \subseteq Y$, so $Z, X \in (X,Y)$, we may conclude that $\psi$ is well-defined and one to one. Next, $\psi$ is onto, for if $Y' \subseteq Y \setminus X$, then $Z = Y' \cup X$ will be an element of $(X,Y)$ satisfying $Z \psi = Y'$. Finally, the fact that $Z \subseteq Z'$ if and only if $Z \setminus X \subseteq Z' \setminus X$ shows that $\psi$ preserves the partial order.

It follows that if $X \subseteq Y \subseteq R$, with $|X| = i$ and $|Y| = j$, then $\langle X, Y \rangle$ has dimension $j-i-1$, that is, $\dim(\langle X, Y \rangle) = |Y| - |X| - 1$. In particular, note that $\dim X = \dim(\emptyset, X) = |X| - |\emptyset| - 1 = |X| - 1$.

Now a $d$-simplex $\mathcal{R}$ satisfies $|\mathcal{R}| = \binom{d+1}{i+1}$, since $|\mathcal{R}|$ is the number of $(i+1)$-subsets of a $(d+1)$-set.
Theorem 4.1.5: \( \mathcal{A} \) is Euler.

Proof: Any section of \( \mathcal{A} \) is a \( k \)-simplex \( \mathcal{S} \) for some \( k \), and therefore satisfies Euler's condition, since

\[
\sum_{j=0}^{k+1} \frac{(-1)^j |\mathcal{S}|}{j!} = \sum_{j=0}^{k+1} \frac{(-1)^j (-1)^{k+1}}{j!} = -\left[ (1 + (-1))^{k+1} \right],
\]

which equals zero.

We also have the following.

Theorem 4.1.6: The Schläfli Symbol of a \( d \)-simplex is \( [3] \ldots [3] \).

Proof: Any \( 2 \)-section \( \mathcal{S} \) satisfies \( \mathcal{S}_1 = (\binom{k+1}{2+1}) = 3 \), so all the entries of the Schläfli Symbol are 3.

Note finally that since the geometric simplex is self-dual, Theorem 3.5.7 yields the following corollary.

Corollary 4.1.7: The combinatorial \( d \)-simplex is self-dual.

Proof: See Theorems 3.5.7 and 4.1.1.

4.2 Combinatorial Cubes and Halfcubes

In this section, we let \( R \) be a \( d \)-dimensional vector space over the field \( \mathbb{Z}_2 \), and let \( R \) be spanned by \( T = \{e_1, \ldots, e_d\} \). We let \( \mathcal{A} \) be the set consisting of the empty set, and all cosets of those subspaces of \( R \) spanned by subsets of \( T \). That is, \( \mathcal{A} = \{\emptyset\} \cup \{x + \text{lin} A : x \in R \text{ and } A \subseteq T\} \). As an example, suppose \( d \) were equal to 2, so \( T = \{e_1, e_2\} \). Then \( T \) has four subsets, yielding the four subspaces \( \text{lin} \emptyset = \{0\} \), \( \text{lin} \{e_1\} = \{0, e_1\} \), \( \text{lin} \{e_2\} = \{0, e_2\} \), and \( \text{lin} T = R = \{0, e_1, e_2, e_1 + e_2\} \). This does not exhaust the subspaces of \( R \). As an example, \( \text{lin} \{e_1 + e_2\} = \{0, e_1 + e_2\} \) is a subspace of \( R \), but not of the form in which we are interested.

If we take all possible sets of our four subspaces, we obtain a total of nine cosets. The subspace \( \{0\} \) will yield four cosets, namely \( \{0\} \) itself, \( \{e_1\} \), \( \{e_2\} \), and \( \{e_1 + e_2\} \). Each of the one-dimensional subspaces yields two cosets; namely \( \{0, e_1\} \) and \( \{e_1 + e_2\} \) from \( \text{lin} \{e_1\} \), and \( \{0, e_2\} \) and \( \{e_1 + e_2\} \) from \( \text{lin} \{e_2\} \). Finally, \( \text{lin} T \) has only one coset, namely itself. The set \( \mathcal{A} \) will therefore have ten elements - the nine cosets listed above, and the empty set. This set \( \mathcal{A} \), when equipped with the partial order \( \subseteq \), shall be called a \( \text{(combinatorial) } d \)-cube.

Note that the structure of the \( d \)-cube will depend only on \( d \), and not on the particular \( d \)-dimensional vector space \( R \) chosen. This is because if \( \mathcal{A} \) is the cube defined from the vector space \( R \) with basis \( T \), and if \( \mathcal{A}' \) is the one defined from \( R' \) with basis \( T' \), then any bijection from \( T \) to \( T' \) will induce an isomorphism from \( \mathcal{A} \) to \( \mathcal{A}' \). Thus, for fixed \( d \), all combinatorial \( d \)-cubes are isomorphic.

It would be possible to give an analysis of the cubes the same form that our analysis of the simplices took. That is, we could prove that a combinatorial cube is the combinatorial counterpart of a geometric cube, and then use this result to show that the combinatorial cube is a combinatorially regular incidence polytope and so forth. There are two reasons for not using this approach here.

Firstly, such a proof would make the results less self-contained - for example, the proof that the combinatorial counterpart is strongly connected relied on the fact that the graph of a polytope is a connected graph, but the latter result is not proven in detail in this thesis. Since the knowledge of cubes becomes more critical in the final classification than the knowledge of the simplices does, this consideration weighs more heavily on the author's mind in this section than it did in the last.

The second, and perhaps more important consideration, is that the alternative approach used here allows another class of polytopes, the halfcubes, to be attacked simultaneously with the cubes. The halfcubes are almost as critical as the cubes to the overall classification, and do not lend themselves to the geometrical analysis used in the last section.

For these reasons, we shall first give an outline proof of the link between geometric and combinatorial cubes, then define the halfcubes and begin the more combinatorial analysis.

Theorem 4.2.1: The combinatorial counterpart of a geometric \( d \)-cube is a combinatorial \( d \)-cube.

Proof: (Outline\(^{34}\)). Let \( Q \) be the geometric cube given by \( \{2\}^d \) (specifically Table 2.5.5). Note that the vertices of \( Q = \{2\}^d \) will have coordinates consisting entirely of zeroes and ones, and in fact any such point will be a vertex of \( Q' \). This suggests an obvious map \( \psi' \) taking subsets of the vertex set of \( Q' \) to subsets of \( \{2\}^d \). This may be shown to map the vertex set of \( Q \) to a coset of \( \{2\}^d \) spanned by a subset of the standard basis \( \{2\}^d \). That is, \( \psi' \) is a well-defined map from \( \{\text{vert } q : q \in \text{nonempty face of } Q\} \) to \( \{\emptyset\} \cup \mathcal{A} \). Given, then, the combinatorial counterpart \( \mathcal{A} \)

\(^{34}\) A full proof is in fact not necessary here, as the result will follow very easily from the theory to be developed in Chapter 5 (see Corollary 5.3.7 and Theorem 6.1.5).
of $Q$, we define the map $\psi : \mathcal{S} \to \mathcal{R}$ via $y' = (\eta y)y'$ if $q \neq \emptyset$, and $\emptyset y = \emptyset$. It may then be shown that $\psi$ is an isomorphism from $\mathcal{S}$ to $\mathcal{R}$.

The half-cubes may be defined in terms of the cubes, and are interesting because they turn out to be indecomposable, yet are not the combinatorial counterpart of any geometric polytope (compare this with Theorem 3.5.9). The half-cubes are defined by taking a cube, and (roughly speaking) identifying pairs of opposite faces — "squashing" the cube so that pairs of opposite faces become single faces.

More formally, let $\tau$ be the transformation taking a subset $X$ of $R$ to the set $\zeta + X = \{\zeta + x : x \in X\}$, where $\zeta$ is the vector $\sum_{\eta \in T} e_\eta$. We note some properties of this transformation. Firstly, $\tau^2$ is the identity map, that is, for any $X \subseteq R$, we have $(\tau X) \cap X = X$. Secondly, if $X \in \mathcal{A}$, then $X \cap X = \emptyset$ also. Thirdly, if $X \subseteq Y$, then $X \subseteq Y$. Incidentally, this shows that $\tau$ is an automorphism of $\mathcal{A}$. A fourth property warrants being stated as a lemma.

Lemma 4.2.2: For $X \subseteq \mathcal{R}$, if $X \neq \emptyset$, then $X \cap X = \emptyset$.

Proof: Assume $X \subseteq \mathcal{R}$. If $X \neq \emptyset$, then $X \cap X = \emptyset$.

Now, we define an equivalence relation $\sim$ amongst the elements of $\mathcal{S}$ via $X \sim X'$ if and only if $X = X'$ or $X = X' \cap X$. Then $\sim$ partitions $\mathcal{S}$ into equivalence classes. Let $\mathcal{B}$ be the set of these equivalence classes, and for $[X], [Y] \in \mathcal{B}$ define $\leq$ via $[X] \leq [Y]$ if and only if $X \subseteq X'$ or $X \subseteq X' \cap Y$. Note that each equivalence class except $[\emptyset]$ and $[R]$ has exactly two elements. More explicitly, $[X] = (X, X \cap Y)$, while $[\emptyset] = \{\emptyset\}$ and $[R] = \{R\}$.

Definition: We call $\mathcal{B}$ a half-cube.

Now that we know what a half-cube is, let us discover some of its properties.

Lemma 4.2.3: The $d$-halfcube $\mathcal{B}$ is a partially ordered set, and has unique minimal and maximal elements $[\emptyset]$ and $[R]$ respectively.

Proof: Given $[x] \in \mathcal{B}$, note that $x \subseteq x$, so $[x] \subseteq [x]$. Also, if $[x] \subseteq [y]$ and $[y] \subseteq [z]$, then $x \subseteq y'$ where $y'$ is either $y$ or $y$. We also have $y' \subseteq x$ or $y' \subseteq x'$, since $[y] = [y']$. In the case $y' \subseteq x'$, we get $x \subseteq x'$, whence $x \subseteq x'$ since $x = x'$, whence $y' \subseteq x'$. But if $y' \subseteq x$, we have $x = y'$, so $[x] = [y']$. Thus $\leq$ is symmetric. Finally, let $[x] \subseteq [y]$ and $[y] \subseteq [z]$. Then $x \subseteq y'$, where $y'$ is equal to either $y$ or $y'$. Now $[y'] = [y']$, so $[y] \subseteq [z']$, so either $y' \subseteq x$ or $y' \subseteq x'$. Thus we have either $x \subseteq z'$ or $x \subseteq x'$, whence $[x] \subseteq [z']$ as required for $\leq$ to be transitive. Thus $\leq$ is a partial order. Finally, note that for all $[x] \in \mathcal{B}$ we have $[\emptyset] \subseteq [x] \subseteq [R]$, making $[\emptyset]$ and $[R]$ respectively the minimum and maximum, and hence also the unique minimal and maximal elements of $\mathcal{B}$.

This result also holds for the cubes.

Lemma 4.2.4: The $d$-cube $\mathcal{R}$ satisfies II.

Proof: The empty set will be the only minimal element of $\mathcal{R}$, being a proper subset of all the other elements. Likewise, $R$ (properly) contains all the other elements of $\mathcal{R}$, and so (being therefore a maximum) is the unique maximal element.

It will be helpful to have a number of small results about the coets under consideration. It is important to keep in mind that since we are working in $\mathbb{Z}_2$, $x = -x$ for all $x \in \mathcal{R}$, or in other words, $x + x = 0$ for all $x \in \mathcal{R}$.

Lemma 4.2.5: If $x + \text{lin}C \subseteq y + \text{lin}D$ for $C, D \subseteq \mathcal{R}$, then $y + \text{lin}D = x + \text{lin}D$.

Proof: Since $x + \text{lin}C \subseteq y + \text{lin}D$ and $0 \in \text{lin}C$, it follows that $x \subseteq y + \text{lin}D$. This yields $x + y \in \text{lin}D$, so $y + \text{lin}D = y + (x + y) + \text{lin}D = x + \text{lin}D$ as required.

Lemma 4.2.6: If $x + \text{lin}C \subseteq y + \text{lin}D$ for $C, D \subseteq \mathcal{T}$, then $C \subseteq D$.

Proof: If $x + \text{lin}C \subseteq y + \text{lin}D$, then $y + \text{lin}D = x + \text{lin}D$. Thus $\text{lin}C = x + z + \text{lin}C \subseteq x + z + \text{lin}D = \text{lin}D$. Now, let $e_i \in \mathcal{T}$, so $e_i \in \text{lin}D$. Since we then have $e_i \in \text{lin}D$, we can express $e_i$ as a linear combination of elements of $D$. But both $e_i \in \mathcal{T}$ and $D \subseteq \mathcal{T}$. Thus, since $\mathcal{T}$ is linearly independent (being a basis for $R$), it must be that $e_i \in D$. We conclude that $C \subseteq D$, as required.

In order to simplify the analysis which follows, note that the cube may also be regarded as a set of equivalence classes $\mathcal{S} = \{(X) : X \in \mathcal{R}\}$, under the equivalence relation $X \sim Y$ if and only if $X = Y$. We shall denote by $\mathcal{P}$ the poset $\{(X) : X \in \mathcal{S}\}$ (with partial order $X \leq Y$) if and only if there exists

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$Y' \in \{Y\}$ such that $X \subseteq Y'$) when it is not desired to distinguish whether we are referring to a cube or a halfcube.

Now, let us classify the chains of $\mathcal{P}$. Define $\psi$, a map from $\mathcal{M}$ to $\mathcal{P}$ via $c \psi = [c]$. Certainly $\psi$ will be well-defined and onto. For a subset $S$ of $\mathcal{M}$, we also define $S \psi = \{ x \psi : x \in S \}$. We can therefore also treat $\psi$ as a map from the set of subsets of $\mathcal{M}$ to the set of subsets of $\mathcal{P}$. (Strictly speaking, we have defined two different maps. Calling them both $\psi$ should not lead to any confusion, however.)

Note that for $X, Y \in \mathcal{P}$, if $X \psi = Y \psi$, then either $X = Y$ or $X = Y'$. This statement holds even if $\mathcal{P}$ is a cube. It follows then that if $X \psi \subseteq Y \psi$, then $X \subseteq Y + \epsilon'$ for some $\epsilon \in \{0, 1\}$.

We now classify the chains of $\mathcal{P}$.

**Lemma 4.2.7:** The chains of $\mathcal{P}$ are of the form $C = D \cup \{ [x + \lambda A_i] : i \in I \}$, where $x \in R$, where $D = \emptyset$ or $\{[0]\}$, where $I \subseteq \{0, 1, \ldots, d\}$, and where the $A_i \subseteq T$ satisfy $|A_i| = i$ for each $i$, and $A_0 \subseteq A_j$ whenever $i \leq j$.

**Proof:** Let $D = \{ [0] \}$ or $\emptyset$ depending on whether or not $C$ contains $[0]$. Note if $[x + \lambda A_0]$ and $[x' + \lambda A']$ are elements of a chain of $\mathcal{P}$, then without loss of generality,

$$x + \lambda A \subseteq (x' + \lambda A') \epsilon'$$

This implies that $A \subseteq A'$, by Lemma 4.2.6. If $|A| = |A'|$, it must therefore be that $A = A'$, whence (by Lemma 4.2.5), $x + \lambda A = x + \lambda A' = x' + \epsilon' + \lambda A'$, whence $[x + \lambda A] = [x + \lambda A']$. Thus we may index the elements $[x + \lambda A]$ of the chain according to the sizes of the sets $A$, as claimed. Let $C = D \cup \{ [x_i + \lambda A_i] : i \in I \}$, and let $x = x_k$ where $k$ is the least element of $I$. For each $i \in I$, note that $[x + \lambda A_i] \subseteq [x + \lambda A_k]$, so $x + \lambda A_i \subseteq x_i + \epsilon_i' + \lambda A_i$, whence $x + \lambda A_i = x_i + \epsilon_i' + \lambda A_i$, by Lemma 4.2.5, whence $[x_i + \lambda A_i] = [x + \lambda A_i]$, as required.

**Lemma 4.2.8:** A chain $C$ of $\mathcal{P}$ is a flag if and only if $I = \{0, 1, \ldots, d\}$ and $D = \{ [0] \}$.

**Proof:** Let $C = D \cup \{ [x + \lambda A_i] : i \in I \}$ be a chain of $\mathcal{P}$, where $D$, $x$, $I$ and the $A_i$ are as in statement of the previous lemma. For each $j \in \{0, 1, \ldots, d\}$, if $j \notin I$, let $A_j$ be such that $|A_j| = j$, and for all $i \in I$, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Then, $\{[0]\} \cup \{ [x + \lambda A_i] : i \in \{0, 1, \ldots, d\} \}$ will be a chain of $\mathcal{P}$, and unless $D = \{ [0] \}$ and $I = \{0, 1, \ldots, d\}$, it will properly contain $C$. Thus flags must be of the form given. Conversely, it is easy to see that any chain of the form given will be a flag (not having any proper superchains).

We have in fact shown the following.

Corollary 4.2.9: $\mathcal{P}$ satisfies $\text{I}_2$, and has dimension $d$.

**Proof:** The previous proof shows how to find a flag containing any given chain, and also shows that flags contain $d + 2$ elements.

Let us now classify the sections of $\mathcal{P}$.

**Lemma 4.2.10:** If $S = \text{link} A$ has dimension $i < d$, and $y \in R$, then $\{[0], [y + \lambda A]\}$ is isomorphic to a combinatorial cube.

**Proof:** Let $\mathcal{P}$ be the $i$-cube defined from the vector space $S$ with basis $A$. I claim that the map $\phi$ defined via $[x + \lambda B] \psi = [y + x + \lambda B]$ is an isomorphism from $\mathcal{P}$ to $\{[0], [y + S]\}$. It is easy enough to show that it is a well-defined map, and if $[x + \lambda B] \in \{[0], [y + S]\}$, then $x + \lambda B \subseteq (y + \epsilon' + \lambda A) \psi$ for some $\epsilon \in \{0, 1\}$. This being so, it is not hard to show that $x = y + x' + \epsilon'$ is an element of $\mathcal{P}$ such that $(x + \lambda B) \psi = [y + \lambda B]$. Now if $x + \lambda B \subseteq x' + \lambda B'$, then clearly $x + \lambda B \subseteq y + x' + \epsilon' + \lambda B'$. Conversely, if $x + \lambda B \subseteq y + \lambda B'$, then $x + \lambda B \subseteq y + \epsilon' + \lambda B'$ for some $\epsilon \in \{0, 1\}$. The $\phi$ may be cancelled from this expression, and since $x, x' \in \mathcal{P}$ and $B, B' \subseteq A$, it follows that $\phi \in \mathcal{P}$. This can only be the case if $A = T$ (which would contradict our choice of $S$ as having dimension less than $d$), or if $\epsilon = 0$. It follows that $x + \lambda B \subseteq x' + \lambda B'$ if and only if $(x + \lambda B) \psi \subseteq (x' + \lambda B') \psi$. This results that the map is structure-preserving, and may also be used to show that it is one to one.

Thus the proper faces $\{[0], [x + \lambda A]\}$ of $\mathcal{P}$, where $|A| = i$, are $i$-cubes. Knowing that the $i$-cube is an $i$-dimensional polytope, we discover the dimensions of all the elements of $\mathcal{P}$: specifically, $\text{dim}_{\mathcal{P}}[x + \lambda A] = \text{dim}[\{0\}, [x + \lambda A]] = i$. We continue our examination of the sections of $\mathcal{P}$.

**Lemma 4.2.11:** If $x \in R = \text{link} T$ and $|A| = i$, then $\{[x + \lambda A], [\text{link} T]\}$ is isomorphic to a $(d - i - 1)$-simplex.

**Proof:** The set $T \cup A$ will have $d - i$ elements. Let $\mathcal{P}$ be the set of subsets of $T \cup A$, with the partial order $\subseteq$; that is, $\mathcal{P}$ is the combinatorial $(d - i - 1)$-simplex defined from $T \cup A$. Define a map $\phi$ from $\mathcal{P}$ to $\{[x + \lambda A], [\text{link} T]\}$ as follows: for any $B \subseteq T \cup A$, let $C_B = [x + \lambda (A \cup B)]$. It is straightforward, but tedious, to show that this map is a well-defined, structure-preserving bijection.

The above lemmas show what the faces and figures of the cubes and halfcubes are. Any section of $\mathcal{P}$ which is neither a face nor a figure will be a proper face of a proper figure, or alternatively, a proper figure of a proper face of $\mathcal{P}$. It follows, either way, that such a section is a simplex (from the above Lemmas, or from Theorem 4.1.4). Specifically, if $|A| = i$ and $|B| = j$, and $[x + \lambda A] \subseteq [y + \lambda B]$, then $([x + \lambda A], [y + \lambda B])$ will be a $(j - i - 1)$-simplex.

This knowledge of the sections of $\mathcal{P}$ is very useful.
Lemma 4.2.12: \( \mathcal{P} \) is strongly connected, satisfying 13.

**Proof:** Note that by Lemma 4.1.6, simplices are weakly connected. Also, cubes and halfcubes are, for if \( x + \text{lin} A \), \( x' + \text{lin} A' \in \mathcal{R} \), then by letting \( x' - x = \sum_{i=1}^n e_i \), and then \( u_k = x + \sum_{j=1}^{n-1} e_j \), we may construct a sequence

\[ [x + \text{lin} A], [u_0], [u_0 + \text{lin} e_{i_0}], [u_1], \ldots, [u_{n-1} + \text{lin} e_{i_{n-1}}], [u_n], [x' + \text{lin} A] \]

of elements of \( \mathcal{P} \), which shows that \( \mathcal{P} \) is weakly connected, whether \( \mathcal{P} \) is a cube or halfcube. Since all proper sections of \( \mathcal{P} \) are either simplices or cubes, it follows that all sections are simplices, cubes, or possibly halfcubes, and thus all sections of \( \mathcal{P} \) are weakly connected (see Corollary 4.1.3). This completes the proof.

Lemma 4.2.13: If \( d \neq 1 \), and \( [X],[Y] \in \mathcal{P} \) are such that \( [X] \subseteq [Y] \) and \( \dim[X] + 1 = \dim[Y] \), then the set \( K = \{ [Z] : [X] < [Z] < [Y] \} \) has exactly two elements.

**Proof:** If \( d > 2 \), the set \( K \cup \{ [X],[Y] \} \) is the 1-section \( ([X],[Y]) \), which must be either a 1-cube or a 1-simplex (being a proper section of \( \mathcal{P} \)). However, both the 1-cube and the 1-simplex may be shown to have exactly 4 elements, giving \( K \) the required 2. Note that if \( d = 0 \) the lemma is satisfied trivially.

Note that if \( \mathcal{P} \) is a cube, we can remove the restriction \( d \neq 1 \) from the above.

Finally, we can show that cubes and halfcubes are combinatorially regular.

Theorem 4.2.14: The d-cube \( \mathcal{R} \) (for all \( d \)) and the d-halfcube \( \mathcal{L} \) (for \( d \neq 1 \)) are combinatorially regular \( d \)-incidence polytopes.

**Proof:** Lemmas 4.2.3, 4.2.4, Corollary 4.2.9, and Lemmas 4.2.13 and 4.2.14 show that the cubes and halfcubes are \( d \)-incidence polytopes when \( d \) is as given. Now \( \mathcal{P} \) is also combinatorially regular, since if \( ([X],[Y]) \) and \( ([X'],[Y']) \) are corresponding sections of \( \mathcal{P} \), with \( \dim[X] = \dim[X'] = i \) and \( \dim[Y] = \dim[Y'] = j \), then either \( X = [0] \), \( X' = [0] \), whence either both sections are \( j \)-cubes or \( j = d \) and both sections are \( \mathcal{P} \) itself, or \( X,X' \neq [0] \) and both sections are \( (j-1) \)-simplices. This shows that corresponding sections are isomorphic, so that \( \mathcal{P} \) is combinatorially regular.

Let us turn our attention to the cubes for the moment. Note that \( \mathcal{R}_{d-1} = 1 \).

For \( i \geq 0 \), there are \( \binom{2^i}{2} \) subsets \( A \) of \( T \) with \( |A| = i \), and so there are \( \binom{2^i}{2} \) \( i \)-dimensional subspaces of \( R \) of the form \( \text{lin} A \) with \( A \subseteq T \). Since each of these subspaces has \( 2^i \) elements, and \( R \) has \( 2^d \) elements, it follows each such subspace has \( \frac{2^d}{2^i} = 2^{d-i} \) cosets. Furthermore, the cosets of any one such subspace are distinct from the cosets of the other such subspaces (as may be shown, using for example Lemma 4.2.6). Thus \( \mathcal{R} \) has \( 2^d \binom{2^i}{2} \) elements of dimension \( i \), that is, \( |\mathcal{R}_i| = 2^{d-i} \binom{2^i}{2} \). As in the case of the simplices, we can use this information to discover that \( \mathcal{R} \) is Euler, and also to calculate its Schlaffi Symbol.

On the other hand, \( \mathcal{L}_i = \{ [a] \in \mathcal{L} : \dim[a] = i \} \), where \( [a] \) denotes the set \( \{a,ar\} \). Note that \( \mathcal{L}_i = \{ [a] : a \in \mathcal{R}_i \} \), so for \( 0 \leq i \leq d-1 \), \( \mathcal{L}_i \) is a partition of \( \mathcal{R}_i \) into 2-element sets, so we can only have \( a = ar \) if \( a = 0 \) or \( a = R \). This helps us to calculate \( \mathcal{L}_i \), in these cases, for

\[ |\mathcal{L}_i| = \sum_{a \in \mathcal{R}_i} |[a]| = \sum_{a \in \mathcal{R}_i} 2 = 2|\mathcal{L}_i| \]

So \( |\mathcal{L}_i| = \frac{1}{2} |\mathcal{R}_i| = 2^{d-i} \binom{2^i}{2} \) if \( 0 \leq i \leq d-1 \). Note also that \( |\mathcal{L}_{d-1}| = |([0])| = 1 \) and \( |\mathcal{L}_d| = |([R])| = 1 \).

We can now discover when the cube and the halfcube will be Eulerian. Recall the definitions of Euler and sub-Euler – that a polytope is Euler if all its sections satisfy Euler's condition, and is sub-Euler if all its proper sections do (see §3.2).

Theorem 4.2.15: \( \mathcal{P} \) is sub-Euler, and is Euler if and only if \( d \) is even or \( \mathcal{P} \) is a cube.

**Proof:** We check to see when \( \mathcal{P} \) satisfies Euler’s condition. Note that for \( 0 \leq i \leq d-1 \), we have \( |\mathcal{P}_i| = \alpha_i |\mathcal{P}_i| = \alpha_i \cdot 2^{d-i} \binom{2^i}{2} \), where \( \alpha = 0 \) or \( \frac{1}{2} \) depending on whether \( \mathcal{P} \) is a cube or a halfcube. We therefore have

\[ \sum_{i=1}^{d} (-1)^i |\mathcal{P}_i| = (-1)^{-1} + \sum_{i=0}^{d-1} (-1)^i \cdot \alpha_i \cdot 2^{d-i} \binom{2^i}{2} + (-1)^d \]

Letting \( j = d-i \), and noting that \( (-1)^d = (-1)^{-1} \), this becomes

\[ -1 + \left[ (-1)^d \alpha \sum_{j=1}^{d} (-2)^{j} \binom{d}{j} \right] + (-1)^d \]

which equals \( -1 + \left[ (-1)^d \alpha \binom{1}{2} - 1 \right] + (-1)^d \), by the binomial theorem. This in turn equals \( -1 + \alpha \cdot (-1)^{2d} - \alpha \cdot (-1)^{d} + (-1)^d \) which equals zero if and only if \( \alpha = 1 \) or \( d \) is even. Thus in particular cubes satisfy Euler’s relation, and we already know that simplices do. It follows that all proper sections of \( \mathcal{P} \) satisfy the relation, and so \( \mathcal{P} \) is sub-Euler. It will then be Euler if and only if \( \mathcal{P} \) itself satisfies the condition, that is, if and only if it is a cube, or if \( d \) is even.
Theorem 4.2.16: The Schl"afli Symbol of \( P \) is \( [4][3] \ldots [3] \) if \( d \geq 3 \).

Proof: Let the Schl"afli Symbol be \( [q_1] \ldots [q_d] \ldots [q_{d-1}] \). Then, if \( \dim(X) = 0 \), the Schl"afli Symbol of \( ([X],[R]) \) is \( [q_1] \ldots [q_{d-1}] \), by Theorem 3.4.12. But \( ([X],[R]) \) is a \( (d-1) \)-simplex (Lemma 4.2.11), and so by Theorem 4.1.6, we have \( q_1 = \ldots = q_{d-1} = 3 \). To find \( q_1 \), let \([X] \in P \) have dimension 2. Then \( ([X],[X]) \) will be a 1-cycle, and so will have \( q_1 \) elements of dimension 1. But \( ([X],[X]) \) is also isomorphic to a 2-cube, and thus has \( 2^{d-1}(1) = 4 \) such elements, making \( q_1 = 4 \) as required.

It may also be shown that the 2-cube has Schl"afli Symbol \( [4] \), and the 2-halfcube has Schl"afli Symbol \( [2] \).

Note that except for the case \( d = 2 \), the \( d \)-halfcubes are indecomposable (Theorem 3.4.14). In Theorem 3.5.9, we saw that given any geometric polytope \( P \), its combinatorial counterpart was indecomposable. Our next result, however, shows that the converse of that theorem does not hold—that is, there exist indecomposable incidence polytopes which are not the combinatorial counterpart of any geometric polytope.

Theorem 4.2.17: For \( d \geq 2 \), there is no geometric polytope whose combinatorial counterpart is a \( d \)-halfcube.

Proof: Let \( \mathcal{L} \) be a \( d \)-halfcube, and let \([A] \in \mathcal{L}_{d-1} \). Then \([A],[A]\) is a \( (d-1) \)-cube, and so has \( 2^{d-1}(1) = 2^{d-1} \) vertices. However, \( \mathcal{L} \) itself has \( |\mathcal{L}| = \frac{1}{2} 2^{d-1}(d) = 2^{d-1} \) vertices also. Now any vertex of the facet \([A],[A]\) is also a vertex of \( \mathcal{L} \), so the vertex set of any facet of \( \mathcal{L} \) coincides with the vertex set of the entire halfcube. Now, let \( P \) be a geometric polytope whose combinatorial counterpart is a halfcube. Then \( P \) will be the convex hull of its vertices (Theorem 2.3.8). Now if \( F \) is any facet of \( P \), then \( F \) is the convex hull of its vertices (Theorem 2.3.8 again, and Theorem 2.3.4). But as we have just seen, these are the same set of vertices, requiring \( P \) to equal \( F \), which is impossible (for if \( P \) has dimension \( d \), then \( P \) has dimension \( d-1 \)).

So geometric polytopes do not exactly correspond with Euler incidence polytopes, even when the problem of decomposability has been taken care of.

Note that this shows (for \( d \geq 2 \)) that cubes and halfcubes are not isomorphic\(^\text{35}\).

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4.3 The Crosses and Halfcrosses

Towards the end of \( \S 4.1 \), we saw that the combinatorial \( d \)-simplex was self-dual (Corollary 4.1.7). No similar theorem was proved in \( \S 4.2 \), for the simple reason that the cube and halfcube are not self-dual. It is their duals that we consider here.

Definition: The dual of a combinatorial \( d \)-cube shall be called a (combinatorial) \( d \)-cross, and that of a \( d \)-halfcube, a \( d \)-halfcross.

Most of the results from the previous two sections yield immediate corollaries here, by virtue of what we know about the duals of polytopes (see \( \S 3.3.3 \)).

Corollary 4.3.1: The proper cofaces of a \( d \)-cross or a \( d \)-halfcross are crosses. All other proper sections are simplices.

Proof: If \((X,Y)\) is a section of the dual of a polytope, then it is dual to the section \((Y,X)\) of the original polytope (Lemma 3.3.14). Therefore, if \((X,Y)\) is a proper coface of a cross or halfcross, it is the dual of a proper face of a cube or halfcube, that is, it is the dual of a cube. The coface itself is therefore a cross. Any other proper section of a cube or halfcube is a simplex. Since simplices are self-dual (Corollary 4.1.7), it follows that any proper section that is not a coface of a cross or halfcross is a simplex (being dual to a simplex—see Theorems 4.2.11 and the notes following it, and Lemma 3.3.14).

Corollary 4.3.2: The \( d \)-cross and, for \( d \geq 2 \) the \( d \)-halfcross, are combinatorially regular incidence polytopes.

Proof: Under these conditions, the \( d \)-cube and \( d \)-halfcube are combinatorially regular incidence polytopes (see Theorem 4.2.14), and hence their duals are also (Theorems 3.3.15 and 3.4.7).

Corollary 4.3.3: The \( d \)-cross is Euler for all \( d \).

Proof: This follows because the \( d \)-cube is Euler (that is, \( d \)-Euler) for all \( d \) (see Theorem 4.2.15), and because if an incidence polytope is \( k \)-Euler, so is its dual (Corollary 3.3.17).

Corollary 4.3.4: Let \( d \geq 2 \). The \( d \)-halfcross is sub-Euler for all such \( d \), and is Euler if and only if \( d \) is even.

Proof: This follows from Theorem 4.2.15 and Corollary 3.3.17.
Corollary 4.3.5: The Schläfli Symbols of the d-cross and the d-halfcross are \(\{3\ldots 3 \{4\} \ldots \{4\}\} \) if \(d > 2\), and are \(\{4\}\) and \(\{2\}\) respectively if \(d = 2\).

Proof: This follows from the fact that the Schläfli Symbol of the dual of a polytope is just the reverse of the Schläfli Symbol of the polytope itself (Theorem 3.4.13), and from what we know of the Schläfli Symbols of the cube and halfcube (Theorem 4.2.16).

Corollary 4.3.6: Let \(d \geq 2\). There is no geometric polytope whose combinatorial counterpart is the d-halfcross.

Proof: If there were, there would have to be one whose combinatorial counterpart was the d-halfcube (Theorem 3.5.7 and see also §2.3.3), and Theorem 4.2.17 has already told us that this is not so.

Theorem 4.3.7: A geometric polytope whose combinatorial counterpart is the combinatorial d-cross is the geometric d-cross.

Proof: Use Theorems 4.2.1, 3.5.7, and Table 2.5.4.

4.4 The Lattices

These are polytopes with no combinatorial counterpart, and have Schläfli Symbol \(\{4\ldots 4 \{3\} \ldots \{3\}\} \). We shall see, in Chapter 6, that there are also polytopes with this Schläfli Symbol which are not lattices.

Let \(Z = Z^{d-1}\) be generated by a “basis” \(T = \{e_1, \ldots, e_{d-1}\} \). For any subset \(I\) of \(T\), let \(S_I = \{ \sum_{e_i \in I} e_i : J \subseteq T\} \). Thus, for example,

\[ S_{\{e_1\}} = \{0, e_1\}, \]

and

\[ S_{\{e_1, e_2\}} = \{0, e_1, e_2, e_1+e_2\}. \]

We define \(S_I\) to equal \(\emptyset\). Note that \(S_T\) is the set of all elements of \(Z\) whose coordinates consist solely of 0’s and 1’s. We define the universal d-lattice \(\mathcal{Z}\) to be the set \(\{0, Z\} \cup \{x + S_I : x \in Z, I \subseteq T\}\), equipped with the partial order \(\leq\).

Example: Let \(d = 3\). A typical element of \(\mathcal{Z}\) might be

\[ \frac{3}{2} + S_{\{e_1\}} = \left\{ \frac{3}{2}, \frac{4}{2} \right\}, \]

or

\[ \frac{3}{2} + S_{\{e_1, e_2\}} = \left\{ \frac{3}{2}, \frac{4}{2}, \frac{3}{2}, \frac{4}{2} \right\}. \]

Note then that for all \(A \in \mathcal{Z}\), we have \(\emptyset \subseteq A \subseteq Z\). Thus we have

Lemma 4.4.1: The universal d-lattice \(\mathcal{Z}\), with the relation \(\leq\), is a poset with unique minimal and maximal elements \(\emptyset\) and \(Z\) respectively.

It would in fact be possible to prove that the universal lattices are infinite combinatorially regular polytopes with Schläfli Symbol \(\{4\ldots 4 \{3\} \ldots \{3\}\} \). However, it is more economical to first define a more general class of partially ordered sets, of which the universal lattice is a special case.

Note that as a group, \(Z\) acts on \(\mathcal{Z}\) via addition, with \((A) = h + A\). Now let \(K\) be a subgroup of \(Z\), and define \(\mathcal{Y} = \{A: A \in \mathcal{Z}\}\) to be the set of orbits of \(K\) as it acts on \(\mathcal{Z}\), that is \(\mathcal{Y} = \{A = k + A : k \in K\}\). We define \(\mathcal{Y} \leq \mathcal{Z}\) if and only if \(A \subseteq B\) for some \(B \in \mathcal{Z}\). Note that if \(K = \{0\}\), then \(\mathcal{Y} \cong \mathcal{Z}\).

Lemma 4.4.2: \(\leq\) is a well-defined partial order of \(\mathcal{Y}\), and under it, \(\mathcal{Y}\) has unique maximal and minimal elements \(\{Z\}\) and \(\{0\}\) respectively.

Proof: We show first that \(\leq\) is well defined. Let \(A, B \in \mathcal{Y}\) be such that \(A \subseteq B\) for some \(B \in \mathcal{Z}\), and let \(A' = A[4].\) Then \(A' = h + A,\) and \(B' = k + B,\) for some \(h, k \in K.\) Letting \(B'' = h + k + B,\) we deduce that \(A' \subseteq B''\) for some \(B'' \in \mathcal{Z}\). This shows that, indeed, \(\leq\) is well defined. Since \(A \subseteq Z\), we also know that \(\mathcal{Y} \leq \mathcal{Z}\), making \(\leq\) reflexive. It is almost as easy to show that it is transitive, not showing it to be antisymmetric takes a little more work. Suppose then that \(\mathcal{Y} \leq \mathcal{B}\) and \(\mathcal{B} \leq \mathcal{A}\), so there exists \(h, h' \in K\) such that \(A \subseteq h + B\) and \(B \subseteq h' + A.\) If \(A = \emptyset\) or \(A = Z,\) it follows easily that \(A = B.\) Otherwise, we may still conclude that \(A \subseteq (h + h') + A,\) whence also \(\mathcal{A} \subseteq (h + h') + A.\) Letting \(A' = a + S_I,\) for some \(a \in Z,\) and \(I \subseteq T,\) we obtain \(A' \subseteq (h + h') + S_I.\) Now \(0 \in S_I,\) hence in fact \(-2(h + h') \in S_T.\) But \(0\) is the only element of \(S_I\) whose coordinates are all even, so it must be that \(-2(h + h') = 0,\) that is, \(h = -h'.\) Thus \(A \subseteq h + B,\) and \(h + B \subseteq h + h' + A = A,\) so \(A = h + B\) and \(\mathcal{Y} \leq \mathcal{Z}\) as required. Finally, note that \(\emptyset \leq \mathcal{Y} \leq \mathcal{Z}\), so \(\{Z\}\) and \(\{0\}\) are as claimed.

In actual fact, it is not so useful to examine the \(\mathcal{Y}\) for arbitrary subgroups \(K\) of \(Z\). Let us restrict attention to those subgroups \(K\) satisfying the following condition.

\((K1)\) For any \(x, y \in S_T,\) \(x - y \in K\) if and only if \(x = y.\)

That is, \(K \cap (S_T - S_T) = \{0\}.\) Essentially, \(K1\) ensures that no nonzero element of \(K\) has coordinates which are all 0’s, 1’s, and -1’s. We could say (very loosely) that it ensures that the elements of \(K\) don’t crowd too closely together. Note that the trivial subgroup \(K = \{0\}\) satisfies \(K1.\)

Given \(\mathcal{Y}\) defined from a subgroup \(K\) satisfying \(K1,\) we call it a d-lattice. Note that the technique used to construct \(\mathcal{Y}\) from \(\mathcal{Z}\) is similar to that used to
construct the halfspaces from the cubes — roughly, a group was found which acted on the original polytope in a 'nice' way, and then the orbits of that group action were collected together to form the new partially ordered set. The group was chosen in such a way that the new partially ordered set is in fact a polytope. This construction will in fact be generalised when, in Chapter 5, we examine the concept of a quotients polytope.

Let us discover the chains and flags of $\mathcal{Y}$. This is easier if we characterise the chains of $\mathcal{Y}$ first. Recall the example given earlier, how it was noted that $(\frac{3}{2}) + S(e_1) \subseteq \left( \frac{1}{2} \right) + S(e_1, e_2)$, in the universal 3-lattice. Observe that $(e_1) \subseteq (e_1, e_2)$ and that $(\frac{3}{2}) + (\frac{3}{1}) \in S(e_1, e_2) \setminus \{e_1\}$. In general, we have the following result.

**Lemma 4.4.3:** Let $a + S_I \subseteq b + S_J$ if and only if $a + S_I$ is in fact an element of $S_J$. Thus $a + S_I$ is an arbitrary element of $a + S_I$, then it follows that $a + S_I \subseteq b + S_J$, as required.

Now, assume that $a + S_I \subseteq b + S_J$. It follows, in particular, that $a + b \in S_J$, so $a + b \in S_J$. Also, for any $e_i \in I$, we have $a + e_i \in b + S_J$, so $a + b + e_i \in S_J$. Since $a + b \in S_J$, all its coordinates are either 0's or 1's. If its $i$th coordinate is 1, then the $i$th coordinate of $a + b + e_i$ must be 2. This is impossible, since $a + b + e_i \in S_J$. Thus the $i$th coordinate of $a + b$ is 0. If we write $a + b = \sum e_i$, where $J'$ is some subset of $J$, then $e_i \in J'$. This will be the case for any element $e_i$ of $I$. So in fact $J'$ is a subset of $J\setminus I$, and $a + b \in S_{J\setminus I}$ as claimed. Also, note that $a + b + e_i = \sum e_i$ (where $J'$ is a subset of $J'$), then $e_i \in J'$ (since in fact $J' = J' \cup \{e_i\}$), and so $e_i \notin I$. It follows that $I \subseteq J$, as claimed. This completes the proof.

Now, for elements of $\mathcal{Y}$, we have a slightly different condition.

**Lemma 4.4.4:** Let $a + S_I \subseteq b + S_J$ if and only if $I \subseteq J$ and $a - b \in S_{J \setminus I} + K$.

**Proof:** Let $a + S_I \subseteq b + S_J$, then $a + S_I \subseteq b + S_J$, that is, and only if there exists $h \in K$ such that $I \subseteq J$ and $a - b - h \in S_{J \setminus I}$, that is, $a - b \in S_{J \setminus I} + h$. This is equivalent to saying $I \subseteq J$ and $a - b \in S_{J \setminus I} + K$, as required.

Note that if $a + S_I = b + S_J$, then $[a + S_I] \subseteq [b + S_J]$ and $[b + S_J] \subseteq [a + S_I]$. Thus in particular, $I \subseteq J$ and $J \subseteq I$, so $I = J$. Also, it will be the case that $a - b \in S_{J \setminus I} + K = S_K + K = \{0\} + K = K$, so $a = b + h$ for some $h \in K$. If we know further that $a + S_I = b + S_I$, then of course $a = b$.

The chains of $\mathcal{Y}$ may now be described.

**Lemma 4.4.5:** Any chain $C$ of $\mathcal{Y}$ may be written in the form $D \cup E$, where $D \subseteq \{0, 1, \ldots, d - 1\}$, and $E = \{i + S_I : i \in J\}$, where $J \subseteq \{0, 1, \ldots, d - 1\}$, and for all $i \in I$, $i \leq T$ and $|I| = |J| = 1$, and whenever $i < j$, we have $i \leq j$ and $z_i - z_j \in S_{J \setminus I}$.

**Proof:** This follows as a corollary to Lemma 4.4.3.

So any chain may be written in the form given. Furthermore, if we are given $J \subseteq \{0, 1, \ldots, d - 1\}$ and $D \subseteq \{0, 1, \ldots, d - 1\}$, and if $I_i$ and $z_i$ are defined for all $i \in J$ and satisfy the conditions given, it is not hard to show that $C = D \cup \{i + S_I : i \in J\}$ is totally ordered, thus we have characterised the chains of $\mathcal{Y}$. The characterisation may not be particularly neat, but we only really need it for the next two results.

**Lemma 4.4.6:** The flags of $\mathcal{Y}$ are just those chains with $D \subseteq \{0, 1, \ldots, d - 1\}$ and $J \subseteq \{0, 1, \ldots, d - 1\}$.

**Proof:** The proof of this is analogous to that of Lemma 4.2.8, i.e., only an outline is given here. Given a chain $C = D \cup \{i + S_I : i \in J\}$ we shall construct $F = \{0, 1, \ldots, d - 1\}$ containing $C$, defining $I_j$ and $z_j$ for $j \neq J$ in such a way that $F$ retains the properties listed is the statement of the previous Lemma. The $I_j$ for $j \neq J$ are easy enough to construct. We may construct the $z_j$ in the case where there exists $l, k \in J$ with $l < j < k$ (but there does not exist $i \in J$ with $l < i < k$) as follows. Let $I' \subseteq \{k \} \setminus I$, be such that

$$z_{I'} = \sum_{e_i \in I'} e_i.$$

Then define

$$z_j = z_I - \sum_{e_i \in I' \setminus I_j} e_i.$$
This $z_j$ will then have the required properties, since firstly $x_1-z_j \in S_T(x_j)$, and $I \cap l_j \subseteq [x_1, z_j)$, and $I \cap l_j \subseteq [x_1, z_j)$, so $z_1-z_j \in S_T(x_j)$, as required. Similarly, it may be shown that $x_1 - z_k \in S_T(k)$, whence (since $I \cap l_j \subseteq [x_1, z_j)$, and $I \cap l_j \subseteq [x_1, z_j)$, we have $z_1 - z_k \in S_T(k)$. We may similarly construct suitable $x_j$ for other $j \neq I$. Thus $\mathcal{P} = ([0], [2)), [x_1, z_1) : 0 \leq i \leq d - 1$), which contains $C$, is indeed a chain, and clearly there can be no larger chain containing it. Thus $C$ has no proper superchains if and only if $D = ([0], [2))$ and $J = (0, 1, \ldots, d - 1)$.\]

Lemma 4.4.7: A d-lattice $\mathcal{W}$ satisfies 12 and has dimension d.

Proof: The above proof actually showed how to construct a flag containing any given chain. Now if $F$ is a flag, then $|F|$ will equal $|([0], [2))| + |([x_1, z_1)| : 0 \leq i \leq d - 1)$, which will equal $2 + d$ as required.\]

Like the simplices, the lattices are self-dual.

Theorem 4.4.8: There is a structure-reversing bijection from $\mathcal{W}$ to itself.

Proof: Let $\phi: \mathcal{W} \to \mathcal{W}$ be defined via $[x + S_T] \psi = [-z + S_T] \psi$. We show first that $\phi$ is well-defined. Now, if $[x + S_T] = [x' + S_T]$, then by the note following Lemma 4.4.4, we have $I = I'$ and $x = z' + h$ for some $h \in K$. Now $[x + S_T] = [-z + S_T] = [-z' + S_T] = [-x' + S_T] = [z + S_T]$, as required. Now $\phi$ is a bijection, being its own inverse. It is also structure reversing, for if $[x + S_T] \subseteq [z + S_T]$, then $I \subseteq I'$ and $x' \in S_T \cup I$. Therefore, $T \cap I \subseteq T \cap I'$, and since $I \cap (T \cap I') = [-2 + S_T] \subseteq [-2 + S_T]$, as required. Thus $\phi$ is indeed a structure reversing bijection from $\mathcal{W}$ to itself.\]

This result will be helpful in classifying the sections of a lattice. First of all, we classify the facets.

Lemma 4.4.9: Let $[x + S_T] \in \mathcal{W}$. Then $([0], [x + S_T])$ is isomorphic to a $(d - 1)$-cube.

Proof: (Outline only). Let $T = \{e_1, \ldots, e_d\}$, and let $\psi: Z^{d-1} \to Z^{d-1}$ be defined by $e_i \mapsto (z_{i+1} \mod 2)$. We extend $\psi$ to a map from subsets of $Z^{d-1}$ to subsets of $Z^{d-1}$ in the obvious way. Note then that $(A + B) \psi = A \psi + B \psi$, and that for any $I, (S_T) \psi$ is a subset of $Z^{d-1}$ spanned by a subset $T \psi$ of the spanning set $T \psi$ of $Z^{d-1}$. Therefore, for any $x' + S_T \in \mathcal{W}$, it will be the case that $x' + S_T \psi$ is a cotet of such a subspace, that is, it will be an element of the cube. Now, we define a map $\phi$ from $([0], [x + S_T]$ to the cube as follows.

For any $[x' + S_T] \subseteq [x + S_T]$, it must be that $x' - z = h + h$ for some $h \in K$ and $x \in x + S_T$. This $h$ and $s$ can be shown to be unique, given $x$ and $x'$. We then set $[x' + S_T] \phi = (s + S_T) \psi = (s + T) \psi$, which, if we are willing to stretch the notion of that far, may be said to equal $\phi \psi + (s + T) \psi$. It is important to remember that $\phi \psi = \lim(\psi)$ when working over $Z_2$. Now $\psi$ is well defined, since it can be shown that $\psi$ depends neither on $x$ nor on $x'$, but only on $[x + S_T]$ and $[x' + S_T]$.

To show that it is onto, note that any element of the $(d-1)$-cube may be written $c + lin(\mathcal{I})$, where $\mathcal{I}$ is a subset of the basis $T \psi$ used for the cube. Note then that $\mathcal{I} = \phi \psi$ for some $I \subseteq T$, and that $\lim(\mathcal{I}) = (S_T) \psi$. It is possible to choose $c$ such that $c \in \lim(\mathcal{I}) \psi = (S_T) \psi$, and then there will exist $s \in S_T$ such that $s \psi = c$. Then $(x + S_T) \psi = c + lin(\mathcal{I})$, so $x + s \psi \in X \psi + lin(\mathcal{I})$. It may then be checked that $(x + S_T)$ is an element of $([0], [x + S_T])$, so there exists $x + s \psi \in ([0], [x + S_T]$ with $x + s \psi = c + lin(\mathcal{I})$. In fact we have just specified how to obtain an inverse for $\phi$. Since $\phi$ is invertible, it follows that it is one to one. We shall prove in more detail that $\psi$ is structure-preserving. To this end, let $[a + S_T], [b + S_T] \subseteq [x + S_T]$ and assume without loss that $a = x + s$ and $b = x + s'$, where $s \in S_T$ and $s' \in S_T$. Now $[a + S_T] \subseteq [b + S_T]$ if and only if $I \subseteq J$ and $a - b \in S_T$, that is, if $a + S_T \subseteq [b + S_T]$. On the other hand, $[a + S_T] \subseteq [b + S_T]$ if and only if $a + S_T \subseteq [b + S_T]$, that is, if and only if $I \subseteq J$. Since $J$ and $J'$ are disjoint subsets of $S_T$, then certainly $a + S_T \subseteq [b + S_T]$ if and only if $a + S_T \subseteq [b + S_T]$. Conversely, let $a + S_T \subseteq [b + S_T]$. Since $a + S_T \subseteq [b + S_T]$, all the coordinates of $s - s'$ will be either 0 or 1. We may write

\[s - s' = \left( \sum_{e_i \in \mathcal{L}} e_i \right) - \left( \sum_{e_i \in \mathcal{L}} e_i \right).

Where $L_-$ and $L_+$ are disjoint subsets of $T$. Now $L_- \subseteq T \setminus I$, since we cannot get the $j$-th coordinate of $s - s'$ to be $1$ unless the $j$-th coordinate of $s'$ is 1. Likewise, $L_+ \subseteq T \setminus I$. However, consider $(s - s') \psi$, recalling that when working over $Z_2$, we have $-a = a$.

\[(s - s') \psi = \left( \sum_{e_i \in \mathcal{L}_+} e_i \psi \right) - \left( \sum_{e_i \in \mathcal{L}_-} e_i \psi \right) = \left( \sum_{e_i \in \mathcal{L}_+} e_i \psi \right) + \left( \sum_{e_i \in \mathcal{L}_-} e_i \psi \right) = \left( \sum_{e_i \in \mathcal{L}_-} e_i \psi \right).

That is, regarding $Z^{d-1}$ as the quotient of $Z^{d-1}$ by $Z^{d-1}$, we let $x \psi$ be the coset $x + Z^{d-1}$. Each such coset has a (unique) "canonical" element $x'$ whose ith coordinate $x'_i$ is 0 if $x_i$ is even, and 1 if $x_i$ is odd.
since $L_+$ and $L_-$ are disjoint. However, $(s - s')\psi \in \text{lin}(J\psi)$, so $L_+ \cup L_- \subseteq J$. It follows that $L_-$ is empty (since only the empty set can simultaneously be a subset of both $J$ and $T\setminus J$), and that $L_+ \subseteq J \cap I$. Thus $s - s' = \left( \sum_{e_t \in E_+} e_t \right) \in S_{J \cap I}$, as required. Thus $\phi$ preserves the partial orders, and hence $(\emptyset, [z + S_{\gamma}])$ is indeed isomorphic to a $(d-1)$-cube.

**Theorem 4.4.10:** For $i < d$, the $i$-faces of a lattice are $i$-cubes, the $i$-cofaces are $i$-crosses, and all other $i$-sections are $i$-simplices.

**Proof:** We have seen that the $(d-1)$-faces (the facets) are $(d-1)$-cubes. It follows that a $(d-1)$-coface (that is, a vertex figure) must be a $(d-1)$-cross, by duality (Theorems 4.4.8 and 3.3.14). Since any other proper section is a section of either a facet or a vertex figure, Corollary 4.3.1 and the notes following 4.2.11 give the result.

We can use this to conclude a number of useful facts about the lattices.

**Lemma 4.4.11:** The lattices are strongly connected.

**Proof:** Let $\mathcal{F}$ be a lattice. All its proper sections, being cubes, crosses and simplices, are strongly, and hence also weakly connected (See Lemmas 4.2.12 and 4.1.3, and note that crosses satisfy 13 because (by Corollary 4.3.2) they are incidence polytopes). We only need to show that $\mathcal{F}$ itself is weakly connected to show that all the sections of $\mathcal{F}$ are weakly connected, yielding the desired result. To show that $\mathcal{F}$ is weakly connected, let $[a + S_{\gamma}]$ and $[a' + S_{\gamma}]$ be elements of $\mathcal{F}$, and let $a - a' = u = \sum_{i=1}^{d-1} u_i e_i$. Certainly, we can construct a sequence $0 = v_0, v_1, \ldots, v_m = u$ such that for all $j$, there exists $i_j$ such that $v_{j+1} - v_j = \pm e_{i_j}$.

For any $0 \leq j \leq m$, let $Y_j = a + v_j + S_{\gamma}$, and for $0 \leq j \leq m - 1$, let $Y_j = X_j \cap X_{j+1}$. Note that

$$Y_j = (a + v_j + S_{\gamma}) \cap (a + v_{j+1} + S_{\gamma}) = a + v_j + S_{\gamma} \cap ([a+e_{i_j} + S_{\gamma}]) = a + v_j + S_{\gamma}(e_{i_j}),$$

where $v'_{j+1} - v_j$ or $v_{j+1} + e_{i_j}$. It follows that

$$[a + S_{\gamma}], [X_0], [Y_0], [X_1], \ldots, [Y_{m-1}], [X_m], [a' + S_{\gamma}]$$

is a sequence of the required form.

**Lemma 4.4.12:** Let $\mathcal{F}$ be a d-lattice, $d \geq 2$. Let $[A], [B] \in \mathcal{F}$ satisfy $[A] \subseteq [B]$ and $\dim [A] + 1 = \dim [B] - 1 = i$. Then the set $\{X \in \mathcal{F} : [A] \subset [X] \subset [B]\}$ has exactly two elements.

**Proof:** Suppose it has $k$ elements. Then the 1-section $([A], [B])$ has exactly $k + 2$. But Theorem 4.4.10 tells us that this section is either a 1-cube, a 1-cross, or a 1-simplex, each of which has exactly 4 elements (see the notes preceding Theorem 4.1.5 and Lemma 4.2.15). Thus $k + 2 = 4$, and $k = 2$ as required.

Finally, we can show when $\mathcal{F}$ is a polytope.

**Theorem 4.4.13:** Let $d \geq 2$, and let our subgroup $K$ satisfy $K$. Then the d-lattice $\mathcal{F}$ is a combinatorially regular incidence polytope.

**Proof:** By Lemmas 4.4.2, 4.4.7, 4.4.11 and 4.4.12, $\mathcal{F}$ is a d-incidence polytope. It is also combinatorially regular, since if $\mathcal{F} = ([A], [B])$ is a section of $\mathcal{F}$, then the 'isomorphism type' of $\mathcal{F}$ depends only on whether or not $[A] = [B]$ and whether or not $[B] = [Z]$ (see Theorem 4.4.10). That is, it depends only on the dimensions of $[A]$ and $[B]$, and not of $[A]$ and $[B]$ themselves.

**Theorem 4.4.14:** If $d \geq 3$, then $\mathcal{F}$ has Schl"afli Symbol $4\{3\} \ldots \{3\}$.

**Proof:** Let its Schl"afli Symbol be $\{q_1, \ldots, q_{d-2}\}$. Now its facets are $(d-1)$-cubes, so have Schl"afli Symbol $4\{3\} \ldots \{3\}$ (Theorem 4.2.16), so we have $q_1 = 4$, $q_{d-3} = 3$, and $q_1 = \ldots = q_{d-2} = 3$. But since the lattice is dual to itself (Theorem 4.4.8), its Schl"afli Symbol must also be $\{q_1, \ldots, q_1\}$ (see Theorem 3.4.13). Thus $q_{d-1} = q_1 = 4$, and the Schl"afli Symbol of $\mathcal{F}$ is as claimed.

Let us also attempt to gather some information about the sizes of the $\mathcal{F}_i$.

**Lemma 4.4.15:** $|\mathcal{F}_i| = \binom{d-1}{i}|Z : K|$, for $0 \leq i \leq d - 1$.

**Proof:** Let $i$ be as stated, let $I \subseteq T$ be such that $|I| = i$, and let $Y_I = \{[a + S_{\gamma}] : a \in Z\}$. Note that $\mathcal{F}_i$ is the disjoint union of all such $Y_I$ for $|I| = i$. Now $Z$ will act transitively on $Y_I$ via the action $x + [a + S_{\gamma}] = [x + a + S_{\gamma}]$. For any $[A] \in Y_I$, the stabilizer $Z_{\gamma}(A)$ of the action will equal $\{x \in Z : x + [A] = [A]\}$, which equals $K$. A theorem sometimes called the "orbit-stabilizer theorem" (§1 Thm. 3.2) then tells us that the length of the single orbit is $|Z : K|$, the index of $K$ in $Z$, so $|Y_I| = |Z : K|$. Now given $i$, there are $\binom{d-1}{i}$ such $Y_I$, hence $\mathcal{F}_i$ has $\binom{d-1}{i}|Z : K|$ elements, as claimed.

This result allows us to determine when, if ever, the lattices are Eulerian.
Theorem 4.4.16: The d-lattice $\mathfrak{F}$ is sub-Euler, and is Euler if and only if $[Z : K]$ is finite and $d$ is even.

Proof: $\mathfrak{F}$ is sub-Euler, since all its proper sections are cubes, facets and simplices, and so satisfy Euler's relation (see Theorems 4.1.5, 4.2.15 and 4.3.3). Thus $\mathfrak{F}$ will be Euler if and only if $\mathfrak{F}_k$ itself also satisfies this relation, that is, if and only if the $\mathfrak{F}_k$ are all finite and

$$\sum_{i=1}^{d} (-1)^i |\mathfrak{F}_i| = 0.$$ 

By the previous result, the $\mathfrak{F}_i$ will be finite if and only if $[Z : K]$ is finite. Also, in this circumstance we can perform the following calculations:

$$\sum_{i=1}^{d} (-1)^i |\mathfrak{F}_i| = (-1)^{-1} |\mathfrak{F}_{-1}| + \sum_{i=0}^{d-1} (-1)^i [Z : K] \left( \begin{array}{c} d-1 \\ i \end{array} \right) + (-1)^d |\mathfrak{F}_d|$$

$$= -1 + [Z : K] \sum_{i=0}^{d-1} (-1)^i \left( \begin{array}{c} d-1 \\ i \end{array} \right) + (-1)^d$$

$$= -1 + (-1)^d + [Z : K]((1 + (-1)^d)^{d-1}$$

$$= -1 + (-1)^d + 0$$

which will equal zero if and only if $(-1)^d = 1$, that is, $d$ is even.

4.5 Sporadic Examples

Here we shall see some examples of $d$-polytopes for particular small $d$.

4.5.1 The 1-polytope:

In Theorem 3.2.2, an example was given of a 1-incidence polytope. It was proved in Theorem 3.3.9 and the notes following it that this is the only 1-incidence polytope up to isomorphism, and in §3.4.1 it was noted that this polytope is also regular (and hence combinatorially regular). The 1-polytope is 1-Euler (being a polytope, rather than a complex), and so is Euler. See Figure 4.5.1.

4.5.2 The 2-polytopes:

In the notes preceding Theorem 3.2.3, the $n$-cycles were defined (for $n \geq 2$). The theorem itself states that for finite $n$, these are Euler incidence polytopes, and Theorem 3.3.10 tells us that every 2-incidence polytope is isomorphic to an $n$-cycle, for some $n$. Furthermore, in Theorem 3.4.5 it was shown that the $n$-cycles, and hence all 2-polytopes, are are combinatorially regular. The fact that all 2-polytopes are 2-cycles was leaned upon when the Schlafli Symbol of a combinatorially regular incidence polytope was defined in §3.4.3. See Figure 3.2.1 for a depiction of a 7-cycle.

4.5.3 3- and 4-polytopes:

Here, we shall draw on the fact that any regular geometric polytope yields a regular incidence polytope via its combinatorial counterpart (Theorems 3.5.5 and 3.5.8). Recall (from §2.5.2) that there were five regular geometric 3-polytopes, namely the tetrahedron (or geometric 3-simplex), the (geometric) cube and the octahedron (or geometric 3-cross), as well as the dodecahedron and the icosahedron. These yield five combinatorially regular 3-incidence polytopes, all of which are Euler, and three of which have already been described in this chapter\(^\text{17}\). It will be seen in Chapter 6 that these are the only combinatorially regular Euler 3-incidence polytopes, although there are many combinatorially regular 3-incidence polytopes which are only sub-Euler.

Recall also (§2.5.3) that there were six regular geometric 4-polytopes, with Schlafli Symbols \{3\{3\}, 3\{3\}4\}, \{4\{3\}\}, \{3\{4\}\}, \{3\{5\}\} and \{5\{3\}\}. These provide six examples of combinatorially regular Euler 4-incidence polytopes (including the 4-simplex, the 4-cross and the 4-cube) via their combinatorial counterparts. Unlike the case of the 3-polytopes, this construction does not prove exhaustive. In §4.4, we examined the lattices – the 4-lattices provide an infinite number of examples of Euler 4-incidence polytopes, and so certainly cannot all (if any) be included in this list of six! Also, in §4.2 and §4.3 we encountered the halfcube and the halfcross, which by Theorem 4.2.17 and Corollary 4.3.6 are not combinatorial counterparts of any geometric polytopes. Many more such examples will be uncovered in Chapter 6, where it will also be shown that if any combinatorially regular 4-incidence polytope is sub-Euler, it is also Euler (Theorem 6.3.8).

Now, the halfcross was defined as the dual of a halfcube, and the halfcube was defined from the cube $\mathfrak{A}$ by "identifying" certain pairs of elements of $\mathfrak{A}$, specifically pairs $\{a, ar\}$ where $r$ was a particular automorphism of $\mathfrak{A}$. In

\(^{17}\) These three are combinatorial counterparts of the tetrahedron, the cube, and the octahedron, which are isomorphic (respectively) to the combinatorial 3-simplex, 3-cube, and 3-cross. See Theorems 4.1.1, 4.2.1 and 4.3.7.
fact, \( r \) was chosen so that for each \( a \in \mathcal{A} \), \( a \) and \( ar \) would be "opposite" each other. In a similar way, we could define a "half-icosahedron" and a "half-dodecahedron", as well as a "half-24-cell", with Schl"afli Symbol \( \{3\{4\}3\} \) but with only 12 vertices instead of 24, and also a "half-120-cell" and a "half-600-cell", with Schl"afli Symbols \( \{5\}3\}3\} \) and \( \{3\{5\}6\} \) respectively\(^{38}\).

Going back to the cubes, note that \( r^2 \) is the identity automorphism of \( \mathcal{A} \), and so the sets of the form \( \{a, ar\} \) are just the orbits of the group \( (r) \leq \text{Aut}(\mathcal{A}) \) as it acts on \( \mathcal{A} \). This then begs certain questions - what subgroups of \( \text{Aut}(\mathcal{A}) \) are suitable for such a construction? Can all polytopes be constructed in such a way? The next chapter will answer these questions. The techniques developed there will bring all the "half-polytopes" mentioned above under one umbrella, and also uncover a great many other incidence polytopes which we have missed here.

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38 Incidentally, it is not possible to construct a "half-simplex" in this way, because the faces of a simplex cannot be paired up in a suitable way. This is not a trivial assertion, and it will not be until §6.1.1 that a proof of it shall be given.

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CHAPTER 5

Groups and Polytopes

"You boil it in sawdust: you salt it in glue:
You condense it with locusts and tape:
Still keeping one principal object in view --
To preserve its symmetrical shape."

Lewis Carroll, 1876

5.1 Coxeter Groups

5.1.1 Introduction:

The Coxeter groups arise naturally in many branches of mathematics, particularly Geometry, and Lie group theory. Since they are, in a sense, very "natural" groups, they have often been studied in the past by mathematicians investigating the geometrical objects on which they act - for example, while studying the symmetries of a regular solid. Their usefulness in the classification of the combinatorially regular Euler polytopes is what motivates their introduction here. Possibly the best reference on Coxeter groups is [5], written by Bourbaki\(^{39}\). There are also treatments in English, for example the book "Reflection Groups and Coxeter Groups" by J. Humphreys [26]. The treatment here will be somewhat informal.

Definition: Let \( S = \{s_0, \ldots, s_{d-1}\} \) be any set, and let \( M = (m_{ij}) \) be a symmetric \( d \times d \) matrix, with \( i \) and \( j \) ranging from 0 to \( d - 1 \). Let the matrix further satisfy \( m_{ij} \in \mathbb{Z}^+ \cup \{0\} \), \( m_{ij} \geq 2 \) for all \( i, j \) with \( i \neq j \), and \( m_{ii} = 1 \). We let \( W \) be the group with presentation \( \langle s_0, \ldots, s_{d-1}; \ (s_is_j)^{m_{ij}} = 1, m_{ij} \neq \infty \rangle \).

Then \( (W, S) \) is called a Coxeter system, and \( W \) itself is called a Coxeter group\(^{40}\).

Note that for each \( i, s_i^2 = 1 \). The matrix \( M \) is called the Coxeter matrix for \( W \).

The Coxeter matrix completely determines the Coxeter group, but is not the best way of representing such a group, since it is possible to have two different matrices yielding isomorphic groups. There exists, however, a graphical way of representing a Coxeter group which does not suffer as much from this problem. Let \( (W, S) \) be a Coxeter system with Coxeter matrix \( M \). We define an edge-labelled graph on the \( n \) nodes \( \{s_0, \ldots, s_{n-1}\} \) via \( s_i \sim s_j \) if and only if \( m_{ij} \geq 3 \),

\(^{39}\) Bourbaki is a pseudonym for a group of French mathematicians working at the University of Nice.

\(^{40}\) We will sometimes call it a \( d \)-Coxeter group.
and if \( s_i \sim s_j \), we label this edge \( m_{ij} \). (Actually, by convention, any edges for which \( m_{ij} = 3 \) are left unlabelled). So if the Coxeter matrix were
\[
\begin{pmatrix}
1 & 3 & 2 \\
3 & 1 & 4 \\
2 & 4 & 1
\end{pmatrix}
\]
then the graph so defined would be as shown in Figure 5.1.1(a). This graph is called the Coxeter graph of the group.

![Figure 5.1.1(a)](image)

![Figure 5.1.1(b)](image)

Note that the two matrices
\[
\begin{pmatrix}
1 & 3 & 2 \\
3 & 1 & 4 \\
2 & 4 & 1
\end{pmatrix}
\quad\quad\quad
\begin{pmatrix}
1 & 4 & 3 \\
4 & 1 & 2 \\
3 & 2 & 1
\end{pmatrix}
\]
although different, actually yield isomorphic Coxeter graphs, such as in Figure 5.1.1(a) and 5.1.1(b).

Certainly, any Coxeter system determines a unique Coxeter graph. Likewise, any Coxeter graph uniquely determines a Coxeter system – the generating set \( S \) for the group \( W \) being the set of its vertices, and the relations for the group being given by the labels on the graph’s edges. Since there is a one to one correspondence between Coxeter systems and Coxeter graphs, the graphs are in ways more useful than the matrices for communicating information about the groups.

It should be noted that if \( W \) and \( W' \) are isomorphic, it is no guarantee that the actual Coxeter systems \((W, S)\) and \((W', S')\) will be isomorphic. Humphreys, in [26, Ex 2.2], leads his readers to discover that

![6 and 6](image)

actually yield isomorphic groups, although the systems evidently are different, having different numbers of generators.

As mentioned earlier, one of the first contexts in which Coxeter groups arose was in the study of groups generated by reflections, and in particular of the symmetry groups of regular figures. In fact, it is the case that if \( P \) is a regular geometric \( d \)-polytope with Schl"{a}fli symbol \( \{q_1 \cdots q_{d-2} \} \), then the symmetry group of \( P \) is the Coxeter group whose Coxeter graph is shown in Figure 5.1.2.

The reader interested in seeing this proof in its entirety should consult [14, §9.4], which states the results for every case that needs to be considered, and gives a reference to where the case was first covered.

More generally, whenever the Coxeter group \( W \) (whether or not it is the symmetry group of a polytope) has a Coxeter graph such as depicted in Figure 5.1.2, we shall denote the group \( W_{\langle s_1 \cdots s_{d-1} \rangle} \).

We now examine some of the abstract properties of Coxeter groups. Let \((W, S)\) be a Coxeter system. Given any element \( w \) of \( W \), there are many ways of writing \( w \) as a product of the \( s_i \) – that is, there are many different words in the \( s_i \) which correspond to \( w \). Some of these words will have minimal length. We define the length of the group element \( w \) to be this minimal length, and denote it by \( \ell(w) \). Define \( \ell(1) \) to be 0. This function could be defined for any group, given a set of generators for it, but in the case of a Coxeter system, it has some particularly nice properties. One such property is that if \( I \subseteq S \), and \( W_I \) is the subgroup \(\langle s_i : s_i \in I \rangle \) of \( W \), then we have the following.

**Theorem 5.1.1:** For any subset \( I \) of \( S \), \((W_I, I)\) is a Coxeter system satisfying the following. If \( n_{ij} \) is the least positive integer such that \((s_1 s_j)^{n_{ij}} = 1 \) in \( W_I \), then \( n_{ij} = m_{ij} \), and the length function \( \ell_I \) on \( W_I \) is the restriction to \( W_I \) of the length function \( \ell \) on \( W \).

**Proof:** See [25, Thm 5.5(a)], or [5, IV Thm 1.8.2].

The subgroups \( W_I \) of \( W \) shall be called parabolic subgroups, and will play an important role in the results of this chapter. In some works, the parabolic subgroups of a Coxeter group \( W \) are defined to include the conjugates of the \( W_I \), and the \( W_I \) are called the special parabolic subgroups of \( W \). In this thesis, however, our interest will exclusively be on the \( W_I \) (rather than their conjugates) and so we reserve the simpler term for them. Some very nice results may be proved about the \( W_I \) (see below) which would not be true if \( W \) were an arbitrary finitely presented group.
Theorem 5.1.2: $W_T \cap S = I$.

Proof: This is also proved in [5, IV Cor 1.8.2]. Consider any element $w$ of $W_T \cap S$. Since $w \in S$, we have $\ell(w) = 1$, and hence $\ell_T(w) = 1$, by the preceding theorem. But if $\ell_T(w) = 1$, it must be that $w \in I$ also, so $W_T \cap S \subseteq I$. Finally, the definition of $W_T$ yields $I \subseteq W_T \cap S$, hence $W_T \cap S = I$ as required.

For any particular element $w$ of $W$, there may exist many words corresponding to it of minimal length on the elements of $S$. We call these words reduced words. It turns out that the set of letters in a reduced word for $w$ is independent of the particular word chosen. That is, if $w = s_{i_1} \cdots s_{i_r}$, then $\ell(w) = r$, and $\{s_{i_1}, \ldots, s_{i_r}\} \subseteq S$. The proof of this (given in [26, Cor 5.10]) requires a concept called the Bruhat ordering which we will not examine here. We can use this fact to prove the following theorem, which will be used in the next section.

Theorem 5.1.3: $W_T \cap W_J = W_{T \cap J}$.

Proof: This is also shown in [26, Cor 5.10b] and [5, Thm 1.8.2ii]. Certainly $W_{T \cap J} \subseteq W_T, W_J$, and hence $W_{T \cap J} \subseteq W_T \cap W_J$. Now consider $w \in W_T \cap W_J$. Note first that $w \in W_T$. Since $\ell_T(w) = \ell(w)$, there exist reduced words for $w$ consisting entirely of letters from $T$, and hence any reduced word for $w$ consists entirely of letters from $I$. By the same token, any reduced word for $w$ consists entirely of letters from $J$, since $w \in W_J$ also. We deduce, therefore, that they must consist entirely of letters from $T \cap J$, and so $w \in W_{T \cap J}$ as required.

As we mentioned earlier, Coxeter groups were first studied in the context of symmetry groups. We shall soon see some of the strong links between Coxeter groups and reflection groups, links which may be used to great effect, and in some surprising contexts. Let us begin by examining reflections in Euclidean space, and some of the concepts that relate them to the groups.

Given any nonzero vector $\alpha$ in $\mathbb{R}^n$, let $s_\alpha$ be the reflection through a hyperplane orthogonal to $\alpha$, so $s_\alpha(\beta) = \beta - 2\alpha \cdot \beta \alpha$, where here $\alpha \cdot \beta$ denotes the vector dot product of $\alpha$ and $\beta$. Then, we define a root system to be a finite set $\Phi$ of nonzero vectors such that:

(R1) For all $\alpha \in \Phi$, we have $\Phi \subseteq \text{lin} \{\alpha, -\alpha\}$, and

(R2) For all $\alpha \in \Phi$, we have $s_\alpha(\Phi) = \Phi$.

Thus, an example of a root system would be $\{\pm e_i\}$, or even $\{\pm e_i : 1 \leq i \leq n\}$. Less trivially, $\{\pm 2e_1, \pm e_1 \pm \sqrt{3}e_2\}$ is also a root system. Given any root system $\Phi$, there is a finite reflection group $W$ associated with it, namely $\{s_\alpha : \alpha \in \Phi\}$. Conversely, it can be shown that any finite reflection group may be realized in this way, for some root system (see [26, §1.2]).

The concept of a root system is a fairly broad one, and in ways, unwieldy. Another concept which [26] introduces is that of a simple system. A simple system $\Delta$ is a subset of a root system $\Phi$ such that $\Delta$ is a vector space basis for $\text{lin} \Phi$, and for all $\alpha \in \Phi$, $\alpha$ may be written as an linear combination of elements of $\Delta$ in such a way that either all of $\alpha$'s coefficients are non-negative, or they are all non-positive.

Example: $\Delta_1 = \{2e_1, -e_1, -\sqrt{3}e_2\}$ is a simple system for $\Phi = \{\pm 2e_1, \pm e_1 \pm \sqrt{3}e_2\}$, whereas $\Delta_2 = \{2e_1, e_1 + \sqrt{3}e_2\}$ would not be, for even though both $\Delta_1$ and $\Delta_2$ are subsets of $\Phi$, and are bases for $\mathbb{R}^2 = \text{lin} \Phi$, if we write $-e_1 + \sqrt{3}e_2 \in \Phi$ as a linear combination of elements of $\Delta_2$, it comes to $-1(2e_1) + (1)(e_1 + \sqrt{3}e_2)$, whose coefficients are $-1$ and $1$; the coefficients are neither all non-negative, nor all non-positive. No such problem arises when considering $\Delta_1$.

Note that a simple system $\Delta$ is not a root system. It cannot satisfy $\Delta \subseteq \text{lin} \{\alpha, -\alpha\}$ for any $\alpha \in \Delta$, because if it did, it would fail to be linearly independent, and so would fail to be a basis. In this respect, the terminology can be slightly confusing.

Humphreys, in [26, Thm 1.3], shows that any root system has simple systems, and in [26, Thm 1.4] demonstrates that given $\Phi$, the simple system is, in a sense, unique. (Specifically, if $\Delta$ and $\Delta'$ are simple systems, there is an element $w$ of the corresponding reflection group $W$ such that $w(\Delta) = \Delta'$.)

Recall that for $\Delta$ to be a simple system, we required that when any element of $\Phi$ is written as a linear combination of the elements of $\Delta$, its coefficients should either be all non-negative, or all non-positive. Thus, a simple system yields a natural way to partition a root system into two subsets, namely a set of positive roots $\Pi$, whose coefficients are all non-negative, and a set of negative roots, whose coefficients are all non-positive. Note that the set of negative roots is just $-\Pi$ so $\Pi \cup (-\Pi) = \Phi$.

Let $\Phi$ be a root system with an associated reflection group $W$, and suppose that $\Delta \subseteq \Phi$ is a simple system with $|\Delta| = d$. It can also be shown that $W$ is generated by the reflections $\{s_\alpha : \alpha \in \Delta\}$ for any simple system $\Delta$. In general, $|\Delta|$ may be much smaller than $|\Phi|$, so this result is quite useful.

43 That is, the positive roots are those whose nonzero coefficients are positive.

44 [26, Thm 1.5]
Once Humphreys (in [26]) proves it, he proceeds to discover what relations the generators of $W$ satisfy, and so obtains an abstract presentation of the group $W$. His conclusion, in [26, Thm 1.9] is that $W$ is in fact a Coxeter group, satisfying relations $(s_i s_j)^{m_{i,j}}$ for certain values of $m_{i,j}$, and no other relations apart from the consequences of these. Hence any results which may be deduced about Coxeter groups may be applied to the study of finite reflection groups. The converse, that any finite Coxeter group may be expressed as a reflection group for some $\Phi$, is also true (see [26, Thm 6.4] for example), allowing results about finite reflection groups to be applied to the study of finite Coxeter groups as well (as will be done, for example, in the proof of Lemma 5.2.11).

Here are two interesting theorems that illustrate how the geometrical and the abstract views of a (finite) Coxeter group can interact. The first tells us about the number of positive roots a given $w \in W$ maps to negative roots.

**Theorem 5.1.4:** If $n(w) = |\Pi \cap w^{-1}(-\Pi)|$, then $n(w) = \ell(w)$.

**Proof:** See [26, Cor 1.7].

**Theorem 5.1.5:** There is a unique $w_0 \in W$ such that $w_0(\Pi) = -\Pi$, (so $\Pi = w_0^{-1}(-\Pi)$). This element has maximal length in $W$.

**Proof:** See the notes after [26, Thm 1.8].

Note that the above geometric treatment only deals with finite Coxeter groups. Humphreys (in [26]) also develops some theory whereby any Coxeter group may be regarded as a "reflection" group, by dispensing with the usual Euclidean dot product, and substituting a more general bilinear form in its place. See [26, §5.3] onwards for more details of this. The method he describes is not the only way of gaining a geometrical interpretation of the Coxeter groups, as indeed he points out, referring his readers to [57] and [58].

### 5.1.2 Other Notes:

Here, we shall introduce some notation which will be used later on, and prove a few results about those Coxeter groups that are of particular use in this thesis. In the rest of this chapter, we will not be concerned with arbitrary Coxeter groups, but only with those that satisfy certain fairly stringent conditions, which we examine below.

---

(C1') The Coxeter graph is a path, so that if $j \neq i, i \pm 1$, we have $m_{ij} = 2$, and $m_{i-1,i} = q_i$.

We will also often require

(C2') If $I$ is a proper subset of $S$, $W_I$ is finite.

The finite Coxeter groups have been classified, and the classification is usually expressed in terms of the Coxeter graph (see for example [5, VI Thm 4.1.1] or the comment at the top of [26, p30]). Similarly, we can classify the groups we are interested in in terms of their graphs. There is a natural connection between the graph for $(W, S)$ and those for its special parabolic subgroups $W_I$ for $I \subseteq S$.

**Theorem 5.1.6:** Let $(W, S)$ be a Coxeter system, and let $G$ be its Coxeter graph. Let $I$ be any subset of $S$. Then the Coxeter graph of $W_I$ is just the subgraph of $G$ induced on the vertices $I$.

**Proof:** This follows as a corollary to the first half of Theorem 5.1.1.

As a consequence of this result, we can re-write C2' as:

(C2'') If any vertices are removed from the Coxeter graph for $(W, S)$, the resulting graph describes a finite Coxeter group.

But this may be simplified still further:

**Theorem 5.1.7:** Let $(W, S)$ be a Coxeter system whose graph $G$ is a path. For any nonempty subset $J$ of $S$, let $I = S \setminus J$ and let $G_J$ be the graph obtained from $G$ by removing the vertices in $J$, leaving $I$. Then $G_J$ describes a finite Coxeter group, for all such $J$, if and only if both $G$ and $G_{(4-1)}$ describe finite Coxeter groups.

**Proof:** If for any subset $J$ of $S$, the graph $G_J$ describes a finite Coxeter group, then certainly $G_{(0)}$ and $G_{(4-1)}$ do so. Conversely, let $G_{(0)}$ and $G_{(4-1)}$ describe finite groups. Note that removing a set of vertices $J$ from $G$, which is a path, will cause it to fall into a collection of mutually disconnected components, each of which is isomorphic to a subgraph of at least one of the graphs $G_{(0)}$ and $G_{(4-1)}$. We deduce, by Theorem 5.1.6 that each of the components represents a group which is a subgroup of a finite group, since both $G_{(0)}$ and $G_{(4-1)}$ describe finite groups. But [26, Prop 6.1] tells us that $G_J$ represents a finite Coxeter group if and only if each of the connected components does. Thus $G_J$ represents a finite group, as required.

Thus, we are interested in Coxeter groups satisfying, at least the first, and often also the second, of the two conditions below.

---

43 A bilinear form $B$ is a function taking two vectors $x$ and $y$ and yielding a single real number in such a way that for any constant $z$, $B(zx, y)$ is a linear function of $y$, and for any constant $y$, $B(x, zy)$ is a linear function of $x$. 

---

Section 5.1: Coxeter Groups
(C1) The Coxeter graph of $W$ is a path.

(C2) If either of the endpoints of this graph is removed, the resulting graph describes a finite group.

Certainly, given a classification of the Coxeter graphs describing finite groups, it is easy to check if a given path satisfies these conditions, and not much harder to classify those paths that do, for first of all, if $W$ itself is finite and satisfies the first condition, it will satisfy the second also. Then for each such finite group, we perform the following operation. To each end in turn, attach another edge, labelled $m$. Then test to see whether this new graph satisfies condition C2, by removing the other end vertex, and comparing the graph so obtained with the list of Coxeter graphs of finite groups, to see what (if any) values of $m$ are appropriate. A complete list of the Coxeter groups satisfying these conditions is given in Tables A.1 and A.2.

In the course of the next few sections, we will find that we are often dealing with certain parabolic subgroups $W_I$, namely those where $I$ has the form $S^i\{s_i\}$ for some $i$. This is inconvenient to express in our current notation, so let us write $H_I = W_{S^i\{s_i\}}$ for all $I \subseteq S$. In particular write $H_i = H_{\{i\}}$ and $H_{ij} = H_{\{i,j\}}$.

Note that because of the particular form which our groups have, namely that $m_{ij} = 2$ if $i \neq j$, $j \pm 1$, we have $(s_1s_2)^k = 1$ for $i < k < j$. Thus, $H_i = W_{S^i\{s_i\}} = W_{\{s_i\} j = 1} = W_{\{s_i\} k = 1} = W_{S^i\{s_i\} k = 1} = W_{S^i\{s_i\} k = 1}$. Denote $W_{S^i\{s_i\} k = 1}$ by $W_{S^i\{s_i\} k = 1}$, and denote $W_{S^i\{s_i\} k = 1}$ by $W_{S^i\{s_i\} k = 1}$, so $H_i = W_{S^i\{s_i\} k = 1} = W_{S^i\{s_i\} k = 1} = W_{S^i\{s_i\} k = 1} = W_{S^i\{s_i\} k = 1}$. Note that $W_{S^i\{s_i\} k = 1}$ is a subgroup of $H_{S^i\{s_i\} k = 1}$, and that $W_{S^i\{s_i\} k = 1}$ is a subgroup of $H_{S^i\{s_i\} k = 1}$. (In fact, $W_{S^i\{s_i\} k = 1}$ is a subgroup of $H_{S^i\{s_i\} k = 1}$ for any $k \geq i$, and a similar result holds for $W_{S^i\{s_i\} k = 1}$.)

The following lemma will prove invaluable in the next section.

Lemma 5.1.8: Let $W$ be a Coxeter group whose graph is a path, and let the subgroups $H_i$ be defined as above. Further, let $0 \leq i_1 < i_2 < \ldots < i_m \leq d - 1$, and let $u_i$ be elements of $W$ such that for all $k$, the cosets $u_{i_k}H_{i_k}$ and $u_{i_k+1}H_{i_k+1}$ have non-empty intersection. Then there exists some $u \in W$ such that $u_iH_i = uH_i$ for all $i$.

Proof: By induction on $m$. Certainly, if $m = 1$, it holds. Now, assume it holds whenever $m < n$, and consider the case $m = n$. Let $I' = \{i_1, \ldots, i_m\}$, and $I = \{i_1\} \cup I'$. Note that by Theorem 5.1.3, $H_{i_1} \cap H_{i_1} \cap \ldots \cap H_{i_m} = H_{I'}$. Then

$$u_{i_1}H_{i_1} \cap u_{i_2}H_{i_1} \cap \cdots \cap u_{i_m}H_{i_m} = u_{i_1}H_{i_1} \cap (u_{i_2}H_{i_2} \cap \ldots \cap u_{i_m}H_{i_m}) = u_{i_1}H_{i_1} \cap (u' H_{i_2} \cap \ldots \cap u' H_{i_m}) = u_{i_1}H_{i_1} \cap u' H_{I'}$$

by the inductive hypothesis. On the one hand, because $u_{i_1}H_{i_1}$ and $u_{i_2}H_{i_2}$ intersect, there exists some $v \in W$ such that $u_{i_1}H_{i_1} = vH_{i_1}$ and $u_{i_2}H_{i_2} = vH_{i_2}$. (Any $v$ in their intersection will do.) Since therefore $u_{i_1}H_{i_1} = vH_{i_1}$, it follows that $H_{i_1} = v^{-1}u_{i_1}H_{i_1}$, so $v^{-1}u_{i_1} = v^{-1}u_{i_1}H_{i_1} = W_{c_{i_1}} \times W_{d_{i_1}}$. Write $v^{-1}u_{i_1}$ as $gh$, where $g \in W_{c_{i_1}}$ and $h \in W_{d_{i_1}}$. Note that $gh = hg$. Then,

$$vH_{i_1} \cap u' H_{I'} = v(H_{i_1} \cap v^{-1}u' H_{I'}) = v(H_{i_1} \cap hg H_{I'}) = v(H_{i_1} \cap H_{I'})$$

(since $g \in W_{c_{i_1}} \subseteq H_{I'}$). But $h$ will be an element of $hH_{I'}$, certainly, and since $W_{d_{i_1}} \subseteq H_{i_1}$, it follows that $h$ will also be an element of $H_{i_1}$. Thus $vh$ is an element of the intersection of all the $u_iH_i$, and so setting $v = vh$ yields $u_{i_1}H_{i_1} = uH_{i_1}$ for each $i$, as required to complete the induction.

In the next subsection we shall establish our first explicit link between polytopes and Coxeter groups.

5.1.3 Coxeter Groups and the Flag Action:

Now that we know a little about Coxeter groups, it is time to apply this knowledge. We do this by demonstrating one way a Coxeter group can act on the set of flags of a polytope. The action we define here will turn out to have a very nice relationship with the action of the polytope's automorphism group (see Corollary 5.1.18). Let $\mathcal{X}$ be a 1-Euler $d$-incidence complex, and let $W$ be the Coxeter group generated by $S = \{s_0, \ldots, s_{d-1}\}$, with Coxeter graph

$$\begin{align*}
&\text{P}_1 & &\text{P}_1 & &\text{P}_{d-2} & &\text{P}_{d-1}
\end{align*}$$

where the $p_i \geq 2$ and may be infinite. At this stage we shall not suppose any particular relationship between $W$ and $\mathcal{X}$.

For any $F \subseteq \mathcal{F}(\mathcal{X})$ and $s_i \in S$, let $F^{s_i}$ be the unique flag of $\mathcal{X}$ differing from $F$ only by an element of dimension $i$, so $(F) = (F^{s_i})$, if and only if $i \neq j$. Note that $(F^{s_i})_j$ depends only on $(F)_{i-1}, (F)_i$ and $(F)_{i+1}$. Given any word $w = s_{i_1} \ldots s_{i_m}$ on the $s_i$, define $F^w$ inductively via $F^{s_{i_1} \ldots s_{i_m}} = (F^{s_{i_1}})^{s_{i_2} \ldots s_{i_m}}$. So in fact for any two words $x$ and $y$ on the $s_i$, we will have $F^{xy} = (F^x)^y$, where $xy$ is the concatenation of $x$ and $y$.

Of course, this does not define an action of $W$ on the set of flags, unless in fact for any relation $r = r'$ of $W$, and any flag $F$ of $\mathcal{X}$, we have $F^r = F^{r'}$; for only then can we be sure that any two distinct words $u$ and $u'$ which are equal as elements of $W$ will satisfy $F^u = F^{u'}$. If the action is well defined, we call it
the flag action of \( W \) on \( \mathbb{F}(\mathcal{X}) \). The next few results aim to determine exactly when \( W \) has a flag action on \( \mathcal{X} \).

Now the defining relations of \( W \) fall into three categories. There are those of the form \( s_i^2 = 1 \), those of the form \( (s_is_j)^3 = 1 \) where \( i \neq j, j \pm 1 \), and finally those of the form \( (s_is_{i+1})^p = 1 \). We consider each of them in turn.

**Lemma 5.1.9:** For any flag \( F \) and any \( s_i \in S \), we have \( F^{s_i} = F \).

**Proof:** Consider the set \( A_i \) of flags \( G \) such that \( (G)_j = (F)_j \) whenever \( j \neq i \). This set has two elements, since \( \mathcal{X} \) is 1-Euler. Certainly, \( F \in A_i \). But by definition, \( F^{s_i} \in A_i \) also, and so \( A_i = \{ F, F^{s_i} \} \). Now consider \( F^{s_i} \). Again by definition, we have \( (F^{s_i})_j = (F)_j \) for all \( j \neq i \), and so \( F^{s_i} \in A_i \). But \( F^{s_i} \neq F \), so it follows that \( F^{s_i} = F \) as required. 

**Lemma 5.1.10:** \( F^{s_is_j} = F \) whenever \( j \neq i, i \pm 1 \).

**Proof:** If \( k \neq i, j \), then \((F^{s_is_j})^k = (F)_k = (F^{s_is_j})_k \). We have \( (F^{s_is_j})_i = ((F^{s_is_j})^i)_i = (F^i)_i \). However, for any flag \( G, (G^n)_j \) depends only on \( (G)_{j-1}, (G)_{j+1} \), and \( (G)_{j+1} \). Note then that \( (F^{s_is_j})_{j+1} = ((F^{s_is_j})_{j+1})_{j+1} = (F^i)_{j+1} \), and \( (F^{s_is_j})_{j+1} = (F^i)_{j+1} \), since \( j \neq i, i \pm 1 \). Thus \( (F^{s_is_j})_j = (F^i)_j \), so \( (F^{s_is_j})_j = (F^{s_is_j})_j \), as required. Similarly \( (F^{s_is_j})_j = (F^i)_j \), so that \( (F^{s_is_j})_j = (F^{s_is_j})_j \) for all \( k \), giving \( F^{s_is_j} = F \) as required.

So relations of the two types examined above will never pose any problems. The third type sometimes will, and the next lemma tells us when.

**Lemma 5.1.11:** Let \( F \) be a flag such that \( (F)_{j-1}, (F)_{j+2} \) is an \( l \)-cycle. Then \( F^{s_is_{i+1}} = F \) if and only if \( l \) divides \( p_i \).

**Proof:** Let \( \{ (F)_{j-1}, (F)_{j+2} \} = \{ (F)_{j-1}, (F)_{j+2} \} \cup \{ a_j, A_j : j \in \mathbb{Z} \} \) where \( a_j < A_j \) if and only if \( j \equiv k \) or \( k + 1 \) (mod \( l \)). Without loss of generality, let \( (F)_i = A_0 \) and \( (F)_{i+1} = A_0 \). I claim that \( (F^{s_is_{i+1}})^n = A_0 \) and \( (F^{s_is_{i+1}})^n = A_0 \), and for any \( n \geq 0 \). These equalities will certainly hold whenever \( n = 0 \). Assume now that

\[
(F^{s_is_{i+1}})^{k-1}, i = a_{(k-1)} \mod l \quad \text{and} \quad (F^{s_is_{i+1}})^{k+1} = A_{(k-1)} \mod l.
\]

Then,

\[
(F^{s_is_{i+1}})^{k} = (F^{s_is_{i+1}})^{k+1}.
\]

For brevity, we will sometimes refer to it as the flag action of \( W \) on \( \mathcal{X} \), although strictly speaking this is an abuse of the terminology.

As defined in §3.2.

which will satisfy

\[
(F^{s_is_{i+1}})^{k-1}, i < (F^{s_is_{i+1}})^{k+1}, i < (F^{s_is_{i+1}})^{k+1}, i+1,
\]

which may be re-written

\[
(F^{s_is_{i+1}})^{k+1}, i < A_{(k-1)} \mod l.
\]

Thus, in fact, \( (F^{s_is_{i+1}})^{k+1}, i \) is equal to either \( q_{(k-1) \mod l} \) or \( q_{(k-1+1) \mod l} \).

It cannot equal the former, since it must be different from \( (F^{s_is_{i+1}})^{k-1}, i \). Thus

\[
(F^{s_is_{i+1}})^{k+1}, i = A_{(k-1) \mod l}.
\]

A similar argument then shows that

\[
(F^{s_is_{i+1}})^{k+1}, i + 1 = A_{(k-1) \mod l},
\]

as required for the induction to proceed. Now \( F^{s_is_{i+1}} = F \) if and only if \( (F^{s_is_{i+1}})^{k+1}, i = (F)_j \) for all \( j \), and hence if and only if \( (F^{s_is_{i+1}})^{k+1}, i = (F)_j \), and \( (F^{s_is_{i+1}})^{k+1}, i+1 = (F)_j \). But this will occur if and only if \( p_i \) or \( p_j \) is a multiple of \( l \), as required. 

Now we can determine exactly when the flag action is well defined.

**Theorem 5.1.12:** The action of \( W \) on \( R(\mathcal{X}) \) given by

\[
(F^{s_is_{i+1}})^{k, i} = (F)_{j} \quad \text{if and only if} \quad j \neq i, j \pm 1, \quad \text{and} \quad F^{s_is_{i+1}} = F^{s_is_{i+1}}\]

is well defined if and only if, for all \( i \) and for all flags \( F \) of \( \mathcal{X} \), \( p_i \) is a multiple of \( l \), where \( (F)_{i-1}, (F)_{i+2} \) is an \( l \)-cycle.

**Proof:** Certainly, if all such \( p_i \) are multiples of all such \( k \), then we have \( F^{s_is_{i+1}} = F^{s_is_{i+1}} \) for all flags \( F \) of \( \mathcal{X} \) and all relations \( r = r' \) of \( W \) (by the preceding three lemmas) and so the action is well defined. Conversely, if for some flag \( F \), \( F \) does not divide \( p_i \), then \( F^{s_is_{i+1}} = F^{s_is_{i+1}} \) will not equal \( F \), so the action fails to be well defined.

Note that the Coxeter group with Coxeter graph

![Coxeter Graph](image)

will act on every \( d \)-polytope. It will generally be more useful to work with less complicated Coxeter groups, specifically those where the \( p_i \) are as small as possible. Note in particular that when \( \mathcal{X} \) is combinatorially regular with Schläfi Symbol \( \{q_1 \ldots q_n\} \), then the Coxeter group with \( p_i = q_i \) for all \( i \) will act on the set of flags of \( \mathcal{X} \). As it happens, the \( q_i \) we are examining will generally be rather small, so that we can restrict our attention to reasonably simple Coxeter groups.

We shall now obtain some important properties of the flag action: firstly that it is transitive.

Section 5.1: Coxeter Groups
Theorem 5.1.13: Let \( \mathcal{X} \) be a d-incidence polytope, and let \( W \) be a Coxeter group (with \( d \) generators) acting on it. Then for any flags \( F \) and \( F' \) of \( \mathcal{X} \), there exists \( w \in W \) such that \( F^w = F' \).

Proof: Recall that any complex is flag-connected (by property I4), so that there exists a sequence \( F = G_1, G_2, \ldots, G_n = F' \) of flags of \( \mathcal{X} \) such that for any \( j \), the flags \( G_j \) and \( G_{j+1} \) differ by exactly one element. Let that element have dimension \( i_j \). Then \( G_{j+1} = G_j^{\eta_i} \). Thus, in fact, \( G_n = G_1^{\eta_{i_1}} \cdot \eta_{i_2} \cdots \cdot \eta_{i_{n-1}} \) and so the element \( w = \eta_{i_1} \cdot \eta_{i_2} \cdots \cdot \eta_{i_{n-1}} \) of \( W \) has the required property.

Using known results of permutation group theory, we obtain the following result.

Corollary 5.1.14: If \( W \) is finite, then \( |\text{Flag}(\mathcal{X})| \) is finite and divides \( |W| \).

Proof: See, for example, [61, Thm 3.2].

This theorem is useful, particularly for combinatorially regular incidence polytopes, where we are more likely (as it happens) to be able to find a finite group \( W \) to act on the polytope.

Let us gain a taste of how useful these group-theoretic considerations can be. Consider a combinatorially regular polytope with Schlӓfli symbol \([3,3,3]\). According to [11], the order of the Coxeter group \( W_{[3,3,3]} \) is 24, and so \( |\text{Flag}(\mathcal{X})| \) divides 24. However, we can calculate \( |\text{Flag}(\mathcal{X})| \) another way, since for any combinatorially regular \( \mathcal{X} \) we may partition \( \text{Flag}(\mathcal{X}) \), so that each subset in the partition consists of flags containing some fixed element \( x_{d-1} \) of \( x_d \). To any flag \( F \) of \( \mathcal{X} \) containing \( x_{d-1} \) there corresponds a flag \( F' \cap (\emptyset, x_{d-1}) \) of \( \emptyset, x_{d-1} \), so

\[
|\text{Flag}(\mathcal{X})| = \sum_{x_{d-1} \in \mathcal{X}_{d-1}} |\text{Flag}(\emptyset, x_{d-1})| = |\mathcal{X}_{d-1}| \cdot |\text{Flag}(\emptyset, x_{d-1})| = |\mathcal{X}_{d-1}| \cdot \sum_{x_{d-1} \in (\emptyset, x_{d-1})} |\text{Flag}(\emptyset, x_{d-1})| = \cdots = |\mathcal{X}_{d-1}| \cdot |\emptyset, x_{d-1}| \cdot \cdots \cdot |\emptyset, x_{d-1}| = 2(d-1)!.
\]

In the case where \( \mathcal{X} \) has Schlӓfli symbol \([3,3,3]\), this tells us that \( |\text{Flag}(\mathcal{X})| = |\mathcal{X}_2| \cdot |\emptyset, x_2| \cdot |\emptyset, x_1| = |\emptyset, x_1| \cdot 3! \). Now \( \emptyset, x_2 \) is a 3-cycle, and \( \emptyset, x_1 \) is a 1-polytope, so \( |\emptyset, x_2| = 3 \) and \( |\emptyset, x_1| = 2 \). This yields \( |\text{Flag}(\mathcal{X})| = |\mathcal{X}_2| \cdot 3! = 6|\mathcal{X}_2| \). Thus we deduce that \( |\mathcal{X}_2| \) divides 4. Now, let \( x \in \mathcal{X}_2 \) and consider the set \( \{x, K\} \cap \mathcal{X}_2 = \{h \in \mathcal{X} : x < h \leq K\} \) (where \( K \) is the maximal element of \( \mathcal{X}_2 \)). This has 2 elements, since \( \mathcal{X}_2 \) is a polytope. Thus \( |\mathcal{X}_2| \geq 2 \). If \( |\mathcal{X}_2| = 2 \), then \( x < h \) for all \( x \in \mathcal{X}_2 \), implying that \( \mathcal{X} \) is decomposable, which would require its Schlӓfli symbol to have the form \([m, m, m]\). Therefore, we may assume \( |\mathcal{X}_2| > 2 \).

would be a contradiction. Thus \( |\mathcal{X}_2| > 2 \), and so in fact \( |\mathcal{X}_2| = 4 \). From this it can also be shown (for example, using Lemma 3.4.15) that \( |\mathcal{X}_1| = 6 \) and \( |\mathcal{X}_0| = 4 \), whence \( \mathcal{X} \) is Euler.

Thus we have deduced a reasonably strong combinatorial result about a certain class of incidence complexes, using only fairly elementary results about a particular group action. This should be regarded as merely a taste of the power that group theory has to aid our classification. Some much more powerful results will be obtained towards the end of this chapter. These results will use the flag action, hence its introduction now.

The notation needs clarification. Given a d-polytope \( \mathcal{X} \) and a Coxeter group \( W = \langle s_0, \ldots, s_{d-1} \rangle \), we define \( \phi_{\mathcal{X}, W}(F, s_i) \) to be the flag \( G \) of \( \mathcal{X} \) with \( (G_j) = (F_j) \) for all \( j \neq i \), and \( (G_i) \neq (F_i) \). Hence in fact \( \phi_{\mathcal{X}, W}(F, s_i) \neq F' \). Note that \( (F')_i \) depends only on \( (F)_i \) and \( (F)_j \) for \( j 
eq i \). We attempted to extend this map to an action of \( W \) on \( \text{Flag}(\mathcal{X}) \), via \( \phi_{\mathcal{X}, W}(F, s_i) = \phi_{\mathcal{X}, W}(F, s_i, F) \). We have seen (Theorem 5.1.12) that this extension is possible if and only if for all \( i \), and for all \( x, y \in \mathcal{X} \), \( x_i y + 1 = x_m y - 2 = i \), the section \( (x, y) \) is an l-cycle for some \( l \) dividing the order of the element \( s_{m+i} \) of \( W \). Now let \( \sigma = (a, b) \) be a section of \( \mathcal{X} \) satisfying \( 2a = 1 \) and \( 2b = j + 1 \) (so \( |\sigma| = j + 1 \)) and let \( H \) be the subgroup \( \langle s_{j+i}, \ldots, s_i \rangle \). Write \( \sigma = \rho_l \cdot \eta_i \), so \( H = \langle \rho_l, \eta_i \rangle \). In this case, define \( (F, s_i) \) and \( (F, s_{j-i}) \) the maps \( (a, F, h) = \phi_{\mathcal{X}, W}(F, h, F) \), and \( (b, F, h) = \phi_{\mathcal{X}, W}(F, h, F) \), where \( F' \) is a flag of \( \mathcal{X} \) which contains \( F \).

Theorem 5.1.15: Suppose that the action of \( W \) on \( \mathcal{X} \) is well defined. Then the maps \( \alpha \) and \( \beta \) are well defined actions of the group \( H \) on the set of flags of \( \mathcal{X} \) and are in fact equal.

Proof: To show that \( \alpha \) is well defined, we need to show that \( \phi_{\mathcal{X}, W}(F, H) \) is well defined. That is, we need to show that for any \( x, y \in \mathcal{X} \) with \( \text{dim}_x = \text{dim}_y = 2 \), the section \( (x, y) \) is an l-cycle for some \( l \) dividing the order of \( s_{i+m+i} \) in \( H \). But if \( \text{dim}_x = \text{dim}_y = 2 \), then \( \text{dim}_x = \text{dim}_y = l = 1 \), and so \( \phi_{\mathcal{X}, W}(F, H) \) is well defined \( l \) divides the order of \( s_{i+m+i} \). But \( s_{i+m+i} = s_i = a_i + 1 \), \( s_{i+m+i} = s_{i+m+i} \), so this just what we require.

To show that \( \beta \) is well defined, we show, for each of the generators \( r_i \) \((1 \leq i \leq (j-1) \cap \mathcal{X})\), that \( \beta(F, r_i) = \mathcal{X} \cap \phi_{\mathcal{X}, W}(F', s_{i+m+i}) \) does not depend on the flag \( F' \) of \( \mathcal{X}' \). To this end, let \( F' \) and \( G' \) be flags of \( \mathcal{X}' \) satisfying \( \mathcal{X}' = F' \cap \mathcal{X}' \cap G' \). Let \( F'' = \phi_{\mathcal{X}, W}(F', s_{j+m+i}) \) and \( G'' = \phi_{\mathcal{X}, W}(G', s_{j+m+i}) \). We wish to show that \( (F'')_i = (G'')_i \) whenever \( 1 \leq i \leq j-1 \). There are two cases. Note first that if \( i = j+i \), then \( (F'')_i = (F''_i) = (G''_i) = (G'')_i \). But then we obtain our result from the fact that \( (F'')_i = (G'')_i \), since \( F' \cap \mathcal{X}' = G' \cap \mathcal{X}' \).
and $i \leq l \leq j + 1$. Secondly, if $l = k + i + 1$, then $(F^o)^i$ depends only on $(F^o)^i$; and $(F^o)^k$. But $l = k + i + 1 \leq j + 1$, and $k + i + 2 \leq j + 1$, so $i \leq l - 1 < l < i + 1 \leq j + 1$. From this it follows that $(F^o)^i = (G^o)^i$, for each of $m = l$ and $m = l + 1$, and hence $(F^o)^i = (G^o)^i$ in this case also. Thus $\beta(F, h)$ does not depend on the $F^o$ chosen, and is so well defined.

It remains to be shown that given a flag $F$ of $Y$ and an element $h$ of $H$, we have $\alpha(F, h) = \beta(F, h)$. It suffices to show this for the case $F$ is a generator $r_1 = s_{k+1} s_1$. Let $r_k = \alpha(F, r_k)$, and $F^o = \beta(F, r_k)$. Then, if $l \neq k$ we will have $(F^o)^i = (F^o)^i = (F^o)^i$, and so we restrict our attention to the case $l = k$; that is, we show that $(F^o)^k = (F^o)^k$. Consider the set $\{x \in Y : (F^o)^k < x < (F^o)^{k+1}\}$. This set is equal to $\{x \in Y : (F^o)^k < x < (F^o)^{k+1}\}$, and has exactly two elements. By definition of $\alpha$, these two elements are $(F^o)^k$ and $(F^o)^k$, but by definition of $\beta$, they are $(F^o)^k$ and $(F^o)^k$. Thus $(F^o)^k = (F^o)^k$, whence $F^o = F^o$ as required.

This result is useful because it means that any results we prove about the flag action of a Coxeter group $W$ on the set of flags of a polytope $\mathcal{X}$, may still be applied when considering the induced action of an appropriate parabolic subgroup of $W$ on the set of flags of the corresponding section of $\mathcal{X}$.

We have already encountered the automorphism group of a complex, and seen how it too may be regarded as acting on the set of flags of the complex (§3.4.1). We saw also that for many complexes, in particular for polytopes, the automorphism group may be regarded as the set of those permutations of $Y(\mathcal{X})$ with the property of "consistent projections". We can use this property to show that automorphisms commute with the flag action — that is, if $a$ is an automorphism of a combinatorially regular polytope $\mathcal{X}$, and $w$ is any element of the Coxeter group $W$ acting on $\mathcal{X}$, then for any flag $F$ of $\mathcal{X}$, we have $(F^o)^w = (F^o)^a$. This result shall not be stated explicitly as a theorem here — instead, we shall prove a somewhat more general result, about isomorphisms.

Theorem 5.1.16: If $\mathcal{X}$ and $\mathcal{Z}$ are isomorphic combinatorially regular incidence polytopes, acted on by the Coxeter groups $W$ and $\psi$, and $\psi$ is an isomorphism from $\mathcal{X}$ to $\mathcal{Z}$, then for any $w \in W$ and any $F \in \mathcal{X}(\mathcal{X})$, we have $(F^o)^w = (F^o)^{\psi}$.

Proof: Note that if $\mathcal{X}$ and $\mathcal{Z}$ are isomorphic, they have the same Schlaffi Symbol (Theorem 3.4.11), and so if the flag action is well-defined on one, it will be on the other as well. We prove our theorem by induction on the length of $w \in W$. If $w = 1$, there is nothing to prove. Let $w$ be a generator $s_i$ of $W$. Now, if $j \neq i$ then, by definition, $(F^o)^w = (F^o)^i$, and so $(F^o)^w \psi = (F^o)^i \psi = (F^o)^i$. But we also have $(F^o)^w = (F^o)^i$, so $(F^o)^w \psi = ((F^o)^w \psi)^i$. As required. Now we calculate $(F^o)^w \psi$ and $(F^o)^w \psi$. Consider the set $\{x^i \in \mathcal{Z} : (F^o)^i - 1 < x < (F^o)^{i+1}\}$. This set will equal $\{(F^o)^i, (F^o)^{i+1}\}$. But since $\psi$ is an isomorphism from $\mathcal{X}$ to $\mathcal{Z}$, the set is also equal to $\{x^i : (F^o)^i - 1 < x < (F^o)^{i+1}\}$, which is equal to $\{(F^o)^i, (F^o)^{i+1}\}$, which equals $\{(F^o)^i, (F^o)^{i+1}\}$. Thus $((F^o)^w \psi) = ((F^o)^w \psi)$, as required. We have shown the theorem to be true for elements of length 0 or 1 of $W$. Now, assume that the result holds for elements of length less than $m$, and let $w = s_{i_1} s_{i_2} \ldots s_{i_m}$. We can write $w = s_{i_1} v$, where $v$ has length at most $m - 1$. Then, $(F^o)^w = (F^o)^{i_1} = (F^o)^{i_1} \psi = ((F^o)^{i_1} \psi)$ as required to complete the induction.

As stated earlier, it is a corollary of this that the action of the automorphism group commutes with the flag action. This corollary has a converse. Before we prove the converse, however, it is helpful to prove the following lemma.

Lemma 5.1.17: Let $F$ and $G$ be flags of $\mathcal{X}$, and let $I \subseteq \{0, 1, \ldots, d-1\}$, and $H_I = \{g_i : i \notin I\}$. Then $(F)_I = (G)_I$ for all $j \in I$ if and only if there exists some $h \in H_I$ such that $G = F^h$.

Proof: First, suppose that $h \in H_I$ is such that $G = F^h$. Let $h = s_{i_1} \ldots s_{i_n}$, where for each $k$, $i_k \notin I$. Let $F_0 = F$, and for each $k$, let $F_k = F_0 s_{i_k}$, so in particular $F_n = F_0$. Note that for each $j \in I$, we have $(F_0)_j = (F_0)_j$ (since in fact $F = F_0$). Assume now that $k \geq 1$, and that $(F_k)_j = (F_{k-1})_j$ for all $j \in I$. Then for each such $j$, we have $(F_k)_j = (F_{k-1})_j$, which equals $(F_{k-1})_j$, since $i_k \notin I$ (so $i_k \notin j$), and so in fact $(F_k)_j = (F_0)_j$. Thus, the principle of mathematical induction informs us in particular that $(F_k)_j = (F_0)_j = (G)_j$. This proves one direction of the implication. To prove the converse, we use induction on the size of the set $I$. By Theorem 5.1.13, the result holds if $I$ is the empty set. Next, let $k \geq 1$, assume that the result holds whenever $I$ has less than $k$ elements, and let $|I| = k$. Choose $i_k \notin I$, and let $(F_k)_j = (G)_j$. Consider the flags $(F_{k-1}, (F_0)_j, \ldots, (F_0)_j)$ and $(G_{k-1}, (G_0)_j, \ldots, (G_0)_j)$ of $(\mathcal{X}, \mathcal{Z})$. Now by Theorem 5.1.15, the group $(s_{i_1}, \ldots, s_{i_k})$ acts on the set of flags of $(\mathcal{X}, \mathcal{Z})$ via the flag action, and by the inductive hypothesis, there in fact exists some element $u$ of $H = \{s_{i_1} : i \in I\}$ such that $u \cdot F = F^u$. Note that since $H \leq W$, the element $u$ also acts on $\mathcal{X}(\mathcal{X})$, and will in fact satisfy $(F^u)_j = \{g_i : j \notin I\}$, $j \geq 1$. Similarly, we can find an element $v$ of $H = \{s_{i_1} : i \in I\}$, such that $(F^v)_j = \{g_i : j \in I\}$, $j \geq 1$. That is, $(F^v)_j = (G)_j$, which is a contradiction. Thus $u^a$ is an element of $W_{\mathcal{X}, \mathcal{Z}} \cap H_I$, $(W_{\mathcal{X}, \mathcal{Z}} \cap H_I) \subseteq (W_{\mathcal{X}, \mathcal{Z}} \cap H_I)$, $H_I = H_I$. This completes the induction, and hence the proof.

Now we can easily prove that the automorphisms of $\mathcal{X}$ are just those permutations of $\mathcal{X}(\mathcal{X})$ which commute with the flag action.
Theorem 5.1.18: If $\alpha \in \text{Sym}(F(\mathcal{X}))$ satisfies $(F^w)^\alpha = (F^w)^\alpha$ for every $w \in W$ and $F \in F(\mathcal{X})$, then $\alpha$ has consistent projections, and so is a permutation of $F(\mathcal{X})$ induced by an automorphism of $\mathcal{X}$.

Proof: The term "consistent projections" was defined in §3.4.1. Let $\alpha$ be as given, and let $F$ and $F'$ be such that $(F_i) = (F'_i)$, for some $i$. Then, by the previous lemma, there exists $h \in H_i$, such that $F = F'$. Since $h \in H_i$, it follows that $(G_i) = (G_i^h)$, for all flags $G$ of $\mathcal{X}$, and so in particular, $(F^w) = ((F^w)^h)$. This in turn will equal $((F^w)^h)^\alpha$, that is, $(F^w)^\alpha$, since $\alpha$ commutes with the action of $W$ on the flags of $\mathcal{X}$. But $F$, $F'$ and $i$ were chosen arbitrarily (with $(F_i) = (F'_i)$), and so it follows that $\alpha$ has consistent projections, and so (by Lemma 3.4.5) is induced by (or induces) an automorphism of $\mathcal{X}$.

We have established a strong link between the automorphism group of a polytope and the Coxeter group that acts on the polytope. This is very good, because previously we knew little about the automorphism group. We will in fact be able to use this link to characterise the automorphisms for some of the examples to be constructed in the next section. These examples will be a lot more general than those constructed in the previous chapter. Before we begin on that work, let us make the link between the automorphism group and the flag action just a little clearer.

Any element $w$ of the Coxeter group $W$ induces a permutation $w \in F(\mathcal{X})$ of $F(\mathcal{X})$. In fact, the map $w \mapsto w(\mathcal{X})$ is a group homomorphism from $W$ to $\text{Sym}(F(\mathcal{X}))$. We shall call its image, $W(\mathcal{X})$, the flag group of $\mathcal{X}$, and denote it $\Gamma(\mathcal{X})$. For any element $\gamma_i = w^i \in \Gamma(\mathcal{X})$, and any flag $F$ of $\mathcal{X}$, we have $F^\gamma_i = F^w_i$. We obtain, then, a corollary from Theorems 5.1.16 and 5.1.18.

Corollary 5.1.19: $\text{Aut}(\mathcal{X})$ is the centraliser, in $\text{Sym}(F(\mathcal{X}))$, of $\Gamma(\mathcal{X})$.

Proof: Let $\alpha \in \text{Aut}(\mathcal{X})$. Then, since $(F^w)^\alpha = (F^w)^\alpha$ for all $w \in W$ and all $F \in F(\mathcal{X})$ (Theorem 5.1.16), it follows that $\alpha$ commutes with $\gamma_i$ for all $\gamma_i \in \Gamma(\mathcal{X})$. Conversely, let $\alpha \in \text{Sym}(F(\mathcal{X}))$ commute with $\gamma_i$ for every $\gamma_i \in \Gamma(\mathcal{X})$. Then for any $w \in W$ and $F \in F(\mathcal{X})$ we have $(F^w)^\alpha = F^{w^\alpha} = F^{w^\alpha}$, and so by Theorem 5.1.18, $\alpha$ is an automorphism of $\mathcal{X}$, as required.

5.2 Complexes Constructed from Groups

In this section we show two ways in which a complex (indeed a polytope) may be constructed from a given Coxeter group. One construction (that of the "universal" polytopes) will involve just the group itself, whereas the other (that of the "quotient" polytopes) will involve also a subgroup of the Coxeter group. These constructions will turn out to be particularly useful. In fact, we shall show that any combinatorially regular Euler polytope whose facets and vertex figures are universal, is isomorphic to a particular kind of quotient polytope. These results shall be used to great effect in Chapter 6, and it will become apparent, when all is said and done, that in fact any indecomposable combinatorially regular Euler polytope has facets and vertex figures of the required form.

5.2.1 Universal Complexes:

It will be useful at this point to recall the notation introduced in §5.1.2.

Let $S = \{s_0, \ldots, s_{d-1}\}$, and let $W$ be the group $\langle s_0, \ldots, s_{d-1} | (s_i s_j)^{m_{ij}} = 1 \rangle$, where $m_{ii} = 1$ for all $i$, and $m_{ij} = 2$ if $j \neq i, i \pm 1$. Let $m_{i-1,i} = q_i$. Thus $W$ is the Coxeter group with the Coxeter graph shown below.

Let $\mathcal{M}$ be the set of all left cosets of the parabolic subgroup $H_i$ of $W$. That is, $\mathcal{M} = \{wH_i : w \in W\}$. Let $\mathcal{M} = \bigcup_{i=0}^{d-1} \mathcal{M}_i \cup \mathcal{M}_d$, be equipped with the relation $\leq$, defined by $\mathcal{M}_i \leq \mathcal{M}_j \leq M$ for all $i = \mathcal{M}$ (and these are the only relationships in $\mathcal{M}$ and other elements of $\mathcal{M}$), and $wH_i \leq wH_j$ if and only if $i \leq j$ and $wH_i \cap wH_j$ is nonempty. We call $\mathcal{M}$ the universal $d$-complex based on $W$. The universal $d$-polytope is in fact a special case of a so-called coset geometry, as defined for example in [55, Ex 2.3.4], and examined further in [8] and the references therein. The notation and terminology used here is clearly anticipated certain results about the objects, which are shown below. First, let us consider two examples, which should help clarify the definition.

Example: Let $W = W_{21} = \langle s_0 : s_0^2 = 1 \rangle$. Note that $W = \{1, s_0\}$, so $H_0 = \{1\}$, and so $\mathcal{M}_0 = \{\{1\}, \{s_0\}\}$. Then, $\mathcal{M}$ will just be the set $\{\{1\}, \{s_0\}, M\}$, with the partial order $\leq$ satisfying $\emptyset \leq \{1\}, \{s_0\}$, and $\{1\}, \{s_0\} \leq M$. Note then (by Theorem 3.2.2) that $\mathcal{M}$ is an Euler $1$-polytope, in fact (by Theorem 3.3.9) it is the only $1$-polytope, up to isomorphism. Thus in particular, any $1$-polytope is universal.

Example: Let $W = W_{31} = \langle s_0, s_1 : s_0^2 = s_1^2 = (s_0 s_1)^2 = 1 \rangle$. It can be shown that $W = \{1, s_0, s_1, s_0 s_1, s_1 s_0, s_0 s_1 s_0 s_1\}$. Let us write $s_0 = A$ and $s_1 = B$, to make things easier to read, so $W = \{1, A, B, AB, BA, ABA\}$. Now $H_0 = \{B\} = \{1\}$, and similarly, $H_1 = \{1, A\}$. Then $\mathcal{M}_0$ has three elements: $\mathcal{M}_0 = \{\{1\}, \{A, AB\}, \{BA, BAB\}\}$ (note that $BAB = ABA$). Also, $\mathcal{M}_1 = \{\{1, A\}, \{B, BA\}, \{AB, ABA\}\}$, and it is easy to see that the relation $\leq$ is: for example we have $\{1\} \leq \{1, A\}, \{B, BA\}$ (since neither $\{1\} \cap \{1, A\}$ nor $\{B, BA\}$).  

In fact, J. Tits in [55] gives the name "polyèdre régulier" or regular polyhedron to what we have called a universal complex.
Lemma 5.2.2: Any chain $C$ of $\mathcal{M}$ may be written $D \cup \{uH_i : i \in I\}$ for some $D \subseteq \{\emptyset, M, E\}$, $I \subseteq \{0, 1, \ldots, d-1\}$, and $u \in W$.

Proof: Let $C$ be a chain. Note first that we may write it as $D \cup E$, where $D \subseteq \{\emptyset, M, E\}$ and $E = \{uH_i : i \in I\}$. But without loss, we may assume that all the $i_n$ are distinct, for if $i_k = i_l$ for some $k \neq l$, then $u_H \cap u_L$ is nonempty, implying that in fact the two cosets $u_H$ and $u_L$ are equal, and that one of them may be struck from our description of $E$. This implies in particular that $E$ is a finite set, since the $i_n$ are distinct integers between 0 and $d-1$ (inclusive). Let us further assume that $i_1 < i_2 < \ldots < i_m$, where $m$ is the size of $E$. Now if this is the case, then $uL_i \cap \ldots \subseteq uL_{i_m}$. Let $I = \{i_1, \ldots, i_m\}$. Then we have $uH_k \cap uL_j$ is nonempty for any $k, l \in I$. Then, Lemma 5.1.8 shows that there exists some $u \in W$ such that $uH_j = uH_j$ for each $j \in I$. Thus in fact the set $E$ may be written as $\{uH_j : j \in I\}$, where $I$ is a subset of $\{0, 1, \ldots, d-1\}$. Reminding ourselves that our chain $C$ was equal to $D \cup E$, where $D \subseteq \{\emptyset, M, E\}$, completes the proof.

Lemma 5.2.3: Let $C = D \cup \{uH_i : i \in I\}$ be a chain of $\mathcal{M}$, where $D \subseteq \{\emptyset, M, E\}$, $I \subseteq \{0, 1, \ldots, d-1\}$, and $u \in W$. Then $C$ is a maximal chain if and only if $I = \{0, 1, \ldots, d-1\}$ and $D = \{\emptyset, M, E\}$.

Proof: If $D \neq \{\emptyset, M, E\}$ or $I \neq \{0, 1, \ldots, d-1\}$, then $C$ will not be maximal, since it will be properly contained in the chain $\{\emptyset, M, \ldots, uH_i : 0 \leq i \leq d-1\}$. Conversely, if $D = \{\emptyset, M, E\}$ and $I = \{0, 1, \ldots, d-1\}$, then $C$ is maximal, since any chain properly contained it would have at least $d+3$ elements, and since by Lemma 5.2.2, any chain of $\mathcal{M}$ has at most $d+2$ elements.

Lemma 5.2.4: $\mathcal{M}$ satisfies I2, and has dimension $d$.

Proof: We have deduced that the flags of $\mathcal{M}$ are those chains of the form $\{\emptyset, M, \ldots, uH_i : 0 \leq i \leq d-1\}$. But any chain that is not already a flag will be contained in a chain of this form, as was noted during the proof of the previous lemma. Thus all chains are contained in flags. Note also that all flags have exactly $2^d + d$ elements, and hence have the same length, $d+1$. This completes the proof.

Lemma 5.2.5: $\mathcal{M}$ satisfies I3.

Proof: We show that $\mathcal{M}$ is flag connected. Let $\phi_u$ denote the flag $\{\emptyset, M, \ldots, uH_i : 0 \leq i \leq d-1\}$. By Lemma 5.2.3, any flag of $\mathcal{M}$ may be written in this form. Now, let $\phi_u$ and $\phi_v$ be two arbitrary flags of $\mathcal{M}$. We wish to show that there is a sequence $\phi_{u_0}, \phi_{u_1}, \ldots, \phi_{u_m} = \phi_v$ of flags of $\mathcal{M}$ such that for any $k$, $F_k$ and $F_{k+1}$ differ by exactly one element. To this end, let $u^{-1}v = s_1s_2\ldots s_m$, and let $u_k = u_{s_1}u_{s_2}\ldots u_{s_m}$ and let $F_k = \phi_{u_k}$. Then $\phi_u = F_0, \phi_v = F_m$, and for

Figure 5.2.1

\[ \{1, B\} \cap \{B, BA\} = \emptyset, \{A, AB\} \leq \{1, A\}, \{AB, ABA\} \]
each $k$, the only elements of $F_k$ and $F_{k+1}$ which differ are those of dimension $i_{k+1}$, namely $u_k H_{i_{k+1}}$ and $u_{k+1} H_{i_{k+1}} = u_k H_{i_{k+1}}$. □

We shall soon characterise the sections of $\mathfrak{M}$, but first, we make the following definition.

Definition: For $w \in W$, let $\alpha_w : M \rightarrow \mathfrak{M}$ be defined by $\Theta_{\mathfrak{M}} \alpha_w = \Theta_{\mathfrak{M}}, \quad \alpha_w = M, \quad \text{and} \quad (u H_i) \alpha_w = w_i H_i$.

Lemma 5.2.6: The map $\alpha : w \mapsto \alpha_w^{-1}$ is a one to one homomorphism from $W$ to $\text{Aut} \mathfrak{M}$.

Proof: We first prove that $w_0 \alpha \in \text{Aut} \mathfrak{M}$ for any $w \in W$. Now $\alpha_{w_0^{-1}}$ is a bijection, so $\alpha_{w_0^{-1}}$ will serve as an inverse to $\alpha_w^{-1}$. It also preserves the partial order, since $w^{-1} H_i \cap w^{-1} H_j = w^{-1} H_i \cap H_{j} = w^{-1} H_i \cap H_{j}$ is nonempty if and only if $H_i \cap H_j$ is. Thus $\alpha_w$ is a well-defined map from $W$ to $\text{Aut} \mathfrak{M}$. It is a homomorphism, since for any $u H_i \in \mathfrak{M}$, we have $(u H_i) \alpha (w^{-1}) = w^{-1} w_0^{-1} u H_i = u H_i \alpha_w^{-1} = u H_i \alpha_w u_0^{-1}$, so $(w^{-1}) \alpha = \alpha \alpha_w^{-1} = \alpha_{w_0^{-1}} \alpha_w = (w_0 \alpha)(w_0 \alpha)$. Finally, if $w_0 \alpha = w_0 \alpha$, then in particular, $w^{-1} H_i = w_0^{-1} H_i$, for all $i$, so $\{w^{-1}\} = \bigcap_{i=0}^{1} w^{-1} H_i = \bigcap_{i=0}^{1} H_i = \{w^{-1}\}$, hence $w = w_0$ as required. □

We will examine the map $\alpha$ again in Lemma 5.2.18 and find that it is also onto, making it an isomorphism between $W$ and $\text{Aut} \mathfrak{M}$.

Lemma 5.2.7: The facets and vertex figures of a universal complex are universal complexes, based on $H_{d-1}$ and $H_0$, respectively.

Proof: We do the proof for the vertex figures – the proof for the facets will be very similar. Any vertex figure of $\mathfrak{M}$ will be of the form $(u H_0, M)$ for some $u \in W$. By the above lemma and by Theorem 3.2.2, this will be isomorphic to $(H_0, M)$ (since $\alpha_{u}$ restricted to $(H_0, M)$ will be an isomorphism from $(H_0, M)$ to $(H_0, M)$). Now if $H_0 \leq v H_0$, then $v H_0 \cap H_0$ is not the empty set, and so there exists $h \in H_0$ such that $h \in v H_0$, whence $v H_0 = h H_0$. Thus any element $u H_0$ of $(H_0, M)$ may be written as $h H_0$, for some $h \in H_0$. Conversely, for any $h \in H_0$, we can show that $H_0 \leq H_0 h$, and so conclude that $(H_0, M) = \{H_0, M\} \cup \{h H_0 : h \in H_0, 0 \leq i \leq d - 1\}$. Now we construct a map $\psi : (H_0, M)$ to the universal complex $\mathfrak{M}$ based on $H_0$, via $\psi : h H_0 \mapsto h H_0$, for any $h \in H_0$, and also $\psi = N$ and $H_0 \psi = \emptyset$. This map will be well defined, for if $h H_0 = h' H_0$ for $h, h' \in H_0$, then $h H_0 \cap H_0 = h' H_0 \cap H_0$, which implies that $H_0 h_0 = H_0 h_0$. The map is also one to one, for if $h H_0 = h' H_0$, then $h H_0 = h' H_0$. Thus in particular $h' H_0 = H_0$, and so $h H_0 = h' H_0$ as required. Thirdly, $\psi$ is also onto, since for any $h H_0$, we will have $H_0 h_0 = (h H_0) \psi$. Finally, note that for any $h H_0, h' H_0 \in (H_0, M)$, we have $h H_0 \leq h' H_0$ if and only if $i \leq j$ and $h H_0 \cap h' H_0$ is nonempty. However $H_0 \cap H_0$ is also nonempty, whence (by Lemma 5.1.8) $H_0 \cap H_0 = \emptyset$ is nonempty. This latter set is equal to $(H_0 \cap h H_0)$, which equals $h H_0$, which is nonempty if and only if $H_0 h_0 \leq h' H_0$ in $\mathfrak{M}$. We have shown, therefore, that $\psi$ is an isomorphism from $(H_0, M)$ to $\mathfrak{M}$. It follows that any vertex figure of $\mathfrak{M}$ is isomorphic to $\mathfrak{M}$, for $(u H_0, M)$ is such a vertex figure, then $\psi^{-1} \alpha_{u}$ is an isomorphism from the universal complex $\mathfrak{M}$ to $(u H_0, M)$. □

Theorem 5.2.8: The sections of $\mathfrak{M}$ are universal complexes, based on parabolic subgroups of $W$.

Proof: (See also [55, Thm 3.4.3].) This could be shown from Lemmas 5.2.7, since any $i$-section is a facet or vertex figure of an $(i+1)$-section.

In fact, we could show that if $\mathfrak{M} = (x, y)$ is a section with dim $x = i$ and dim $y = j$, then $\mathfrak{M}$ is based on the Coxeter group $(A_k : i + 1 \leq k \leq j - 1) = W(i+1, i+2, \ldots, j-1)$. These will turn out to be very useful results. Note in particular that it allows us to calculate the dimensions in $\mathfrak{M}$ of the elements of $\mathfrak{M}$. Consider $(\Theta_{\mathfrak{M}}, u H_{d-1})$. This will be a universal $(d-1)$-complex. Its facets, of the form $(\Theta_{\mathfrak{M}}, u H_{d-2})$, will be universal $(d-2)$-complexes. Inductively, we could show that $(\Theta_{\mathfrak{M}}, u H_0)$ is a universal $i$-complex, for any $i$, and therefore that it has dimension $i$. It follows that dim $\Theta_{\mathfrak{M}} u H_0 = \text{dim}(\Theta_{\mathfrak{M}}, u H_0) = i$. This vindicates our use of the symbol $\mathfrak{M}$ for the set $\{u H_0 : u \in W\}$.

The theorem also makes it simple to completely checking that $\mathfrak{M}$ satisfies all of the axioms required for it to be an incidence polytope.

Theorem 5.2.9: For any $x, y \in \mathfrak{M}$, if $x \leq y$ and dim $x + 1 = \text{dim} y - 1$, then the set $\{z : x < z \leq y\}$ has exactly 2 elements, and hence the universal complex is in fact a $d$-incidence polytope.

Proof: From Theorem 5.2.8, there is an isomorphism from $(x, y)$ to the universal complex $\mathfrak{M}$, based on the Coxeter group $(a) = \{1, s\}$. Under this isomorphism, the set $\{z : x < z \leq y\}$ maps to $\{1, s\}$, and thus has two elements. From this, and from Lemmas 5.2.1, 5.2.4 and 5.2.5, the result follows. □

Thus we are justified in calling these objects universal polytopes. We call them universal complexes, since it shall turn out that any polytope may be regarded as some kind of "quotient" of a universal complex, so the universal complex is a kind of "universal cover" for the polytope. We shall at times use the word universal as an adjective.

Let us obtain some more information about these polytopes.
Theorem 5.2.10: For each \( i \neq -1, d, |\mathcal{A}_i| = |W : H_i| \).

Proof: This follows trivially from the fact that \( \mathcal{A}_i \) is the set of left cosets of \( H_i \) in \( W \).

In this thesis, the group \( W \) is usually chosen so that the \( H_i \) are all finite. In this circumstance, the \( \mathcal{A}_i \) are finite if and only if \( W \) is, in which case we have \( |\mathcal{A}_i| = |W|/|H_i| \). Tables A.7 lists the values of \( |W| \) and \( |\mathcal{A}_i| = |W|/|H_i| \) for various particular Coxeter groups \( W \), and certain particular \( i \).

Lemma 5.2.11: If \( W \) is finite, then \( \mathcal{A}_i \) satisfies Euler’s relation.

Proof: Let \( W = W_{\{g_1, \ldots, g_{d-1}\}} \). If \( W \) is finite, then all the \( \mathcal{A}_i \) are also finite. Now \( |\mathcal{A}_{-1}| = |\emptyset \mathcal{A}| = 1 \) and \( |\mathcal{A}_d| = |\{M\}| = 1 \). For \( 0 \leq i \leq d - 1 \), the size of the set \( \mathcal{A}_i \) will be the index of \( H_i \) as a subgroup of \( W \). As \( W \) is finite, this index will equal \( |W|/|H_i| \). Now \( H_i = W_{\{g_i\}} \times W_{\{g_j\}} \), which will equal \( W_{\{g_1, \ldots, g_{i-1}\}} \times W_{\{g_{i+1}, \ldots, g_{d-1}\}} \), so the order of \( H_i \) is the product of the orders of these two finite Coxeter groups. Since [11] and [14] give the orders of all the finite Coxeter groups, we could prove the theorem by calculating all of the \( |\mathcal{A}_i| \) for every possible \( \{g_1, \ldots, g_{d-1}\} \). (There are only five or six individual cases and three or four infinite families.) However, the following argument is shorter.

Coxeter tells us, in Equation 7.63 of [12], that given a regular geometric polytope with Schläfli Symbol \( \{q_1, \ldots, q_{d-1}\} \), the number of its \( d \)-faces is

\[
\frac{|W_{\{g_1, \ldots, g_{i-1}\}}|}{|W_{\{g_1, \ldots, g_{i-1}\}}|/|W_{\{g_1, \ldots, g_{d-1}\}}|}.
\]

It should be noted that Coxeter uses quite different notation. Now this expression is just what we obtained for \( |\mathcal{A}_i| \), so the number of \( d \)-faces of a regular geometric polytope with Schläfli Symbol \( \{q_1, \ldots, q_{d-1}\} \) is \( |\mathcal{A}_i| \). We may then apply Theorem 2.3.9 (which gives Euler’s formula for geometric polytopes) to yield

\[
\sum_{i=0}^{d-1} (-1)^i |\mathcal{A}_i| = (-1)^{-1} |\mathcal{A}_{-1}| + \sum_{i=0}^{d-1} (-1)^i |\mathcal{A}_i| = (-1)^d |\mathcal{A}_d| = (-1)^d + (1 - (-1)^d) + (-1)^d = 0
\]

as required.

From this, we obtain some immediate corollaries.

Corollary 5.2.12: If \( W \) is finite, then \( \mathcal{A}_i \) is Euler.

Proof: This will be so, since all of the sections of \( \mathcal{A}_i \) will be universal polytopes based on (finite) parabolic subgroups of the (finite) group \( W \).

Corollary 5.2.13: If \( H_0 \) and \( H_{d-1} \) are finite, then \( \mathcal{A}_i \) is sub-Euler.

Proof: Any \( i \)-section (\( 0 \leq i \leq d-1 \)) of \( \mathcal{A}_i \) will be a universal polytope based on some subgroup of either \( H_0 \) or \( H_{d-1} \) (if not both). Thus, since this subgroup will be finite, the section will satisfy Euler’s relation. Thus \( \mathcal{A}_i \) is (\( d-1 \))-Euler, as required.

We close this subsection by investigating the flag group and the automorphism group of a universal polytope. Let \( \phi_u \) denote the set \( \{0, \mathcal{A}_i \} \cup \{uH_i : 0 \leq i \leq d-1\} \). However, the following argument is shorter.

Theorem 5.2.14: There is a one to one correspondence \( \psi \) between the group \( W \) and the set of flags of \( \mathcal{A}_i \), via \( \psi : u \to \phi_u \).

Proof: \( \psi \) is onto, since as we saw in Lemma 5.2.3, any flag of \( \mathcal{A}_i \) may be written as \( \phi_u \) for some \( u \in W \). Now, consider the case where \( u \psi = v \psi \).

Then, in fact, \( uH_i = vH_i \) for all \( i \), and so in particular \( \bigcap_{i=0}^{d-1} uH_i = \bigcap_{i=0}^{d-1} vH_i \).

But \( \bigcap_{i=0}^{d-1} uH_i = \bigcap_{i=0}^{d-1} vH_i = uH_{\{g_1, \ldots, g_{d-1}\}} = u \{1\} \), and so we must have \( uH_{\{g_1, \ldots, g_{d-1}\}} = vH_{\{g_1, \ldots, g_{d-1}\}} \), which is \( u = v \). This tells us that the map is one to one, and hence is a one to one correspondence, as stated.

From this there follows another very useful theorem.

Theorem 5.2.15: \( \mathcal{A}_i \) is flag regular.

Proof: Let \( F = \phi_u \) and \( G = \phi_v \). Then, recall that \( \alpha_{g_{d-1}} \) will be an automorphism of \( \mathcal{A}_d \), where for any \( wH_i \in \mathcal{A}_i \), we have \( (wH_i)\alpha_{g_{d-1}} = w^{-1}H_i \). Then, \( (uH_i)\alpha_{g_{d-1}} = u^{-1}H_i = vH_i \), so in fact \( (\mathcal{A}_i)\alpha_{g_{d-1}} = G \). Since \( F \) and \( G \) were arbitrary, we must conclude that Aut.\( \mathcal{A}_i \) acts transitively on \( F(\mathcal{A}_i) \).

Since the universal polytope is regular, it is also combinatorially regular, and so has a well-defined Schläfli Symbol.
Theorem 5.2.16: The Schläfli Symbol of the universal polytope based on the Coxeter group $W = W_{\{e_1, ..., e_{k-1}\}}$ is $\{q_1 | ..., q_{k-1}\}$.

Proof: Let $x \leq y$ with $\dim x + 1 = i$ and $y = \dim y - 1$. From the notes following Theorem 5.2.8, the universal $2$-polytope $\mathcal{F}$, based on the Coxeter group $(s_1, ..., s_k)$, which has order $2k$, (see [11, 14, or Table A.1]). The section is therefore isomorphic to a $q_i$-cycle (since $\mathcal{F}_0 = \mathcal{F}_1 = \mathcal{F}_2$ by Theorem 5.2.10), so the $i$th entry of the Schläfli Symbol of $\mathcal{M}$ is $q_i$.

Given a Coxeter group with Coxeter graph

```
\begin{center}
  q_1 \rightarrow \cdots \rightarrow q_{k-1}
\end{center}
```

we have defined from it a polytope $\mathcal{M}$, which turned out to be regular, and to have Schläfli Symbol $\{q_1 | ..., q_{k-1}\}$. It follows that the flag action of $W$ on $\mathcal{M}$ is well-defined. In fact, we have the following result.

Theorem 5.2.17: Let $\phi_v$ be a flag of $\mathcal{M}$, and let $v \in W$. Then $(\phi_v)^v = \phi_v$.

Proof: We do the proof inductively on the length of $v$. Certainly the result will hold if $v$ is the identity element of $W$. Now, let $v = s_i$ (a generator of $W$), and consider the flag $\phi_v = \{\emptyset, M\} \cup \{uH_0, ..., uH_{k-1}\}$. Since $uH_0 = H_i$ if and only if $i \neq j$, it follows that $\phi_v$ is equal to $\{\emptyset, M\} \cup \{uH_0, ..., uH_{i-1}, uH_i, uH_{i+1}, ..., uH_{k-1}\}$, and thus $\phi_v$ is a flag of $\mathcal{M}$ which differs from $\phi_0$ by exactly one element, $uH_i$, of dimension $i$. It follows that $(\phi_v)^v = \phi_v$. Now, assume that for any element $v'$ of $W$ whose length is less than $k$, we have $(\phi_v)^v = \phi_v$, and let $v = s_i, ..., s_j$. Then $(\phi_v)^v = (\phi_0)^v$ (by definition) equal to $(\phi_{s_j})^{s_{j-1}}...^{s_{i+1}}$. This will then be equal to $(\phi_{s_j})^{s_{j-1}}...^{s_{i+1}}$, which by the inductive hypothesis equals $\phi_{s_j, ..., s_i}$. This completes the induction, and gives the desired result.

So $(\phi_v)^v = \phi_v$. Note however that the automorphism $\alpha_v$ will act differently on the set of flags, for $(\phi_v)^{\alpha_v} = \phi_v$. We now proceed to characterise the automorphisms of $\mathcal{M}$, using the link between the automorphism action and the flag action that was established in Corollary 5.1.19.

Lemma 5.2.18: The map $\alpha': W \rightarrow \text{Aut.}\mathcal{M}$ taking $w$ to $\alpha_w^{-1}$ is onto.

Proof: Let $\beta$ be an automorphism of $\mathcal{M}$, and write $(\phi_1)^\beta = \phi_w$. By Theorem 5.2.14, this $w \in W$ exists and is unique, since all flags of $\mathcal{M}$, including $(\phi_1)^\beta$, in particular, may be written in this form. We shall show that $\beta = \alpha_w = w^{-1}\alpha'$. To this end, we let $uH_i \in \mathcal{M}$, and compare $(uH_i)^\beta$ with $(uH_i)\alpha_w$. The latter is easy to calculate: $(uH_i)\alpha_w = uH_i$. To calculate $(uH_i)^\beta$, let us consider a flag containing $uH_i$, namely $\phi_w$. Now $(uH_i)^\beta$ is equal to $((\phi_1)^\beta)_i$, which equals $(((\phi_1)^\beta)_i)$. But $((\phi_1)^\beta)_i = ((\phi_1)^\beta)_i$, since the actions of $\text{Aut.}\mathcal{M}$ and $W$ on $F(\mathcal{M})$ commute (Theorem 5.1.16). Also, we know that $(\phi_1)^\beta = \phi_w$, so we have $((\phi_1)^\beta)_i = ((\phi_1)^\beta)_i = (\phi_w)_i$, by the preceding theorem. Thus $(uH_i)^\beta = uH_i = (uH_i)\alpha_w$ for any $uH_i \in \mathcal{M}$, and so $\beta = \alpha_w = w^{-1}\alpha'$ as required.

From this, and from Lemma 5.2.6, we immediately obtain the following.

Theorem 5.2.19: $\text{Aut.}\mathcal{M}$ is isomorphic to $W$.

This is the first result we have obtained for characterising the automorphism group of a significant class of combinatorially regular Euler polytopes. It is not the last, however. In the next section, we define the quotient polytopes. We will be able to say a great deal about the polytopes in this class as well.

5.2.2 Quotient Polytopes:

It may not be totally obvious, but in essence, when we define a quotient polytope, we are partitioning a universal polytope into certain equivalence classes, and treating each equivalence class as an element of a partially ordered set. This is a common technique in many branches of algebra. It was used in the previous chapter to obtain the half cubes from the cubes, and the lattices from the universal lattices.

Note that [39] also examines quotients of polytopes by subgroups of a group acting on them. A few of the ideas and results here appear there also there, but most, and particularly the focus on sparse subgroups, are new.

Let $\mathcal{M}$ be the universal polytope based on a Coxeter group $W$ with $d$ generators. The objective in this section will be to take a kind of "quotient" of $\mathcal{M}$ by some subgroup $A$ of $W$, and so obtain another polytope. The definition is as follows. Let $\mathcal{A} = \{\emptyset, M\}$, $\mathcal{Q} = \{Q\}$, and for $0 \leq i \leq d - 1$, let $\mathcal{A}_i = \{AuH_i : u \in W\}$, where $AuH_i$, the double coset of $A$ and $H_i$, is defined to be $\{auh : a \in A, h \in H_i\}$. It will be helpful to know a little about these double cosets.

Lemma 5.2.20: $AuH_i = AuH_i$ if and only if $v \in AuH_i$.

Proof: Note that if $AuH_i = AuH_i$, then $v = 1.v.1 \in AuH_i$, so $v \in AuH_i$. Likewise, if $v \in AuH_i$, then there exists $a \in A$ and $h \in H_i$ such that $v = ah$. But then $AuH_i = AauhH_i = AuH_i$, since $Ah = A$ and $hh_i = h_i$. 


Lemma 5.2.21: If $AuH_i = AuH_j$ for any $j \neq i$, then $AuH_i = W$.

Proof: By Lemma 5.2.20, $u \in AuH_i$, so $u \in AuH_j$. Again by this lemma, we have $AuH_j = AuH_j$, whence $AuH_i = AuH_j$. Multiplying on the right by $H_j$ yields $AuH_i H_j = AuH_j = AuH_i$. But then, multiplying $AuH_i H_j = AuH_i$ on the right by $H_j$ yields $AuH_i = AuH_i H_j H_i$. By an inductive argument we could show that $AuH_i = Au(H_i)^m H_j$ for all $k \geq 1$. Now if $w \in W$, then we can write $u^{-1}w = s_1 a_1 \ldots s_t a_t$, which in turn may be written $u^{-1} w = v_1 a_1 v_2 a_2 \ldots v_m a_m v_{m+1}$ for some $m \in Z^+$, where each $a_i$ is a word in $\{s_1, \ldots , s_t, \ldots , s_1\}$, and is therefore an element of $H_i$. Thus $u^{-1}w \in (H_i)^m H_i$, and so $w \in u(H_i)^m H_i$, which is a subset of $AuH_i$. Thus $W \subseteq AuH_i$, yielding $W = AuH_i$ as required.

Note that under this circumstance, $\mathcal{D}_i$ and $\mathcal{D}_j$ contain exactly one element each. Returning to the $\mathcal{D}_i$, we form our quotient by taking the "disjoint union" $\mathcal{D}_i$ of all the $\mathcal{D}_i$, and equipping $\mathcal{D}_i$ with the relation $\leq$ satisfying $\theta \leq \leq Q$ for all $\theta \in \mathcal{D}_i$, and $AuH_i \leq AuH_j$ if and only if $i \leq j$ and $AuH_i \cap AuH_j$ is nonempty.

By disjoint union in this context, it is intended that the sets $\mathcal{D}_i$, be treated as disjoint – even if, in a strict set-theoretic sense, they are not. We could accomplish this by labelling each element of $\mathcal{D}_i$, with the number $i$ of the set it came from (so then $AuH_i$ would become $(AuH_i, i)$) but this is notationally very messy. An alternative way of accomplishing the same thing would be to note that there is a natural correspondence between the $AuH_i$ and the orbits of the action of $\mathcal{A}$ restricted to the group $A$ (defined via $(uH_i)^a = a^{-1}uH_i$), and so the $\mathcal{D}_i$ may be regarded as collections of orbits rather than as collections of complexes, two orbits satisfying $(uH_i)^a \leq (vH_j)^b$ if and only if $i \leq j$ and there exists $a, a' \in A$ such that $(uH_i)^a$ and $(vH_j)^b$ have nonempty intersection. In any event, we shall not be interested in the case where this need arises.

For if $AuH_i = AuH_j$ for $i \neq j$, the set $\mathcal{D}_i = \bigcup_{i=1}^{d} (\mathcal{D}_i \times \{i\})$ can form a complex – the set would (at the very least) fail to satisfy axiom I4, as the set $\{x \in \mathcal{D}_i \times \{i\} : x < x \leq y\}$ (where dim $x + 1 = i = dim y - 1$ and $x \leq y$) would have at most one element, less than the requisite minimum of two.

So to recap – the set $\mathcal{D}_i$ is the disjoint union of the sets $\mathcal{D}_i$, where $\mathcal{D}_i = \{\theta \}$, $\mathcal{D}_d = \{Q\}$, and $\mathcal{D}_j = \{AuH_i : u \in W\}$. We equip $\mathcal{D}_i$ with a relation $\leq$ which makes $\theta$ and $Q$ a minimum and a maximum respectively, and for other elements of $\mathcal{D}_i$, the relation $AuH_i \leq AuH_j$ holds if and only if $i \leq j$ and $AuH_i \cap AuH_j$ is nonempty. $\mathcal{D}_i$ will at times be denoted $\mathcal{D}_i/A$, for example when it is necessary to distinguish the subgroup $G$ being used. We call $\mathcal{D}_i$ the quotient of $\mathcal{A}$ by $A$. A complex isomorphic to $\mathcal{D}/A$ for some $\mathcal{A}$ and $A$ will be called a quotient complex, if it is a polytope, a quotient polytope.

It will be helpful to have a result for double cosets similar to Lemma 5.1.8.

Lemma 5.2.22: Let $0 \leq i_1 < i_2 < \ldots < i_m \leq d - 1$, and let $w_1, \ldots , w_m \in W$ be such that for any $j \neq m$, $(AuH_{i_j} \cap AuH_{i_{j+1}})$ is nonempty. Then there exists some $u \in W$ such that $AuH_{i_j} = AuH_{i_{j+1}}$ for all $j$.

Proof: By induction on $m$. If $m = 1$, there is nothing to prove. If $m = 2$, we may let $u \in AuH_{i_1} \cap AuH_{i_2}$ and then Lemma 5.2.20 yields the result. Now, assume that the lemma is true for $m < k$, and let $m = k \geq 3$. Now since the sequence $AuH_{i_1}, \ldots , AuH_{i_k}$ satisfies all the conditions of the lemma, yet has less than $k$ elements, we deduce that there exists $w \in W$ such that $AuH_{i_j} = AuH_{i_{j+1}}$ for all $j \geq 2$. Likewise, there exists $w \in W$ such that $AuH_{i_j} = AuH_{i_{j+1}}$ for all $j \leq 2$. In particular, note that $AuH_{i_1} = AuH_{i_2}$. Thus, $v \in AuH_{i_1}$, and there exists some $a \in A$ and $h \in H_{i_1}$ such that $v = auh$. But $H_{i_1} = W_{i_1}gW_{i_1}$, so we can write $h = gh$, where $g \in W_{i_2}$ and $h \in W_{i_1}$. Note that $W_{i_2} \leq W_{i_1} \leq H_{i_1}$, and for all $j \geq 2$, we have $W_{i_2} \leq W_{i_1} \leq H_{i_1}$, so $h \in H_{i_1}$ if $j \geq 2$ and $g \in H_{i_1}$ if $j \leq 2$. Let $u = v_{i_2}$. Then for $j \geq 2$, we have $AuH_{i_j} = AuH_{i_{j+1}} = AuH_{i_j} = AuH_{i_{j+1}} = AuH_{i_j}$, and for $j \leq 2$, we have $AuH_{i_j} = AuH_{i_{j+1}} = AuH_{i_j} = AuH_{i_{j+1}}$. Hence we have found an element $u \in W$ as required to complete the induction.

Now, we can begin to tackle $\mathcal{D}$ in earnest.

Lemma 5.2.23: Under $\leq$, $\mathcal{D}$ is a poset, with unique minimal and maximal elements $\emptyset_0$ and $Q$ respectively.

Proof: The relation is antisymmetric, by definition. It is also reflexive. We may show it is transitive using the previous lemma. Suppose $AuH_{i_j} \leq AuH_{i_{j+1}}$, and $AuH_{i_{j+1}} \leq AuH_{i_{j+2}}$. Then $0 \leq i_1 < i_2 < i_3 < d - 1$, and neither $AuH_{i_1} \cap AuH_{i_2}$ nor $AuH_{i_2} \cap AuH_{i_3}$ is empty. Thus there exists $w \in W$ such that $AuH_{i_1} = AuH_{i_2}$ for $1 \leq j \leq 3$. But then, $u \in AuH_{i_1} \cap AuH_{i_2}$, so this intersection is nonempty, which shows that $AuH_{i_1} < AuH_{i_2}$ as required. Showing transitivity for the cases involving $\emptyset_0$ or $Q$ is trivial. Thus $\leq$ is a partial order. Finally, since $\emptyset_0$ and $Q$ are a minimum and a maximum respectively in $\mathcal{D}$, they must also be the unique minimal and maximal elements of $\mathcal{D}$.

Let us examine, by way of example, the case when $A = \{1\}$.
Lemma 5.2.25: Any chain of $\mathcal{L}$ may be written as $D \cup \{AuH_i : i \in I\}$ where $I \subseteq \{0, 1, \ldots, d-1\}$ and $D \subseteq \{\emptyset, A\}$.

Proof: If $AuH_i \neq AuH_j$, then $AuH_i \cap AuH_j$ is empty, and so $AuH_i$ and $AuH_j$ cannot be distinct elements of the one chain. Thus we can write our chain as $D \cup \{AuH_i : 1 \leq i \leq m\}$, where $D$ is as required, and $0 \leq i_1 < \cdots < i_m \leq d-1$. But then Lemma 5.2.22 tells us that without loss, we can assume that all the $u_{i_j}$ are identical. Setting $I = \{i_1, i_2, \ldots, i_m\}$ yields the required result.

Lemma 5.2.26: Any flag of $\mathcal{L}$ is of the form $(\emptyset, Q) \cup \{AuH_i : 0 \leq i \leq d-1\}$.

Proof: The proof of this lemma is analogous to that of Lemma 5.2.3.

Lemma 5.2.27: $\mathcal{L}$ satisfies I2, and has dimension $d$.

Proof: Certainly every chain is contained in a flag, for if $D \cup \{AuH_i : i \in I\}$ (where $D \subseteq \{\emptyset, Q\}$) is a chain, it is contained in the flag $(\emptyset, Q) \cup \{AuH_i : 0 \leq i \leq d-1\}$. Also, all the flags have exactly $d+2$ elements.

Lemma 5.2.28: $\mathcal{L}$ satisfies I3.

Proof: The proof of this is akin to the proof of Lemma 5.2.5.

Knowing that $\mathcal{L}$ satisfies I2, we can talk about the sections of $\mathcal{L}$, and the dimensions of elements of $\mathcal{L}$, and so forth. However, it is difficult to classify the sections of quotients by arbitrary subgroups $A$ of $W$, and (as it happens) such strong results are not really needed for the classification of combinatorially regular Euler incidence complexes. We therefore, let us introduce the notion of a "sparse" subgroup of $W$.

Definition: A subgroup $A$ of $W$ is said to be sparse if for all $w \in W$, we have $A \cap wKw^{-1} = \{1\}$, where $K = H_0H_{d-1}$ if $d > 1$, or $K = W = \{1, w_0\}$ if $d = 1$.

Note that in the latter case ($d = 1$), the Coxeter group has only two subgroups, only one of which is sparse, namely $\{1\}$. The following results will be useful.

Lemma 5.2.29: For $i < j$, $tH_iH_jt^{-1} \subseteq tH_0H_{d-1}t^{-1}$.

Proof: We have but to show that $H_iH_j \subseteq H_0H_{d-1}$. Now if $i < j$, then $H_iH_j = W_{x_i}W_{x_j}W_{x_i} < W_{x_i}W_{x_j}W_{x_j}C_i \subseteq W_{x_i}W_{x_j}W_{x_i}C_i$. But $W_{x_i}W_{x_j} \subseteq H_0$, and $W_{x_i}W_{x_j} \subseteq H_0$. Thus $H_iH_j \subseteq H_0H_{d-1}$, and $tH_iH_jt^{-1} \subseteq tH_0H_{d-1}t^{-1}$ as required.

This tells us that if $A$ is sparse, then for any $i$ and $j$ with $i < j$ and for any $t \in W$, we have $A \cap tH_iH_jt^{-1} = \{1\}$.

Recall that even if $AuH_i = AuH_j$ for some $i \neq j$, we are treating $AuH_i$ and $AuH_j$ as distinct.

 Lemma 5.2.30: Let $u, v \in W$. If $A$ is sparse, and $i \neq j$, then $AuH_i \neq AuH_j$.

Proof: Assume without loss that $i < j$. If $AuH_i = AuH_j$, then in fact $u \in AuH_j$, so $AuH_i = AuH_j = AuH_j$. This being so, we find that for all $h \in H_i$, it is the case that $uh_i \in AuH_j$, so that for all $h_i \in H_i$, there exists $h_j \in H_j$, $a \in A$ such that $uh_i = auh_j$, that is, $(a = u)h_i^{-1}u^{-1}$, whence $a = 1$ by the previous lemma. Thus, for all $h \in H_i$, there exists $h_j \in H_j$ such that $h \in u^{-1}uh_j = h_j$, so in fact $H_i \subseteq H_j$. We could similarly show that $H_j \subseteq H_i$. This is not possible (as could be shown, for example, from Theorem 5.1.2), give that $i \neq j$.

Lemma 5.2.31: Let $x, y \in W$, and let $A \leq W$ be sparse. If, for every $i \in \{0, 1, \ldots, d-1\}$ there exists $a_i \in A$ such that $axH_i = yH_i$, then all the $a_i$ are identical, equal to $a$, say. If, in addition, $I = \{0, 1, \ldots, d-1\}$, then $ax = y$.

Proof: If $d = 1$ there is nothing to prove for the first part of the lemma, since there is only one $a_i$ in that case. If $d > 1$, consider $yH_i \cap yH_j = axH_i \cap xH_j$. This set contains $y$. Assume without loss that $i < j$. Then there exists $h \in H_j$ and $h_j \in H_j$ such that $axh_i = ah_jh_i$, and so $a_i = a$, say. It follows that for all $k \in I$, $a_k = a$, as required. This proves the first part of the lemma. Continuing this train of thought (which from now is also valid when $d = 1$), note that $axH_i = yH_i$ for all $i \in I$. If in fact $I = \{0, 1, \ldots, d-1\}$, then $y = H_0 \cap \cdots \cap H_{d-1} = H_0 \cap \cdots \cap H_{d-1} = \{ax\}$, yielding $x = ax$ as required.

Now, we consider the map $\psi : A \rightarrow \mathcal{L}$ defined via $\emptyset \psi = \emptyset$, $M \psi = Q$, and $(uH_i) \psi = AuH_i$, which we shall call the projection from $A$ onto $\mathcal{L}$. We have the following.

Lemma 5.2.32: For any $x \in \mathcal{L}$, there exists $x \in A$ such that $\psi x = x'$. Furthermore, if $x', y' \in \mathcal{L}$, are such that $x' \neq y'$, then for all $x \in A$ with $\psi x = x'$, there exists $y \in A$ with $\psi y = y'$ and $x \neq y$.

Proof: If $AuH_i \in \mathcal{L}$, then $AuH_i = (uH_i) \psi$, and $\emptyset \psi = \emptyset$ and $Q \psi = Q$, so the first part of the lemma holds. It also means that there is no difficulty proving the second part if either $x = \emptyset$ or $y = Q$. Now, let $x' = AuH_i$ and (without loss of generality) $x = uH_i$. Also, let $y' = AuH_j$, and let $w \in \mathcal{L}$ (of $\mathcal{L}$). Then $\psi x' = AuH_j$, so that $w = uH_j$, and $x \neq y'$, for all $x \in A$. Let $\psi x'' = \psi y''$, and let $w = \psi y''$. Then since $x = uH_i$, we have $x \neq y$, and also $(\psi y) = \psi uH_j = \psi uH_iH_j = \ψ uH_i = \psi y'$ as required.

From now on, $A$ will be sparse, unless otherwise stated. We can prove the following very strong result about the sections of the quotient $A / A$. 


Theorem 5.2.32: Let \( \mathcal{M} \) be the universal polytope based on the Coxeter group \( W \), and let \( \mathcal{B} = \mathcal{M}/A \) be a quotient of \( \mathcal{M} \) by some sparse subgroup \( A \) of \( W \). Further, let \( \psi \) be the projection of \( \mathcal{M} \) onto \( \mathcal{B} \). When restricted to a proper section \((x, y)\) of \( \mathcal{M} \), \( \psi \) becomes an isomorphism from \((x, y)\) to \((x', y')\).

Proof: Let us show it first for the vertex figures of \( \mathcal{M} \). That is, we wish to show that for any \( u \in W \), the restriction \( \psi' \) of \( \psi \) to \( \mathcal{D} = (uH_0, M) \) is an isomorphism from \( \mathcal{D} \) to \( (uH_0, Q) \). Firstly, \( \psi' \) will be well-defined, for if \( uH_0 = wH_0 \), then \( v \in wH_0 \), so \( v \in AuH_0 \), whence \( (wH_0)\psi' = AuH_0 = AuH_0 = (wH_0)\psi' \), and also, if \( uH_0 \cap AuH_0 \) is nonempty, then \( AuH_0 \cap AuH_0 \) is nonempty, so indeed \( (wH_0)\psi' \in (AuH_0, Q) \) for all \( uH_0 \in \mathcal{D} \).

Next, let \( (vH_0)\psi' = (wH_0)\psi' \), so \( AuH_0 = AuH_0 \). This implies first of all that all \( t = j \), by Lemma 5.2.30. Then, since \( uH_0, vH_0 \in \mathcal{D} \), we have \( uH_0 \cap vH_0 \) and \( uH_0 \cap wH_0 \) are nonempty, and so there exist \( t, t' \in W \) such that \( uH_0 = tH_0 \) and \( vH_0 = t' H_0 \) and \( wH_0 = t'' H_0 \) (Lemma 5.1.8). Let \( h_0 \in H_0 \) be such that \( t'' = th_0 \). Then \( uH_0 = th_0 H_0 \). But if \( (wH_0)\psi' = (wH_0)\psi' \), it follows that \( AuH_0 = AuH_0 \), whence \( th_0 = th_0 \) (Lemma 5.2.20), so there exists \( h_1 \in H_1 \) such that \( th_0 = th_0 \), whence \( th_0 = th_0 \). But \( th_0 \) is also an element of \( tH_0 \) if \( tH_0 \) is a subset of \( tH_0 \) if \( tH_0 \) is a subset of \( tH_0 \). From this, and from the fact that \( A \) is sparse, it follows that \( th_0 = th_0 \). But if this is the case, then \( uH_0 = th_0 H_0 = th_0 H_0 = th_0 H_0 = th_0 H_0 \), so as required, \( \psi' \) is one to one. Likewise, it is onto, for if we let \( x = uH_0 \) and \( x' = AuH_0 \), then the previous lemma tells us that for any \( y' \in (x', Q) \), there exists \( y \in (x, M) \) with \( y' = y' \).

To show that \( \psi' \) is structure preserving, let \( vH_0 \cap wH_0 \) be nonempty. Certainly then \( AuH_0 \cap AuH_0 \) is likewise nonempty, so \( vH_0 \leq wH_0 \) implies \((wH_0)\psi' \leq (wH_0)\psi' \). Conversely, let \( vH_0 \) and \( wH_0 \) be elements of \( \mathcal{D} \) such that \( AuH_0 \cap AuH_0 \) is nonempty. Then also \( AuH_0 \cap AuH_0 \) is nonempty, there exists \( t' \in W \) such that \( AuH_0 = At' H_0 \), \( AuH_0 = At' H_0 \), and \( AuH_0 = At' H_0 \) (Lemma 5.2.22). From the latter equality, we conclude that \( u \in At' H_0 \), so there exists \( a \in A \) such that \( u = at' H_0 \) and \( h_0 \in H_0 \) such that \( a = at' H_0 \). Let \( t' = at' \). Then we have \( uH_0 = at' H_0 = H_0 \), so \( th_0 = th_0 \) if \( t' \in \mathcal{D} \). Then, \((wH_0)\psi' = AuH_0 = AuH_0 = AuH_0 \), and \((wH_0)\psi' = AuH_0 = AuH_0 = AuH_0 \). Since \( \psi' \) is a bijection from \( \mathcal{D} \) to \( (uH_0, Q) \), the fact that \((wH_0)\psi' = (wH_0)\psi' \) implies that \( vH_0 \leq wH_0 \). Similarly, \( uH_0 \leq wH_0 \) if and only if \((wH_0)\psi ' \leq (wH_0)\psi ' \), and \( \psi' \) is structure preserving.

This completes the proof that \((uH_0, M) \) and \((AuH_0, Q) \) are isomorphic.

We could similarly show that \((uH_0, uH_0) \) and \((uH_0, AuH_0) \) are isomorphic. These results, combined with the fact that any proper section of \( \mathcal{M} \) satisfying \( \Pi \) and \( \Pi \) is a section of either one of the facets or one of the vertex figures of \( \mathcal{M} \), allow us to use Theorem 3.5.2 to conclude that \((x, y) \) is isomorphic to \( (x', y') \) for any proper section \((x, y) \) of \( \mathcal{M} \).

Corollary 5.2.34: Any proper section of \( \mathcal{M}/A \) is universal.

Proof: This follows from Lemma 5.2.32, from the above result, and from Theorem 5.2.5.

This shall turn out to be a very useful result. We shall use it to prove a number of results about quotients.

Let us examine the dimensions of the sections and elements of \( \mathcal{M}/A \), where \( A \) is sparse. We have already seen that \( \dim \mathcal{B} = d \), since its flags have \( d + 2 \) elements. Now consider \((AuH_0, AuH_0) \). This will be isomorphic to \((uH_0, vH_0) \), where \( uH_0 \), \( vH_0 \) \( \in \mathcal{M} \). It follows that \( \dim (AuH_0, AuH_0) = \dim (uH_0, vH_0) = j - i - 1 \). Also, \( \dim (AuH_0, Q) = d - i - 1 \) and \( \dim (uH_0, vH_0) = j \). In particular this tells us that \( \dim AuH_0 = j \), thus justifying the notation \( \mathcal{B} \) for the set \((AuH_0 : v \in W) \).

We shall now obtain some results based on this knowledge of the sections of \( \mathcal{B} \).

Lemma 5.2.35: For any \( x, y \in \mathcal{B} \) with \( x < y \) and \( y \leq x \) = \( y \leq x \), the set \( \{x \in \mathcal{B} : x < y \text{ has two elements.} \}

Proof: If \( d > 1 \), then \((x, y) \), having dimension 1, is a proper section of \( \mathcal{B} \). Thus it is isomorphic to a 1 dimensional universal polytope (Theorem 5.2.34), and the result follows immediately (see the first example preceding Lemma 5.2.1). If instead \( d = 1 \), then \( A = \{1\} \) (being sparse), whence \((x, y) \leq \mathcal{B} \leq \mathcal{M} \) (by Lemma 5.2.24), and again the result follows.

Theorem 5.2.36: If \( A \) is a (sparse) subgroup of \( W \), the \( \mathcal{B} \) is an incidence structure.

Proof: This follows directly from Lemmas 5.2.23, 5.2.27, 5.2.28 and 5.2.35.

Bu: that's not all. We can also prove the following.

Theorem 5.2.37: If \( A \leq W \) is sparse, then \( \mathcal{B} = \mathcal{M}/A \) is a combinatorially regular polytope with the same Schl"afli Symbol as \( \mathcal{M} \). Also, if \( H_0 \) and \( H_{d-1} \) are finite, then \( \mathcal{B} \) is sub- Euler.

Proof: Let \( \mathcal{D} = (x, y) \) and \( \mathcal{D}' = (y', x') \) be corresponding (proper) sections of \( \mathcal{B} \), with \( \dim x = \dim y' \) and \( \dim y = \dim x' \). Then, by Theorems 5.2.32 and 5.2.33 (see also the notes following Theorem 5.2.8), \( \mathcal{D} \) and \( \mathcal{D}' \) are (isomorphic to the same universal polytope, and are therefore isomorphic, making \( \mathcal{D} \) combinatorially regular. If \((x, y) \) is a 2-section, it will again be isomorphic to a "corresponding" 2-section of \( \mathcal{M} \), and so the Schl"afli Symbols of \( \mathcal{B} \) and \( \mathcal{M} \) are the same. Finally, let \( H_0 \) and \( H_{d-1} \) are finite. Now \((x, y) \) will be a universal
polytope based on some proper parabolic subgroup of $W$ (Theorems 5.2.33 and 5.2.8). Since all such subgroups are also subgroups of either $H_0$ or $H_{d-1}$ (or both), and since these are finite, we conclude that $(x, y)$ is isomorphic to a universal polytope based on a finite Coxeter group. According to Lemma 5.2.11, all such universal polytopes satisfy Euler’s condition. Therefore, all proper sections of $\mathcal{Z}$ satisfy Euler's condition, making $\mathcal{Z}$ sub-Euler, as claimed.

We have discovered that if $A$ is a sparse subgroup of a Coxeter group $W$, and if $\mathcal{M}$ is a universal polytope based on $W$, then $\mathcal{M}/A$ is a combinatorially regular incidence polytope. It may be noted that there are subgroups $A$ of $W$ which, although not sparse, nonetheless yield well-defined quotient polytopes $\mathcal{Z}$. In [99], a result is proved (Proposition 13) which classifies exactly which subgroups of $W$ yield quotients which are incidence polytopes, but does not characterize when these polytopes will be combinatorially regular. The result is stated below.

**Definition:** Let $W$ be a Coxeter group, and $A$ a subgroup of $W$. Then $A$ shall be called semisparsel if and only if the following two conditions hold.

1. For each $i, j, k$ with $-1 < i < j < k < d$, and for each $w \in W$, we have
   \[ H_k W_i \cap w A \cap H_k W_j w A = H_k (W_i \cap W_j) w A. \]
2. For each $k = 0, \ldots, d-1$, and for each $w A$, we have
   \[ H_k w A \cap (w^{-1} A w) = \emptyset. \]

Then we have the following result.

**Theorem 5.2.38:** $\mathcal{M}/A$ is an incidence polytope if and only if $A$ is semisparsel.

**Proof:** See [99, Prop 13].

Let us return to the case where $A$ is sparse. Recall that the flags of $\mathcal{Z}$ are of the form $\{U, Q\} \cup \{AuH_i : 0 \leq i \leq d-1\}$ (Lemma 5.2.26). Denote this flag by $F_w$. It is, of course, quite conceivable that $F_u$ could equal $F_v$ for some $u \neq v$. The next lemma tells us exactly when this will occur.

**Lemma 5.2.39:** The map $\psi : Au \rightarrow F_w$ is a bijection from the set of right cosets of $A$ to the set of flags of $\mathcal{Z}$.

**Proof:** It is a well-defined map, for if $Au = Av$, then $AuH_i = AvH_i$ for all $i$, and so $F_u = F_v$. It is onto, for any flag may be written in the form $F_u = (Au) \psi$ for some $u \in W$ (Lemma 5.2.26). Finally, it is one to one, for if $F_u = F_v$, then $AuH_i = AvH_i$ for each $i$, whence $v \in uH_i$ for each $i$, and so for each $i$, there exists some $a_i \in A$ such that $v \in a_i u H_i$, whence $H_i = a_i u H_i$. Then Lemma 5.2.31 tells us that in fact there exists $a \in A$ such that $v = au$, so $Au = Aa u = Au$ as required.

It would conceivably be better to denote the flag $F_w$ with the symbol $F_{A,w}$ instead. This will in fact sometimes be done—especially when we simultaneously consider more than one sparse subgroup of $W$. From the above result we immediately obtain the following theorem.

**Theorem 5.2.40:** $|\mathcal{Z}| = |W : A|$.

We can also calculate the sizes of the individual sets $\mathcal{Z}_i$.

**Theorem 5.2.41:** Let $W$ be such that $H_0$ and $H_{d-1}$ are finite. If $|W : A|$ is infinite, then all the $\mathcal{Z}_i$ for $0 \leq i \leq d-1$ are infinite. On the other hand, if $|W : A|$ is finite, $|\mathcal{Z}_i| = \frac{|W|}{|H_i|}$ for these $i$.

**Proof:** Consider the action of $H_i$ on the set $\{Au : u \in W\}$ of cosets of $A$, defined via $(Au)^h = Auh$. Note that $\mathcal{Z}_i = \{AuH_i : u \in W\}$ is in one to one correspondence with the set of orbits $\{\{Auh : h \in H_i\} : u \in W\}$ of this action. The stabiliser of any coset $Au$ under this action will be $H_i \cap u^{-1}Au$, since $Au = Auh$ if and only if $u^{-1}Au = u^{-1}Auh$. However, $A$ is sparse, so $H_i \cap u^{-1}Au = (1)$. The "orbit-stabiliser theorem" ([81 Thm 3.2]) then tells us that each orbit $\{Auh : h \in H_i\}$ contains $|H_i|$ distinct cosets of $A$. Since $A$ has $|W : A|$ cosets, there will be $\frac{|W|}{|H_i|}$ such orbits if $|W : A|$ is finite, or infinitely many otherwise.

We know that $\mathcal{Z}$ is sub-Euler. We can now determine (for finite $W$) when it will be Euler as well.
Theorem 5.2.42: Let $W$ be finite. Then $\mathcal{B} - A/A$ is Euler if and only if either $A = \{1\}$ or $d$ is even.

Proof: Being sub-Euler already, $\mathcal{B}$ will be Euler if and only if it satisfies Euler's condition. Now

$$\sum_{i=1}^{d} (-1)^i |a_i| = -1 + (-1)^d \sum_{i=0}^{d-1} (-1)^i |a_i|$$

$$= -1 + (-1)^d \sum_{i=0}^{d-1} (-1)^i \frac{|W|}{|H_i|}$$

$$= -1 + (-1)^d + \frac{1}{|A|} \sum_{i=0}^{d-1} (-1)^i \frac{|W|}{|H_i|}$$

$$= [-1 + (-1)^d] \left[ 1 - \frac{1}{|A|} \right] + \frac{1}{|A|} \sum_{i=0}^{d-1} (-1)^i |a_i|$$

Now $\sum_{i=1}^{d} (-1)^i |a_i|$ will equal zero, since $A$ satisfies Euler's condition, $W$ being finite (see Theorem 5.2.11). Thus, $\mathcal{B}$ will satisfy Euler's condition if and only if $A = \{1\}$ or $d$ is even (since otherwise Euler's formula would fail to yield an integer).

Note that an argument along similar lines would show that for $W$ finite and $d$ odd, a sparse subgroup of $W$ must have order 1 or 2 (since otherwise Euler's formula would fail to yield an integer).

Our knowledge of the sizes of the $a_i$ will be useful later. For now, let us follow a different line of argument, and consider the action of the group $W$ on $\mathcal{B}$.

5.2.3 Groups Acting on Quotients:

We continue to assume that $A$ is a sparse subgroup of $W$. Note that the flag action of $W$ on $\mathcal{B}$ (discussed in §5.1.3) is well-defined, since the Schl"{a}fli Symbol of $\mathcal{B}$ is of the required form (see Theorems 5.2.37 and 5.2.16, and the notes following Theorem 5.1.12). In fact, we have the following theorem.

Theorem 5.2.43: For all $w \in W$, and for all flags $F_u$ of $\mathcal{B}$, we have $(F_u)^w = F_w u$.

Proof: The proof will be done by induction on the length of $w$. Certainly it holds for the element of length 0. Now if $w$ has length 1, it is a generator of $W$. Let $w = s_i$. Then $(F_u)^w$ will equal $F_{wu}$ for some $u \in W$. This $u$ will satisfy $A H_i = A H_i$ for all $j \neq i$. But for all such $j$, $H_j = s_j H_j$, so in fact $A H_i = A s_i H_i$ for all $j \neq i$. The set $\{A H_i : (F_u)^{A H_i} < A H_i < (F_u)^{A H_i} + 1\}$ will have two elements, since $\mathcal{B}$ is a polytope. These two elements will be $((F_u)^{A H_i}) = A H_i$, and $((F_u)^{A H_i}) = A H_i$, by definition of the flag action (see §5.1.3). However, it will also be the case that $A s_i H_i$ will be an element of this set, by virtue of the fact that $A s_i H_i = A H_i = A s_i H_i = (F_u)^{A s_i H_i}$ for all $j \neq i$ (and so in particular $A s_i H_i \cap A H_i$ is nonempty for $j = i = \pm 1$). I claim that $A s_i H_i$ cannot equal $A H_i$. If it does, then $a s_i \in A H_i$, by Lemma 5.2.20, that is, there exists some $a \in A$ and $h \in H_i$ such that $a s_i = a h u$, whence $a^{-1} = u h^{-1} a^{-1}$. If $i \neq d - 1$, then $s_i \in H_i$, so $a^{-1} = u h^{-1} a^{-1} \subseteq u H_i H_i$, whence $a = 1$ since $A$ is sparse. If, on the other hand, $i = d - 1$, then $s_i = s_{i+1} \in H_0$, so $a = (a s_{d-1} u^{-1})^{-1} = u s_{d-1} u^{-1} \subseteq u H_0 H_{d-1} u^{-1}$, and so once again $a = 1$. Given then that $a = 1$, we have $s_i = 1$, so $s_i \in H_i$, which is a contradiction (see Theorem 5.1.2), and so the claim that $A H_i \neq A s_i H_i$ is vacuous. Thus $\{A H_i : (F_u)^{A H_i} < A H_i < (F_u)^{A H_i} + 1\} = \{A H_i, A s_i H_i\}$, which follows that $A s_i H_i = A H_i$, so $A H_i = A s_i H_i$ for all $j \neq 0 \leq j \leq d - 1$, and so $(F_u)^w = F_{w u}$, as required. This completes the first step of the induction. Now assume that the theorem holds for all elements of $W$ with length less than $n$, and let $w = s_i \ldots s_n$. We can write $w = s_i v$, where $v$ has length at most $n - 1$. Then $(F_u)^w = (F_u)^{s_i v} = (F_u)^w v = (F_u)^w$ which equals $F_w u$.

Consider now the group $\Gamma(\mathcal{B}) = W^{w(\mathcal{B})}$ of permutations of $F(\mathcal{B})$ induced by $W$, namely the image of $W$ under the map $\psi : w \mapsto w(\mathcal{B})$, where $w(\mathcal{B})$ is the permutation of $F(\mathcal{B})$ induced when $w$ acts on elements of $F(\mathcal{B})$ via the flag action. Since the flag action is well-defined, it follows that $\psi$ is a well defined group homomorphism from $W$ to $\text{Sym}(F(\mathcal{B}))$. We can describe $\Gamma(\mathcal{B})$ in terms of $A$.

Theorem 5.2.44: The kernel of the map $\psi$ is isomorphic to the core of $A$ in $W$, whence $\Gamma(\mathcal{B}/A) \cong W/\text{Core}(W/A)$.

Proof: Now $w$ will be an element of ker$\psi$ if and only if $(F_u)^w = F_u$ for all $u \in W$. This can be the case if and only if $Au = Au w$ for all $u \in W$, that is, $\text{Core}(A)$ in $W$ is the intersection of all $W$, or $w^{-1} A w$. It may be shown that this is also the largest subgroup of $A$ which is normal in $W$. 

\footnote{The core of a subgroup $A$ in a group $W$ is the intersection, over all $w \in W$, of $w^{-1} A w$. It may be shown that this is also the largest subgroup of $A$ which is normal in $W$.}
\( u^{-1}Au = w^{-1}Auw \), which is the case if and only if \( w \in u^{-1}A \) for all \( u \in W \). It follows that \( w \in \ker \psi \) if and only if \( w \in \bigcap u^{-1}A \). This latter expression is just the core of \( A \) in \( W \). That \( \Gamma(\mathcal{M}/A) \) is isomorphic to \( W/\text{Core}_W(A) \) then follows from the so-called first isomorphism theorem, which may be found in [42, Thm. 2.12].

Let us, for now, move away from examining the flag action of \( W \) on \( \mathcal{M}/A \), and begin to consider isomorphisms and automorphisms of quotient polytopes. First, we consider the question of when \( \mathcal{M}/A \) and \( \mathcal{M}/B \) will be isomorphic.

**Theorem 5.2.45:** Let \( \mathcal{M}/A \) and \( \mathcal{M}/B \) be quotients of \( \mathcal{M} \) by sparse subgroups \( A \) and \( B \) of \( W \), and let \( \psi \) be an isomorphism from \( \mathcal{M}/A \) to \( \mathcal{M}/B \). Then there exists some \( w \in W \) such that for all \( AuH_i \in \mathcal{M}/A \), we have \( (AuH_i)\psi = BwH_i \). Furthermore, \( B = wAu^{-1} \).

**Proof:** First, consider the flag \( F_A = F_{A_1} \) of \( \mathcal{M}/A \). Then \( (F_A)\psi \) will be a flag of \( B \), call it \( F_{BuH_i} \). Now let \( AuH_i \in \mathcal{M}/A \), and consider the flag \( F_{Bu} \). Note that \( (AuH_i)\psi = (F_{Bu})\psi = (F_{BuH_i}) \). But \( (F_{Bu})\psi = (F_{BuH_i}) \) (by Theorem 5.1.10), and this equals \( (F_{BuH_i})\psi \). Thus \( (F_{Bu})\psi = (F_{BuH_i}) \). Whence it follows that \( (AuH_i)\psi = BwH_i \), as required. We now show that \( B = wAu^{-1} \). Note that for any \( a \in A \), for all \( i \) we have \( Ah_i = A \), and hence \( (A\bar{a})\psi = (A\bar{a})\psi \). But that implies that \( Bu = BuH_i \). It follows that \( w = BuH_i \), so for all \( i \) there exists \( b \in B \) such that \( wa = bH_i \), so \( wAu = wAu^{-1} \). Then, Lemma 5.2.31 tells us that all the \( b_i \) will be the same element \( b \) of \( B \), and in fact that \( bu = w \). Thus for any \( a \in A \), there exists \( b \in B \) such that \( b = w^{-1}a \), and so \( wAu^{-1} \in B \). Similarly, by considering the isomorphism \( \psi^{-1} \) (since \( (F_{Bu})\psi = (F_{BuH_i}) \)), we obtain \( w^{-1}Bu \leq A \), that is, \( B \subseteq wAu^{-1} \), yielding the required equality \( B = wAu^{-1} \).

Now we can identify exactly when two such quotients are isomorphic.

**Theorem 5.2.46:** For sparse \( A, B \leq W \), the quotients \( \mathcal{M}/A \) and \( \mathcal{M}/B \) are isomorphic if and only if \( A \) and \( B \) are conjugate.

**Proof:** Theorem 5.2.45 shows that if \( \mathcal{M}/A \cong \mathcal{M}/B \) then \( A \) and \( B \) are conjugate. It remains to be shown that if \( A \) and \( B \) are conjugate, then \( \mathcal{M}/A \) and \( \mathcal{M}/B \) are isomorphic. To this end, let \( B = wAu^{-1} \), and define \( \psi \) via \( (AuH_i)\psi = BuH_i \). We show first that \( \psi \) is well-defined. Let \( AuH_i = AuH_i \). Then \( v = aH_i \) for some \( a \in A \) and \( h \in H_i \), and so we require that \( BwH_i = BwH_i \) should equal \( BuH_i \). Now \( BwH_i = BwH_i = Bw^{-1}uH_i = Bw^{-1}uH_i \) since \( w^{-1}uH_i \in \mathcal{M} \). Also, \( \psi \) is one to one, for if \( (AuH_i)\psi = (AuH_i)\psi \), then \( BwH_i = BwH_i \), so \( w \in BwH_i \).

12 Or any other elementary group theory text.

which means that \( v = w^{-1}buw \) for some \( b \in B \) and \( u \in H_i \). Thus \( AuH_i = AuH_i \) from \( H_i \), which equals \( AuH_i \) since \( H_i \) and \( w^{-1}buw \in \mathcal{M} \). Likewise, \( \psi \) is onto, for if \( BuH_i \in \mathcal{M}/B \), then \( BuH_i = (Au^{-1}uH_i)\psi \), and \( w^{-1}bH_i \in \mathcal{M}/A \). Finally, we show that \( \psi \) preserves the partial order. Let \( AuH_i \leq AuH_j \). Now \( AuH_i \cap AuH_j \) will be equal to \( w^{-1}BuH_i \cap AuH_j \), which equals \( w^{-1}BuH_i \cap AuH_j \), which in turn equals \( w^{-1}(AuH_i)\psi \cap (AuH_j)\psi \). We may conclude that \( (AuH_i)\psi \cap (AuH_j)\psi \) is nonempty if and only if \( (AuH_i)\psi \cap (AuH_j)\psi \) is nonempty, so \( AuH_i \leq AuH_j \) if and only if \( (AuH_i)\psi \leq (AuH_j)\psi \).

The above two results are very powerful. We have not only classified exactly when two quotient polytopes will be isomorphic, but also we have classified exactly what the isomorphisms between them will be: they are the maps \( \psi \), taking \( AuH_i \) to \( BuH_i \), for just those \( w \in W \) which satisfy \( B = wAu^{-1} \).

When one applies these considerations to the case when \( A = B \), they yield very strong results about the automorphism group of the quotient polytope. For example, it tells us that any automorphism of \( \mathcal{M}/A \) may be written in the form \( a_{\omega} \) for some \( \omega \in \mathcal{M} \) (where \( (AuH_i)a_{\omega} = AuH_i \) for any \( AuH_i \in \mathcal{M}/A \), and that \( a_{\omega} \) is a well-defined automorphism if and only if \( \omega \) satisfies \( a_{\omega} = a_{w} = w^{-1}a_{w} \).

The set \( \{w \in W : wAu^{-1} = A \} \) is called the normaliser of \( A \) in \( W \), and is denoted \( N_W(A) \). It can be shown to be the largest subgroup of \( W \) in which \( A \) is normal. Let \( \psi : N_W(A) \to \text{Aut}(\mathcal{M}/A) \) be defined by \( \psi_{a_{\omega}} \).

**Theorem 5.2.47:** The map \( \psi \) is a well-defined onto group homomorphism.

**Proof:** It is well-defined, since if \( \omega \in N_W(A) \), then \( w^{-1} \omega \in N_W(A) \) also, and so \( a_{\omega} \) is a well-defined isomorphism from \( \mathcal{M}/A \) to \( \mathcal{M}/A \), by Theorem 5.2.45. The map \( \psi \) is onto, since again by Theorem 5.2.45, any automorphism of \( \mathcal{M}/A \) is of the form \( a_{\omega} \) for some \( \omega \) satisfying \( \omega^{-1}A = A \). Finally, we show that \( \psi \) preserves the group multiplication. Consider \( (AuH_i) \in \mathcal{M}/A \).

Now \( (AuH_i)\psi = (Au_{\omega}H_i) \) (where \( (AuH_i)a_{\omega} = AuH_i \) for any \( AuH_i \in \mathcal{M}/A \), and that \( a_{\omega} \) is a well-defined automorphism if and only if \( \omega \) satisfies \( a_{\omega} = a_{w} = w^{-1}a_{w} \).

Thus \( \psi \) is a group homomorphism as required.

**Theorem 5.2.48:** The kernel of \( \psi \) is \( A \), and hence \( \text{Aut}(\mathcal{M}/A) \cong N_W(A)/A \).

**Proof:** Let \( \psi_{a_{\omega}} = a_{\omega}^{-1} = 1 \). Then, for all \( i \), for all \( u \in W \), we have \( Au^{-1}H_i = AuH_i \). Thus \( u \in Au^{-1}H_i \), so there exists \( a_{\omega} \) such that \( a_{\omega}^{-1}uH_i = H_i \), whence \( uH_i = q_{\omega}^{-1}uH_i = q_{\omega}^{-1}uH_i \). It follows from Lemma 5.2.31 that for all \( u \in W \), there exists \( a \in A \) such that \( u = e^{-u}a \), that is, there exists \( a \in A \) such that \( a = w^{-1}a \). So \( \omega \) implies \( w \in A \). Conversely, if \( w \in A \), then for all \( AuH_i \in \mathcal{M}/A \), we have \( (AuH_i)\psi = Au^{-1}H_i = AuH_i \), and so \( \omega = 1 \). Thus \( \ker \psi = A \), as required. The first isomorphism theorem ([42, Thm. 2.12]) then yields the formula given for \( \text{Aut}(\mathcal{M}/A) \).

We can use these results about \( \text{Aut}(\mathcal{M}/A) \) to determine when our combinatorially regular quotient polytope will also be flag regular.
Theorem 5.2.49: For sparse $A \subseteq W$, $\mathcal{M}/A$ is regular if and only if $A$ is normal in $W$.

Proof: First, let $A$ be normal in $W$. Then $uAw^{-1} = A$ for all $w \in W$, and so for all $u^{-1}w^{-1} \in W$, $\alpha_{u^{-1}w^{-1}}$ is a well-defined automorphism of $\mathcal{M}/A$. Thus, for any flags $F_{A^u}$ and $F_{A^w}$ of $\mathcal{M}/A$, there is an automorphism $\alpha$ (namely $\alpha = \alpha_{u^{-1}w^{-1}}$) which satisfies $(F_{A^u})^\alpha = F_{A^w}$ and so $\Aut(\mathcal{M}/A)$ is flag-transitive, as required. Next, let $\mathcal{M}/A$ be regular, so in particular, for any $u \in W$, there is an automorphism $\alpha$ with $(F_A)^\alpha = F_A$. Writing $\alpha = \alpha_u$ (which must be possible), we conclude that $F_{A^u} = F_{A^v}$, for some $v \in \mathcal{N}_W(A)$. But if $F_{A^u} = F_{A^v}$, then $Au = Av$ (Lemma 5.2.39), and so there exists some $a \in A$ such that $u = au$. Then, $uAu^{-1} = auAu^{-1}a^{-1}$, which equals $aAA^{-1}$ (since $v \in \mathcal{N}_W(A)$), which in turn equals $A$ (since $a \in A$). Thus $uAu^{-1} = A$ for all $u \in W$, which means that $A$ is a normal subgroup of $W$.

5.3 Applications of the Constructions

We have spent considerable time examining the polytopes which may be defined from certain rather special subgroups of certain rather special Coxeter groups. In this section it shall be seen that the time was well spent—first of all because any polytope can be expressed as a quotient, and more particularly because any combinatorially regular polytope whose facets and vertex figures are universal may be expressed as a quotient by a sparse subgroup of the relevant Coxeter group. This second point is more useful than it appears—we shall see by the end of Chapter 6) that any indecomposable combinatorially regular Euler incidence polytope will have universal facets and vertex figures, and so is isomorphic to such a quotient.

Let $\mathcal{K}$ be a $d$-incidence polytope, and let $W$ be a Coxeter group acting on $\mathcal{P}(\mathcal{K})$ via the flag action. Let $\mathcal{M}$ be the universal complex based on $W$. Select and fix a flag $B$ of $\mathcal{K}$. We may consider $B$ to be a "base flag" for what follows. Finally, let $A$ be equal to $W_B$, the stabiliser of $B$ in the group $W$, so $A = \{w \in W : B^w = B\}$. It is a well-known result of the theory of group actions that $A$ is a subgroup of $W$, and this being the case, we can define the quotient $\mathcal{M}/A$. Now we prove a lemma which leads to our main result.

Let $\Psi^*$ be a map taking elements of $\mathcal{K}\setminus\{0_\mathcal{K}, K\}$ to certain subsets of $W$, as follows. Let $w$ be an element of $f^\Psi^*$ (where $f \in \mathcal{K}\setminus\{0_\mathcal{K}, K\}$) if and only if $f \in B^w$, that is, if $\dim f = i$, we have $w \in f^\Psi^*$ if and only if $f = (B^i)$. Our aim will be to use $\Psi^*$ to construct an isomorphism $\Psi$ from $\mathcal{K}$ to $\mathcal{M}/A$. The following lemma tells us what the subsets $f^\Psi^*$ where $f \in \mathcal{K}$ will be.
polytopes (not necessarily Euler). Such a broad line of research is sadly beyond the scope of this thesis, for reasons explained elsewhere.

We therefore prove the following important result. Let \( \mathcal{X} \) be a combinatorially regular \( d \)-incidence polytope (\( d \geq 1 \)), let \( W \) be a Coxeter group acting on \( \mathcal{F}(\mathcal{X}) \) via the flag action, and let \( A \) be the subgroup of \( W \) which stabilizes the base flag \( B \).

**Theorem 5.3.3**: *If the facets and vertex figures of \( \mathcal{X} \) are universal, based on \( H_{d-1} \) and \( H_0 \) respectively, then \( A \) is sparse in \( W \).*

**Proof**: The case \( d = 1 \) is easy to dispose of. In this case, \( W = \{1, s_0\} \).

By definition of the flag action, \( B^{s_0} \neq B \), so by definition of \( A \), \( s_0 \notin A \).

It follows that \( A = \{1\} \), which is sparse. Now let \( d \geq 2 \), let \( \mathcal{X} \) be as stated, and let \( \mathcal{X} \cap u H_0 H_{d-1} w^{-1} \) for some \( u \in W \). Then, \( u \in A \), so \( B^u = B \).

But \( u \in u H_0 H_{d-1} w^{-1} \), so there exists \( h_0 \in H_0 \) and \( h_{d-1} \in H_{d-1} \) such that \( u = \omega h_0 h_{d-1} w^{-1} \). Letting \( F = B^u \), we can conclude that \( F = (\mathcal{F}(\mathcal{X})_{k_0}) = (\mathcal{F}(\mathcal{X})_{k_0}) = f_{k_0} \), and note that \( (\mathcal{F}(\mathcal{X}))_{k_0} = (\mathcal{F}(\mathcal{X}))_{k_0} = f_{k_0} \), and likewise, \( (\mathcal{F}(\mathcal{X}))_{k-1} = (\mathcal{F}(\mathcal{X}))_{k-1} = f_{k-1} \). We make the following definitions.

Knowing that \( W = \{s_0, \ldots, s_{d-1}\} \),

\[
W' = \{s_0, \ldots, s_{d-2}\} = H_{d-1},
\]

\[
W'' = \{s_1, \ldots, s_{d-2}\} = H_0,
\]

and \( W^* = \{s_1, \ldots, s_{d-2}\} = H_{d-1} \).

Denote the parabolic subgroups \( H_{d-1} \) of \( W \) by \( H' \). Note that \( W' \) acts on the section \( \mathcal{F}' = \mathcal{F}(\mathcal{X}, f_{k-1}) \) of \( \mathcal{X} \), and likewise \( W' \) acts on \( \mathcal{F}'' = \mathcal{F}(f_0, K) \), and \( W^* \) on \( \mathcal{F}^* = \mathcal{F}(f_0, f_{k-1}) \) (see Theorem 5.1.12). Now, by Lemma 5.1.15, the action of \( W' \) on \( \mathcal{F}' \) satisfies \( (F \cap \mathcal{F}') = (F \cap \mathcal{F}') \), for all \( F' \in W' \), so in particular, \( (F \cap \mathcal{F}')_{k-1} = (F \cap \mathcal{F}')_{k-1} \). Recall that \( \mathcal{F}' \) is isomorphic to the universal polytope \( \mathcal{M}' \) based on \( W' \), so there is an isomorphism \( \phi' \) from \( \mathcal{F}' \) to \( \mathcal{M}' \). Let \( (F \cap \mathcal{F}')_{k-1} \) be the flag \( \phi' \) on \( \mathcal{F}' \), and let \( (F \cap \mathcal{F}')_{k-1} \) be equal to \( \phi' \). Then, since the flag action commutes with isomorphisms (see Theorem 5.1.16), we have \( \phi' \) = \( (F \cap \mathcal{F}')_{k-1} \) = \( (F \cap \mathcal{F}')_{k-1} \) = \( (F \cap \mathcal{F}')_{k-1} \) = \( (F \cap \mathcal{F}')_{k-1} \) = \( (F \cap \mathcal{F}')_{k-1} \). Thus, \( u = u' \) and \( v = v' \) (Theorem 5.2.14). But it will also be the case, since \( (F \cap \mathcal{F}')_{k-1} = (F \cap \mathcal{F}')_{k-1} \), that \( (F \cap \mathcal{F}')_{k-1} \) is isomorphic to \( \mathcal{F}' \), so there exists \( h' \in H' \) with \( (F \cap \mathcal{F}')_{k-1} = (F \cap \mathcal{F}')_{k-1} \). This tells us that \( u = u' \) and \( v = v' \), and in fact \( h_{d-1} = h' \), and we can conclude that \( h_{d-1} \in H_0 = H_{d-1} \).

A similar argument, involving the section \( \mathcal{F}'' \) and the universal polytope \( \mathcal{M}'' \), would show that \( h_0 = h_{d-1} \). Now, elements of \( H_{d-1} = W' \) act on the flags of the section \( \mathcal{F}' \) of \( \mathcal{X} \). But \( \mathcal{F}' \) is in fact a universal complex, it being a vertex figure of \( \mathcal{X} \) and a facet of \( \mathcal{F}'' \), both themselves universal polytopes. Let \( \psi' \) be an isomorphism from \( \mathcal{F}' \) to \( \mathcal{M}' \). Under this isomorphism, \( (F \cap \mathcal{F}') \) correspond to \( \phi' \), so \( (F \cap \mathcal{F}') = \phi' \). But then, \( \phi' \) will equal \( (\mathcal{F}'_{k-1})_{k-1} \) which in turn will equal \( (F \cap \mathcal{F}')_{k-1} \), which equals \( (F \cap \mathcal{F}')_{k-1} \), because the flag action commutes with isomorphisms (see Theorem 5.1.16). This latter expression equals \( (F \cap \mathcal{F}')_{k-1} \), by Theorem 5.1.15, which is equal to \( (F \cap \mathcal{F}')_{k-1} \). Reversing the above argument shows that this in turn equals \( (\mathcal{F}'_{k-1})_{k-1} \), which is equal to \( \phi'_{k-1} \). We can therefore conclude (Theorem 5.2.14) that \( u_h h_0 = u_h h_{d-1} \), and so \( h_0 = h_{d-1} \). But recall how \( h_0 \) and \( h_{d-1} \) were originally defined. They were elements of \( H_0 \) and \( H_{d-1} \) respectively such that \( u = u_h h_{d-1} w^{-1} \), where \( u \) was an arbitrary element of \( A(\mathcal{X}, H_0 H_{d-1} w^{-1}) \). It follows that for any such \( u \), we have \( u = u_h h_{d-1} w^{-1} = 1 \). Thus \( A \cap u H_0 H_{d-1} w^{-1} = \{1\} \) for all \( u \in W \), so \( A \) is again sparse.

So any combinatorially regular incidence polytope whose facets and vertex figures are universal is isomorphic to a quotient of a universal \( d \)-polytope by a sparse subgroup of the Coxeter group that the universal incidence polytope is based on. But we have already seen (Theorem 5.2.37 and Corollary 5.2.34) that any quotient by a sparse subgroup is a combinatorially regular incidence polytope with universal facets and vertex figures. Hence we have the following corollary, which we shall call a theorem since it is one of the main results of the thesis.

**Theorem 5.3.4**: *A partially ordered set \( \mathcal{X} \) is a combinatorially regular incidence polytope with Schl"{a}fli Symbole \( \{p_1, \ldots, p_{d-1}\} \) and with universal facets and vertex figures, if and only if it is a quotient of the universal polytope based on the Coxeter group \( W = W_{\{p_1, \ldots, p_{d-1}\}} \) by some sparse subgroup of \( W \).*

This theorem is the main work of this chapter, and in fact the climax of the whole thesis. The next chapter is devoted to applying this theorem to specific Coxeter groups, and so obtaining classification results for the corresponding specific Schl"{a}fli Symbols. We close with a few corollaries based on these results.

Let \( \mathcal{X} \) be a combinatorially regular incidence polytope whose facets and vertex figures are universal, and let \( \mathcal{M} \) be the universal polytope with the same Schl"{a}fli Symbol as \( \mathcal{X} \). Furthermore, let this Schl"{a}fli Symbol be such that the Coxeter group acting on \( \mathcal{X} \) and \( \mathcal{M} \) has all parabolic subgroups finite.

**Corollary 5.3.5**: \( \mathcal{X} \) is sub-Euler, and will be Euler if and only if it is even or \( \mathcal{X} \) is universal.

**Proof**: This follows from the above theorem, along with Theorems 5.2.37 and 5.2.42, and Lemma 5.2.24.
Corollary 5.3.6: If, for some $i$, we have $|X_i| = |\mathcal{M}_i|$ is finite, then $X$ and $\mathcal{M}$ are isomorphic.

Proof: By Theorem 5.3.4, we know that $X$ is isomorphic to a quotient of $\mathcal{M}$. Now $\mathcal{M}_i$ cannot be finite unless $W$ is also (by Theorem 5.2.10, since we are assuming that all the $H_i$ are finite). Let $X \cong \mathcal{M}/A$. Then Theorem 5.2.41 tells us that $|X_i| = \frac{|W|}{|A_i|} \cdot \frac{|W_i|}{|H_i|}$. But we also have $|X_i| = |\mathcal{M}_i| - \frac{|W_i|}{|H_i|}$ (Theorem 5.2.10 again), so $|W| = \frac{|W_i|}{|A|}$. It follows that $|A| = 1$, so $A = \{1\}$. Lemma 5.2.24 then gives the desired result.

Corollary 5.3.7: The combinatorial counterpart of a regular geometric polytope is a universal complex.

Proof: See also [55, §3.1]. In the proof of Lemma 5.2.11, we saw that the number of $i$-faces of a regular geometric polytope $P$ was the same as the number of elements of $\mathcal{M}_i$, where $\mathcal{M}$ was the universal complex whose Schläfli Symbol was the same as that of $P$. Therefore, if $\mathcal{P}$ is the combinatorial counterpart of $P$, then $|\mathcal{P}_i| = |\mathcal{M}_i|$, and $|\mathcal{P}|$ is finite (Theorem 2.3.6) for any $i$ with $0 \leq i \leq d - 1$, so the above result may be applied.

CHAPTER 6

Some Classifications

Art...does not classify objects...

Benedetto Croce, 1959

The aim of this chapter is to prove a number of classification theorems about combinatorially regular incidence polytopes, which will be used in a kind of grand classification theorem in the next chapter. Note that in some cases (particularly when examining the 4-incidence polytopes), a complete classification will not be given.

6.1 General Classes

6.1.1 The Simplices:

The aim of this subsection is to prove the following

Theorem 6.1.1: If $\mathcal{P}$ is a combinatorially regular incidence polytope whose Schläfli Symbol is $[3] \ldots [3]$, then $\mathcal{P}$ is a d-simplex.

This can be (and will be) proven very easily using the group theoretic results of Chapter 5. However, I shall start with a purely combinatorial proof. The latter is included for two reasons – firstly to show that such proofs are possible in the study of these polytopes, and secondly to give a better appreciation of just how powerful the group theory really is.

Proof: By induction on $d$. Certainly, if $d = 1$, the result is true, as there is only one 1-incidence polytope up to isomorphism (see Theorem 3.3.9). Also, if $d = 2$, the Schläfli Symbol of $\mathcal{P}$ is $[3]$, so $\mathcal{P}$ is a 2-cycle (Theorem 3.3.10). But the 2-simplex is also a 2-cycle (Theorem 4.1.6), and so $\mathcal{P}$ is a 2-simplex. Now let $d \geq 3$ and assume for all $k < d$ that all combinatorially regular $k$-incidence polytopes with Schläfli Symbol $[3] \ldots [3]$ are $k$-simplices. Let $\mathcal{P}$ be a $d$-incidence polytope with Schläfli Symbol $[3] \ldots [3]$. Its $i$-sections ($i \leq d - 1$) will have Schläfli Symbol $[3] \ldots [3]$ (Theorem 3.4.12), so by the inductive hypothesis, they are all $i$-simplices. We now prove a number of claims.
Claim I: in a $k$-simplex $\mathcal{A}$, there is a structure-preserving bijection between the elements of $\mathcal{A}$, and subsets of $\mathcal{A}$. Recall that a $k$-simplex was defined (in §4.1) to be (isomorphic to) the set of subsets of some $(k+1)$-set $R$, with the partial order $\subset$. Thus, we define a map $\phi$ taking the subset $\{r_1\}, \{r_2\}, \ldots, \{r_k\}$ of $\mathcal{R}_0$ to the element $\{r_1, r_2, \ldots, r_k\} \subseteq R$ of $\mathcal{A}$. It is easy to show that $\phi$ is one to one, onto, and preserves the inclusion relation.

Claim II: if $X \in \mathcal{A}$, then $\{x \in \mathcal{R}_0 : x \leq X\}$ has $i + 1$ elements. This follows easily from the fact that $\emptyset, X$ is an $i$-simplex (see the notes preceding Theorem 4.1.5).

Claim III: let $B_1, B_2 \in \mathcal{P}_{d-1}$, and let $C_1 \in \mathcal{P}_{d-2}$ satisfy $C_1 \leq B_1, B_2$. Then $\{x \in \mathcal{R}_0 : x \leq B_1\} = \{x \in \mathcal{R}_0 : x \leq B_2\}$ can only occur if in fact $B_1 = B_2$. To prove this, let $B_1 \neq B_2$ and let $\{x \in \mathcal{R}_0 : x \leq B_1\} = \{x \in \mathcal{R}_0 : x \leq B_2\} = \{x_1, x_2, \ldots, x_{d+1}\}$, and let $\{x \in \mathcal{R}_0 : x \leq C_1\} = \{x_3, \ldots, x_{d+4}\}$. Now, $(\emptyset, B_1)$ is a $(d-1)$-simplex, with vertex set $\{x_1, x_2, x_3, \ldots, x_{d+4}\}$. As such, there exists (by our Claim I) a structure-preserving bijection $\phi$ from the set of subsets of $\{x_1, x_2, x_3, \ldots, x_{d+4}\}$ to the simplex $(\emptyset, B_1)$ and some $C_2 \in (\emptyset, B_1)$ such that $C_2 \leq \{x_1, x_2, x_3, \ldots, x_{d+4}\}$. We may also deduce that there exists some $D \in (\emptyset, B_1)$ such that $D \leq \{x_1, x_2, x_3, \ldots, x_{d+4}\}$. Note also that $C_2 \leq \{x_1, x_2, x_3, \ldots, x_{d+4}\}$. We may deduce, since $D \leq \{x_1, x_2, x_3, \ldots, x_{d+4}\}$, that $D \leq C_1, C_2$, and since $|D \setminus \{x_1, x_2, x_3, \ldots, x_{d+4}\}| = d - 2$ and $|C_2 \setminus \{x_1, x_2, x_3, \ldots, x_{d+4}\}| = d - 1$ for $i = 1$ or $2$ that $(\emptyset, D)$ is a $(d-3)$-simplex and $(\emptyset, C_1)$ and $(\emptyset, C_2)$ are $(d-2)$-simplices, or more particularly, that $D \subseteq (\emptyset, C_1)$ and $C_2 \subseteq (\emptyset, C_2)$. This being the case, $(D, P)$ will be a 3-cycle (where $P$ is the maximal element of $\mathcal{P}$). Let $C_3 \in \mathcal{P}_{d-3}$ and $B_3 \in \mathcal{P}_{d-1}$ be such that $(D, P) = \{D, C_1, C_2, C_3, B_1, B_2, B_3, P\}$. Note that $C_3 \leq B_1, B_2$ and $C_2 \leq B_3$ (see Figure 6.1.1).

Consider $(\emptyset, C_3) = (\emptyset, C_4) \cap \mathcal{R}_0 = \{x \in \mathcal{R}_0 : x \leq C_3\}$. Certainly, since $x_4, \ldots, x_{d+4} \leq D \leq C_3$, we have $\{x_4, \ldots, x_{d+4}\} \subseteq (\emptyset, C_3)$. But since $(\emptyset, C_3)$ is a $(d-2)$-simplex, we have $|(\emptyset, C_3)| = d - 1$. Thus $(\emptyset, C_3) = \{x^*, x_4, \ldots, x_{d+4}\}$ for some $x^* \in \mathcal{R}_0, x^* \notin \{x_4, \ldots, x_{d+4}\}$. But $C_3 \leq B_1, B_2$, whence either $x^* = x_2$ or $x^* = x_3$. But since $C_1$ and $C_2$ are distinct elements of the $(d-1)$-simplex $(\emptyset, B_1)$, it follows that their vertex sets $C_1 \setminus \{x^*_1\} = (\emptyset, C_3) \setminus C_1$ and $C_2 \setminus \{x^*_2\} = (\emptyset, C_3) \setminus C_2$ are different sets. That is, $(x_2, x_4, \ldots, x_{d+4}) \notin (x_1, x_4, \ldots, x_{d+4})$, and $x^*$ cannot equal $x_2$. However a similar argument, examining $C_2$ and $C_3$ as elements of the $(d-1)$-simplex $(\emptyset, B_3)$ would reveal that $x^*$ cannot equal $x_3$. Thus we have a contradiction, and so we reject our initial assumption, that $\{x \in \mathcal{R}_0 : x \leq B_1\}$ and $\{x \in \mathcal{R}_0 : x \leq B_2\}$ could be equal for distinct $B_1$ and $B_2$ satisfying the conditions specified. From now on, let $B_1 \in \mathcal{P}_{d-1}$, let $C_1 \in \mathcal{P}_{d-2}$ with $C_1 \leq B_1$, and let $B_2$ be such that $\{X \in \mathcal{P}_{d-1} : C_1 \subseteq X\} = \{B_1, B_2\}$. Note that $B_2$ will be uniquely defined given $B_1$ and $C_1$, since $(C_1, P)$ is a 1-polytope.

Claim IV: $|(\emptyset, B_1) \cup (\emptyset, B_2)| = d + 1$. To show this, note that $\|(\emptyset, B_1) \cup (\emptyset, B_2)\| = |(\emptyset, B_1)| + |(\emptyset, B_2)| - |(\emptyset, B_1) \cap (\emptyset, B_2)| = d + 1 - |(\emptyset, B_1) \cap (\emptyset, B_2)| = d + 1 - |x \in \mathcal{R}_0 : x \leq B_1 \cap (\emptyset, B_2)|$. Now $\{x \in \mathcal{R}_0 : x \leq B_1\} \neq \{x \in \mathcal{R}_0 : x \leq B_2\}$, by our Claim III, so $|(\emptyset, B_1) \cup (\emptyset, B_2)| < |(\emptyset, B_2)| = d$. On the other hand, $C_1 \subseteq \{x \in \mathcal{R}_0 : x \leq B_1\} \subseteq \{x \in \mathcal{R}_0 : x \leq B_2\}$, so $|(\emptyset, B_1) \cup (\emptyset, B_2)| \geq |(\emptyset, C_1)| = d - 1$. It follows that $|(\emptyset, B_1) \cup (\emptyset, B_2)| = d - 1$, so $|(\emptyset, B_1) \cup (\emptyset, B_2)| = 2d - (d - 1) = d + 1$, as claimed. Note that we can also deduce from the above argument that for such $B_1, B_2$ and $C_1$.

Claim V: $(\emptyset, B_1) \cup (\emptyset, B_2) = (\emptyset, B_1) \cap (\emptyset, B_2)$. Let us continue with the proof.

Claim VI: Let $R = \{x \in \mathcal{R}_0 : x \leq B_1 \} \cup \{x \in \mathcal{R}_0 : x \leq B_3\}$, that is, $R = (\emptyset, B_1) \cup (\emptyset, B_3)$. Then, $R = \mathcal{R}_0$. To show this, let $x \in \mathcal{R}_0$, and let $B \in \{X \in \mathcal{P}_{d-1} : x \notin X\}$. Then $x \in \mathcal{R}_0 : x \leq B = (\emptyset, B)$. Now we show that $(\emptyset, B) \subseteq R$, for this would tell us that $p \in R$, so $\mathcal{R}_0 \subseteq R$, whence $R = \mathcal{R}_0$ as required. We apply $(d - 1, d - 2)$-Connectivity (Theorem 3.3.20) to $\mathcal{P}$ to obtain a sequence $(B_1, C_1), B_2, C_2, \ldots, B_m = B$.

Note that $B_1$ will not be affected by this "trimming". However, if $B_i = B_1$ for any $i > 2$, then $C_1$ and $B_2$ may be affected. In this case, first remove any duplicates of $B_2$ from the sequence (as above), and then, without loss of generality, replace the original sequence with $B_3, C_1, B_2, C_1, B_3, C_1, \ldots$, and then continue trimming as before, finally obtaining a sequence with elements alternately from $\mathcal{P}_{d-1}$ and $\mathcal{P}_{d-2}$, the first and third terms of which are the $B_i$ and the $B_i$ from which $R$ was defined.

Note 44: We...
know that $(\emptyset, B_1), (\emptyset, B_2) \subseteq R$. Assume now that $(\emptyset, B_{i-1}), (\emptyset, B_i) \subseteq R$ for some $i \geq 2$. Note that since $C_{i-1} \leq B_{i-2}, B_i$, we know, from Claim V, that $(\emptyset, C_{i-1}) = (\emptyset, B_{i-1}) \cap (\emptyset, B_i) \subseteq R$. Similarly, $(\emptyset, C_i) = (\emptyset, B_i) \cap (\emptyset, B_{i+1}) \subseteq R$. Now $(\emptyset, B_i)$ is a $(d-1)$-simplex, with $C_{i-1}, C_i \in (\emptyset, B_i)$. As in the proof of Claim III, we can find some element of $\mathcal{P}_{d-1}$, call it $D_i$, such that $D_i \subseteq C_{i-1}, C_i$. Then $(D_i, P)$ will be a 3-cycle, with $B_{i-1}, B_i, B_{i+1} \subseteq (D_i, P)$. Let $C_i \in \mathcal{P}_{d-2}$ be such that $(D_i, P) \cap \mathcal{P}_{d-2} = \{C_{i-1}, C_i, C_i^\circ\}$ (see Figure 6.1.2).

![Figure 6.1.2](image)

Note that $C_i^\circ \leq B_{i-1}, B_{i+1}$, and so again, $(\emptyset, C_i^\circ) = (\emptyset, B_{i-1}) \cap (\emptyset, B_{i+1}) \subseteq R$. Now $|(\emptyset, C_i^\circ)| = d - 1$ and $|(\emptyset, B_{i+1})| = d$, so there exists some $a \in \mathcal{P}_0$ such that $(\emptyset, B_{i+1}) = \{a\} \cup (\emptyset, C_i^\circ)

\subseteq \{a\} \cup (\emptyset, B_{i-1})$

Similarly, there exists some $b \in \mathcal{P}_0$ such that

$(\emptyset, B_{i+1}) = \{b\} \cup (\emptyset, C_i^\circ)

\subseteq \{b\} \cup (\emptyset, B_{i-1})$

It cannot be that $a = b$, for this would imply that $(\emptyset, C_i^\circ) = (\emptyset, C_i^\circ)$, that is, $\mathcal{C}_{i-1}^\circ = \mathcal{C}_{i-1}^\circ$ (where $\circ$ is a bijection $\circ$ is Claim I from subsets of $(\emptyset, B_{i-1})$ to $(\emptyset, B_{i+1})$), and this in turn would imply that $C_i = C_i^\circ$, which would contradict our choice of $C_i^\circ$. Thus $a \neq b$. Then, since $\{a\} \cup (\emptyset, B_{i-1}) \neq \{b\} \cup (\emptyset, B_{i-1})$, we have $a \in (\emptyset, B_{i-1})$. We conclude therefore that $a \in R$ (since $(\emptyset, B_{i-1}) \subseteq R$). It follows from this that $(\emptyset, B_{i+1}) = \{a\} \cup (\emptyset, B_{i-1}) \subseteq R \cup R = R$. We have shown, $(\emptyset, B_{i-1}) \subseteq R$ and $(\emptyset, B_{i-1}) \subseteq R$. Since we know that $(\emptyset, B_{i-1}) \subseteq R$ and $(\emptyset, B_{i-1}) \subseteq R$, it follows by induction that $(\emptyset, B) \subseteq R$, vindicating the claim.

Claim VI: For distinct $B, B' \in \mathcal{P}_{d-1}$, there exists $C \in \mathcal{P}_{d-1}$ such that $C \leq B, B'$. To show this, first note that $(\emptyset, B) \cap (\emptyset, B') = \emptyset$, then $d + 1 = |R| \geq |(\emptyset, B) \cup (\emptyset, B')| = |(\emptyset, B)| + |(\emptyset, B')| - 0 = 2d$,

which cannot be, since $d \geq 3$. Let $a \in (\emptyset, B)_0 \cap (\emptyset, B')_0$. Then $B$ and $B'$ are $(d-2)$-dimensional elements of the $(d-1)$-simplex $(a, P)$. It follows (as many times before) that there exists some $(d-3)$-dimensional element $C$ of $(a, P)$ with $C \subseteq B, B'$. Thus we have found an element of $\mathcal{P}_{d-2}$ with the required property.

Note that using this claim, and Claim III, we obtain as a corollary that if $B, B' \in \mathcal{P}_{d-1}$, then $B \neq B'$, then $(\emptyset, B) \neq (\emptyset, B')$. To complete the proof of the theorem, we define a map $\psi$ from $\mathcal{P}$ to the $d$-simplex $\mathcal{A}$ based on $R$, as follows. For $X \in \mathcal{P}$, let

$$X\psi = \begin{cases} \text{the empty set} & \text{if } X = \emptyset, \\ \{x \in \mathcal{P}_0 : x \leq X\} & \text{otherwise}. \end{cases}$$

Now $\psi$ is certainly a well-defined map, since $\mathcal{P}$ consists of all subsets of $R = \mathcal{P}_0$, and since $X\psi$ is always a subset of $\mathcal{P}_0$. The map is also one to one. To show this, let $X, Y \in \mathcal{P}$ with $X\psi = Y\psi$. Now $X$ has dimension $i$ if and only if $|X\psi| = i + 1$ (by Claims II and VI), so if $|X\psi| = |Y\psi| = i + 1$, then $\dim_{\mathcal{A}} X = \dim_{\mathcal{A}} Y = i$. If $i = -1$ or $i = d$, there is nothing to prove, for we (respectively) have $X = \emptyset$ or $X = Y = P$. If $i = d$, then we already know, by the corollary to Claim VII, that $X = Y$, since if $B, B' \in \mathcal{P}_{d-1}$, then $B\psi = B\psi$ can only occur if $B = B'$. We will use this fact further down. Now, let $X\psi = Y\psi = \{a_0, \ldots, a_i\}$, where $0 \leq i < d - 1$. Note that for any $B \in \mathcal{P}_{d-1}$ with $X \subseteq B$, we have $a_i \subseteq B'$ for all $j$, and so $a_j \subseteq B'$ for all $j$. Now consider the set $\{B \in \mathcal{P}_{d-1} : X \subseteq B\}$. Since $(X, P)$ is a $(d-1)$-simplex, this set has $d - i$ elements. Since each $B$ with $X \subseteq B$ also satisfies $a_i \subseteq B$ for all $j$, each element of $\{B \in \mathcal{P}_{d-1} : X \subseteq B\}$ will also be an element of the set $\{B \in \mathcal{P}_{d-1} : \{a_0, \ldots, a_i\} \subseteq B\}$, so $\{B \in \mathcal{P}_{d-1} : X \subseteq B\} \subseteq \{B \in \mathcal{P}_{d-1} : \{a_0, \ldots, a_i\} \subseteq B\}$. This latter set will have the same number of elements as $\{B \psi : B \in \mathcal{P}_{d-1} \text{ and } \{a_0, \ldots, a_i\} \subseteq B\}$, which is a subset of the set $\{B \in \mathcal{P}_0 : |T| = d \text{ and } \{a_0, \ldots, a_i\} \subseteq T\}$, which latter set has $d(d+i-1) = d - i$ elements, so we have

$$d - i = \left|\{B \in \mathcal{P}_{d-1} : X \subseteq B\}\right| \leq \left|\{B \in \mathcal{P}_{d-1} : \{a_0, \ldots, a_i\} \subseteq B\}\right| \leq \left|\{T \subseteq \mathcal{P}_0 : |T| = d \text{ and } \{a_0, \ldots, a_i\} \subseteq T\}\right|$$

$$= d - i.$$
Thus $\psi$ is a structure-preserving monomorphism from the $d$-polytope $\mathcal{P}$ to the $d$-polytope $\mathcal{Q}$, and therefore, by Corollary 3.3.25, it is an isomorphism also. Thus $\mathcal{P}$ is a $d$-simplex. Recall that we assumed only that $\mathcal{P}$ was a $d$-incidence polytope with Schlafli Symbol $[3] \ldots [3]$, and that for $k < d$, all $k$-incidence polytopes with Schlafli Symbol $[3] \ldots [3]$ were k-simplices. We have therefore completed the inductive step of the proof, and therefore the whole proof, as we had already noted that the theorem holds for $d = 1$ and $d = 2$. 

We have shown that all polytopes with Schlafli Symbol $[3] \ldots [3]$ are simplices. Since, also, all simplices have $[3] \ldots [3]$ as their Schlafli Symbol, we obtain the following corollary.

**Corollary 6.1.2**: Let $\mathcal{P}$ be a combinatorially regular incidence polytope. Then $\mathcal{P}$ has Schlafli Symbol $[3] \ldots [3]$ if and only if $\mathcal{P}$ is a combinatorial simplex.

However, the results of Chapter 5 yield a far simpler method of classifying polytopes with this Schlafli Symbol, as follows.

**Theorem 6.1.3**: If a combinatorially regular $d$-polytope has Schlafli Symbol $[3] \ldots [3]$, then it is universal.

**Proof**: By induction on $d$. Note that it is true for $d = 1$, that is, the polytope with Schlafli Symbol $\{1\}$ is universal (see the first example in §5.2.1). Now assume that $d > 1$, and that for all $k < d$, the only combinatorially regular $k$-incidence polytope with Schlafli Symbol $[3] \ldots [3]$ is the universal one. Let $\mathcal{P}$ be a polytope with Schlafli Symbol $[3] \ldots [3]$. Note that the facets and vertex figure of $\mathcal{P}$ have Schlafli Symbol $[3] \ldots [3]$, and are therefore universal. Theorem 5.3.4 then tells us that $\mathcal{P}$ is a quotient $\mathcal{M}/A$ of the universal polytope $\mathcal{M}$ based on the $d$-Coxeter group $W = W_{[3] \ldots [3]}$, where $A$ is a sparse subgroup of $W$. Now the facets of $\mathcal{M}/A$, being universal $(d - 1)$-polytopes with Schlafli Symbol $[3] \ldots [3]$, will have $d$ vertices each (refer to Lemma 5.2.10, and Table A.7). However, $\mathcal{M}/A$ itself, by Theorem 5.2.41 or Table A.6, will have $\frac{1}{d!}(d+1)$ vertices. It follows that $\frac{1}{d!} \leq d$, so $|A| \leq 1 + \frac{1}{d}$. Since $d > 1$, we conclude that $|A| = 1$, so (by Lemma 5.2.24), $\mathcal{P} \cong \mathcal{M}/A \cong \mathcal{M}$ as required. 

An alternative proof of Theorem 6.1.1 would then be as follows.

**Proof**: A simplex has Schlafli Symbol $[3] \ldots [3]$, and is therefore universal. Therefore, any other polytope with this Schlafli Symbol, being universal also, must be isomorphic to it.

It is important to note that simplices, being universal, are not only combinatorially regular (cf Corollary 4.1.2), but (by Theorem 5.2.15) are also regular (that is, flag regular).

### 6.1.2 Cubes and Halfcubes:

In this section, the aim is to classify all those combinatorially regular Euler incidence polytopes having a certain Schlafli Symbol, in particular, to show that the only combinatorially regular Euler $d$-incidence polytopes having Schlafli Symbol $[4] \ldots [3]$ are the $d$-cube, and (if $d$ is even), the $d$-halfcube. First, we prove a lemma.

**Lemma 6.1.4**: There is no indecomposable combinatorially regular Euler incidence polytope whose facets are halfcubes and whose vertex figures are simplices.

**Proof**: By contradiction. Let $\mathcal{K}$ be such a $d$-polytope. Now since $\mathcal{K}$ is Euler, its facets (like all its other sections) satisfy Euler's relation. But a $(d - 1)$-halfcube satisfies Euler's relation if and only if $d - 1$ is even (see Theorem 4.2.15), and we conclude therefore that $d$ is odd. If $d = 3$, the facets of $\mathcal{K}$ are 2-halfcubes, and so have Schlafli Symbol $\{2\}$ (See the note following Theorem 4.2.16). Therefore, $\mathcal{K}$ has a two in its Schlafli Symbol (Theorem 3.4.12), and so is decomposable (see Theorem 3.4.14). Thus $d$, being odd, is at least 5. We can calculate the $|\mathcal{K}|$ as follows. Note that $|\mathcal{K}| = 1$. Now for all $x \in \mathcal{K}$, there exists some $B \in \mathcal{R}_{d-1}$ such that $x \leq B$, so in fact $\mathcal{R}_0 = \bigcup_{B \in \mathcal{R}_{d-1}} ((0, B) \cap \mathcal{R}_0) = \bigcup_{B \in \mathcal{R}_{d-1}} (0, B)$. Now, let $B, B^* \in \mathcal{R}_{d-1}$.

By $(d - 1, d - 2)$-connectivity (Theorem 3.3.20), there exists some sequence $B = B_0, C_0, B_1, \ldots, C_{d-1}, B_k = B^*$ such that $C_i \in \mathcal{R}_{d-2}, B_i \in \mathcal{R}_{d-1}$, and...
$C_i \leq B_i, B_{i+1}$ for all i. Now $(\emptyset, B_{i+1})$ is a $(d-1)$-halflcub, so $(\emptyset, C_{i+1})$ is a $(d-2)$-cube. It follows that $|\emptyset, C_i \cap \mathcal{R}_0| = 2^{d-2}$, and that $|\emptyset, B_i \cap \mathcal{R}_0| = 2^{d-1-2}$ (see the notes preceding Theorem 4.2.15), that is, $(\emptyset, C_i \cap \mathcal{R}_0)$ and $(\emptyset, B_i \cap \mathcal{R}_0)$ have the same size. Hence also, they are equal as sets, one being a subset of the other (for if $x \in \mathcal{R}_0$ satisfies $x \leq C_i$, then certainly $x \leq B_i$, also, since $C_i \leq B_i)$. Similarly, $(\emptyset, C_i \cap \mathcal{R}_0) = (\emptyset, B_{i+1}) \cap \mathcal{R}_0$. Therefore, $(\emptyset, B_i) \cap \mathcal{R}_0 = (\emptyset, B_{i+1}) \cap \mathcal{R}_0$, and so by an inductive argument, $(\emptyset, B_i) \cap \mathcal{R}_0 = (\emptyset, B^*_i) \cap \mathcal{R}_0$, and so, for any $B^*_i \in \mathcal{R}_{i-1}$,

$$\mathcal{R}_0 = \bigcup_{B \in \mathcal{R}_{i-1}} (\emptyset, B) \cap \mathcal{R}_0 = \bigcup_{B \in \mathcal{R}_{i-1}} (\emptyset, B^*_i) \cap \mathcal{R}_0 = (\emptyset, B^*_i) \cap \mathcal{R}_0 = (\emptyset, B^*_0).$$

Thus $\mathcal{R}$ has the same number of vertices as its facet $(\emptyset, B^*_i)$. Since this facet is a $(d-1)$-halflcub, we obtain $|\mathcal{R}| = 2^{d-1-1} = 2^d$ (see again the notes preceding Theorem 4.2.15). Now assume that we know $|\mathcal{R}_i|$, where $i \geq 0$. Then $|\mathcal{R}_{i+1}|$ may be calculated using Theorem 3.4.1, so

$$|\mathcal{R}_{i+1}| = |\mathcal{R}_i| \times \left\{ \begin{array}{ll}
(a_{i+1}, R^*_i) \cap \mathcal{R}_{i+1} & \text{if } a_{i+1} \cap \mathcal{R}_{i+1} \\
(\emptyset, a_{i+1}) \cap \mathcal{R}_{i+1} & \text{if } a_{i+1} \cap \mathcal{R}_{i+1} = \emptyset
\end{array} \right\},$$

where $a_{i+1}$ and $a_{i+1}$ are arbitrary elements of $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ respectively. Now the section $(a_{i+1}, R)$ is a $(d-i-1)$-simplex (being a section of a vertex figure of $\mathcal{R}$ — see Theorem 4.1.4), and $(a_{i+1}, R) \cap \mathcal{R}_{i+1}$ is the set of its vertices, so $|(a_{i+1}, R) \cap \mathcal{R}_{i+1}| = d - i$ (see the notes preceding Theorem 4.1.5). Also, $(\emptyset, a_{i+1})$ will be an $(i+1)$-halflcub (being a facet of $\mathcal{R}$) if $i + 1 = d - 2$, or an $(i+1)$-cube (being a facet of a halflcub) if $i + 1 < d - 1$. $(\emptyset, a_{i+1}) \cap \mathcal{R}_{i+1}$ will be the number of facets of this object, so will equal $2(i + 1)$ if $i < d - 2$, or $(i+1)$ if $i = d - 2$.

Thus $|\mathcal{R}_0| = 2^d$, and $|\mathcal{R}_{i+1}| = |\mathcal{R}_i| \times 2^{d-i}$ if $0 \leq i < d - 2$, and finally $|\mathcal{R}_{d-1}| = |\mathcal{R}_{d-2}| 2^{d-1}$. From this information, we could show, by an inductive argument, that

$$|\mathcal{R}| = \begin{cases}
2^{d-1-1} & \text{if } 0 \leq i \leq d - 2, \\
\frac{d}{2} & \text{if } i = d - 1, \text{ or} \\
0 & \text{if } i = d - 1, \text{ or} \\
1 & \text{if } i = 0 \text{ or } d.
\end{cases}$$

Since $\mathcal{R}$ is Euler, we require that

$$0 = \sum_{i=0}^{d} (-1)^i |\mathcal{R}_i| = -|\mathcal{R}_{d-1}| + \sum_{i=0}^{d-1} (-1)^i |\mathcal{R}_i| + (-1)^{d-1}|\mathcal{R}_{d-1}| + (-1)^d|\mathcal{R}_d|.$$
CHAPTER 6: Some Classifications

Theorem 6.1.6: Let \( d \geq 2 \). If \( \mathcal{P} \) is a combinatorially regular \( d \)-incidence polytope whose facets are \((d-1)\)-cubes, and whose vertex figures are \((d-1)\)-simplices, then \( \mathcal{P} \) is a \( d \)-cube or a \( d \)-halfcube.

Proof: Let \( S = \{ s_1, \ldots, s_{d+1} \} \) be a generating set for the \( d \)-Coxeter group \( W = W_{d+1} \). Note first that \( \mathcal{P} \) has Schl"afli Symbol \( \{ 4 \} \ldots \{ 4 \} \), since its facets have Schl"afli Symbol \( \{ 4 \} \ldots \{ 4 \} \) and its vertex figures \( \{ 4 \} \ldots \{ 4 \} \) (see Theorems 4.2.16 and 4.1.6). Note second that its facets and vertex figures are universal (Theorem 6.1.5, and Corollary 6.1.2 and Theorem 6.1.3), and so \( \mathcal{P} \) is a quotient \( \mathcal{A} \hookrightarrow \mathcal{A} \) by some sparse subgroup \( A \) of \( W \) (by Theorem 5.3.4). Let \( X \in \mathcal{P} \). Then \( (X, X) \) is a \((d-1)\)-cube, so \( (X, X) \in \mathcal{P} = (X, X) \), so \( \mathcal{P} \) has \( 2^{d-1} \) elements (see Table A.7 or the notes preceding Theorem 4.2.15), so \( \mathcal{P} \) itself has at least this many. But \( \mathcal{P} \) being a quotient by a sparse subgroup, we may apply Theorem 5.2.41 to obtain \( |\mathcal{P}| = \frac{|\mathcal{A}|}{|\mathcal{A}|} \), which by Table A.6 equals \( \frac{2^{d+1}}{|\mathcal{A}|} \). Thus \( |\mathcal{A}| \geq \frac{2^{d-1}}{d} \), whence \(|A| = 1 \) or \( d \). If \(|A| = 1 \), then \( \mathcal{P} \) is universal, by Lemma 5.2.24. If \(|A| = 2 \), then \( \mathcal{P} = \mathcal{A} \) for some order 2 element \( a \) of \( W \). Now in Theorem 8.3 of [26], we learn that any element of order 2 in a Coxeter group \( W \) is conjugate to some \( w \) for some subset \( S \) of \( W \), where \( w \) is of maximal length in the (finite) parabolic subgroup \( W_a \) of \( W \), so we may write \( a = w_b \), where \( b \in W \). Let us first assume that \( I \neq S \), so there exists some \( s_i \in S \) such that \( s_i \notin I \). Then \( W_I \leq H_I \leq W_C \leq \leq H_{d-1} \) (since \( W_I = \{ s_j : s_j \in I \} \), and \( H_I = \{ s_j : s_j \notin I \} \), and we may write \( w_I = w_b \) for some \( u \in B \) and \( v \in H_{d-1} \). But then \( a = w'u^{-1} v^{-1} \), and so \( H_I \partial H_{d-1}^{-1} = \mathcal{P} \), and so \( A \) is an \( \mathcal{A} \) of \( H_I \). However, this is a contradiction, so we conclude that in fact \( \mathcal{A} = \mathcal{A} \), so \( \mathcal{P} \) is of maximal length in \( W \). Recall that in [5.1.1], it was noted how \( W \) could be regarded as the reflection group of some root system \( \Phi = \Pi(-\Pi) \), in which case Theorem 5.1.5 tells us that \( \mathcal{P} \) is the unique element of \( W \) satisfying \( w_\mathcal{P} = -I \). Since \( w_\mathcal{P} \) is unique, and since \( A \) is conjugate to \( w_\mathcal{P} \), it follows for any two sparse subgroups \( A \) and \( B \) with \(|A| = |B| = 2 \) that \( A \) and \( B \) are conjugate (both being conjugate to \( w_\mathcal{P} \)) and therefore that \( \mathcal{A} \) is isomorphic to \( \mathcal{A} / \mathcal{P} \). Thus \( \mathcal{P} \) is isomorphic to \( \mathcal{A} / \mathcal{P} \). Hence we conclude that \( \mathcal{P} \) is a \( d \)-halfcube. This completes the proof.

We have incidentally shown that \( w_\mathcal{P} \) is of maximal length in \( W \), and \( w_\mathcal{P} \) is of order 2, and \( \mathcal{P} \) is a \( d \)-halfcube. This completes the proof.

Theorem 6.1.7: The group \( \mathcal{P} \) is normal in \( W \), hence \( \mathcal{P} \) is a \( d \)-halfcube.

Proof: Recall that \( \mathcal{P} \) was the unique element of \( W \) to satisfy \( \mathcal{P} = -I \). Keeping this in mind, let us turn to [14, §9.6], where we discover that the group \( W \) in question contains a particular element \( r \) which reverses every vector in \( \mathcal{P} \), and also commutes with every element of \( W \). Since \( r \) reverses every element of \( \mathcal{P} \), in particular \( r \mathcal{P} = -\mathcal{P} \). However, \( r \mathcal{P} \) was unique in satisfying \( \mathcal{P} = -I \), so it must be that \( r = -\mathcal{P} \). Thus \( r \mathcal{P} \) commutes with every element of \( W \), that is, for all \( r \in W \), we have \( r(\mathcal{P}) = (r \cdot \mathcal{P}) = (r \cdot \mathcal{P}) \mathcal{P} \). So the group \( \mathcal{P} \) is indeed normal in \( W \), and so by Theorem 5.2.49, the halfcube \( \mathcal{P} / \mathcal{P} \) is regular.

Now we classify the combinatorially regular Euler polytopes with Schl"afli Symbol \( \{ 4 \} \ldots \{ 4 \} \).

Theorem 6.1.8: Let \( d \geq 2 \). If \( \mathcal{P} \) is a combinatorially regular Euler \( d \)-incidence polytope with Schl"afli Symbol \( \{ 4 \} \ldots \{ 4 \} \), then either \( \mathcal{P} \) is a \( d \)-cube, or \( d \geq 3 \) is even and \( \mathcal{P} \) is a \( d \)-halfcube.

Proof: If \( d = 2 \), then \( \mathcal{P} \) has Schl"afli Symbol \( \{ 4 \} \), so satisfies \( |\mathcal{P}| = |\mathcal{P}| = 4 \) (see §3.5), and so is a 4-cycle (Theorem 3.3.10), and hence also a cube. Now, let \( d \geq 3 \) and assume that the result holds for combinatorially regular Euler \( k \)-polytopes, where \( k < d \). Now if \( \mathcal{P} \) is as per the statement of the theorem, then the vertex figures of \( \mathcal{P} \) have Schl"afli Symbol \( \{ 4 \} \ldots \{ 4 \} \), and hence are \( d \)-cubes \( \mathcal{P} \), \( d \geq 3 \) is even and \( \mathcal{P} \) is a \( d \)-halfcube. This completes the proof.

The surface had little to do with the Coxeter group in question. This is a rather pointed illustration of how intricately tied together geometry, combinatorics and group theory can sometimes be.

Let us proceed. We know that cubes, being universal, are regular (that is, flag regular; see Theorem 5.2.15). We can, by extending the arguments of the above proof, show that halfcubes are also (flag) regular.

\[ \text{Section 6.1: General Classes} \]
6.1.3 Crosses and Halfcrosses:

All the hard work needed to classify combinatorially regular Euler incidence polytopes with Schlӓfli Symbol \( \{3\ldots3[4] \} \) has already been done in the previous subsection, thanks to the properties of duality. Hence we move immediately on to the following theorem.

**Theorem 6.1.9:** If \( \mathcal{P} \) is a combinatorially regular Euler \( d \)-incidence polytope with Schlӓfli Symbol \( \{3\ldots3[4] \} \), then either \( \mathcal{P} \) is a \( d \)-cross, or \( d \geq 3 \) is even and \( \mathcal{P} \) is a \( d \)-halfcross.

**Proof:** Note that by Theorem 3.4.7, the dual \( \mathcal{P}^* \) of \( \mathcal{P} \) is combinatorially regular, and by Theorems 3.3.17 and 3.4.13 is Euler and has Schlӓfli Symbol \( \{4[3\ldots3] \} \). Thus from Theorem 6.1.8, either \( \mathcal{P}^* \) is a \( d \)-cube, or \( d \geq 3 \) is even and \( \mathcal{P}^* \) is a \( d \)-halfcube. Thus the dual of \( \mathcal{P}^* \), that is, \( \mathcal{P}^{**} = \mathcal{P} \) (Lemma 3.3.13), is either a \( d \)-cross, or \( d \geq 3 \) is even, and it is a \( d \)-halfcross (by the definitions of the cross and the halfcube given in §4.3).

**Theorem 6.1.10:** The \( d \)-cross and the \( d \)-halfcross are (flag) regular.

**Proof:** See Theorems 6.1.5, 5.2.15, 6.1.7 and 3.4.7.

**Theorem 6.1.11:** The \( d \)-cross is universal.

**Proof:** This follows from the note following Lemma 5.2.1, since (by Theorem 6.1.5) the \( d \)-cube is universal.

6.2 The Lattices and Related Polytopes

6.2.1 Preliminaries:

Let \( \mathcal{R} \) be a combinatorially regular Euler \( d \)-incidence polytope with Schlӓfli Symbol \( \{4[3\ldots3[4] \} \), \( d \geq 3 \). We saw some examples of these in §4.4, namely the lattices. In the course of this section, it should become clear that the lattices are not the only such polytopes.\(^{57}\) Note that since \( \mathcal{R} \) is Euler and combinatorially regular, so are its facets and vertex figures (if every section of \( \mathcal{P} \) satisfies Euler's condition, then certainly every section of a facet or vertex figure of \( \mathcal{P} \) does, and likewise if corresponding sections of \( \mathcal{P} \) are isomorphic, then corresponding sections of its facets and vertex figures are isomorphic (see §3.2 and §3.4.2)). Since also the Schlӓfli Symbol of the facets must be \( \{4[3\ldots3[4] \} \), that of the vertex figures must be \( \{3\ldots3[4] \} \) (Theorem 3.4.12), it follows that the facets are either cubes or halfcubes, and the vertex figures are either crosses or halfcrosses (see Theorem 6.1.8 and Corollary 6.1.9). We have the following two lemmas.

**Lemma 6.2.1:** The vertex figures are not halfcubes.

**Proof:** By contradiction. Assume that the vertex figures of \( \mathcal{R} \) are indeed \( (d-1) \)-halfcubes. Since \( (d-1) \)-halfcubes are only Euler if \( (d-1) \geq 2 \) is even (Corollary 4.3.4), it follows that \( d \) is odd and at least 3. But if \( d = 3 \), then the \( (d-1) \)-halfcube has Schlӓfli Symbol \( \{2\} \) (by Corollary 4.3.5), which contradicts our choice of \( \mathcal{R} \). It follows that \( d \) is odd and at least 5. Now consider the \( \mathcal{R} \). These must be finite, since \( \mathcal{R} \) is Euler. Also, by Lemma 3.4.15, we can write \( |\mathcal{R}| = a_0|\mathcal{R}_0| \), the \( a_i \) (for \( i > 0 \)) being given by

\[
a_i = \frac{(a_0, R) \cap \mathcal{R}_i}{|\emptyset, a_i| \cap \mathcal{R}_0|} = a_i = (a_0, R) \cap \mathcal{R}_i \cap \mathcal{R}_0,
\]

where \( a_0 \in \mathcal{R}_0 \) and \( a_i \in \mathcal{R}_i \) are arbitrary, and \( \emptyset \) and \( R \) are the minimal and maximal elements of \( \mathcal{R} \). Now \( a_0 = 1 \). To calculate \( a_i \) for \( i > 0 \), note that \( (a_0, R) \) is a vertex figure of \( \mathcal{R} \), so \( (a_0, R) \cap \mathcal{R}_i \) is the number of \( (i-1) \)-faces of a \( (d-1) \)-halfcube, that is, the number of \( (d-1) \)-faces of a \( (d-1) \)-halfcube, which is \( 2^{d-1}(d-1) \). Thus, the halfcube is a quotient of the \( (d-1) \)-cube by the \( d \)-cube (see the notes preceding Theorem 4.2.15, or in fact Table A.6, since the halfcube is a quotient by an order 2 group). Also, \( (\emptyset, a_i) \) is an \( i \)-face of a facet of \( \mathcal{R} \) if \( i < d-1 \), and hence is an \( i \)-cube. \( (\emptyset, a_i) \cap \mathcal{R}_0 \) will then be the number of vertices of this \( i \)-cube, that is, \( 2^i \) (see the notes preceding Theorem 4.2.15 or Table A.7). It follows then that for \( 0 < i < d-1 \), we have

\[
a_i = \frac{2^{d-1}d!}{2^i i!} = \frac{1}{2} \left( \frac{d-1}{i} \right).
\]

Note finally that

\[
a_{d-1} = \frac{2d-2}{2^{d-1}(d-1)!} \frac{2d-2}{d!} = \frac{d-1}{i}.
\]

Now \( \mathcal{R} \) is Euler, so we have

\[
0 = \sum_{i=1}^{d-1} (-1)^i |\mathcal{R}_i|
\]

\[
= (-1)^1 + (-1)^d + |\mathcal{R}_0| \left[ a_0 + (-1)^{d-1} a_{d-1} + d-2 \sum_{i=1}^{d-2} (-1)^i \frac{1}{2} \left( \frac{d-1}{i} \right) \right].
\]

\(\footnote{In fact, see Appendix B, Section B.1.}\)
But \( d \) is odd and \( a_0 = 1 \), so this becomes

\[
(-1) + (-1) + |\mathcal{R}_d| \left[ 1 + a_{d-1} + \frac{1}{2} \sum_{i=0}^{d-2} (-1)^i \binom{d-1}{i} \right] = 0,
\]

that is,

\[
\left[ 1 + a_{d-1} + \frac{1}{2} \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} \right] = \frac{2}{|\mathcal{R}_d|}
\]

which becomes

\[
\left[ 1 + a_{d-1} + \frac{1}{2} (1 - 1)^{d-1} - 1 - \frac{1}{2} \right] = \frac{2}{|\mathcal{R}_d|},
\]

that is, \( a_{d-1} = \frac{1}{|\mathcal{R}_d|} \). But \( a_{d-1} = \frac{2^{d-2}}{|\mathcal{R}_d|} \), and hence \( |\mathcal{R}_d| \cap (\mathbb{Z}, a_{d-1})| = 2^{d-3} |\mathcal{R}_d| \). However, since \( |\mathcal{R}_d| \cap (\mathbb{Z}, a_{d-1})| \subseteq |\mathcal{R}_d| \), it follows that \( 2^{d-3} \leq 1 \), whence \( d \leq 3 \). Since we have already concluded that \( d \geq 5 \), this is a contradiction, and we have completed the proof.

**Lemma 6.2.2:** The facets of \( \mathcal{R} \) are not halfcubes.

**Proof:** By contradiction. If the facets of \( \mathcal{R} \) were halfcubes, then the vertex figures of a dual \( \mathcal{R}^* \) of \( \mathcal{R} \) would be halfcubes (see Section 3.3.14, and the definition of halfcubes). However, \( \mathcal{R}^* \) would still be a combinatorially regular Euler incidence polytope, and would still have the same Schl"{a}fli Symbol as \( \mathcal{R} \) (see Corollary 3.3.17 and Theorems 3.3.18, 3.4.7 and 3.4.13). This would then contradict the previous lemma.

Now, let \( \mathcal{P} \) be a combinatorially regular \( d \)-incidence polytope with Schl"{a}fli Symbol \( \{4|3| \ldots |3|4 \} \). Further, let \( \mathcal{P} \) be sub-Euler.

**Lemma 6.2.3:** \( \mathcal{P} \) is Euler if and only if \( d \) is even, the \( \mathcal{P}_i \) are finite, and the facets and vertex figures are cubes and crosses respectively.

**Proof:** (Outline). If \( \mathcal{P} \) is Euler, the previous two lemmas tell us what the facets and vertex figures are not, and hence what they are. Then, Euler's condition immediately tells us that the \( \mathcal{P}_i \) are finite, and after some algebraic manipulations, that \( d \) must be even. Conversely, if \( d \) is even, the \( \mathcal{P}_i \) are finite, and the facets and vertex figures are as specified, Euler's condition may be shown to hold, ensuring that our sub-Euler polytope is also Euler.

### 6.2.2 The Coxeter Group:

Recall that cubes and crosses are universal polytopes. It follows (from Theorem 5.3.4) that all the combinatorially regular Euler incidence polytopes with Schl"{a}fli Symbol \( \{4|3| \ldots |3|4 \} \) are in fact quotients, as defined in §5.2.2. It will thus be very helpful to examine the structure of the Coxeter group with Coxeter graph given below.

![Coxeter Graph](image)

This group is known as the affine Weyl group \( \tilde{C}_d \). It is generated by generators \( S = \{s_0, \ldots, s_{d-1} \} \) with relations \( s_t^2 = 1 \) for all \( t \), \( (s_0 s_j)^4 = (s_{2d-2} s_{d-1})^4 = 1 \), \( s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \) if \( i \neq 0, d-2, \) and \( s_i s_j = s_j s_i \) if \( i \neq j, j \pm 1 \), and may representations of it are known. We shall construct a few special elements of \( \tilde{C}_d \), which will give us a strong handle on its structure, and help show how \( \tilde{C}_d \) is isomorphic to \( (\mathbb{Z}/4\mathbb{Z}) \cdot \tilde{C}_d \) (where \( \tilde{C}_d \) is the Coxeter group \( \tilde{W}_{\{4|3| \ldots |3|4\}} \)). That \( \tilde{C}_d \) is isomorphic to this semidirect product is not new, but it will be helpful to express this isomorphism explicitly in terms of the \( s_i \), for then we may link it to the definition of a sparse subgroup of a Coxeter group.

Let

\[
v_1 = s_0 s_1 \ldots s_{d-2} s_{d-1} s_{d-2} \ldots s_0 s_1.
\]

Then, let \( v_2 = s_1 v_1 s_1 \), so

\[
v_2 = s_1 s_0 s_1 \ldots s_{d-2} s_{d-1} s_{d-2} \ldots s_2.
\]

and in fact for \( j = 1, \ldots, d-2 \), let \( v_{j+1} = s_j v_j s_j \), so

\[
v_j = s_{d-1} s_{d-2} \ldots s_1 s_0 s_1 \ldots s_{d-2} s_{d-1} \ldots s_j.
\]

and

\[
v_{d-1} = s_d s_{d-2} s_{d-3} \ldots s_1 s_0 s_1 \ldots s_{d-2} s_{d-1}.
\]

We do not define \( v_d \).

First, let us learn how to multiply the \( s_i \) by the \( v_j \). Specifically, we shall find the conjugates \( s_i v_j s_i \) for arbitrary \( i \) and \( j \).
Lemma 6.2.4: We have the following:
(i) \( s_i v_i = v_i s_i \) for \( 1 \leq i \leq d-2 \), but \( s_{d-1} v_d = v_d s_{d-1} = v_d \).
(ii) \( s_{d-2} v_d = v_d s_{d-2} = v_d \).

Proof: (i) By definition of the \( v_i \), we have \( s_i v_i s_i = v_i + 1 \) for \( 1 \leq i \leq d-2 \), using which \( s_{d-1} v_d = s_d \).
Now \( s_d v_d = s_d 1 = v_d + 1 \),
which equals \( v_d \) as required.

(ii) Given the relation \( s_i v_i s_i = v_i + 1 \) for \( 1 \leq i \leq d-1 \), conjugating both sides by \( s_i \) yields \( s_i v_i s_i = s_i v_i s_i \), thus \( v_i = v_i \) as required. Also,
\[
\begin{align*}
    s_0 v_0 s_0 &= s_0(s_0 v_0 s_0) = s_0 v_0 s_0, \\
    &= (s_0 s_0 v_0)(s_0 v_0 s_0), \\
    &= (s_0)(s_0 v_0 s_0), \\
    &= v_0 s_0,
\end{align*}
\]
which equals \( v_0 \) as required.

(iii) First, let \( j \geq 2 \). Then, since \( s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \),
\[
\begin{align*}
    s_0 v_j &= s_0(s_j-2 \ldots s_j s_{j-1} \ldots s_2 s_1) \ldots 2 \ldots s_0 v_j, \\
    &= (s_j-1 \ldots s_0 s_1)(s_0 s_1 s_2 \ldots s_2 s_1 \ldots s_{j-2} s_{j-1} s_j) \ldots s_0 v_j, \\
    &= (s_j-1 \ldots s_0 s_1)(s_0 s_1 s_2 \ldots s_2 s_1 \ldots s_{j-2} s_{j-1} s_j) \ldots s_0 v_j, \\
    &= v_j s_0.
\end{align*}
\]
So \( s_0 v_j s_0 = v_j \). More generally, if \( 0 < i < j-1 \), we have
\[
\begin{align*}
    s_i v_i s_i &= s_i(s_{i+1} s_{i+2} \ldots s_{i+1} s_{i+2} \ldots s_j s_{j-2} s_{j-1}) \ldots 2 \ldots s_0 v_i, \\
    &= s_i(s_{i+1} s_{i+2} \ldots s_{i+1} s_{i+2} \ldots s_j s_{j-2} s_{j-1} s_j) \ldots 2 \ldots s_0 v_i, \\
    &= s_i(s_{i+1} s_{i+2} \ldots s_{i+1} s_{i+2} \ldots s_j s_{j-2} s_{j-1} s_j) \ldots 2 \ldots s_0 v_i, \\
    &= v_i s_0,
\end{align*}
\]
which equals \( v_i \). Similar arguments show that \( s_{d-1} v_j s_{d-1} = v_j \) if \( j < d-1 \),
and \( s_j v_j = v_j \) when \( j = d-1 \).

Theorem 6.2.5: For all \( i, j \), we have \( v_i v_j = v_j v_i \).

Proof: If \( i = j \) there is nothing to prove. Assume without loss of generality that \( i < j \). Then
\[
\begin{align*}
    v_i v_j &= (s_i-1 \ldots s_i s_{i+1} \ldots s_d-1 \ldots s_{d-1} s_d \ldots 2 \ldots s_0 v_j, \\
    &= (s_i-1 \ldots s_i s_{i+1} \ldots s_d-1 \ldots s_{d-1} s_d \ldots 2 \ldots s_0 v_j, \\
    &= (s_i-1 \ldots s_i s_{i+1} \ldots s_d-1 \ldots s_{d-1} s_d \ldots 2 \ldots s_0 v_j, \\
    &= (s_i-1 \ldots s_i s_{i+1} \ldots s_d-1 \ldots s_{d-1} s_d \ldots 2 \ldots s_0 v_j, \\
    &= v_j.
\end{align*}
\]
Therefore, the group generated by the \( v_i \) is an Abelian group, and hence a quotient of \( Z^{d-1} \). We will see later that \( V = \langle v_1, \ldots, v_{d-1} \rangle \) is in fact isomorphic to \( Z^{d-1} \) itself. Now \( V \) is also normal in \( W \), since by an extension of Lemma 6.2.4, we have \( s_i v_j = v_j s_i \) for all \( u \) in \( V \) and \( s_i \) in \( S \). Let \( H = H_{d-1} = \langle s_0, \ldots, s_{d-2} \rangle \).

Theorem 6.2.6: Any element of \( W \) may be written in the form \( u H \) for some \( u \) in \( V \) and some \( h \) in \( H \).

Proof: Let \( w = s_0 s_1 \ldots s_{d-2} \) in \( W \). Note that this may be written
\[
h_1 s_c \ldots s_{d-1} h_2 s_{d-1} h_3 \ldots s_1 s_0 h_2 \ldots s_0 s_1 h_0 h_1 s_1 h_2 \ldots h_2 h_1 h_3 \ldots h_{d-1} h_{d-2} \ldots h_1 h_2 \ldots h_0.
\]
for some \( h_i \in \langle s_0, \ldots, s_{d-2} \rangle = H \). Letting \( h^* = s_0 s_1 \ldots s_{d-2} \in H \), we have \( s_{d-1} = h^* v_{d-1} \), so we may write \( w \) in the form
\[
w = h_1 h^* v_{d-1} h^* v_{d-2} \ldots h_3 h_2 h_1 v_{d-1} h_2 \ldots h_0.
\]
However, \( (h_1 h^*) v_{d-1} (h_1 h^*)^{-1} \) is in \( V \), since \( v_{d-1} \in V \) and \( V \) is normal in \( W \). Let \( (h_1 h^*) v_{d-1} (h_1 h^*)^{-1} = v_{d-1} \). Then
\[
w = u_1 h^* v_{d-1} h^* v_{d-2} \ldots h_3 h_2 h_1 v_{d-1} h_2 \ldots h_0.
\]
Similarly, we let \( (h_1 h^*) v_{d-2} (h_1 h^*)^{-1} = u_2, \) so
\[
w = u_1 u_2 h^* v_{d-2} h^* v_{d-3} \ldots h_3 h_2 h_1 v_{d-1} h_2 \ldots h_0.
\]
This process may be repeated, to obtain
\[
w = (u_1 u_2 u_3 \ldots u_{d-1}) (h_1 h^* h^* \ldots h_{d-1} h^* h_0),
\]
which is of the required form.
Thus \( W = V \cdot H \). The question remains whether for all \( w \in W \), writing \( w = u \cdot h \) yields a unique \( u \in V \) and \( h \in H \). This question is answered by the next few results.

We first show that \( V \cong \mathbb{Z}^d \). In fact, it is already known that \( W \) is a semidirect product of \( \mathbb{Z}^d \) by \( H \), where \( H \) is the Weyl group \( C_\Delta \). To see this, turn to [26, Prop 4.2], which tells us that the affine Weyl group \( C_\Delta \) is isomorphic to the semidirect product of \( L \) by \( C_\Delta \), where \( L \) is (isomorphic to) the group of translations generated by the coroots of \( C_\Delta \) (see [26, p39] for a discussion of coroots). The group \( L \) turns out to be isomorphic to \( \mathbb{Z}^d \), since the coroots of \( C_\Delta \) are \( \{ \pm e_1, \pm e_2, \pm e_3, \pm e_i : 1 \leq i, j \leq d - 1 \text{ and } i \neq j \} \). Note in particular that \( W \) is infinite (as could also have been deduced by observing that \( C_\Delta \) is absent from the table of finite Coxeter groups).

Let \( \phi \) be the map from \( \mathbb{Z}^d \) to \( V \) defined via \((\sum a_i e_i) \phi = \prod v_i^{a_i} \).

**Lemma 6.2.7:** \( \phi \) is an isomorphism.

**Proof:** It is trivial to show that \( \phi \) is a well-defined onto group homomorphism. This being the case, it follows by the first isomorphism theorem ([42, Thm 2.12]) that \( V \cong \mathbb{Z}^d / \ker \phi \).

Let \((a_1, \ldots, a_d) \in \ker \phi \), so \( \prod v_i^{a_i} = 1 \). Assume that \( a_j \neq 0 \) for some \( j \). Let \( h_j = a_j \cdot \ldots \cdot a_1 a_2 \cdot \ldots \cdot a_d \). It may be shown that \( h_j v_i h_j^{-1} = v_i \) if \( i \neq j \), and that \( h_j v_i h_j^{-1} = v_j \) if \( i = j \), so \( h_j \prod v_i^{a_i} h_j^{-1} v_i \) equals 1 if \( i \neq j \) and equals \( v_j \) otherwise. Now if \( \prod v_i^{a_i} = 1 \), we also have \( h_j (\prod v_i^{a_i}) h_j^{-1} = 1 \), that is, \( \prod (h_j v_i h_j^{-1}) = \prod v_i^{a_i} \), whence \( \prod (h_j v_i h_j^{-1} v_i^{a_i}) = 1 \). All the terms of this product will be 1, except the term where \( i = j \), which will be \( v_j^{2a_j} \), so we have \( v_j^{2a_j} = 1 \). But if this is the case, then

\[
\begin{align*}
v_j^{2a_j} &= (s_1 s_2 \ldots s_{j-1} s_2 s_1 \ldots s_2) h_j^{2a_j} \\
&= (s_1 s_2 \ldots s_{j-1}) h_j^{2a_j} (s_2 s_1) \\
&= (s_1 s_2 \ldots s_{j-1}) (s_{j-1} \ldots s_2)
\end{align*}
\]

and so \( v_j^{2a_j} = 1 \). Thus for any \( i \) we have

\[
v_i^{2a_i} = (s_1 s_2 \ldots a_1 a_2 \ldots a_d) v_i^{2a_i} \]

which will likewise equal 1. Thus \( v_i^{2a_i} = 1 \) for all \( i \), whence \( 2a_i e_i \in \ker \phi \) for all \( i \). It follows that \( 2a_i \mathbb{Z}^d \leq \ker \phi \), and so (since \( a_i \neq 0 \), \( V \cong \mathbb{Z}^d / \ker \phi \) is finite, having order at most \( (2a_i) \)). This would imply that \( W = V \cdot H \) is also finite, which is a contradiction. Thus for any \((a_1, \ldots, a_d) \in \ker \phi \), we cannot have \( a_j \neq 0 \) for any \( j \), so \( \ker \phi = \{0\} \). It follows that \( \phi \) is one to one, and so is an isomorphism, as required.

The map \( \phi \) from \( \mathbb{Z}^d \) to \( V \) defined by \( (\sum a_i e_i) \phi = \prod v_i^{a_i} \) is an isomorphism, so \( \phi^{-1} \) is a well-defined map. For each \( h \in H \) we may define a transformation \( \Delta_H \) of \( \mathbb{Z}^d \) via \( \Delta_H u = h(\psi u) h^{-1} \phi^{-1} \). Although the usual convention for maps is that they operate from the right, in the case of the \( \Delta_H \) we will soon be regarding them as matrices operating on column vectors, the standard notation for which is that they multiply on the left.

**Lemma 6.2.8:** For any \( h \in H \), the map \( \Delta_H \) is well defined, and preserves addition and scalar multiplication of the elements of \( \mathbb{Z}^d \).

**Proof:** It is well defined, for if \( u = v \in \mathbb{Z}^d \) and \( h \in H \), then \( \psi u = \psi v \), whence \( h(\psi u) h^{-1} = g(\psi v) g^{-1} \). But then, \( h(\psi u) h^{-1} \phi^{-1} = g(\psi v) g^{-1} \phi^{-1} \) as required for \( \Delta_H u \) to equal \( \Delta_H v \). Note also that for any \( h \in H \), we have \( \Delta_H (u + v) = h((u + v) \phi) h^{-1} \phi^{-1} \), which equals \( h((u \phi)(v \phi)) h^{-1} \phi^{-1} \), since \( \phi \) is a group homomorphism. But this is just equal to \( (h(\psi u)h^{-1})(h(\psi v)h^{-1}) \phi^{-1} \), which equals \( h(\psi u)h^{-1} \phi^{-1} + h(\psi v)h^{-1} \phi^{-1} \), which is just \( \Delta_H u + \Delta_H v \), as required. Finally, if \( n \) is any integer, the identity \( \Delta_H nu = n \Delta_H u \) follows immediately from the fact that \( nu \) is a sum of \( n \) copies of \( u \).

Thus \( \Delta_H \) is an endomorphism\(^{58}\) of the module\(^{59}\) \( \mathbb{Z}^d \). It follows that we may regard \( \Delta_H \) as a matrix over \( \mathbb{Z} \), whose \( i \)th column is the image of \( e_i \) under \( \Delta_H \). The set of invertible endomorphisms form a group under composition of maps (that is, under matrix multiplication). We denote this group \( \text{GL}(\mathbb{Z}^d) \). The next lemma is important.

**Lemma 6.2.9:** The map \( \delta : H \to \text{GL}(\mathbb{Z}^d) \) defined by \( \delta : h \mapsto \Delta_H \) is a group homomorphism.

**Proof:** We need to show first that it is a well-defined map, that is, that \( \Delta_H \in \text{GL}(\mathbb{Z}^d) \). This may be shown by noting that \( \Delta_H \) is invertible, with inverse \( \Delta_H^{-1} \). To see that \( \delta \) preserves the group multiplication, note that for \( g, h \in H \), we have \( \Delta_H \cdot \Delta_H = \Delta_H h(\psi u) h^{-1} \phi^{-1} \), which equals \( g((h(\psi u)h^{-1}) \phi^{-1} \phi^{-1} \), yielding \( g(\psi u)h^{-1} \phi^{-1} \phi^{-1} = \Delta_H \cdot \Delta_H \) as required.

Let \( M \) be the image of \( \delta \), that is, \( M = \{ \Delta_H : h \in H \} \).

---

\(^{58}\) An endomorphism is a transformation preserving the operations of addition and scalar multiplication.

\(^{59}\) Like a vector space, but over a ring instead of over a field.
Lemma 6.2.10: If \( h \in H \) is such that \( h u h^{-1} = u \) for all \( u \in V \), then \( h = 1 \).

Proof: (Outline) Now \( h u h^{-1} = u \) for all \( u \in V \) if and only if \( \Delta u = u \) for all \( u \in \mathbb{Z}^{d-1} \). Thus we are trying to show that the map sending \( h \) to \( \Delta h \) has trivial kernel, that is, that \( \Delta = I \) implies that \( h = 1 \). Now \( M \) will be generated by the elements \( \Delta_i, \) for \( 0 \leq i \leq d-2 \). The \( \Delta_i \) are easy to calculate: for example, \( \Delta_{0}e_{1} = s_{0}(e_{1}e_{0})s_{0}^{-1} = s_{0}v_{1}s_{0}^{-1} \), which equals \( v_{1}^{-1} \), and equals \( v_{1}^{-1}e_{1} = e_{1} \) otherwise (see Lemma 6.2.4). Thus

\[
\Delta e_{1} = \begin{pmatrix}
-1 & 0 \\
1 & 0 \\
0 & \ddots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

Likewise, for \( i > 0 \), \( \Delta_i \) will be a matrix which interchanges \( e_i \) and \( e_{i+1} \) and leaves the other \( e_j \) fixed. It may be shown that the \( \Delta_i \) (for \( 0 \leq i \leq d-2 \)) generate exactly the set of matrices of the form \( E.S \), where \( E \) is a diagonal matrix whose diagonal entries are all \( \pm 1 \), and \( S \) is a permutation matrix\(^{60}\); that is, \( M \) is a set of permutation matrices with some of the \( +1 \)'s changed to \( -1 \)'s. It follows that the set \( M \) has \( 2^{d-1}(d-1)! \) elements. However, it is already well known that \( H \) has \( 2^{d-1}(d-1)! \) elements (see Table A.6 or A.7). But since the map \( \delta \) sending \( h \) to \( \Delta h \) is a group homomorphism, the first isomorphism theorem \(^{42}, \text{Thm 2.12}\) tells us that \( M \cong H/\ker \delta \), so \( |M| = \frac{|H|}{|\ker \delta|} \). Thus \( \ker \delta = 1 \), as required.\(^{61}\)

Note that since the kernel of \( \delta \) is trivial, \( \delta \) must be one to one, so \( \Delta = \Delta_\Lambda \) if and only if \( g = h \).

Finally, we have the following.

Theorem 6.2.11: \( V \cap H = \{1\} \), and hence \( W \) is a semidirect product of \( V \) by \( H \).

Proof: Let \( v \notin V \). Then \( v v u v^{-1} = u \) for all \( u \in V \) (since \( V \) is Abelian, by Theorem 6.2.5). Thus if \( v \notin H \), then Lemma 6.2.10 tells us that \( v = 1 \) as claimed. Since we also know that \( V \) is normal in \( W \) (this follows from Lemma 6.2.4) and that \( W = V.H \), it follows by the definition of a semidirect product (see [42, p13]) that \( W \) is a semidirect product of \( V \) and \( H \).

As has been mentioned, the fact that \( \hat{\mathcal{C}}_{d-1} \) is a semidirect product of \( \mathbb{Z}^{d-1} \) by \( \mathcal{C}_d \) is not new. However, we have now obtained a concrete realization of \( W \)

\[^{60}\] That is, a matrix which permutes the \( e_i \).

\[^{61}\] This representation of the group \( \mathcal{C}_d \) as the set of matrices of the form \( E.S \) is nothing new. See for example [19, p8].
Lemma 6.2.13: $\Delta_p$ maps $e_{d,-1}$ to $-e_{d,-1}$, and leaves $e_i$ fixed for all $i \leq d - 2$.

Proof: The facts that $pup^{-1} = v_i$ for $i \leq d - 2$ and that $pup^{-1}p^{-1} = v_i'x$ were shown during the proof of Lemma 6.2.7.

Let $p = p_{d-1}, \ldots, p_1, p_0, \ldots, p_{-1}$, so $p_{d-1} = (e_{d-1}, \Delta_p)$. Note (by way of example) that $(-e_{d-1}, \Delta_p) = (-e_{d-1} + \Delta_p(-e_{d-1}), \Delta_p) = (-e_{d-1} - e_{d-1}, \Delta_p) = (0, I)$.

Let $S_T = \{x \in Z^{d-1} : \text{all coordinates of } x \text{ are either 0 or } 1\}$. Consider the set $P = \{p \in S_T \times M : \{x, \Delta_p \} : x \in S_T \text{ and } h \in H\}$, so $(u, \Delta) \in P$ if and only if all coordinates of $u$ are either 0 or 1.

Lemma 6.2.14: Let $1 \leq i \leq d - 1$. If $(x, \Delta_p) \in P$, then $(s_i\psi)(x, \Delta_p) \in P$.

Proof: Consider the case $i = d-1$. Now $(s_{d-1}\psi)(x, \Delta_p) = (e_{d-1}, \Delta_p)(\Delta_p, x)$ which in turn equals $(-e_{d-1} + \Delta_p, \Delta_p, x)$. Writing $x = (x_1, \ldots, x_{d-1})$, the $i$th coordinate of $-e_{d-1} + \Delta_p, x$ will be $x_i$ if $i \neq d - 1$, and the $(d-1)$th coordinate will be $-x_{d-1} - 1$. Since $-e_{d-2}$ is either 0 or 1, the latter expression will be either $-d-1 - 1$ or $(d-1) - 1 = 0$. Thus all coordinates of $-e_{d-1} + \Delta_p, x$ are either 0 or 1, so $(-e_{d-1} + \Delta_p, \Delta_p, x) \in P$. On the other hand, if $1 \leq i \leq d - 2$ we have $(s_i\psi) = (0, \Delta_p)$, so $(s_i\psi)(x, \Delta_p) = (\Delta_p, x, \Delta_p)$. However, as was noted in the proof of Lemma 6.2.10 (see also Lemma 6.2.4), $\Delta_p$ merely transposes the coefficients of $e_i$ and $e_{d-1}$. Thus, if all was said of $\Delta_p, x$, and we conclude that $(\Delta_p, x, \Delta_p, x) \in P$ as required.

Lemma 6.2.15: For all $z \in S_T$, there exists $g \in H$ such that $(z, \Delta_p) \in (H_0)\psi$.

Proof: (Outline) Let $z = -\sum_{i \in I} e_i$ for some $I \subseteq T$. We may write this as $z = \sum_{j=1}^k c_j$, with $i_j \neq d - 1$ if $j \neq k$. Then, let $p_{d-1} = \ldots, p_{d-k+1}, p_{d-k+1}, \ldots, p_d$, and let $p_{d-k+1} = 1$. Note that for each $i$, we have $p_{d-k+1} \in H$, so $p_{d-k+1} = (0, \Delta_p)$. It can be shown (for $i > 0$) that $\Delta_p$ interchanges $e_{d-1}$ and $e_i$, but leaves $e_j$ fixed if $j \neq i, d - 1$. Now

\[
(p_{d-k+1}, \ldots, p_{d-k+1}) = (0 - \Delta_p, e_{d-1} - \Delta_p, p_{d-1})
\]

which equals $(-e_{d-1}, \Delta_p, p_{d-1})$. In particular, we have $(p_{d-k+1}, \ldots, p_{d-k+1}) = (-e_{d-1}, \Delta_p, p_{d-1})$.

Now, assume that

\[
(p_{d-k+1}, \ldots, p_{d-k+1})\psi = (-e_{d-1}, -e_{d-1}, \Delta_p, p_{d-1})
\]

for some $l$ with $1 \leq l \leq k - 1$. Then

\[
(p_{d-k+1}, \ldots, p_{d-k+1})\psi = (0, \Delta_p)(-e_{d-1}, \Delta_p, p_{d-1}) = (-e_{d-1}, \Delta_p, p_{d-1})
\]

which is just $(z, \Delta_p)$, where $g = p_{d-k+1}, p_{d-k+1}, \ldots, p_{d-k+1}$ is an element of $H$. Note finally that since $i_j \geq 1$ for all $j$, $(p_{d-k+1}, \ldots, p_{d-k+1})$ is an element of $H_0$, and so $(z, \Delta_p)$ is indeed an element of $(H_0)\psi$.

Theorem 6.2.16: $(H_0H_d, \psi) = P$.

Proof: Let $(z, \Delta_p) \in P$. Then there exists $g$ such that $(z, \Delta_p) \in (H_0)\psi$. By the previous Lemmas. But then $(z, \Delta_p) = (0, \Delta_p) = \Delta_p$ which is an element of $(H_0)\psi = P$, which is the end of the proof.

If $\alpha \in M$ and $\beta \in H$, then $\alpha \psi = \beta \psi$ implies $\alpha = \beta$. This is the required conjugation.

Lemma 6.2.17: $(u, \Delta) \in \Delta$ if and only if $(I + \Delta + D^2 + \ldots + \Delta^n)^m = 0$, where $n$ is the order of $\Delta$ in $M$.

Proof: Now $(u, \Delta)$ has finite order if only if there exists some $m \in Z^+$ such that $(u, \Delta)^m = (0, I)$. As noted, this is equivalent to saying $(u + \Delta + \ldots + \Delta^{m-1}u, \Delta^m) = (0, I)$ for some $m$. Certainly, $m$ is at most the order of $\Delta$ in $M$. Let $m = nk$, where $k \in Z^+$. Then it may be shown that $(I + \Delta + \ldots + \Delta^m) = k(I + \Delta + \ldots + \Delta^m)$, so $(u, \Delta)$ has finite order if and only if $(I + \Delta + \ldots + \Delta^m)^m = 0$, which is the case if and only if $(I + \Delta + \ldots + \Delta^m)^m = 0$, as claimed.

If $(u, \Delta) \in (wH_0H_d, \psi)$ for some $w \in W$, we shall call it a nonparsec element of $X$. 
Lemma 6.2.18: \((u, \Delta_k)\) is nonsparse if and only if there exist \(z^*, w \in \mathbb{Z}^{k-1}\) such that all coordinates of \(z^*\) are 0 or \(\pm 1\), and \((I - \Delta_k)w = u + z^*\).

Proof: \((u, \Delta_k)\) is nonsparse if and only if there exists \((w, \Delta_k) \in X\) and \((-z, \Delta_k) \in P\) such that \((u, \Delta_k) = (w, \Delta_k)(-z, \Delta_k)(w, \Delta_k)^{-1}\), that is, if and only if there exists \(z \in S_T\), \(w \in \mathbb{Z}^{k-1}\) and \(h, k \in H\) such that
\[
(u, \Delta_k) = (w, \Delta_k)(-z, \Delta_k)(-\Delta_{k-1}w, \Delta_{k-1}) = (w - \Delta_k z - \Delta_k \Delta_{k-1}w, \Delta_k \Delta_{k-1}) = (w - \Delta_k z - \Delta_k \Delta_{k-1}w, \Delta_k \Delta_{k-1} - 1).
\]
Evidently, we must be that \(k = h^{-1}gh\), whence \((u, \Delta_k)\) is nonsparse if and only if there exists \(h \in H\) and \(z \in S_T\) such that \(u = (I - \Delta_k)w = \Delta_k z\), that is, such that \((I - \Delta_k)w = u + \Delta_k z\). Now since \((\Delta_k z : h \in H\) and \(z \in S_T\)\) \((z^* : \text{every coordinate of } z^* \text{ is either } 0 \text{ or } \pm 1\)) \(\subseteq S_T - S_T\), we therefore have \((u, \Delta_k)\) is nonsparse if and only if there exists \(w \in \mathbb{Z}^{k-1}\) and \(z^* \in S_T - S_T\) such that \((I - \Delta_k)w = u + z^*\).

We now work towards establishing a link between the nonsparse elements and those of finite order.

Lemma 6.2.19: Let \(D\) be a \(k \times k\) matrix of the form
\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]
where \(\chi = \pm 1\). Further, let \(n\) be such that \(D^n = I\). Then for \(u \in \mathbb{Z}^k\) the following two conditions are equivalent.

(i) There exists \(z \in \mathbb{Z}^k\), all of whose coordinates are 0 or \(\pm 1\), such that \((I + D + \ldots + D^{n-1})(u + z) = 0\), and
(ii) There exists \(z' \in \mathbb{Z}^k\), all of whose coordinates are 0 or \(\pm 1\), and \(w \in \mathbb{Z}^n\), such that \((I - D)w = u + z'\).

Proof: It is easy to prove that (ii) implies (i), for if there exists \(z'\) with all coordinates 0 or \(\pm 1\), and \(w \in \mathbb{Z}^n\) such that \(u + z' = (I - D)w\), then
\[
(I + D + \ldots + D^{n-1})(u + z') = (I + D + \ldots + D^{n-1})(I - D)w = (I - D^n)w
\]
which, since \(D^n = I\), is equal to 0.

To show that (i) implies (ii), note first that \(D^k = \chi I\). Thus the order of \(D\) is \(k\) if \(\chi = 1\) and \(2k\) if \(\chi = -1\). In particular, since \(D^n = I\), we conclude that \(k\) divides \(n\). Hence we may write
\[
(I + D + \ldots + D^{n-1}) = (I + D^k + \ldots + D^{k\frac{n-1}{k}})(I + D + \ldots + D^{n-1}) = (I + \chi I + \ldots + \chi^{k-1})I = (I + D + \ldots + D^{n-1}) = (I + \chi I + \ldots + \chi^{k-1}I = (I + D + \ldots + D^{n-1}).
\]

If \(\chi = -1\), then \(2k\) divides \(n\), so \(k\) is even. Thus \((1 + \chi + \ldots + \chi^{k-1}) = 0\), so \(I + D + \ldots + D^{n-1} = 0\). On the other hand, if \(\chi = 1\), it may be shown that
\[
(I + D + \ldots + D^{n-1}) = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{pmatrix}
\]
which is denoted \(J\). Also, \(1 + \chi + \ldots + \chi^{k-1} = 0\), so \(I + D + \ldots + D^{n-1} = \frac{1}{k}J\).

We are assuming that there exists \(z \in \mathbb{Z}^k\) with coordinates all 0 or \(\pm 1\) such that \((I + D + \ldots + D^{n-1})(u + z) = 0\). If \(\chi = -1\) this will be the case for any \(u \in \mathbb{Z}^k\), so we need to prove that for all \(u \in \mathbb{Z}^k\), there exists \(z'\) (with coordinates 0 and \(\pm 1\)) and \(w\) such that \((I - D)w = u + z'\). It may be shown that \((I - D)\) is invertible (over \(Q\)), with inverse
\[
(I - D)^{-1} = \frac{1}{2}
\]
so if such a \(w\) exists, it will equal \((I - D)^{-1}(u + z')\), which equals
\[
\begin{pmatrix}
+(u_1 + z'_1) - (u_2 + z'_2) - \ldots - (u_{k-1} + z'_{k-1}) - (u_k + z'_k) \\
+(u_1 + z'_1) + (u_2 + z'_2) + \ldots + (u_{k-1} + z'_{k-1}) - (u_k + z'_k) \\
\vdots \\
+(u_1 + z'_1) + (u_2 + z'_2) + \ldots + (u_{k-1} + z'_{k-1}) + (u_k + z'_k)
\end{pmatrix}
\]

which will be an element of \(\mathbb{Z}^k\) if all the \(u_i + z'_i\) are even. This however is easy to ensure, by careful choice of the \(z'_i\). Thus there exists \(z'\) with all coordinates either 0 or \(\pm 1\), such that there exists \(w \in \mathbb{Z}^n\) with \((I - D)w = u + z'\).
Now, let us consider the case \( x = +1 \). We are assuming that there exists \( z \in \mathbb{Z}^k \), all coordinates 0 or \( \pm 1 \), such that \( (I + D + \ldots + D^{n-1})(u + z) = 0 \), that is, \( \frac{1}{2} \sum (u_i + z_i) = 0 \), that is, \( \sum (u_i + z_i) = 0 \). Let \( w = (w_1, \ldots, w_k) \) be such that
\[
w_j = \sum_{i=1}^{j} (u_i + z_i),
\]
so \( w_1 = u_1 + z_1 \) and \( w_k = (u_1 + z_1) + \ldots + (u_k + z_k) = 0 \). Note that for \( 1 \leq j \leq k - 1 \) we have \( w_j - w_{j-1} = u_j + z_j \). Now \( D(w_1, \ldots, w_{k-1}, w_k)^T = (w_2, w_3, \ldots, w_k)^T \), so
\[
(I - D)w = (w_1 - w_k, w_2 - w_1, \ldots, w_k - w_{k-1})^T
= (u_1 + z_1 - u_2 - z_2, \ldots, u_k + z_k - u_{k-1})^T
= u + z.
\]
as required. This completes the proof.

**Theorem 6.2.20:** An element \((u, \Delta) \in X \times S_T - S_T\) is non-base if and only if there exists \( z^* \in S_T - S_T \) such that \((u + z^*, \Delta) \in \text{finite order in } X\).  

**Proof:** First, let \((u, \Delta) \in S_T - S_T\). Then there exists \( w \in \mathbb{Z}^{d-1} \) and \( z^* \in S_T - S_T \), such that \( u + z^* = (I - \Delta)w \) (Lemma 6.2.18). Let \( \Delta \) have order \( n \) in the finite group \( M \). Then \((I + \Delta + \Delta^2 + \ldots + \Delta^{n-1})(I - \Delta) = I - \Delta^n \), which equals the zero matrix \( O_n \) since \( \Delta^n = I \). Thus, \((I + \Delta + \Delta^2 + \ldots + \Delta^{n-1})(u + z^*) = O_{n}w = 0\), and so by Lemma 6.2.17 \((u, z^*, \Delta) \in \text{finite order in } X\), as required. Conversely, let us now assume that there exists \( z^* \in S_T - S_T \) such that \((I + \Delta + \Delta^2 + \ldots + \Delta^{n-1})(u + z^*) = 0\). Now \( \Delta \) will permute the sets \( \{ \pm e_1 \}, \{ \pm e_2 \}, \ldots, \{ \pm e_{d-1} \} \). This permutation may be written as a product of disjoint cycles
\[
\{ \pm e_1 \}, \{ \pm e_2 \}, \ldots, \{ \pm e_{d-1} \} \\
\{ \pm e_{d+1} \}, \{ \pm e_{d+2} \}, \ldots, \{ \pm e_{d+k} \}.
\]

Let us define \( \varepsilon_j \in \{ \pm 1 \} \) via
\[
\Delta e_i = \varepsilon_{i,1} e_i \quad \Delta(e_i \varepsilon_{i,2}) = \varepsilon_{i,2} e_i \quad \ldots \quad \Delta(e_i \varepsilon_{i,n-1}) = \varepsilon_{i,n-1} e_i \\
\Delta e_{i+1} = \varepsilon_{i+1,1} e_{i+1} \quad \Delta e_{i+2} = \varepsilon_{i+2,1} e_{i+2} \quad \ldots \quad \Delta(e_k \varepsilon_{k,n-1}) = \varepsilon_{k,n-1} e_k
\]
and so forth. It may be shown that there exists \( Q \in M \) such that \( Q e_1 = e_1, Q e_2 = \varepsilon_{1,1} e_2, \ldots, Q e_k = \varepsilon_{k-1,1} e_k, \) and \( Q e_{k+1} = e_{k+1,1} \) and so forth. Then let \( D = Q^{-1} \Delta Q \). This matrix will satisfy \( \Delta e_1 = e_2, \Delta e_2 = e_3, \ldots, \Delta e_{k+1} = e_{k+2,1} \). Now, if \( z^* \in S_T - S_T \) is such that \((I + \Delta + \Delta^2 + \ldots + \Delta^{n-1})(u + z^*) = 0\), then we also have \( Q(I + \Delta + \Delta^2 + \ldots + \Delta^{n-1})Q^{-1}(u + z^*) = 0\). Letting \( u' = Q^{-1} u \) and \( z' = Q^{-1} z^* \), we find this to be equivalent to the statement that there exists \( z' \in S_T - S_T \) such that \((I + D + \ldots + D^{n-1})(u' + z') = 0\). Writing \( u' \) and \( z' \) in block form, we have
\[
\begin{pmatrix}
0 & 0 & \chi_1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]
where each \( D_i \) is a \( k_i \times k_i \) matrix of the form
\[
\begin{pmatrix}
D_i & 0 \\
0 & D_i
\end{pmatrix}
\]
and \( \chi_i = \varepsilon_{k_i} = \pm 1 \).

Now, if \( x = +1 \) and \( z^* \in S_T - S_T \) is such that \((I + \Delta + \Delta^2 + \ldots + \Delta^{n-1})(u + z^*) = 0\), then we also have \( Q(I + \Delta + \Delta^2 + \ldots + \Delta^{n-1})Q^{-1}(u + z^*) = 0\). Letting \( u' = Q^{-1} u \) and \( z' = Q^{-1} z^* \), we find this to be equivalent to the statement that there exists \( z' \in S_T - S_T \) such that \((I + D + \ldots + D^{n-1})(u' + z') = 0\). Writing \( u' \) and \( z' \) in block form, we have
\[
\begin{pmatrix}
0 & 0 & \chi_1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]
where each \( D_i \) is a \( k_i \times k_i \) matrix of the form
\[
\begin{pmatrix}
D_i & 0 \\
0 & D_i
\end{pmatrix}
\]
and \( \chi_i = \varepsilon_{k_i} = \pm 1 \).
and so, since $Qz \in S_T - S_T$, this completes the proof.

In particular, a sparse subgroup of $C_{d-1}$ has no elements of finite order (except the identity). We are still a long way from classifying these sparse subgroups. It would appear that completing this job will be a major work in its own right. This being the case, I shall close this section by pointing to one or two techniques that may provide a starting point, and by linking the results of this section to the lattices, which were described in §4.4.

Let $W = W_{4(3)}$, and let $X \cong W$ be the group $X = Z_d - M = \{ (a, \Delta) : z \in Z - M, \Delta \in M \}$, as described above. First, note the following lemma.

Lemma 6.2.21: If $G$ is any group, and $H$ is a finite index subgroup of $G$, then for all $g \in G$, there exists some $m \in Z^+$ such that $g^m \in H$.

Proof: Since $H$ has finite index, the coets of the form $g^iH$ cannot all be distinct. Letting $i$ and $j$ be such that $i < j$ and $g^iH = g^jH$, we conclude that $H = g^{j-i}H$ as required.

If $A$ is a finite index sparse subgroup of $X$, it follows that for each $i$, there exists some $m_i \in Z^+$ such that $(e_i, I)^{m_i} = (m_i e_i, I) \in A$, which leads to the next result.

Lemma 6.2.22: Let $A$ be a finite index sparse subgroup of $X$. Then there exists some $m \in Z^+$ such that $\{(z, I) : z \in mZ - 1 \}$ is a subgroup of $A$.

Proof: Let $m$ be any common multiple of the $m_i$, and write $m = m_i k_i$ for all $i$. Then if $z \in mZ - 1$, we have $z = \sum k_i m_i e_i$ for some $k_i$, which in turn is equal to $\sum m_i k_i e_i$, so $(z, I) = \prod (m_i e_i, I)^{k_i}$, whence $(z, I) \in A$ as required.

From now on, let $m$ be the least positive integer such that $\{(z, I) : z \in mZ - 1 \}$ is a subgroup of $A$. We shall simplify our notation a little, denoting $(z, I) : z \in mZ - 1$ by $mZ - 1$. This $mZ - 1$ can be shown to be normal in $X$ (hence also in $A$). Now if $(z, \Delta) \in A$, then $(mz + \Delta, \Delta) \in A$ for all $mz \in mZ - 1$, that is, $mZ - 1 (z, \Delta) \subseteq A$. It follows that if we know all elements of $A' = mZ - 1$, then we know the group $A$ itself, so the structure of $A$ is completely determined by the structure of $A' \leq X' = mZ - 1$ and of $mZ - 1$. The problem of classifying the sparse subgroups of $A$ becomes that of classifying those subgroups $A'$ of $X'$ for which $A = (mZ - 1)A'$. This problem is to be somewhat easier than the original problem, because of the next lemma.

Lemma 6.2.23: $X'$ is a finite group, having order $(2m)^{d-1}(d-1)!$.

Proof: Note that $X' = \{ (mZ - 1 + z, \Delta) : z \in Z - 1, \Delta \in M \}$, which in turn is equal to $\{ (z, \Delta) : z' \in Z - 1/mZ - 1, \Delta \in M \}$. The size of this set will equal $\lfloor (mZ - 1)/mZ - 1 \rfloor |M|$. But $Z - 1/mZ - 1$ is just $Z - 1$, and $M$ is isomorphic to the Coxeter group $C_{d-1}$ (which has order $2^{d-1}(d-1)!$), from Table A.7, so $|X'| = |Z - 1| |C_{d-1}| = m^{d-1} 2^{d-1}(d-1)!$, as claimed.

Further, we have the following.

Theorem 6.2.24: $A$ is a sparse subgroup of $X$ if for every element $(z, \Delta)$ of $A' = A/mZ - 1$ where $\Delta$ has order $n$, there is no $z' \in Z - 1$ with coordinates all 0 or $\pm 1$ such that $(I + \Delta + \ldots + \Delta^{n-1})(z' + z) \equiv 0 \pmod m$.

Proof: This follows easily enough from Theorem 6.2.20 and Lemma 6.2.17.

The $mZ - 1 = \{ (z, I) : z \in mZ - 1 \}$ are not the only normal subgroups of $X = \{ (z, \Delta) : z \in Z - 1, \Delta \in M \}$, and it should be noted that the considerations that followLemma 6.2.22 may be applied to any finite index normal subgroup $N$ of $X$. That is, the problem of classifying those sparse subgroups of $A$ containing $N$ becomes the problem of classifying certain subgroups $A'$ of the finite group $X/N$.

The technique was illustrated with the groups $mZ - 1$ since these groups have a particularly simple structure, and because (as Lemma 6.2.22 shows), every finite index sparse subgroup contains such an $N$, and so has the potential to be discovered in this way. We obtain three more results.

Lemma 6.2.25: Let $A$ be a sparse subgroup of $X$, and let $mZ - 1 < A$. Let $A' = A/mZ - 1$, and $X' = X/mZ - 1$. Then the index $[X : A]$ of $A$ in $X$ satisfies $[X : A] = \frac{1}{2m} (2m)^{d-1}(d-1)!$.

Proof: By the so-called “Correspondence Theorem” [42, Thm 2.17] we have $[X : A] = [X' : A']$, which equals $\frac{1}{2m} [X' : X]$ since $X'$ and $A'$ are finite groups. Applying Lemma 6.2.23 yields the required result.

Corollary 6.2.26: Let $A$ be a finite index sparse subgroup of $X$, and let $mZ - 1 \leq A$. Further, let $A' = A/mZ - 1$. Then $|A'| = mZ - 1$. Proof: We know that $[X : A] = \frac{1}{2m} (2m)^{d-1}(d-1)!$. Since $A$ is sparse in $X$, the polytope $P = \mathcal{A}/A$ is well defined (Theorem 5.2.36) and from Theorem 5.2.41 satisfies $[P_0] = \frac{|X'|}{|Z'|} m^{d-1} (2m)^{d-1}(d-1)! = m^{d-1} [A']$. Since $P_0$ must be an integer, it follows that $[A'] = mZ - 1$ as claimed.
This corollary restricts (to a greater or lesser degree, depending on \( m \)) the class of subgroups of \( \mathbb{Z}^d \) that need to be examined. The next few considerations have the potential to restrict this class still further.

Note that there exists a homomorphism \( \phi \) from \( H \) to \( \mathbb{Z}^d \), defined via \( \phi(0, \Delta) = \Delta \). If we restrict \( \phi \) to \( A \), it remains a homomorphism. Now Theorem 6.2.20 and Lemma 6.2.17 tell us that there exist certain elements of \( M \) that cannot be elements of \( A \) (the image of \( A \) under \( \phi \)) if \( A \) is to remain sparse. For example, an element of order \( n \) cannot be such that \( I - \Delta = \sum \Delta = 0 \), and this may be shown to be equivalent to saying that the determinant of \( I - \Delta \) must be zero. One way of classifying the sparse subgroups of \( M \) may be to classify the subgroups \( L \) for which \( \{ \det(I - \Delta) : \Delta \in \mathbb{Z}^d \} = \{0\} \), and see how they may be extended to form sparse subgroups of \( X \).

Combining the two methods has the potential to yield some nice results, for example the following.

**Theorem 6.2.27:** Let \( d \geq 3 \), and let \( A \) be a sparse subgroup of \( X \). If \( m \) has no prime factors less than \( d \), and is such that \( m2^{d-1} \leq A \), then \( A \subseteq \mathbb{Z}^{d-1} \).

**Proof:** We are essentially trying to prove that \( A = \{0\} \). Assume this is not the case, and let \( (a, \Delta) \in A \) be such that \( \Delta \neq 1 \). If the order of \( \Delta \) is \( n(\neq 1) \), then \( n \) must divide the order of \( (a + m2^{d-1}, \Delta) = (a', \Delta) \) in \( A' = A/m2^{d-1} \). However, the order of \( (a', \Delta) \) divides the order of \( A' \), which in turn divides \( m2^{d-1} \). Thus \( n \) divides \( m2^{d-1} \). Let \( p \) be a prime dividing \( n \), so \( p \) also divides \( m \). Now \( n \), being the order of an element of \( M \), must divide the order of \( M \) itself, which is \( 2^{d-1}(d - 1) \). All the prime factors of this number (including \( p \)), are less than \( d \). It follows that \( m \) has a prime factor less than \( d \), which contradicts the hypotheses of the theorem.

It would then remain to classify those subgroups of \( \mathbb{Z}^{d-1} \) containing no nontrivial elements of \( S_T - S_T \) (that is, no non-sparse elements), and we would have enumerated a great many of the sparse subgroups of \( X \).

We have glimpsed a few methods with which one might begin the attack on the combinatorially regular Euler incidence polytopes with Schl"{a}fli Symbol \( \{4|3| \ldots |3|4\} \). It should be remembered, of course, that these are not the only courses of action available, and perhaps in the end these methods will founder, and other quite different methods succeed.

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6.2.4 The Link With the Lattices:

We have discovered a great deal regarding combinatorially regular Euler polytopes with Schl"{a}fli Symbol \( \{4|3| \ldots |3|4\} \), finding them to be quotients of the universal polytope \( \mathcal{M} \) with this Schl"{a}fli Symbol by sparse subgroups of the Coxeter group \( W_{4|3| \ldots |3|4} \). However, in §4.4 we examined a class of polytopes, the \( d \)-lattices, which turned out also to be (if \( d \) is even) combinatorially regular Euler polytopes with this Schl"{a}fli Symbol. It follows that the lattices must be quotients of \( \mathcal{M} \) also, but it is not immediately obvious which sparse subgroups they are associated with. The aim of this subsection is to make it obvious.

Recall that the universal \( d \)-lattice \( \mathcal{L} \) was defined as follows. Letting \( Z = \mathbb{Z}^{d-1} \) be generated by the \('basis'\) \( T = \{e_1, \ldots, e_{d-1}\} \), we define the subsets \( S_i \) of \( Z \) via \( S_i = \{ \sum e_j : J \subseteq I \} \). Then \( \mathcal{L} \) was defined to be the set \( \{0, Z\} \cup \{S + S_j : \} \). With the partial order \( \leq \). Letting \( K \) be a subgroup of \( Z \), we define the combinatorial \( d \)-lattice \( \mathcal{V} \) by defining first an action of \( K \) on \( \mathcal{L} \) via \( (X)^k = k + X \), and letting \( \mathcal{V} = \{[X] : X \in \mathcal{L} \} \), where \( [X] = \{k + X : k \in K \} \) is the orbit of \( K \) containing \( X \).

By Theorems 4.4.13 and 4.4.16, \( \mathcal{V} \) will be a combinatorially regular sub-Euler incidence polytope (Euler if \( d \) is even) if \( K \) satisfies the condition (labelled K1) that for all \( x, y \in \mathcal{V} \), we have \( x - y \in K \) if and only if \( x = y \); that is, \( K \) has no element (except 0), whose coordinates are all 0's and ±1's. Also, in this circumstances, we find (for \( d \geq 3 \)), that the Schl"{a}fli Symbol of \( \mathcal{V} \) is \( \{4|3| \ldots |3|4\} \) (see Theorem 4.4.14). It follows (Theorem 5.1.12) that the \( d \)-lattices have a well-defined action on the set of flags \( \mathcal{F}(\mathcal{V}) \) of \( \mathcal{V} \), as discussed in §5.1.3. This being the case, Theorem 5.3.2 tells us that if we fix some flag \( F \) of \( \mathcal{V} \), then \( \mathcal{V} \) will be isomorphic to the quotient polytope \( \mathcal{M}/A \), where \( A = \{w \in W : F^w = F \} \), the stabiliser of \( F \) in \( W \). In fact, \( A \) will be sparse (by Theorem 5.3.3), since the facets and vertex figures of a lattice are cubes and crosses (see Theorem 4.4.10), which are universal (Theorems 6.1.5 and 6.1.11). It is important, for our purpose here, to study more closely how the elements of \( W \) act on the flags of \( \mathcal{V} \), for then we will be able to determine exactly which sparse subgroup the lattices are quotients of.

By Lemmas 4.4.5 and 4.4.6, any flag of \( \mathcal{V} \) may be written in the form \( F = \{0\} \cup \{F_i : 0 \leq i \leq d - 1\} \), where the \( F_i \) satisfy \( |F_i| = i \) and \( F_0 \subseteq F_1 \subseteq \ldots \subseteq F_{d-1} \subseteq T \) and the \( F_i \) satisfy \( F_i - F_j \in F_{|i-j|} \) for all \( i, j \). It will prove more useful to express \( F \) in a slightly different form.
Lemma 6.2.28: Any flag $F$ of $\mathcal{B}$ may be written in the form $F = \{\emptyset, [Z]\} \cup \{[x+y, S_{j,i}]: 0 \leq i \leq d-1\}$, where $I_i = \{e_{j,i}: 1 \leq j \leq i\}$ and $y_i = e_{d-1} - \sum_{j=i+1}^{d-1} \zeta_{e_{j,i}}e_{j,i}$, for some $z \in Z$, some $\zeta_e \in \{0,1\}$, and some permutation $\sigma$ of $\{1, \ldots, d-1\}$.

Proof: Let $F = \{\emptyset, [Z]\} \cup \{[x_i, S_{j,i}]: 0 \leq i \leq d-1\}$, and let $z = x_{d-1}$. Now for each $i \geq 1$, we have $|I_i| = 1$, so define $e_i$ via $I_i = \{e_i\}$. This $\sigma$ is easily shown to be a well-defined permutation of the set $\{1, \ldots, d-1\}$. Finally, let $y_i = z - x_i$, for each $i$. To show that $y_i$ are the required form, note that $y_{j,i} - y_i = (x_{j,i} - z) - (x_i - z) = x_{j,i} - x_i$, and that $x_{j,i} - z \in S_{j,i} \cup \{0, e_i\}$. Thus $y_{j,i} - y_i$ may be written $\zeta_{e_{j,i}}e_{j,i}$, for some $\zeta_e \in \{0,1\}$, and since also $y_{d-1} = z - x = 0$, our result follows.

Note also that for any $z_i$ and $\sigma_i$, $(\sigma_i)_{i=1}^{d-1}$, if $I_i = \{e_i: i + 1 \leq j \leq d - 1\}$ and $I_i = \{e_i: 1 \leq j \leq d - 1\}$, then $I_i = \{e_i: 1 \leq j \leq d - 1\}$ is a flag of $\mathcal{B}$. We shall denote this flag via $F(z_i, \sigma_i, \zeta_e)$). Let us consider the action of $W$ on these flags. Recall the definition of the flag action, as given in §5.1.3. We shall first of all discover how the generators $s_1, \ldots, s_{d-1}$ of $W$ affect the flags $F(z_i, \sigma_i, \zeta_e)$ of $\mathcal{B}$.

Lemma 6.2.29: $F(z_i, \sigma_i, \zeta_e) = F(z_i', \sigma_i', \zeta_e')$, where $z_i' \in Z + K$, $\sigma_i'$ = $\sigma_i$, $\zeta_e' = 1 - \zeta_e$, and $\zeta_e'^{\sigma} = \zeta_e$ whenever $i > 1$.

Proof: $F(z_i, \sigma_i, \zeta_e) = F(z_i', \sigma_i', \zeta_e')$ will satisfy $F(z_i, \sigma_i, \zeta_e) \cap \mathcal{B} = F(z_i', \sigma_i', \zeta_e') \cap \mathcal{B}$, for all $i \neq 0$. Let $F(z_i, \sigma_i, \zeta_e)$ equal $F(z_i', \sigma_i', \zeta_e')$. Then we have $[x_i + y_i, S_{j,i}] = [x_i' + y_i', S_{j,i}]$ for all $i > 0$, that is, $I_i = I_i'$ and $y_i + y_i' = z + y_i + K$ (see the note following Lemma 4.4.4). Since $|I_0| = |I_0'| = 0$, it follows likewise that $I_0 = I_0'$ in all, whence in fact $\sigma = \sigma'$. We next aim to show that the $\zeta_i$ are as claimed. Consider $i = d-1$. Now $x_{d-1} + y_{d-1} \in Z + y_{d-1} + K$. However, $y_{d-1} + K = Z$, whence $z' \in Z$. For $i > 0$, we thus also have $y_i' \in Z + y_i$. But since $y_i' \in Z + y_i$ and $K \cap (Z + y_i) = \{0\}$, it follows that $y_i' = y_i$. For $i > 1$, we have $(y_i + y_j + S_{j,i}) = \{z_i + y_i + 1\}$, that is, $y_i = 0$. Now we also know that $[x_i + y_i + S_{j,i}] = [x_i + y_i' + S_{j,i}]$. Since $I_i = I_i'$ and $y_i = y_i'$, it must be that $y_i' \neq y_i$. However, $y_i \in \zeta_e e_{i+1} + y_i = \zeta_e e_{i+1} + y_i'$, and $y_i = \zeta_e e_{i+1} + y_i'$. If these two are to be unique, it must be that $z_i = 1 - \zeta_i$. Since both $\zeta_i = \zeta_i$ and $\zeta_i^{\sigma} = \zeta_i$ are elements of $\{0,1\}$, it follows that $\zeta_i = 1 - \zeta_i$. This completes the proof.

Lemma 6.2.30: If $1 \leq i \leq d - 2$, then $F(z_i, \sigma_i, \zeta_e) = F(z_i', \sigma_i', \zeta_e')$, where $z_i' \in Z + K$, $\zeta_e' = \zeta_i$ for all $i$, and $\sigma_i' = (i, i+1). \sigma$.

Proof: In this case, $[x_i + y_i + S_{j,i}] = [x_i' + y_i' + S_{j,i}']$ whenever $j \neq i$, and so if $j \neq i$ we have $I_j = I_i$ and $z_i' + y_i' \in Z + y_i + K$. In particular, if $j = d - 1$ this tells us that $z_i' \in Z + K$, whence as in the proof of the previous theorem we also have $y_i' = y_i$ whenever $j \neq i$. Now $y_i - y_i' = y_i - y_i + y_i + y_i' = y_i + y_i' + y_i$. However, $y_i = y_i'$ and $y_i' = y_i$, so $y_i - y_i' = y_i - y_i' = \zeta_{e_{i+1}}e_{i+1}$. If $j \neq i$, then $z_i' + y_i' \in \{e_i': 1 \leq j \leq d - 1\} = \{e_i': 1 \leq j \leq d - 1\}$, that is, $\zeta_{e_i} = \zeta_i$ whenever $j = d - 1$, that is, $\zeta_i = \zeta_i'$ for all $i$. We now calculate $\sigma_i$. If $j \neq i$, then $I_i$ is given by $e_{j,i} = I_j \cup I_i = I_j \cup I_i = \{e_{j,i}\}$, so $\sigma_i = \sigma_i'$. Also, $e_{i+1} \zeta_{e_{i+1}}$ equals $(I_{i+1} \cup I_i) \cup I_{i+1} = I_{i+1}$, which will equal $I_{i+1}$ whenever $i = 1$. This in turn equals $I_{i+1} \cup I_{i+1} = \{e_{i+1}\}$, which equals $\{e_{i+1}\}$. Thus $e_i$ is either $e_i$ or $e_i$, and $e_{i+1} \zeta_{e_{i+1}}$ (respectively) either $e_{i+1} \zeta_{e_{i+1}}$ or $e_{i+1} \zeta_{e_{i+1}}$, so $\sigma_i$ is either $\sigma_i$ or $(i, i+1) \sigma$. However, since we already know that $z_i' \in Z + K$, $\zeta_i = \zeta_i'$ for all $i$, so $\sigma_i$ is $\sigma_i$ whenever $j \neq i$, that is, $\sigma_i = \sigma_i'$ whenever $j \neq i$. This contradicts our assumption, so $\sigma_i = (i, i+1) \sigma$, as claimed.

By similar arguments, we could also prove the following.

Lemma 6.2.31: $F(z_i, \sigma_i, \zeta_e) = F(z_i', \sigma_i', \zeta_e')$, where $z_i' = z_i + 2(\zeta_{i+1} - 1)d_{i+1} + 1$, and $z_i' \in Z + K$. We shall now look at an example. We shall choose a particular flag of a lattice, and see how the $s_i$ act on it.

Example: Let $d = 3$, so $T = \{e_1, e_2\}$, and let $F$ be the flag $F = \{\emptyset, [Z]\} \cup \{[S_0, 1] \cup [S_0, 1] \cup [S_0, 1] \cup [S_0, 1] \cup [S_0, 1] \cup [S_0, 1]\}$. so $F = F(z, \sigma, \zeta_e)$, where $z = \frac{1}{2}, \sigma$ is the identity permutation, and $\zeta_e$ is the sequence $(\zeta_1, \zeta_2) = (0, 0)$. According to Lemma 6.2.29, $F(z, 1, 0, 0) = F(z, 1, 1, 0)$, so

$F^{\sigma} = \{[0], [Z]\} \cup \{[S_0, 1] \cup [S_0, 1] \cup [S_0, 1] \cup [S_0, 1] \cup [S_0, 1] \cup [S_0, 1]\}$.

Then Lemma 6.2.30 yields $F^{\sigma^2} = F(z, 0, 1, 0) = F(z, 1, 2, 0)$. Unfortunately, the notation at this point has become a little confusing. In this expression for $F^{\sigma^2}$, the $z$ is the vector $e_1 + e_2$, the $1$ is the permutation interchanging $1$ and $2$, and the $0$ is the sequence $(\zeta_1, \zeta_2) = (0, 0)$. Written in full, $F^{\sigma^2}$ equals $F^{\sigma^2} = \{[0], [Z]\} \cup \{[S_0, 1] \cup [S_0, 1] \cup [S_0, 1] \cup [S_0, 1] \cup [S_0, 1] \cup [S_0, 1]\}$. 


Finally, let us calculate $F^{a_0, a_2} = F(s, \sigma, (\zeta)^2)$. Lemma 6.2.31 tells us that this will equal $F(a', \sigma', (\zeta'))$, where $\sigma' = \sigma = (1, 2)$, and $\zeta' = \zeta_1 = \zeta_2 = 0$, and $\zeta' = \zeta_1 = \zeta_2 = 1 - \zeta_2 = 1 = 0$, and also $s' = s + (2\zeta_2 - 1)e_{s_2} + K = (\zeta + e_1 + K) = \zeta + K$. Thus

$$F^{a_0, a_2} = F(s', \sigma', (\zeta'))$$

$$= \{0, [Z] \cup [(\zeta + S_{a_0}, [\zeta] + S_{e_1}], [\zeta] + S_{T_j})].$$

We could also show that

$$F^{a_0, a_2} = \{0, [Z] \cup [(\zeta + S_{a_0}, [\zeta] + S_{e_1}], [\zeta] + S_{T_j})].$$

Compare this with

$$F = \{0, [Z] \cup [(\zeta + S_{a_0}, [\zeta] + S_{e_1}], [\zeta] + S_{T_j}],$$

and note that $a_0 S_{a_0}$ and $a_2 S_{a_2}$, acting on $F$, has had the effect of adding $e_1$ to $x$. This result may be generalised. Recall we defined elements $v_1, \ldots, v_{-1}$ of $W$ via $v_1 = s_0, s_2, s_4, \ldots, s_1$, and then $v_{i+1} = s_1 v_i n$. I state two further lemmas, without full proof.

Lemma 6.2.32: If $F = F(s, \sigma, (\zeta))$ satisfies $\sigma = 1$ and $\zeta = 0$ for all $l$, then $F^v = F(s + e_i, \sigma, (\zeta))$.

Proof: This may be shown through repeated application of Lemmas 6.2.29 to 6.2.31, as in the example above.

Lemma 6.2.33: If $F = F(s, \sigma, (\zeta))$, and $h \in (s_0, \ldots, s_{4-2})$, then $F^h = F(s', (\zeta'))$ satisfies $s' = s + K$, and we have $\sigma = \sigma'$ and $\zeta = \zeta'$ for all $l$ if and only if $h = 1$.

Proof: (Partial Outline) For the first part, note from Lemmas 6.2.29 to 6.2.31 that the only $s_i$, whose action affects $s' = s$ (mod $K$) is in fact $s_{4-1}$, so no effect of element $H = (s_0, \ldots, s_{4-2})$ can have any such effect. One way of showing the second part would be to show that for any $s \in Z^{4-1}$, the action of $H$ is transitive on the set $F(s, \sigma, (\zeta)) : \sigma \in \sum(1, 2, \ldots, d - 1), \zeta \in [0, 1])$, and that this set has the same (finite) order as $H$.

Now we have the tools needed to classify the combinatorial $d$-lattices as quotient polytopes. Let $B = F(0, 1, \ldots, 0)$ be a base flag for the $d$-lattice based on $K \leq Z^{d-1}$. Write $V = \{v_i : 1 \leq i \leq d - 1\} \leq W$, and recall that $V \cong Z^{d-1}$ via the isomorphism $\varphi : \prod v_i^1 \to \sum k_i e_i$ (Lemma 6.2.7). We have the following.

**Theorem 6.2.34**: If $A = \{w \in W : B^w = B\}$, then $A \leq V$, and $A \phi = K$.

Proof: Let $w \in A$, and write $w = v_h$, where $v \in V$ and $h \in H = (s_0, \ldots, s_{d-2})$ (see Lemma 6.2.6). Then $B^w = B$, so $B^w = B^h$. Write $B = F(s, \sigma, (\zeta))$, so $h = 0$, and $\zeta = 0$ for all $l$, and let $B^w = B^h = F(s', (\zeta'))$. Lemma 6.2.32 tells us that $\sigma' = \sigma = 1$, and $\zeta' = \zeta = 0$ for each $l$. Then, Lemma 6.2.33 tells us that in fact $h' = 1 = (s = h), \zeta' = \zeta = 0 \in K$. Thus $v \in V$, whence in particular, $A \leq V$. Now if $v = \prod v_i^{h_i}$, then $F^{a_0, a_1}$ may be used again, to tell us that $B^v = F(s + k_i e_i, \sigma, (\zeta))$, which is just $F(s + \psi, (\zeta))$. Thus $s' = s + \psi$, and we may conclude that $\psi \in K$, so $A \phi = K$. To show that $\phi \in A$, let $x \in K$. Then, writing $x = \sum k_i e_i$, note that $v = \prod v_i^{h_i}$ will satisfy $B^v = F(s, \sigma, (\zeta))F(x, \sigma, (\zeta))$. Since $\zeta = 0$ for each $l$ and $\sigma = 1$, we have $B^v = [x + S_{e_i}]$, where $I_i = \{e_1, \ldots, e_i\}$. Similarly, $B^v = \{0 + S_{e_1}, \ldots, e_{d-1}\}$, which is equal to $[x + S_{e_1}, \ldots, e_{d-1}]$, since $x \in K$. Thus $B^{d-1} = B$ for all $x \in K$, so $\phi \in A$, yielding equality, as required.

So if $\mathcal{F}$ is a lattice, based on $K \leq Z^{d-1}$, then $\mathcal{F} = \mathcal{A}/A$, where $A = K \phi^{-1}$. Note in particular the following.

**Corollary 6.2.35**: The universal $d$-lattice $\mathcal{F}$ is isomorphic to the universal $d$-polytope $\mathcal{A}/A$, based on $W$.

Proof: It has been noted (see the notes preceding Lemma 6.4.2) that $\mathcal{F}$ is the lattice based on $K = (0)$. As such, it will be isomorphic to $\mathcal{A}/A$, where $A = K \phi^{-1} = (0) \phi^{-1} = (1)$. Consulting Lemma 5.2.24, we conclude that $\mathcal{A}/(1) \cong \mathcal{F}$, so $\mathcal{F} \cong \mathcal{A}$ as claimed.

Thus the terminology “universal lattice” is justified. We have succeeded in bringing the lattices under the umbrella of quotient polytopes. These results open up a number of possible lines for further investigation. First of all, we could seek for sparse subgroups of $W$ which are not subgroups of $V$, or we could attempt to classify the flag regular polytopes, by classifying the sparse normal subgroups of $W$. In fact, Appendix B at least makes a start at all of these, exhibiting a sparse subgroup of $W$ not contained in $V$, and also proving some very nice results about flag regular polytopes. On another tack, the lattices themselves could be examined using the results of §5.2. For example, we could (using Theorem 5.2.46) research the circumstances under which two lattices will be isomorphic.

However, as interesting as these results would be, and as incomplete as this section perhaps becomes through their omission, it is time to move on – to focus our attention on some of the other Schläfli Symbols which arise in our search. So far, the best classification for the Schläfli Symbol $\{4, 3, 3, 3\}$ is that stated
at the start of §6.2.3, and hence perhaps the omission of certain results may be justified on the grounds of brevity. Although they would certainly appear in a complete classification, a complete classification is exactly what we do not (at this stage) have. One thing that would seem absolutely necessary before such a classification can go ahead is a much deeper understanding of the structures of the groups \( \mathcal{C}_k \) and \( \mathcal{C}_k^{-1} \).

### 6.3 Particular Dimensions

We now start classifying the indecomposable combinatorially regular Euler \( d \)-incidence polytopes for various specific small \( d \). The results of the previous sections will be a great help, allowing us to quickly dispose of certain cases. In this section we will see unfold a number of theorems stating exactly what Schl"{a}fi Symbols may occur as the Schl"{a}fi Symbols of indecomposable combinatorially regular Euler incidence polytopes, and this will define exactly what the bounds of our search must be.

#### 6.3.1 The 1- and 2-polytopes:

See also §4.5.1 and §4.5.2. We have already seen (see the notes following Theorem 3.3.9) that there is only one 1-incidence polytope up to isomorphism. In Theorem 3.3.10, it was shown that any 2-incidence polytope \( \mathcal{P} \) is isomorphic to an \( n \)-cycle for some \( n \geq 2 \) (specifically, \( n = |\mathcal{P}_0| = |\mathcal{P}_1| \)). It is worth noting that the \( n \)-cycle is therefore isomorphic to the universal polytope based on the dihedral group\(^2\) \( I_2(n) \), which has Coxeter matrix

\[
\begin{pmatrix}
1 & n \\
1 & 1
\end{pmatrix}.
\]

This tells us (amongst other things) that all 2-polytopes are universal.

#### 6.3.2 The 3-polytopes:

Let \( \mathcal{P} \) be a combinatorially regular 3-incidence polytope, and let it have Schl"{a}fi Symbol \( \{p|q| \} \). Its facets and vertex figures will be 2-polytopes, and hence will be universal. It follows that to classify the combinatorially regular 3-incidence polytopes, we have but to classify the sparse subgroups of the Coxeter groups \( W(p,q) \) with Coxeter diagram given below.

\[\text{(Diagram of Coxeter group)}\]

By Theorem 3.4.14, we do not need to consider the case when either \( p \) or \( q \) is 2. Since only five Schl"{a}fi Symbols \( \{p|q| \} \) with \( p,q \geq 3 \) are such that \( W(p,q) \) is finite, the main difficulty in classifying the combinatorially regular 3-incidence polytopes would be in analysing the subgroup structures of the vast array of infinite Coxeter groups that arise. This task being somewhat beyond the scope of this thesis, let us return once again to the case where the polytope is Euler. We then have the following.

**Theorem 6.3.1:** Let \( \mathcal{P} \) be an indecomposable combinatorially regular Euler 3-incidence polytope with Schl"{a}fi Symbol \( \{p|q| \} \). Then \( \{p|q| \} \) is one of \( \{3|3| \}, \{3|4| \}, \{4|3| \}, \{3|5| \} \) or \( \{5|3| \} \).

**Proof:** Note first that \( p,q \geq 3 \), since \( \mathcal{P} \) is indecomposable (see Theorem 3.4.14). Then, for any \( a_0 \in \mathcal{P}_0, a_1 \in \mathcal{P}_1 \) and \( a_2 \in \mathcal{P}_2 \), we have \( |\{a_0,a_2| \cap \mathcal{P}_0| = p \) (since by Theorem 3.4.12, it has Schl"{a}fi Symbol \( \{l| \} \)), \( |\{a_0,P| \cap \mathcal{P}_1| = |\{a_0,P| \cap \mathcal{P}_2| = q \) (for similar reasons), and also \( \{a_1| \) is a 1-incidence polytope, so (by the note following Theorem 3.3.9), \( |\{a_1| \cap \mathcal{P}_1| = 2 \). It follows from Lemma 3.4.15 that

\[
|\mathcal{P}_0| = \frac{|\{a_0,P| \cap \mathcal{P}_1|}{|\{a_0,a_1| \cap \mathcal{P}_0| = \frac{q}{2} \mathcal{P}_0|}
\]

and \( |\mathcal{P}_2| = \frac{|\{a_0,P| \cap \mathcal{P}_2|}{|\{a_1,a_2| \cap \mathcal{P}_0| = \frac{q}{p} \mathcal{P}_0|}.
\]

Since \( \mathcal{P} \) is Euler, we require that

\[
(-1)^{|\mathcal{P}_0|} + (-1)^{|\mathcal{P}_1|} + (-1)^{|\mathcal{P}_2|} = 0,
\]

that is,

\[
-1 + \frac{q}{2} \mathcal{P}_0| + \frac{q}{p} \mathcal{P}_0| = 1.
\]

Therefore \( 1 + \frac{q}{2} + \frac{q}{p} \leq 1 \). Since \( |\mathcal{P}_0| \geq |\{a_0,a_1| \cap \mathcal{P}_0| = p \), it follows that \( 0 < 1 + \frac{q}{2} + \frac{q}{p} \leq 4 \), whence \( 0 < 2p - q + 2q - 4 \leq 4 \). If we subtract each term of this inequality from 4, we obtain \( 4 - 0 > 2q - 2q + 4 \geq 4 - 4 \), that is, \( 4 > (p - 2)(q - 2) \geq 4 \). Since also \( p,q \geq 3 \), the above inequality has only five solutions, namely those listed in the statement of the theorem.

This proof could also have been done using the formulas

\[
|\mathcal{P}| = \frac{|W:A|}{|H|}
\]
given by Theorem 5.2.41.

We could take the five Coxeter groups \(W_{[3]}, W_{[4]}, W_{[5]}, W_{[6]}, \) and so forth, and individually search each one for sparse subgroups. However, this is not necessary if we restrict attention to Euler polytopes (as indeed we are).

From now on, let \( \zeta_4 = \frac{1}{2} + \frac{1}{4} - \frac{1}{2} \).

**Theorem 6.3.2:** For any combinatorially regular Euler 3-incidence polytope \( \mathcal{P} \), we have \(|\mathcal{P}_0| = \frac{1}{\zeta_4}, |\mathcal{P}_1| = \frac{1}{\zeta_4} \) and \(|\mathcal{P}_2| = \frac{1}{2\zeta_4} \).

**Proof:** Since we must have (see the proof of the previous result) \((1 - \frac{1}{4} + \frac{1}{2})|\mathcal{P}_0| = q|\mathcal{P}_0| = 2 \), it follows that \(|\mathcal{P}_0| = \frac{1}{\zeta_4} \), as required. Also, since \(|\mathcal{P}_1| = \frac{3}{4}|\mathcal{P}_0| \) (as was argued in the previous proof from Theorem 3.4.15), \(|\mathcal{P}_1| = \frac{1}{\zeta_4}, \) and \(|\mathcal{P}_2| = \frac{1}{2\zeta_4} \).

**Theorem 6.3.3:** Any indecomposable combinatorially regular Euler 3-incidence polytope is universal.

**Proof:** Let \( \mathcal{P} \) be any such polytope, and let its Schl"afli Symbol be \( \{p|q|r\} \). Theorem 6.3.1 tells us that \( \{p|q|\} \) is one of \( \{3|3\}, \{3|4\}, \{3|5\}, \{4|3\} \) or \( \{5|3\} \). It should be noted that all five of the Coxeter groups \( W_{[3]}, W_{[4]}, W_{[5]}, W_{[6]} \) and \( W_{[7]} \) have finite order (see Table A.1 or A.7). Now the universal polytope with this Schl"afli Symbol is also combinatorially regular (by Theorem 5.2.15), and Euler (by Theorem 5.2.12), and so satisfies the conditions of the previous theorem. It follows from Theorem 6.3.2 that \(|\mathcal{P}_0| = |\mathcal{P}_0| = \frac{1}{\zeta_4} \). Then, Corollary 5.3.6 tells us that in fact \( \mathcal{P} \) and \( \mathcal{A} \) are isomorphic.

Combining these results yields the following.

**Theorem 6.3.4:** There are (up to isomorphism) only five indecomposable combinatorially regular Euler 3-incidence polytopes, namely the five described in §4.5.3.

**Proof:** Any indecomposable combinatorially regular Euler 3-incidence polytope has Schl"afli Symbol either \( \{3|3\}, \{3|4\}, \{3|5\}, \{4|3\} \) and \( \{5|3\} \), and is also universal. Since for each of the groups \( W_{[3]}, W_{[4]}, \) and \( W_{[5]}, \) and so on, the universal polytope both exists and is unique, and is also combinatorially regular and Euler (by Theorem 5.2.9, Corollary 5.2.12 and the note following Theorem 5.2.15), and these five polytopes are pairwise nonisomorphic by Theorem 3.4.11, it follows that as claimed there are exactly five indecomposable combinatorially regular Euler 3-incidence polytopes. That these polytopes are isomorphic to those described in §4.5.3 may be seen from Corollary 5.3.7.

So far in our search for combinatorially regular Euler incidence polytopes, there have been no surprises— all the polytopes examined have been universal polytopes, and combinatorial counterparts to (the familiar) regular geometric polytopes. This is all about to change.

### 6.3.3 The 4-polytopes:

Let us quickly obtain some useful results about the case \( d = 4 \). Let \( \mathcal{P} \) be an indecomposable combinatorially regular sub-Euler 4-polytope with Schl"afli Symbol \( \{p|q|r|s\} \).

**Theorem 6.3.5:** \( \{p|q|r\} \) is one of \( \{3|3|3\}, \{3|3|4\}, \{4|3|3\} \) or \( \{5|3|3\} \), where \( x \) is 3, 4 or 5.

**Proof:** If \( \mathcal{P} \) is sub-Euler with Schl"afli Symbol \( \{p|q|r\} \), its facets and vertex figures are Euler with Schl"afli Symbols \( \{p|q\} \) and \( \{q|r\} \) (respectively (Theorem 5.2.12). Then, Theorem 6.3.1 tells us that \( \{p|q\} \) and \( \{q|r\} \) each must be one of \( \{3|3\}, \{3|4\}, \{4|3\}, \{3|5\} \) or \( \{5|3\} \), leading to the given combinations.

It follows that if \( \mathcal{P} \) is sub-Euler with Schl"afli Symbol \( \{p|q|r\} \), then neither \( \zeta_5 \) nor \( \zeta_4 \) is zero.

**Theorem 6.3.6:** Let \( \mathcal{M} \) be the universal polytope based on \( W = W_{[p|q|r]} \). Then \( \mathcal{P} \) is a quotient of \( \mathcal{M} \) by some sparse subgroup \( A \) of \( W \).

**Proof:** This follows immediately from Theorem 5.3.4, since Theorem 6.3.3 tells us that the facets and vertex figures of \( \mathcal{P} \) are universal.

We move now to some theorems which require longer proofs. Recall Theorem 6.3.2, where it was found that a combinatorially regular Euler 3-incidence polytope \( \mathcal{P} \) with Schl"afli Symbol \( \{p|q|\} \) satisfies \( |\mathcal{P}_0| = \frac{2}{2\zeta_4}, |\mathcal{P}_1| = \frac{1}{2\zeta_4}, \) and \( |\mathcal{P}_2| = \frac{1}{2\zeta_4} \), where \( \zeta_6 = \frac{1}{2} + \frac{1}{4} - \frac{1}{2} \).

**Theorem 6.3.7:** If \( \mathcal{P} \) is a combinatorially regular 4-incidence polytope, then the \( \mathcal{P}_i \) satisfy \( |\mathcal{P}_0| = \frac{1}{\zeta_4}|\mathcal{P}_0|, |\mathcal{P}_1| = \frac{1}{\zeta_4}|\mathcal{P}_0| \) and \( |\mathcal{P}_2| = \frac{1}{2\zeta_4}|\mathcal{P}_0| \).

**Proof:** Let \( a_0, a_1, a_2 \) and \( a_3 \) be elements of \( \mathcal{P} \) with \( a_i \in \mathcal{P}_i \) for each \( i \). Then from Lemma 3.4.15, we have

\[
|\mathcal{P}_1| = \left| \{a_0, a_i \} \cap \mathcal{P}_0 \right| \mathcal{P}_0.
\]
However, \((a_0, P)\) is an Euler polytope with Schl"{a}fi Symbol \(\{q/r\}\), and \([[(a_0, P) \cap \mathcal{S}]\) is the number of its vertices, that is, \(\frac{1}{r_{\mathcal{S}}}\) (see Theorem 6.3.2). Also, \((\emptyset, a_1)\) is a 1-polytope, so \(\{[(\emptyset, a_1) \cap \mathcal{S}]\) = 2. Thus we have

\[
|\mathcal{S}_1| \left(\frac{2/r_{\mathcal{S}}}{}\right) \cdot |\mathcal{S}|_0 = \frac{1}{r_{\mathcal{S}}}|\mathcal{S}|_0.
\]

By similar reasoning, we can obtain

\[
|\mathcal{S}_2| = \left(\frac{a_0, P \cap \mathcal{S}}{\emptyset, a_2} \cap \mathcal{S}_0\right) = \left(\frac{1}{2} |\mathcal{S}|_0 + \frac{1}{r_{\mathcal{S}}} |\mathcal{S}|_0\right) = \frac{1}{r_{\mathcal{S}}}|\mathcal{S}|_0
\]

and

\[
|\mathcal{S}_3| = \left(\frac{a_0, P \cap \mathcal{S}}{\emptyset, a_3} \cap \mathcal{S}_0\right) = \left(\frac{2/2r_{\mathcal{S}}}{2/2r_{\mathcal{S}}}|\mathcal{S}|_0\right) = \frac{2}{r_{\mathcal{S}}}|\mathcal{S}|_0
\]

As required.

Again, this proof could have been accomplished using Theorem 5.2.41.

Theorem 6.3.8: If \(\mathcal{P}\) is a combinatorially regular 4-incidence polytope, then \(\mathcal{P}\) is Euler if and only if it is sub-Euler and the \(\mathcal{P}\) are finite.

Proof: Certainly, if it is Euler, it will be sub-Euler and the \(\mathcal{P}\) are finite. Now, let \(\mathcal{P}\) be sub-Euler, and let the Schl"{a}fi Symbol of \(\mathcal{P}\) be \(\{p|q|r\}\). We need to show that \(\mathcal{P}\) itself satisfies Euler’s relation, that is,

\[|\mathcal{P}_1| + |\mathcal{P}_0| - |\mathcal{P}_2| - |\mathcal{P}_3| = 0.
\]

This expression may be evaluated, since the \(\mathcal{P}\) are finite. Also, since \(|\mathcal{P}_1| = |\mathcal{P}_4| = 1\), it may be rearranged to yield \(|\mathcal{P}_0| - |\mathcal{P}_1| + |\mathcal{P}_2| - |\mathcal{P}_3| = 0\). Using the previous theorem, we obtain

\[
|\mathcal{P}_0| = \frac{1}{\zeta_{\mathcal{S}}} |\mathcal{P}_1| - \frac{1}{\zeta_{\mathcal{S}}} |\mathcal{P}_2| - \frac{1}{\zeta_{\mathcal{S}}} |\mathcal{P}_3|
\]

which equals zero, as required.

Thus, in contrast with 6.3.2, here we find that Euler’s condition places no restriction whatsoever on \(|\mathcal{P}_0|\). Other considerations do place restrictions, however, such as the ratios we have found between the \(|\mathcal{P}_i|\), and the fact that \(|\mathcal{P}_0| > \max\{\emptyset, a_3\} \cap \mathcal{P}_0\). Taking these into account yields Table 6.3.1.

Table 6.3.1: Overview of Combinatorially Regular 4-Polytopes.

| Case | Dual | Schl"{a}fi Symbol | \(|\mathcal{P}_0|\) | \(|\mathcal{P}_1|\) | \(|\mathcal{P}_2|\) | \(|\mathcal{P}_3|\) | Restrictions |
|------|------|------------------|----------------|----------------|----------------|----------------|---------------|
| I    | I    | \{3\} \{3\}      | Q             | 2Q             | 2Q             | Q              | Q \(\geq 4\)   |
| II   | III   | \{3\} \{3\}      | Q             | 3Q             | 4Q             | Q              | Q \(\geq 4\)   |
| III  | II    | \{4\} \{3\}      | Q             | 2Q             | \(\frac{3}{2}\)Q | \(\frac{3}{2}\)Q | Q \(\geq 8, 2|Q|\)
| IV   | IV    | \{3\} \{3\}      | Q             | 4Q             | 4Q             | Q              | Q \(\geq 6\)   |
| V    | VI    | \{3\} \{3\}      | Q             | 6Q             | 10Q            | 5Q             | Q \(\geq 4\)   |
| VI   | V     | \{3\} \{3\}      | Q             | 2Q             | \(\frac{3}{2}\)Q | \(\frac{3}{2}\)Q | Q \(\geq 20, 5|Q|\)
| VII  | VII   | \{4\} \{3\}      | Q             | 3Q             | 3Q             | Q              | Q \(\geq 8\)   |
| VIII | IX    | \{3\} \{3\}      | Q             | 6Q             | \(\frac{3}{2}\)Q | \(\frac{3}{2}\)Q | Q \(\geq 8, 2|Q|\)
| IX   | VIII  | \{3\} \{3\}      | Q             | 3Q             | \(\frac{12}{5}\)Q | \(\frac{12}{5}\)Q | Q \(\geq 20, 5|Q|\)
| X    | X     | \{3\} \{3\}      | Q             | 10Q            | 10Q            | Q              | Q \(\geq 12\)  |
| XI   | XI    | \{3\} \{3\}      | Q             | 6Q             | 6Q             | Q              | Q \(\geq 20\)  |

Table 6.3.2: Some 4-Polytopes from 6.1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Schl&quot;{a}fi Symbol</th>
<th>Q</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>{3} {3}</td>
<td>5</td>
<td>Simplex</td>
</tr>
<tr>
<td>II</td>
<td>{3} {3}</td>
<td>8</td>
<td>Cross</td>
</tr>
<tr>
<td>III</td>
<td>{4} {3}</td>
<td>16</td>
<td>Cube</td>
</tr>
<tr>
<td>IIIi</td>
<td>{4} {3}</td>
<td>8</td>
<td>Halfcube</td>
</tr>
<tr>
<td>IV</td>
<td>{4} {3}</td>
<td>many</td>
<td>Lattices and Others</td>
</tr>
</tbody>
</table>

We have already examined the cases \{3\} \{3\}, \{3\} \{3\}, \{4\} \{3\} and to some extent \{4\} \{3\}, in §6.1 to §6.2. These yielded results which are summarised in Table 6.3.2.

In the case where \(W = W_{\{p|q|r\}}\) is finite, there are computer programs which can find all the subgroups of \(W\) and check each one to determine whether or not it is sparse. This would then classify the combinatorially regular Euler incidence polytopes with the particular Schl"{a}fi Symbol in question.
This operation was in fact programmed into CAYLEY version 3.8.3, and performed for the Schläfli Symbols \{3\langle3\rangle\} and \{3\langle3\rangle3\}. The results of course confirmed the results of Section 6.1.1 and Section 6.1.3. It was also performed for \{3\langle4\rangle\}, yielding three conjugacy classes of sparse subgroups. Hence we have the following theorem.

Theorem 6.3.9: Up to isomorphism, there are exactly three combinatorially regular Euler incidence polytopes with Schläfli Symbol \{3\langle4\rangle\}.

Of these three conjugacy classes, one was a class of non-normal subgroups, and hence we have a combinatorially regular polytope that is not regular. Some information about the sparse subgroups \(A\) and the quotient polytopes \(\mathcal{M}/A\) derived from them is displayed in Table 6.3.3. The generators shown in the table were derived from those given by CAYLEY, using the known relations between the \(s_i\).

Table 6.3.3: Combinatorially Regular 4-Polytopes with Schläfli Symbol \{3\langle4\rangle\}.

| Case | \(|A|\) | \(|\mathcal{M}/A|\) | \(|\text{Aut}(\mathcal{M}/A)|\) | Regular? |
|------|--------|-----------------|----------------------------|--------|
| IV.i | \{1\}  | 24              | 1152                       | Yes.   |
| IV.ii| \(\langle s_0, s_1, s_2, s_3 \rangle^4\) | 12              | 576                        | Yes.   |
| IV.iii| \(\langle s_0, s_1, s_2, s_3 \rangle^6\)  | 8               | 48                         | No.    |

Unfortunately, CAYLEY's subgroups command lacked the power required to calculate all the subgroups of \(W_{\langle3\langle3\rangle\rangle}\) (which has order 14400). However this does not mean that it was impossible to classify the combinatorially regular polytopes with Schläfli Symbol \{3\langle3\rangle3\}.

Theorem 6.3.10: There are exactly 10 combinatorially regular Euler incidence polytopes with Schläfli Symbol \{3\langle3\rangle3\}, and these are listed in Table 6.3.4.

Proof: Let \(\mathcal{P} = \mathcal{M}/A\), where \(\mathcal{M}\) is the universal polytope based on \(W = W_{\langle3\langle3\rangle\rangle}\) and \(A\) is sparse in \(W\). Then \(|\mathcal{P}| = |W| |\langle A \rangle/|A| = 14400 / 120 = 120\). Since \(|\mathcal{P}|\) is an integer, it follows that \(|A|\) divides 120 (the values of \(W_{\langle3\langle3\rangle\rangle}\) and \(W_{\langle3\rangle}\) coming from [11] or Table A.7. See also Table A.4). Also, \(|\mathcal{P}| \geq 4\) (see Table 6.3.1), so \(|A| \leq 30\). This gives a number of possible values of \(|A|\), and hence of the index \(W : A\) of \(A\) in \(W\). The CAYLEY package may then be instructed to find all the subgroups of \(W\) which have these indices, and to

Note that not all of the values of \(|A|\) mentioned in the proof actually yield examples in practice (for example, there are no sparse subgroups of \(W\) with orders 10, 15, 20 or 30, even though each of these numbers divides 120 and is less than or equal to 30). On the other hand, it was found that there are two conjugacy classes of order 5 sparse subgroups, yielding two nonisomorphic combinatorially regular incidence polytopes, with the same \(|\mathcal{P}|\) for all \(i\). Thus, even if two combinatorially regular Euler incidence polytopes have the same Schläfli Symbol, and the same number of \(i\)-faces for any \(i\), we still cannot conclude that they are isomorphic.

Only two of the ten polytopes are actually regular, the rest being only combinatorially regular. This contrasts strongly with the case in lower dimensions, where we found all indecomposable combinatorially regular Euler incidence polytopes to be regular as well, and indeed universal.

We do not need to examine the case where the Schläfli Symbol is \{5\langle3\rangle\}, since any such polytope will be dual to one of the ten listed above. The only cases left to examine are \{4\langle3\rangle\} (and the dual \{6\langle3\rangle\}), \{5\langle3\rangle3\}, and \{3\langle3\rangle3\}. The corresponding Coxeter groups are infinite, so we should not expect CAYLEY's general-purpose group theory algorithms to be able to classify their subgroups, let alone check them for sparseness.

Peter Lorimer, in a private communication, [31], stated that these Coxeter groups have a very rich subgroup structure, and conjectured that each would
have a great many subgroups of the required form. Marston Conder (in [10], another private communication) gave some examples of such subgroups, demonstrating the existence of combinatorially regular incidence polytopes with each of the given Schl"{a}fli Symbols. The examples he gave, being normal in the various $W$, in fact yield flag regular polytopes. The following Lemma will help demonstrate the method Conder used.

**Lemma 6.3.11:** Let $(W,S)$ be a Coxeter system (whose graph is a path), and where $W$ is finite for all proper subsets $I$ of $S$. If $f$ is any homomorphism from $W$ to $W$ then the kernel of $f$ in $W$ is sparse if and only if $f([H_0]) = [H_0], f([H_0] \cap [H_1]) = [H_0 \cap H_1].$

**Proof:** Let $w \in \ker f \cap wH_0H_1w^{-1}$. Then, $w^{-1}w = w^{-1}(\ker f)w \cap H_0H_0H_1 = \ker f \cap H_0H_1$ where $w$ is normal in $W$. Let $w^{-1}aw = hh$, where $h \in H_0$ and $h \in H_1$. Then $f(h)f(k) = f(hk) = 1$, since $f$ is a homomorphism, and since $w^{-1}aw \in \ker f$, it follows that $f(h) = f(k^{-1})$. We show that $f$ restricted to $H_0 \cap H_1$ is a one to one map. Consider $f(H_0 \cap H_1) = f(H_0) \cup f(H_1)$. Since $H_0$ and $H_1$ are finite, we can deduce that $f(H_0 \cap H_1) = [f(H_0) \cup f(H_1)] \cap [f(H_0) \cap f(H_1)]$. But this will in turn be equal to $|H_0| + |H_1| - |H_0 \cap H_1|$, by the conditions of the lemma. This is just $|H_0 \cap H_1|$, so $f([H_0 \cap H_1]) = [H_0 \cap H_1]$. Since these sets are finite, we deduce that $f$ restricted to $H_0 \cap H_1$ is one to one. Thus, since $f(h) = f(k^{-1})$, we deduce that $h = k^{-1}$, so $a = wkhkw^{-1} w^{-1}w^{-1} = 1$ as required.

Conversely, let $\ker f$ be sparse, and let $x, y \in H_0H_1$ be such that $f(x) = f(y)$. Then $f(xy^{-1}) = f(x) f(y)^{-1} = f(x) f(y^{-1}) = 1$, so $xy^{-1} \in \ker f$. Let $x = h_kh_y$, $y = h_yh_x$, where $h_x, h_y \in H_0$, and $h_y, h_x \in H_1$. Then $xy^{-1} = h_yh_xh_yh_xh_y^{-1}h_x^{-1}h_y^{-1} \in H_0H_0H_1H_1$. But we are assuming that $\ker f$ is sparse, so this tells us that $xy^{-1} = 1$, whence $x = y$. Thus $f$ restricted to $H_0H_1 = 1$ one to one. But $H_0H_1 \subseteq H_0H_1$, and $f$ will also be one to one when restricted to those, yielding in particular $f([H_0]) = [H_0]$ and $f([H_1]) = [H_1]$. Likewise, $f([H_0 \cap H_1]) = [f(H_0) \cap f(H_1)]$, whence $[H_0 \cap H_1] = [f(H_0) + [H_1] - [H_0 \cap H_1] = [f(H_0) + f(H_1)] - [f(H_0) \cap f(H_1)]$, also, completing the proof.

Conder (in [10]) obtains his examples by constructing explicit homomorphisms from $W$, and checking to see whether or not they satisfy the conditions of the lemma. He does this by taking a low index subgroup of $W$ (call it $K$), and examining the action of $W$ on the (right) cosets of $K$. The action of any element $w$ of $W$ permutes the cosets of $K$ by right multiplication, and thus induces a homomorphism from $W$ to the group of permutations of the cosets. It is possible to write a CAYLEY routine to do this, and when this is done, it yields numerous examples of polytopes, some of them very large. For example, there is a subgroup of $W(3|3|3)$ of index 39 where $W$ induces every possible permutation of the subgroup's cosets. The number of vertices of the corresponding strongly regular polytope is a number with thirty-one digits! Some smaller examples are listed in Table 6.3.5.

**Table 6.3.5:** Some Combinatorially Regular Polytopes Corresponding to Infinite Coxeter Groups.

| Case | S.Sym. | $|W : K|$ | $|W : \ker f|$ | $Q = (|\ker f|)_{|\ker(f)|}$ | Regular? |
|------|--------|----------|----------------|------------------|--------|
| VIII | (4|3|5) | 12 | 3840 | 32 = 2⁵ | Yes. |
| VIII | (4|3|5) | 12 | 2793500 | 23232 = 2⁴.3⁶ | Yes. |
| X    | (3|5|3) | 17 | 8160 | 68 = 2⁵.17 | Yes. |
| X    | (3|5|3) | 12 | 259200 | 2160 = 2⁸.3⁵ | Yes. |
| X    | (3|5|3) | 22 | 675840 | 5632 = 2⁷.11 | Yes. |
| X    | (3|5|3) | 22 | 1351680 | 11264 = 2⁹.11 | Yes. |
| XI   | (3|5|3) | 10 | 7200 | 60 = 2³.5.15 | Yes. |
| XI   | (3|5|3) | 17 | 8160 | 68 = 2³.17 | Yes. |

We have found some examples of flag regular incidence polytopes with Schl"{a}fli Symbols (4|3|5), (3|5|3) and (6|3|5). In fact, [35, Cor 4.1] tells us that there will exist infinitely many such polytopes. Since our primary interest in this thesis is in combinatorially regular incidence polytopes, we turn our attention to a method for checking an arbitrary subgroup of these groups for sparseness. This method once again uses the action of $W$ on the cosets of on its subgroups. Let $A \leq W$, and let $\Omega$ be the set of right cosets of $A$. Let $f$ be the natural homomorphism from $W$ to $\Sym(\Omega)$, defined by right multiplication. Then

**Lemma 6.3.12:** $A$ is sparse if and only if for all $x \in H_0H_1, x \neq 1$, the permutation $f(x)$ fixes no element of $\Omega$.

**Proof:** Assume first that $A$ is sparse. Then, let $x \in H_0H_1, x \neq 1$, and $w \in W$. Thus $wxw^{-1} \notin A$, and so $Axw^{-1} \neq A$, yielding $Axw \neq A$. Thus $f(w) \neq \Id_{\Omega}$ for any $w \in W$, as required. Conversely, assume that for all $x \in H_0H_1$, if $x \neq 1$ then $f(x)$ fixes no elements of $\Omega$. Now let $y \in H_0H_1H_1w^{-1} \quad \Omega$, for some (arbitrary) $w \in W$. Then $w^{-1}wy \in H_0H_1H_1$. Now consider the effect of $f(w^{-1}wy)$ on the coset $Aw$. We have $Aw(w^{-1}wy) = \ldots$
$Ayw = Aw$, so $f(w^{-1}yw)$ fixes $Aw$, an element of $\Omega$. Thus it must be that $w^{-1}yw = 1$, so $y = 1$. Thus $wH_0H_\ell w^{-1} \cap A = \{1\}$ for all $w \in W$, as required.

If, further, $ker f$ is known to be sparse (and certainly if $ker f$ is not sparse, then $A$ can not be either), then the lemma may be re-stated.

**Lemma 6.3.13:** If $ker f$ is sparse, then $A$ is sparse if and only if for all $y \in f(H_0)/f(H_\ell - 1)$, $y \neq 1$, the permutation $y$ fixes no elements of $\Omega$.

**Proof:** It suffices to show that $f(H_0H_\ell - 1) = f(H_0)/f(H_\ell - 1)$, and the result follows from Lemma 6.3.12. Now $y \in f(H_0H_\ell - 1)$ if and only if there exists $z \in H_0H_\ell - 1$ such that $f(z) = y$, which is the case if and only if there exists $h \in H_0$ and $k \in H_\ell - 1$ such that $y = f(hk) = f(h)f(k)$. This will be the case if and only if $y \in f(H_0)(f(H_\ell - 1))$.

This form is the more useful one when using CAYLEY to examine finite index subgroups $A$ of $W$ where $W$ is an infinite Coxeter group, since if the program were asked to compute $H_0H_\ell - 1$, it would attempt to calculate all the elements of $W$ first (which is difficult on computers with only finite amount of memory). This problem does not occur when calculating $f(H_0)/f(H_\ell - 1)$. Thus we have an implementable algorithm for checking whether or not a particular subgroup $A$ of $W$ is sparse or not – we first find the homomorphism $f$ from $W$ onto $Sym(\Omega)$, where $\Omega$ is the set of (right) cosets of $A$ in $W$. Then check (via Lemma 6.3.11) whether $ker f$ is sparse. If it is, we check that no nontrivial element of $f(H_0)(f(H_\ell - 1))$ has fixed points.

The next lemma greatly restricts the domain one has to search to find examples of the subgroups in question.

**Lemma 6.3.14:** Let $W = W_{\{p_1, p_2, \ldots, p_\ell\}}$. If $A \subseteq W$ is sparse, then the index $[W : A]$ of $A$ in $W$ is at least $\frac{|W_{\{p_1, p_2, \ldots, p_\ell\}}|}{|W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}|}$, and is a multiple of both $|W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}|$ and $|W_{\{p_1, p_2, \ldots, p_{\ell - 2}\}}|$.

**Proof:** Let $\mathcal{P} = \mathcal{A}/A$, where $\mathcal{A}$ is the universal polytope corresponding to $W$. We know $\mathcal{P}$ will be a well-defined polytope, because $A$ is sparse (Theorem 5.2.36). Let $z \in \mathcal{P}_{\ell - 1}$. Now by Theorem 5.2.7, $(\{z\}, \mathcal{P})$ will be the universal polytope corresponding to the group $H_{\ell - 1} = W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}$ so by Theorem 5.2.10, $(\{z\}, \mathcal{P}) | \mathcal{P}_{\ell - 1} = [H_{\ell - 1} : H_{\ell - 1} = W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}]$.

On the other hand, Theorem 5.2.41 tells us that $\mathcal{P}_0 = \frac{|W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}|}{|W_{\{p_1, p_2, \ldots, p_{\ell - 2}\}}|}$. But certainly $\mathcal{P}_0 \geq [(\{z\}, \mathcal{P}) | \mathcal{P}_{\ell - 1}]$, so $|W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}| \geq \frac{|W_{\{p_1, p_2, \ldots, p_{\ell - 2}\}}|}{|W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}|}$, which when re-arranged yields the required inequality. For the second half of the result, consider $\mathcal{P}_0$.

In the case we are considering, $W = W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}$, this tells us that $[W : A] \geq \frac{|W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}|}{|W_{\{p_1, p_2, \ldots, p_{\ell - 2}\}}|}$, which is a multiple of $\text{LCM}(|W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}|, |W_{\{p_1, p_2, \ldots, p_{\ell - 2}\}}|)$. In the particular cases we are concerned with, this yields the results of Table 6.3.6. For interest, some of the previously examined Schl"afli Symbols are also included in this table.

### Table 6.3.6: Some Restrictions on Sparse Subgroups $A$ of Certain Coxeter Groups $W$.

| Case | $|p_\ell|/|p_\ell|/|p_\ell|$ | $|W_{\{p_1, p_2, \ldots, p_{\ell - 1}\}}|$ | $|W_{\{p_1, p_2, \ldots, p_{\ell - 2}\}}|$ | $|W_{\{p_1, p_2, \ldots, p_1, p_2, \ldots, p_\ell\}|$ | Hence? |
|------|------------------|------------------|------------------|------------------|---------|
| VIII | \(4/3/5\) | 48 | 120 | 6 | $W : A = 240k$, where $k \geq 4$ |
| X    | \(3/2/3\) | 120 | 120 | 10 | $W : A = 120k$, where $k \geq 12$ |
| XI   | \(5/3/5\) | 120 | 120 | 6 | $W : A = 120k$, where $k \geq 20$ |
| I    | \(3/2/3\) | 24 | 24 | 6 | $W : A = 24k$, where $k \geq 4$ |
| II   | \(3/2/4\) | 24 | 48 | 6 | $W : A = 48k$, where $k \geq 4$ |
| IV   | \(3/3/4\) | 48 | 48 | 8 | $W : A = 48k$, where $k \geq 6$ |
| V    | \(3/2/3\) | 24 | 120 | 6 | $W : A = 120k$, where $k \geq 4$ |
| VII  | \(4/2/4\) | 48 | 48 | 6 | $W : A = 48k$, where $k \geq 8$ |

The above results are extremely useful for recognizing whether or not a particular subgroup of a Coxeter group is sparse and yields a combinatorially regular sub-Euler 4-incidence polytope. Unfortunately, for the cases that needed to be considered, CAYLEY's low index subgroups and related commands were not fast enough on the available computer architecture, to yield any significant results in a reasonable time.

#### 6.3.4 The 5-polytopes:

Now since many of the 4-polytopes are quotients but not universal polytopes, we cannot, immediately, apply the results of §5.2 and §5.3 to the 5-polytopes. However, since 5 is odd, Euler's formula becomes $|\mathcal{P}_0| + \cdot |\mathcal{P}_0| = |\mathcal{P}_0| + |\mathcal{P}_0| = 2$, and so there is the promise that once we have used Lemma 3.4.15 to find ratios between the $|\mathcal{P}_0|$, we will be able (as with the 3-polytopes) to restrict the possible values of $|\mathcal{P}_0|$. It will turn out, in many cases, that when we do this, we totally eliminate a particular Schl"afli Symbol as admissible. However,
it will be a lot of work, mainly due to the fact that there are a large number of Schläfli Symbols to check. We need to check all possible \( \{p|q|r|s\} \) where \( \{p|q|r\} \) and \( \{q|r|s\} \) appear in the statement of Theorem 6.3.5.

**Theorem 6.3.15:** If \( \mathcal{P} \) is an indecomposable combinatorially regular sub-Euler 5-incidence polytope, either its Schläfli Symbol, or its dual's Schläfli Symbol, is one of \( \{3|3|3|3\}, \{3|3|3|4\}, \{3|3|3|5\}, \{3|3|4|3\}, \{3|3|5|3\}, \{3|4|3|4\}, \{3|4|3|5\}, \{3|5|3|4\}, \{3|5|3|5\}, \{4|3|3|4\}, \{4|3|3|5\} \) or \( \{5|3|5\} \).

**Proof:** If \( \mathcal{P} \) has Schläfli Symbol \( \{p|q|r|s\} \), its facets and vertex figures have Schläfli Symbols \( \{p|q|r\} \) and \( \{q|r|s\} \) respectively. Since these facets and vertex figures are in fact indecomposable combinatorially regular Euler 4-polytopes, it follows that \( \{p|q|r\} \) and \( \{q|r|s\} \) must both be listed in the statement of Theorem 6.3.5. This leads to 21 possibilities for \( \{p|q|r|s\} \). For all of these possibilities, either \( \{p|q|r|s\} \) or \( \{s|p|r|q\} \) is listed above.

Let \( \mathcal{P} \) be a combinatorially regular Euler 5-polytope, with Schläfli Symbol \( \{p|q|s|t\} \). Let \( a_1, \ldots, a_n \in \mathcal{P} \) be such that \( a_i \in \mathcal{P} \), for each \( i \), and let \( |\mathcal{P}| = S \). Now \( \langle a_0, P \rangle \) and \( \langle a_0, q \rangle \) are both Euler 4-polytopes. Let \( |\langle a_0, P \rangle \cap \mathcal{P}_k| = Q \) and \( |\langle a_0, q \rangle \cap \mathcal{P}_k| = R \).

**Lemma 6.3.16:** In this case, Euler's Formula becomes

\[
QS \left[ \frac{1}{Q} + \frac{1}{R} + \frac{2}{ps} \frac{\zeta_{rs} + \zeta_{st}}{2\zeta_{er}} \right] = 2
\]

where \( \zeta_{rs} = \frac{1}{Q} + \frac{1}{R} - \frac{1}{Q} \).

**Proof:** In this proof, Lemma 3.4.15 will be worked overtime. If \( |\mathcal{P}| = S \), then \( |\mathcal{P}_k| = |\langle a_0, P \rangle \cap \mathcal{P}_k| |\mathcal{P}_0| = \frac{Q}{R} S \). Now \( \langle a_0, a_k \rangle \) is a combinatorially regular Euler 4-polytope, thus for \( i \neq 0 \), \( |\langle a_0, a_k \rangle \cap \mathcal{P}_n| = |\langle a_0, a_k \rangle \cap \mathcal{P}_k| |\langle a_0, a_k \rangle \cap \mathcal{P}_n| = |\langle a_0, a_k \rangle \cap \mathcal{P}_n| \). But \( \langle a_0, a_k \rangle \) will be a 3-polytope with Schläfli Symbol \( \{q|r|s\} \), and so (by Theorem 6.3.2) will satisfy

\[
|\langle a_0, a_k \rangle \cap \mathcal{P}_n| = \frac{2}{\zeta_{er}},
\]

\[
|\langle a_0, a_k \rangle \cap \mathcal{P}_n| = \frac{1}{\zeta_{er}}, \text{ and}
\]

\[
|\langle a_0, a_k \rangle \cap \mathcal{P}_n| = \frac{2}{\zeta_{er}}.
\]

Also, \( |\langle 0, a_k \rangle \cap \mathcal{P}_n| = \frac{2}{\zeta_{er}} \) for similar reasons. Note also that \( |\langle 0, a_k \rangle \cap \mathcal{P}_n| = p \) and \( |\langle 0, a_0 \rangle \cap \mathcal{P}_n| = 2 \). Thus, we deduce that

\[
|\langle 0, a_0 \rangle \cap \mathcal{P}_n| = \frac{2}{\zeta_{er}},
\]

\[
|\langle 0, a_0 \rangle \cap \mathcal{P}_n| = \frac{1}{\zeta_{er}}, \text{ and}
\]

\[
|\langle 0, a_0 \rangle \cap \mathcal{P}_n| = \frac{2}{\zeta_{er}}.
\]

Note also that \( |\langle a_1, P \rangle \cap \mathcal{P}_n| = \frac{2}{\zeta_{er}}, |\langle a_1, P \rangle \cap \mathcal{P}_n| = s, \) and \( |\langle a_1, P \rangle \cap \mathcal{P}_n| = 2 \).

The above information is useful because it can be used to calculate the ratios between the various \( |\mathcal{P}_n| \). We already know that \( |\mathcal{P}_k| = S \) and that \( |\mathcal{P}_n| = \frac{Q}{R} S \). From this, we deduce

\[
|\mathcal{P}_n| = \frac{\frac{1}{Q} + \frac{1}{R} + \frac{2}{ps} \frac{\zeta_{rs} + \zeta_{st}}{2\zeta_{er}}}{2} = 2
\]

Thus Euler's formula, \( |\mathcal{P}_0| - |\mathcal{P}_n| + |\mathcal{P}_n| - |\mathcal{P}_k| = 2 \), becomes

\[
S = \frac{SQ \zeta_{rs} + SQ \zeta_{st}}{2\zeta_{er}} + \frac{SQ \zeta_{rs} + SQ \zeta_{st}}{2\zeta_{er}} = 2.
\]

Taking out a common factor of \( SQ \) and rearranging the terms yields

\[
S \left[ \frac{1}{Q} + \frac{1}{R} + \frac{1}{\zeta_{er}} \frac{1}{ps} \frac{\zeta_{rs} + \zeta_{st}}{2\zeta_{er}} \right] = 2,
\]

As required.

**Theorem 6.3.17:** Up to isomorphism, there are only three indecomposable combinatorially regular Euler 5-incidence polytopes.

**Proof:** Note that Euler's Formula may be written \( SQ \left[ \frac{1}{Q} + \frac{1}{R} + \frac{1}{\zeta_{er}} \frac{1}{ps} \frac{\zeta_{rs} + \zeta_{st}}{2\zeta_{er}} \right] \) depends only on \( p, q, r, s \), and \( s \), and not at all on \( Q, R, \) and \( S \). We examine below (in Table 6.3.7) the values of \( s \) for all the Schläfli Symbols \( \{p|q|r|s\} \) listed in Theorem 6.3.15. We do not need to consider the reversals \( \{s|r|p|q\} \) of these, as any polytope with Schläfli Symbol \( \{s|r|p|q\} \) will be dual to one with Schläfli Symbol \( \{p|q|r|s\} \). Note that \( \zeta_{33} = \frac{1}{2}, \zeta_{34} = \zeta_{43} = \frac{1}{2}, \) and \( \zeta_{53} = \zeta_{54} = \frac{3}{2} \).
Table 6.3.7: Analysing Euler’s Condition for 5-Polytopes.

| (p|q|r|s) | \a \ | \b \ | \c \ | \d \ |
|---|---|---|---|
| \(\{3\}{\{3\}}{\{5\}}\) | 6 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |
| \(\{3\}{\{3\}}{\{4\}}\) | 6 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |
| \(\{3\}{\{3\}}{\{5\}}\) | 6 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |
| \(\{3\}{\{3\}}{\{3\}}\) | 12 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |
| \(\{3\}{\{3\}}{\{5\}}\) | 30 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |
| \(\{4\}{\{3\}}{\{4\}}\) | 12 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |
| \(\{4\}{\{3\}}{\{5\}}\) | 12 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |
| \(\{4\}{\{3\}}{\{4\}}\) | 30 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |
| \(\{4\}{\{3\}}{\{5\}}\) | 30 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |
| \(\{4\}{\{3\}}{\{4\}}\) | 6 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |
| \(\{4\}{\{3\}}{\{5\}}\) | 6 \(\frac{5}{15}\) | \(\frac{1}{4}\) | \(\frac{5}{6}\) | \(\frac{5}{6}\) |

Note that if \(\alpha \geq 0\), then \(\text{SQ}[\frac{5}{6}, \frac{3}{4}, \alpha, \gamma] \geq \text{SQ}[\frac{5}{6}, \frac{3}{4}, \gamma] \geq \text{SQ}[\frac{5}{6}, \frac{3}{4}] = Q+R > 2\). Thus there can be no regular combinatorially 5-polytopes with Schl"{a}fli Symbols yielding \(\alpha \geq 0\). Also, the cases \(\{3\}{\{3\}}{\{5\}}, \{3\}{\{3\}}{\{4\}}\) (and \(\{4\}{\{3\}}{\{5\}}\)) were considered in 6.1 to 6.2, yielding exactly one example each for \(\{3\}{\{3\}}{\{5\}}\) and \(\{3\}{\{3\}}{\{4\}}\) (and \(\{4\}{\{3\}}{\{5\}}\)), and (since 5 is odd) no examples at all of 5-polytopes with Schl"{a}fli Symbol \(\{4\}{\{3\}}{\{5\}}\) (Corollary 6.1.2, Theorems 6.1.8 and 6.1.9 and Lemma 6.2.3). Thus, there are only three further cases to investigate, which are listed in Table 6.3.8, below.

Table 6.3.8: Further Analysis of Euler’s Condition.

<table>
<thead>
<tr>
<th>Case</th>
<th></th>
<th>Euler’s Formula</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>({3}{{3}}{{5}})</td>
<td>(\text{SQ}[\frac{5}{6}, \frac{3}{4}, \gamma] = 2)</td>
<td>(S \geq R = 5, Q \in X)</td>
</tr>
<tr>
<td>II</td>
<td>({3}{{3}}{{4}})</td>
<td>(\text{SQ}[\frac{5}{6}, \frac{3}{4}, \gamma] = 2)</td>
<td>(S \geq R = 4) or (Q \in {8, 12, 24})</td>
</tr>
<tr>
<td>III</td>
<td>({4}{{3}}{{5}})</td>
<td>(\text{SQ}[\frac{5}{6}, \frac{3}{4}, \gamma] = 2)</td>
<td>(S \geq R = 8) or (Q \in X)</td>
</tr>
</tbody>
</table>

In this table, the set \(X\) equals \(\{25, 50, 75, 100, 120, 150, 200, 300, 600\}\). The restrictions given are gleaned from the previous subsection (see in particular Table 6.3.2, 6.3.3 and 6.3.4, along with row V of 6.3.1). Now \(\alpha < 0\) in each of these cases. If \(R \leq -\frac{5}{6}\), then \(\frac{5}{6} \geq -\alpha\), so \(\frac{5}{6} \geq 0\). But if this is so, then \(2 = \text{SQ}[\frac{5}{6}, \frac{3}{4}, \alpha] \geq \text{SQ}[\frac{5}{6}, \frac{3}{4}] = S\). Since none of our cases allow \(S = 2\), this eliminates I and III, and in case II, forces \(R\) to equal 8. That is, our polytope must have Schl"{a}fli Symbol \(\{3\}{\{4\}}{\{5\}}\), and have vertices, faces, not halfcrosses. But in this case, Euler’s formula becomes \(\text{SQ}[\frac{5}{6}, \frac{3}{4}, \gamma] = 2\), that is, \(S(1 - \frac{2}{5}) = 2\). Now the only possible values for \(Q\) are 8, 12 and 24, since \(Q\) is the number of vertices of a polytope with Schl"{a}fli Symbol \(\{3\}{\{4\}}{\{5\}}\) (see Table 6.3.3). If \(Q = 24\), then \(S(1 - \frac{2}{5}) = 8\), not 2. On the other hand, if \(Q = 12\), then \(1 - \frac{2}{5} \geq 1\). Since also \(S \geq R = 8\), we have \(S(1 - \frac{2}{5}) \geq 8\), so Euler’s formula cannot be satisfied in this case either, and case II is eliminated.

It remains, therefore, that there are only three indecomposable combinatorially regular Euler 5-polytopes up to isomorphism. They are the 5-simplex, with Schl"{a}fli Symbol \(\{3\}{\{3\}}{\{3\}}\), the 5-cross, with Schl"{a}fli Symbol \(\{3\}{\{3\}}{\{4\}}\), and its dual, the 5-cube, with Schl"{a}fli Symbol \(\{4\}{\{3\}}{\{3\}}\).

Note that all three of these polytopes are universal (Theorems 6.1.3, 6.1.5 and 6.1.11). Admittedly, there will be many non-Euler indecomposable combinatorially regular 5-polytopes, and an attempt to analyse even the sub-Euler polytopes would undoubtedly be very interesting and challenging, and would uncover numerous very nice examples.

6.3.5 From Then On:

We saw in the last section that there were only three indecomposable combinatorially regular Euler 5-incidence polytopes, those with Schl"{a}fli Symbols \(\{3\}{\{3\}}{\{3\}}\), \(\{3\}{\{3\}}{\{4\}}\), and \(\{4\}{\{3\}}{\{3\}}\). This result allows us to prove the following result.

Lemma 6.3.18: If \(d \geq 5\), the only possible Schl"{a}fli Symbols for an indecomposable combinatorially regular Euler \(d\)-incidence polytope are \(\{3\}{\{3\}}\), \(\{3\}{\{3\}}{\{4\}}\), \(\{4\}{\{3\}}{\{3\}}\), and if and only if \(d\) is even, \(\{4\}{\{3\}}{\{3\}}\).

Proof: We do the proof by induction on \(d\). We already know that the theorem is true for \(d = 5\). Now assume it is true for \(d = k - 1\) (\(k \geq 6\)). Then, if \(\{p_1, \ldots, p_{k-1}\}\) is a Schl"{a}fli Symbol for a regular combinatorial \(k\)-polytope, it must be the case that both \(\{p_1, \ldots, p_{k-2}\}\) and \(\{p_1, \ldots, p_{k-1}\}\) have the form \(\{3\}{\{3\}}\), \(\{3\}{\{3\}}{\{4\}}\), \(\{4\}{\{3\}}{\{3\}}\), or possibly \(\{4\}{\{3\}}{\{3\}}\). Note that if \(k \geq 6\) is any one of these, then \(r_2 = \ldots = r_{k-3} = 3\). This tells us, in particular, that \(r_2 = \ldots = r_{k-3} = 3\), and also that \(p_3 = \ldots = p_{k-3} = 3\).
We can similarly deduce that $p_1$ is either 3 or 4, and that $p_{k-1}$ is either 3 or 4, whence we deduce that \( \{p_1 \ldots p_{k-1}\} \) is one of $\{3\} \ldots \{3\}$, $\{3\} \ldots \{3\}$, $\{4\} \ldots \{3\}$, or $\{4\} \ldots \{3\}$. However, Lemma 6.2.3 tell us that it cannot have the form $\{4\} \ldots \{3\}$ unless $k$ is, in fact, even. Using the principle of mathematical induction completes the proof.

Using the results of §6.1 and §6.2 yield the following.

**Corollary 6.3.19**: For $d \geq 5$ the only indecomposable combinatorially regular Euler $d$-incidence polytopes are the Simplices, the Cubes, the Crosses, and (if $d$ is even) the Halfcubes, the Halffcrosses, and quotients of the universal lattice.

**Proof**: See Corollary 6.1.2, Theorems 6.1.8 and 6.1.9, and the notes following Lemma 6.2.3.

Some information about these is summarised in Table 6.3.9. Much of it also appears in Tables A.6 and A.7.

### Table 6.3.9: Combinatorially Regular Euler $d$-Polytopes for $d \geq 5$

<table>
<thead>
<tr>
<th>Name</th>
<th>Schl&quot;afli Symbol</th>
<th>Exists</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$d$-Simplex</td>
<td>${3} \ldots {3}$</td>
<td>for all $d$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d-1 \geq 3$</td>
<td></td>
</tr>
<tr>
<td>II.i</td>
<td>$d$-Cross</td>
<td>${3} \ldots {3} {4}$</td>
<td>for all $d$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d-2 \geq 3$</td>
<td></td>
</tr>
<tr>
<td>II.ii</td>
<td>$d$-Halffcross</td>
<td>${3} \ldots {3}$</td>
<td>if $d$ even</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d-2 \geq 3$</td>
<td></td>
</tr>
<tr>
<td>III.i</td>
<td>$d$-Cube</td>
<td>${4} \ldots {3}$</td>
<td>for all $d$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d-2 \geq 3$</td>
<td></td>
</tr>
<tr>
<td>III.ii</td>
<td>$d$-Halffcube</td>
<td>${3} \ldots {3}$</td>
<td>if $d$ even</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d-2 \geq 3$</td>
<td></td>
</tr>
<tr>
<td>IV*</td>
<td>$d$-Polypolytope</td>
<td>${4} \ldots {3}$</td>
<td>if $d$ even</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d-2 \geq 3$</td>
<td></td>
</tr>
</tbody>
</table>

*Actually an infinite family of regular polytopes.*

### 7.1 Summary

We have seen, in the previous chapter and elsewhere, a large number of classification theorems about combinatorially regular Euler polytopes. In this section, it will be shown that these polytopes have a particularly neat characterisation. A table (Table 7.1.1) is included at the end of the chapter: this table gives an overview of all indecomposable combinatorially regular Euler $d$-incidence polytopes for all the various values of $d$ and the Schl"afli Symbols that arise. The table shows, for each case, the number of examples which occur, whether these examples are flag regular (or how many are, if not all of them), and whether it has been possible in this thesis to find a neat classification in each particular case. The reader is also given references to places in the thesis where each case is examined. Alternatively, one may wish to refer directly to the many tables in the thesis which give more details for each particular case. The $d$-polytopes with $d \geq 4$ are covered by Tables 6.3.1 to 6.3.6, and 6.3.9. The 3-polytopes, all being combinatorial counterparts (Theorem 6.3.4) are adequately covered by Tables 2.5.1 and 2.5.2, and for the 1- and 2-polytopes, the reader is referred to §4.5.1 and §4.5.2.

A number of interesting points may be noted from the classification. Throughout this section, let $W$ be a Coxeter group, whose graph is a path, and whose parabolic subgroups are finite, and let $\mathcal{A}$ be the universal polytope based on $W$.

Perhaps the most surprising point to note is the following.
Theorem 7.1.1: Any indecomposable combinatorially regular Euler d-incidence polytope is a quotient of a universal polytope by a finite index sparse subgroup of an appropriate Coxeter group.

Proof: Let $P$ be such a $d$-polytope. If $d = 2$ or $d$ is odd, then $P$ is in fact universal. This follows, in the case $d = 1$, from the fact that there exists a universal 1-polytope (based on the Coxeter group $(a_2) = (1, a_2)$), and that all 1-polytopes are isomorphic (see the notes following Theorem 3.3.9). For $d = 2$, this was noted at the end of §6.3.2 and may be proven from Theorems 3.3.10, 5.2.9, 5.2.10 and Corollary 5.2.12). For $d = 3$ it is given by Theorem 6.3.3, and for odd $d \geq 5$, it follows from Corollary 6.3.19, when one notes that simplices, cubes and crosses are all universal (Theorems 6.1.1 and 6.1.3 for the simplices, 6.1.5 for the cubes, and 6.1.11 for the crosses). Since $P$ is universal, it is, by Lemma 5.2.34, a quotient by the trivial group $\{1\}$, which is sparse.

Now if $d \geq 4$ is even, the facets and vertex figures of $P$ are indecomposable combinatorially regular Euler $(d - 1)$-polytopes (they are polytopes by Theorem 3.1.4. Euler since sections of sections of $P$ are themselves sections of $P$, combinatorially regular by Theorem 3.4.6, and indecomposable by Theorems 3.4.14 and 3.4.12). This being so, they are universal, since $d - 1$ is odd. Theorem 5.3.4 then tells us that $P$ is a quotient by a sparse subgroup of the corresponding Coxeter group.

Note that the theorem says nothing directly about sub-Euler polytopes, nor about decomposable polytopes. In fact, the next theorem demonstrates the existence of decomposable polytopes which are not quotient polytopes by sparse subgroups.

Theorem 7.1.2: There exist decomposable combinatorially regular Euler incidence polytopes which are not quotient polytopes by sparse subgroups.

Proof: Let $\mathcal{K}$ be the 4-halfcube and $\mathcal{P}$ be the 1-polytope of Theorem 3.2.2. Both $\mathcal{K}$ and $\mathcal{P}$ are combinatorially regular Euler polytopes (see Theorem 4.2.14 for the halfcube, and note that the 1-polytope is almost trivially combinatorially regular). It follows, from Theorems 3.3.29, 3.3.32 and 3.4.8, that the composition $\mathcal{K} \circ \mathcal{P}$ is a combinatorially regular Euler incidence polytope. However, its facets are not universal, being isomorphic to $\mathcal{K}$, hence by Theorem 5.3.4 it is not a quotient by a sparse subgroup of the corresponding Coxeter group.

Let us return now, to examining the indecomposable polytopes. We have found that all indecomposable combinatorially regular Euler polytopes may be written in the form $\mathcal{K}/A$ for some sparse subgroup $A$. Since Theorem 5.2.42 tells us exactly when such a quotient is Euler, we obtain Theorems 7.1.3 and 7.1.4, which may be regarded as a classification of the polytopes we have examined. They are easier to state if we make the following definition.

Definition: A $d$-CS pair\(^{68}\) is an ordered pair $((W, S), C)$, where $W$ is a Coxeter system with $d$ generators and finite parabolic subgroups, whose graph is a path, and $C$ is a conjugacy class of finite index sparse subgroups of $W$.

Theorem 7.1.1 tells us in fact that every indecomposable combinatorially regular Euler incidence polytope corresponds in a natural way to a $d$-CS pair. Let $\Xi$ be the map from the set of such polytopes to the set of $d$-CS pairs.

Theorem 7.1.3: If $d$ is even, there exists a one to one correspondence between isomorphism classes of indecomposable combinatorially regular Euler $d$-incidence polytopes and $d$-CS pairs.

Proof: We show that $\Xi$ is a one to one correspondence. Certainly, it is one to one, for if two polytopes yields the same $d$-CS pair $(W, S, C)$, then both will be isomorphic to $\mathcal{K}/A$, where $A$ is a representative of $C$, and $\mathcal{K}$ is the universal polytope based on $W$. On the other hand, it is onto, for if $(W, S, C)$ is a $d$-CS pair, then since $d$ is even, Theorem 5.2.42 tells us that $\Xi = \mathcal{K}/A$ is Euler, and satisfies $(\Xi) = ((W, S), C)$.

Theorem 7.1.4: If $d$ is odd, there is a one to one correspondence between isomorphism classes of indecomposable combinatorially regular Euler $d$-incidence polytopes and finite Coxeter systems (with $d$ generators) whose graph is a path.

Proof: Let $P$ be a combinatorially regular Euler $d$-incidence polytope, where $d$ is odd. Theorem 7.1.1 tells us that there is a unique $d$-CS pair $((W, S), C)$ corresponding to $P$. However, Theorem 5.2.42 informs us that there is only one choice for $C$, namely the conjugacy class of the trivial subgroup of $W$, so any such polytope defines a unique Coxeter system $(W, S)$ whose graph is a path. Conversely, any such Coxeter system with an odd number of generators will define a unique such polytope, the universal polytope based on $W$.

Finally, we can complete the classification as follows.

Theorem 7.1.5: There is a one to one correspondence between combinatorially regular Euler incidence polytopes, and finite sequences of indecomposable combinatorially regular Euler incidence polytopes.

Proof: Theorem 3.3.38 tells us that any incidence polytope $P$ may be written $P \equiv P^{(1)} \circ P^{(2)} \circ \ldots \circ P^{(k)}$ for some unique sequence $P^{(1)}, P^{(2)}, \ldots, P^{(k)}$ of indecomposable incidence polytopes. We know conversely that any such sequence yields a well defined incidence polytope. By Theorems 3.3.32 and 3.4.8, $P$ will be combinatorially regular and Euler if and only if all the $P^{(i)}$ are.

---

\(^{68}\) From “Coxeter-Sparse”.

---

Section 7.1: Summary
7.2 What Follows

The three theorems above, taken together, may be regarded as a classification theorem for combinatorially regular Euler incidence polytopes, so in a sense, we have succeeded in our original goals. The reason some of the rows of Table 7.1.1 are marked as not being completely classified is that the above results do not give a terribly explicit description of the structures they refer to — classifying the sparse subgroups of a Coxeter group is by no means a trivial task in general, although it is undoubtedly a simpler task than that of constructing combinatorial polytopes from scratch (compare, for example, the proofs of Theorems 6.1.1 and 6.1.3). Note in fact that for Schläfli Symbols \( \{3\{3\} \}, \{4\{3\} \}, \{5\{3\} \} \) and \( \{3\{5\} \} \), we have not demonstrated the existence of a single polytope with these Schläfli Symbols that is combinatorially regular but not flag regular, however reasonable it might be to conjecture their existence. The other cases where we have only partial results is that of the polytopes with Schläfli Symbol \( \{4\{3\} \} \). The Coxeter groups corresponding to these cases are almost begging for deeper study.

Another direction that might yield fruit is the following. Recall that in Theorem 6.2.38, a result from [9] was quoted, namely that for a subgroup \( A \) (not necessarily sparse) of the Coxeter group \( W \), the quotient \( W/A \) is a polytope if and only if \( A \) is 'semisparse'. If we were to analyse more deeply the class of semisparse subgroups, we may well be able to relax our Eulerian condition, and still make reasonable headway into the study of combinatorially regular polytopes. A good starting point would be to try to determine which semisparse subgroups yield combinatorially regular polytopes, by examining which aspects of the structure of \( A \) correspond to interesting parts (such as the sections, and so forth) of the polytope's structure.

We have stolen a glimpse into the world of combinatorially regular polytopes, and our Eulerian condition has served as a wedge to open the crack through which we peeked. In exploring this world, there would be multitudes of avenues down which to venture — some obvious, some less so. This, however, would always be true, no matter how far the work was taken — and so we close, for now, with what we have.

<table>
<thead>
<tr>
<th>( d )</th>
<th>S.Sym.</th>
<th>H.M. (^{69})</th>
<th>P.R. (^{70})</th>
<th>C'fed?</th>
<th>For more details see:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{}</td>
<td>1</td>
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<td>Yes</td>
<td>Th 3.3.9, §6.3.1.</td>
</tr>
<tr>
<td>2</td>
<td>( {4, k \geq 3 } ) (^{71})</td>
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<td>All</td>
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</tr>
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<td>Yes</td>
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</tr>
<tr>
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<td>Yes</td>
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<tr>
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<td>Yes</td>
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<td>Yes</td>
<td>Co 6.1.2, §6.3.3, Tb 6.3.2.</td>
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</tr>
<tr>
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<td>Th 6.1.9, §6.3.3, Tb 6.3.2.</td>
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<tr>
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<td>Yes</td>
<td>§6.3.3, Tb 6.3.3.</td>
</tr>
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<td>Yes</td>
<td>Yes</td>
<td>§6.3.3, Tb 6.3.4.</td>
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<td>Yes</td>
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<td>No</td>
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<td>No</td>
<td>No</td>
<td>§6.3.3, Tb 3.4.13</td>
</tr>
<tr>
<td></td>
<td>( {4{5} )</td>
<td>2</td>
<td>No</td>
<td>No</td>
<td>§6.3.3, Tb 6.3.5</td>
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<td>2</td>
<td>No</td>
<td>No</td>
<td>§6.3.3, Tb 6.3.5</td>
</tr>
<tr>
<td></td>
<td>( {4{5} )</td>
<td>2</td>
<td>No</td>
<td>No</td>
<td>§6.3.3, Tb 6.3.5</td>
</tr>
</tbody>
</table>

\( \geq 5 \), \( \{3\ldots \{3\} \) | 1          | Yes       | Yes    | §6.1.1, Co 6.3.19.      |

odd \( \{4\ldots \{3\} \) | 1          | Yes       | Yes    | §6.1.2, Co 6.3.19.      |

\( \{3\ldots \{3\} \) | 1          | Yes       | Yes    | §6.1.3, Co 6.3.19.      |

\( \{3\ldots \{3\} \) | 1          | Yes       | Yes    | §6.1.1, Co 6.3.19.      |

even \( \{4\ldots \{3\} \) | 2          | Both      | Yes    | §6.1.2, Co 6.3.19.      |

\( \{3\ldots \{3\} \) | 2          | Both      | Yes    | §6.1.3, Co 6.3.19.      |

\( \{4\ldots \{3\} \) | 2          | Yes       | Yes    | §6.2, Co 6.3.19.        |

Any. Other. | 0          | 0         | Yes    | Th 6.3.1, 6.3.5, Lm 6.3.18.

\(^{69}\) This column shows how many polytopes exist of each Schläfli Symbol.

\(^{70}\) This column shows how many are flag regular.

\(^{71}\) One for each value of \( k \), that is.
APPENDIX A

Information on Selected Coxeter Groups

This appendix lists the Coxeter groups which are particularly useful in this thesis, that is, those groups \( W = W_{(p_1 \ldots p_{n-1})} \) satisfying the conditions C1 and C2 which were given in §5.1.2, and which are repeated below.

(C1) The Coxeter graph of \( W \) is a path.

(C2) If either of the endpoints of this graph are removed, the resulting graph describes a finite group.

Table A.1, adapted from [5, VI Thm 4.1.1] (or see [26, Fig 2.1] or [14, Table 10]) lists the finite Coxeter groups satisfying C1 (and C2). The condition C1 excludes certain finite Coxeter groups that appear in other tables, specifically \( B_4 \), \( E_6 \), \( E_7 \) and \( E_8 \). Note also that \( G_2 = I_2(6) \).

<table>
<thead>
<tr>
<th>Group</th>
<th>S.Sym.</th>
<th>Coxeter Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_4 )</td>
<td>{3}</td>
<td><img src="image1" alt="Graph" /></td>
</tr>
<tr>
<td>( B_4 = C_4 )</td>
<td>{3}[3][4]</td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td>( B_4 = C_4 )</td>
<td>{4}[3][3]</td>
<td><img src="image3" alt="Graph" /></td>
</tr>
<tr>
<td>( F_4 )</td>
<td>{3}[4][3]</td>
<td><img src="image4" alt="Graph" /></td>
</tr>
<tr>
<td>( H_4 )</td>
<td>{3}[3][5]</td>
<td><img src="image5" alt="Graph" /></td>
</tr>
<tr>
<td>( H_4 )</td>
<td>{5}[3][3]</td>
<td><img src="image6" alt="Graph" /></td>
</tr>
<tr>
<td>( H_3 )</td>
<td>{3}[3]</td>
<td><img src="image7" alt="Graph" /></td>
</tr>
<tr>
<td>( H_3 )</td>
<td>{5}[3]</td>
<td><img src="image8" alt="Graph" /></td>
</tr>
<tr>
<td>( I_2(n) )</td>
<td>{n}</td>
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</tr>
<tr>
<td>( I_2(\infty) )</td>
<td>{}</td>
<td><img src="image10" alt="Graph" /></td>
</tr>
</tbody>
</table>

In this table, certain groups (such as \( B_4 \)) have been listed more than once. This is because it is useful to distinguish between (say) \( W_{(3\ldots3)[4]} \) and \( W_{(4\ldots3)[3]} \), as they actually yield two different classes of polytopes. Members of one such class will be dual to members of the other.

Given the graphs listed in Table A.1, we may construct further groups satisfying C1 and C2 as follows. Take one of the graphs from the table, and add an edge (labelled \( m \)) on to one of the ends. The new group certainly satisfies C1. To check if it satisfies C2, remove the vertex from the other end, and examine the resulting graph to see if it appears in Table A.1.

As was intimated in §5.1.2, this construction may be used as the basis for an algorithm to classify the groups satisfying C1 and C2. When this is done, it yields the additional graphs listed in Table A.2.

Table A.2: Infinite Coxeter Groups Satisfying C1 and C2.

<table>
<thead>
<tr>
<th>Group</th>
<th>S.Sym.</th>
<th>Coxeter Graph</th>
</tr>
</thead>
<tbody>
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<td>{3}[4][3]</td>
<td><img src="image14" alt="Graph" /></td>
</tr>
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<td>{3}[4][3][3]</td>
<td><img src="image15" alt="Graph" /></td>
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<td>{4}[3][5]</td>
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<td></td>
<td>Other ( {p</td>
<td>q} )</td>
</tr>
<tr>
<td>( \hat{A}_1 )</td>
<td>{\infty}</td>
<td><img src="image22" alt="Graph" /></td>
</tr>
</tbody>
</table>
The Schl"{a}fli Symbols listed in Tables A.1 and A.2 are therefore the only possible Schl"{a}fli Symbols for an indecomposable sub-Euler incidence polytope with universal facets and vertex figures (see Theorem 5.2.37 and 5.3.4, and note that if \( W \) does not satisfy C2, either the facets or the vertex figures will be infinite (see the note following Theorem 5.2.10), whence the polytope is not sub-Euler).

Let \( W = W(\rho_1,...,\rho_r) \) be one of the Coxeter groups given in Tables A.1 and A.2, let \( \mathcal{M} \) be the universal polytope based on \( W \), and let \( \mathcal{S} = \mathcal{M}/A \) be a quotient of \( \mathcal{M} \) by a sparse subgroup \( A \) of \( W \). Tables A.3 to A.6 list some information about the orders of \( W \) and of the \( \mathcal{S}_i \). This information is calculated from [14, Table 10], which gives the orders of the various finite Coxeter groups, and from Theorem 5.2.41, which tells us that \( |\mathcal{S}_i| = \frac{|W:A|}{|A|} \), which if \( W \) is finite equals \( \frac{1}{|A|} \frac{|W|}{|A|} \). It should be noted that in some cases there exist no nontrivial sparse subgroups (see for example Theorem 6.1.3) and so the information given is a little superfluous.

| S.Sym. | \( |W| \) | \( |\mathcal{S}_0| \) | \( |\mathcal{S}_1| \) | \( |\mathcal{S}_2| \) |
|--------|----------|----------|----------|----------|
| \{}    | 2        | \( |W:A| = 2 \) | \( \frac{1}{2} |W:A| = 1 \) | \( \frac{1}{2} |W:A| = 1 \) |
| \{\eta\} | 2n       | \( \frac{1}{2} |W:A| = \frac{n}{2} \) | \( \frac{1}{2} |W:A| = \frac{n}{2} \) | \( \frac{1}{2} |W:A| = \frac{n}{2} \) |
| (\infty) | \infty   | \( \frac{1}{2} |W:A| = \infty \) | \( \frac{1}{2} |W:A| = \infty \) | \( \frac{1}{2} |W:A| = \infty \) |
| (3|3)   | 24       | \( \frac{1}{2} |W:A| = 12 \) | \( \frac{1}{2} |W:A| = 6 \) | \( \frac{1}{2} |W:A| = 6 \) |
| (3|4)   | 48       | \( \frac{1}{2} |W:A| = 12 \) | \( \frac{1}{2} |W:A| = 12 \) | \( \frac{1}{2} |W:A| = 12 \) |
| (4|3)   | 48       | \( \frac{1}{2} |W:A| = 12 \) | \( \frac{1}{2} |W:A| = 12 \) | \( \frac{1}{2} |W:A| = 12 \) |
| (3|5)   | 120      | \( \frac{1}{2} |W:A| = 20 \) | \( \frac{1}{2} |W:A| = 20 \) | \( \frac{1}{2} |W:A| = 20 \) |
| (5|3)   | 120      | \( \frac{1}{2} |W:A| = 20 \) | \( \frac{1}{2} |W:A| = 20 \) | \( \frac{1}{2} |W:A| = 20 \) |
| Other \( \{p|q\} \) | \( \infty \) | \( \frac{1}{2} |W:A| = \infty \) | \( \frac{1}{2} |W:A| = \infty \) | \( \frac{1}{2} |W:A| = \infty \) |

Recall that if \( A = \{1 \} \), we have \( \mathcal{S} = \mathcal{M} \) (see Lemma 5.2.24). Information about the \( \mathcal{M} \), may therefore be obtained from these tables, alternatively, we could apply Theorem 5.2.10. The sizes of the \( \mathcal{M} \) (for those cases where \( \mathcal{M} \) is finite) has been collected into Table A.7.

Note that many of the cases listed in Tables A.3 to A.6 are not considered in this thesis. This is because the thesis aims only to classify the Euler combinatorially regular incidence polytopes, and this restriction excludes these cases from consideration (see for example the proof of Theorem 6.3.17).
Table A.6: Quotient Polytopes for \( d \geq 6 \).

| S.Sym.          | \(|W|\) | \(|\mathcal{D}_0|\) | \(|\mathcal{D}_1|\) | \(|\mathcal{D}_{d-1}|\) | \(|\mathcal{D}_d|\) |
|-----------------|--------|----------------|----------------|----------------|--------|
| \(\{3\ldots3\}\) | \((d+1)!\) | \(\frac{2d+1}{2}\) | \(\frac{(d+1)d}{2}\) | \(\frac{d^2}{2}\) | \(\frac{(d+1)}{2}\) |
| \(d = 3\ 3\'s\) |        |                |                |                |        |
| \(\{3\ldots3\ | \{4\}\) | \(2^d d!\) | \(\frac{2d^2}{2}\) | \(\frac{2d(d+1)}{2}\) | \(\frac{d^2}{2}\) | \(\frac{2d^{d+1}}{2}\) |
| \(d = 3\ 3\'s\) |        |                |                |                |        |
| \(\{4\|3\ldots3\) | \(2^{d-1} d!\) | \(\frac{2d^2}{2}\) | \(\frac{2d(d+1)}{2}\) | \(\frac{d^2}{2}\) | \(\frac{2d^{d-1}}{2}\) |
| \(d = 2\ 3\'s\) |        |                |                |                |        |
| \(\{4\ldots3\ | \{4\}\) | \(\infty\) | \(\frac{2d^{d-1}}{2}\) | \(2^{d+1} d!\) | \(\frac{d^2}{2}\) | \(\frac{2d^{d-1}}{2}\) |
| \(d = 3\ 3\'s\) |        |                |                |                |        |

Table A.7: Finite Universal Polytopes.

| S.Sym.          | \(|W|\) | \(|\mathcal{M}_0|\) | \(|\mathcal{M}_1|\) | \(|\mathcal{M}_2|\) | \(|\mathcal{M}_3|\) | \(|\mathcal{M}_4|\) |
|-----------------|--------|----------------|----------------|----------------|----------------|--------|
| \(\{\}\)        | 2      | 2              |                |                |                |        |
| \(\{n\}\)       | 2n     | n              | n              |                |                |        |
| \(\{3\|5\)       | 120    | 12             | 30             | 20             |                |        |
| \(\{5\|3\)       | 120    | 20             | 30             | 12             |                |        |
| \(\{3\|4\|3\)    | 1152   | 24             | 96             | 96             | \(2^2\)        |        |
| \(\{3\|3\|5\)    | 14400  | 120            | 720            | 1200           | 600            |        |
| \(\{5\|3\|3\)    | 14400  | 600            | 1200           | 720            | 120            |        |
| \(\{3\ldots3\ | \{4\}\) \(d = 1\) | \(d + 1\) | \(\frac{(d+1)d}{2}\) | \(\frac{(d+1)}{2}\) |                |        |
| \(d = 3\ 3\'s\) |        |                |                |                |                |        |
| \{3\ldots3\ | \{4\}\) \(d = 1\) | \(2d!\) | \(2d\) | \(2d(d+1)\) | \(2^{d+1}\) \(\frac{(d+1)}{2}\) |        |
| \(d = 3\ 3\'s\) |        |                |                |                |                |        |
| \{4\|3\ldots3\|3\) \(d = 1\) | \(2^d d!\) | \(2^d\) | \(2^{d-1}d\) | \(2^{d-1}\) \(\frac{(d+1)}{2}\) |        |
| \(d = 3\ 3\'s\) |        |                |                |                |                |        |

**APPENDIX B**

Non-Lattices

This appendix examines some results about non-lattices. The first section demonstrates a sparse subgroup of the \(d\)-Coxeter group \(W = W_{(4\ldots3\|4)}\), chosen in such a way that the quotient polytope generated is not isomorphic to a lattice. This polytope turns out to be combinatorially regular but not flag regular. The second section shows that this is not surprising, by showing that the only flag regular Euler incidence polytopes with Schl"afli Symbol \(\{3\|3\ldots3\|4\)\) are lattices. Recall the results of §4.4 and §6.2, where polytopes with this Schl"afli Symbol were examined.

In §6.2, we defined the elements \(v_i\) of \(W\) such that

\[
(\text{for } 1 \leq i \leq d - 1).
\]

Let \(V = (v_1, \ldots, v_{d-1})\) and \(H = H_{d-1} = \langle v_0, \ldots, v_{d-2} \rangle\), we found that \(W\) is a semidirect product of \(V\) by \(H\) (Theorem 6.2.11). We also found that there is an isomorphism \(\phi\) from \(Z^{d+1}\) to \(Y\) taking \(\sum q_i e_i\) to \(\prod v_i^{q_i}\) (see Lemma 6.2.7). This being the case, each element of \(h\) induces a transformation \(\Delta_h\) of \(Z^{d+1}\), via \(\Delta_h u = (h(u))h^{-1}\phi^{-1}\), and this transformation may be expressed as a matrix, in fact a permutation matrix with some of the +1 entries changed to -1 (as was pointed out within the proof of Lemma 6.2.10).

Let \(M = \{\Delta_h : h \in H\}\). Then the set \(X = \{(v, \Delta) : v \in \mathbb{Z}^{d+1}, \Delta \in M\}\) with multiplication given by

\[
(v, \Delta)(v', \Delta') = (v + \Delta v', \Delta \Delta')
\]

is a group, isomorphic to \(W\) via the isomorphism

\[
(vh)v = (v\phi^{-1}, \Delta_h)
\]

for \(v \in V\) and \(h \in H\) (see Theorem 6.2.13).

Let \(\mathcal{M}\) be the universal polytope based on \(W\). To find combinatorially regular Euler incidence polytopes with Schl"afli Symbol \(\{4\|3\ldots3\|3\|4\)\), it suffices, by Lemma 6.2.3 (see the notes at the start of §6.2.2) to find sparse subgroups of \(W\), or equivalently, to find subgroups of \(X\) with no nontrivial non-sparse elements.
B.1 The Example

By Lemma 6.2.18, an element \((y, \Delta)\) of \(X\) is non sparse if and only if there exists \(x, w \in \mathbb{Z}^{d-1}\), where all the coordinates of \(x\) are 0 or \(\pm 1\), such that \((I - \Delta)w = u + x\). We define our first sparse subgroup as follows. Let \(k \geq 2\), and for \(1 \leq i \leq d - 2\), let \(a_i = (ke_{d-1}, 0)\). Further, let \(a_{d-1} = (ke_{d-1}, \Delta_{d-1})\). Then, let \(A = \langle a_1, a_2, \ldots, a_{d-2} \rangle\). Recalling that \(\Delta_{d-1}e_1 = -e_1\), and that \(\Delta_{d-1}e_i = e_i\) if \(i \geq 2\), note that

\[
\begin{align*}
a_{d-1}^2 &= (ke_{d-1}, \Delta_{d-1})^2 \\
&= (ke_{d-1} + \Delta_{d-1}ke_{d-1}, \Delta_{d-1}^2) \\
&= (2ke_{d-1}, I),
\end{align*}
\]

and for \(1 \leq i \leq d - 2\),

\[
\begin{align*}
a_i^2 &= (ke_{d-1}, I)^2 \\
&= (2ke_{d-1}, I),
\end{align*}
\]

whence \(A\) contains the group \(\{(a, I) : a \in 2k\mathbb{Z}^{d-1}\}\). It follows in particular that \(A\) has finite index in \(X\). The next two lemmas show that \(A\) is sparse.

**Lemma B.1.1:** \(A\) is a subset of \(T = \{ \sum k_i e_i, \Delta_{d-1} \in \mathbb{Z} : i \in \mathbb{Z} \}\).

**Proof:** (Outline) For any of the \(a_i\), it may be shown that \(a_i T \subseteq T\), and that \(a_i^{-1} T \subseteq T\) (so in fact \(a_i T \subseteq T\)). Also, \((0, I) \in T\). Thus, for any \(a - a_i^0 a_i^1 \cdots a_i^n \in A\), we have \(a \in (0, I) \subseteq a_i T = a_i^0 a_i^1 \cdots a_i^n T \subseteq T\). Thus \(a \in T\), and we conclude that \(A \subseteq T\) as required.

In fact it would be possible to prove that \(A = T\), but we shall not do so here.

**Lemma B.1.2:** \(A\) has no nontrivial non sparse elements.

**Proof:** Assume \(a \in A\) is non sparse. Then \(a \in T\) also, so \(T\) has a non sparse element, say \(a = (kz, \Delta_{d-1})\), where \(z = \sum z_i e_i\). Assume first that \(z_{d-1}\) is odd, so \(\Delta_{d-1}^2 = \Delta_{d-1}\). By Lemma 6.2.20 and 6.2.17, this will be non sparse if and only if there exists \(z^* \in \mathbb{Z}^{d-1}\), all of whose coordinates are 0 or \(\pm 1\), such that \((I + \Delta_{d-1})(kz + z^*) = 0\). Since \((I + \Delta_{d-1})e_i = 2e_i\) whenever \(i \geq 2\), if \(z\) is non sparse, then in particular \(2(kz_{d-1} + z^*_{d-1}) = 0\), so \(kz_{d-1} = -z^*_{d-1}\). Since \(k \geq 2\) and \(z^*_{d-1} = 0\) or \(\pm 1\), it must in fact be that \(z_{d-1} = z^*_{d-1} = 0\), which contradicts our assumption that \(z_{d-1}\) be odd.

Now, consider the case where \(z_{d-1}\) is even. This time, \(a = (kz, I)\), which will be non sparse if and only if it is equal to \((-z^*, I)\) for some \(z^*\) whose coordinates are all 0's and \(\pm 1\)'s (Lemmas 6.2.20 and 6.2.17 again). It follows, again since \(k \geq 2\), that \(z_i = z^*_i = 0\) for all \(i\), and so \(a = (0, I)\). Thus the only non sparse element of \(T\), and therefore of \(A\) (Lemma B.1.1), is the identity.

\(A\) is thus sparse, so yields a well-defined quotient polytope, with universal facets and vertex figures, and Schl"{a}fli Symbol \(\{4\}[3] \ldots [3]4\}. By Lemma 6.2.3, it will be Euler if and only if \(d\) is even. The polytope will also be combinatorially regular (Theorem 5.2.37). It will not be flag regular, however, since the group \(A\) is not normal in \(W\) (see Theorem 5.2.49). For example,

\[
(0, \Delta_{d-2})^{-1}(ke_{d-1}, \Delta_{d-1})(0, \Delta_{d-2}) = (0, \Delta_{d-2})(ke_{d-1}, \Delta_{d-1} \Delta_{d-2}) = (\Delta_{d-2}ke_{d-1}, \Delta_{d-2} \Delta_{d-1} \Delta_{d-2}) = (ke_{d-2} + 0e_{d-1}, \Delta_{d-1} \Delta_{d-2} e_{d-2}),
\]

which is not an element of \(T\), hence nor of \(A\), since \(\Delta_{d-1} \Delta_{d-2} e_{d-2} \notin \{I, \Delta_{d-1}\}\).

Finally, the polytope is not a lattice.

**Theorem B.1.3:** There exist combinatorially regular Euler incidence polytopes with Schl"{a}fli Symbol \(\{4\}[3] \ldots [3]4\} which are not lattices.

**Proof:** If a polytope \(\mathcal{M}\) is a lattice, then it is a quotient polytope of \(\mathcal{M}\) by some sparse subgroup \(B\) of \(X\) of the form \(\{(a, I) : a \in K\}\), where \(K\) was the subgroup of \(\mathbb{Z}^{d-1}\) from which the lattice was defined (see Theorem 6.2.34). The group \(A\) we have been examining cannot be of this form, or even conjugate to any group to any group of this form, since it contains the element \((ke_{d-1}, \Delta_{d-1})\), and so \(\mathcal{M}/A\) is a polytope of the required form.

In terms of the \(s_i\) and the \(v_i\), we have \(A^{-1} = \langle v_1^k, v_2^k, \ldots, v_{d-2}^k, v_{d-1}^k, v_1, v_2, \ldots, v_{d-2} \rangle\).

B.2 The Non-Existence

Let \(W = W_{\{4\}[3] \ldots [3]4\}} = V.H\), where \(V = \langle v_1, \ldots, v_{d-4} \rangle\) and \(H = \langle s_0, \ldots, s_{d-2} \rangle\). Recall that \(W \cong X\) via the isomorphism \(\psi\) satisfying \(\psi(v_i) = (v_i, I)\), where \(\phi\) is the isomorphism from \(\mathbb{Z}^{d-1}\) to \(V\) given by \((\sum u_i e_i) \phi = [u_1^e]\) (Theorem 6.2.12).
Theorem B.2.1: Let $A$ be a sparse subgroup of $W = V.H$. If $A$ is normal in $W$, then $A$ is a subgroup of $V$.

Proof: We show that $A\psi$ has no elements of the form $(v, \Delta)$, where $\Delta \neq I$, as follows. If $a = (v, \Delta)$ is an element of a normal subgroup of $W$, then so is $(-e_i, I)^{-1}a(-e_i, I)$, and hence so is $(-e_i, I)^{-1}a(-e_i, I)^{-1}$, for any $i$. But

$$(e_i, I)^{-1}a(-e_i, I)^{-1} = (e_i, I)^{-1}(e_i, \Delta)(-e_i, I)(v, \Delta)^{-1}$$

$$= (e_i + v, \Delta)(-e_i, I)^{-1}$$

$$= (e_i + v + \Delta(-e_i - \Delta^{-1}v), \Delta^{-1})$$

$$= (v - v + (I - \Delta)e_i, I),$$

which equals $(I - \Delta)e_i$. If $A\psi$ is to be sparse, then this cannot be non-sparse unless it is zero. However, by Lemma 6.2.18, it will be non-sparse if all the coordinates of $(I - \Delta)e_i$ are $0$ or $\pm 1$. It follows that if $A\psi$ is to be sparse, then for all $i$, either $\Delta(e_i) = -e_i$ (in which case $(I - \Delta)e_i = 2e_i$) or $\Delta(e_i) = +e_i$, (in which case $(I - \Delta)e_i = 0$). Now, let $\Delta$ and $i$ be such that $\Delta(e_i) = -e_i$ (such an $i$ will exist unless $\Delta = I$). It follows that $(2e_i, I) \in A\psi$, that is, $v^\psi \in A$. Since in fact all the $v_i$ are conjugate (see Lemma 6.2.4), and since $A$, being normal, is closed under conjugation, we deduce that $(2e_i, I) \in A\psi$ for all $j$, so in fact $(2w, I) \in A\psi$ for all $w \in \mathbb{Z}^{d-1}$. Recall also that $(v, \Delta) \in A\psi$, and write $v = 2u + z$, where all the coordinates of $z$ are $0$ or $1$. Then, since $(-2u, I) \in A\psi$, we have $(-2u, I)(v, \Delta) = (z, \Delta) \in A\psi$. Then, Theorem 6.2.20 tells us that $(z, \Delta)$ is non-sparse, since $(z + (-z), \Delta)$ has finite order in $X$, so if $\Delta \neq I$, we have found a nontrivial non-sparse element of $A\psi$. It follows that $A\psi$ must be a subgroup of $\{(v, I) : v \in \mathbb{Z}^{d-1}\} = V\psi$, whence $A$ is a subgroup of $V$, as required.

It follows (by Theorems 6.2.34 and 5.2.49) that the only flag regular Euler polytopes with Schläfi Symbol $[4|3] \ldots [3|4]$ are fact lattices. It is also possible to classify the subgroups of $V$ which are normal in $W$, and so determine exactly which lattices are flag regular (for not all are).

Lemma B.2.2: Let $K$ be a subgroup of $\mathbb{Z}^{d-1}$, and let $A$ be the lattice defined from it. Then $A\psi$ is flag regular if and only if $K$ is closed under sign changes and permutations of its coordinates.

Proof: (Outline) By Theorem 6.2.34, $A\psi$ is isomorphic to a quotient polytope $\mathcal{A}/A$, where $A \leq V$ and $A = K_\psi$. Theorem 5.2.49 then tells us that $A\psi$ is flag regular if and only if $A$ is normal in $W$, which will be the case if and only if $v_1^\psi A v_1 = A$ and $s_j A s_j = A$ for all the generators $v_i$ of $V$ and all the generators $s_j$ of $H = H_d$. The former equality will always hold, since $v_1 \in V$ and $A \leq V$, and $V$ is an Abelian group (see Theorem 6.2.5). Thus $A\psi$ is flag regular if and only if $s_j A s_j = A$ for all $s_j \in \{e_0, e_1, \ldots, e_{d-2}\}$.

Assume first that this is in fact the case, so in fact $h A h^{-1} = A$ for all $h \in H$. It follows that for all $v \in K$, we have $v \psi \in A$, so $h(v \psi)h^{-1} \in A$, whence in fact $\Delta v = (h(v \psi)h^{-1})^{-1} \in K$. Thus $K$ is closed under the operation of elements of $M = \{\Delta : \hat{h} \in H\}$. It may be shown that the elements of $M$ are just those matrices which induce permutations and sign changes of the coordinates of the vectors they operate on, so $K$ is closed under such changes, as required.

Conversely, let us now assume that $K$ is closed under permutations and sign changes of the coordinates of its elements, that is, for any $\Delta \in M$ and any $v \in K$ we have $\Delta v \in K$. In particular, $\Delta v \in K$ for each generator $s_j$ of $H$. But any $a \in A$ may be written in the form $a = v \psi$ for some $v \in K$, so $s_j a s_j = s_j (v \psi) s_j^{-1} = (\Delta v) \psi \in K \psi = A$. It follows that $s_j A s_j \subseteq A$, from which it is easy to deduce that $A = s_j A s_j$ as required.

We now give three examples of subgroups of $\mathbb{Z}^{d-1}$ which are closed under such operations, and hence yield flag regular polytopes. Let $k \geq 2$.

Example: (i) Vertex-Centred Cubic: Let $K = k \mathbb{Z}^{d-1} = \{k e_0, k e_1, \ldots, k e_{d-2}\}$. It is easy to see that this set is closed under permutations and sign changes of its elements' coordinates. We require $k \geq 2$ to ensure that $K$ satisfies condition $K_1$, and then we have an example of a strongly regular lattice.

Example: (ii) Body-Centred Cubic: Let $K = k(2 \mathbb{Z}^{d-1} + \{\sum e_i\})$. This may be written as $2k e_0, 2k e_1, \ldots, 2k e_{d-2} + k e_0 + k e_1 + \ldots + k e_{d-2}$, and may be regarded as the union of the two cosets $2k \mathbb{Z}^{d-1}$ and $k \mathbb{Z}^{d-1}$. It may be shown that each of these cosets is closed under permutations and sign changes of its elements' coordinates, and so therefore will be the whole set. Again, $K_1$ is satisfied if $k \geq 2$, and we have a strongly regular lattice.

Example: (iii) Face-Centred Cubic: Let $K = k(2e_0, e_1, e_2, \ldots, e_{d-1})$. If we define the weight of $\sum v_i e_i$ to be $\sum v_i$, this is $k$ times the set of vectors of even weight. Module 2, the weight of a vector is not affected by the operation of permuting its coordinates, nor by that of changing the signs of some of them. From this observation, it follows without much work that this example also yields a flag regular lattice when $k \geq 2$.

We then have the following theorem.
Theorem B.2.3: The only flag regular d-lattices are the universal lattice, and those determined by the subgroups \( K \) described in examples (i), (ii) and (iii) above.

Proof. (Outline) Let \( F \neq \{0\} \) be a subgroup of \( \mathbb{Z}^{d-1} \) which is closed under permutations and sign changes of its elements' coordinates, and let \( \mathbf{v} \in \mathbb{K} \) be nonzero. Writing \( \mathbf{v} = \sum_i v_i \mathbf{e}_i \), we may deduce that for each \( i \), the vector \( 2v_i \mathbf{e}_i \) is an element of \( K \), by subtracting from \( \mathbf{v} \) the vector obtained by changing the sign of the \( i \)-th coordinate of \( \mathbf{v} \). This being the case, we in fact have \( 2v_i \mathbf{e}_i \in K \) for all \( i \) and \( j \), whence in fact \( 2k_i \mathbf{e}_j \in K \) for all \( j \), where \( k_i \) is the greatest common divisor of \( \{v_i\} \). The set \( \{k_i : \mathbf{w} \in K \setminus \{0\}\} \) will have a least (positive) element, call it \( k \). It may be shown that \( k \) divides \( k_i \) for all \( \mathbf{w} \in K \setminus \{0\} \) (by considering the elements \( k \mathbf{e}_1 \) and \( k \mathbf{e}_1 \mathbf{e}_1 \) of \( K \) and showing that \( k_i \mathbf{e}_1 \mathbf{e}_i \in K \) where \( l \) is the least common divisor of \( k_i \) and \( k \)). Thus in fact, any vector \( \mathbf{v} \) may be written \( k \mathbf{u} \) for some \( \mathbf{u} \in \mathbb{Z}^{d-1} \). Denote by \( S_k \) the set of all vectors whose coordinates are all \( 0 \)'s and \( 1 \)'s. We may then write \( \mathbf{u} = 2u_i + z \) for some \( z \in \mathbb{Z}^d \). Note then that \( k \mathbf{u} \in K \). In fact, if \( L \) is the set \( k \mathbb{Z} \mathbb{Z}^{d-1} \), we have \( k \mathbb{Z}^{d-1} \in L \). Let us examine the structure of \( L \). Certainly \( 0 \in L \), since \( 0 \in K \) and \( 0 \in S_k \). Also, \( L \) has nonzero elements, for if it does not, we can show this to contradict our choice of \( k \). If \( k \sum e_i \) is the only nonzero element, then \( L = \{0, k \sum e_i \} \), then \( K \) is body-centred cubic, since \( K = 2k \mathbb{Z}^{d-1} + L = (2k \mathbb{Z}^{d-1}) \cup (k \sum e_i + 2k \mathbb{Z}^{d-1}) \) (see example (ii), above). Now, let \( k \mathbf{u} \) be an element of \( L \) with both a zero and a nonzero coordinate, say the \( i \)-th and \( j \)-th coordinates respectively. Let \( \mathbf{z}' \) be the vector obtained by exchanging these coordinates of \( \mathbf{u} \). Since \( \mathbf{z} \in K \), we have \( \mathbf{z}' \) and hence \( \mathbf{z}' - \mathbf{z} \) are elements of \( K \). Suppose \( \mathbf{z}' - \mathbf{z} = k(\mathbf{e}_i + \mathbf{e}_j) \), so in fact \( k(e_i + e_j) \) are elements of \( K \), and therefore \( k(e_i + e_j) \in K \) for any \( i \) and \( j \). Suppose that \( L \) contains a vector with an odd number of nonzero coordinates. It quickly follows from this that \( k \mathbf{e}_i \in L \) for some \( i \), and so in fact \( k \mathbf{e}_i \in L \) for all \( k \) (for example, if \( k(e_i + e_j + e_k) \in K \), it follows that \( k \mathbf{e}_i = k(e_i + e_j + e_k) - k(e_j + e_k) \in K \). This tells us that \( K \) is of the form of example (i), that is, vertex-centred cubic. Likewise, if \( L \) contains no such element, \( K \) may be shown to be face-centred cubic. We have considered all cases, and so these three are the only possibilities.

It would appear then that flag regularity is a rare condition amongst the combinatorially regular Euler polytopes with Schl"{a}fli Symbol \( \{4\}3 \ldots \{3\}4 \).

APPENDIX C

CAYLEY Code

This appendix contains three programs for CAYLEY v3.8.3, which performs certain searches for sparse subgroups in certain Coxeter groups (see §6.3.3 for more details). The code is included here with explanatory comments. For more information on the CAYLEY language, refer to [4].

C.1 The Naive Algorithm

The following program searches for subgroups of the Coxeter group \( W[1[4][3]] \), and checks each one to see if it is sparse.

```
set workspace=1;

The set workspace=1; command merely ensures that CAYLEY has enough memory to handle this particular group. For larger groups, a larger workspace may be required. For smaller groups this command may be omitted entirely.

W:=free(s0,s1,s2,s3);
W.relations:=s0^2=s1^2=s2^2=s3^2=(s0*s2)^2=(s0*s3)^2=(s1*s3)^2,
             (s0*s1)^3,(s1*s2)^4,(s2*s3)^3;
```

This is where the Coxeter group itself is defined, via its generators and relations.

If a different Coxeter group is to be examined, these lines must be edited.

```
H0:=<s1,s2,s3>;
H3:=<s0,s1,s2>;
K:=H0^H3;
KK:=K;
for each g in W do
    KK:=KK join (K^g);
end;
```

The set \( KK \) now contains all the nonsparse elements of \( W \). This piece of code will only need to be changed if one wishes to change the dimension of the polytopes being sought.

```
S:=subgroups(W);
I:=<identity of W>;
A:=empty;
```
for j = 1 to length(S) do
    if (S[j] meet KK) eq 1 then
        A:=append(A, S[j]);
    end;
end;

The subgroups command finds all the subgroups of \( W \). For each subgroup \( S[j] \) in turn, the program intersects (or "meets") it with \( KK \), and checks whether or not this intersection is the identity. If it is, the subgroup is sparse, so the program stores it in the array \( A \).

print 'List of subgroups which produce quotient polytopes';
print 'Also some additional information';
print 'There are', length(A), 'subgroups listed.';
for j = 1 to length(A) do
    print ' ';
    print 'A[',j,'] ', A[j];
end;

The array \( A \) contains our list of sparse subgroups, so we display each one in turn. In the above lines, the print command displays the order of each group, as well as a list of its generators.

if invariant(W, A[j]) then
    print 'A is Normal in W, hence N/A is flag regular';
end;

The invariant command checks whether or not \( A[j] \) is a normal subgroup of \( W \). Theorem 5.2.49 tells us that if it is, the quotient polytope corresponding to it will be flag regular.

print 'Normaliser: (Aut(N/A) is isomorphic to N/A)';
print normaliser(W, A[j]);
print 'Core: (Gamma(N/A) is isomorphic to W/C)';
print core(W, A[j]);
end;
save 'w343';
quit;

These last few statements print out the normaliser and the core of each sparse subgroup discovered, before closing the for loop, saving our information, and quitting CAYLEY. The core and normaliser are interesting for the reasons given in the program (see Theorems 5.2.48 and 5.2.44).

That completes the exposition of the first program. In actual fact, its algorithm is not terribly useful, not only because (as mentioned in §6.3.3) the subgroups command of CAYLEY v3.8.3 cannot handle certain groups. In Theorem 5.2.46, it was shown that two quotient polytopes \( N/A \) and \( N/B \) are isomorphic if and only if \( A \) and \( B \) are conjugate. The above program in fact generates eight distinct index three subgroups of \( W_{343} \), and it is not immediately obvious whether or not they are all conjugate. Ideally, we would instruct the computer to give us only one representative from each conjugacy class of subgroups. Fortunately, CAYLEY's low index subgroups command enables us to do exactly that. It was pointed out by one of the examiners of this thesis, that CAYLEY's subgroup lattice command also returns just a single member of each conjugacy class of subgroups, and so this command may be used in place of the low index subgroups command in the next section's algorithm. This was done during the revision of the thesis, to compare the utility of the two commands.

It was found that subgroup lattice resulted in a faster algorithm than low index subgroups, but that it still failed to classify the subgroups of the Coxeter group \( W_{343} \), since CAYLEY v3.8.3 does not have the 'soluble residuum' of this group catalogued.

Hence, to the subgroup lattice command enables us to more efficiently classify the combinatorially regular Euler incidence polytopes with Schlafli Symbol (3|4|3), but we still require the low index subgroups command, along with the theory discussed in the next section, to classify those with Schlafli Symbol (9|9|9).

C.2 Refining the Search

This program uses the low index subgroups command to find one representative from each conjugacy class of sparse subgroups of \( W_{343} \).

set workspace=2;
W:=W343(s0,s1,s2,s3);
W:=relations:s0^2,s1^2,s2^2,s3^2,(s0*s2)^2,(s0*s3)^2,(s1*s3)^2,
       (s0*s1)^3,(s1*s2)^3,(s2*s3)^5;
c:=order(W);

This time, having defined the group, we calculate its order. If the group \( W \) is infinite, this operation will fail, and the program will halt. The order of the group is useful later on.

H0:=<s1,s2,s3>
H3:=<s0,s1,s2>
K:=H0^H3
KK:=K
for each j in W do
    KK:=KK join (R_{j});
end;
SS:=seq(seq(o/30,o/30),seq(o/24,o/24),seq(o/20,o/20),
    seq(o/15,o/15),seq(o/12,o/12),seq(o/10,o/10),seq(o/8,o/8),
    seq(o/6,o/6),seq(o/5,o/5),seq(o/4,o/4),seq(o/3,o/3),
    seq(o/2,o/2),seq(o,0));

In the proof of Theorem 6.3.10, it was noted that the order of a sparse subgroup of W_{2} must be a factor of 120, and must be less than or equal to 30. These figures were obtained by considering the number of vertices of the quotient polytope (which by Theorem 5.2.41 is \frac{1}{120} [W_{2}], whence |A| divides \frac{1}{120} [W_{2}]), and noting that a polytope has at least as many vertices as its facets (each of which, by 5.2.10, has \frac{1}{120} [H_{w-1}] vertices, whence |A| \leq \frac{1}{120} [H_{w-1}]). Knowing the possible orders of our sparse subgroups, their indices are calculated by dividing them into the order of the Coxeter group. These indices are used below, by the long index subgroups command, which receives them as a CAYLEY sequence, in the form given above.

An alternative method of calculating the allowable indices is could be derived from Theorem 6.3.14.

S:=low index subgroups(W,0,SS; al:tc, genset=1);

The low index subgroups is quite a versatile one. The above format instructs it to calculate those subgroups of W with indices in the sequence SS, using the so-called Todd-Coxeter algorithm (al:tc), and to return a set of generators for one member of each conjugacy class of subgroups (genset=1). For more information on this command, the reader is referred to [4].

The remainder of the program is not materially different from the previous one. The main difference is that the S is now a sequence of sets of generators for subgroups of W, instead of a sequence of the subgroups themselves. Hence we need to use expressions such as <S[j]> meet KK instead of S[j] meet KK to calculate the intersection between the group and the set of non-sparse elements.

I:=<identity of W>;
A:=empty;
for j = 1 to length(S) do
    if ((<S[j]> meet KK) eq I) then
        A:=append(A,<S[j]>);
    end;
end;
end;

The section of the program which displays the results of the search and quits CAYLEY is exactly the same as for the previous algorithm, so we omit it here.

C.3 An Infinite Search

The program in this section finds sparse normal subgroups of the Coxeter group. The algorithm is based on Lemma 6.3.11, and is useful in that it can function even when the Coxeter group being examined is infinite.

It takes low index subgroups K = K of W = W, and finds the homomorphism f = f from W to the set of permutations of cosets of K, defined via f(x) : K \rightarrow K\cdot w, and prints out those K for which the kernel of f is sparse. Note that the kernel of f is the core of K, since x \in ker f if and only if K\cdot w = K\cdot w for all w \in W, which occurs if and only if w\cdot w^{-1} \in K for all w \in W, that is, x = \cap_{w} w^{-1} K\cdot w. The disadvantages of the algorithm are primarily that it cannot classify the sparse subgroups (although this is no fault of the algorithm itself), secondarily that it does not find non-normal sparse subgroups, and that it appears to find each of its examples several times over. The code, with explanatory notes, follows.

W:=free(s0,s1,s2,s3);
W.relations:o^2,s1^2,s2^2,s3^2, (s0*s2)^2,(s1*s3)^2,(s0*s3)^2,
    (s0*s1)^4,(s1*s2)^3,(s2*s3)^5;

Again we first define the group. Note that W_{430} is an infinite group (see Table A.2).

A:=empty;
i:=0;
while (length(A) \leq 7) do
    W:=low index subgroups(W,0,seq(i,i);genset=1);
    if (length(K) ne 0) then
        print 'Found',length(K),'subgroups of index',i,
    end;
    i:=i+1;
end;

72 Except for the save statement.
The variable \( i \) is the index of the subgroups \( K \) being examined. Having found an array \( K \) of index \( i \) subgroups, we check them one by one.

```plaintext
for \( j = 1 \) to length(\( K \)) do
  \( f, \text{im}, \ker = \text{coset homomorphism}(W, \langle k[j] \rangle) \);

The command \( f, \text{im}, \ker = \text{coset homomorphism}(W, \langle k[j] \rangle) \) sets up the homomorphism \( f \) from \( W \) to the group of permutations of cosets of \( \langle k[j] \rangle \). It stores the homomorphism in \( f \), its image in \( \text{im} \), and if possible, its kernel in \( \ker \).

The kernel of \( f \), which is the core of \( \langle k[j] \rangle \) in \( W \), is the group we will be checking for sparseness. Unfortunately, with infinite groups, it cannot be calculated directly, and the variable \( \ker \) is left unassigned.

\[ f(0) = \langle f(a1), f(a2), f(a3) \rangle; \]
\[ f(3) = \langle f(a0), f(a1), f(a2) \rangle; \]
\[ f(0) = f(0) \mod f(3); \]
if \( \text{order}(f(3)) \equiv 48 \) and \( \text{order}(f(0)) \equiv 120 \)
  and \( \text{order}(f(0)) \equiv 6 \) then

Here, we calculate the subgroups \( f(0) = f(H_{d-1}) \), \( f(0) = f(H_0) \) and \( f(0) = f(0) \mod f(3) \) of \( f(W) \), and compare their orders with the (known) orders of \( H_{d-1} \) (of order 48), \( H_0 \) (of order 120) and \( H_0 \cap H_{d-1} = H_0, d-1 \) (of order 6). If it is desired to analyze a different Coxeter group, the 48, 120 and 6 above would need to be changed. The appropriate values may either be gleaned from the tables in Appendix A (in particular Table A.7), or (if \( d = 4 \)) from Table 3.6.

```plaintext
A := append(A, K[j]);
print 'Core of f, j, is sparse, of index', order(\text{im});
```

Lemma 6.3.11 tells us that \( \ker \) is sparse if and only if \( |f(H_{d-1})| = |H_{d-1}| \), \( |f(H_0)| = |H_0| \), and \( |f(H_0) \cap f(H_{d-1})| = |H_0 \cap H_{d-1}| \). We therefore store the group we have found, and print a little information about it.

```plaintext
end;
end;
```

Having found several groups with sparse cores, we print out generating sets for them all, save the information for future reference, and quit.

This particular program does not give much by way of output, but it serves to illustrate the algorithm, and can easily be adapted to become more informative.

\[ \text{Bibliography} \]


[45] L. Schlaffi, "On The Multiple Integral \( \int \int \cdots \int dx \, dy \cdots dz \), Whose Limits Are \( p_1 = a_1 \xi + b_1 \eta + \cdots + h_1 z > 0, p_2 > 0, \ldots, p_n > 0 \); And \( z^2 + y^2 + \cdots + x^2 < 1 \)", Quarterly Journal Of Pure And Applied Mathematics 2 (1858) 260–301, 3 (1860) 54–68, 97–108.


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